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ON DERIVATION OF EULER-LAGRANGE EQUATIONS FOR AREA-PRESERVING ENERGY-MINIMIZERS

NIRMALENDU CHAUDHURI AND ARAM L. KARAKHANYAN

ABSTRACT. Derivation of the system of Euler-Lagrange equations for volume-preserving, energy-minimizing $W^{1,2}$ -deformations and establishing the existence of an integrable pressure associated with the volume constraint is an open problem. In this article we consider this problem for the case $n = 2$. For an area-preserving, elastic energy-minimizing deformation \mathbf{u} with $|\nabla \mathbf{u}|^2$ in the Hardy space \mathcal{H}^1 , we establish an explicit representation of the associated pressure $p \in L^1_{\text{loc}}$ via Calderón-Zygmund type singular integral operators. We then derive the system of Euler-Lagrange equations for $W^{1,r}_{\text{loc}}(\Omega, \mathbb{R}^2)$, $r \geq 3$ area-preserving local minimizers and prove partial regularity under smallness assumption on pressure.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a smooth, bounded and simply connected domain. The classical Stokes problem in hydrodynamics involves minimizing the potential energy

$$I[\mathbf{w}] := \int_{\Omega} \frac{1}{2} |\nabla \mathbf{w}|^2 + \langle \mathbf{f}, \mathbf{w} \rangle$$

for all divergence free velocity fields $\mathbf{w} \in W_0^{1,2}(\Omega, \mathbb{R}^n)$ for a given force field $\mathbf{f} \in L^2(\Omega, \mathbb{R}^n)$. It follows that the problem has a unique incompressible minimizer $\mathbf{u} \in W_0^{1,2}(\Omega, \mathbb{R}^n)$. The linear incompressible constraint $\text{div } \mathbf{u} = 0$ ensures the existence of a hydrostatic pressure $p \in L^2_{\text{loc}}(\Omega)$ and the pair (\mathbf{u}, p) satisfies the following system of Euler-Lagrange equations

$$(1.1) \quad \begin{cases} \Delta \mathbf{u}(x) = \nabla p(x) - \mathbf{f}(x), & \text{in } \Omega \\ \text{div } \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

in the weak sense, see for example [Ev 98, pp 472-474]. The regularity of (\mathbf{u}, p) is well understood and detailed analysis can be found in [Ga 94, Chapter IV].

An analogue of this problem appears in nonlinear elasticity. In such context, \mathbf{w} represents the displacement of an incompressible elastic body which has the rest configuration $\Omega \subset \mathbb{R}^n$. For incompressible neo-Hookean materials [Ba 77], [TO 81], [Og 84], such as vulcanized rubber, in the equilibrium state, one is interested in

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minimizing the elastic energy

$$(1.2) \quad E[\mathbf{w}] := \int_{\Omega} L(\nabla \mathbf{w}(x)) dx,$$

for incompressible $W^{1,2}$ -deformations $\mathbf{w} : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, subject to its own boundary condition and corresponding to a given bulk energy $L : \mathbb{M}^{n \times n} \rightarrow \mathbb{R}$. The simplest L is the Dirichlet energy, given by $L(X) = \frac{1}{2}|X|^2 := \frac{1}{2}\text{tr}(X^t X)$. Let us denote the admissible set of deformations

$$(1.3) \quad \mathcal{A} := \{ \mathbf{w} \in W^{1,2}(\Omega, \mathbb{R}^n) : \text{cof } \nabla \mathbf{w} \in L^2(\Omega, \mathbb{M}^{n \times n}), \det \nabla \mathbf{w} = 1 \text{ a.e.} \},$$

where $W^{k,p}$ denotes the usual *Sobolev spaces* [Ad 75] and $\text{cof } P$ is the *cofactor matrix*, whose ij -th entries is the determinant of $(n-1) \times (n-1)$ submatrix obtained by deleting the i -th row and the j -th column from the $n \times n$ matrix P . We call $\mathbf{u} \in \mathcal{A}$ to be a *local minimizer* of $E[\cdot]$ if and only if

$$(1.4) \quad E[\mathbf{u}] \leq E[\mathbf{w}] \quad \text{for all } \mathbf{w} \in \mathcal{A} \text{ and } \text{supp}(\mathbf{w} - \mathbf{u}) \subset \Omega.$$

Under the hypothesis that the energy density L is quasiconvex [Mo 52] and have quadratic growth, using direct methods in the calculus of variations together with weak continuity of determinant, Ball [Ba 77] proved the existence of local minimizers $\mathbf{u} \in \mathcal{A}$ of the energy $E[\cdot]$. However the derivation of the system of Euler-Lagrange equations for such minimizers and proving the existence of an integrable pressure associated with the volume constraint is a challenging open problem.

We will be concerned in this paper with the derivation of Euler-Lagrange equations for the area-preserving local minimizers and the existence of a locally integrable pressure in the planar case $n = 2$. Our main results are as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth, simply connected and bounded domain. Assume that $\mathbf{u} \in W_{\text{loc}}^{1,r}(\Omega, \mathbb{R}^2) \cap \mathcal{A} = \{ \mathbf{w} \in W^{1,2}(\Omega, \mathbb{R}^2) : \det \nabla \mathbf{w}(x) = 1, \text{ a.e. in } \Omega \}$, for some $r \geq 3$ is a local minimizer of $E[\cdot]$. Then there exists a scalar function $q \in L_{\text{loc}}^{r/2}(\mathbf{u}(\Omega))$ such that the pair (\mathbf{u}, p) satisfies*

$$(1.5) \quad \int_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx = \int_{\Omega} p(x) \text{cof}(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx,$$

for all $\phi \in C_0^1(\Omega, \mathbb{R}^2)$, where $p := q \circ \mathbf{u} \in L_{\text{loc}}^{r/2}(\Omega)$ and $A : B := \sum_{ij} a_{ij} b_{ij}$, for $A, B \in \mathbb{M}^{2 \times 2}$. In other words, the pair (\mathbf{u}, p) satisfies the system of Euler-Lagrange equations

$$(1.6) \quad \text{div} [DL(\nabla \mathbf{u}(x)) - p(x) \text{cof}(\nabla \mathbf{u}(x))] = 0 \quad \text{in } \Omega,$$

in the sense of distribution, where the divergence is taken in each rows.

Under the stronger hypothesis that the local minimizers of $E[\cdot]$ are classical, namely in Sobolev spaces $W^{2,r}$, $r > 2$, Tallec and Oden [TO 81] established the above system of equations. Whereas, our approach to establish the existence of a pressure $p \in L^{r/2}$ associated with the local minimizer \mathbf{u} , we only require $\mathbf{u} \in W_{\text{loc}}^{1,r}$, $r > 2$ and to derive the system of equilibrium equations (1.6) for (\mathbf{u}, p) in Ω we need $r \geq 3$.

Recall that for $f \in L^1(\mathbb{R}^n)$ the maximal function Mf is defined by

$$(Mf)(x) := \sup_{\rho > 0} \frac{1}{\text{meas } B_{\rho}(x)} \int_{B_{\rho}(x)} |f(y)| dy.$$

From the classical results in singular integrals due to Stein [St 69, Theorem 1] or [St 70, pp 23], it follows that if $f \in L^1(\mathbb{R}^n)$ and is supported on a finite ball $B \subset \mathbb{R}^n$, then $Mf \in L^1(B)$ is and only if

$$\begin{aligned} f \in L \log L &:= \left\{ g : B \rightarrow \mathbb{R} : \int_B |g| \log^+ |g| dx < \infty \right\} \\ &\equiv \left\{ g : B \rightarrow \mathbb{R} : \int_B |g| \log(2 + |g|) dx < \infty \right\}, \end{aligned}$$

where $\log^+ |x| = 0$ for $0 < |x| \leq 1$ and $\log^+ |x| = \log |x|$ for $|x| > 1$. A standard result states that a positive function f is in the *Hardy space* \mathcal{H}^1 (the pre dual of BMO) if and only if $f \in L \log^+ L$. Notice that without any further higher integrability assumption on $\nabla \mathbf{u}$, we cannot ensure integrability of the maximal function $M|\nabla \mathbf{u}|^2$. However, under the additional assumption that $M|\nabla \mathbf{u}|^2$ is integrable, which is equivalent to $|\nabla \mathbf{u}|^2 \in \mathcal{H}^1$, we prove that the pressure q on the deformed domain $\mathbf{u}(\Omega)$ is locally integrable and (\mathbf{u}, q) satisfies the same system of differential equations a very weak sense. More precisely, we prove the following theorem.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a smooth, simply connected, bounded domain. Assume that $\mathbf{u} \in \mathcal{A}$ is a local minimizer of $E[\cdot]$ such that $|\nabla \mathbf{u}|^2 \in \mathcal{H}_{\text{loc}}^1(\Omega)$. Then there exists $q \in L_{\text{loc}}^1(\mathbf{u}(\Omega))$ such that the pair (\mathbf{u}, q) satisfies the integral identity*

$$(1.7) \quad \int_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v} \circ \mathbf{u}) dx = \int_{\mathbf{u}(\Omega)} q(z) \operatorname{div} \mathbf{v}(z) dz$$

for all $\mathbf{v} \in C_0^\infty(\mathbf{u}(\Omega), \mathbb{R}^2)$.

