



THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

## Jacobson radical non-nil algebras of Gelfand-Kirillov dimension 2

**Citation for published version:**

Smoktunowicz, A & Bartholdi, L 2013, 'Jacobson radical non-nil algebras of Gelfand-Kirillov dimension 2', *Israel journal of mathematics*, vol. 194, no. 2, pp. 597-608. <https://doi.org/10.1007/s11856-012-0073-5>

**Digital Object Identifier (DOI):**

[10.1007/s11856-012-0073-5](https://doi.org/10.1007/s11856-012-0073-5)

**Link:**

[Link to publication record in Edinburgh Research Explorer](#)

**Document Version:**

Peer reviewed version

**Published In:**

Israel journal of mathematics

**General rights**

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

**Take down policy**

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact [openaccess@ed.ac.uk](mailto:openaccess@ed.ac.uk) providing details, and we will remove access to the work immediately and investigate your claim.



## ON A CONJECTURE OF GOODEARL: JACOBSON RADICAL NON-NIL ALGEBRAS OF GELFAND-KIRILLOV DIMENSION 2

AGATA SMOKTUNOWICZ AND LAURENT BARTHOLDI

ABSTRACT. For an arbitrary countable field, we construct an associative algebra that is graded, generated by finitely many degree-1 elements, is Jacobson radical, is not nil, is prime, is not PI, and has Gelfand-Kirillov dimension two. This refutes a conjecture attributed to Goodearl.

### 1. INTRODUCTION

Consider an algebra  $R$  over a field  $\mathbb{K}$ , generated by a finite-dimensional subspace  $V$ . The *Gelfand-Kirillov dimension*, or *GK-dimension*, of  $R$  is the infimal  $d$  such that  $\dim(V + V^2 + \cdots + V^n)$  grows slower than  $n^d$  as  $n \rightarrow \infty$ . For example,  $\mathbb{K}[t_1, \dots, t_d]$  has GK-dimension  $d$ . Which constraints does an associative algebra of finite Gelfand-Kirillov dimension have to obey? For example, if  $R$  is a group ring, then the group has polynomial growth, so is virtually nilpotent by Gromov's celebrated theorem [5], so  $R$  is noetherian. For elementary properties of the Gelfand-Kirillov dimension, see [7].

However, various flexible constructions have produced quite exotic examples of finitely generated associative algebras (*affine algebras* in the sequel) of finite GK-dimension [3], and it has been hoped at least that algebras of GK-dimension 2 would enjoy some sort of classification — algebras of GK-dimension  $< 2$  are well understood, and are essentially polynomials in at most one variable, by Bergman's gap theorem [4], and graded domains of GK-dimension 2 are essentially twisted coordinate rings of projective curves [1].

An element  $x$  in a ring  $R$  is *quasi-regular* if there exists  $y \in R$  with  $x + y + xy = 0$ . This happens, for instance, if  $x$  is nilpotent (take  $y = -x + x^2 - x^3 + \cdots$ ). Conversely, if  $R$  is graded, then homogeneous quasi-regular elements are nilpotent. The *Jacobson radical*  $J(R)$  of  $R$  is the largest ideal all of whose elements are quasi-regular. A ring is *radical* if it is equal to its Jacobson radical; note then, in particular, that it may not contain a unit (in fact, not even a non-trivial idempotent:  $x^2 = x, -x + y - xy = 0 \Rightarrow -x^2 + xy - x^2y = -x^2 = -x = 0$ ).

A typical result showing the connection between nillicity and the structure of the Jacobson radical is:  $R$  is artinian, then  $J(R)$  is nilpotent. The following structural result was expected:

**Conjecture** (Goodearl, [3, Conjecture 3.1]). *If  $R$  is an affine algebra of GK-dimension 2, then its Jacobson radical  $J(R)$  is nil.*

We disprove this conjecture, by constructing for every countable field  $\mathbb{K}$  an algebra  $R$  over  $\mathbb{K}$ , which is

- graded by the natural numbers;

---

*Date:* 12 February 2011.

*2010 Mathematics Subject Classification.* 16N40, 16P90.

*Key words and phrases.* Goodearl conjecture, Nil algebras, the Jacobson radical, growth of algebras, Gelfand-Kirillov dimension.

The research of the first author was supported by Grant No. EPSRC EP/D071674/1.

- generated by finitely many degree-1 elements;
- prime;
- of Gelfand-Kirillov dimension 2;
- equal to its Jacobson radical;
- not PI (i.e. does not satisfy a polynomial identity);
- not nil.

