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FOURIER RESTRICTION TO POLYNOMIAL CURVES I: A GEOMETRIC INEQUALITY

SPYRIDON DENDRINOS AND JAMES WRIGHT

ABSTRACT. We prove a Fourier restriction result for general polynomial curves in \mathbb{R}^d . Measuring the Fourier restriction with respect to the affine arclength measure of the curve, we obtain a universal estimate for the class of all polynomial curves of bounded degree. Our method relies on establishing a geometric inequality for general polynomial curves which is of interest in its own right. Applications of this geometric inequality to other problems in euclidean harmonic analysis have recently been established.

1. INTRODUCTION

Recently there has been considerable attention given to certain euclidean harmonic analysis problems associated to a surface or curve (for example, the problems of Fourier restriction and the smoothing effects of generalised Radon transforms) where the underlying euclidean surface or arclength measure is replaced by the so-called affine surface or arclength measure. See [1], [3], [5], [8], [10], [12], [14], [15], [16], [18], [19], [20] and [21]. This has the effect of making the problem affine invariant as well as invariant under reparametrisations of the underlying variety. For this reason there have been many attempts to obtain universal results, establishing uniform bounds over a large class of surfaces or curves. The affine surface or arclength measure also has the mitigating effect of dampening any curvature degeneracies of the surface or curve and therefore the expectation is that the universal bounds one seeks will be the same as those arising from the most non-degenerate situation.

In this paper we establish such a result for the problem of Fourier restriction to a general polynomial curve in \mathbb{R}^d . More specifically, if $\Gamma : I \rightarrow \mathbb{R}^d$ parametrises a smooth curve in \mathbb{R}^d on an interval I , set

$$L_\Gamma(t) = \det(\Gamma'(t) \cdots \Gamma^{(d)}(t));$$

this is the determinant of a $d \times d$ matrix whose j th column is given by the j th derivative of Γ , $\Gamma^{(j)}(t)$. The affine arclength measure $\nu = \nu_\Gamma$ on Γ is defined on a test function ϕ by

$$\nu(\phi) = \int_I \phi(\Gamma(t)) |L_\Gamma(t)|^{\frac{2}{d(d+1)}} dt;$$

one easily checks that this measure is invariant under reparametrisations of Γ .

A basic problem in the theory of Fourier restriction is to determine the exponents p and q so that the apriori estimate

$$\|\widehat{f}\|_{L^q(\Gamma, d\nu)} \leq C \|f\|_{L^p(\mathbb{R}^d)} \tag{1}$$

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holds uniformly for a large class of curves Γ . This problem was first considered by Sjölin in [22] where he showed that (1) holds uniformly over all smooth convex curves in the plane if and only if $p' = 3q$ (here p' denotes the conjugate exponent to p ; $p' = p/(p-1)$) and $1 \leq p < 4/3$. See also [19]. The convexity assumption implies that $L_\Gamma(t)$ remains single-signed and Sjölin produced a plane curve Γ where L_Γ rapidly changes sign and (1) fails for any $p' = 3q$ and $1 < p < 4/3$ (Sjölin's argument establishing (1) for convex curves works for any smooth plane curve as long as the number of sign changes of L_Γ remains bounded).

By considering the non-degenerate example $\Gamma(t) = (t, t^2, \dots, t^d)$ where $L_\Gamma \equiv \text{constant}$, one sees that in order for (1) to hold with a uniform constant C independent of the interval I where $\Gamma : I \rightarrow \mathbb{R}^d$, we must have $p' = \frac{d(d+1)}{2}q$ and $1 \leq p < \frac{d^2+d+2}{d^2+d}$. The former restriction follows from a simple scaling argument whereas the latter restriction follows from work of Arkhipov, Kuratsuba and Chubarikov [2]. Furthermore, Drury [13] showed that these restrictions on p and q are sufficient for (1) to hold for this non-degenerate example (see also the recent work of Bak, Oberlin and Seeger [3]). We note here that on the critical line $p' = \frac{d(d+1)}{2}q$, (1) becomes affine invariant; that is, (1) remains unchanged if we consider any affine transformation of Γ .

In higher dimensions the problem of understanding when (1) holds was first considered by Drury and Marshall [15] (see also [14] and [16]). Recently Bak, Oberlin and Seeger [3] have shown that if $p' = \frac{d(d+1)}{2}q$ and $1 \leq p < \frac{d^2+d+2}{d^2+d}$, then (1) holds for $\Gamma(t) = (|t|^{a_1}, \dots, |t|^{a_d})$ where a_1, \dots, a_d are any real numbers and the constant C may be taken to depend only on p and d ; in particular, it may be taken to be independent of the exponents (a_1, \dots, a_d) . Our purpose here is to initiate an extension of the theory to general polynomial curves $\Gamma(t) = (P_1(t), \dots, P_d(t))$ where each component P_j is a real polynomial. We consider the curve as parametrised over the entire real line.

Theorem 1.1. *The inequality (1) holds for all polynomial curves of bounded degree if $p' = \frac{d(d+1)}{2}q$ and $1 \leq p < \frac{d^2+d+2}{d^2+d}$. More precisely, the constant C may be taken to depend only on p, d and the degrees of the polynomials defining Γ .*

Remarks 1.2.

- A preliminary (more restrictive) result was obtained earlier in [11].
- We expect Theorem 1.1 to remain true in the larger range $1 \leq p < \frac{d^2+d+2}{d^2+d}$.
- By considering the class of polynomial curves with bounded degree, we control the number of sign changes of L_Γ which seems natural in light of Sjölin's counterexample. Furthermore as remarked above, Sjölin's argument extends to cover the two dimensional case of Theorem 1.1 although his argument differs from ours in this special case.

Notation: Let A, B be complex-valued quantities. We use the notation $A \lesssim B$ or $A = O(B)$ to denote the estimate $|A| \leq C|B|$ where C depends only on d and the degrees of the polynomials defining the curve Γ . We use $A \sim B$ to denote the estimates $A \lesssim B \lesssim A$. Finally we will be making various decompositions of \mathbb{R} into disjoint intervals $\{I\}$ and it will be convenient to keep all intervals I open; therefore, a decomposition $\mathbb{R} = \cup I$ will in fact mean $\mathbb{R} = \cup \bar{I}$.

We wish to express our gratitude to Fulvio Ricci for many valuable discussions at an earlier stage of this project.

2. OUTLINE OF PROOF

By following Christ's argument in [9] which establishes (1) for the non-degenerate case $\Gamma(t) = (t, t^2, \dots, t^d)$ in the range $1 \leq p < \frac{d^2+2d}{d^2+2d-2}$, matters are reduced to establishing two properties about

$$\Phi_\Gamma(t_1, \dots, t_d) = \Gamma(t_1) + \dots + \Gamma(t_d) :$$

Key properties

(a) Φ_Γ is 1-1;

(b) $|J_{\Phi_\Gamma}(t_1, \dots, t_d)| \geq C \prod_{j=1}^d |L_\Gamma(t_j)|^{\frac{1}{d}} \prod_{j < k} |t_j - t_k|$

where $J_{\Phi_\Gamma}(t_1, \dots, t_d) = \det(\Gamma'(t_1) \cdots \Gamma'(t_d))$ is the determinant of the Jacobian matrix for the mapping Φ_Γ .

Even in the non-degenerate case $\Gamma(t) = (t, t^2, \dots, t^d)$, Φ_Γ is not quite 1-1 but it is $d!$ to 1 off a set of measure zero. Furthermore in this case, the geometric inequality (b) alluded to in the abstract is an equality.

For polynomial curves both (a) and (b) are false in general. However we will find a decomposition of $\mathbb{R} = \cup I$ into a bounded number (depending only on d and the degrees of the polynomials defining Γ) of disjoint open intervals so that on each I^d , Φ_Γ is $d!$ to 1 off a set of measure zero and the geometric inequality (b) holds. Therefore, by restricting the original operator to each I and applying Christ's argument, we obtain a proof of the theorem.

The decomposition is produced in two stages. The first stage produces an elementary decomposition of $\mathbb{R} = \cup J$ so that on each interval J , various polynomial quantities (more precisely, certain determinants of minors of the $d \times d$ matrix $(\Gamma'(t) \cdots \Gamma^{(d)})$, including L_Γ) are single-signed. This allows us to write down a formula relating J_{Φ_Γ} and L_Γ . When $d = 2$ this formula is particularly simple; namely,

$$J_{\Phi_\Gamma}(s, t) = P_1'(s)P_1'(t) \int_s^t \frac{L_\Gamma(w)}{P_1'(w)^2} dw$$

for any $s, t \in J$ (here $\Gamma = (P_1, P_2)$). From this, using an argument of Steinig [23], one can establish the injectivity of Φ_Γ on $\{(t_1, \dots, t_d) \in J^d : t_1 < \dots < t_d\}$. Next we decompose each $J = \cup I$ further so that on each I^d , (b) holds. More precisely, we have

$$|J_{\Phi_\Gamma}(t_1, \dots, t_d)| \geq C \prod_{j=1}^d |L_\Gamma(t_j)|^{\frac{1}{d}} \prod_{j < k} |t_j - t_k| \quad (2)$$

for all $(t_1, \dots, t_d) \in I^d$ where C depends only on d and the degrees of the polynomials defining Γ . This is the geometric inequality referred to above which has already found applications in the theory of generalised Radon transforms; see [12].

This second stage decomposition $J = \cup I$ is much more technical and derived from a certain algorithm which uses two further decomposition procedures generated by individual polynomials; one of these decomposition procedures has been used in other problems and first appeared in [6]. The algorithm exploits in a crucial way the affine invariance of the inequality (2); that is, the inequality is invariant under replacement of Γ by $A\Gamma$ for any invertible $d \times d$ matrix A .

The paper is organised as follows. The next section will reduce the proof of Theorem 1.1 to the **Key properties** above; most notably, to the geometric inequality (2). The following two sections will set up the initial decomposition on which the injectivity property of the Key properties will then be established on each subinterval in Section 6. The next three sections are devoted to developing and implementing an algorithm to carry out the secondary decomposition into subintervals on which the geometric inequality (2) of the Key properties will hold. Sections 10 and 11 will reduce the geometric inequality to two combinatorial lemmas and the final section of the paper is devoted to these combinatorial issues.

3. REDUCTION TO THE GEOMETRIC INEQUALITY

In this section we will sketch the argument of Christ in [9] and show how it quickly reduces matters to the **Key properties** in the previous section. More precisely we will assume that we have achieved the decomposition of $\mathbb{R} = \cup I$ into 0(1) disjoint open intervals so that the following two properties hold for each I :

(P1) for each permutation π of $\{1, \dots, d\}$, the map

$$\Phi_\Gamma(t_1, \dots, t_d) = \Gamma(t_1) + \dots + \Gamma(t_d)$$

is 1-1 on the region

$$D_\pi = \{(t_1, \dots, t_d) \in I^d : t_{\pi(1)} < \dots < t_{\pi(d)}\};$$

(P2) for $(t_1, \dots, t_d) \in I^d$,

$$|J_{\Phi_\Gamma}(t_1, \dots, t_d)| \geq C \prod_{j=1}^d |L_\Gamma(t_j)|^{\frac{1}{d}} \prod_{j < k} |t_j - t_k|$$

where C depends only on d and the degrees of the polynomials defining Γ . Recall that $J_{\Phi_\Gamma}(t_1, \dots, t_d) = \det(\Gamma'(t_1) \cdots \Gamma'(t_d))$ is the determinant of the Jacobian matrix for the mapping Φ_Γ .

To prove Theorem 1.1 we see by duality that it suffices to show

$$\|\widehat{gd\nu}\|_{p'} \lesssim \|g\|_{q'(d\omega)}, \quad (3)$$

where

$$d\nu(\phi) = \int_I \phi(P(s)) |L(s)|^{\frac{2}{d(d+1)}} ds \quad \text{and} \quad d\omega(\phi) = \int_I \phi(s) |L(s)|^{\frac{2}{d(d+1)}} ds.$$

Now, with $gd\nu * \dots * gd\nu$ denoting the d -fold convolution of $gd\nu$ with itself, we have

$$\|\widehat{gd\nu}\|_{p'}^d = \|\widehat{gd\nu}^d\|_{p'/d} = \|gd\nu * \dots * gd\nu\|_{p'/d} \leq \|gd\nu * \dots * gd\nu\|_r, \quad (4)$$

where $dr' = p'$ by the Hausdorff-Young inequality. Note that because $1 \leq p < \frac{d(d+2)}{d(d+2)-2}$, we have $1 \leq r \leq 2$. Now

$$gd\nu * \dots * gd\nu(\phi) = \int_{I^d} \phi\left(\sum_{i=1}^n P(t_i)\right) \prod_{i=1}^d g(t_i) |L(t_i)|^{\frac{2}{d(d+1)}} dt,$$

where $t = (t_1, \dots, t_d)$. If S_d denotes the permutations of $\{1, \dots, d\}$,

$$\begin{aligned} g d\nu * \dots * g d\nu(\phi) &= \sum_{\pi \in S_d} \int_{D_\pi} \phi \left(\sum_{i=1}^d P(t_i) \right) \prod_{i=1}^d g(t_i) |L(t_i)|^{\frac{2}{d(d+1)}} dt \\ &= \sum_{\pi \in S_d} \int_{\Delta_\pi} \phi(x) \prod_{i=1}^d g(t_i) |L(t_i)|^{\frac{2}{d(d+1)}} \frac{1}{|J(t)|} dx, \end{aligned}$$

where in the second inequality we perform the change of variables $x = (x_1, \dots, x_d)$

$$= \Gamma(t_1) + \dots + \Gamma(t_d) \quad \left[\text{so that for each } 1 \leq k \leq d, \quad x_k = \sum_{i=1}^d P_k(t_i) \right]$$

separately on each region D_π , and which is well defined by (P1). Here Δ_π is the image of the region D_π under the transformation defined by the change of variables and $J = J_{\Phi_\pi}$. Hence

$$g d\nu * \dots * g d\nu = \sum_{\pi \in S_d} \prod_{i=1}^d g(t_i) |L(t_i)|^{\frac{2}{d(d+1)}} \frac{1}{|J(t)|} \chi_{\Delta_\pi}.$$

Therefore

$$\begin{aligned} \|g d\nu * \dots * g d\nu\|_r &\leq \sum_{\pi \in S_d} \left\| \prod_{i=1}^d g(t_i) |L(t_i)|^{\frac{2}{d(d+1)}} \frac{1}{|J(t)|} \chi_{\Delta_\pi} \right\|_r \\ &= \sum_{\pi \in S_d} \left(\int_{D_\pi} \prod_{i=1}^d |g(t_i)|^r |L(t_i)|^{r \frac{2}{d(d+1)}} \frac{1}{|J(t)|^{r-1}} dt \right)^{\frac{1}{r}}, \end{aligned}$$

by changing variables back. From the geometric inequality (P2) it follows that

$$\begin{aligned} &\|g d\nu * \dots * g d\nu\|_r \\ &\leq \sum_{\pi \in S_d} \left(\int_{D_\pi} \prod_{i=1}^d |g(t_i)|^r |L(t_i)|^{r \frac{2}{d(d+1)} - \frac{r-1}{d}} \prod_{k < l} |t_l - t_k|^{1-r} dt \right)^{\frac{1}{r}}. \end{aligned}$$

Finally we use the following result of Christ [9].

Proposition 3.1. *If $0 \leq \gamma$ then*

$$\int \prod_{i=1}^d f(x_i) \prod_{1 \leq i < j \leq d} |x_i - x_j|^{-\gamma} dx_1 \dots dx_d \leq C \|f\|_p^d,$$

for all f , if and only if $\gamma < 2/d$, $1 \leq p < d$ and $p^{-1} + \gamma(nd - 1)/2 = 1$.

We use this proposition with $\gamma = r - 1$. One can easily check that $r - 1 < 2/d$ since $dr' = p'$ and $p < \frac{d(d+2)}{d(d+2)-2}$. As a result, we obtain

$$\|g d\nu * \dots * g d\nu\|_r \lesssim \left(\int_I (|g(t)|^r |L(t)|^{\frac{1}{d} + r(\frac{2}{d(d+1)} - \frac{1}{d})})^{\tilde{p}} dt \right)^{\frac{d}{\tilde{p}r}},$$

where

$$\frac{1}{\tilde{p}} + (r - 1) \frac{d - 1}{2} = 1. \quad (5)$$

By (4) we see that the required relations for (3) to hold are

$$\tilde{p}r = q' \quad \text{and} \quad \frac{\tilde{p}}{d} + r\tilde{p} \left(\frac{2}{d(d+1)} - \frac{1}{d} \right) = \frac{2}{d(d+1)}.$$

These can be verified by algebraic calculations, using (5), $dr' = p'$ and $\frac{1}{q} = \frac{d(d+1)}{2} \frac{1}{p'}$.

4. PRELIMINARIES FOR THE INITIAL DECOMPOSITION

Our main goal is to produce a decomposition of $\mathbb{R} = \cup I$ into $0(1)$ disjoint open intervals so that properties (P1) and (P2) in the previous section hold for each interval I . As indicated in Section 2 this will be carried out in two stages. The first stage is elementary and this section is devoted to the necessary preliminaries needed for this initial decomposition.

Our analysis will be based on examining the polynomial

$$L_\Gamma(t) = \det(\Gamma'(t) \cdots \Gamma^{(d)}(t))$$

introduced in the definition of the affine arclength measure of Γ . Without loss of generality we may assume that L_Γ is not the zero polynomial since otherwise the estimates (1) are trivial. From this assumption our initial goal is to deduce that various polynomials formed via the determinant of certain minors of the $d \times d$ matrix $(\Gamma'(t) \cdots \Gamma^{(d)}(t))$ are also not the zero polynomial.

Notation: For any k -tuple of real polynomials $Q = (Q_1, \dots, Q_k)$, set

$$L_{Q_1 \dots Q_k}(t) = \det(Q'(t) \cdots Q^{(k)}(t)).$$

In particular, if $\Gamma = (P_1, \dots, P_d)$ is our original polynomial curve, $L_\Gamma = L_{P_1 \dots P_d}$.