The proof of Theorem 1.2 is quite delicate. The main ideas in our proof are to localize the *mollified pressure* on the deformed domain $\mathbf{u}(\Omega)$, its explicit representation using Green's function of the unit disc in \mathbb{R}^2 and finding its uniform bound by using Calderón-Zygmund estimate [CZ 52]. Finally we show that the pressure on $\mathbf{u}(\Omega)$ is locally represented as the sum of certain singular integral operators of $|\nabla \mathbf{u}|^2$ involving Calderón-Zygmund type kernels (see equation (4.17) in Section 4) [CZ 52].

Theorem 1.3. [CZ 52, **Calderón-Zygmund Theorem**] *Let $f \in L \log^+ L$ and let Γ be a C^1 function on $\mathbb{R}^n \setminus \{0\}$ homogeneous of degree 0 with mean value 0 over the unit sphere \mathbb{S}^{n-1} , that is*

$$(1.8) \quad \int_{\mathbb{S}^{n-1}} \Gamma(x) dS(x) = 0.$$

Then the function defined as

$$(1.9) \quad f^*(x) := \lim_{\delta \rightarrow 0} \int_{|x-y| \geq \delta} \frac{\Gamma(x-y)}{|x-y|^n} f(y) dy$$

exists a.e. and integrable. Furthermore,

$$(1.10) \quad \int_K |f^*| dy \leq C \int_{\mathbb{R}^n} |f| \left(1 + \log^+ \left((\operatorname{meas} K)^{\frac{n+1}{n}} |f| \right) \right) dy + C(\operatorname{meas} K)^{-\frac{1}{n}},$$

for all measurable subset K of \mathbb{R}^n with finite measure.

For $n = 2$, through a series of papers, Bauman, Owen and Phillips [BOP 91], [BOP 91a], [BOP 92] proved that any $W^{2,r}$, $r > 2$ solutions of (1.6) are smooth solutions. In 1999, Evans and Gariepy [EG 99] proved that any *non-degenerate*, Lipschitz area-preserving

local minimizers of $E[\cdot]$ are $C^{1,\alpha}(\Omega_0)$, for some $0 < \alpha < 1$ for a dense open subset $\Omega_0 \subset \Omega$. However, as a consequence of the Euler-Lagrange equations (1.6) together with the standard elliptic estimates [GM 79] we prove the following theorem.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and $L : \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$ be a smooth and uniformly convex function having quadratic growth. Assume that $\mathbf{u} \in \mathcal{A} \cap W_{\text{loc}}^{1,3}(\Omega, \mathbb{R}^2)$ be a local minimizer of $E[\cdot]$ and $q(z) \in C^\alpha$ for some positive α . Then \mathbf{u} has Hölder continuous first derivatives in subdomain Ω_0 . Moreover*

$$|\Omega \setminus \Omega_0| = 0.$$

In a forthcoming paper [CHK 08] we will discuss the regularity of $W_{\text{loc}}^{1,r}$, $r > 2$ - area-preserving local minimizers and the derivation of system of Euler-Lagrange equations for the case $n \geq 3$.

2. THE FIRST VARIATION OF ENERGY

In the study of regularity of finite energy deformations, Šverák [Sv 88] proved that for any $W^{1,n}$ -deformation \mathbf{w} with $\det \nabla \mathbf{w}(x) > 0$, a.e., there exists a continuous function ω on \mathbb{R} with $\omega(0) = 0$ such that

$$|\mathbf{w}(x) - \mathbf{w}(y)| \leq \omega(|x - y|), \quad \text{for any } x, y \in \Omega \subset \subset \mathbb{R}^n.$$

In connection to the study of quasi-regular maps for $n = 2$, Iwaniec and Šverák [IS 93] proved that any $W^{1,2}$ -deformation \mathbf{w} with the *distortion* function $K(\cdot, \mathbf{w}) := |\nabla \mathbf{w}(\cdot)|^2 / \det \nabla \mathbf{w}(\cdot)$ being integrable, \mathbf{w} is a homeomorphism. Thus in particular, area-preserving $W^{1,2}$ -deformations in the plane are continuous and open maps. For $n \geq 3$, it is still unknown whether a map $\mathbf{u} \in \mathcal{A}$ is a homeomorphism.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a smooth bounded domain. Let $L : \mathbb{M}^{n \times n} \rightarrow \mathbb{R}$ be a smooth function and $\mathbf{u} \in \mathcal{A}$ be a local minimizer of $E[\cdot]$. For $n \geq 3$, we further assume that \mathbf{u} is a continuous and an open map. Then \mathbf{u} satisfies the following integral identity*

$$(2.1) \quad \int_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v} \circ \mathbf{u})(x) dx = 0,$$

for all smooth, compactly supported and divergence free vector fields \mathbf{v} on $\mathbf{u}(\Omega)$, where $A : B := \text{tr}(A^t B) = \sum_{i,j=1}^n a_{ij} b_{ij}$ is the scalar product on $\mathbb{M}^{n \times n}$.

Proof: Let $\mathbf{v} \in C_0^\infty(\mathbf{u}(\Omega), \mathbb{R}^n)$ be a vector field with $\text{div } \mathbf{v} = 0$. For each $y \in \mathbf{u}(\Omega)$, consider the unique smooth flow $\phi(y, \cdot) : \mathbb{R} \rightarrow \mathbf{u}(\Omega)$ given by

$$(2.2) \quad \frac{d\phi}{dt}(y, t) = \mathbf{v}(\phi(y, t)) \quad \text{in } \mathbb{R}, \quad \phi(y, 0) = y.$$

Using the relations $\frac{\partial}{\partial P_{ij}} \det P = (\text{cof } P)_{ij}$ and $P(\text{cof } P)^t = I_n \det P$, by a direct calculations we observe that

$$(2.3) \quad \frac{d}{dt} (\det \nabla_y \phi(y, t)) = \det \nabla_y \phi(y, t) \text{div } \mathbf{v} = 0.$$

Since $\det \nabla_y \phi(y, 0) = 1$, from (2.3) it follows that $\det \nabla_y \phi(y, t) = 1$ for all $t \in \mathbb{R}$ and $y \in \mathbf{u}(\Omega)$. Consider the map $\mathbf{w} : \Omega \times \mathbb{R} \rightarrow \mathbf{u}(\Omega)$ defined by

$$\mathbf{w}(x, t) := \phi(\cdot, t) \circ \mathbf{u}(x) = \phi(\mathbf{u}(x), t) \quad \text{for any } t \in \mathbb{R}, x \in \Omega.$$

Let $V := \text{supp } \mathbf{v} \subset \mathbf{u}(\Omega)$, then $\mathbf{v}(\mathbf{u}(x)) = 0$ for $\mathbf{u}(x) \notin V$. This in conjunction with the uniqueness of ϕ implies that $\phi(\mathbf{u}(x), t) = \mathbf{u}(x)$ for all points x such that $\mathbf{u}(x) \notin V$. Since Ω is bounded, \mathbf{u} is continuous and V is compact, $\Omega' = \mathbf{u}^{-1}(V)$ is a compact subset of Ω . Hence $\text{supp}(\mathbf{w}(x, t) - \mathbf{u}(x)) \subset \Omega'$. Furthermore, $\det \nabla_x \mathbf{w}(x, t) = \det \nabla_y \phi(y, t) \det \nabla \mathbf{u}(x) = 1$. Therefore, $\mathbf{w}(\cdot, t) \in \mathcal{A}$ and $\text{supp}(\mathbf{u} - \mathbf{w}(\cdot, t)) \subset \Omega$ for all $t \in \mathbb{R}$. Since \mathbf{u} is a local minimizer of $E[\cdot]$,

$$E[\mathbf{u}] \leq E[\mathbf{w}(\cdot, t)] \quad \text{for all } t \in \mathbb{R}.$$

Thus in particular,

$$\begin{aligned} 0 &= \left. \frac{d}{dt} \int_{\Omega} L(\nabla \mathbf{w}(x, t)) dx \right|_{t=0} \\ &= \sum_{i,j=1}^2 \int_{\Omega} L_{ij}(\nabla \mathbf{w}(x, t)) \left. \frac{d}{dt} \left(\frac{\partial w^i}{\partial x_j}(x, t) \right) dx \right|_{t=0} \\ &= \sum_{i,j=1}^2 \int_{\Omega} L_{ij}(\nabla \mathbf{w}(x, t)) \left. \frac{\partial}{\partial x_j} \left(\frac{d\phi^i}{dt}(\mathbf{u}(x), t) \right) dx \right|_{t=0} \\ &= \sum_{i,j=1}^2 \int_{\Omega} L_{ij}(\nabla \mathbf{w}(x)) \left. \frac{\partial}{\partial x_j} (v^i(\phi(\mathbf{u}(x), t))) dx \right|_{t=0} \\ &= \sum_{i,j=1}^2 \int_{\Omega} L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial}{\partial x_j} (v^i(\mathbf{u}(x))) dx \\ &= \int_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v} \circ \mathbf{u})(x) dx, \end{aligned}$$

for all smooth, compactly supported and divergence free vector fields on $\mathbf{u}(\Omega)$, where $L_{ij}(P) := \frac{\partial L}{\partial P_{ij}}(P)$. This proves the Theorem. \square

