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Applications of Logic Constrained Equilibria to Traffic Networks and to Power Systems with Storage*

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Abstract

We study equilibria in traffic networks and in power system networks with storage in the presence of logic constraints. These constraints consist of binary variables that are added to complementarity-based equilibrium models. Although these models have been thoroughly studied, the addition of logic constraints can provide additional benefits for practical applications. The main contribution of this work is to demonstrate, using two specific examples of applications, that logic constraints can render classical equilibrium models more realistic by allowing the inclusion of useful features such as equity in network flows or threshold events. Specifically, for the traffic equilibrium problem, we show how logic constraints can introduce some equity in the assignment of traffic when more than one equilibrium exists. For power system networks, we show that the presence of a storage operator acting as a service provider will not only support the operation of a power grid, but will also help stabilize the price of electricity and avoid the well-documented price-shifting effect. Unlike previous works, our model considers the storage operator as a service provider rather than a competitor to the producers. We also consider the minimum power output of production. We present results illustrating the expanded capabilities and insights provided by these new paradigms.

Keywords: Mixed linear complementarity problems, mixed integer linear optimization, traffic equilibrium, energy storage, power markets.

1 Introduction

Complementarity problems have become an important tool for modeling equilibria. In energy markets for example, they can be used to compute the production level of each Nash-Cournot producer and the price of energy in order to satisfy demand. Complementarity problems have also been used to model the flow pattern in a traffic network given a set of travel demands between the origin-destination pairs. Other applications of complementarity problems can be found in the literature of game theory and market equilibrium in economics, see for example (Gabriel et al. 2012).

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The objective of this paper is to demonstrate that logic constraints can render classical equilibrium models more realistic by allowing the inclusion of useful features such as equity in network flows or threshold events. We do this by considering two specific applications of equilibria. For the traffic equilibrium problem, we show that logic constraints can be used to introduce some equity in the assignment of traffic when more than one equilibrium exists. Logic constraints also make it possible to model threshold effects jointly with the equilibrium conditions. As a second application, we show that in a power system network, the presence of a storage operator in the market operating as a service provider will not only support the generation, transmission, and distribution functions of a power grid, but will also help stabilize the price of electricity and avoid the price-shifting effect that is well documented in the literature (Sioshansi et al. 2009).

One of the equilibrium problems studied in this paper is the traffic equilibrium problem (TEP). This is a classical problem in complementarity modeling which consists of predicting steady state flows of vehicles along a congested road. Considering a set of origins, a set of destinations and intermediate nodes, which can relate to intersection points that form a road network, the objective is to predict how many vehicles will be using the individual paths in the network if the 'cost' of each particular path is taken into account in the decision process of each vehicle. It is generally agreed that drivers consider a number of criteria (such as time, money, distance, safety, route complexity, etc.) when selecting paths for their journeys. Presumably, these criteria are then combined in some manner to define a cost (more generally a disutility) for each particular path. Wardrop (Wardrop 1952) stated an equilibrium where no driver had an incentive to deviate from a particular chosen path resulting in the following two conditions: (a) All paths serving the same origin-destination pair with positive flow must have the same costs, (b) paths with costs higher than the minimum must have zero flow. These equilibrium conditions can be formulated as a complementarity problem (see (Wardrop 1952, Aashtiani and Magnanti 1981)).

In the first part of our study, we consider using logic (binary) constraints whose aim is to more equitably distribute equilibrium flows where more than one TEP solution exists. This is one of several applications for a transportation grid operator to better manage the network. Additional logic constraints for traffic equilibria are also possible. For example, threshold effects can also be modeled for traffic equilibrium whereby if the traffic level rises beyond a certain point an additional cost could be incurred.

In the context of energy markets, complementarity problems are often obtained by combining the Karush-Kuhn-Tucker (KKT) conditions of the optimization problems solved by players in this market together with market-clearing constraints (Gabriel et al. 2012). In the presence of binary variables which may represent the ON/OFF statuses of power generators, it is not obvious how one can achieve an equilibrium as the KKT conditions are not valid. However, (Gabriel, Conejo, Ruiz and Siddiqui 2013, Gabriel, Siddiqui, Conejo and Ruiz 2013) suggest that one can attempt to find an equilibrium solution by first relaxing the binary-constrained variables to their continuous analogues, taking the KKT conditions for this relaxed problem, converting these conditions to disjunctive-constraints form (Fortuny-Amat and McCarl 1981), and then solving them along with the original integer (binary) restrictions. The solutions obtained in this way are not guaranteed to be optimal for individual players' optimization problems, but rather are the binary-constrained solutions to the overall complementarity system being modeled. As such, we could consider them as stationary points for each of the players. The approach has since then yielded several studies of the discretely-constrained mixed linear complementarity problem (DC-MLCP), see for example (Ruiz et al. 2012, Fomeni et al. 2015).

In the second part of this paper, we consider the equilibrium problem of power markets with the inclusion of a storage operator. The model considers a multi-period power supply-demand balance with on-peak and off-peak demand periods. It then computes the optimal schedules for the production level of each producer and the storage level, as well as the prices of electricity and of storing the power for each time period, while ensuring that the demand is always satisfied. Our contribution in this part of the paper is in showing that the presence of a storage operator in the market is beneficial in terms of stabilizing the price of electricity during peak demand periods. More precisely, we consider the storage operator as a service provider, whose role is to support the production and the transmission. This means that this operator only stores electric power sent by the producers and then releases it in the future to be sold by the same producers. We also consider the alternative that instead of resorting directly to storage the producers could increase their production capacity in order to meet the demand. Therefore, we analyze the profitability of the storage operator in two different scenarios. In one scenario, we assume that the producers can respond to the demand by increasing their production capacity. In the other scenario, we assume that it would be a lot more expensive for the producers to expand their capacity than to use the storage unit. Our computational results show that in both cases it is possible for the storage operator to be profitable while having a stabilizing impact on the prices.

The rest of this paper is organized as follows. In Section 2 we review some of the literature relevant to this research. In Section 3 we present how the logic constraints can be added to the traffic equilibrium problem. In Section 4 we study the impact of a storage operator in a power system network. Finally, we present some conclusions and discussions in Section 5.