Our strategy is to adapt a construction of the first author, see [13], by showing that it may yield non-nil algebras. Some tools are also borrowed from the second author's paper [2]; however, the construction given there is not correct, and indeed not yield a radical algebra. One of the goals of this paper is therefore to give a correct solution to the problem raised by Goodearl.

## 2. THE CONSTRUCTION

We begin by constructing the following algebra  $P$ ; the proof of this theorem will be split over the next three sections.

**Theorem 2.1.** *Over every countable field  $\mathbb{K}$  of characteristic zero, there exists a radical algebra  $P$ , such that the polynomial ring  $P[X]$  is not radical.*

*Moreover,  $P$  may be chosen to have Gelfand-Kirillov dimension two, be  $\mathbb{N}$ -graded and generated by two elements of degree one.*

We then show that a sufficiently large ring of matrices over such a  $P$  is not nil:

**Proposition 2.2.** *Let  $P$  be a radical algebra such that the polynomial ring  $P[X]$  is not radical. Then there is a natural number  $n$  such that the algebra  $M_n(P)$  of  $n$  by  $n$  matrices over  $P$  is not nil.*

*Proof.* Suppose that  $P$  is radical and that, for every  $n \in \mathbb{N}$ , the ring  $M_n(P)$  is nil. Write  $R = P[X]$  and  $\mathcal{S} = XR$ ; we will deduce that  $R$  is radical. Observe that  $M_n(XP)$  is nil for all  $n \in \mathbb{N}$ , and  $\mathcal{S} = XP + (XP)^2 + \dots$ ; therefore, by [12, Theorem 1.2], the ring  $\mathcal{S}$  is radical. Notice then that  $\mathcal{S}$  is an ideal in  $R$ , and  $R/\mathcal{S} = P$  is radical. Now, if both  $\mathcal{S}$  and  $R/\mathcal{S}$  are radical, then so is  $R$ .  $\square$

**Lemma 2.3.** *Let  $R$  be a non-nil ring. Then there exists a quotient  $R/\mathcal{S}$  that is non-nil and prime. If  $R$  is graded, then  $R/\mathcal{S}$  may also be taken to be graded.*

*Proof.* Let  $a \in R$  be non-nilpotent. Let  $\mathcal{S}$  be a maximal ideal in  $R$  subject to being disjoint with  $\{a^n : n = 1, 2, \dots\}$ . Then  $R/\mathcal{S}$  is still not nil. Consider ideals  $\mathcal{P}, \mathcal{Q} \supsetneq \mathcal{S}$  with  $\mathcal{P}\mathcal{Q} \subseteq \mathcal{S}$ . By maximality of  $\mathcal{S}$ , we have  $a^m \in \mathcal{P}$  and  $a^n \in \mathcal{Q}$  for some  $m, n \in \mathbb{N}$ ; but then  $a^{m+n} \in \mathcal{S}$ , a contradiction. Therefore,  $R/\mathcal{S}$  is prime.

If  $R$  is graded, let  $\mathcal{S}$  be a maximal *homogeneous* ideal subject to being disjoint with  $\{a^n : n = 1, 2, \dots\}$ . We claim that  $\mathcal{S}$  is a prime ideal in  $R$ . Suppose the contrary; then there are elements  $p, q \notin \mathcal{S}$  such that  $prq \in \mathcal{S}$  for all  $r \in R$ . Write  $p = p_1 + \dots + p_d$  and  $q = q_1 + \dots + q_e$  in homogeneous components, and let  $p_i$  and  $q_j$  denote those summands, for minimal  $i, j$ , that do not belong to  $\mathcal{S}$ .

By assumption,  $prq \in \mathcal{S}$  for all homogeneous  $r \in R$  (say of degree  $k$ ); so, by considering the component of degree  $i+k+j$  of  $prq$ , we see that  $p_i r q_j$  belongs to  $\mathcal{S}$  for all homogeneous  $r \in R$  (because  $\mathcal{S}$  is graded), whence  $p_i r q_j \in \mathcal{S}$  for all  $r \in R$ .

Let now  $\mathcal{P}$  be the ideal generated by  $p_i$  and  $\mathcal{S}$ ; and, similarly, let  $\mathcal{Q}$  be the ideal generated by  $q_j$  and  $\mathcal{S}$ . Then, by maximality of  $\mathcal{S}$ , we have  $a^m \in \mathcal{P}$  and  $a^n \in \mathcal{Q}$  for some  $m, n \in \mathbb{N}$ ; but then  $a^{m+n} \in \mathcal{P}\mathcal{Q} \subseteq \mathcal{S}$ , a contradiction. Therefore,  $R/\mathcal{S}$  is prime.  $\square$

Combining these results, we deduce:

**Corollary 2.4.** *Over any countable field  $\mathbb{K}$ , there exists a non-nil non-PI radical prime algebra  $R$ , of Gelfand-Kirillov dimension two,  $\mathbb{N}$ -graded and generated by finitely many elements of degree one.*

*Proof.* Let  $P$  be as in Theorem 2.1. By Proposition 2.2, the ring  $R_0 = M_n(P)$  is radical and non-nil for  $n$  large enough. By Lemma 2.3, some quotient  $R$  of  $R_0$  is radical and prime. Because  $P$  is radical, its ring of matrices  $R_0$  is also radical, and so is its quotient  $R$ . Because  $P$  has GK-dimension  $\leq 2$ , so do  $R_0$  and  $R$ . If  $R$  has GK-dimension  $< 2$ , it would have dimension  $\leq 1$  by Bergman's gap theorem [4], so would be finitely generated as a module over its centre by [10], so  $R$ 's radical would be nilpotent, a contradiction; therefore,  $R$  has GK-dimension exactly 2.

Since  $P$  is generated by 2 elements of degree 1, the rings  $R_0$  and  $R$  are generated by finitely many elements of degree 1 (the elementary matrices).

Finally,  $R$  is not PI; indeed, by the Razmyslov-Kemer-Braun theorem [6, §2.5], if  $R$  were PI then its radical would be nilpotent.  $\square$

### 3. NOTATION AND PREVIOUS RESULTS

Our notation closely matches that of [13]. In what follows,  $\mathbb{K}$  is a countable field and  $A$  is the free associative  $\mathbb{K}$ -algebra in three non-commuting indeterminates  $x, y, z$ . The set of monomials in  $\{x, y\}$  is denoted by  $M$  and, for  $n \geq 0$ , the set of monomials of degree  $n$  is denoted by  $M(n)$ . In particular,  $M(0) = \{1\}$  and for  $n \geq 1$  the elements in  $M(n)$  are of the form  $x_1 \cdots x_n$  with  $x_i \in \{x, y\}$ . The *augmentation ideal* of  $A$ , consisting of polynomials without constant term, is denoted by  $\bar{A}$ .

The  $\mathbb{K}$ -subspace of  $A$  spanned by  $M(n)$  is denoted by  $A(n)$ , and elements of  $A(n)$  are called *homogeneous polynomials of degree  $n$* . More generally, if  $S$  is a subset of  $A$ , then its homogeneous part  $S(n)$  is defined as  $S \cap A(n)$ .

The *degree*,  $\deg f$ , of  $f \in A$ , is the least  $d \geq 0$  such that  $f \in A(0) + \cdots + A(d)$ . Any  $f \in A$  can be uniquely written in the form  $f = f_0 + f_1 + \cdots + f_d$ , with  $f_i \in A(i)$ . The elements  $f_i$  are the *homogeneous components* of  $f$ . A (right, left, two-sided) ideal  $\mathcal{S}$  of  $A$  is *homogeneous* if, for every  $f \in \mathcal{S}$ , all its homogeneous components belong to  $\mathcal{S}$ .

**Lemma 3.1** ([13, Lemma 6]). *Let  $\mathbb{K}$  be a countable field, and let  $\bar{A}$  be as above. Then there exists a subset  $Z \subset \{5, 6, \dots\}$ , and an enumeration  $\{f_i\}_{i \in Z}$  of  $\bar{A}$ , such that*

$$i > 3^{2\deg(f_i)+2}(\deg(f_i) + 1)^2 \text{ for all } i \in Z.$$

Define the sequence  $e(i) = 2^{2^{2^i}}$ , and set

$$S = \bigcup_{i \geq 5} \{e(i) - i - 1, e(i) - i, \dots, e(i) - 1\}.$$

**Lemma 3.2** ([13, Theorem 9]). *Let  $Z$  and  $\{f_i\}_{i \in Z}$  be as in Lemma 3.1. Fix  $m \in Z$ , and set  $w_m = 2^{e(m)+2}$ . Then there is a two-sided ideal  $\mathcal{P}_m \leq \bar{A}$  such that*

- the ideal  $\mathcal{P}_m$  is generated by homogeneous elements of degrees larger than  $10w_m$ ;
- there exists  $g_m \in \bar{A}$  such that  $f_m - g_m + f_m g_m \in \mathcal{P}_m$ ;
- there is a linear  $\mathbb{K}$ -space  $F_m \subseteq A(2^{e(m)})$  such that  $\mathcal{P}_m \subseteq \sum_{k=0}^{\infty} A(w_m k) F_m A$  and  $\dim_{\mathbb{K}}(F_m) < m$ .