3. DERIVATION OF EULER-LAGRANGE EQUATIONS FOR $n = 2$

Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded and simply connected domain. Assume that the bulk energy $L : \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$ is smooth such that, $|L(P)| \leq C(1 + |P|^2)$, $|DL(P)| \leq C(1 + |P|)$ and $|D^2L(P)| \leq C$ for all $P \in \mathbb{M}^{2 \times 2}$, for some $C > 0$. Since $|\text{cof } P| = |P|$ for $P \in \mathbb{M}^{2 \times 2}$, the area-preserving maps in the plane \mathcal{A} defined in (1.3) is equivalent to the family $\{\mathbf{w} \in W^{1,2}(\Omega, \mathbb{R}^2) : \det \nabla \mathbf{w}(x) = 1, \text{ a.e. in } \Omega\}$. Let $\mathbf{u} \in \mathcal{A}$ be a local minimizer of $E[\cdot]$. Then $\mathbf{u} : \Omega \rightarrow \mathbf{u}(\Omega)$ is an open map and a local homeomorphism [Sv 88], [IS 93]. Throughout this section we denote $V \subset\subset \mathbf{u}(\Omega)$, a smooth and simply connected sub-domain, C is a generic absolute constant depending only on Ω , V , and L . Its value can vary from line to line, but each line is valid with C being a pure positive number.

Let $\mathbf{v} = (v^1, v^2) \in C_0^\infty(V, \mathbb{R}^2)$ such that $\text{div } \mathbf{v} = 0$. Let ρ be the usual mollification kernel. For $0 < \varepsilon < \text{dist}(V, \partial \mathbf{u}(\Omega))$, let $\mathbf{v}_\varepsilon := (v_\varepsilon^1, v_\varepsilon^2)$ be the mollification of \mathbf{v} , where

$$v_\varepsilon^i(y) := (v^i * \rho_\varepsilon)(y) = \int_{\mathbb{R}^2} \rho_\varepsilon(y - z) v^i(z) dz = \int_V \rho_\varepsilon(y - z) v^i(z) dz, \quad y \in \mathbf{u}(\Omega).$$

Thus $\mathbf{v}_\varepsilon \in C_0^\infty(\mathbf{u}(\Omega), \mathbb{R}^2)$ and $\operatorname{div} \mathbf{v}_\varepsilon = 0$. Hence by testing the identity (2.1) with $\mathbf{v} = \mathbf{v}_\varepsilon$, we obtain

$$\sum_{i,j=1}^2 \int_{\Omega} L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial}{\partial x_j} (v_\varepsilon^i \circ \mathbf{u})(x) dx = 0,$$

or in more explicitly

$$(3.1) \quad \sum_{i,j,k=1}^2 \int_{\Omega} L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial v_\varepsilon^i}{\partial y_k}(\mathbf{u}(x)) \frac{\partial u^k}{\partial x_j}(x) dx = 0.$$

From the definition of mollification, by taking $y = \mathbf{u}(x)$, for $x \in \Omega$, we obtain

$$(3.2) \quad \frac{\partial v_\varepsilon^i}{\partial y_k}(\mathbf{u}(x)) = \int_V \frac{\partial \rho_\varepsilon}{\partial y_k}(\mathbf{u}(x) - z) v^i(z) dz.$$

Therefore by plugging (3.2) into (3.1) and Fubini's Theorem yields

$$(3.3) \quad \begin{aligned} 0 &= \sum_{i,j,k=1}^2 \int_{\Omega} L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial u^k}{\partial x_j}(x) \left(\int_V \frac{\partial \rho_\varepsilon}{\partial y_k}(\mathbf{u}(x) - z) v^i(z) dz \right) dx \\ &= \sum_{i,j,k=1}^2 \int_V \left(\int_{\Omega} L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial \rho_\varepsilon}{\partial y_k}(\mathbf{u}(x) - z) \frac{\partial u^k}{\partial x_j}(x) dx \right) v^i(z) dz \\ &= \sum_{i,j=1}^2 \int_V \left(\int_{\Omega} L_{ij}(\nabla \mathbf{u}(x)) \sum_{k=1}^2 \frac{\partial \rho_\varepsilon}{\partial y_k}(\mathbf{u}(x) - z) \frac{\partial u^k}{\partial x_j}(x) dx \right) v^i(z) dz \\ &= \sum_{i=1}^2 \int_V \left[\sum_{j=1}^2 \int_{\Omega} L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial}{\partial x_j} \left(\rho_\varepsilon(\mathbf{u}(x) - z) \right) dx \right] v^i(z) dz. \end{aligned}$$

Let us define the smooth function $g_\varepsilon^i : V \rightarrow \mathbb{R}$, for $i = 1, 2$ by

$$(3.4) \quad g_\varepsilon^i(z) := \sum_{j=1}^2 \int_{\Omega} L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial}{\partial x_j} \left(\rho_\varepsilon(\mathbf{u}(x) - z) \right) dx.$$

Then $\mathbf{g}_\varepsilon = (g_\varepsilon^1, g_\varepsilon^2) \in C^\infty(V, \mathbb{R}^2)$ and

$$\begin{aligned} |\mathbf{g}_\varepsilon(z)| &\leq \sum_{ij} \int_{\Omega} \left| L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial \rho_\varepsilon}{\partial y_k}(\mathbf{u}(x) - z) \frac{\partial u^k}{\partial x_j}(x) \right| dx \\ &\leq \frac{C}{\varepsilon^3} \left((\operatorname{meas} \Omega)^{1/2} + \|\nabla \mathbf{u}\|_{L^2(\Omega)} \right) \|\nabla \mathbf{u}\|_{L^2(\Omega)}. \end{aligned}$$

Thus combing (3.3) and (3.4) we get

$$(3.5) \quad \int_V \langle \mathbf{g}_\varepsilon(z), \mathbf{v}(z) \rangle dz = 0 \quad \text{for } \mathbf{v} \in C_0^\infty(V, \mathbb{R}^2) \text{ such that } \operatorname{div} \mathbf{v} = 0 \text{ in } V,$$

where $\langle \cdot, \cdot \rangle$ the usual scalar product in \mathbb{R}^2 . Let $\phi \in C_0^\infty(V)$ and define $\mathbf{v}(z) := J \nabla \phi(z)$ for $z \in V$, where J be the 90° planar rotation given by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then it follows that $\operatorname{div} \mathbf{v} = 0$ and hence by testing (3.5) with this particular choice of \mathbf{v} and integrating by parts we obtain,

$$\begin{aligned} 0 &= \int_V \langle \mathbf{g}_\varepsilon(z), J\nabla\phi(z) \rangle dz \\ &= \int_V \langle J^t \mathbf{g}_\varepsilon(z), \nabla\phi(z) \rangle dz \\ &= - \int_V \phi(z) \operatorname{div}(J^t \mathbf{g}_\varepsilon(z)) dz \quad \text{for all } \phi \in C_0^\infty(V). \end{aligned}$$

Hence $\operatorname{curl} \mathbf{g}_\varepsilon := \frac{\partial g_\varepsilon^1}{\partial z_2} - \frac{\partial g_\varepsilon^2}{\partial z_1} = \operatorname{div}(J^t \mathbf{g}_\varepsilon) = 0$ in V . Since V is simply connected, there exists $q_\varepsilon \in C^\infty(V)$, such that

$$(3.6) \quad \mathbf{g}_\varepsilon(z) = -\nabla q_\varepsilon(z), \quad \text{for all } z \in V,$$

modulo translation of a constant.