Lemma 4.1. *If Q_1, \dots, Q_k, P, R are $k+2$ real polynomials so that $L_{Q_1 \dots Q_k P}$ is not the zero polynomial, then*

$$\left(\frac{L_{Q_1 \dots Q_k R}}{L_{Q_1 \dots Q_k P}} \right)' = \frac{L_{Q_1 \dots Q_k P R} L_{Q_1 \dots Q_k}}{[L_{Q_1 \dots Q_k P}]^2}. \quad (6)$$

Remark: This lemma can be viewed as a generalisation of the quotient rule for differentiation. In fact, the $k=0$ case simply states

$$\left(\frac{L_R}{L_P} \right)' = \left(\frac{R'}{P'} \right)' = \frac{R'' P' - R' P''}{P'^2} = \frac{L_{PR} \cdot 1}{L_P^2}.$$

Proof The formula (6) is a consequence of a well-known determinant identity. If A denotes an $\ell \times \ell$ matrix, let $[r_1, \dots, r_k; c_1, \dots, c_k] = \det B$ where B is the submatrix of A obtained by deleting the rows r_1, \dots, r_k and columns c_1, \dots, c_k of A . The following is an instance of Sylvester's Determinant Identity (see for example, [4])

$$[\ell-1, \ell; \ell-1, \ell] \cdot \det A = [\ell; \ell][\ell-1; \ell-1] - [\ell; \ell-1][\ell-1; \ell]. \quad (7)$$

To establish (6) it suffices to show

$$(L_{Q_1 \dots Q_k R})' L_{Q_1 \dots Q_k P} - L_{Q_1 \dots Q_k R} (L_{Q_1 \dots Q_k P})' = L_{Q_1 \dots Q_k R P} L_{Q_1 \dots Q_k} \quad (8)$$

and we will do this by applying (7) to the $(k+2) \times (k+2)$ matrix A defining $L_{Q_1 \dots Q_k R P}$ so that $\det A = L_{Q_1 \dots Q_k R P}$. Note that $[L_{Q_1 \dots Q_k R}(t)]'$

$$\begin{aligned} &= \frac{d}{dt} \left[\det \begin{pmatrix} Q'_1 & \cdots & Q_1^{(k)} & Q_1^{(k+1)} \\ \vdots & & \vdots & \vdots \\ Q'_k & \cdots & Q_k^{(k)} & Q_k^{(k+1)} \\ R' & \cdots & R^{(k)} & R^{(k+1)} \end{pmatrix} \right] \\ &= \det \begin{pmatrix} Q'_1 & \cdots & Q_1^{(k)} & Q_1^{(k+2)} \\ \vdots & & \vdots & \vdots \\ Q'_k & \cdots & Q_k^{(k)} & Q_k^{(k+2)} \\ R' & \cdots & R^{(k)} & R^{(k+2)} \end{pmatrix} = [k+2; k+1]. \end{aligned}$$

Similarly $L'_{Q_1 \dots Q_k P} = [k+1; k+1]$. Since $L_{Q_1 \dots Q_k} = [k+1, k+2; k+1, k+2]$, we see that (8) follows from (7). \blacksquare

Using Lemma 4.1 we now show that various polynomials associated to $L_\Gamma = L_{P_1 \dots P_d}$ are nonzero if our basic assumption that L_Γ is not the zero polynomial is in force.

Lemma 4.2. *If $L_\Gamma = L_{P_1 \dots P_d}$ is not the zero polynomial, then for every distinct k -tuple $(j_1, \dots, j_k), 1 \leq j_r \leq d, L_{P_{j_1} \dots P_{j_k}}$ is not the zero polynomial.*

Proof We proceed by induction on k ; the case $k=1$ being trivial since if $L_{P_j} = P'_j \equiv 0$ for some $1 \leq j \leq d$, then the j th row $P'_j P''_j \cdots P_j^{(d)}$ in the $d \times d$ matrix $(\Gamma'(t) \cdots \Gamma^{(d)}(t))$ vanishes identically.

We argue by contradiction; fix (j_1, \dots, j_k) , set $(Q_1, \dots, Q_k) = (P_{j_1}, \dots, P_{j_k})$ and suppose $L_{Q_1 \dots Q_k}$ is the zero polynomial. By the induction hypothesis $L_{Q_1 \dots Q_{k-2} Q_{k-1}}$ is not the zero polynomial and so Lemma 4.1 implies

$$\frac{d}{dt} \left(\frac{L_{Q_1 \dots Q_{k-2} Q_k}}{L_{Q_1 \dots Q_{k-2} Q_{k-1}}} \right) \equiv 0.$$

Hence $L_{Q_1 \dots Q_{k-2} Q_k}(t) = c L_{Q_1 \dots Q_{k-2} Q_{k-1}}(t)$ for some absolute constant c (not depending on t) and so $L_{Q_1 \dots Q_{k-2} (Q_k - c Q_{k-1})} \equiv 0$ which by Lemma 4.1 implies

$$\frac{d}{dt} \left(\frac{L_{Q_1 \dots Q_{k-3} (Q_k - c Q_{k-1})}}{L_{Q_1 \dots Q_{k-3} Q_{k-2}}} \right) \equiv 0.$$

Therefore,

$$L_{Q_1 \dots Q_{k-3} (Q_k - c Q_{k-1})} = b L_{Q_1 \dots Q_{k-3} Q_{k-2}}$$

or

$$L_{Q_1 \dots Q_{k-3} Q_k} = c L_{Q_1 \dots Q_{k-3} Q_{k-1}} + b L_{Q_1 \dots Q_{k-3} Q_{k-2}}$$

for some absolute constants b and c .

Continuing in this way we obtain, for each $0 \leq j \leq k-2$, absolute constants c_1, \dots, c_{k-j-1} so that

$$L_{Q_1 \dots Q_j Q_k} = c_1 L_{Q_1 \dots Q_j Q_{k-1}} + \cdots + c_{k-j-1} L_{Q_1 \dots Q_j Q_{j+1}}.$$

The case $j=0$ yields $L_{Q_k} = c_1 L_{Q_{k-1}} + \cdots + c_{k-1} L_{Q_1}$ or $Q'_k = c_1 Q'_{k-1} + \cdots + c_{k-1} Q'_1$. Differentiating gives

$$P_{j_k}^{(\ell)} = c_1 P_{j_{k-1}}^{(\ell)} + \cdots + c_{k-1} P_{j_1}^{(\ell)}$$

for every $\ell \geq 1$ and so the j_k th row of the matrix $(\Gamma' \cdots \Gamma^{(d)})$ is a linear combination (as polynomials) of the j_1 st, \dots , j_{k-1} th rows and this implies $L_\Gamma = L_{P_1 \dots P_d}$ is the zero polynomial, contradicting our basic assumption. \blacksquare

5. THE INITIAL DECOMPOSITION

We are now in a position to describe the initial decomposition of $\mathbb{R} = \cup J$ into $0(1)$ disjoint open intervals. We will see that on each interval J , the one-to-oneness property (P1) holds. Later, we will decompose each J further in order to establish the geometric inequality, property (P2).

Notation: For any sequence (finite or infinite) of real polynomials $Q = (Q_1, Q_2, \dots)$, we set for any $j \geq 1$,

$$L_{Q,j} = L_{Q_1 \dots Q_j}$$

so that in particular, $L_\Gamma = L_{\Gamma,d} = L_{P_1 \dots P_d}$ for our original curve $\Gamma = (P_1, \dots, P_d)$. For convenience we will often denote $L_{\Gamma,j}$ simply by L_j for $1 \leq j \leq d$.

A particular instance of Lemma 4.2 implies that each $L_j = L_{\Gamma,j}$, $1 \leq j \leq d$, is a nonzero polynomial under our basic assumption that L_Γ is nonzero. The real roots of all the polynomials $\{L_{\Gamma,j}\}_{j=1}^d$ give our initial decomposition $\mathbb{R} = \cup J$ into $0(1)$ disjoint open intervals so that on each J , every $L_{\Gamma,j}$ is either strictly positive or strictly negative.

The main goal of this section is to establish a formula relating the determinant of the Jacobian matrix for the mapping $\Phi_\Gamma(t) = \Gamma(t_1) + \cdots + \Gamma(t_d)$, $J_{\Phi_\Gamma}(t) = \det(\Gamma'(t_1) \cdots \Gamma'(t_d))$, and the polynomials $L_j = L_{\Gamma,j}$, $1 \leq j \leq d$. This formula will be valid only on each interval J separately. We will write J_{Φ_Γ} as a series of nested iterated integrals. To this end we define a sequence of multi-variate functions $\{I_r\}_{r=1}^d$; for each $1 \leq r \leq d$, $I_r = I_r(x_1, \dots, x_r)$ will be a function of r variables which will be well-defined on J^r for each interval J arising in the initial decomposition. We define this sequence inductively. For $r = 1$ we set $I_1(x) = L_{d-2}(x)L_d(x)/[L_{d-1}(x)]^2$ and then inductively, define

$$I_r(x_1, \dots, x_r) =$$

$$\prod_{s=1}^r \frac{L_{d-r-1}(x_s)L_{d-r+1}(x_s)}{[L_{d-r}(x_s)]^2} \int_{x_1}^{x_2} \cdots \int_{x_{r-1}}^{x_r} I_{r-1}(y_1, \dots, y_{r-1}) dy_1 \cdots dy_{r-1}.$$

In order to make sense of I_{d-1} and I_d we set $L_0 = L_{-1} \equiv 1$. Our goal is to show that on each J^d ,

$$J_{\Phi_\Gamma}(t_1, \dots, t_d) = I_d(t_1, \dots, t_d). \quad (9)$$

We begin with an elementary lemma in differential calculus.

Lemma 5.1. *Let $\{g_i\}_{i=1}^n$ be smooth functions on an open interval $J \subset \mathbb{R}$ so that g_1 never vanishes on J . If $f_i = \frac{g_i}{g_1}$, $2 \leq i \leq n$, then for $(t_1, \dots, t_d) \in J^n$,*

$$\begin{aligned} & \det \begin{pmatrix} g_1(t_1) & \cdots & g_1(t_n) \\ \vdots & & \vdots \\ g_n(t_1) & \cdots & g_n(t_n) \end{pmatrix} \\ &= \prod_{i=1}^n g_1(t_i) \int_{t_1}^{t_2} \cdots \int_{t_{n-1}}^{t_n} \det \begin{pmatrix} f'_2(x_1) & \cdots & f'_2(x_{n-1}) \\ \vdots & & \vdots \\ f'_n(x_1) & \cdots & f'_n(x_{n-1}) \end{pmatrix} dx_1 \cdots dx_{n-1}. \end{aligned}$$

Proof Without loss of generality assume $t_1 < \cdots < t_n$. By factoring $g_1(t_i)$ out of every column we write

$$\det \begin{pmatrix} g_1(t_1) & \cdots & g_1(t_n) \\ \vdots & & \vdots \\ g_n(t_1) & \cdots & g_n(t_n) \end{pmatrix} = \prod_{i=1}^n g_1(t_i) \det \begin{pmatrix} 1 & \cdots & 1 \\ f_2(t_1) & \cdots & f_2(t_n) \\ \vdots & & \vdots \\ f_n(t_1) & \cdots & f_n(t_n) \end{pmatrix}.$$

Then by conducting column operations the determinant involving the f_i 's is equal to

$$\begin{aligned} & \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ f_2(t_1) & f_2(t_2) - f_2(t_1) & \cdots & f_2(t_n) - f_2(t_1) \\ \vdots & \vdots & & \vdots \\ f_n(t_1) & f_n(t_2) - f_n(t_1) & \cdots & f_n(t_n) - f_n(t_1) \end{pmatrix} \\ &= \int_{t_1}^{t_2} \cdots \int_{t_1}^{t_n} \det \begin{pmatrix} f'_2(x_1) & \cdots & f'_2(x_{n-1}) \\ \vdots & & \vdots \\ f'_n(x_1) & \cdots & f'_n(x_{n-1}) \end{pmatrix} dx_1 \cdots dx_{n-1}. \end{aligned}$$

For fixed x_1, x_2, \dots, x_{n-1} except x_l and x_m with $1 \leq l < m \leq n-1$, consider

$$I_k := \int_{t_k}^{t_{k+1}} dx_l \int_{t_k}^{t_{k+1}} dx_m \det \begin{pmatrix} f'_2(x_1) & \cdots & f'_2(x_{n-1}) \\ \vdots & & \vdots \\ f'_n(x_1) & \cdots & f'_n(x_{n-1}) \end{pmatrix}.$$

By interchanging the l th with the m th column we see that I_k is equal to

$$- \int_{t_k}^{t_{k+1}} dx_l \int_{t_k}^{t_{k+1}} dx_m \det \begin{pmatrix} f'_2(x_1) & \cdots & f'_2(x_m) & \cdots & f'_2(x_l) & \cdots & f'_2(x_{n-1}) \\ \vdots & & \vdots & & \vdots & & \vdots \\ f'_n(x_1) & \cdots & f'_n(x_m) & \cdots & f'_n(x_l) & \cdots & f'_n(x_{n-1}) \end{pmatrix}.$$

Thus $I_k = -I_k$ and so $I_k = 0$. So finally

$$\begin{aligned} & \prod_{i=1}^n g_1(t_i) \int_{t_1}^{t_2} \cdots \int_{t_1}^{t_n} \det \begin{pmatrix} f'_2(x_1) & \cdots & f'_2(x_{n-1}) \\ \vdots & & \vdots \\ f'_n(x_1) & \cdots & f'_n(x_{n-1}) \end{pmatrix} dx_1 \cdots dx_{n-1} \\ &= \prod_{i=1}^n g_1(t_i) \int_{t_1}^{t_2} \cdots \int_{t_{n-1}}^{t_n} \det \begin{pmatrix} f'_2(x_1) & \cdots & f'_2(x_{n-1}) \\ \vdots & & \vdots \\ f'_n(x_1) & \cdots & f'_n(x_{n-1}) \end{pmatrix} dx_1 \cdots dx_{n-1}, \end{aligned}$$

concluding the proof of Lemma 5.1. \blacksquare

To see (9) define $f_{i,0} = P_i$ for $1 \leq i \leq d$ and for $1 \leq k \leq d$ define inductively,

$$f_{i,k} = \frac{f'_{i,k-1}}{f'_{k,k-1}}$$

for $k+1 \leq i \leq d$. If we set $F_k = (f_{k+1,k}, \dots, f_{d,k})^T$ for each $1 \leq k \leq d-1$, then Lemma 5.1 shows

$$\det\left(F'_{k-1}(x_1) \cdots F'_{k-1}(x_{d-k+1})\right) = \quad (10)$$

$$\prod_{s=1}^{d-k+1} f'_{k,k-1}(x_s) \int_{x_1}^{x_2} \cdots \int_{x_{d-k}}^{x_{d-k+1}} \det\left(F'_k(y_1) \cdots F'_k(y_{d-k})\right) dy_1 \cdots dy_{d-k}.$$

Now an application of Lemma 4.1 shows that

$$f'_{i,k} = \left(\frac{L_{P_1 \dots P_{k-1} P_i}}{L_{P_1 \dots P_k}}\right)' = \frac{L_{P_1 \dots P_{k-1}} L_{P_1 \dots P_k P_i}}{L_{P_1 \dots P_k}^2} \quad (11)$$

for $k+1 \leq i \leq d$. In fact to see (11) we proceed by induction on k . For $k=1$

$$f'_{i,1} = \left(\frac{L_{P_i}}{L_{P_1}}\right)' = \frac{L_{P_1 P_i}}{L_{P_1}^2}.$$

If (11) is true for $k = m-1$ then

$$\begin{aligned} f'_{i,m} &= \left(\frac{f'_{i,m-1}}{f'_{m,m-1}}\right)' = \left(\frac{L_{P_1 \dots P_{m-2}} L_{P_1 \dots P_{m-1} P_i} / L_{P_1 \dots P_{m-1}}^2}{L_{P_1 \dots P_{m-2}} L_{P_1 \dots P_m} / L_{P_1 \dots P_{m-1}}^2}\right)' \\ &= \left(\frac{L_{P_1 \dots P_{m-1} P_i}}{L_{P_1 \dots P_m}}\right)' = \frac{L_{P_1 \dots P_{m-1}} L_{P_1 \dots P_m P_i}}{L_{P_1 \dots P_m}^2}, \end{aligned}$$

where the last inequality follows from Lemma 4.1. Now combining (11) and (10) iteratively gives (9).

6. PROPERTY (P1) – INJECTIVITY OF Φ_Γ

In this section we establish property (P1) for $\Phi_\Gamma(t) = \Gamma(t_1) + \cdots + \Gamma(t_d)$ on each interval J in the initial decomposition of \mathbb{R} . To do this we will use the fact that $J_{\Phi_\Gamma}(t_1, \dots, t_d)$ is single-signed and nonzero on $D_J = \{t = (t_1, \dots, t_d) \in J^d : t_1 < \cdots < t_d\}$; this follows from Lemma 4.2, (9) and our basic assumption that L_Γ is not the zero polynomial. In fact we will need to use this fact for truncations of Γ ; that is, if $Q = (P_1, \dots, P_\mu)$, $1 \leq \mu \leq d$ is a truncation of $\Gamma = (P_1, \dots, P_d)$, then

$$\det\left(Q'(t_1) \cdots Q'(t_\mu)\right) \text{ is single – signed and nonzero on } D_{J,\mu} \quad (12)$$

where $D_{J,\mu} = \{t = (t_1, \dots, t_\mu) \in J^\mu : t_1 < \cdots < t_\mu\}$. This follows in exactly the same way as for Γ .

More precisely we have the following proposition whose proof follows an argument of Steinig, [23] (although rediscovered on several occasions, see e.g., [7]) and clearly establishes property (P1).