2 Literature review

2.1 The Traffic Equilibrium Problem

Many different mathematical formulations have been presented in the literature for the TEP initially proposed by Wardrop (Wardrop 1952). One of the early references is by Dafermos (Dafermos 1980) who formulated it as a variational inequality (VI). Aashtiani and Magnanti (Aashtiani and Magnanti 1981) showed that under the assumption of positive cost functions, the Wardrop TEP conditions could be formulated as the following complementarity problem (CP): find h and u such that:

$$0 \leq C(h) - \Gamma u \perp h \geq 0, \tag{1a}$$

$$0 \leq \Gamma^T h - D(u) \perp u \geq 0, \tag{1b}$$

where $\Gamma = [\gamma_{pi}]$ is the path-OD incidence matrix with $\gamma_{pi} = 1$ if path p serves OD pair i and zero otherwise, $h = (\ldots, h_p, \ldots)$ the vector of flows on each path p, $u = (\ldots, u_i, \ldots)$ the vector of minimum costs for each OD pair i, $D_i(u)$ the demand function of OD pair i, and $C_p(h)$ the cost function of path p.

Some models of the TEP (Aashtiani and Magnanti 1981, Magnanti 1984, Florian 1986) assume that the cost of each path is simply the sum of the costs of the arcs on this path (the additive cost model). In other words, $C(h) = \Delta^T c(f)$, where $c(f)$ is the cost function of the arcs, $f = (\ldots, f_a, \ldots)$ the vector of flows on each arc a and $\Delta = [\delta_{ap}]$ the arc-path incidence matrix with $\delta_{ap} = 1$ if arc a is on path p and zero otherwise. Note that if both functions $c(.)$ and $D(.)$ are linear, then (1) is equivalent to a linear complementarity problem (LCP).

It has been noted in (Gabriel and Bernstein 1997, Bernstein and Gabriel 1997) that there are many real-life applications, such as toll pricing, where the cost of a path is not necessarily the sum of the costs of each arc along the path, i.e. that the additive cost model may not be appropriate. Various alternative forms of the function $C(h)$ have been proposed in the literature, see for example (Gabriel and Bernstein 1997, Lo and Chen 2000*b*, Larsson et al. 2002). Several studies of traffic equilibria have been built around the non-additive cost models, we refer the interested reader to (Suwansirikul et al. 1987, Meng and Yang 2002, Lo and Chen 2000*b*,*a*, Caggiani and Ottomanelli 2011, Chen et al. 2010, 2001, Agdeppa et al. 2007, Chen et al. 1999) for more details.

In this paper, we propose new ways of balancing the flows in a traffic network by finding more equitable equilibrium solutions when there is more than one solution. There are many benefits to incorporating such logic constraints in a traffic equilibrium. First, as described earlier, some transportation planners could attempt to more equitably balance the equilibrium flows with these sorts of constraints. Additionally, threshold effects could also be modeled for traffic equilibrium whereby if the traffic level rose beyond a certain point an additional cost could be incurred or some other behavioral action modeled.

2.2 Power Systems with Storage

The increase in prices and volatility of electricity have raised a huge interest in the potential economic opportunity of electricity storage. A recent report by the U.S. Department of Energy (DoE 2013) highlights how energy storage can be used to support the generation, distribution and transmission needs of a power grid. Several other works have also been conducted to show the profitability of electricity storage. A review by Zucker *et al.* (Zucker et al. 2013) assesses the value of storage in electricity markets in different ways. This review points out that most of the literature in this direction has focused on studying the profitability from a storage investor's point of view whereby the main concern is in buying inexpensive electricity available during off-peak demand periods and selling it back at a higher price during on-peak demand periods. Sioshansi *et al.* (Sioshansi et al. 2009) consider the impact of large storage devices and examine how the use of power storage can decrease on-peak and increase off-peak prices, which diminishes the value of arbitrage. This price-shifting effect is related to the fact that the storage operator has the role of temporal arbitrage, that is, buying power during the off-peak demand periods and selling it during the on-peak demand periods (instead of just providing the storage services to support the supply as we consider in the current paper). Walawalkar *et al.* (Walawalkar et al. 2007) analyze the economic case of the installation of some specific storage technology in New York State's power market. Similar analyses are conducted in (Chacra et al. 2005, Sugihara et al. 2013). Barton and Infield (Barton and Infield 2004) focus on the use of storage to maximize the penetration of intermittent renewable generation in the grid. Other evaluation of the benefits of energy storage include (Bayod-Rújula 2009, Wade et al. 2010, Zucker et al. 2013). A summary of different existing applications of storage in energy markets is given in (Sioshansi et al. 2012).

In our research, we consider an energy storage operator as a service provider located at a node of a power system network that might experience congestion. This storage operator only stores power sent by energy producers and then releases it in the future to be sold by the same producers. Though the storage operator is a profit maximizer, the network operator clears the price of storing the power as well as how much power from each producer should be stored. We show that the presence of such a storage operator supports the operation of the power grid, and also helps stabilize the price of electricity and avoid the price-shifting effect noted in (Sioshansi et al. 2009).

2.3 Solving Binary-Constrained Linear Complementarity Problems

Both the problems of traffic equilibrium and power market with storage will be shown to be instances of a binary-constrained mixed linear complementarity problem (BC-MLCP) which we now explain. More formally, the mathematical formulation of the BC-MLCP is described as follows. Given the vector $q = \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} \in \mathbb{R}^n$ and the partitioned matrix $A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \in \mathbb{R}^{n \times n}$, find $\begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where $n_1 + n_2 = n$, such that:

$$0 \leq q^1 + \begin{pmatrix} A^{11} & A^{12} \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \perp z^1 \geq 0 \tag{2a}$$

$$0 = q^2 + \begin{pmatrix} A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}, \; z^2 \text{ free,} \tag{2b}$$

where some components of the vector $\begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$ are constrained to be binary. z^1 and z^2 are called the nonnegative complementarity variables and the free variables, respectively.

In (Fomeni et al. 2015), the authors developed a solution method for the BC-MLCP considering the fact that solving this problem is equivalent to finding a vector that satisfies both the complementarity and the binary constraints. Therefore, we define a set $\mathcal{F}$ to be the set all the vectors that satisfy the above constraints (2) with the additional requirement that some components of the vector $\begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$ are constrained to be binary. Given that the complementarity constraints in this set can be seen as quadratic equations, we can then, following the idea of the RLT (Reformulation and Linearization Technique) approach (Sherali and Adams 1990, 1994, 1998), introduce new variables to replace all the quadratic terms in these equations. We then relax the quadratic requirements by replacing them with the McCormick inequalities (McCormick 1979). This yields a new set $\tilde{\mathcal{F}}$, which can now be seen as the RLT relaxation of set $\mathcal{F}$. The nice feature about this latter set, is that, it is convex, and one can more easily solve any optimization problem over it rather than $\mathcal{F}$.