Lemma 3.1. *Consider the family \mathbf{g}_ε be given by (3.4). Then $\mathbf{g}_\varepsilon \rightharpoonup \mathbf{g}$ weakly in the dual space $(C_0^1(V, \mathbb{R}^2))^*$.*

Proof: Since ρ_ε is radially symmetric

$$(3.7) \quad \frac{\partial \rho_\varepsilon}{\partial y_k}(|y-z|) = \rho_\varepsilon'(|y-z|) \frac{y_k - z_k}{|y-z|} = -\frac{\partial \rho_\varepsilon}{\partial z_k}(|y-z|).$$

Therefore from the definition of g_ε^i in (3.4), we have

$$\begin{aligned} (3.8) \quad g_\varepsilon^i(z) &= - \sum_{j,k=1}^2 \int_\Omega L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial u^k}{\partial x_j}(x) \frac{\partial \rho_\varepsilon}{\partial z_k}(\mathbf{u}(x) - z) dx \\ &= - \sum_{k=1}^2 \int_\Omega \sigma_{ik}(x) \frac{\partial}{\partial z_k} \left(\rho_\varepsilon(\mathbf{u}(x) - z) \right) dx, \end{aligned}$$

where

$$(3.9) \quad \sigma_{ik}(x) := \sum_{j=1}^2 L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial u^k}{\partial x_j}(x) \quad \text{for } x \in \Omega.$$

Since \mathbf{u} is a $W^{1,2}$ area-preserving homeomorphism, $\nabla \mathbf{u}^{-1}(\mathbf{u}(x)) = (\operatorname{cof} \nabla \mathbf{u}(x))^t$. Thus it follows that $\mathbf{u}^{-1} \in W^{1,2}(\mathbf{u}(\Omega), \Omega)$. Using the structural assumptions on L in (3.9), we get

$$\int_{\mathbf{u}(\Omega)} |(\sigma_{ik} \circ \mathbf{u}^{-1})(z)| dz = \int_\Omega |\sigma_{ik}(x)| dx \leq C \int_\Omega |\nabla \mathbf{u}(x)|^2 dx < \infty,$$

and hence $\tilde{\sigma}_{ik} := \sigma_{ik} \circ \mathbf{u}^{-1} \in L^1(\mathbf{u}(\Omega))$, for $i, k = 1, 2$. Now observe that for any test function $\mathbf{v} \in C_0^\infty(V, \mathbb{R}^2)$, using Fubini, integration by parts and change of variable

$\xi = \mathbf{u}(x)$ we obtain

$$\begin{aligned}
(3.10) \quad \int_V \langle \mathbf{g}_\varepsilon(z), \mathbf{v}(z) \rangle dz &= - \sum_{i,j=1}^2 \int_\Omega \sigma_{ij}(x) \left(\int_V \frac{\partial}{\partial z_j} (\rho_\varepsilon(\mathbf{u}(x) - z)) v^i(z) dz \right) dx \\
&= \sum_{i,j=1}^2 \int_\Omega \sigma_{ij}(x) \left(\int_V \rho_\varepsilon(\mathbf{u}(x) - z) \frac{\partial v^i}{\partial z_j}(z) dz \right) dx \\
&= \sum_{i,j=1}^2 \int_V \frac{\partial v^i}{\partial z_j}(z) \left(\int_\Omega \sigma_{ij}(x) \rho_\varepsilon(\mathbf{u}(x) - z) dx \right) dz \\
&= \sum_{i,j=1}^2 \int_V \frac{\partial v^i}{\partial z_j}(z) \left(\int_{\mathbf{u}(\Omega)} \sigma_{ij}(\mathbf{u}^{-1}(\xi)) \rho_\varepsilon(\xi - z) d\xi \right) dz \\
&= \sum_{i,j=1}^2 \int_V \frac{\partial v^i}{\partial z_j}(z) (\tilde{\sigma}_{ij})_\varepsilon(z) dz,
\end{aligned}$$

where

$$(3.11) \quad (\tilde{\sigma}_{ij})_\varepsilon(z) := ((\sigma_{ij} \circ \mathbf{u}^{-1}) * \rho_\varepsilon)(z) = \int_{\mathbf{u}(\Omega)} \sigma_{ij}(\mathbf{u}^{-1}(\xi)) \rho_\varepsilon(\xi - z) d\xi,$$

is the usual mollification of $\sigma_{ij} \circ \mathbf{u}^{-1}$. Since $(\sigma_{ij} \circ \mathbf{u}^{-1}) * \rho_\varepsilon \rightarrow \sigma_{ij} \circ \mathbf{u}^{-1}$ in $L^1(\mathbf{u}(\Omega))$ as $\varepsilon \rightarrow 0$, by passing through the limit as $\varepsilon \rightarrow 0$ in (3.10) we conclude that

$$(3.12) \quad \int_V \langle \mathbf{g}_\varepsilon(z), \mathbf{v}(z) \rangle dz \rightarrow \sum_{i,j=1}^2 \int_V \sigma_{ij}(\mathbf{u}^{-1}(z)) \frac{\partial v^i}{\partial z_j}(z) dz \quad \text{as } \varepsilon \rightarrow 0$$

for all $\mathbf{v} \in C_0^\infty(V, \mathbb{R}^2)$. Now let us define the functional $\mathbf{g} : C_0^1(V, \mathbb{R}^2) \rightarrow \mathbb{R}$ as

$$(3.13) \quad \langle \mathbf{g}, \mathbf{v} \rangle := \lim_{\varepsilon \rightarrow 0} \int_V \langle \mathbf{g}_\varepsilon(z), \mathbf{v}(z) \rangle dz = \int_V \sigma(\mathbf{u}^{-1}(z)) : \nabla \mathbf{v}(z) dz,$$

for $\mathbf{v} \in C_0^1(V, \mathbb{R}^2)$, where $\sigma(x) := (\sigma_{ij}(x)) \in \mathbb{M}^{2 \times 2}$. Then from (3.13) it follows that

$$(3.14) \quad |\langle \mathbf{g}, \mathbf{v} \rangle| \leq C \|\sigma\|_{L^1(\Omega)} \|\nabla \mathbf{v}\|_{L^\infty(\mathbf{u}(\Omega))} \leq C \|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 \|\nabla \mathbf{v}\|_{L^\infty(\mathbf{u}(\Omega))},$$

for any $\mathbf{v} \in C_0^1(\mathbf{u}(\Omega), \mathbb{R}^2)$. Hence \mathbf{g} is a continuous linear functional on $C_0^1(\mathbf{u}(\Omega), \mathbb{R}^2)$. Therefore, from the definition of \mathbf{g}_ε in (3.4), it follows that $\mathbf{g}_\varepsilon \rightharpoonup \mathbf{g}$ weakly in the dual space $(C_0^1(V, \mathbb{R}^2))^*$. This proves the lemma. \square

Lemma 3.2. *Assume that $\mathbf{u} \in W_{\text{loc}}^{1,r}(\Omega, \mathbb{R}^2) \cap \mathcal{A}$ for some $r > 2$. Then the family q_ε defined by $-\nabla q_\varepsilon = \mathbf{g}_\varepsilon$ in (3.6) is uniformly bounded in $L_{\text{loc}}^{r/2}(\mathbf{u}(\Omega))$.*

Proof Since $\mathbf{u} \in W_{\text{loc}}^{1,r}(\Omega, \mathbb{R}^2)$ for some $r > 2$, from the definition of σ_{ij} in (3.9) and the growth condition on L , it follows that for any $V \subset \subset \mathbf{u}(\Omega)$

$$(3.15) \quad \int_V |(\sigma_{ij} \circ \mathbf{u}^{-1})(z)|^{r/2} dz = \int_{\mathbf{u}^{-1}(V)} |\sigma_{ij}(x)|^{r/2} dx \leq C \int_{\mathbf{u}^{-1}(V)} |\nabla \mathbf{u}(x)|^r dx,$$

and hence $\tilde{\sigma}_{ij} := \sigma_{ij} \circ \mathbf{u}^{-1} \in L^{r/2}(V)$, for $i, j = 1, 2$. Let $f_\varepsilon : V \rightarrow \mathbb{R}$ be defined as $f_\varepsilon(z) := q_\varepsilon(z) |q_\varepsilon(z)|^{\frac{r}{2}-2}$, $z \in V$, so that for any $1 < s < \infty$,

$$\int_V |f_\varepsilon(z)|^s dz = \int_V |q_\varepsilon(z)|^{s(\frac{r}{2}-1)} dz = \| |q_\varepsilon|^{\frac{r}{2}-1} \|_{L^s(V)}^s.$$

Translating f_ε to $f_\varepsilon - \frac{1}{\text{meas } V} \int_V f_\varepsilon(z) dz$, if necessary, so that $\int_V f_\varepsilon(z) dz = 0$. In view of this normalization, there exists a smooth vector field $\mathbf{w}_\varepsilon : V \mapsto \mathbb{R}^2$, such that

$$(3.16) \quad \begin{cases} \operatorname{div} \mathbf{w}_\varepsilon = f_\varepsilon & \text{in } V \\ \mathbf{w}_\varepsilon = 0 & \text{on } \partial V. \end{cases}$$

Furthermore we have the estimate

$$(3.17) \quad \|\mathbf{w}_\varepsilon\|_{W^{1,s}(V)} \leq C \|f_\varepsilon\|_{L^s(V)} = C \| |q_\varepsilon|^{\frac{r}{2}-1} \|_{L^s(V)},$$

for $C > 0$ independent of ε , see Dacorogna-Moser [DM 90]. Then for sufficiently small $\varepsilon > 0$

$$\begin{aligned} \int_V |q_\varepsilon(z)|^{r/2} dz &= \int_V q_\varepsilon(z) |q_\varepsilon(z)|^{r/2-2} q_\varepsilon(z) dz \\ &= \int_V q_\varepsilon(z) \operatorname{div} \mathbf{w}_\varepsilon(z) dz && \text{by (3.16)} \\ &= - \int_V \langle \nabla q_\varepsilon(z), \mathbf{w}_\varepsilon(z) \rangle dz \\ &= \int_V \langle \mathbf{g}_\varepsilon(z), \mathbf{w}_\varepsilon(z) \rangle dz && \text{by (3.6)} \\ &= \sum_{i,j=1}^2 \int_V ((\sigma_{ij} \circ \mathbf{u}^{-1}) * \rho_\varepsilon)(z) \frac{\partial w_\varepsilon^i}{\partial z_k}(z) dz && \text{by (3.10)} \\ &\leq C \sum_{i,j=1}^2 \left(\int_V |\sigma_{ij}(\mathbf{u}^{-1}(z))|^{r/2} dz \right)^{2/r} \left(\int_V \left| \frac{\partial w_\varepsilon^i}{\partial z_k}(z) \right|^{r/(r-2)} dz \right)^{(r-2)/r} \\ &\leq C \| |q_\varepsilon|^{\frac{r}{2}-1} \|_{L^{r/(r-2)}(V)} \sum_{i,j=1}^2 \|\sigma_{ij} \circ \mathbf{u}^{-1}\|_{L^{r/2}(V)} && \text{by (3.17)} \\ &= C \left(\int_V |q_\varepsilon(z)|^{r/2} dz \right)^{1-2/r} \|\sigma\|_{L^{r/2}(\mathbf{u}(\Omega), \mathbb{M}^{2 \times 2})} \\ &\leq C \left(\int_V |q_\varepsilon(z)|^{r/2} dz \right)^{1-2/r} \|\nabla \mathbf{u}\|_{L^r(\Omega)}^2. && \text{by (3.15)} \end{aligned}$$