**Lemma 3.3** ([13, Theorem 10]). *Let  $Z$  and  $F_m$  be as in Lemma 3.2. There are  $\mathbb{K}$ -linear subspaces  $U(2^n)$  and  $V(2^n)$  of  $A(2^n)$  such that, for all  $n \in \mathbb{N}$ ,*

- (1)  $\dim_{\mathbb{K}} V(2^n) = 2$  if  $n \notin S$ ;
- (2)  $\dim_{\mathbb{K}} V(2^{e(i)-i-1+j}) = 2^{2^j}$ , for all  $i \geq 5$  and all  $j \in \{1, \dots, i-1\}$ ;
- (3)  $V(2^n)$  is spanned by monomials;

- (4)  $F_i \subseteq U(2^{e(i)})$  for every  $i \in Z$ ;
- (5)  $V(2^n) \oplus U(2^n) = A(2^n)$ ;
- (6)  $A(2^n)U(2^n) + U(2^n)A(2^n) \subseteq U(2^{n+1})$ ;
- (7)  $V(2^{n+1}) \subseteq V(2^n)V(2^n)$ ;
- (8) if  $n \notin S$  then there are monomials  $m_1, m_2 \in V(2^n)$  such that  $V(2^n) = \mathbb{K}m_1 + \mathbb{K}m_2$  and  $m_2A(2^n) \subseteq U(2^{n+1})$ .

#### 4. NEW RESULTS

Consider the polynomial ring  $A[X]$  in an indeterminate  $X$ . Consider the elements  $(x + Xy)^n$ . Write

$$w(n, i) = \sum_{\substack{m \in M(n) \\ \deg_y m = n-i, \deg_x m = i}} m,$$

and observe that  $(x + Xy)^{2^n} = \sum_{i=0}^{2^n} w(2^n, 2^n - i)X^i$ . Let  $W(n)$  denote the linear span of all  $w(n, i)$  with  $i \in \{0, \dots, n\}$ .

We extend the results of the previous section by imposing additional conditions on the  $U(n)$  and  $V(n)$  constructed in Lemma 3.3. Throughout this section, we use the notation

$$T(2^{n+1}) = A(2^n)U(2^n) + U(2^n)A(2^n).$$

**Proposition 4.1.** *There exist subspaces  $U(2^n), V(2^n) \subseteq A(2^n)$  satisfying all assumptions from Lemma 3.3, with the additional property that*

- (9) for all  $n \in \mathbb{N}$ , if  $i \in \mathbb{N}$  be such that  $\{n, n-1, \dots, n-i\} \subset S$ , then

$$\dim_{\mathbb{K}}(W(2^n) + U(2^n)) \geq \dim_{\mathbb{K}}U(2^n) + 2 + i;$$

- (10)  $z \in U(2^0) = U(1)$ .

**Lemma 4.2.** *If  $\dim_{\mathbb{K}}(W(2^n) + U(2^n)) \geq \dim_{\mathbb{K}}U(2^n) + 2$  and  $m_1, m_2 \in V(2^n)$  are linearly independent, then there exists  $h \in \{1, 2\}$  such that*

$$\dim_{\mathbb{K}}(W(2^{n+1}) + T(2^{n+1}) + m_hV(2^n)) \geq \dim(T(2^{n+1}) + m_hV(2^n)) + 2.$$

*Proof.* Let  $i \geq 0$  be minimal such that  $w(2^n, i)$  does not belong to  $U(2^n)$ , and let  $j > i$  be minimal such that  $w(2^n, j)$  does not belong to  $U(2^n) + \mathbb{K}w(2^n, i)$ . By the inductive assumption such elements can be found. By permuting  $m_1$  and  $m_2$  if necessary, we may assume that  $w(2^n, i)$  is not a multiple of  $m_2$ , and we choose  $h = 2$ . We have

$$w(2^{n+1}, 2i) = \sum_{k=-i}^i w(2^n, i+k)w(2^n, i-k),$$

and either

- 1.  $k = 0$ ,
- or 2.  $k < 0$ , in which case  $w(2^n, i+k) \in U(2^n)$ ,
- or 3.  $k > 0$ , in which case  $w(2^n, i-k) \in U(2^n)$ .

Consequently, we get

$$(1) \quad w(2^{n+1}, 2i) \equiv w(2^n, i)w(2^n, i) \pmod{T(2^{n+1})}.$$

Consider now

$$w(2^{n+1}, i+j) = \sum_{k=-i}^j w(2^n, i+k)w(2^n, j-k);$$

then either

- 1.  $k < 0$ , in which case  $w(2^n, i+k) \in U(2^n)$ ,

- or 2.  $0 < k < j - i$ , in which case  $w(2^n, i + k) \in U(2^n) + \mathbb{K}w(2^n, i)$  and  $w(2^n, j - k) \in U(2^n) + \mathbb{K}w(2^n, i)$ ,  
or 3.  $k = 0$  or  $k = j - i$ ,  
or 4.  $k > j - i$ , in which case  $w(2^n, j - k) \in U(2^n)$ .