Proposition 6.1. *For each interval J in our initial decomposition of \mathbb{R} , the map $\Phi_\Gamma(t) = \Gamma(t_1) + \cdots + \Gamma(t_d)$ is 1-1 on D_J .*

Proof Suppose, in order to obtain a contradiction, that $\Phi_\Gamma(s) = \Phi_\Gamma(t)$ or

$$\Gamma(s_1) + \cdots + \Gamma(s_d) = \Gamma(t_1) + \cdots + \Gamma(t_d)$$

for some $s \neq t$ and $s, t \in D_J$. Hence

$$0 = \sum_{j=1}^{\nu} \epsilon_j \Gamma(u_j)$$

where $2 \leq \nu \leq 2d$ is even, $u_1 < u_2 \cdots < u_\nu$, each $u_j \in J$, $\epsilon_j = \pm 1$ and $\sum_{j=1}^{\nu} \epsilon_j = 0$. Let $\alpha_k = \sum_{j=1}^k \epsilon_j$, $1 \leq k \leq \nu$.

Key observation: $\{\alpha_k\}_{k=1}^{\nu}$ has at most $d - 1$ changes of sign.

Thus

$$0 = \sum_{j=1}^{\nu} \epsilon_j \Gamma(u_j) = \sum_{j=1}^{\nu-1} \alpha_j [\Gamma(u_j) - \Gamma(u_{j-1})] = \int_{u_1}^{u_\nu} \varphi(u) \Gamma'(u) du \quad (13)$$

where φ is a step function with at most $d - 1$ changes of sign. Let I_ℓ , $1 \leq \ell \leq \mu$, be the ordered, maximal intervals on which φ is constant and nonzero. Thus $\boxed{\mu \leq d}$.

Consider the $\mu \times \mu$ matrix $A = (a_{\ell,k})$,

$$a_{\ell,k} = \int_{I_\ell} |\varphi(u)| P_k'(u) du.$$

The rows of A are linearly dependent by (13). Hence $0 = \det A$ and so

$$0 = \int_{I_1} \cdots \int_{I_\mu} |\varphi(u_1)| \cdots |\varphi(u_\mu)| \det(Q'(u_1) \cdots Q'(u_\mu)) du_1 \cdots du_\mu$$

where $Q = (P_1, \dots, P_\mu)$. But this contradicts (12). \blacksquare

7. TWO DECOMPOSITION PROCEDURES

We now embark down the road of setting up the less elementary (and more technical) secondary decomposition of \mathbb{R} . To date we have established the initial decomposition $\mathbb{R} = \cup J$ where the property (P1) holds for each J . Simple examples show that the geometric inequality, property (P2), may not hold on some interval¹ J from the initial decomposition and therefore the secondary decomposition is necessary. Consider for the moment the two dimensional case where our original polynomial curve is, say, $\Gamma = (P, Q)$. In this situation the geometric inequality on an interval J becomes

$$\left| \frac{1}{b-a} \int_a^b R(t) dt \right| \gtrsim \sqrt{|R(b)R(a)|} \quad (14)$$

for any $a, b \in J$, where $R = (Q'/P)'$ is a rational function whose numerator L_Γ and denominator $[P']^2$ is assumed not to vanish anywhere on J . If we choose polynomials P and Q so that $Q'(x) = x$ and $P'(x) = (x + \epsilon)(x - 1 + \epsilon)$ for some small $\epsilon > 0$ then the intervals $(-\infty, -\epsilon)$, $(-\epsilon, 1 + \epsilon)$ and $(1 + \epsilon, \infty)$ comprise the initial decomposition and one can easily check that (14) does not hold for $a = 0$ and $b = 1$, uniformly in $\epsilon > 0$. Nevertheless if we further split the interval $(0, 1) = (0, \sqrt{\epsilon}) \cup (\sqrt{\epsilon}, 1) := J_1 \cup J_2$, then (14) does in fact hold on each J_1 and J_2 separately.

¹We thank Tony Carbery for pointing this out to us.

We will see that the secondary decomposition procedure described below will insist on a splitting at $\sqrt{\epsilon}$.

Furthermore observe that (14) cannot hold for general rational functions R . There is no decomposition of \mathbb{R} into $0(1)$ intervals so that (14) holds for $R(t) = 1/t$ but such a rational function does not arise as the derivative of another rational function. To avoid this example one might try to verify (14) for the class of rational functions $R = S/T$ in reduced form where T has no simple zeros which includes the class of derivatives of rational functions. But unfortunately the example $R(t) = (t + 2\epsilon)/(t + \epsilon)^2$ shows (14) fails in a similar way as for $1/t$. The fact that (14) holds (after possibly a further decomposition of J into $0(1)$ intervals) for the derivative of a general rational function suggests that there is a significant amount of cancellation occurring from this derivative which we must exploit more fully. We will do this via a secondary decomposition, refining the initial decomposition of $\mathbb{R} = \cup J$.

We will develop an algorithm that generates this further decomposition. The algorithm depends on two decomposition procedures associated to individual polynomials which we will describe in this section. Given a polynomial Q , these procedures allow us to decompose any open interval $J = \cup I$ into $0(1)$ disjoint open intervals so that on each I , $Q(t) \sim A(t - b)^k$ looks like a centred monomial. Recall that the basic geometric inequality in (P2) bounds the determinant J_{Φ_Γ} of the derivative map of $\Phi_\Gamma = \Gamma(t_1) + \dots + \Gamma(t_d)$ from below and by (9), this reduces to bounding from below a series of nested iterated integrals involving the functions I_r , $1 \leq r \leq d$ and thus the polynomials $L_j = L_{\Gamma,j}$, $1 \leq j \leq d$. The idea of the algorithm is to treat each L_j in (9) as an independent polynomial in the first instance and use the two decomposition procedures in tandem to reduce to intervals on which each polynomial L_j behaves like a centred monomial (in fact we will be able to achieve this with the same centre for all L_j !). Therefore we will reduce to intervals on which J_{Φ_Γ} becomes a concrete series of nested iterated integrals involving only monomials with various exponents. Although concrete and explicit, the desired bound from below for this concrete series of nested iterated integrals is *false* in general. For instance, in the two dimensional case, by treating $L_2 = L_\Gamma = Q''P' - Q'P''$ and $L_1 = P'$ independently in this way (applying the two decomposition procedures), one can arrive at the rational function $R(t) = 1/t$ in (14). Therefore it will be essential to recover and exploit the intimate relationships among the various polynomials L_j , $1 \leq j \leq d$ and thereby avoiding certain bad exponents arising in the reduction to centred monomials. We will do this using the affine-invariance of the geometric inequality.

We now turn to the two decomposition procedures associated to individual polynomials. We fix an arbitrary open interval J over which we will attempt to describe the behaviour of a polynomial. The first procedure is more elementary and has the advantage of describing the polynomial over the entire interval J .

[D1] Given a real polynomial Q .

Then $J = \cup I$ can be decomposed into $0(1)$ open disjoint intervals so that on each

I : $Q(t) \sim A(t - b)^k$ for some $A = A_I \neq 0$, an integer $k = k_I \geq 0$ and $b = b_I$, the real part of a root of Q .

To simplify matters we will assume that all the roots of Q are real, the general case requiring only minor adjustments. To see **[D1]** factor $Q(t) = C \prod (t - \eta_j)^{a_j}$

where the η_j are the distinct roots of Q and consider the preliminary decomposition $J = \cup S_i$ where

$$S_i = \{t \in J : |t - \eta_i| < |t - \eta_j| \forall j \neq i\}.$$

We will decompose each S_i further. Without loss of generality we will describe this for $i = 1$. Order the η_j so that $|\eta_1 - \eta_2| \leq \dots \leq |\eta_1 - \eta_{d'}|$ and set $T_i = \{t \in S_1 : |t - \eta_1| < 1/2|\eta_1 - \eta_i|\}, 2 \leq i \leq d'$ so that $T_2 \subset T_3 \subset \dots \subset T_{d'} \subset S_1$. Finally set $T_1 = \emptyset, T_{d'+1} = S_1$ and $I_k = T_{k+1} \setminus \overline{T_k}$ so that $S_1 = \cup_{k=1}^{d'} I_k$. We now make two simple observations.

$$t \in T_k \Rightarrow 1/2|\eta_1 - \eta_{k'}| \leq |t - \eta_{k'}| \leq 3/2|\eta_1 - \eta_{k'}|, k' \geq k. \quad (15)$$

In fact if $t \in T_k, |t - \eta_{k'}| \geq |\eta_{k'} - \eta_1| - |t - \eta_1| \geq 1/2|\eta_{k'} - \eta_1|$ for any $k' \geq k$. The other inequality follows in the same manner. The second observation is

$$t \notin T_k \Rightarrow |t - \eta_1| \leq |t - \eta_{k'}| \leq 3|t - \eta_1|, k' \leq k. \quad (16)$$

In fact if $t \notin T_k, |\eta_1 - \eta_k| \leq 2|t - \eta_1|$ and so $|t - \eta_{k'}| \leq |t - \eta_1| + |\eta_1 - \eta_{k'}| \leq |t - \eta_1| + |\eta_1 - \eta_k| \leq |t - \eta_1| + 2|t - \eta_1|$ for any $k' \leq k$. Again the other inequality follows in the same way. From (15) and (16) we see that if $t \in I_k,$

$$\prod_{j=1}^{d'} (t - \eta_j)^{a_j} \sim (t - \eta_1)^{a_1 + \dots + a_k} \prod_{j=k+1}^{d'} (\eta_1 - \eta_j)^{a_j}$$

and so $Q(t) \sim A(t - \eta_1)^K$ where $K = a_1 + \dots + a_k$.

Finally we turn to the second decomposition procedure which not only depends on a polynomial Q but also depends on a given centre b . Here we will attempt to describe Q on most of J as monomials (with varying exponents) but with a fixed centre b . This decomposition also has the advantage of being able to avoid certain exponents arising in the expressions of Q as centred monomials.

[D2] *Given a real polynomial Q and a centre $b \in \mathbb{R}$.*

Then $J = \cup I$ can be decomposed into $0(1)$ open disjoint intervals which fall into two classes: \mathcal{G} (gaps) and \mathcal{D} (dyadic).

On $I \in \mathcal{G}$: $Q(t) \sim A(t - b)^k$ for some $A = A_I \neq 0$ and an integer $k = k_I \geq 0$.

On $I \in \mathcal{D}$: $(t - b) \sim D$ for some $D = D_I \neq 0$.

*Furthermore, if $Q(t + b) = \sum c_k t^k$ and $c_{k_0} = 0$ then **no** gaps $I \in \mathcal{G}$ exist on which $Q(t) \sim A(t - b)^{k_0}$.*

As mentioned in section 2 this decomposition appears in [6] and so we shall be brief in its description. Factor $Q(t + b) = \sum c_j t^j = B \prod (t - \beta_j)$ (the roots β_j may be repeated) and order the roots so that $|\beta_1| \leq |\beta_2| \leq \dots$. Fix a large constant $C = 0(1)$. Our gap intervals will arise from intervals of the form $G_j = [C|\beta_j|, 1/C|\beta_{j+1}|]$ or $[-1/C|\beta_{j+1}|, -C|\beta_j|]$ and our dyadic intervals arise from intervals of the form $D_j = [1/C|\beta_j|, C|\beta_j|]$ or $[-C|\beta_j|, -1/C|\beta_j|]$.

Clearly on $D_j, t \sim \beta_j$. On $G_j,$

$$Q(t + b) \sim B t^j \prod_{k \geq j+1} \beta_k \sim c_j t^j$$

if C is chosen large enough. The latter bounds $B \prod_{k \geq j+1} \beta_k \sim c_j$ for C large enough can be found in [17]. Therefore c_j cannot be zero! Finally we make a translation by b so that our gap intervals are of the form $(G_j + b) \cap J$ and our dyadic intervals are of the form $(D_j + b) \cap J$.

8. THE TWO DIMENSIONAL CASE

As discussed above we will build an algorithm, using the two decomposition procedures in the previous section on each polynomial $L_j = L_{\Gamma,j}$, $1 \leq j \leq d$, appearing in (9) to bound $J_{\Phi_{\Gamma}}$ from below. As this algorithm and its implementation is somewhat involved in general, we choose in this section to illustrate how it works in the two dimensional case where everything simplifies greatly. If our curve is given by $\Gamma = (P, Q)$, then $L_1 = P'$ and $L_2 = P'Q'' - P''Q'$ and (9) simply states

$$J_{\Phi_{\Gamma}}(s, t) = L_1(s)L_1(t) \int_s^t \frac{L_2(w)}{L_1^2(w)} dw \quad (17)$$

for any $s, t \in J$ where J is an interval from our initial decomposition of \mathbb{R} .

Our goal is to decompose $J = \cup I$ into $0(1)$ disjoint open intervals so that for each I ,

$$s, t \in I \Rightarrow |J_{\Phi_{\Gamma}}(s, t)| \gtrsim \sqrt{|L_2(s)||L_2(t)||s-t|}. \quad (18)$$

The algorithm is carried out in d steps (hence only two steps in this section). We first treat the polynomials L_2 and L_1 independently and then we will go back and adjust the steps of the algorithm taking into account the relationship between these two polynomials.

Step 0: Use $\boxed{D1}$ with respect to L_2 to decompose $J = \cup K$ into $0(1)$ disjoint open intervals so that on each

$$K: L_2(w) \sim A(w-b)^k \quad \text{for some } A \neq 0, \text{ an integer } k \geq 0 \text{ and } b \in \mathbb{R}.$$

Step 1: This step will decompose each interval K from the previous step into intervals of two types T_0 and T_1 (in general Step n will produce intervals of 2^n types $\{T_{\mathbf{r}}\}_{\mathbf{r} \in \mathcal{B}_n}$ where \mathcal{B}_n is the collection of 0-1 bitstrings of length n , each interval of a particular type from Step $(n-1)$ being decomposed into intervals of two new types). Hence this step will decompose each $K = \cup I$ into $0(1)$ disjoint open intervals where either $I \in T_0$ or $I \in T_1$.

Fix an interval K from Step 0 and use $\boxed{D2}$ with respect to the polynomial L_1 and centre b to decompose $K = \cup I$ into $0(1)$ disjoint open intervals where each I is either a gap (\mathcal{G}) interval or a dyadic (\mathcal{D}) interval. The gap intervals $I \in \mathcal{G}$ are our intervals of type T_0 . Note that on

$$I \in \mathcal{G}: L_1(w) \sim A_0(w-b)^{k_0} \quad \text{for some } A_0 \neq 0 \text{ and an integer } k_0 \geq 0.$$

More accurately we should write $A_0 = A_{K,0}$, $k_0 = k_{K,0}$, etc... but we wish to emphasise the dependence on the type (T_0 in this case) of the particular interval K in the decomposition at this stage in the algorithm. For notational convenience we also write $b = b_0$ for the given centre to emphasise that it is a centre associated to an interval of type T_0 .

On the dyadic intervals $I \in \mathcal{D}$ we have $(w - b) \sim D$ and therefore $L_2(w) \sim AD^k$ for $w \in I$. We now decompose each $I \in \mathcal{D}$ further, using $\boxed{D1}$ with respect to L_1 , so that $I = \cup I'$ where on each

$$I': L_1(w) \sim A_1(w - b_1)^{k_1} \quad \text{for some } A_1 \neq 0, \text{ an integer } k_1 \geq 0 \text{ and } b_1 \in \mathbb{R}$$

(again using the convention above to tag the various parameters A, k and b which occur with the associated interval type). The intervals I' arising here are our intervals of type T_1 .

Let us recapitulate: we have decomposed each interval $J = \cup I$ into $0(1)$ disjoint intervals of two types T_0 and T_1 . On an interval I of type T_0 we have

$$L_2(w) \sim A(w - b)^k, \quad L_1(w) \sim A_0(w - b_0)^{k_0} \quad \text{for some } A, A_0 \neq 0, \text{ integers } k, k_0 \geq 0 \text{ and } b = b_0 \in \mathbb{R}.$$

And on an interval I of type T_1 we have

$$L_2(w) \sim AD^k, \quad L_1(w) \sim A_1(w - b_1)^{k_1} \quad \text{for some } D, A_1 \neq 0, \text{ integer } k_1 \geq 0 \text{ and } b_1 \in \mathbb{R}.$$

Finally we decompose each I further in order to avoid the collection of numbers $\{b_0\}$ and $\{b_1\}$ and arrive at the desired decomposition of \mathbb{R} . Recall that we aim to prove (18) for each interval I in this final decomposition. Let us begin with intervals I of type T_1 . From (17) and above we see that for $s, t \in I$,

$$J_{\Phi_{\Gamma}}(s, t) \sim (s - b_1)^{k_1}(t - b_1)^{k_1} \int_s^t \frac{AD^k}{(w - b_1)^{2k_1}} dw.$$

Therefore to prove (18) it suffices to establish

$$|s - b|^{k_1}|t - b_1|^{k_1} \int_s^t \frac{1}{|w - b_1|^{2k_1}} dw \gtrsim |s - t|.$$

We have the following simple lemma which is a special case of a more general result which we will establish later.

Lemma 8.1. *For $b \notin [s, t]$,*

$$\int_s^t \frac{1}{|w - b|^\sigma} dw \gtrsim \frac{|s - t|}{|s - b|^{\sigma/2}|t - b|^{\sigma/2}}$$

holds if and only if $\sigma \leq 0$ or $\sigma \geq 2$.

Since k_1 is a nonnegative integer, $2k_1 = 0$ or $2k_1 \geq 2$, and this establishes (18) for intervals of type T_1 .

Now suppose I is an interval of type T_0 . From (17) and above we see that for $s, t \in I$,

$$J_{\Phi_{\Gamma}}(s, t) \sim (s - b_0)^{k_0}(t - b_0)^{k_0} \int_s^t \frac{A}{(w - b_0)^{2k_0 - k}} dw.$$

Therefore to prove (18) it suffices to establish

$$\int_s^t \frac{1}{|w - b_0|^{2k_0 - k}} dw \gtrsim \frac{|s - t|}{|s - b_0|^{k_0 - k/2}|t - b_0|^{k_0 - k/2}}$$

for $s, t \in I$. By Lemma 8.1 this will be the case if and only if $2k_0 - k \neq 1$! Therefore we need to avoid the situation where k is odd and $k_0 = (k+1)/2$. We will accomplish this by adjusting the transition from Step 0 to Step 1. It is important to observe that the desired geometric inequality (18) is affine-invariant; that is, the inequality remains unchanged if we replace $\Gamma = (P, Q)$ by $\mathcal{A}\Gamma$ where \mathcal{A} is any invertible 2×2 constant matrix.