Hence there exists a constant $C > 0$, independent of ε such that

$$(3.18) \quad \|q_\varepsilon\|_{L^{r/2}(V)} \leq C \|\nabla \mathbf{u}\|_{L^r(\Omega)}^2.$$

Since $r > 2$, there exists a function $q \in L^{r/2}(V)$, such that $q_\varepsilon \rightharpoonup q$ weakly in $L^{r/2}(V)$. This proves the lemma. \square

Proof of Theorem 1.1 Using the change of variables, recalling the definitions of \mathbf{g} in (3.13), and σ_{ij} in (3.9), we obtain

$$\begin{aligned}
(3.19) \quad \langle \mathbf{g}, \mathbf{v} \rangle &= \sum_{i,j=1}^2 \int_V \sigma_{ij}(\mathbf{u}^{-1}(z)) \frac{\partial v^i}{\partial z_j}(z) dz \\
&= \sum_{i,j=1}^2 \int_{\mathbf{u}^{-1}(V)} \sigma_{ij}(x) \frac{\partial v^i}{\partial z_j}(\mathbf{u}(x)) dx \\
&= \sum_{i,k=1}^2 \int_{\mathbf{u}^{-1}(V)} L_{ik}(\nabla \mathbf{u}(x)) \left(\sum_{j=1}^2 \frac{\partial v^i}{\partial z_j}(\mathbf{u}(x)) \frac{\partial u^j}{\partial x_k}(x) \right) dx \\
&= \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v} \circ \mathbf{u})(x) dx \quad \text{for } \mathbf{v} \in C_0^1(V, \mathbb{R}^2).
\end{aligned}$$

Since $\mathbf{u}^{-1} \in W^{1,r}(V, \mathbf{u}^{-1}(V))$, for any $\phi \in C_0^1(\mathbf{u}^{-1}(V), \mathbb{R}^2)$, the composition $\phi \circ \mathbf{u}^{-1} \in W_0^{1,r}(V, \mathbb{R}^2)$. Hence there exists $\mathbf{v}_\delta \in C_0^1(V, \mathbb{R}^2)$ such that $\mathbf{v}_\delta \rightarrow \psi := \phi \circ \mathbf{u}^{-1}$ strongly in $W^{1,r}(V, \mathbb{R}^2)$ as $\delta \rightarrow 0$. Then Hölder inequality yields

$$\begin{aligned}
\int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \left(\nabla(\mathbf{v}_\delta \circ \mathbf{u})(x) - \nabla(\psi \circ \mathbf{u})(x) \right) dx \\
= \int_{\mathbf{u}^{-1}(V)} (\nabla \mathbf{u}(x))^t DL(\nabla \mathbf{u}(x)) : \left(\nabla_z \mathbf{v}_\delta(\mathbf{u}(x)) - \nabla_z \psi(\mathbf{u}(x)) \right) dx \\
\leq C \|\nabla \mathbf{u}\|_{L^{2r'}(\mathbf{u}^{-1}(V))} \|\nabla(\mathbf{v}_\delta - \psi)\|_{L^r(V)},
\end{aligned}$$

where $r' = r/(r-1)$. Notice that $r \geq 3$ yields $2r' \leq r$ and hence $\nabla \mathbf{u} \in L_{\text{loc}}^r(\Omega) \subseteq L_{\text{loc}}^{2r'}(\Omega)$. Therefore, from (3.19) we obtain

$$\begin{aligned}
(3.20) \quad \langle \mathbf{g}, \mathbf{v}_\delta \rangle &= \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v}_\delta \circ \mathbf{u})(x) dx \\
&\rightarrow \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla(\phi \circ \mathbf{u}^{-1} \circ \mathbf{u})(x) dx \quad \text{as } \delta \rightarrow 0 \\
&= \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx.
\end{aligned}$$

Now define the linear functional $\mathbf{g} \circ \mathbf{u} : C_0^1(\mathbf{u}^{-1}(V), \mathbb{R}^2) \rightarrow \mathbb{R}$ by

$$(3.21) \quad \langle \mathbf{g} \circ \mathbf{u}, \phi \rangle := \langle \mathbf{g}, \phi \circ \mathbf{u}^{-1} \rangle = \lim_{\delta \rightarrow 0} \langle \mathbf{g}, \mathbf{v}_\delta \rangle = \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx,$$

for any $\phi \in C_0^1(\mathbf{u}^{-1}(V), \mathbb{R}^2)$. Hence $\mathbf{g} \circ \mathbf{u}$ defines a continuous linear functional on $W_0^{1,2}(\mathbf{u}^{-1}(V), \mathbb{R}^2)$. On the other hand, since $q_\varepsilon \rightharpoonup q$ weakly in $L^{r/2}(V)$, using the definition of \mathbf{g} , the representation of $\mathbf{g}_\varepsilon = -\nabla q_\varepsilon$ and integration by parts we conclude

that

$$\begin{aligned}
(3.22) \quad \langle \mathbf{g}, \mathbf{v}_\delta \rangle &= \lim_{\varepsilon \rightarrow 0} \int_V \langle \mathbf{g}_\varepsilon(z), \mathbf{v}_\delta(z) \rangle dz \\
&= - \lim_{\varepsilon \rightarrow 0} \int_V \langle \nabla q_\varepsilon(z), \mathbf{v}_\delta(z) \rangle dz \\
&= \lim_{\varepsilon \rightarrow 0} \int_V q_\varepsilon(z) \operatorname{div} \mathbf{v}_\delta(z) dz \\
&= \int_V q(z) \operatorname{div} \mathbf{v}_\delta(z) dz \\
&= \int_V q(z) \operatorname{tr}(\nabla_z \mathbf{v}_\delta(z)) dz.
\end{aligned}$$

The area constraint $\det \nabla \mathbf{u}(x) = 1$ a.e., and $\nabla(\mathbf{v} \circ \mathbf{u})(x) = \nabla_z \mathbf{v}(\mathbf{u}(x)) \nabla \mathbf{u}(x)$, yields $\nabla_z \mathbf{v}(\mathbf{u}(x)) = \nabla(\mathbf{v} \circ \mathbf{u})(x) (\operatorname{cof} \nabla \mathbf{u}(x))^t$. Using $\mathbf{u} \in W_{\operatorname{loc}}^{1,r}(\Omega, \mathbb{R}^2)$ together with the fact that $|\operatorname{cof} P| = |P|$ for any $P \in \mathbb{M}^{2 \times 2}$, we conclude that $\operatorname{cof} \nabla \mathbf{u} \in L_{\operatorname{loc}}^r(\Omega, \mathbb{M}^{2 \times 2})$. Since $q \in L^{r/2}(V)$ and $L_{\operatorname{loc}}^{r/2} \subseteq L_{\operatorname{loc}}^{r/(r-1)}$ for $r \geq 3$, applying change of variables in (3.22), we obtain

$$\begin{aligned}
(3.23) \quad \langle \mathbf{g}, \mathbf{v}_\delta \rangle &= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \operatorname{tr}(\nabla_z \mathbf{v}_\delta(\mathbf{u}(x))) dx \\
&= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \operatorname{tr}(\nabla(\mathbf{v}_\delta \circ \mathbf{u})(x) (\operatorname{cof} \nabla \mathbf{u}(x))^t) dx \\
&= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \operatorname{cof}(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v}_\delta \circ \mathbf{u})(x) dx, \\
&\rightarrow \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \operatorname{cof}(\nabla \mathbf{u}(x)) : \nabla(\phi \circ \mathbf{u}^{-1} \circ \mathbf{u})(x) dx \quad \text{as } \delta \rightarrow 0 \\
&= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \operatorname{cof}(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx.
\end{aligned}$$