Consequently, we get

$$(2) \quad w(2^{n+1}, i + j) \equiv w(2^n, i)w(2^n, j) + w(2^n, j)w(2^n, i) \pmod{T(2^{n+1}) + \mathbb{K}w(2^n, i)w(2^n, i)}.$$

Recall now that we have

$$w(2^n, i) \equiv t_{i1}m_1 + t_{i2}m_2 \pmod{U(2^n)}, \quad w(2^n, j) \equiv t_{j1}m_1 + t_{j2}m_2 \pmod{U(2^n)}$$

for some  $t_{i1}, t_{i2}, t_{j1}, t_{j2} \in \mathbb{K}$ . Furthermore,  $t_{i1} \neq 0$ , and the vectors  $(t_{i1}, t_{i2})$  and  $(t_{j1}, t_{j2})$  are linearly independent over  $\mathbb{K}$ . Write  $Q = T(2^{n+1}) + m_2V(2^n)$ , so that  $Q$  contains  $m_2m_2$  and  $m_2m_1$ .

It follows from (1) that  $w(2^{n+1}, 2i) \equiv t_{i1}^2m_1m_1 + t_{i1}t_{i2}m_1m_2 \pmod{Q}$ ; and, because  $t_{i1} \neq 0$ , we have  $w(2^{n+1}, 2i) \notin Q$ .

Similarly, from (2) we get  $w(2^{n+1}, i + j) \equiv 2t_{i1}t_{j1}m_1m_1 + (t_{j1}t_{i2} + t_{i1}t_{j2})m_1m_2 \pmod{Q + \mathbb{K}w(2^{n+1}, 2i)}$ ; and, because the vectors  $(t_{i1}, t_{i2})$  and  $(t_{j1}, t_{j2})$  are linearly independent, so are  $(2t_{i1}t_{j1}, t_{j1}t_{i2} + t_{i1}t_{j2})$  and  $(t_{i1}^2, t_{i1}t_{i2}) = t_{i1}(t_{i1}, t_{i2})$ , so we have  $w(2^{n+1}, i + j) \notin Q + \mathbb{K}w(2^{n+1}, 2i)$ .

We then get  $\dim_{\mathbb{K}}(W(2^{n+1}) + Q) \geq \dim_{\mathbb{K}}Q + 2$  as required.  $\square$

**Lemma 4.3.**  $\dim_{\mathbb{K}}(W(2^{n+1}) + T(2^{n+1})) \geq \dim_{\mathbb{K}}(W(2^n) + T(2^n)) + 1$ .

*Proof.* Let there be  $k_1, k_2, \dots, k_j \in \mathbb{N}$  such that

$$w(2^n, k_1), w(2^n, k_2), \dots, w(2^n, k_j)$$

are linearly independent modulo  $T(2^n)$ . We may assume that the sequence  $(k_1, \dots, k_j)$  is minimal with this property in the lexicographical ordering. We claim that the elements  $w(2^{n+1}, 2k_j)$  and  $w(2^{n+1}, k_1 + k_m)$  for  $1 \leq m \leq j$  are linearly independent modulo  $T(2^{n+1})$ . There are  $j + 1$  such elements, as required. As in (1) we observe

$$w(2^{n+1}, 2k_1) \equiv w(2^n, k_1)w(2^n, k_1) \pmod{T(2^{n+1})},$$

and similarly, for each  $m \in \{1, \dots, j\}$  we have

$$w(2^{n+1}, k_1 + k_m) \equiv w(2^n, k_1)w(2^n, k_m) + w(2^n, k_m)w(2^n, k_1) \pmod{T(2^{n+1}) + \sum_{\substack{1 \leq p < m \\ 1 \leq q < m}} \mathbb{K}w(2^n, k_p)w(2^n, k_q)}.$$

Therefore,  $w(2^{n+1}, k_1 + k_m)$  contains the summand  $w(2^n, k_1)w(2^n, k_m) + w(2^n, k_m)w(2^n, k_1)$  which no  $w(2^{n+1}, k_1 + k_p)$  with  $p < m$  contains.