The proposed solution approach for the BC-MLCP proceeds in three main steps. The first step consists of reducing the upper bounds of each complementarity variable using the bound refinement procedure of Sherali and Tuncbilek (Sherali and Tuncbilek 1997). This step enables us to get tighter relaxations of the problem. More precisely, we find the maximum value of each complementarity variable by maximizing it over the set $\tilde{\mathcal{F}}$. The solution of such an LP enables us to have a new upper bound of the complementarity variable. In the second step, we solve a series of LPs over the set $\tilde{\mathcal{F}}$ which enables us to replace some of the complementarity constraints with linear equations. Finally, we solve a mixed integer linear optimization problem (MILP) that is equivalent to the original BC-MLCP with a reduced number of complementarity constraints. We refer the interested reader to (Fomeni et al. 2015) for more details of the methods and for the full description of the algorithm.

It is important to mention that there are at least two other methods that can be used to solve BC-MLCP instances. The first one is the disjunctive constraints approach (or big-M method) that consists of replacing each complementarity constraint with two disjunctive constraints and solve the resulting problem as a MILP with a zero objective function (Fortuny-Amat and McCarl 1981). The other approach is the RLT-based MILP reformulation approach of Sherali et al. (Sherali et al. 1998). This approach consists of linearizing the complementarity constraints using the RLT approach and then solving the resulting MILP where the total violation of complementarity is minimized (this will be zero if the problem is feasible). A thorough comparative study of the three methods for solving the BC-MLCP was already presented in (Fomeni et al. 2015). This comparison showed that, in terms of computational time, the RLT-based MILP approach in (Sherali et al. 1998) is dominated by both the approach in (Fomeni et al. 2015) and by the disjunctive constraints approach (Fortuny-Amat and McCarl 1981). On the other hand, the disjunctive constraints approach (Fortuny-Amat and McCarl 1981) dominates the approach in (Fomeni et al. 2015) on most BC-MLCP instances, but there are some instances where the disjunctive constraints approach (Fortuny-Amat and McCarl 1981) is dominated by the approach in (Fomeni et al. 2015). Furthermore the efficiency of the disjunctive constraints approach (Fortuny-Amat and McCarl 1981) depends to a large extent on the application-specific choice of the parameter M. Considering the overall results reported in (Fomeni et al. 2015), we used the algorithm proposed in that paper for this study. Our computational experience shows that the performance of the technique in (Fomeni et al. 2015) is competitive with that of the disjunctive constraints approach (Fortuny-Amat and McCarl 1981) for the types of instances considered in this paper.

3 Logic Constraints in Traffic Equilibrium Models

In this section, we show how one can add logic constraints to a complementarity model of the traffic equilibrium problem described earlier. Such an approach can have several important applications such as balancing equilibrium flows more equitably or modeling threshold events to name just two. We use the small example given below as illustration.

For the sake of simplicity, we assume that the cost of each path is the sum of the costs of the arcs along this path, i.e. $C(h) = \Delta^T c(f)$ in (1). In all the examples in this section we choose the cost and demand functions to be defined as follows: $c_a(f) = f_a - l$ and $D_i(u) = s_i - u_i$, where l and s are some (given) parameters. With these choices, the MLCP formulation (1) of the problem is:

$$0 \leq Ax + q \perp x \geq 0,$$

where

$$A = \begin{pmatrix} \Delta^T \Delta & -\Gamma \\ \Gamma^T & I \end{pmatrix}, \qquad q = \begin{pmatrix} -\Delta^T l \\ -s \end{pmatrix} \qquad \text{and} \qquad x = \begin{pmatrix} h \\ u \end{pmatrix}$$

3.1 Small Example

We consider the following small but illustrative example. The network for this example, shown in Figure 1, is taken from (Lo and Chen 2000*b*). The set of arcs is $\mathcal{A} = \{1, 2, 3, 4, 5\}$, where arc 1 goes from node 1 to node 3, etc. The set of all OD pairs is $\mathfrak{J} = \{1-5, 2-5\}$ and the set of paths (we use arc numbers in the names of paths instead of node numbers)

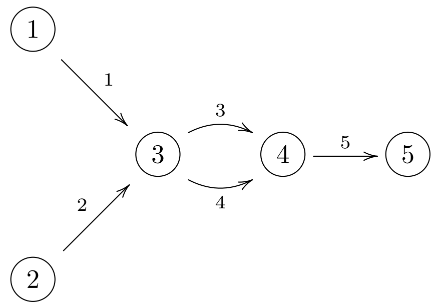

Figure 1: Traffic network 1

connecting the OD pairs is $\mathfrak{P} = \{1 \to 3 \to 5, 1 \to 4 \to 5, 2 \to 3 \to 5, 2 \to 4 \to 5\}$, while the arc-path and path-OD incidence matrices are, respectively:

$$\Delta = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \Gamma = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Four preliminary tests with this network example were carried out for different choices of l and s corresponding to different values of $q = \begin{pmatrix} -\Delta^T l \\ -s \end{pmatrix}$ as given below.

- Test 1: $q = -(24, 24, 24, 24, 30, 40)^T$,
- Test 2: $q = -(24, 24, 24, 24, 40, 40)^T$,
- Test 3: $q = -(40, 20, 15, 20, 40, 40)^T$,
- Test 4: $q = -(40, 20, 15, 20, 40, 80)^T$.

For Test 1 and Test 2, the cost functions are the same for all the paths, but the demand constant of the OD pair $1 - 5$ is lower than that of the OD pair $2 - 5$. While, for Test 3 we make the paths $1 \to 3 \to 5$ and $2 \to 4 \to 5$ more expensive than their alternative paths and equal demand for the two OD pairs. Finally, the cost functions for Test 4 are the same as for Test 3, but the demand is higher for the OD pair $2 - 5$.