Hence from (3.21) and (3.23) we obtain

$$\int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx = \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \operatorname{cof}(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx,$$

for any $\phi \in C_0^1(\mathbf{u}^{-1}(V), \mathbb{R}^2)$. Finally choose a sequence of smooth, simply connected sets $V_k \subset\subset V_{k+1} \subset\subset \mathbf{u}(\Omega) = \cup_{k=1}^\infty V_k$. Utilizing the foregoing arguments and lemmas 3.1-3.2, there exists $q_k \in L^{r/2}(V_k)$, $k \geq 1$ such that

$$(3.24) \quad \int_{\mathbf{u}^{-1}(V_k)} DL(\nabla \mathbf{u}(x)) : \nabla \phi(x) = \int_{\mathbf{u}^{-1}(V_k)} q_k(\mathbf{u}(x)) \operatorname{cof}(\nabla \mathbf{u}(x)) : \nabla \phi(x),$$

for $\phi \in C_0^1(\mathbf{u}^{-1}(V_k), \mathbb{R}^2)$. Since \mathbf{u} is locally area-preserving homeomorphism, $\Omega = \cup_{k=1}^\infty \mathbf{u}^{-1}(V_k)$ is an open covering of Ω and $\mathbf{u}^{-1}(V_k) \subset\subset \mathbf{u}^{-1}(V_{k+1})$. Using the identity $\operatorname{div} \operatorname{cof} \nabla \mathbf{u}(x) = 0$ and invertibility of $\nabla \mathbf{u}(x)$, from (3.24) it follows that q_k is unique up to a translation of a constant. Thus adding constant terms as necessary to each q_k , we deduce from (3.24) that for each fixed $k \geq 1$

$$q_i(z) = q_k(z) \quad \text{for } z \in V_i, \quad 1 \leq i \leq k.$$

We finally define $q : \mathbf{u}(\Omega) \rightarrow \mathbb{R}$ as $q(z) := q_k(z)$, for $z \in V_k$, so that $q \in L_{\text{loc}}^{r/2}(\mathbf{u}(\Omega))$. This proves that for any $\phi \in C_0^1(\Omega, \mathbb{R}^2)$, the pair (\mathbf{u}, q) satisfies

$$\int_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx = \int_{\Omega} q(\mathbf{u}(x)) \operatorname{cof}(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx.$$

Now let us define the pressure p on Ω by

$$p(x) := q(\mathbf{u}(x)) \quad \text{for } x \in \Omega.$$

Then for any $k \geq 1$,

$$\int_{\mathbf{u}^{-1}(V_k)} |p(x)|^{r/2} = \int_{\mathbf{u}^{-1}(V_k)} |q(\mathbf{u}(x))|^{r/2} dx = \int_{V_k} |q(z)|^{r/2} dz < \infty,$$

and hence $p \in L_{\text{loc}}^{r/2}(\Omega)$ and the pair (\mathbf{u}, p) satisfies

$$(3.25) \quad \int_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx = \int_{\Omega} p(x) \operatorname{cof}(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx,$$

for any $\phi \in C_0^1(\Omega, \mathbb{R}^2)$. In other words, (\mathbf{u}, p) satisfies the system of Euler-Lagrange equations

$$\operatorname{div} [DL(\nabla \mathbf{u}(x)) - p(x) \operatorname{cof}(\nabla \mathbf{u}(x))] = 0, \quad \text{in } \Omega.$$

in the sense of (3.25). This completes the proof. \square

4. LOCAL L^1 -ESTIMATE AND THE INTEGRAL REPRESENTATION OF THE PRESSURE

In this section we establish an explicit representation of the pressure on the deformed domain $\mathbf{u}(\Omega)$ in terms of Calderón-Zygmund type singular integral operator of the energy $|\nabla \mathbf{u}|^2$. Our main ideas in the proof are to localize the mollified pressure on the deformed domain $\mathbf{u}(\Omega)$, finding its explicit representation using Green's function of the unit disc in \mathbb{R}^2 and finding an uniform estimate by using Calderón-Zygmund Theorem [CZ 52] for $L \log^+ L$ functions.

Proof of Theorem 1.2. Let us assume that $\mathbf{u} \in \mathcal{A}$ minimizes the energy $E[\cdot]$ and $|\nabla \mathbf{u}|^2 \in L \log^+ L$. Let $V \subset \subset \mathbf{u}(\Omega)$ be a smooth and simply connected sub-domain of $\mathbf{u}(\Omega)$. Without loss of generality let us assume that $0 \in V$ and $V = B_1 := \{z \in \mathbb{R}^2 : |z| < 1\}$ be the unit disc. Recall the family $(\mathbf{g}_\varepsilon)_{\varepsilon > 0}$ defined by (3.4) and the family $(q_\varepsilon)_{\varepsilon > 0}$ in (3.6) represented by

$$(4.1) \quad -\nabla q_\varepsilon = \mathbf{g}_\varepsilon,$$

modulo an additive constant. Applying the divergence operator to the both sides of the above equation, we obtain

$$(4.2) \quad -\Delta q_\varepsilon = \operatorname{div} \mathbf{g}_\varepsilon.$$

Now our idea is to localize the equation (4.2) and find appropriate uniform estimates for the localized q_ε . Let $\eta \in C_0^\infty(B_1)$, $0 \leq \eta \leq 1$ be a cut-off function such that $\eta \equiv 1$ in $B_{2/3}$. Let $\bar{q}_\varepsilon = \eta q_\varepsilon$ be the localized pressure. Then \bar{q}_ε is the solution to the Dirichlet problem

$$(4.3) \quad \begin{cases} -\Delta \bar{q}_\varepsilon = f_\varepsilon & \text{in } B_1 \\ \bar{q}_\varepsilon = 0 & \text{on } \partial B_1, \end{cases}$$

where $f_\varepsilon := \eta \Delta q_\varepsilon + 2 \langle \nabla q_\varepsilon, \nabla \eta \rangle + q_\varepsilon \Delta \eta$. Therefore \bar{q}_ε is the Green's potential of f_ε in B_1 . In other words,

$$(4.4) \quad \bar{q}_\varepsilon(y) = \int_{B_1} G(z-y) f_\varepsilon(z) dz,$$

where $G(z, y)$ Green's function of the unit disc $B_1 \subset \mathbb{R}^2$ given by

$$(4.5) \quad G(z, y) := -\frac{1}{2\pi} \log |z-y| + \frac{1}{2\pi} \log(|y||z-\hat{y}|), \quad \hat{y} := \frac{y}{|y|^2}.$$

Using (4.1), (4.2) and (4.5) in (4.4), we obtain

$$(4.6) \quad \begin{aligned} \bar{q}_\varepsilon(y) &= -\frac{1}{2\pi} \int_{B_1} \left(\eta \Delta q_\varepsilon + 2 \langle \nabla q_\varepsilon, \nabla \eta \rangle + q_\varepsilon \Delta \eta \right) \log |z-y| dz \\ &\quad + \frac{1}{2\pi} \int_{B_1} f_\varepsilon(z) \log(|y||z-\hat{y}|) dz \\ &= \frac{1}{2\pi} \int_{B_1} \left(\eta \operatorname{div} \mathbf{g}_\varepsilon + 2 \langle \mathbf{g}_\varepsilon(z), \nabla \eta(z) \rangle - q_\varepsilon \Delta \eta \right) \log |z-y| dz \\ &\quad + \frac{1}{2\pi} \int_{B_1} f_\varepsilon(z) \log(|y||z-\hat{y}|) dz \\ &= \frac{1}{2\pi} I_\varepsilon^1(y) + \frac{1}{\pi} I_\varepsilon^2(y) + \frac{1}{2\pi} I_\varepsilon^3(y) + \frac{1}{2\pi} I_\varepsilon^4(y) \end{aligned}$$

where

$$\begin{aligned} I_\varepsilon^1(y) &:= \int_{B_1} \eta(z) \log |z-y| \operatorname{div} \mathbf{g}_\varepsilon(z) dz \\ I_\varepsilon^2(y) &:= \int_{B_1} \langle \mathbf{g}_\varepsilon(z), \nabla \eta(z) \rangle \log |z-y| dz \\ I_\varepsilon^3(y) &:= - \int_{B_1} q_\varepsilon(z) \Delta \eta(z) \log |z-y| dz \\ I_\varepsilon^4(y) &:= \int_{B_1} f_\varepsilon(z) \log |y| (|z-\hat{y}|) dz. \end{aligned}$$

We now establish an uniform local L^1 -estimate for q_ε through the following steps.