Finally,

$$w(2^{n+1}, 2k_j) \equiv w(2^n, k_j)w(2^n, k_j) \pmod{T(2^{n+1}) + \sum_{p=1}^{j-1} w(2^n, k_p)A(2^n) + A(2^n)w(2^n, k_p)},$$

so  $w(2^{n+1}, 2k_j)$  contains the summand  $w(2^n, k_j)w(2^{n+1}, k_j)$  which none of the previous elements contains. It follows that the  $j + 1$  elements we exhibited are linearly independent modulo  $T(2^{n+1})$ .  $\square$

*Proof of Proposition 4.1.* We adapt the proof of [13, Theorem 10] to show how the additional assumptions may be satisfied. In fact, (10) is already part of the construction.

Recall that the proof of [13, Theorem 10] constructs sets  $U(2^{n+1})$  and  $V(2^{n+1})$  by induction. The following cases are considered:

1.  $n \in S$  and  $n + 1 \in S$ .
2.  $n \notin S$ .
3.  $n \in S$  and  $n + 1 \notin S$ .

We modify cases 2 and 3, while not changing case 1, which we repeat for convenience of the reader:

**Case 1:  $n \in S$  and  $n + 1 \in S$ .** Define  $U(2^{n+1}) = T(2^{n+1})$  and  $V(2^{n+1}) = V(2^n)V(2^n)$ . Conditions (6,7) certainly hold. If, by induction, Conditions (3,5) hold for  $U(2^n)$  and  $V(2^n)$ , they hold for  $U(2^{n+1})$  and  $V(2^{n+1})$  as well. Moreover,  $\dim_{\mathbb{K}} V(2^n) = (\dim_{\mathbb{K}} V(2^n))^2$ , inductively satisfying Condition (2). Finally, Condition (9) follows directly from Lemma 4.3.

**Case 2:  $n \notin S$ .** We begin as in the original argument:  $\dim_{\mathbb{K}} V(2^n) = 2$ , and is generated by monomials, by the inductive hypothesis. Let  $m_1, m_2$  be the distinct monomials that generate  $V(2^n)$ . Then  $V(2^n)V(2^n) = \mathbb{K}m_1m_1 + \mathbb{K}m_1m_2 + \mathbb{K}m_2m_1 + \mathbb{K}m_2m_2$ . By Lemma 4.2, there exists  $h \in \{1, 2\}$  such that

$$\dim_{\mathbb{K}}(W(2^{n+1}) + T(2^{n+1}) + m_hV(2^n)) \geq \dim(T(2^{n+1}) + m_hV(2^n)) + 2.$$

Permuting  $m_1$  and  $m_2$  if necessary, we assume  $h = 2$ , and set

$$U(2^{n+1}) = T(2^{n+1}) + m_2V(2^n), \quad V(2^{n+1}) = \mathbb{K}m_1m_1 + \mathbb{K}m_1m_2.$$

It is clear that Conditions (1,3,6,7,9) hold, and Condition (5) follows from

$$\begin{aligned} A(2^{n+1}) &= A(2^n)A(2^n) \\ &= U(2^n)U(2^n) \oplus U(2^n)V(2^n) \oplus V(2^n)U(2^n) \oplus m_1V(2^n) \oplus m_2V(2^n) \\ &= U(2^{n+1}) \oplus V(2^{n+1}). \end{aligned}$$

**Case 3:  $n \in S$  and  $n + 1 \notin S$ .** We begin as in the original argument: we have  $n = e(i) - 1$  for some  $i > 0$ . By the inductive hypothesis, we have  $\dim_{\mathbb{K}}(W(2^n) + T(2^n)) \geq \dim_{\mathbb{K}} T(2^n) + i + 1$ . One more application of Lemma 4.3 gives

$$\dim_{\mathbb{K}}(W(2^{n+1}) + T(2^{n+1})) \geq \dim_{\mathbb{K}} T(2^{n+1}) + i + 2.$$

So as to treat simultaneously the cases  $i \in Z$  and  $i \notin Z$ , we extend Condition (4) to all  $i \in \mathbb{N}$  by taking  $F_i = 0$  and  $s = 0$  if  $i \notin Z$ .

We know that  $F_i$  has a basis  $\{f_1, \dots, f_s\}$  for some  $f_1, \dots, f_s \in A(2^{e(i)})$  and  $s < i$ . Write each  $f_j$  as  $f_j = \bar{f}_j + g_j$  for  $\bar{f}_j \in V(2^n)V(2^n)$  and  $g_j \in T(2^{n+1})$ . Since  $V(2^n)V(2^n) \cap T(2^{n+1}) = 0$ , this decomposition is unique.