The results of these tests are presented in Table 1. In these results, the equilibrium solutions for Test 1 suggests that all the vehicles going from 1 to 5 only use the path $1 \to 4 \to 5$, while for the OD pair $2 - 5$ a large proportion of vehicles are using the path $2 \to 3 \to 5$ vs. few vehicles that are using the alternative path $2 \to 4 \to 5$ for the same cost. A similar observation can be done for Test 2 where, all the vehicles are simply ignoring the existence of the alternative paths. Ideally for such examples one would like to have a more equitable equilibrium or division of the flow if at all possible. This is where it can be helpful to add some logic constraints to the system for equity balancing. In some cases it might not be possible to impose such constraints without increasing the costs for some vehicles. For example, in Test 3 and Test 4 we can see that the paths with zero flow have costs that are larger than the minimum costs (this depicts the Wardrop equilibrium condition). Therefore redirecting some vehicles onto those paths will be penalizing them in terms of the costs. Meng and Yang (Meng and Yang 2002) also noted this kind of effect in network capacity improvement. In such situations, it could be beneficial for the network operator to propose a discount to vehicles to use the alternative paths in order to achieve the desired equity. A thorough explanation of how we add logic constraints to the model is presented below in Subsection 3.2.

Table 1: Results of preliminary test with Network 1

Test Num.	$h_{1\to3\to5}$	$h_{1\to4\to5}$	$h_{2\to3\to5}$	$h_{2\to4\to5}$	u_{1-5}	u_{2-5}
Test 1	0.0	9.3	11.8	2.5	20.7	25.7
Path cost	20.7	20.7	25.7	25.7		
Test 2	0.0	12.8	12.8	0.0	27.2	27.2
Path cost	27.2	27.2	27.2	27.2		
Test 3	17.3	0.0	0.0	10.7	22.7	29.3
Path cost	22.7	36	41	29.3		
Test 4	14.2	0.0	1.2	20.5	25.7	58.2
Path cost	25.7	50.7	58.2	58.2		

3.2 Imposing Equity Constraints on the Small Example

In this subsection, we consider Test 1 in Table 1 which showed that all the vehicles on the OD pair $1-5$ only use the path $1 \to 4 \to 5$. However, one may wish to redirect some of the equilibrium flow to the path $1 \to 3 \to 5$ without compromising travel cost. Therefore, it makes sense to plan to have a certain proportion of the traffic for this OD pair redirected to the path $1 \to 3 \to 5$. In order to do this, we add the following logic constraints to the problem (1). We define the binary variable

$$x_{1-5}^{1\to4\to5} = \begin{cases} 1 & \text{if} \quad h_{1\to4\to5} > 0 \\ 0 & \text{if} \quad h_{1\to4\to5} = 0. \end{cases}$$

This binary condition can then be enforced in the problem by the following complementarity condition

$$0 \leq h_{1\to4\to5} \perp 1 - x_{1-5}^{1\to4\to5} \geq 0. \tag{3}$$

In order to include the logic constraints as stated earlier, we add the following constraints

$$\alpha^- x_{1-5}^{1\to4\to5}(h_{1\to3\to5} + h_{1\to4\to5}) \leq x_{1-5}^{1\to4\to5} h_{1\to3\to5} \leq \alpha^+ x_{1-5}^{1\to4\to5}(h_{1\to3\to5} + h_{1\to4\to5}), \tag{4}$$

where α^- and α^+ are parameters that represent respectively the lower and upper admissible proportion of the traffic to be redirected to the path $1 \to 3 \to 5$. Here $0 \leq \alpha^- \leq \alpha^+ \leq 1$. These parameters could be defined by the network operators depending on their equity preferences. It can be seen from these conditions that if the flow on path $1 \to 4 \to 5$ is zero, then the above constraints are not active. To see this assume that $h_{1\to4\to5} = 0$, then (4) reduces to

$$\alpha^- x_{1-5}^{1\to4\to5} h_{1\to3\to5} \leq x_{1-5}^{1\to4\to5} h_{1\to3\to5} \leq \alpha^+ x_{1-5}^{1\to4\to5} h_{1\to3\to5},$$

or

$$(\alpha^- - 1) x_{1-5}^{1\to4\to5} h_{1\to3\to5} \leq 0 \tag{5a}$$

$$(1 - \alpha^+) x_{1-5}^{1\to4\to5} h_{1\to3\to5} \leq 0. \tag{5b}$$

- For (5a), since $(\alpha^- - 1) \leq 0 \implies x_{1-5}^{1\to4\to5} h_{1\to3\to5} \geq 0$ which is always the case as $x_{1-5}^{1\to4\to5} \in \{0,1\}$ and $h_{1\to3\to5} \geq 0$.
- For (5b), since $(1 - \alpha^+) \geq 0 \implies x_{1-5}^{1\to4\to5} h_{1\to3\to5} \leq 0$ which in light of $x_{1-5}^{1\to4\to5} \in \{0,1\}$ and $h_{1\to3\to5} \geq 0$ means that $x_{1-5}^{1\to4\to5} 0$ or $h_{1\to3\to5} = 0$ or both. In the former case, via (3) we see that $h_{1\to4\to5} = 0$ which is already true. In the latter case, $h_{1\to3\to5} = 0$ so that means there is no flow for OD pair $1-5$ which can reasonably be excluded as an extreme case.

Conversely, of $h_{1\to4\to5} > 0$ then by (3), we know that $x_{1-5}^{1\to4\to5} = 1$ and that (4) reduces to

$$\alpha^- \leq \frac{h_{1\to3\to5}}{h_{1\to3\to5} + h_{1\to4\to5}} \leq \alpha^+$$

as desired. Thus, (3) and (4) guarantee that for the OD pair $1-5$, there is a better equity of the flow between the tow paths $1\to3\to5$ and $1\to4\to5$, both serving this OD pair.

We have implemented this model for several values of α^- and α^+ and the results presented in Table 2 show how the model redirects some of the flow for OD pair $1-5$ to the path $1\to3\to5$. Consider for example when $\alpha^- = 0.1, \alpha^+ = 1.0$. Then, the total flow for OD pair $1-5$ is 9.30. 10% of this equilibrium flow must go along the path $1\to3\to5$ (assuming that $h_{1\to4\to5} > 0$); this corresponds to the value $h_{1\to3\to5} = 0.93$ and it is achieved without increasing the minimum cost of 20.7 for OD pair $1-5$. This pattern is repeated for all the changing α^- values in Table 4, namely $h_{1\to3\to5}$ takes the value of $\alpha^-(h_{\to3\to5} + h_{\to4\to5})$. It is also important to note the changes that are occurring also on the paths for the OD pair $2-5$ even though we are enforcing equity only on a path of the OD pair $1-5$. This kind of indirect interaction can also be noticed in our second TEP example presented in Subsection 3.3. Note that the above example can easily be generalized.