Step 1: Limits of I_ε^3 and I_ε^4 Let us fix $|y| < 1/2$. Since $\Delta \eta = 0$ for $|z| < 2/3$, both the integrals $I_\varepsilon^3(y)$ and $I_\varepsilon^4(y)$ are well defined for $|y| < 1/2$. Since q_ε is determined up to a constant, we can add a constant to $z \mapsto \Delta \eta(z) \log |z-y|$, if necessary, to ensure that it has vanishing integral. For each fixed $|y| < 1/2$, let $\mathbf{v}_y : B_1 \rightarrow \mathbb{R}^2$ be the solution of the Dirichlet problem

$$(4.7) \quad \begin{cases} \operatorname{div} \mathbf{v}_y(z) = \Delta \eta(z) \log |z-y| & \text{for } z \in B_1 \\ \mathbf{v}_y = 0 & \text{on } \partial B_1. \end{cases}$$

Then using (4.7) and (3.13) we obtain

$$\begin{aligned}
(4.8) \quad I_\varepsilon^3(y) &= - \int_{B_1} q_\varepsilon(z) \Delta \eta(z) \log |z - y| dz \\
&= - \int_{B_1} q_\varepsilon(z) \operatorname{div} \mathbf{v}_y(z) dz \\
&= \int_{B_1} \langle \nabla q_\varepsilon(z), \mathbf{v}_y(z) \rangle dz \\
&= - \int_{B_1} \langle \mathbf{g}_\varepsilon(z), \mathbf{v}_y(z) \rangle dz \\
&\rightarrow - \int_{B_1} \sigma(\mathbf{u}^{-1}(z)) : \nabla \mathbf{v}_y(z) dz \quad \text{as } \varepsilon \rightarrow 0 \\
&:= I_0^3(y).
\end{aligned}$$

Since $f_\varepsilon = \Delta(q_\varepsilon \eta)$ and for each fixed $|y| < 1/2$ the function $z \mapsto \Delta \log(|y|(z - \hat{y}))$ is smooth on B_1 . By taking $\mathbf{w}_y : B_1 \rightarrow \mathbb{R}^2$ to be the solution of the Dirichlet problem

$$(4.9) \quad \begin{cases} \operatorname{div} \mathbf{w}_y(z) = \eta(z) \Delta \log(|y||z - \hat{y}|) & \text{for } z \in B_1 \\ \mathbf{w}_y = 0 & \text{on } \partial B_1, \end{cases}$$

and applying the above arguments we obtain

$$\begin{aligned}
(4.10) \quad I_\varepsilon^4(y) &= \int_{B_1} \Delta \left(q_\varepsilon(z) \eta(z) \right) \log |y| (|z - \hat{y}|) dz \\
&= \int_{B_1} q_\varepsilon(z) (\eta(z) \Delta \log(|y||z - \hat{y}|)) dz \\
&\rightarrow \int_{B_1} \sigma(\mathbf{u}^{-1}(z)) : \nabla \mathbf{w}_y(z) dz \quad \text{as } \varepsilon \rightarrow 0 \\
&:= I_0^4(y).
\end{aligned}$$

Step 2: Limit of I_ε^2 Since $\nabla \eta(z) = 0$ for $|z| < 2/3$, the integral $I_\varepsilon^3(y)$ is well-defined for $|y| < 1/2$. Recall that from (3.8) and (3.9)

$$\begin{aligned}
-g_\varepsilon^i(z) &= \sum_{j=1}^2 \frac{\partial}{\partial z_j} \int_{\Omega} \sigma_{ij}(x) \rho_\varepsilon(\mathbf{u}(x) - z) dx \\
&= \sum_{j=1}^2 \frac{\partial}{\partial z_j} \int_{\mathbf{u}(\Omega)} \sigma_{ij}(\mathbf{u}^{-1}(y)) \rho_\varepsilon(\mathbf{y} - z) dy \\
&= \sum_{j=1}^2 \frac{\partial}{\partial z_j} \left((\sigma_{ij} \circ \mathbf{u}^{-1}) * \rho_\varepsilon \right) (z).
\end{aligned}$$

In other words,

$$(4.11) \quad \mathbf{g}_\varepsilon = - \operatorname{div} \tilde{\sigma}_\varepsilon,$$

where the divergence is taken in each rows of matrix $\tilde{\sigma}_\varepsilon := \left((\tilde{\sigma}_{ij})_\varepsilon \right) \in \mathbb{M}^{2 \times 2}$, $(\tilde{\sigma}_{ij})_\varepsilon := (\sigma_{ij} \circ \mathbf{u}^{-1}) * \rho_\varepsilon$. Notice that $(\tilde{\sigma}_{ij})_\varepsilon \rightarrow \tilde{\sigma}_{ij} := \sigma_{ij} \circ \mathbf{u}^{-1}$ in L^1 as $\varepsilon \rightarrow 0$ for each $i, j = 1, 2$.

Using the above representation of \mathbf{g}_ε observe that

$$\begin{aligned}
(4.12) \quad I_\varepsilon^2(y) &= - \int_{B_1} \left\langle \operatorname{div} \tilde{\sigma}_\varepsilon(z), \log |z-y| \nabla \eta(z) \right\rangle dz \\
&= \int_{B_1} \tilde{\sigma}_\varepsilon(z) : \nabla \left(\log |z-y| \nabla \eta(z) \right) dz \\
&= \int_{B_1 \setminus B_{2/3}} \tilde{\sigma}_\varepsilon(z) : \left(\log |z-y| \nabla^2 \eta(z) + \frac{\nabla \eta \otimes (z-y)}{|y-z|^2} \right) dz \\
&\rightarrow \int_{B_1 \setminus B_{2/3}} \tilde{\sigma}(z) : \left(\log |z-y| \nabla^2 \eta(z) + \frac{\nabla \eta \otimes (z-y)}{|y-z|^2} \right) dz \quad \text{as } \varepsilon \rightarrow 0 \\
&:= I_0^2(y).
\end{aligned}$$

Step 3: Limit of $I_\varepsilon^1(y)$ Since we assumed $|\nabla \mathbf{u}|^2 \in \mathcal{H}_{\text{loc}}^1(\Omega)$, from the definition of $\tilde{\sigma}_{ij}$ it follows that $\tilde{\sigma}_{ij} \in L \log^+ L$. Thus the mollification $(\tilde{\sigma}_{ij})_\varepsilon$ converges strongly to $\tilde{\sigma}_{ij}$ in $L \log^+ L$ as $\varepsilon \rightarrow 0$. Integrating by parts twice and using (4.11)

$$\begin{aligned}
I_\varepsilon^1(y) &= \int_{B_1} \operatorname{div} \mathbf{g}_\varepsilon(z) \eta(z) \log |z-y| dz \\
&= - \int_{B_1} \tilde{\sigma}_\varepsilon(z) : \nabla^2(\eta(z) \log |z-y|) dz \\
&= - \int_{B_1 \setminus B_{2/3}} \tilde{\sigma}_\varepsilon(z) : \left(\log |z-y| \nabla^2 \eta(z) + 2 \frac{\nabla \eta(z) \otimes (z-y)}{|z-y|^2} \right) dz \\
&\quad - \int_{B_1} \tilde{\sigma}_\varepsilon(z) : \left(\operatorname{Id} - 2 \frac{(z-y) \otimes (z-y)}{|z-y|^2} \right) \frac{\eta(z)}{|z-y|^2} dz \\
&:= I_\varepsilon^{11}(y) + I_\varepsilon^{12}(y),
\end{aligned}$$

where Id is the 2×2 identity matrix and

$$\begin{aligned}
(4.13) \quad I_\varepsilon^{11}(y) &:= - \int_{B_1 \setminus B_{2/3}} \tilde{\sigma}_\varepsilon(z) : \left(\log |z-y| \nabla^2 \eta + 2 \frac{\nabla \eta \otimes (z-y)}{|z-y|^2} \right) dz \\
&\rightarrow - \int_{B_1} \tilde{\sigma}(z) : \left(\log |z-y| \nabla^2 \eta + 2 \frac{\nabla \eta \otimes (z-y)}{|z-y|^2} \right) dz \quad \text{as } \varepsilon \rightarrow 0 \\
&:= I_0^{11}(y),
\end{aligned}$$

and

$$(4.14) \quad I_\varepsilon^{12}(y) := - \int_{B_1} \tilde{\sigma}_\varepsilon(z) : \left(\operatorname{Id} - 2 \frac{(z-y) \otimes (z-y)}{|z-y|^2} \right) \frac{\eta(z)}{|z-y|^2} dz$$

is the sum of Calderón-Zygmund [CZ 52] type singular integrals with the homogeneous kernel

$$(4.15) \quad G_{ij}(z) := \delta_{ij} - 2 \frac{z_i z_j}{|z|^2}, \quad z \in \mathbb{R}^2 \setminus \{0\}, \quad i, j = 1, 2.$$

Observe that each G_{ij} satisfies all the conditions of the Calderón-Zygmund Theorem 1.3 [CZ 52]. Since $\sigma_{ij} \in L \log^+ L$, the following sum of singular integrals

$$(4.16) \quad I_0^{12}(y) := - \int_{B_1} \tilde{\sigma}(z) : \left(\operatorname{Id} - 2 \frac{(z-y) \otimes (z-y)}{|z-y|^2} \right) \frac{\eta(z)}{|z-y|^2} dz$$

exists for almost every $|y| < 1/2$ and is integrable.

Claim: $I_\varepsilon^{12} \rightarrow I_0^{12}$ strongly in $L^1(B_{1/2})$.