We begin with Step 0 as before which reduces to intervals on which $L_{\Gamma,2}(w) \sim A(w-b)^k$. Now assume k is odd (otherwise we do not need any further adjustments). Before proceeding to Step 1 we prepare our polynomials P and Q accordingly; looking ahead into Step 1 we see that intervals of type T_0 arise from gap intervals when we employ $\boxed{D2}$ with respect to $L_{\Gamma,1} = P'$ and b . Recall that the ‘bad’ exponent $k_0 = (k+1)/2$ can be avoided in the decomposition procedure $\boxed{D2}$ if the polynomial $L_{\Gamma,1}(t+b) = \sum c_j t^j$ has **no** $c_{(k+1)/2}$ coefficient. Therefore, before moving to Step 1 we apply a certain invertible linear transformation

$$\mathcal{A} = \begin{pmatrix} a_1 & a_2 \\ * & * \end{pmatrix} \in GL_2(\mathbb{R})$$

to $\Gamma = (P, Q)$ where the row vector $\vec{a} = (a_1, a_2)$ will be chosen to be nonzero and the remaining entries chosen to make the matrix invertible. We now check how this transformation affects our two polynomials $L_2 = L_{\Gamma,2}$ and $L_1 = L_{\Gamma,1}$. The polynomial L_2 is only changed by a constant; $L_{\mathcal{A}\Gamma,2} = (\det \mathcal{A})L_{\Gamma,2}$. On the other hand, if $P'(t+b) = \sum c_j^1 t^j$ and $Q'(t+b) = \sum c_j^2 t^j$, then

$$L_{\mathcal{A}\Gamma,1}(t+b) = \sum (a_1 c_j^1 + a_2 c_j^2) t^j = \sum \vec{a} \cdot \vec{c}_j t^j$$

where $\vec{c}_j = (c_j^1, c_j^2)$. Therefore we simply choose a nonzero vector \vec{a} so that $\vec{a} \cdot \vec{c}_{k_0} = 0$ where $k_0 = (k+1)/2$.

We carry out Step 1 proceeding exactly as before except now using the transformed polynomials $\mathcal{A}\Gamma$. Observe by homogeneity the inequality (18) is unaffected by changing $L_2 = L_{\Gamma,2}$ by a constant. In fact (18), and more generally (2), is unaffected by changing any of the polynomials $L_j, 1 \leq j \leq d$, by a multiplicative constant. Therefore, for notational convenience, we will systematically suppress all multiplicative constants arising when reducing to intervals on which our various polynomials behave like centred monomials; for example, we will write $L_{\Gamma,j}(t) \sim (t-b)^k$ when in fact we mean $L_{\Gamma,j}(t) \sim A(t-b)^k$ for some $A \neq 0$. Furthermore we will suppress the constants arising from ‘dyadic’ intervals; for example, we will write $(t-b) \sim 1$ when in fact we mean $(t-b) \sim D$ for some $D \neq 0$. On such intervals the fact that $(t-b) \sim D$ will have the effect of making various polynomials behave like constants (for instance, in the discussion above, this makes L_2 behave like a constant) and by homogeneity of (18) or (2), these constants always cancel out.

9. THE ALGORITHM – GENERAL CASE

We now set out the general algorithm, valid in all dimensions, to decompose an interval J from the initial decomposition of \mathbb{R} into $0(1)$ intervals on which (2) holds. As discussed in the previous section this will be carried out in d steps.

Step 0: Use $\boxed{D1}$ with respect to $L_{\Gamma,d}$ to decompose $J = \cup K$ into $0(1)$ disjoint open intervals so that on each

K : $L_{\Gamma,d} \sim (t-b)^k$ for some nonnegative integer $k = k(K)$ and real number $b = b(K)$

(recall that we are systematically suppressing all multiplicative constants – we in fact mean $L_{\Gamma,d} \sim A(t-b)^k$ for some $A \neq 0$ – which cancel out in (2)). Before we proceed to the next step we transform Γ in order to avoid certain bad exponents when we apply $\boxed{D2}$. To this end we introduce

$$A = \begin{pmatrix} a_1 & \cdots & a_d \\ & & * \end{pmatrix} \in GL_d(\mathbb{R})$$

where the row vector $\vec{a} = (a_1, \dots, a_d) \in \mathbb{R}^d$ will be chosen momentarily to be nonzero and the remaining entries chosen to guarantee that A is invertible but otherwise chosen in an arbitrary fashion. We note that

$$L_{A\Gamma,d} = (\det A) L_{\Gamma,d}$$

is only changed by a multiplicative constant and so will not affect (2). On the other hand, if $\Gamma = (P_1, \dots, P_d)$ and $P'_k(t+b) = \sum c_j^k t^j$, then

$$L_{A\Gamma,1}(t+b) = \sum \vec{a} \cdot \vec{c}_j t^j$$

where $\vec{c}_j = (c_j^1, \dots, c_j^d)$. We choose $\vec{a} = (a_1, \dots, a_d)$ to be any nonzero vector which is orthogonal to the hyperplane $H = \text{span}\{\vec{c}_{\lfloor \frac{k}{d} \rfloor + j}\}_{j=1}^{d-1}$.

Step 1: In this step we will decompose each $K = \cup L$ from the previous step into $0(1)$ disjoint open intervals of two types T_0 and T_1 . For an interval L of type T_0 we will associate an exponent (a nonnegative integer) $k_0 = k_0(L)$ and a centre (a real number) $b_0 = b_0(L)$ so that on

$$L: L_{A\Gamma,1}(t) \sim (t-b_0)^{k_0} \quad \text{and} \quad L_{A\Gamma,d}(t) \sim (t-b)^k. \quad (19)$$

Importantly we will have $b = b_0$ and $k_0 \notin \{\lfloor \frac{k}{d} \rfloor + 1, \dots, \lfloor \frac{k}{d} \rfloor + (d-1)\}$. Furthermore for an interval L of type T_1 we will associate an exponent $k_1 = k_1(L)$ and a centre $b_1 = b_1(L)$ so that on

$$L: L_{A\Gamma,1}(t) \sim (t-b_1)^{k_1} \quad \text{and} \quad L_{A\Gamma,d}(t) \sim 1. \quad (20)$$

Here we will have no control over the values of b_1 and k_1 .

To achieve (19) and (20) we use $\boxed{D2}$ with respect to the polynomial $L_{A\Gamma,1}$ and centre b to decompose $K = \cup L$ into gap (\mathcal{G}) intervals or dyadic (\mathcal{D}) intervals. Note that by construction (19) is satisfied for our gap intervals $L \in \mathcal{G}$ and so these are the intervals of type T_0 . To arrive at our intervals of type T_1 we use $\boxed{D1}$ with respect to the polynomial $L_{A\Gamma,1}$ to decompose each dyadic $L = \cup L'$ further into $0(1)$ disjoint intervals so that on each L' (20) holds (note that on each dyadic $L \in \mathcal{D}$, $L_{A\Gamma,d}(t) \sim 1$ since $(t-b) \sim 1$ on such L). This finishes Step 1.

Step $n \rightarrow$ Step $(n+1)$: We now describe how we pass from Step n to Step $(n+1)$, $1 \leq n \leq d-2$.

The intervals which arise by Step n will be of 2^n types $T_{\mathbf{r}}$, parametrised by 0-1 bitstrings $\mathbf{r} = r_1 \cdots r_n$ of length n . Fix an interval J of type $T_{\mathbf{r}}$; we will have associated to J a centre (real number) $b_{\mathbf{r}} = b_{\mathbf{r}}(J)$ and an exponent (nonnegative integer) $k_{\mathbf{r}} = k_{\mathbf{r}}(J)$. Furthermore J will have a unique parent (and grandparent, etc... all the way back to an interval from Step 0) \tilde{J} from the previous step - Step

$(n-1)$ - of type $T_{\tilde{\mathbf{r}}}$ where $\tilde{\mathbf{r}} = r_1 \cdots r_{n-1}$ and there will have been associated to \tilde{J} a matrix $A_{\tilde{\mathbf{r}}} = A_{\tilde{\mathbf{r}}}(\tilde{J}) \in GL_d(\mathbb{R})$ so that on

$$J : L_{A_{\tilde{\mathbf{r}}}\Gamma, n}(t) \sim (t - b_{\mathbf{r}})^{k_{\mathbf{r}}}. \quad (21)$$

To carry out the decomposition of each interval $J = \cup K$ into intervals of type $T_{\mathbf{r}_0}$ or type $T_{\mathbf{r}_1}$ at Step $(n+1)$, we will need to construct an appropriate invertible matrix $A_{\mathbf{r}} = A_{\mathbf{r}}(J)$ (which in fact will leave the first n components of $A_{\tilde{\mathbf{r}}}\Gamma$ unchanged). For an interval K of type $T_{\mathbf{r}_0}$ we will find a centre $b_{\mathbf{r}_0} = b_{\mathbf{r}_0}(K)$ and an exponent $k_{\mathbf{r}_0} = k_{\mathbf{r}_0}(K)$ so that on

$$K : L_{A_{\mathbf{r}}\Gamma, n+1}(t) \sim (t - b_{\mathbf{r}_0})^{k_{\mathbf{r}_0}}. \quad (22)$$

Importantly we will achieve this with $b_{\mathbf{r}_0} = b_{\mathbf{r}}$ and some $k_{\mathbf{r}_0} \notin \{N_{\mathbf{r}} + 1, \dots, N_{\mathbf{r}} + d - n - 1\}$ where

$$N_{\mathbf{r}} = \lfloor \frac{d-n-1}{d-n} k_{\mathbf{r}} \rfloor \text{ if } \mathbf{r} \neq 0 \cdots 0, \quad N_{\mathbf{r}} = \lfloor \frac{d-n-1}{d-n} k_{\mathbf{r}} + \frac{k}{d-n} \rfloor \text{ if } \mathbf{r} = 0 \cdots 0. \quad (23)$$

For an interval K of type $T_{\mathbf{r}_1}$ we will find a centre $b_{\mathbf{r}_1} = b_{\mathbf{r}_1}(K)$ and an exponent $k_{\mathbf{r}_1} = k_{\mathbf{r}_1}(K)$ so that on

$$K : L_{A_{\mathbf{r}}\Gamma, n+1}(t) \sim (t - b_{\mathbf{r}_1})^{k_{\mathbf{r}_1}}. \quad (24)$$

Here we will have no control over the values of $b_{\mathbf{r}_1}$ and $k_{\mathbf{r}_1}$.

Before we see (22) and (24) we construct the invertible matrix $A_{\mathbf{r}} = A_{\mathbf{r}}(J)$ which will depend on $b_{\mathbf{r}} = b_{\mathbf{r}}(J)$ and $k_{\mathbf{r}} = k_{\mathbf{r}}(J)$ already determined by Step n . In fact

$$A_{\mathbf{r}} = \left(\begin{array}{c|ccc} I_n & & & \mathbf{O} \\ \hline & a_1 & a_2 & \cdots & a_{d-n} \\ \mathbf{O} & & & & * \end{array} \right) A_{\tilde{\mathbf{r}}}$$

for an appropriate choice of $\vec{a} = (a_1, \dots, a_{d-n}) \in \mathbb{R}^{d-n}$ which we now describe. If $A_{\tilde{\mathbf{r}}}\Gamma = (Q_1, \dots, Q_d)$, then for $1 \leq j \leq d-n$, expand $L_{Q_1 \cdots Q_n Q_{n+j}}(t + b_{\mathbf{r}}) = \sum_{\ell} c_{\ell}^j t^{\ell}$ and set $\vec{c}_{\ell} = (c_{\ell}^1, \dots, c_{\ell}^{d-n})$. Here we are using the notation introduced in section 4 for $L_{Q_1 \cdots Q_n Q_{n+j}}$. Hence

$$L_{A_{\mathbf{r}}\Gamma, n+1}(t + b_{\mathbf{r}}) = \sum_{j=1}^{d-n} a_j L_{Q_1 \cdots Q_n Q_{n+j}}(t + b_{\mathbf{r}}) = \sum_{\ell} \vec{a} \cdot \vec{c}_{\ell} t^{\ell}.$$

Now choose a nonzero vector $\vec{a} \in \mathbb{R}^{d-n}$ orthogonal to the subspace spanned by $\{\vec{c}_{N_{\mathbf{r}}+1}, \dots, \vec{c}_{N_{\mathbf{r}}+(d-n-1)}\}$ where $N_{\mathbf{r}}$ is defined in (23).

The procedure to establish (22) and (24) is exactly the same as for (19) and (20); use $\boxed{D2}$ with respect to the polynomial $L_{A_{\mathbf{r}}\Gamma, n+1}$ and centre $b_{\mathbf{r}}$ to decompose $J = \cup K$ into gap (\mathcal{G}) intervals or dyadic (\mathcal{D}) intervals. Note that by construction (22) is satisfied for our gap intervals $L \in \mathcal{G}$ since $L_{A_{\mathbf{r}}\Gamma, n+1}(t + b_{\mathbf{r}}) = \sum c_j t^j$ has the property that the coefficients $c_{N_{\mathbf{r}}+\ell}$ vanish for all $\ell = 1, 2, \dots, d-n-1$. The way we defined $A_{\mathbf{r}}$ guarantees that this is the case. Hence these gap intervals will be our intervals of type $T_{\mathbf{r}_0}$. To arrive at our intervals of type $T_{\mathbf{r}_1}$ we use $\boxed{D1}$ with respect to the polynomial $L_{A_{\mathbf{r}}\Gamma, n+1}$ to decompose each dyadic $K = \cup K'$ further into $0(1)$ disjoint intervals so that each K' (24) holds. This completes the inductive step from Step n to Step $(n+1)$.

⋮

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Step $(d-1)$: We finally arrive at the final step. Let us fix an interval $J_{\mathbf{r}}$, $\mathbf{r} = r_1 \cdots r_{d-1}$, of type $T_{\mathbf{r}}$ at this final step and describe what the algorithm produces on this interval. To this end let $\mathbf{r}_j = r_1 \cdots r_j$ when $1 \leq j \leq d-1$ (so that $\mathbf{r}_{d-1} = \mathbf{r}$) and let \mathbf{r}_0 denote the empty string. We have $d-1$ invertible matrices $\{A, A_{\mathbf{r}_1}, \dots, A_{\mathbf{r}_{d-2}}\}$, d centres $\{b, b_{\mathbf{r}_1}, \dots, b_{\mathbf{r}_{d-1}} = b_{\mathbf{r}}\}$ and d exponents $\{k, k_{\mathbf{r}_1}, \dots, k_{\mathbf{r}}\}$ associated to $J_{\mathbf{r}}$, its parent, grandparent, etc... all the way back to an interval J from the initial decomposition at Step 0 (note there is no matrix $A_{\mathbf{r}}$ as we do not pass from Step $(d-1)$ to Step d). Let $0 \leq m \leq d-1$ be so that $\mathbf{r} = r_1 \cdots r_m 0 \cdots 0$ and $r_m = 1$ ($m=0$ being the case $\mathbf{r} = 0 \cdots 0$). When $m \geq 1$ we have the following properties from our algorithm:

Property 1 On $J_{\mathbf{r}}$,

$$L_{\Gamma, d}(t) \sim 1, \dots, L_{A_{\mathbf{r}_{m-2}} \Gamma, m-1}(t) \sim 1, L_{A_{\mathbf{r}_{m-1}} \Gamma, m}(t) \sim (t - b_{\mathbf{r}_m})^{k_{\mathbf{r}_m}},$$

$$L_{A_{\mathbf{r}_m} \Gamma, m+1}(t) \sim (t - b_{\mathbf{r}_{m+1}})^{k_{\mathbf{r}_{m+1}}}, \dots, L_{A_{\mathbf{r}_{d-2}} \Gamma, d-1}(t) \sim (t - b_{\mathbf{r}})^{k_{\mathbf{r}}}$$

where $k_{\mathbf{r}_m} \geq 0$ is unrestricted but for $m+1 \leq j \leq d-1$,

$$k_{\mathbf{r}_j} \notin \{N_{\mathbf{r}_{j-1}} + 1, \dots, N_{\mathbf{r}_{j-1}} + d - j\} \quad \text{where } N_{\mathbf{r}_{j-1}} = \lfloor \frac{d-j}{d-j+1} k_{\mathbf{r}_{j-1}} \rfloor$$

(the $m=1$ case being interpreted as $L_{\Gamma, d} \sim 1, L_{A_{\Gamma, 1}}(t) \sim (t - b_{\mathbf{r}_1})^{k_{\mathbf{r}_1}}$, etc...).

Property 2 For each $0 \leq j \leq d-2$, $L_{A_{\mathbf{r}_j} \Gamma, j+1} = L_{A_{\mathbf{r}_{d-2}} \Gamma, j+1}$ because of the form of the matrices $A_{\mathbf{r}_j}$. Hence on $J_{\mathbf{r}}$, if $Q = A_{\mathbf{r}_{d-2}} \Gamma$,

$$L_{Q, d}(t) \sim 1, L_{Q, 1}(t) \sim 1, \dots, L_{Q, m-1}(t) \sim 1,$$

$$L_{Q, m}(t) \sim (t - b_{\mathbf{r}_m})^{k_{\mathbf{r}_m}}, \dots, L_{Q, d-1}(t) \sim (t - b_{\mathbf{r}})^{k_{\mathbf{r}}}.$$

Property 3 For $m \leq j \leq d-1$, $b_{\mathbf{r}_j} = b_{\mathbf{r}}$.