Table 2: Adding logic constraint on Network 3

α^-	α^+	$h_{1\to3\to5}$	$h_{1\to4\to5}$	$h_{2\to3\to5}$	$h_{2\to4\to5}$	u_{1-5}	u_{2-5}
0.0	1.0	0.0	9.3	11.8	2.5	20.7	25.7
0.1	1.0	0.93	8.37	10.87	3.43	20.7	25.7
0.2	1.0	1.86	7.44	9.94	4.36	20.7	25.7
0.3	1.0	2.79	6.51	9.01	5.29	20.7	25.7
0.4	1.0	3.72	5.58	8.08	6.22	20.7	25.7
0.5	1.0	4.65	4.65	7.15	7.15	20.7	25.7

More generally, let i be an OD pair in a traffic network. Let $\mathfrak{P}_i$ be the set of paths of the OD pair i with $|\mathfrak{P}_i| \geq 2$. Let $p_0 \in \mathfrak{P}_i$ be a path of the OD pair i from which the flow is to be redirected, also let $p_k \in \mathfrak{P}_i, p_k \neq p_0$ be another path of the OD pair i for which we want the flow to be within a certain range whenever the flow of the path p_0 is positive. We define the following binary variable:

$$x_i^{p_0} = \begin{cases} 1 & \text{if} \quad h_{p_0} > 0 \\ 0 & \text{if} \quad h_{p_0} = 0. \end{cases}$$

Conditions to impose deviation from the path p_0 to the path p_k when connecting the OD pair i are added to traffic equilibrium problem (1) using the following constraints:

$$0 \leq h_{p_0} \perp 1 - x_i^{p_0} \geq 0 \tag{6a}$$

$$x_i^{p_0} h_{p_k} \geq \alpha_{p_0 i}^{-} x_i^{p_0} \left(h_{p_0} + \sum_{\substack{q \in \mathfrak{P}_i \\ q \neq p_0}} h_q \right) \tag{6b}$$

$$x_i^{p_0} h_{p_k} \leq \alpha_{p_0 i}^{+} x_i^{p_0} \left(h_{p_0} + \sum_{\substack{q \in \mathfrak{P}_i \\ q \neq p_0}} h_q \right) \tag{6c}$$

$$x_i^{p_0} \in \{0, 1\}. \tag{6d}$$

Therefore, a traffic equilibrium model with logic constraints to redirect a proportion of the traffic from a path p_0 to a path p_k of the same OD pair $i \in \mathcal{J}$ can be modeled by combining equations (1) and (6).

3.3 Large Example: Sioux Falls Network

In this subsection, we illustrate the application of logic constrained traffic equilibria using the Sioux Falls network (Github 2015) shown in Figure 2. We assume that there is a congested portion of the network where the network operator may need to re-direct a proportion of the traffic to an alternative road with same minimal travel cost.

We assume that the congested portion consists of the red arcs in Figure 2 with the OD pairs and the set of arcs being respectively

$$\mathfrak{J} = \{3-7, 3-18, 12-7\}$$

and

$$\mathcal{A} = \{6, 9, 12, 13, 16, 24, 20, 25, 26, 22, 47, 18, 54, 36, 32, 29, 50, 30, 52\}.$$

For the sake of readability and traceability, we further assume that the admissible paths connecting the three OD pairs are:

$$\text{For OD pair 3-7 :} \begin{cases} p_1 := 6 \to 9 \to 12 \to 16 \to 20 \\ p_2 := 6 \to 9 \to 13 \to 24 \to 20 \\ p_3 := 6 \to 9 \to 12 \to 16 \to 22 \to 50 \to 54 \\ p_4 := 6 \to 9 \to 13 \to 25 \to 29 \to 50 \to 54 \end{cases}$$

$$\text{For OD pair 3-18 :} \begin{cases} p_5 := 6 \to 9 \to 12 \to 16 \to 20 \to 18 \\ p_6 := 6 \to 9 \to 13 \to 25 \to 29 \to 50 \\ p_7 := 6 \to 9 \to 12 \to 16 \to 22 \to 50 \\ p_8 := 6 \to 9 \to 13 \to 24 \to 20 \to 18 \\ p_9 := 6 \to 9 \to 13 \to 24 \to 22 \to 50 \end{cases}$$

$$\text{For OD pair 12-7 :} \begin{cases} p_{10} := 36 \to 32 \to 30 \to 52 \to 50 \to 54 \\ p_{11} := 36 \to 32 \to 29 \to 50 \to 54 \\ p_{12} := 36 \to 32 \to 26 \to 24 \to 20 \\ p_{13} := 36 \to 32 \to 29 \to 47 \to 20 \\ p_{14} := 36 \to 32 \to 30 \to 52 \to 47 \to 20 \end{cases}$$