Proof. Let $\rho > 1/2$ and extend $\tilde{\sigma}_{ij}$ by 0 outside the unit ball B_1 . From the singular integrals (4.14) and (4.16), we have

$$I_\varepsilon^{12}(y) - I_0^{12}(y) = - \sum_{i,j=1}^2 \int_{\mathbb{R}^2} \eta \left((\tilde{\sigma}_{ij})_\varepsilon - \tilde{\sigma}_{ij} \right) \left(\delta_{ij} - 2 \frac{(z_i - y_i)(z_j - y_j)}{|z - y|^2} \right) \frac{dz}{|z - y|^2}.$$

Extend I_ε^{12} and I_0^{12} by 0 outside the ball $B_{1/2}$. Then by using Calderón-Zygmund estimate in Theorem 1.3 and strong convergence of $(\tilde{\sigma}_{ij})_\varepsilon$ in $L \log^+ L$, for any $\rho > 1/2$ we obtain

$$\begin{aligned} \int_{B_{1/2}} |I_\varepsilon^{12}(y) - I_0^{12}(y)| dy &= \int_{B_\rho} |I_\varepsilon^{12}(y) - I_0^{12}(y)| dy \\ &\leq C \sum_{i,j=1}^2 \int_{\mathbb{R}^2} \eta |(\tilde{\sigma}_{ij})_\varepsilon - \tilde{\sigma}_{ij}| dz + C(\text{meas } B_\rho)^{-\frac{1}{2}} \\ &\quad + C \sum_{i,j=1}^2 \int_{\mathbb{R}^2} \eta |(\tilde{\sigma}_{ij})_\varepsilon - \tilde{\sigma}_{ij}| \log^+ \left((\text{meas } B_\rho)^{\frac{3}{2}} \eta |(\tilde{\sigma}_{ij})_\varepsilon - \tilde{\sigma}_{ij}| \right) dz \\ &\leq C(1 + \log^+ \rho) \sum_{i,j=1}^2 \int_{B_1} \eta |(\tilde{\sigma}_{ij})_\varepsilon - \tilde{\sigma}_{ij}| dz + \frac{C}{\rho} \\ &\quad + C \sum_{i,j=1}^2 \int_{B_1} |(\tilde{\sigma}_{ij})_\varepsilon - \tilde{\sigma}_{ij}| \log \left(2 + |(\tilde{\sigma}_{ij})_\varepsilon - \tilde{\sigma}_{ij}| \right) dz \\ &\rightarrow \frac{C}{\rho} \quad \text{as } \varepsilon \rightarrow 0 \\ &\rightarrow 0 \quad \text{as } \rho \rightarrow \infty. \end{aligned}$$

Hence $I_\varepsilon^{12} \rightarrow I_0^{12}$ strongly in $L^1(B_{1/2})$. This proves the claim. \square

Step 4: An explicit representation of the pressure To complete the proof, let us define $q : B_{1/2} \rightarrow \mathbb{R}$ by

$$q(y) := \frac{1}{2\pi} (I_0^{11}(y) + I_0^{12}(y)) + \frac{1}{\pi} I_0^2(y) + \frac{1}{2\pi} (I_0^3(y) + I_0^4(y)).$$

Then from (4.9), (4.11), (4.12), (4.13) and (4.16), we conclude that $q_\varepsilon \rightarrow q$ strongly in $L^1(B_{1/2})$ and q is represented as

$$(4.17) \quad q(y) = \frac{1}{2\pi} \int_{B_1} \sigma(\mathbf{u}^{-1}(z)) : \left(\nabla_z(\mathbf{w}_y(z) - \mathbf{v}_y(z)) - 2\Delta\eta \log|z - y| \right) dz \\ - \frac{1}{2\pi} \int_{B_1} \sigma(\mathbf{u}^{-1}(z)) : \left(Id - 2 \frac{(z - y) \otimes (z - y)}{|z - y|^2} \right) \frac{\eta(z)}{|z - y|^2} dz,$$

where $\sigma(x) := (\sigma_{ij}(x)) \in \mathbb{M}^{2 \times 2}$ given by the equation (3.9). Since q is the strong limit of the family q_ε in ball $B_{1/2}$, it is independent of the choice of the cut-off function η . Following the same arguments as in Section 3, we can extend q to all of $\mathbf{u}(\Omega)$ such that $q \in L^1_{\text{loc}}(\mathbf{u}(\Omega))$ and the pair (\mathbf{u}, q) satisfies the identity (1.7). This completes the proof of Theorem 1.2. \square

5. PARTIAL REGULARITY

Let us denote $L(x, \nabla \mathbf{u}) = \nabla \mathbf{u} - p(x) \nabla \mathbf{u}^{-t}$, then the equation is $\operatorname{div} L(x, \nabla \mathbf{u}) = 0$. First let us examine the ellipticity condition $L_{ij}(x, \xi) \xi_{ij} \geq \lambda |\xi|^2$ for some $\lambda > 0$. Since the deformation is incompressible we obtain

$$(5.1) \quad \nabla \mathbf{u}^{-t} = \begin{pmatrix} u_2^2 & -u_1^2 \\ -u_2^1 & u_1^1 \end{pmatrix}.$$

Introduce $I = L_{ij}(x, \xi) \xi_{ij} = |\xi|^2 - 2p(x) \det \xi$, where ξ is any 2×2 matrix. Then completing squares we get

$$(5.2) \quad \begin{aligned} I &= \xi_{11}^2 + \xi_{12}^2 + \xi_{21}^2 + \xi_{22}^2 - 2p(\xi_{11}\xi_{22} - \xi_{12}\xi_{21}) \\ &= (\xi_{11} - p\xi_{22})^2 + (\xi_{12} - p\xi_{21})^2 + (1 - p^2)(\xi_{22}^2 + \xi_{21}^2) \\ &= (\xi_{22} - p\xi_{11})^2 + (\xi_{21} - p\xi_{12})^2 + (1 - p^2)(\xi_{11}^2 + \xi_{12}^2). \end{aligned}$$

Adding both identities and dividing by 2 we arrive at

$$\begin{aligned} I &= \frac{1}{2}((\xi_{11} - p\xi_{22})^2 + (\xi_{12} - p\xi_{21})^2 + (\xi_{22} - p\xi_{11})^2 + (\xi_{21} - p\xi_{12})^2 + (1 - p^2)|\xi|^2) \\ &\geq \frac{1 - p^2}{2} |\xi|^2. \end{aligned}$$

This computation shows that ellipticity condition

$$L_{ij}(x, \xi) \xi_{ij} \geq \lambda |\xi|^2, \lambda > 0$$

is equivalent to assume that

$$(5.3) \quad p^2 \leq 1 - 2\lambda.$$

Note that p is defined up to addition of arbitrary constant, thus (5.3) is satisfied in subdomain $D \subset \Omega$ if

$$\operatorname{osc}_D p^2 < 1.$$

Next we examine the strong ellipticity condition, i.e.

$$(5.4) \quad L_{ij,kl}(x, \eta) \xi_{ij} \xi_{kl} \geq \lambda |\xi|^2,$$

where η stands as dummy variable for $\nabla \mathbf{u}$. Recall that

$$(5.5) \quad L_{ij}(x, \eta) = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} - p(x) \begin{pmatrix} \eta_{22} & -\eta_{21} \\ -\eta_{12} & \eta_{11} \end{pmatrix}.$$

For instance $L_{11,kl} = \delta_{11,kl} - p\delta_{22,kl}$, and it is easy to check that

$$L_{ij,kl}(x, \eta) \xi_{ij} \xi_{kl} = |\xi|^2 - 2p(x) \det \xi,$$

that is the ellipticity implies strong ellipticity.

In what follows we make the following two assumptions

1 \mathbf{u} is $W^{1,3}(\Omega)$

2 $q(z)$ is α -Hölder continuous with respect to z .

Proposition 5.1. *Under the assumptions 1-2 we have that*

(i)

$$|L_{ij}(x, \nabla \mathbf{u})| \leq L(1 + |\nabla \mathbf{u}|)$$

(ii) for any $x_1, x_2 \in \bar{\Omega}, \eta \in M^{2 \times 2}$

$$\frac{|L_{ij}(x_1, \eta) - L_{ij}(x_2, \eta)|}{1 + |\eta|} \leq C|x_1 - x_2|^\alpha$$

(iii) L_{ij} is differentiable with respect to η with bounded and continuous derivatives

$$|L_{ij,kl}(x, \eta)| \leq L$$

(iv) L_{ij} satisfies to strong ellipticity condition

$$L_{ij,kl}(x, \eta)\eta_{ij}\eta_{kl} \geq \lambda|\eta|^2$$

Proof: Since $\mathbf{u} \in W^{1,3}$, Sobolev imbedding theorem implies that $\mathbf{u} \in C^{1/3}$ then $p(x) = q(\mathbf{u}(x))$ is Hölder continuous and (i)-(ii) follow. (iii)-(iv) follow from (5.3). \square

Remark 5.2. Assumptions (i) – (iv) are stated in [GM 79], in fact they consider more general systems of elliptic equations. Using their theorem 1 we can obtain the following partial regularity result.

Theorem 5.3. Assume that assumptions 1-2 are satisfied, i.e. $\mathbf{u} \in W^{1,3}(\Omega), q \in C^\alpha$. Then the first derivatives of \mathbf{u} are Hölder continuous on an open set Ω_0 . Moreover

$$|\Omega \setminus \Omega_0| = 0.$$

Proof: It follows from proposition then the requirements of theorem 1 in [GM 79] are satisfied and the result follows. \square

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