The case $m=0$ is special; here $\mathbf{r} = 0 \cdots 0$. In this case we have on $J_{\mathbf{r}}$:

$$L_{Q, d}(t) \sim (t - b)^k, L_{Q, 1}(t) \sim (t - b)^{k_{\mathbf{r}_1}}, \dots, L_{Q, d-1}(t) \sim (t - b)^{k_{\mathbf{r}}} \quad (25)$$

where $k \geq 0$ is unrestricted but each $k_{\mathbf{r}_j}$, $1 \leq j \leq d-1$ has the restriction $k_{\mathbf{r}_j} \notin \{M_{\mathbf{r}_{j-1}} + 1, \dots, M_{\mathbf{r}_{j-1}} + d - j\}$ where $M_{\mathbf{r}_{j-1}} = \lfloor \frac{d-j}{d-j+1} k_{\mathbf{r}_{j-1}} + \frac{k}{d-j+1} \rfloor$ (here $k_{\mathbf{r}_0} = 0$).

We are now in a position to describe our final decomposition of $\mathbb{R} = \cup I$ into $0(1)$ disjoint open intervals so that (P1) and (P2) of section 3 hold for each I . The initial decomposition together with the algorithm set out in this section produces a decomposition of $\mathbb{R} = \cup J$ so that Properties 1, 2 and 3 hold on each J (this is the case when $m \geq 1$; property (25) holding for the case $m=0$). Now collect together all the centres $\{b_{\mathbf{r}}\}$ associated to each J , its parent, grandparent, etc... (there are $0(1)$ such centres) and decompose each J into disjoint open intervals avoiding these numbers. Thus we finally arrive at our desired decomposition $\mathbb{R} = \cup I$.

10. PRELIMINARIES FOR THE GEOMETRIC INEQUALITY (P2)

We must verify that (P2) or (2) holds for each I in the final decomposition of $\mathbb{R} = \cup I$ described in the previous section. Fix an interval I in this final decomposition and recall that $I \subset J_{\mathbf{r}}$ for a unique interval $J_{\mathbf{r}}$, $\mathbf{r} = r_1 \cdots r_{d-1}$, arising in the last step of our algorithm. Recall also that it suffices to prove (2) for I with Γ replaced by $B\Gamma$ for any $B \in GL_d(\mathbb{R})$ and we will do so for $B = A_{\mathbf{r}_{d-2}}$. Again we set $Q = A_{\mathbf{r}_{d-2}}\Gamma$ and we will use $I_r, 1 \leq r \leq d$ to denote iterated integrals, originally defined in section 5 with respect to Γ , but now we define the I_r with respect to Q . For $t = (t_1, \dots, t_d)$, we set $J_Q(t) = \det(Q'(t_1) \cdots Q'(t_d))$ and as before we have

$$J_Q(t_1, \dots, t_d) = I_d(t_1, \dots, t_d)$$

for $t = (t_1, \dots, t_d) \in I^d$. Our remaining task is to establish

$$|I_d(t_1, \dots, t_d)| \gtrsim \prod_{r=1}^d |L_{Q,d}(t_r)|^{1/d} \prod_{r < s} |t_r - t_s| \quad (26)$$

for $t = (t_1, \dots, t_d) \in I^d$.

We now use Properties [1](#), [2](#) and [3](#) or (25) to reduce (26) to a concrete inequality. Let us begin with the case $m \geq 1$; to this end we define a concrete series of iterated integrals which arise when we replace the various polynomials $L_{Q,j}$ defining the integrals I_r by their associated centred monomials which describe the behaviour of $L_{Q,j}$ on our interval I . Define $S_1(t_1) = |t_1 - b_{\mathbf{r}}|^{k_{\mathbf{r}_{d-2}} - 2k_{\mathbf{r}}}$ and then inductively for $2 \leq r \leq d - m$,

$$S_r(t_1, \dots, t_r) = \prod_{s=1}^r |t_s - b_{\mathbf{r}}|^{\sigma_r} \int_{t_1}^{t_2} \cdots \int_{t_{r-1}}^{t_r} S_{r-1}(w_1, \dots, w_{r-1}) dw_1 \dots dw_{r-1}$$

where $\sigma_r = k_{\mathbf{r}_{d-r-1}} + k_{\mathbf{r}_{d-r+1}} - 2k_{\mathbf{r}_{d-r}}$ for $2 \leq r \leq d - m - 1$ but $\sigma_{d-m} = k_{\mathbf{r}_{m+1}} - 2k_{\mathbf{r}_m}$. Finally for $r = d - m + 1$ we set $S_{d-m+1}(t_1, \dots, t_{d-m+1}) =$

$$\prod_{s=1}^{d-m+1} |t_s - b_{\mathbf{r}}|^{k_{\mathbf{r}_m}} \int_{t_1}^{t_2} \cdots \int_{t_{d-m}}^{t_{d-m+1}} S_{d-m}(w_1, \dots, w_{d-m}) dw_1 \dots dw_{d-m}. \quad (27)$$

In this case ($m \geq 1$) (26) reduces to

$$S_{d-m+1}(t_1, \dots, t_{d-m+1}) \gtrsim \prod_{1 \leq r < s \leq d-m+1} (t_r - t_s). \quad (28)$$

Recall $k_{\mathbf{r}_m} \geq 0$ is unrestricted but $k_{\mathbf{r}_j} \notin \{N_{\mathbf{r}_{j-1}} + 1, \dots, N_{\mathbf{r}_{j-1}} + d - j\}$ where $N_{\mathbf{r}_{j-1}} = \lfloor \frac{d-j}{d-j+1} k_{\mathbf{r}_{j-1}} \rfloor$ for $m+1 \leq j \leq d-1$.

We end up with a similar inequality to (28) to establish for the case $m = 0$ ($\mathbf{r} = 0 \cdots 0$). Using (25) we see that we need to adjust the formula for S_{d-m+1} above when $m = 1$ slightly; we define \tilde{S}_d exactly as S_d above in (27) with $m = 1$ except we change S_1 to

$$\tilde{S}_1(t_1) = |t_1 - b_{\mathbf{r}}|^{k + k_{\mathbf{r}_{d-2}} - 2k_{\mathbf{r}}}.$$

For $1 \leq j \leq d-1$, $k_{\mathbf{r}_j} \notin \{M_{\mathbf{r}_{j-1}} + 1, \dots, M_{\mathbf{r}_{j-1}} + d - j\}$ where $M_{\mathbf{r}_{j-1}} = \lfloor \frac{d-j}{d-j+1} k_{\mathbf{r}_{j-1}} + \frac{k}{d-j+1} \rfloor$ ($k_{\mathbf{r}_0} = 0$) but $k \geq 0$ is unrestricted. The inequality to establish in this case

($m = 0$) is

$$|\tilde{S}_d(t_1, \dots, t_d)| \gtrsim \prod_{r=1}^d |t_r - b_{\mathbf{r}}|^{k/d} \prod_{1 \leq r < s \leq d} |t_r - t_s|. \quad (29)$$

Hence we simply need to prove d concrete inequalities, (28) (cases $1 \leq m \leq d-1$) and (29) (case $m = 0$). The notation $\mathbf{r} \in \mathcal{B}$ of 0-1 bitstrings to enumerate our exponents $k_{\mathbf{r}}$ and centres $b_{\mathbf{r}}$ served us well for the general algorithm but is no longer needed and we rewrite (28) and (29) with simpler notation. We begin with the cases $1 \leq m \leq d-1$; for $n \geq 2$, we start with any sequence of $n-1$ nonnegative integers k_0, k_1, \dots, k_{n-2} where $k_0 \geq 0$ is unrestricted but for $1 \leq j \leq n-2$,

$$\text{either } k_j \leq \frac{n-j-1}{n-j} k_{j-1} \text{ or } k_j \geq \frac{n-j-1}{n-j} k_{j-1} + (n-j-1). \quad (30)$$

From this sequence k_j we form a new sequence $\sigma_0 = k_1 - 2k_0, \sigma_1 = k_2 + k_0 - 2k_1, \dots, \sigma_{n-2} = k_{n-3} - 2k_{n-2}$ of nonnegative integers. We now write down a nested series of iterated integrals. Set $E_n = E_n(x_1, \dots, x_n, b)$ where

$$E_n = \prod_{r=1}^n |x_r - b|^{k_0} \int_{x_1}^{x_2} \dots \int_{x_{n-1}}^{x_n} \prod_{r=1}^{n-1} |y_r - b|^{\sigma_0} \dots \int_{u_1}^{u_2} |w - b|^{\sigma_{n-2}} dw du_1 \dots dy_{n-1}.$$

Our desired inequality (28) is implied by the following proposition.

Proposition 10.1. *For any $n \geq 2$, $x_1 < x_2 < \dots < x_n$ and $b \notin [x_1, x_n]$,*

$$E_n \gtrsim \prod_{r < s} (x_s - x_r).$$

The notational reformulation of (29) is a slight variant of Proposition 10.1. Here we start with a sequence of d nonnegative integers k_0, \dots, k_{d-2} and k where now $k \geq 0$ is unrestricted but for $0 \leq j \leq d-2$ ($k_{-1} = 0$),

$$k_j \leq \frac{d-j-1}{d-j} k_{j-1} + \frac{k}{d-j} \quad \text{or} \quad k_j \geq \frac{d-j-1}{d-j} k_{j-1} + \frac{k}{d-j} + (d-j-1). \quad (31)$$

We define a sequence $\tilde{\sigma}_j = \sigma_j$ for $0 \leq j \leq d-3$ (where the σ_j are defined above) but we define $\tilde{\sigma}_{d-2} = k + k_{d-3} - 2k_{d-2}$. Finally we define $F_d = F_d(x_1, \dots, x_d, b)$ exactly as we defined E_n with $n = d$ except the sequence $\{\sigma_j\}$ is replaced by $\{\tilde{\sigma}_j\}$. The inequality (29) follows from the next proposition.

Proposition 10.2. *For any $x_1 < x_2 < \dots < x_d$ and $b \notin [x_1, x_d]$,*

$$|F_d| \gtrsim \prod_{r=1}^d |x_r - b|^{k/d} \prod_{r < s} |x_s - x_r|.$$

In the proof of both Propositions 10.1 and 10.2 we may assume, without loss of generality, that x_n or $x_d < b$ and for the proof itself we will need to examine iterated integrals of the form

$$I = \int_{z_1}^{z_2} \dots \int_{z_{\ell-1}}^{z_{\ell}} \prod_{r=1}^{\ell-1} |y_r - b|^{\rho_r} \prod_{r < s} |y_r - y_s| dy_1 \dots dy_{\ell-1}$$

where $z_1 < \dots < z_{\ell} < b$.

Lemma 10.3. *If $y_1 < \dots < y_{k-1} < w < z < y_{k+1} < \dots < y_{\ell-1} < b$, then*

$$\int_w^z |y_k - b|^a \prod_{r < s} |y_r - y_s| dy_k \gtrsim C_{a,\ell,k}(w, z, b) \int_w^z \prod_{r < s} |y_r - y_s| dy_k \quad (32)$$

where

$$C_{a,\ell,k}(w, z, b) = \max\left(|w - b|^a, \frac{|z - b|^{a+\ell-k}}{|w - b|^{\ell-k}}\right).$$

Proof We will assume that $|w - b| > 9|z - b|$; otherwise the proof simplifies. Consider the following two disjoint subsets of $[w, z]$, $L = \{w \leq y_k \leq z : 1/3|w - b| \leq |y_k - b| \leq 1/2|w - b|\}$ and $U = \{w \leq y_k \leq z : 2|z - b| \leq |y_k - b| \leq 3|z - b|\}$. On these subsets it is a simple matter to verify that for $r < k$, $|y_r - y_k| \sim |y_r - b|$, and for $k < s$, $|y_k - y_s| \sim |y_k - b|$. Hence

$$\begin{aligned} & \int_w^z |y_k - b|^a \prod_{r < s} |y_r - y_s| dy_k \\ & \gtrsim \prod_{\substack{r < s \\ r \neq k, s \neq k}} |y_r - y_s| \prod_{r=1}^{k-1} |y_r - b| \left[\int_L + \int_U \right] |y_k - b|^{a+\ell-k-1} dy_k \\ & \sim \prod_{\substack{r < s \\ r \neq k, s \neq k}} |y_r - y_s| \prod_{r=1}^{k-1} |y_r - b| (|w - b|^{a+\ell-k} + |z - b|^{a+\ell-k}) \\ & \gtrsim \left(|w - b|^a + \frac{|z - b|^{a+\ell-k}}{|w - b|^{\ell-k}} \right) \int_w^z \prod_{r < s} |y_r - y_s| dy_k \end{aligned}$$

where in the last inequality we used $|w - b| \geq |w - z|$ and $|w - b| \geq |y_k - y_s|$ for $s > k$. \blacksquare

We will use the notation \downarrow to indicate when we employ the estimate in Lemma 10.3 with $|w - b|^a$ and the notation \uparrow when we use the estimate with $|z - b|^{a+\ell-k}/|w - b|^{\ell-k}$. Furthermore we will use the notation $\downarrow(r) \downarrow \uparrow(s) \uparrow$ etc... to indicate we are using Lemma 10.3 iteratively to estimate an iterated integral (for example, I above), using the estimate \downarrow for the first r integrals and then the estimate \uparrow for the remaining s integrals (so we must have $r + s = \ell - 1$ if we are indeed estimating the iterated integral I above). We single out two special estimates for I which follow from Lemma 10.3:

$$\downarrow(\ell - 1) \downarrow : \quad I \gtrsim \prod_{r=1}^{\ell-1} |z_r - b|^{\rho_r} \prod_{r < s} |z_r - z_s|$$

and

$$\uparrow(\ell - 1) \uparrow : \quad I \gtrsim \frac{1}{|z_1 - b|^{\ell-1}} \prod_{r=2}^{\ell} |z_r - b|^{\rho_{r-1}+1} \prod_{r < s} |z_r - z_s|.$$

We put these two estimates for I together to establish the following useful estimate for I in the case where all the exponents ρ_r appearing in I are equal. This is the extension of Lemma 8.1 promised earlier.

Lemma 10.4. *If $\rho_1 = \rho_2 = \dots = \rho_{\ell-1} = \rho$,*

$$I \gtrsim \prod_{r=1}^{\ell} |z_r - b|^{\frac{\ell-1}{\ell}\rho} \prod_{r<s} |z_r - z_s|$$

holds if and only if $\rho \geq 0$ or $\rho \leq -\ell$.

Proof We will only prove the sufficiency part of the lemma as this is the only part that is needed to establish Propositions 10.1 and 10.2 and leave the necessity to the interested reader.

For $\rho \geq 0$, use $\downarrow (\ell - 1) \downarrow$ to obtain

$$I \gtrsim \prod_{r=1}^{\ell-1} |z_r - b|^{\rho} \prod_{r<s} |z_r - z_s| \geq \prod_{r=1}^{\ell} |z_r - b|^{\frac{\ell-1}{\ell}\rho} \prod_{r<s} |z_r - z_s|$$

since $|z_r - b| \geq |z_{\ell} - b|$ for $1 \leq r \leq \ell - 1$.

For $\rho \leq -\ell$, use $\uparrow (\ell - 1) \uparrow$ to obtain

$$I \gtrsim \frac{1}{|z_1 - b|^{\ell-1}} \prod_{r=2}^{\ell} |z_r - b|^{\rho+1} \prod_{r<s} |z_r - z_s| \geq \prod_{r=1}^{\ell} |z_r - b|^{\frac{\ell-1}{\ell}\rho} \prod_{r<s} |z_r - z_s|$$

since $|z_1 - b| \geq |z_r - b|$ for $2 \leq r \leq \ell$. ■

We now return to E_n and F_d in Propositions 10.1 and 10.2 and prove a preliminary estimate for these nested series of iterated integrals by making repeated use of Lemma 10.4. We start with the innermost integral and apply Lemma 10.4 to it;

$$\int_{u_1}^{u_2} |w - b|^s dw \gtrsim |u_1 - u_2| [|u_1 - b| |u_2 - b|]^{\frac{1}{2}s}$$

holds if and only if $s \geq 0$ or $s \leq -2$. For E_n , $s = \sigma_{n-2} = k_{n-3} - 2k_{n-2}$ and by (30) either

$$i) \quad k_{n-2} \leq \frac{1}{2}k_{n-3} \quad \Rightarrow \quad s = \sigma_{n-2} \geq 0 \quad \text{or}$$

$$ii) \quad k_{n-2} \geq \frac{1}{2}k_{n-3} + 1 \quad \Rightarrow \quad s = \sigma_{n-2} \leq -2.$$

For F_d , $s = \tilde{\sigma}_{d-2} = k + k_{d-3} - 2k_{d-2}$ and by (31) either

$$i) \quad k_{d-2} \leq \frac{1}{2}k_{d-3} + \frac{k}{2} \quad \Rightarrow \quad s = \tilde{\sigma}_{d-2} \geq 0 \quad \text{or}$$

$$ii) \quad k_{d-2} \geq \frac{1}{2}k_{d-3} + \frac{k}{2} + 1 \quad \Rightarrow \quad s = \tilde{\sigma}_{d-2} \leq -2.$$

Observe that when we apply Lemma 10.4 iteratively to each successive nested iterated integral defining either E_n or F_d we end up with an iterated integral with the form I above where all the exponents ρ_r are equal and so Lemma 10.4 can once again be applied. At the $(\ell - 1)$ th application ($2 \leq \ell \leq n$ or d) of Lemma 10.4 we need to estimate

$$I_{\ell} = \int_{z_1}^{z_2} \dots \int_{z_{\ell-1}}^{z_{\ell}} \prod_{r<t} |y_r - y_t| \prod_{r=1}^{\ell-1} |y_r - b|^{\rho_{\ell}} dy_1 \dots dy_{\ell-1}$$

where $\rho_{\ell} = s_{n-\ell} + \frac{\ell-2}{\ell-1}(s_{n-\ell+1} + \dots + \frac{2}{3}(s_{n-3} + \frac{1}{2}s_{n-2}))$ and $s = \sigma$ for E_n and $s = \tilde{\sigma}$ for F_d (and then $n = d$).

Claim 1: For E_n (and so $s = \sigma$), $\rho_\ell = k_{n-\ell-1} - \frac{\ell}{\ell-1}k_{n-\ell}$, $2 \leq \ell \leq n$.