Figure 2: Sioux Falls Network

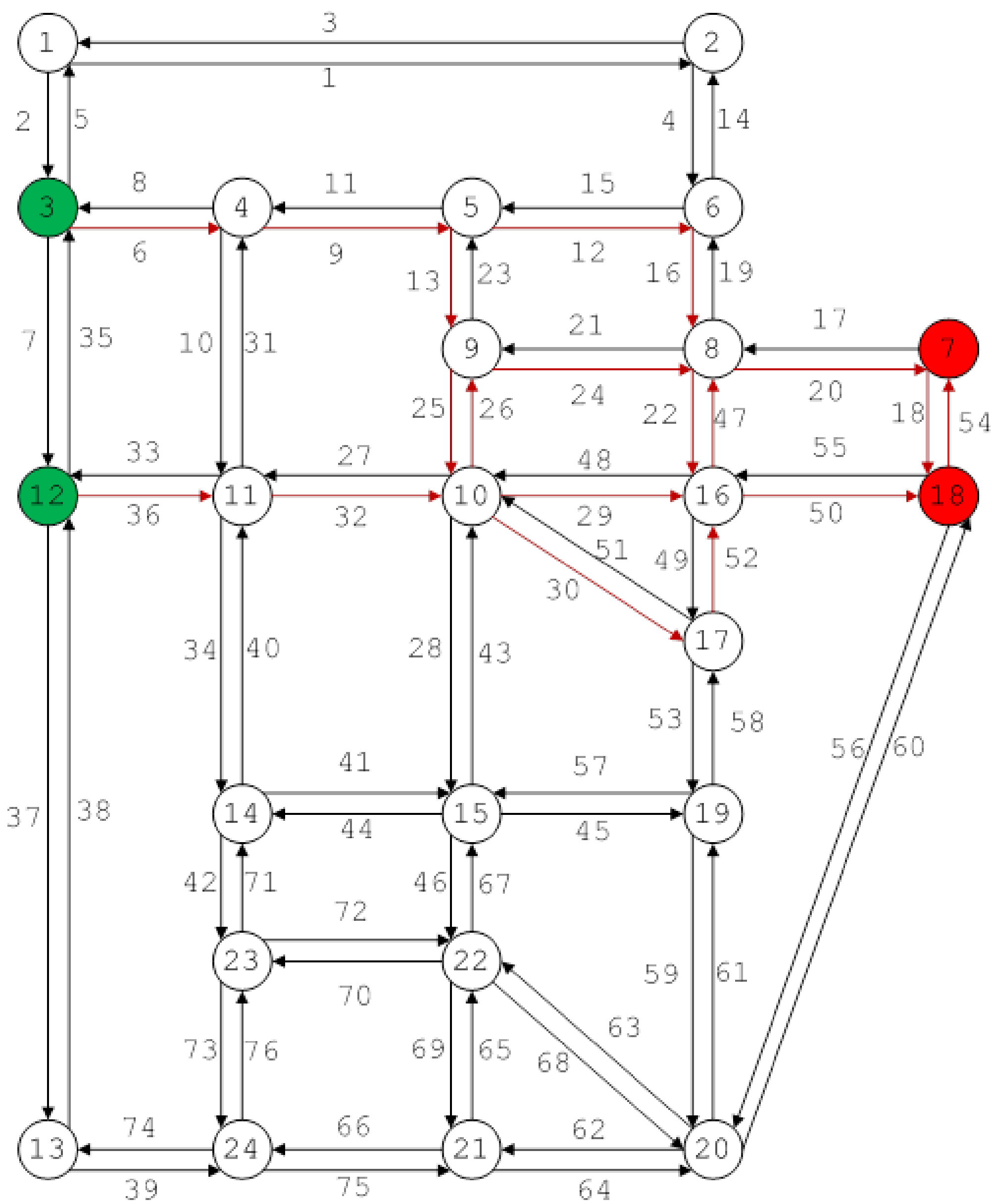

Table 3: Applying logic constraints on a congested portion of the Sioux Falls Network

			OD: 3-7				OD: 3-18					OD: 12-7				
α^-	α^+		h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9	h_{10}	h_{11}	h_{12}	h_{13}	h_{14}
0	1	flow	0.00	0.00	228.72	233.78	224.64	98.73	0.00	0.00	101.64	0.00	0.00	370.9	137.86	235.19
		cost	350.33	350.33	37.49	37.49	74.98	74.98	74.98	74.98	74.98	583.64	583.64	0.00	0.00	0.00
0.1	1	flow	0.00	0.00	129.98	332.52	122.99	0.00	200.37	101.64	0.00	0.00	0.00	370.9	137.86	235.19
		cost	350.33	350.33	37.49	37.49	74.98	74.98	74.98	74.98	74.98	583.64	583.64	0.00	0.00	0.00
0.2	1	flow	0.00	0.00	129.98	332.52	122.99	0.00	200.37	101.64	0.00	0.00	0.00	370.9	137.86	235.19
		cost	350.33	350.33	37.49	37.49	74.98	74.98	74.98	74.98	74.98	583.64	583.64	0.00	0.00	0.00
0.3	1	flow	0.00	0.00	129.98	332.52	122.99	0.00	200.37	101.64	0.00	0.00	0.00	370.9	137.86	235.19
		cost	350.33	350.33	37.49	37.49	74.98	74.98	74.98	74.98	74.98	583.64	583.64	0.00	0.00	0.00
0.4	1	flow	0.00	0.00	129.98	332.52	122.99	0.00	200.37	101.64	0.00	0.00	0.00	370.9	137.86	235.19
		cost	350.33	350.33	37.49	37.49	74.98	74.98	74.98	74.98	74.98	583.64	583.64	0.00	0.00	0.00
0.5	1		INFEASIBLE													
0.4	0.8	flow	0.00	0.00	129.98	332.52	122.99	0.00	200.37	101.64	0.00	0.00	0.00	370.9	137.86	235.19
		cost	350.33	350.33	37.49	37.49	74.98	74.98	74.98	74.98	74.98	583.64	583.64	0.00	0.00	0.00
0.3	0.5	flow			129.98	332.52	195.86	0.00	127.5	28.77	72.87	0.00	0.00	370.9	137.86	235.19
		cost	350.33	350.33	37.49	37.49	74.98	74.98	74.98	74.98	74.98	583.64	583.64	0.00	0.00	0.00
0.3	0.4	flow	0.00	0.00	129.98	332.52	195.86	0.00	127.5	28.77	72.87	0.00	0.00	370.9	137.86	235.19
		cost	350.33	350.33	37.49	37.49	74.98	74.98	74.98	74.98	74.98	583.64	583.64	0.00	0.00	0.00
0.2	0.4	flow	0.00	0.00	129.98	332.52	224.63	0.00	98.73	0.00	101.64	0.00	0.00	370.9	137.86	235.19
		cost	350.33	350.33	37.49	37.49	74.98	74.98	74.98	74.98	74.98	583.64	583.64	0.00	0.00	0.00
0.2	0.3	flow	0.00	0.00	129.98	332.52	224.63	0.00	98.73	0.00	101.64	0.00	0.00	370.9	137.86	235.19
		cost	350.33	350.33	37.49	37.49	74.98	74.98	74.98	74.98	74.98	583.64	583.64	0.00	0.00	0.00
0.2	0.2	flow	0.00	0.00	245.35	217.14	122.99	115.37	85	101.64	0.00	0.00	0.00	370.9	137.86	235.19
		cost	350.33	350.33	37.49	37.49	74.98	74.98	74.98	74.98	74.98	583.64	583.64	0.00	0.00	0.00
0.1	0.2	flow	0.00	0.00	186.21	276.28	224.63	56.23	42.5	0.00	101.64	0.00	0.00	370.9	137.86	235.19
		cost	350.33	350.33	37.49	37.49	74.98	74.98	74.98	74.98	74.98	583.64	583.64	0.00	0.00	0.00
0.1	0.1	flow	0.00	0.00	287.86	174.64	122.99	157.87	42.5	101.64	0.00	0.00	0.00	370.9	137.86	235.19
		cost	350.33	350.33	37.49	37.49	74.98	74.98	74.98	74.98	74.98	583.64	583.64	0.00	0.00	0.00

Although these paths are not the only ones connecting the OD pairs, they may represent the paths used by the vast majority of the drivers.