Since  $s < i$ , there are elements  $w_1, w_2 \in W(2^{e(i)})$  such that

$$(\mathbb{K}w_1 + \mathbb{K}w_2) \cap (T(2^{n+1}) + \mathbb{K}\bar{f}_1 + \dots + \mathbb{K}\bar{f}_s) = 0.$$

Let  $P$  be a linear  $\mathbb{K}$ -subspace of  $V(2^n)V(2^n)$  maximal with the properties that  $(\mathbb{K}w_1 + \mathbb{K}w_2) \cap (P + T(2^{n+1})) = 0$  and  $\bar{f}_j \in P$  for all  $j \in \{1, \dots, s\}$ .

Observe that  $P$  has codimension 2 in  $V(2^n)V(2^n)$ . Since the monomials in  $V(2^n)V(2^n)$  form a basis, there are two such monomials, say  $m_1$  and  $m_2$ , that are linearly independent modulo  $P$ . Define then

$$V(2^{n+1}) = \mathbb{K}m_1 + \mathbb{K}m_2, \quad U(2^{n+1}) = T(2^{n+1}) + P.$$

Conditions (5,6) are immediately satisfied. Since each polynomial  $f_j = g_j + \bar{f}_j$  belongs to  $U(2^{n+1})$ , Condition (4) is satisfied as well.

To end the proof, observe now that  $\{w_1, w_2\}$  are linearly independent modulo  $U(2^{n+1})$ , so  $\dim_{\mathbb{K}}(\mathbb{K}w_1 + \mathbb{K}w_2 + U(2^{n+1})) = \dim_{\mathbb{K}} U(2^{n+1}) + 2$ ; this proves (9).  $\square$

## 5. PROOF OF THEOREM 2.1

We present  $P$  as a quotient  $\bar{A}/\mathcal{E}$  for a suitable ideal  $\mathcal{E}$ ; we follow [13, page 844]. First,  $\mathcal{E}$  is a graded ideal:  $\mathcal{E} = \mathcal{E}(1) + \mathcal{E}(2) + \cdots$ , so it suffices to define  $\mathcal{E}(n)$  for all  $n \in \mathbb{N}$ . By definition,  $\mathcal{E}(n)$  is the maximal subset of  $A(n)$  such that, if  $m \in \mathbb{N}$  be such that  $2^m \leq n < 2^{m+1}$ , then

$$(3) \quad A(j)\mathcal{E}(n)A(2^{m+2} - j - n) \subseteq U(2^{m+1})A(2^{m+1}) + A(2^{m+1})U(2^{m+1})$$

for all  $j \in \{0, \dots, 2^{m+2} - n\}$ ; or, more briefly,  $(A\mathcal{E}A)(2^m) \subseteq T(2^m)$  for all  $m \in \mathbb{N}$ .

**Theorem 5.1.** *The subset  $\mathcal{E}$  is an ideal in  $\bar{A}$ . Moreover,  $P := \bar{A}/\mathcal{E}$  is radical, has Gelfand-Kirillov dimension two, is  $\mathbb{N}$ -graded and generated by two degree-1 elements, and  $P[X]$  is not radical.*

*Proof.* By [13, Theorem 20], the GK-dimension of  $P$  is at most 2; it is in fact exactly 2, by Bergman's gap theorem [4]. Also,  $P$  is radical by [13, Theorem 24]. Moreover,  $z \in U(1) = \mathcal{E}(1)$ , so  $P$  is generated by the images of  $x$  and  $y$  in  $\bar{A}/\mathcal{E}$ .

Recall that  $X$  is a free indeterminate commuting with  $x$  and  $y$ . Consider  $n \geq 2$ . By Proposition 4.1, not all  $w(2^n, i)$  belong to  $U(2^n)$ , so  $(x + Xy)^{2^n} \notin U(2^n) \otimes \mathbb{K}[X]$ , so  $(x + Xy)^{2^{n-2}} \notin \mathcal{E}[X]$  by (3), so  $(x + Xy)^{2^{n-2}} \neq 0$  in  $P[X]$ . Since  $n$  may be taken arbitrarily large, it follows that  $x + Xy$  is not nilpotent.

If  $X$  be now declared to have degree 0, then  $P[X]$  is a graded ring, and  $x + Xy$  is homogeneous and not nilpotent. However, in a graded ring, a homogeneous element belongs to the Jacobson radical if and only if it is nilpotent; it therefore follows that  $P[X]$  is not radical.  $\square$

## 6. FINAL REMARKS

The methods employed here depend crucially on the hypothesis that  $\mathbb{K}$  is countable. We don't know if there are finitely generated radical algebras of Gelfand-Kirillov dimension two over an uncountable field. By Amitsur's theorem, such algebras must be nil.

The argument in Theorem 5.1 requires us, in particular, to construct a ring  $P$  such that  $P[X]$  is not graded nil. We do not know if  $P$  is nil; if so, this would be an improvement over [11], in which Smoktunowicz constructs a nil ring  $R$  such that  $R[X]$  is not nil.