Here we interpret $k_{-1} = 0$. To prove this claim we proceed by induction on ℓ ; the case $\ell = 2$ being clear. By induction, for $3 \leq \ell \leq n$, $\rho_\ell =$

$$\sigma_{n-\ell} + \frac{\ell-2}{\ell-1} \left(k_{n-\ell} - \frac{\ell-1}{\ell-2} k_{n-\ell+1} \right) = k_{n-\ell-1} + k_{n-\ell+1} - 2k_{n-\ell} + \frac{\ell-2}{\ell-1} k_{n-\ell} - k_{n-\ell+1}$$

and so $\rho_\ell = k_{n-\ell-1} - (\ell/\ell-1)k_{n-\ell}$.

By (30) we see that Claim 1 implies that either $\rho_\ell \geq 0$ or $\rho_\ell \leq -\ell$ *only if* $2 \leq \ell \leq n-1$ and so Lemma 10.4 can be applied to these I_ℓ .

Claim 2: For F_d (and so $s = \tilde{\sigma}$), $\rho_\ell = k_{d-\ell-1} - \frac{1}{\ell-1}(\ell k_{d-\ell} - k)$, $2 \leq \ell \leq d$.

The proof of Claim 2 is the same as Claim 1, proceeding by induction on ℓ . Hence by (31), Claim 2 implies that either $\rho_\ell \geq 0$ or $\rho_\ell \leq -\ell$ *for all* $2 \leq \ell \leq d$ and so Lemma 10.4 can be applied to all the iterated integrals defining F_d giving us the desired estimate for F_d , completing the proof of Proposition 10.2.

On the other hand, after the $(n-1)$ st application of Lemma 10.4 to each of the iterated integrals defining E_n we have

$$E_n \gtrsim \prod_{r=1}^n |x_r - b|^{k_0} \int_{x_1}^{x_2} \cdots \int_{x_{n-1}}^{x_n} \prod_{r=1}^{n-1} |y_r - b|^{-\frac{n}{n-1}k_0} \prod_{r < s} |y_r - y_s| dy_1 \cdots dy_{n-1}.$$

Unfortunately the exponent $k_0 \geq 0$ is unrestricted, preventing us to obtain an unconditional estimate for E_n . Nevertheless **if** $k_0 = 0$ or $k_0 \geq n-1$, then Lemma 10.4 can be applied once more to conclude $E_n \gtrsim \prod_{r < s} |x_r - x_s|$, completing the proof of Proposition 10.1 in this case.

It still remains to prove Proposition 10.1 but, from above, we may assume $1 \leq k_0 \leq n-2$.

11. A FURTHER REDUCTION

In this section we will reduce the proof of Proposition 10.1 when $1 \leq k_0 \leq n-2$ to a couple of combinatorial lemmas. To this end we split the analysis into 2^{n-2} cases, depending on the exact relations among the exponents k_j in (30). Here we reintroduce the notation of 0-1 bitstrings, a bitstring $\mathbf{s} = \epsilon_1 \cdots \epsilon_{n-2}$ of length $(n-2)$ denotes the following case:

$$\epsilon_j = 0 : \text{ if } k_j \leq \frac{n-j-1}{n-j} k_{j-1}$$

$$\epsilon_j = 1 : \text{ if } k_j \geq \frac{n-j-1}{n-j} k_{j-1} + (n-j-1).$$

We now give a more refined estimate for E_n , no longer relying on Lemma 10.4, but instead using more involved estimates arising from Lemma 10.3. To this end we enumerate the iterated integrals defining E_n : let $\Lambda_{n-2} = \int_{u_1}^{u_2} |w-b|^{\sigma_{n-2}} dw$, $\Lambda_{n-3} = \int_{v_1}^{v_2} \int_{v_2}^{v_3} [|u_1-b||u_2-b|]^{\sigma_{n-3}} \Lambda_{n-2} du_1 du_2$ and more generally, for $3 \leq r \leq n$,

$$\Lambda_{n-r} = \int_{y_1}^{y_2} \cdots \int_{y_{r-1}}^{y_r} \prod_{s=1}^{r-1} |z_s - b|^{\sigma_{n-r}} \Lambda_{n-r+1} dz_1 \cdots dz_{r-1}$$

so that $E_n = \prod_{s=1}^n |x_s - b|^{k_0} \times \Lambda_0$.

We first estimate $\Lambda_{n-2}, \Lambda_{n-3}, \dots, \Lambda_0$ in succession using estimate $\downarrow (n-j-1) \downarrow$ if $\epsilon_j = 0$, or estimate $\uparrow (n-j-1) \uparrow$ if $\epsilon_j = 1$. Therefore inductively (see the special estimates preceding Lemma 10.4), if

$$\Lambda_{j+1} \gtrsim \int_{y_1}^{y_2} \cdots \int_{y_{n-j-2}}^{y_{n-j-1}} \prod_{r=1}^{n-j-2} |u_r - b|^{\rho_r^{j+1}} \prod_{r < s} |u_r - u_s| du_1 \cdots du_{n-j-2},$$

then $\Lambda_j \gtrsim$

$$\begin{aligned} & \int_{z_1}^{z_2} \cdots \int_{z_{n-j-1}}^{z_{n-j}} \prod_{r=1}^{n-j-1} |y_r - b|^{\sigma_j} \int_{y_1}^{y_2} \cdots \int_{y_{n-j-2}}^{y_{n-j-1}} \prod_{r=1}^{n-j-2} |u_r - b|^{\rho_r^{j+1}} \\ & \quad \prod_{r < s} |u_r - u_s| du_1 \cdots du_{n-j-2} dy_1 \cdots dy_{n-j-1} \\ & \gtrsim \int_{z_1}^{z_2} \cdots \int_{z_{n-j-1}}^{z_{n-j}} |y_1 - b|^{A_j} \left[\prod_{r=2}^{n-j-2} |y_r - b|^{B_{r,j}} \right] |y_{n-j-1} - b|^{C_j} \prod_{r < s} |y_r - y_s| dy_1 \cdots dy_{n-j-1} \end{aligned}$$

where

$$A_j = \sigma_j + (1 - \epsilon_{j+1})\rho_1^{j+1} - \epsilon_{j+1}(n-j-2)$$

$$B_{r,j} = \sigma_j + (1 - \epsilon_{j+1})\rho_r^{j+1} + \epsilon_{j+1}(\rho_{r-1}^{j+1} + 1) \quad \text{and} \quad C_j = \sigma_j + \epsilon_{j+1}(\rho_{n-j-2}^{j+1} + 1).$$

This gives us the following recursive relations for $0 \leq j \leq n-3$: $\rho_1^j = A_j$ and for $2 \leq r \leq n-j-2$, $\rho_r^j = B_{r,j}$, and finally $\rho_{n-j-1}^j = C_j$. Note that $\rho_1^{n-2} = \sigma_{n-2}$. Therefore we arrive at

$$\Lambda_0 \gtrsim \int_{x_1}^{x_2} \cdots \int_{x_{n-1}}^{x_n} \prod_{r=1}^{n-1} |y_r - b|^{\rho_r} \prod_{r < s} |y_r - y_s| dy_1 \cdots dy_{n-1}$$

where $\rho_r = \rho_r^0$.

For this last iterated integral we use the estimate $\downarrow (n-1-k_0) \downarrow \uparrow (k_0) \uparrow$ to obtain

$$\Lambda_0 \gtrsim \left[\prod_{r=1}^{n-k_0-1} |x_r - b|^{\rho_r} \right] \frac{1}{|x_{n-k_0} - b|^{k_0}} \left[\prod_{r=n-k_0+1}^n |x_r - b|^{\rho_{r-1}+1} \right] \prod_{r < s} |x_r - x_s|.$$

Finally we arrive at

$$E_n \gtrsim \frac{\prod_{r=1}^{n-k_0-1} |x_r - b|^{\rho_r+k_0}}{\prod_{r=n-k_0+1}^n |x_r - b|^{-k_0-\rho_{r-1}-1}} \prod_{r < s} |x_r - x_s|.$$

We will prove the following lemma.

Lemma 11.1.

1. For each $1 \leq L \leq n - k_0 - 1$,

$$\rho_1 + \cdots + \rho_L + Lk_0 \geq 0.$$

2. For each $1 \leq L \leq k_0$,

$$-\rho_{n-1} - \cdots - \rho_{n-L} - L(k_0 + 1) \geq 0.$$

Using Lemma 11.1 with our assumption $b > x_n$ (and so $|x_1 - b| \geq \dots \geq |x_n - b|$), we arrive at

$$E_n \gtrsim \frac{|x_{n-k_0} - b|^{\rho_1 + \dots + \rho_{n-k_0-1} + (n-k_0-1)k_0}}{|x_{n-k_0} - b|^{-\rho_{n-1} - \dots - \rho_{n-k_0} - k_0(k_0+1)}} \prod_{r < s} |x_r - x_s|.$$

Thus, finally, the desired estimate $E_n \gtrsim \prod_{r < s} |x_r - x_s|$ for Proposition 10.1 follows from the observation

$$\rho_1 + \dots + \rho_{n-1} + nk_0 = 0. \quad (33)$$

To see (33) we first note that for each $0 \leq j \leq n-3$,

$$\sum_{r=1}^{n-j-1} \rho_r^j = (n-j-1)\sigma_j + \sum_{r=1}^{n-j-2} \rho_r^{j+1}.$$

In fact, from the recursive formulae,

$$\begin{aligned} \sum_{r=1}^{n-j-1} \rho_r^j &= (n-j-1)\sigma_j + (1 - \epsilon_{j+1}) \sum_{r=1}^{n-j-2} \rho_r^{j+1} - \epsilon_{j+1}(n-j-2) \\ &\quad + \epsilon_{j+1} \sum_{r=2}^{n-j-1} (\rho_{r-1}^{j+1} + 1) \\ &= (n-j-1)\sigma_j + \sum_{r=1}^{n-j-2} \rho_r^{j+1} - \epsilon_{j+1} \sum_{r=1}^{n-j-2} \rho_r^{j+1} + \epsilon_{j+1} \sum_{r=1}^{n-j-2} (\rho_r^{j+1} + 1) \\ &\quad - \epsilon_{j+1}(n-j-2) \\ &= (n-j-1)\sigma_j + \sum_{r=1}^{n-j-2} \rho_r^{j+1}. \end{aligned}$$

Hence $\rho_1 + \dots + \rho_{n-1} + nk_0 =$

$$nk_0 + \sum_{j=0}^{n-3} (n-j-1)\sigma_j + \rho_1^{n-2} = nk_0 + \sum_{j=0}^{n-2} (n-j-1)\sigma_j$$

since $\rho_1^{n-2} = \sigma_{n-2}$. But $\sigma_0 = k_1 - 2k_0$, $\sigma_1 = k_2 + k_0 - 2k_1, \dots, \sigma_{n-2} = k_{n-3} - 2k_{n-2}$ implies

$$nk_0 + \sum_{j=0}^{n-2} (n-j-1)\sigma_j = \sum_{j=0}^{n-2} (n-j + n-j-2 - 2(n-j-1))k_j = 0.$$

It remains to prove Lemma 11.1 but (33) allows us to rewrite the two statements in that lemma into one. We use the notation $a_+ = \max\{a, 0\}$.

Lemma 11.2. (*Reformulation of Lemma 11.1*)

For any $1 \leq L \leq n-2$,

$$\rho_1 + \dots + \rho_L + Lk_0 \geq -(L - (n - k_0 - 1))_+.$$

In the final part of this section we reduce the proof of Lemma 11.2 to a couple of combinatorial lemmas. In order to do this we first express $\rho_1 + \dots + \rho_L + Lk_0$ in terms of our exponents $\{k_j\}$ whose restrictions (30), described by the particular case $\mathbf{s} = \epsilon_1 \cdots \epsilon_{n-2}$ under consideration, are given at the outset of this section.

We list the first L' , $0 \leq L' \leq L$ 1's in \mathbf{s} ($L' = L$ if the sequence $\epsilon_1, \epsilon_2, \dots, \epsilon_{n-2}$ has at least L 1's; otherwise L' denotes the total number of 1's in this sequence), $\epsilon_{j_1} = \dots = \epsilon_{j_{L'}} = 1, 1 \leq j_1 \leq \dots \leq j_{L'} \leq n-2$ if $L' \geq 1$. The case $\mathbf{s} = 0 \cdots 0$ corresponds to $L' = 0$. We now list some consequences of the recursive formulae defining the exponents $\{\rho_r^j\}$: for $1 \leq j \leq n-2$ and any $1 \leq S \leq n-j-1$,

1. $\rho_1^{j-1} + \dots + \rho_{n-j}^{j-1} = (n-j)\sigma_{j-1} + \rho_1^j + \dots + \rho_{n-j-1}^j$;
2. if $\epsilon_j = 0$,

$$\rho_1^{j-1} + \dots + \rho_S^{j-1} = S\sigma_{j-1} + \rho_1^j + \dots + \rho_S^j$$
;
3. if $\epsilon_j = 1$,

$$\rho_1^{j-1} + \dots + \rho_S^{j-1} = S\sigma_{j-1} + \rho_1^j + \dots + \rho_S^j - ((n-j-1) - S) - 1.$$

Here we are setting $\rho_{-1} = 0$. Recall that $\rho_1^{n-2} = \sigma_{n-2} = k_{n-3} - 2k_{n-2}$.

If it exists, let $\ell_* \leq L'$ denote the least ℓ_* so that $j_{\ell_*} \geq n-L+\ell_*$.

Case 1: Suppose ℓ_* exists. Then 1., 2. and 3. imply that

$$\begin{aligned} \rho_1 + \dots + \rho_L + Lk_0 &= L(k_0 + \sigma_0 + \dots + \sigma_{j_1-1}) - (n-j_1-1) + L-1 \\ &\quad + (L-1)(\sigma_{j_1} + \dots + \sigma_{j_2-1}) - (n-j_2-1) + L-2 \\ &\quad \vdots \\ &\quad + (L-\ell_*+2)(\sigma_{j_{\ell_*-2}} + \dots + \sigma_{j_{\ell_*-1}-1}) - (n-j_{\ell_*-1}-1) + L - (\ell_*-1) \\ &\quad + (L-\ell_*+1)(\sigma_{j_{\ell_*-1}} + \dots + \sigma_{n-L+\ell_*-2}) + (L-\ell_*)\sigma_{n-L+\ell_*-1} \\ &\quad + \dots + 2\sigma_{n-3} + \sigma_{n-2} \\ &= k_{j_1} - k_{j_1-1} - (n-j_1-1) + L-1 + k_{j_2} - k_{j_2-1} - (n-j_2-1) + L-2 \\ &\quad \vdots \\ &\quad + k_{j_{\ell_*-1}} - k_{j_{\ell_*-1}-1} - (n-j_{\ell_*-1}-1) + L - (\ell_*-1) - k_{n-L+\ell_*-2}. \end{aligned}$$

Hence

$$\begin{aligned} \rho_1 + \dots + \rho_L + Lk_0 &= \sum_{j=1}^{\ell_*-1} [k_{j_{\ell}} - k_{j_{\ell}-1} - (n-j_{\ell}-\ell)] \\ &\quad + (\ell_*-1)(L - (\ell_*-1)) - k_{n-L+\ell_*-2}. \end{aligned} \tag{34}$$

Case 2: Suppose ℓ_* does not exist. Arguing as above we arrive at

$$\begin{aligned} \rho_1 + \dots + \rho_L + Lk_0 &= L(k_0 + \dots + \sigma_{j_1-1}) - (n-j_1-1) + L-1 \\ &\quad \vdots \\ &\quad + (L-L'+1)(\sigma_{j_{L'-1}} + \dots + \sigma_{j_{L'}-1}) - (n-j_{L'}-1) + L-L' \\ &\quad + \rho_1^{j_{L'}} + \dots + \rho_{L-L'}^{j_{L'}}. \end{aligned}$$

Therefore if $L = L'$ (and then one checks that we are necessarily in Case 2), we have

$$\begin{aligned} \rho_1 + \dots + \rho_L + Lk_0 &= L(k_0 + \sigma_0 + \dots + \sigma_{j_1-1}) - (n-j_1-1) + L-1 \\ &\quad \vdots \\ &\quad + (\sigma_{j_{L-1}} + \dots + \sigma_{j_L-1}) - (n-j_L-1) \end{aligned}$$

$$= \sum_{\ell=1}^L [k_{j_\ell} - k_{j_{\ell-1}} - (n - j_\ell - \ell)]. \quad (35)$$

If $L' < L$, set $M = n - j_{L'} - 1 - (L - L') \geq 0$. Then $\rho_1^{j_{L'}} + \cdots + \rho_{L-L'}^{j_{L'}}$

$$\begin{aligned} &= (L - L')(\sigma_{j_{L'}} + \cdots + \sigma_{j_{L'+M}}) + \rho_1^{j_{L'+M+1}} + \cdots + \rho_{L-L'}^{j_{L'+M+1}} \\ &= (L - L')(\sigma_{j_{L'}} + \cdots + \sigma_{j_{L'+M}}) + (L - L' - 1)\sigma_{j_{L'+M+1}} + \cdots + 2\sigma_{n-3} + \sigma_{n-2}. \end{aligned}$$

Hence

$$\begin{aligned} \rho_1 + \cdots + \rho_L + Lk_0 &= \sum_{\ell=1}^{L'} [k_{j_\ell} - k_{j_{\ell-1}} - (n - j_\ell - \ell)] \\ &\quad + L'(L' - 1) - k_{n-(L-L')-1}. \end{aligned} \quad (36)$$

This last equation (36) includes the case $\mathbf{s} = 0 \cdots 0$ in which case we have

$$\rho_1 + \cdots + \rho_L + Lk_0 = -k_{n-L-1}.$$

Furthermore note that (36) is an ‘endpoint’ case of (34); formally taking $\ell_* = L' + 1$. In fact the arguments we develop for Lemma 11.2 in Case 1 work in exactly the same way as in Case 2 when $L' < L$. Therefore we will only prove Lemma 11.2 when $L' = L$, using (35), and in Case 1, using (34).

12. TWO COMBINATORIAL LEMMAS

In this final section we prove two combinatorial lemmas about our exponents $\{k_j\}$ whose restrictions (30) are given by $\mathbf{s} = \epsilon_1 \cdots \epsilon_{n-2}$ as explained at the outset of the previous section.