As in the previous example, the cost and demand functions are given by the formulae $c_a(f) = f_a - l$ and $D_i(u_i) = s_i - u_i$, where the values of the parameter l for the 19 arcs in $\mathcal{A}$ are respectively

$$l = (200, 500, 800, 800, 800, 800, 1000, 800, 800, 800, 800, 500, 500, 800, 500, 400, 300, 500, 300),$$

and $s = 500\,e$ with e equal to the all-ones vector. The computational results we obtained are reported in Table 3 where we denote the flow on path p_i by h_i.

We first solved the standard complementarity-based traffic equilibrium model (1) with no logic constraints and obtained the solution reported in the first row of Table 3 where $\alpha^- = 0.0$ and $\alpha^+ = 1$. In this solution, one can see that for OD pairs $3-7$ and $12-7$ the paths with zero flows have associated costs larger than the minimum. On the other hand, the costs of the paths of the OD-pair $3-18$ are all equal while the flows on the paths p_7 and p_8 are equal to zero.

Next we focused on path p_7 and aimed to re-direct a proportion of the traffic for the OD pair $3-18$ onto that path whenever there is a positive flow on the path p_5. This corresponds to solving an extended version of the TEP model (1) that includes logic constraints as described in Subsection 3.2. We report results using different values of α^+ and α^- as shown in Table 3. Note that when $\alpha^- \geq 0.5$ the problem becomes infeasible, which means that it is not possible to re-direct 50% or more of the flow from p_5 onto the other paths.

We observe that re-directing traffic to path p_7 with no upper limit often results in zero flow on path p_6, which is equivalent to path p_7 in the sense that all their arcs have same values for the parameter l in the cost function. This suggests that some of the flow assigned to path p_7 could equivalently be assigned to path p_6 without violating the Wardrop principle of equilibrium.

The results also suggest that for some values of α^+ and α^- one can more equitably balance the flows between the paths of the OD-pair $3-18$. For example, for $\alpha^+ = 0.2$ and $\alpha^- = 0.2$ the flows are reasonably well balanced between paths p_5, p_6, p_7 and p_8. We also note that while equity is achieved to the desired level (for each choice of α^+ and α^-) for the OD-pair $3-18$, there is also a variation in the flows of the paths of the OD pair $3-7$ (in the four "OD: 3-7" columns in the table). This interaction is certainly due to the fact that these two OD pairs use almost the same arcs.

Although this paper is not concerned with the algorithmic aspects of computing logic-constrained equilibria, we conclude this section with some brief remarks concerning the time required to obtain the results reported in Table 3 so that the reader can have an idea of the computational cost involved. It took on average 0.56 seconds to compute the solution in each of the rows in Table 3 using the approach from (Fomeni et al. 2015) described in Section 2.3, while the disjunctive constraints approach (Fortuny-Amat and McCarl 1981) took on average 0.19 seconds for the same instances. It is however important to note that the approach in (Fomeni et al. 2015) is parameter-free while the disjunctive constraints approach (Fortuny-Amat and McCarl 1981) requires a careful choice of the parameter M. A more thorough comparison of the approaches to compute logic-constrained equilibria can be found in (Fomeni et al. 2015).

4 Equilibria in Power Markets with Storage

In this section we consider an energy storage operator in a power market as a service provider, whose role is to support the generation and transmission as well as to help stabilize the price of electricity during the on-peak demand periods. Previous papers have focused on studying the economic opportunity for a storage operator in a power system network in terms of buying electricity during the off-peak demand periods inexpensively and selling it during the on-peak demand periods to make a profit. This approach diminishes the value of arbitrage as it shifts the load from the on-peak demand periods to off-peak demand periods, see (Sioshansi et al. 2009).

We illustrate such a system using a small example with a network of two nodes, two producers and a storage operator at node 2 as illustrated in Figure 3.

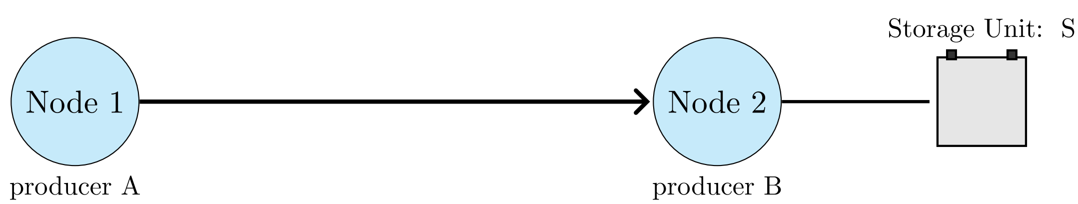

Figure 3: A two-node power system network with a storage operator

In Figure 3, we have two nodes and two producers A and B, and electricity can only be transmitted from node 1 to node 2. There is a storage operator located at node 2 who provides storage facilities to both producers. We assume that the transmission capacity of the network as well as the generation capacity of the producers are limited in terms of satisfying the demand at peak times. We consider two time periods: an off-peak period followed by an on-peak period. During the off-peak period, either producer can send a part of its production to the storage unit which will then be released during the on-peak demand period to be sold on the market at node 2. Given the economic opportunity of storage, the producers could respond by expanding their generation capacity. We investigate the profitability of the storage operator in two scenarios. In the first scenario, we assume that capacity expansion is not an option for the producers, while in the second scenario, we assume that the producers are willing to expand their production capacities if the associated cost can be competitive with the operating cost of storage.

4.1 Model Description

In this example, each producer maximizes its profit by choosing appropriate nonnegative levels of production q^A_{t1} and q^B_{t2}, sales s^A_{t1} and s^B_{t2}, storage and flow variables b^A, b^B and $f^A_{t,12}$ subject to their minimum and maximum production limits, and consistency between sales, production, storage and flow. While the network transportation operator and the storage operator, respectively decide on the transmission level $g_{t,12}$ and the storage level h^S that maximize their profits. More formally, each player solves an optimization problem as follows.