We note that, over any countable field, nil algebras of Gelfand-Kirillov dimension at most three were constructed by Lenagan, Smoktunowicz and Young [8, 9].

It remains an open problem whether there exist affine self-similar algebras satisfying the conditions of Corollary 2.4.

We are also unable to construct an algebra of quadratic growth (i.e. whose growth function is bounded by a polynomial of degree two). The algebras  $R$  constructed here do admit an upper bound on their growth of the form  $\dim_{\mathbb{K}}(R(1) + \cdots + R(n)) \leq Cn^2 \log(n)^3$ , see [13, Theorem 20].

We finally refer to Zelmanov's survey [14] for a wealth of similar problems.

## REFERENCES

- [1] M. Artin and J. T. Stafford, *Noncommutative graded domains with quadratic growth*, Invent. Math. **122** (1995), no. 2, 231–276, DOI 10.1007/BF01231444. MR1358976 (96g:16027)
- [2] Laurent Bartholdi, *Branch rings, thinned rings, tree enveloping rings*, Israel J. Math. **154** (2006), 93–139, available at [arXiv:math.RA/0410226](https://arxiv.org/abs/math.RA/0410226). MR2254535
- [3] Jason P. Bell, *Examples in finite Gelfand-Kirillov dimension*, J. Algebra **263** (2003), no. 1, 159–175. MR1974084 (2004d:16042)
- [4] George M. Bergman, *A note on growth functions of algebras and semigroups*, 1978. unpublished.



- [5] Mikhael L. Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. **53** (1981), 53–73.
- [6] Alexei Kanel-Belov and Louis Halle Rowen, *Computational aspects of polynomial identities*, Research Notes in Mathematics, vol. 9, A K Peters Ltd., Wellesley, MA, 2005. MR2124127 (2006b:16001)
- [7] Günter R. Krause and Thomas H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, Revised edition, Graduate Studies in Mathematics, vol. 22, American Mathematical Society, Providence, RI, 2000.
- [8] T. H. Lenagan and Agata Smoktunowicz, *An infinite dimensional affine nil algebra with finite Gelfand-Kirillov dimension*, J. Amer. Math. Soc. **20** (2007), no. 4, 989–1001 (electronic), DOI 10.1090/S0894-0347-07-00565-6. MR2328713 (2008f:16047)
- [9] Tom H. Lenagan, Agata Smoktunowicz, and Alexander Young, *Nil algebras with restricted growth* (2010), available at [arXiv:math/1008.4461v1](https://arxiv.org/abs/math/1008.4461v1).
- [10] L. W. Small and R. B. Warfield Jr., *Prime affine algebras of Gelfand-Kirillov dimension one*, J. Algebra **91** (1984), no. 2, 386–389, DOI 10.1016/0021-8693(84)90110-8. MR769581 (86h:16006)
- [11] Agata Smoktunowicz, *Polynomial rings over nil rings need not be nil*, J. Algebra **233** (2000), no. 2, 427–436, DOI 10.1006/jabr.2000.8451. MR1793911 (2001i:16045)
- [12] ———, *The Jacobson radical of rings with nilpotent homogeneous elements*, Bull. Lond. Math. Soc. **40** (2008), no. 6, 917–928, DOI 10.1112/blms/bdn086. MR2471940 (2009m:16038)
- [13] ———, *Jacobson radical algebras with Gelfand-Kirillov dimension two over countable fields*, J. Pure Appl. Algebra **209** (2007), no. 3, 839–851, DOI 10.1016/j.jpaa.2006.08.003. MR2298862 (2008c:16032)
- [14] Efim Zelmanov, *Some open problems in the theory of infinite dimensional algebras*, J. Korean Math. Soc. **44** (2007), no. 5, 1185–1195, DOI 10.4134/JKMS.2007.44.5.1185. MR2348741 (2008g:16053)

AGATA SMOKTUNOWICZ:

MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES  
 SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH,  
 JAMES CLERK MAXWELL BUILDING, KING'S BUILDINGS, MAYFIELD ROAD,  
 EDINBURGH EH9 3JZ, SCOTLAND, UK

*E-mail address:* [A.Smoktunowicz@ed.ac.uk](mailto:A.Smoktunowicz@ed.ac.uk)

LAURENT BARTHOLDI:

MATHEMATISCHES INSTITUT  
 GEORG AUGUST-UNIVERSITÄT ZU GÖTTINGEN  
 BUNSENSTRASSE 3–5  
 D-37073 GÖTTINGEN  
 GERMANY

*E-mail address:* [laurent.bartholdi@gmail.com](mailto:laurent.bartholdi@gmail.com)