Key Combinatorial Lemma: *For any $1 \leq \mathcal{L} \leq L'$,*

$$\sum_{\ell=1}^{\mathcal{L}} [k_{j_\ell} - k_{j_{\ell-1}} - (n - j_\ell - \ell)] \geq -(\mathcal{L} - (n - k_0 - 1))_+.$$

As explained in the previous section, by (35), the Key Combinatorial Lemma gives a proof of Lemma 11.2 in the case $L' = L$. Furthermore we will use the Key Combinatorial Lemma to establish Lemma 11.2 in Case 1. of Section 11 (when ℓ_* exists), thus completing the proof of this lemma and hence the proof of our main Theorem 1.1. To do this, it suffices by (34) to prove a second combinatorial lemma.

Combinatorial Lemma - 2: *With $1 \leq \ell_* \leq L' \leq L$ as in Section 11, we have*

$$\begin{aligned} &\sum_{\ell=1}^{\ell_*-1} [k_{j_\ell} - k_{j_{\ell-1}} - (n - j_\ell - \ell)] + (\ell_* - 1)(L - (\ell_* - 1)) - k_{n-L+\ell_*-2} \\ &\geq -(L - (n - k_0 - 1))_+. \end{aligned}$$

The proof of the Key Combinatorial Lemma is based in part on the following lemma.

Lemma 12.1. *If $k_{j_{\ell-1}} \geq M(n - j_{\ell}) - N$, for some M, N where $0 \leq N \leq n - j_{\ell}$, then*

- 1) $k_{j_{\ell-1}} \geq M(n - j_{\ell-1} - 1) - N$;
- 2) $-k_{j_{\ell-1}} + k_{j_{\ell-1}} \geq M(j_{\ell} - j_{\ell-1} - 1)$.

Proof To prove 1) we prove by induction

$$k_{j_{\ell-r}} \geq M(n - j_{\ell} + r - 1) - N, \quad 1 \leq r \leq j_{\ell} - j_{\ell-1}.$$

The case $r = 1$ is our hypothesis and if the statement is true for $r \geq 2$, then by (30),

$$M - \frac{N}{n - j_{\ell} + r - 1} \leq \frac{k_{j_{\ell-r}}}{n - j_{\ell} + r - 1} \leq \frac{k_{j_{\ell-r-1}}}{n - j_{\ell} + r}$$

which implies

$$k_{j_{\ell-r-1}} \geq M(n - j_{\ell} + r) - N - \frac{N}{n - j_{\ell} + r - 1}.$$

But $N \leq n - j_{\ell}$ implies $N < n - j_{\ell} + r - 1$ and so

$$k_{j_{\ell-r-1}} \geq M(n - j_{\ell} + r) - N,$$

completing the proof of 1).

To prove 2) we prove by induction

$$k_{j_{\ell-1}+r} \leq k_{j_{\ell-1}} - rM, \quad 0 \leq r \leq j_{\ell} - j_{\ell-1} - 1. \quad (37)$$

There is nothing to prove when $r = 0$ so suppose the statement holds for $r - 1$; then

$$k_{j_{\ell-1}+r} \leq \frac{n - j_{\ell-1} - r - 1}{n - j_{\ell-1} - r} (k_{j_{\ell-1}} - (r - 1)M) < k_{j_{\ell-1}} - rM + 1.$$

The last inequality is true since it is equivalent to

$$\begin{aligned} (rM - 1)(n - j_{\ell-1} - r) - (r - 1)M(n - j_{\ell-1} - r - 1) &< k_{j_{\ell-1}} \\ \iff (M - 1)(n - j_{\ell-1} - r) + (r - 1)M &< k_{j_{\ell-1}} \\ \iff (M - 1)(n - j_{\ell-1}) + r - M &< k_{j_{\ell-1}} \\ \iff r \leq k_{j_{\ell-1}} - (M - 1)(n - j_{\ell-1}) + M - 1. \end{aligned}$$

But by part 1)

$$\begin{aligned} k_{j_{\ell-1}} - (M - 1)(n - j_{\ell-1} - 1) &\geq M(n - j_{\ell-1} - 1) - N - (M - 1)(n - j_{\ell-1} - 1) \\ &= n - j_{\ell-1} - 1 - N_{\ell} \geq j_{\ell} - j_{\ell-1} - 1 \geq r. \end{aligned}$$

■

For notational convenience we set

$$I_{\ell} = k_{j_{\ell}} - k_{j_{\ell-1}} - (n - j_{\ell} - \ell).$$

Next we give a corollary of Lemma 12.1.

Corollary 12.2. *For any $0 \leq \ell \leq \mathcal{L} - 1$, if*

$$k_{j_{\mathcal{L}-\ell+1}-1} \geq (M - \ell + 1)(n - j_{\mathcal{L}-\ell+1}) - N \quad (38)$$

with $0 \leq N \leq n - j_{\mathcal{L}-\ell+1}$, then we have the following implication:

$$\begin{aligned} I_{\mathcal{L}} + \cdots + I_{\mathcal{L}-\ell+1} \\ \geq (M - \ell + 1)(n - j_{\mathcal{L}-\ell+1}) - \ell(M - \mathcal{L} + 1) - k_{j_{\mathcal{L}-\ell+1}-1} \end{aligned}$$

implies $I_{\mathcal{L}} + \cdots + I_{\mathcal{L}-\ell}$

$$\geq (M - \ell)(n - j_{\mathcal{L}-\ell}) - (\ell + 1)(M - \mathcal{L} + 1) - k_{j_{\mathcal{L}-\ell}-1}.$$

Proof By Lemma 12.1, part 2) and (38),

$$\begin{aligned} & I_{\mathcal{L}} + \cdots + I_{\mathcal{L}-\ell+1} + I_{\mathcal{L}-\ell} \\ & \geq (M - \ell + 1)(n - j_{\mathcal{L}-\ell+1}) - \ell(M - \mathcal{L} + 1) - (n - j_{\mathcal{L}-\ell} - \mathcal{L} + \ell) \\ & \quad - k_{j_{\mathcal{L}-\ell+1}-1} + k_{j_{\mathcal{L}-\ell}} - k_{j_{\mathcal{L}-\ell}-1} \\ & \geq (M - \ell + 1)(n - j_{\mathcal{L}-\ell} - 1) - (n - j_{\mathcal{L}-\ell}) + \mathcal{L} - \ell - \ell(M - \mathcal{L} + 1) - k_{j_{\mathcal{L}-\ell}-1} \\ & = (M - \ell)(n - j_{\mathcal{L}-\ell}) - (M - \ell + 1) + \mathcal{L} - \ell - \ell(M - \mathcal{L} + 1) - k_{j_{\mathcal{L}-\ell}-1} \\ & = (M - \ell)(n - j_{\mathcal{L}-\ell}) - (M - \mathcal{L} + 1)(\ell + 1) - k_{j_{\mathcal{L}-\ell}-1}. \end{aligned}$$

■

We now turn to the proof of the Key Combinatorial Lemma when $1 \leq \mathcal{L} \leq n - k_0 - 1$ and do so by induction on \mathcal{L} .

$$\mathcal{L} = 1: \quad k_{j_1} - k_{j_1-1} - (n - j_1 - 1) \geq -\frac{k_{j_1-1}}{n-j_1} > -1.$$

The first inequality follows from (30) and the last inequality is equivalent to $k_{j_1-1} \leq n - j_1 - 1$ but again by (30),

$$k_{j_1-1} \leq k_0 - j_1 + 1 \leq n - 2 - j_1 + 1 = n - j_1 - 1,$$

completing the case $\mathcal{L} = 1$.

Suppose $\sum_{\ell=1}^{\overline{\mathcal{L}}} I_{\ell} \geq 0$ for every $1 \leq \overline{\mathcal{L}} \leq \mathcal{L} - 1$.

Case A: $k_{j_{\mathcal{L}-1}} \leq \mathcal{L}(n - j_{\mathcal{L}}) - 1$. Then by (30),

$$k_{j_{\mathcal{L}}} - k_{j_{\mathcal{L}-1}} - (n - j_{\mathcal{L}} - 1) \geq -\frac{k_{j_{\mathcal{L}-1}}}{n-j_{\mathcal{L}}} > -\mathcal{L}$$

and this implies that $I_{\mathcal{L}} \geq 0$ and so $\sum_{\ell=1}^{\mathcal{L}} I_{\ell} \geq 0$ by the inductive hypothesis.

Case B: $k_{j_{\mathcal{L}-1}} \geq \mathcal{L}(n - j_{\mathcal{L}})$. Let $1 \leq \ell_0 \leq \mathcal{L} - 1$ be the least integer value of $1 \leq \ell \leq \mathcal{L} - 1$ so that

$$k_{j_{\mathcal{L}-\ell}-1} \leq (\mathcal{L} - \ell)(n - j_{\mathcal{L}-\ell}) - \ell - 1.$$

We observe that such a value of ℓ exists since

$$k_{j_{\mathcal{L}-(\mathcal{L}-1)}-1} = k_{j_1-1} \leq k_0 - j_1 - 1 \leq (\mathcal{L} - (\mathcal{L} - 1))(n - j_{\mathcal{L}-(\mathcal{L}-1)}) - 1 - (\mathcal{L} - 1)$$

since $\mathcal{L} \leq n - k_0 - 1$!

Claim: For $0 \leq \ell \leq \ell_0$,

$$I_{\mathcal{L}} + \cdots + I_{\mathcal{L}-\ell} \geq (\mathcal{L} - \ell)(n - j_{\mathcal{L}-\ell}) - (\ell + 1) - k_{j_{\mathcal{L}-\ell}-1}.$$

We prove this by induction on ℓ .

$$\ell = 0: \quad k_{j_{\mathcal{L}-1}} \geq \mathcal{L}(n - j_{\mathcal{L}}) \implies k_{j_{\mathcal{L}}} \geq (\mathcal{L} + 1)(n - j_{\mathcal{L}} - 1) \implies$$

$$I_{\mathcal{L}} = k_{j_{\mathcal{L}}} - k_{j_{\mathcal{L}-1}} - (n - j_{\mathcal{L}} - \mathcal{L}) \geq \mathcal{L}(n - j_{\mathcal{L}}) - 1 - k_{j_{\mathcal{L}-1}}.$$

The first implication follows from (30).

The inductive step will follow from Corollary 12.2. First we note that for $1 \leq \ell \leq \ell_0$,

$$k_{j_{\mathcal{L}-\ell+1}-1} \geq (\mathcal{L} - \ell + 1)(n - j_{\mathcal{L}-\ell+1}) - \ell + 1.$$

Hence Corollary 12.2 with $M = \mathcal{L}$ and $N = \ell - 1$ (note that $j_{\mathcal{L}-\ell+1} \leq n - 2 - (\ell - 1) = n - \ell - 1$ implies $\ell - 1 = N \leq n - j_{\mathcal{L}-\ell+1}$) proves the inductive step of the Claim.

Using the Claim with $\ell = \ell_0$ gives

$$I_{\mathcal{L}} + \cdots + I_{\mathcal{L}-\ell_0} \geq (\mathcal{L} - \ell_0)(n - j_{\mathcal{L}-\ell_0}) - (\ell_0 + 1) - k_{j_{\mathcal{L}-\ell_0}-1} \geq 0$$

since $k_{j_{\mathcal{L}-\ell_0}-1} \leq (\mathcal{L} - \ell_0)(n - j_{\mathcal{L}-\ell_0}) - (\ell_0 + 1)$. Now the induction hypothesis establishes the Key Combinatorial Lemma when $1 \leq \mathcal{L} \leq n - k_0 - 1$.

Next suppose $\mathcal{L} \geq n - k_0 - 1$ and we will prove the Key Combinatorial Lemma by induction on \mathcal{L} ; the case $\mathcal{L} = n - k_0 - 1$ being done above.

Suppose $\sum_{\ell=1}^{\overline{\mathcal{L}}} I_{\ell} \geq -(\overline{\mathcal{L}} - (n - k_0 - 1))$ for every $n - k_0 - 1 \leq \overline{\mathcal{L}} \leq \mathcal{L} - 1$. If $k_{j_{\mathcal{L}-1}} \leq (\mathcal{L} + 1)(n - j_{\mathcal{L}}) - 1$, then

$$k_{j_{\mathcal{L}}} - k_{j_{\mathcal{L}-1}} - (n - j_{\mathcal{L}} - \mathcal{L}) \geq -\frac{k_{j_{\mathcal{L}-1}}}{n - j_{\mathcal{L}}} + \mathcal{L} - 1 > -2. \quad (39)$$

The last inequality being equivalent to $k_{j_{\mathcal{L}-1}}/(n - j_{\mathcal{L}}) < \mathcal{L} + 1$ which in turn is equivalent to $k_{j_{\mathcal{L}-1}} \leq (\mathcal{L} + 1)(n - j_{\mathcal{L}}) - 1$. By induction we have

$$\sum_{\ell=1}^{\mathcal{L}-1} I_{\ell} \geq -(\mathcal{L} - 1 - (n - k_0 - 1))$$

which together with (39) gives $\sum_{\ell=1}^{\mathcal{L}} I_{\ell} \geq -(\mathcal{L} - (n - k_0 - 1))$ as desired. Hence we may assume

$$\boxed{k_{j_{\mathcal{L}-1}} \geq (\mathcal{L} + 1)(n - j_{\mathcal{L}})}.$$

Case A': $k_{j_{\mathcal{L}-\ell}-1} \leq (\mathcal{L} - \ell + 1)(n - j_{\mathcal{L}-\ell}) - \ell - 1$ for some $0 \leq \ell \leq \mathcal{L} - (n - k_0 - 1)$. Let ℓ_0 denote the least such value of ℓ .

Claim: For $0 \leq \ell \leq \ell_0$,

$$I_{\mathcal{L}} + \cdots + I_{\mathcal{L}-\ell} \geq (\mathcal{L} - \ell + 1)(n - j_{\mathcal{L}-\ell}) - 2(\ell + 1) - k_{j_{\mathcal{L}-\ell}-1}.$$

We prove this by induction on ℓ .

$$\ell = 0: k_{j_{\mathcal{L}-1}} \geq (\mathcal{L} + 1)(n - j_{\mathcal{L}}) \implies k_{j_{\mathcal{L}}} \geq (\mathcal{L} + 2)(n - j_{\mathcal{L}} - 1) \implies$$

$$I_{\mathcal{L}} = k_{j_{\mathcal{L}}} - k_{j_{\mathcal{L}-1}} - (n - j_{\mathcal{L}} - \mathcal{L}) \geq (\mathcal{L} + 1)(n - j_{\mathcal{L}}) - 2 - k_{j_{\mathcal{L}-1}}.$$

The first implication follows from (30).

The inductive step will follow from Corollary 12.2. First we note that for $1 \leq \ell \leq \ell_0$,

$$k_{j_{\mathcal{L}-\ell+1}-1} \geq (\mathcal{L} - \ell + 2)(n - j_{\mathcal{L}-\ell+1}) - \ell + 1.$$

Then Corollary 12.2 with $M = \mathcal{L} + 1$ and $N = \ell - 1$ (note that $\ell - 1 \leq n - j_{\mathcal{L}-\ell+1} \Leftrightarrow j_{\mathcal{L}-\ell+1} \leq n - \ell + 1$ and this follows from (30) and the fact $j_{\mathcal{L}} \leq n - 2$) proves the inductive step of the Claim.

Applying the Claim with $\ell = \ell_0$ gives

$$\sum_{\ell=0}^{\ell_0} I_{\mathcal{L}-\ell} \geq (\mathcal{L} - \ell_0 + 1)(n - j_{\mathcal{L}-\ell_0}) - 2(\ell_0 + 1) - k_{j_{\mathcal{L}-\ell_0}-1}.$$

Using $k_{j_{\mathcal{L}-\ell_0}-1} \leq (\mathcal{L} - \ell_0 + 1)(n - j_{\mathcal{L}-\ell_0}) - \ell_0 - 1$ and the inductive hypothesis gives

$$\begin{aligned} & I_{\mathcal{L}} + \cdots + I_{\mathcal{L}-\ell_0} + \sum_{\ell=1}^{\mathcal{L}-\ell_0-1} I_{\ell} \\ & \geq (\mathcal{L} - \ell_0 + 1)(n - j_{\mathcal{L}-\ell_0}) - 2(\ell_0 + 1) - (\mathcal{L} - \ell_0 - 1 - (n - k_0 - 1)) - k_{j_{\mathcal{L}-\ell_0}-1} \\ & \geq -(\mathcal{L} - (n - k_0 - 1)) + (\mathcal{L} - \ell_0 + 1)(n - j_{\mathcal{L}-\ell_0}) - (\ell_0 + 1) - k_{j_{\mathcal{L}-\ell_0}-1} \\ & \geq -(\mathcal{L} - (n - k_0 - 1)), \text{ finishing Case A'}. \end{aligned}$$

Case B': $k_{j_{\ell}-1} \geq (\ell + 1)(n - j_{\ell}) - (\mathcal{L} - \ell)$, for all $n - k_0 - 1 \leq \ell \leq \mathcal{L}$.

Claim 1: $k_{j_{n-k_0-1}-1} \geq (n - k_0 - 1)(n - j_{n-k_0-1}) - 1$.

In fact, from above with $\ell = n - k_0 - 1$, it suffices to see

$$(n - k_0)(n - j_{n-k_0-1}) - \mathcal{L} + n - k_0 - 1 \geq (n - k_0 - 1)(n - j_{n-k_0-1}) - 1.$$

This is equivalent to $n - j_{n-k_0-1} \geq \mathcal{L} - (n - k_0 - 1) - 1$ which in turn is equivalent to $j_{n-k_0-1} \leq n + (n - k_0 - 1) - \mathcal{L} + 1$. But by (30), since $j_{\mathcal{L}} \leq n - 2$, $j_{n-k_0-1} = j_{\mathcal{L} - (\mathcal{L} - (n - k_0 - 1))}$

$$\leq n - 2 - (\mathcal{L} - (n - k_0 - 1)) = n + n - k_0 - 1 - \mathcal{L} - 2. \quad (40)$$

Next, let ℓ_0 denote the least value of ℓ , $2 \leq \ell \leq n - k_0 - 1$ so that

$$k_{j_{n-k_0-\ell}-1} \leq (n - k_0 - \ell)(n - j_{n-k_0-\ell}) - \ell.$$

Observe that such a value exists since for $\ell = n - k_0 - 1$, (30) implies $k_{j_1-1} \leq k_0 - j_1 + 1$ and $k_0 - j_1 + 1 = n - j_1 - (n - k_0 - 1)$.

Claim 2: For $1 \leq \ell \leq \ell_0$,

$$I_{\mathcal{L}} + \cdots + I_{n-k_0-\ell} \geq (n - k_0 - \ell)(n - j_{n-k_0-\ell}) - [\mathcal{L} - (n - k_0 - \ell) + 1] - k_{j_{n-k_0-\ell}-1}.$$

We prove this by induction on ℓ :

$\ell = 1$: In this case we use the Claim in Case A' with $\ell = \mathcal{L} - (n - k_0 - 1)$ to deduce

$$I_{\mathcal{L}} + \cdots + I_{n-k_0-1} \geq (n - k_0)(n - j_{n-k_0-1}) - 2(\mathcal{L} - (n - k_0 - 1) + 1) - k_{j_{n-k_0-1}-1}.$$

But (40) implies $j_{n-k_0-1} \leq n + (n - k_0 - 1) - \mathcal{L} - 1$ which is equivalent to

$$\begin{aligned} & (n - k_0)(n - j_{n-k_0-1}) - 2(\mathcal{L} - (n - k_0 - 1) + 1) \\ & \geq (n - k_0 - 1)(n - j_{n-k_0-1}) - [\mathcal{L} - (n - k_0 - 1) + 1], \end{aligned}$$

completing the case $\ell = 1$.

We will once again employ Corollary 12.2. But first note, by the definition of ℓ_0 ,

$$k_{j_{n-k_0-\ell+1}-1} \geq (n - k_0 - \ell + 1)(n - j_{n-k_0-\ell+1}) - (\ell - 2)$$

for all $1 \leq \ell \leq \ell_0$. The inductive step of Claim 2 now follows from Corollary 12.2 with $M = \mathcal{L}$ and $N = \ell - 2$ (check $\ell - 2 \leq n - j_{n-k_0-\ell+1}$). This finishes the proof of Claim 2.

We now use Claim 2 with $\ell = \ell_0$: $I_{\mathcal{L}} + \cdots + I_{n-k_0-\ell_0}$

$$\begin{aligned} & \geq (n - k_0 - \ell_0)(n - j_{n-k_0-\ell_0}) - [\mathcal{L} - (n - k_0 - \ell_0) + 1] - k_{j_{n-k_0-\ell_0}-1} \\ & \geq \ell_0 - [\mathcal{L} - (n - k_0 - 1) + \ell_0] = -[\mathcal{L} - (n - k_0 - 1)], \end{aligned}$$

and together with the inductive hypothesis $\sum_{\ell=1}^{n-k_0-\ell_0-1} I_\ell \geq 0$ gives

$$\sum_{\ell=1}^{\mathcal{L}} I_\ell \geq -[\mathcal{L} - (n - k_0 - 1)]$$

for Case B' as well. This completes the proof of the Key Combinatorial Lemma.

Finally we turn to the proof of Combinatorial Lemma - 2. Let us recall the set-up for this lemma: there exists an ℓ_* , $1 \leq \ell_* \leq L'$ which is the smallest value of ℓ with the property $j_\ell \geq n - L + \ell$. We wish to prove

$$\sum_{\ell=1}^{\ell_*-1} I_\ell + (\ell_* - 1)(L - (\ell_* - 1)) - k_{n-L+\ell_*-2} \geq -(L - (n - k_0 - 1))_+.$$

The case $\ell_* = 1$ is interpreted as $-k_{n-L-1} \geq -(L - (n - k_0 - 1))_+$ and this is easily seen to be the case. In fact, by (30), since $n - L + 1 \leq j_1$, $k_{n-L-1} \leq k_0 - (n - L - 1) = L - (n - k_0 - 1)$ from which $-k_{n-L-1} \geq -(L - (n - k_0 - 1))_+$ easily follows. Therefore, from now on, we may assume that $\ell_* \geq 2$.

It will be convenient to shift notation slightly; set $\underline{\ell} = \ell_* - 1$, $T_\ell = k_{j_\ell} - k_{j_{\ell-1}} - (n - j_\ell)$ and $S = -k_{n-L+\underline{\ell}-1} + \underline{\ell}L - \underline{\ell}(\underline{\ell} - 1)/2 + \sum_{\ell=1}^{\underline{\ell}} T_\ell$. Therefore $1 \leq \underline{\ell} \leq L' - 1$,

$$j_{\underline{\ell}} \leq n - L + \underline{\ell} - 1 \leq j_{\underline{\ell}+1} - 2 \leq n - 4 \quad (41)$$

(and so $\boxed{L \geq \underline{\ell} + 3}$), and the inequality we wish to prove becomes

$$S \geq -(L - (n - k_0 - 1))_+. \quad (42)$$

Case 1: $k_{j_{\underline{\ell}}} \leq (\underline{\ell} + 1)(n - j_{\underline{\ell}}) - L - 1$. We split this case into $\underline{\ell}$ subcases, $1 \leq s \leq \underline{\ell}$,

$$(s) \quad s(n - j_{\underline{\ell}} - 1) - L + \underline{\ell} < k_{j_{\underline{\ell}}} \leq (s + 1)(n - j_{\underline{\ell}} - 1) - L + \underline{\ell}.$$

These subcases do indeed divide up Case 1 because $k_{j_{\underline{\ell}}} > n - j_{\underline{\ell}} - L + \underline{\ell} - 1$. In fact since $\boxed{L \geq \underline{\ell} + 3}$,

$$k_{j_{\underline{\ell}}} \geq \frac{n - j_{\underline{\ell}} - 1}{n - j_{\underline{\ell}}} k_{j_{\underline{\ell}-1}} + (n - j_{\underline{\ell}} - 1) \geq n - j_{\underline{\ell}} - 1 > n - j_{\underline{\ell}} - L + \underline{\ell} - 1.$$

The first inequality following from (30). In exactly the same way that (37) was established in the proof of Lemma 12.1 part 2) we have (for $r, r_0 \geq 0$)

$$j_{\underline{\ell}} + r_0 + r \leq n - L + \underline{\ell} - 1 \implies k_{j_{\underline{\ell}}+r_0+r} \leq k_{j_{\underline{\ell}}+r_0} - r s. \quad (43)$$

The restriction on $k_{j_{\underline{\ell}}}$ from below given in (s) serving the same role as the hypothesis in Lemma 12.1.

Each subcase (s), $1 \leq s \leq \underline{\ell}$ splits naturally into two further subcases:

$$\begin{aligned} (s)_1 & \quad s(n - j_{\underline{\ell}} - 1) - L + \underline{\ell} < k_{j_{\underline{\ell}}} \leq s(n - j_{\underline{\ell}}) - s; \\ (s)_2 & \quad s(n - j_{\underline{\ell}}) - s < k_{j_{\underline{\ell}}} \leq (s + 1)(n - j_{\underline{\ell}} - 1) - L + \underline{\ell}. \end{aligned}$$

Each range is nonempty (except if $j_{\underline{\ell}} = n - L + \underline{\ell} - 1$ in (s)₂):

- $s(n - j_{\underline{\ell}}) - s - [s(n - j_{\underline{\ell}} - 1) - L + \underline{\ell}] = L - \underline{\ell} \geq 3$;
- $(s + 1)(n - j_{\underline{\ell}} - 1) - L + \underline{\ell} - [s(n - j_{\underline{\ell}}) - s] = n - j_{\underline{\ell}} - L + \underline{\ell} - 1 \geq 0$.

The last inequality follows from (41).

We now address case $(s)_1$. Using (43) with $r_0 = 0$ and $r = n - L + \underline{\ell} - 1 - j_{\underline{\ell}}$,

$$\begin{aligned} S &\geq s(n - L + \underline{\ell} - 1 - j_{\underline{\ell}}) - k_{j_{\underline{\ell}}-1} - (n - j_{\underline{\ell}}) + \sum_{\ell=1}^{\underline{\ell}-1} T_{\ell} + \underline{\ell}L - \frac{\underline{\ell}(\underline{\ell}-1)}{2} \\ &= (s-1)(n - j_{\underline{\ell}}) + (\underline{\ell} - s)L + s(\underline{\ell} - 1) - \frac{\underline{\ell}(\underline{\ell}-1)}{2} - k_{j_{\underline{\ell}}-1} + \sum_{\ell=1}^{\underline{\ell}-1} T_{\ell}. \end{aligned}$$

We claim that $k_{j_{\underline{\ell}}-1} \leq (s-1)(n - j_{\underline{\ell}})$. Indeed, if $k_{j_{\underline{\ell}}-1} > (s-1)(n - j_{\underline{\ell}})$ then by (30),

$$k_{j_{\underline{\ell}}} \geq \frac{n - j_{\underline{\ell}} - 1}{n - j_{\underline{\ell}}} k_{j_{\underline{\ell}}-1} + (n - j_{\underline{\ell}} - 1) > s(n - j_{\underline{\ell}} - 1).$$

But the restrictions on $k_{j_{\underline{\ell}}}$ given in $(s)_1$ tell us that $k_{j_{\underline{\ell}}} \leq s(n - j_{\underline{\ell}} - 1)$. Hence $k_{j_{\underline{\ell}}-1} \leq (s-1)(n - j_{\underline{\ell}})$ and so

$$S \geq (\underline{\ell} - s)L - (\underline{\ell} - 1)(\underline{\ell}/2 - s) + \sum_{\ell=1}^{\underline{\ell}-1} T_{\ell}.$$

But $\boxed{L \geq \underline{\ell} + 3}$ implies $(\underline{\ell} - s)L \geq (\underline{\ell} - 1)(\underline{\ell} - s)$ which in turn is equivalent to $(\underline{\ell} - s)L - (\underline{\ell} - 1)(\underline{\ell}/2 - s) \geq \underline{\ell}(\underline{\ell} - 1)/2$ and so

$$S \geq \frac{\underline{\ell}(\underline{\ell} - 1)}{2} + \sum_{\ell=1}^{\underline{\ell}-1} T_{\ell} = \sum_{\ell=1}^{\underline{\ell}-1} I_{\ell}.$$

Finally an application of the Key Combinatorial Lemma proves (42) for the case $(s)_1$.

Consider now case $(s)_2$. Again, in the same way that (37) or (43) was proved, we have

$$r \leq k_{j_{\underline{\ell}}} - s(n - j_{\underline{\ell}} - 1) \implies k_{j_{\underline{\ell}}+r} \leq k_{j_{\underline{\ell}}} - r(s+1). \quad (44)$$

Therefore to estimate S we split

$$-k_{n-L+\underline{\ell}-1} + k_{j_{\underline{\ell}}} = -k_{j_{\underline{\ell}}+r_0+r} + k_{j_{\underline{\ell}}+r_0} - k_{j_{\underline{\ell}}+r_0} + k_{j_{\underline{\ell}}}$$

where $r_0 = k_{j_{\underline{\ell}}} - s(n - j_{\underline{\ell}} - 1)$ and $r = n - L + \underline{\ell} - 1 - j_{\underline{\ell}} - r_0$, and then apply (43) to the first difference and (44) to the second difference to see that S is at least

$$\begin{aligned} &s(n - L + \underline{\ell} - 1 - j_{\underline{\ell}} - r_0) + (s+1)r_0 - k_{j_{\underline{\ell}}-1} - (n - j_{\underline{\ell}}) + \sum_{\ell=1}^{\underline{\ell}-1} T_{\ell} + \underline{\ell}L - \frac{\underline{\ell}(\underline{\ell}-1)}{2} \\ &= r_0 - k_{j_{\underline{\ell}}-1} + (s-1)(n - j_{\underline{\ell}}) + \sum_{\ell=1}^{\underline{\ell}-1} T_{\ell} + \underline{\ell}L - \frac{\underline{\ell}(\underline{\ell}-1)}{2} + s(\underline{\ell} - 1 - L) \\ &= k_{j_{\underline{\ell}}} - k_{j_{\underline{\ell}}-1} - (n - j_{\underline{\ell}}) + \sum_{\ell=1}^{\underline{\ell}-1} T_{\ell} + (\underline{\ell} - s)L + s\underline{\ell} - \frac{\underline{\ell}(\underline{\ell}-1)}{2}. \end{aligned}$$

But $\boxed{L \geq \underline{\ell} + 3}$ implies

$$(\underline{\ell} - s)L + s\underline{\ell} - \frac{\underline{\ell}(\underline{\ell}-1)}{2} \geq \frac{\underline{\ell}(\underline{\ell}+1)}{2}$$

and so

$$S \geq \frac{\underline{\ell}(\underline{\ell}+1)}{2} + \sum_{\ell=1}^{\underline{\ell}} T_{\ell} = \sum_{\ell=1}^{\underline{\ell}} I_{\ell}.$$

Another application of the Key Combinatorial Lemma now gives (42) for $(s)_2$ and this finishes Case 1.

Case 2: $k_{j_{\underline{\ell}}} \geq (\underline{\ell}+1)(n-j_{\underline{\ell}}) - L$.

For $1 \leq s \leq \underline{\ell}$, set

$$\mathcal{S}_s = -L + \frac{s(s+1)}{2} - 1 + s(n-j_s) - k_{j_{s-1}} + \sum_{\ell=1}^{s-1} T_{\ell}$$

and we shall prove that $\mathcal{S}_s \geq -(L - (n - k_0 - 1))_+$ by induction on s .

$$s = 1 : \mathcal{S}_1 = -L + 1 - 1 + (n - j_1) - k_{j_1-1} = n - j_1 - L - k_{j_1-1} \geq -(L - (n - k_0 - 1))_+.$$

To see the last inequality, first note that (30) implies $k_{j_1-1} \leq k_0 - j_1 + 1$ and so

$$n - j_1 - L - k_{j_1-1} \geq -(L - (n - k_0 - 1)) \geq -(L - (n - k_0 - 1))_+.$$

Suppose now $\mathcal{S}_{s-1} \geq -(L - (n - k_0 - 1))_+$.

Case i): $k_{j_{s-1}} \leq (s-1)(n-j_s)$.

This inequality in fact implies $k_{j_{s-1}} \leq s(n-j_s) - L + s - 1$. To see this note that $j_s \leq n - L + s - 1$ which follows from the definition of $\underline{\ell} = \ell_* - 1$ and the fact that $s \leq \underline{\ell}$. But this is equivalent to $(s-1)(n-j_s) \leq s(n-j_s) - L + s - 1$. Hence

$$\begin{aligned} \mathcal{S}_s &\geq -L + \frac{s(s+1)}{2} - 1 + s(n-j_s) - [s(n-j_s) - L + s - 1] + \sum_{\ell=1}^{s-1} T_{\ell} \\ &= \frac{s(s+1)}{2} - s + \sum_{\ell=1}^{s-1} T_{\ell} = \sum_{\ell=1}^{s-1} I_{\ell} \geq -(L - (n - k_0 - 1))_+, \end{aligned}$$

the last inequality following from the Key Combinatorial Lemma.

Case ii): $k_{j_{s-1}} \geq (s-1)(n-j_s) + 1$.

By Lemma 12.1 part 2), this inequality implies $-k_{j_{s-1}} + k_{j_{s-1}} \geq s(j_s - j_{s-1} - 1)$ and so

$$\begin{aligned} \mathcal{S}_s &\geq -L + \frac{s(s+1)}{2} - 1 + s(n-j_s) + s(j_s - j_{s-1} - 1) - (n - j_{s-1}) \\ &\quad - k_{j_{s-1}-1} + \sum_{\ell=1}^{s-2} T_{\ell} \\ &= -L + \frac{s(s-1)}{2} - 1 + (s-1)(n-j_{s-1}) - k_{j_{s-1}-1} + \sum_{\ell=1}^{s-2} T_{\ell} = \mathcal{S}_{s-1}; \end{aligned}$$

the inductive hypothesis $\mathcal{S}_{s-1} \geq -(L - (n - k_0 - 1))_+$ gives us the desired estimate in Case ii).

We will use the bound for \mathcal{S}_s with $s = \underline{\ell}$. First though, arguing as in (43), but now using the lower bound $k_{j_{\underline{\ell}}} \geq (\underline{\ell}+1)(n-j_{\underline{\ell}}) - L$ assumed in Case 2, we have

$$k_{j_{\underline{\ell}}+r} \leq k_{j_{\underline{\ell}}} - (\underline{\ell}+1)r, \quad r \leq n - L + \underline{\ell} - 1 - j_{\underline{\ell}}.$$

Hence

$$\begin{aligned}
S &\geq (\underline{\ell} + 1)(n - L + \underline{\ell} - 1 - j_{\underline{\ell}}) - k_{j_{\underline{\ell}}-1} - (n - j_{\underline{\ell}}) + \sum_{\ell=1}^{\underline{\ell}-1} T_{\ell} + \underline{\ell}L - \frac{\underline{\ell}(\underline{\ell} - 1)}{2} \\
&= -L + \frac{\underline{\ell}(\underline{\ell} + 1)}{2} - 1 + \underline{\ell}(n - j_{\underline{\ell}}) - k_{j_{\underline{\ell}}-1} + \sum_{\ell=1}^{\underline{\ell}-1} T_{\ell} \\
&= \mathcal{S}_{\underline{\ell}} \geq -(L - (n - k_0 - 1))_+.
\end{aligned}$$

This proves (42) in Case 2 and thus completes the proof the Combinatorial Lemma - 2.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRISTOL, UNIVERSITY WALK, BRISTOL BS8
1TW, UNITED KINGDOM

E-mail address: S.Dendrinos@bristol.ac.uk

SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, JCMB, KING'S BUILDINGS, MAYFIELD
ROAD, EDINBURGH EH9 3JZ, UNITED KINGDOM

E-mail address: J.R.Wright@ed.ac.uk