- Producer A:

$$\max_{\substack{s^A_{t1}, q^A_{t1}, f^A_{t,12}, E^A_{t1}, \\ b^A, v^A_{t1}, y^A_{t1}}} \quad Rev(A) - Cost(A) \tag{7a}$$

$$\begin{aligned}
\text{s.t.} \quad & v^A_{t1} q^A_{min} \le q^A_{t1} \le v^A_{t1} q^A_{max} && \text{for } t = 1, 2 \quad \left(\lambda^A_{t,min}, \lambda^A_{t,max}\right) && \text{(7b)}\\
& y^A_{t1} E^A_{min} \le E^A_{t1} \le y^A_{t1} E^A_{max} && \text{for } t = 1, 2 \quad \left(\xi^A_{t,min}, \xi^A_{t,max}\right) && \text{(7c)}\\
& s^A_{t1} = q^A_{t1} + E^A_{t1} - f^A_{t,12} - b^A && \text{for } t = 1 \quad \left(\delta^A_{t1}\right) && \text{(7d)}\\
& s^A_{t1} = q^A_{t1} + E^A_{t1} - f^A_{t,12} && \text{for } t = 2 \quad \left(\delta^A_{t1}\right) && \text{(7e)}\\
& y^A_{t1} \le v^A_{t1} && \text{for } t = 1, 2 \quad \left(l^A_{t1}\right) && \text{(7f)}\\
& s^A_{t1}, q^A_{t1}, f^A_{t,12}, b^A, E^A_{t1} \ge 0 && \text{for } t = 1, 2 && \text{(7g)}\\
& v^A_{t1}, y^A_{t1} \in \{0, 1\}. && \text{for } t = 1, 2, && \text{(7h)}
\end{aligned}$$

where

$$Rev(A) = \sum_{t=1}^{2} \left[\pi_{t1} s^A_{t1} + \pi_{t2} f^A_{t,12}\right] + \pi_{22} b^A,$$

and

$$Cost(A) = \sum_{t=1}^{2} \left[\gamma^A_1 q^A_{t1} + \left(\tau_{t,12} + \tau^{reg}_{12}\right) f^A_{t,12} + \rho^A E^A_{t1}\right] + \left(\tau_{1,12} + \tau^{reg}_{12}\right) b^A + \omega^S b^A.$$

- Producer B:

$$
\begin{aligned}
\max_{\substack{s^B_{t2}, q^B_{t2}, E^B_{t2},\\ b^B, v^B_{t2}, y^B_{t2}}} \quad & Rev(B) - Cost(B) && && \text{(8a)}\\
\text{s.t.} \quad & v^B_{t2} q^B_{min} \le q^B_{t2} \le v^B_{t2} q^B_{max} && \text{for } t = 1,2 \quad \left(\lambda^B_{t,min}, \lambda^B_{t,max}\right) && \text{(8b)}\\
& y^B_{t2} E^B_{min} \le E^B_{t2} \le y^B_{t2} E^B_{max} && \text{for } t = 1,2 \quad \left(\xi^B_{t,min}, \xi^B_{t,max}\right) && \text{(8c)}\\
& s^B_{t2} = q^B_{t2} + E^B_{t2} - b^B && \text{for } t = 1 \quad \left(\delta^B_{t2}\right) && \text{(8d)}\\
& s^B_{t2} = q^B_{t2} + E^B_{t2} && \text{for } t = 2 \quad \left(\delta^B_{t2}\right) && \text{(8e)}\\
& y^B_{t2} \le v^B_{t2} && \text{for } t = 1,2 \quad (l^B_{t2}) && \text{(8f)}\\
& s^B_{t2}, q^B_{t2}, b^B, E^B_{t2} \ge 0 && \text{for } t = 1,2 && \text{(8g)}\\
& v^B_{t2}, y^B_{t2} \in \{0,1\}. && \text{for } t = 1,2, && \text{(8h)}
\end{aligned}
$$

where

$$
Rev(B) = \sum_{t=1}^{2} \pi_{t2} s^B_{t2} + \pi_{22} b^B,
$$

and

$$
Cost(B) = \sum_{t=1}^{2} \left[\gamma^B_1 q^B_{t2} + \rho^B E^B_{t2}\right] + \omega^S b^B.
$$

- Transportation System Operator:

$$
\max_{g_{t,12}} \quad \sum_t \left[(\tau_{t,12} + \tau^{reg}_{12}) g_{t,12} - \gamma^{TSO} g_{t,12}\right]
$$

$$
\begin{aligned}
\text{s.t.} \quad & g_{t,12} \le \bar{g}_{12} && \text{for } t = 1,2 \quad (\epsilon_{t,12})\\
& g_{t,12} \ge 0 && \text{for } t = 1,2
\end{aligned}
$$

- Storage Operator:

$$
\max_{h^S} \quad \omega^S h^S - \gamma^S h^S
$$

$$
\begin{aligned}
\text{s.t.} \quad & h^S \le \sigma^S \quad && (\theta^S)\\
& h^S \ge 0
\end{aligned}
$$

In the above definition E^A_{t1} and E^B_{t2} are the amount of power produced by A and B from the expanded production unit during time period t. τ^{Reg}_{12} represents the nonnegative regulated tariff for using the network from node 1 to node 2; this is a fixed parameter. The complementarity model determines $\tau_{t,12}$ and ω^S, respectively the congestion tariff for sending power from node 1 to node 2 at time period t and the unit tariff for using the storage facilities. γ is a parameter representing the unit cost of production with the superscript relating to the particular player. Finally ρ^A and ρ^B are the unit cost of production of the expanded production units of A and B. Note that constraints (7f) and (8f) are included in the model to ensure that the expanded production units cannot be switched on while the existing the production units are off.

In addition to the optimization problems for these four players, there are the following market-clearing conditions that force supply to equal demand:

$$0 = \left[s_{t1}^A\right] - D_{t1}(\pi_{t1}), \qquad \pi_{t1} \text{ free }, t = 1, 2, \tag{9a}$$

$$0 = \left[s_{t2}^B + f_{t,12}^A\right] - D_{t2}(\pi_{t2}), \qquad \pi_{t2} \text{ free, } t = 1, \tag{9b}$$

$$0 = \left[s_{t2}^B + f_{t,12}^A + h^S\right] - D_{t2}(\pi_{t2}), \quad \pi_{t2} \text{ free, } t = 2. \tag{9c}$$

Note that the terms in square brackets are the net supply at each node (assuming no losses) and $D_{tn}(\pi_{tn}), t, n = 1, 2$ are the nodal demands that depend on the price π_{tn}.

The last part of the equilibrium are the market-clearing conditions that balance the network flows as well as the storage flows:

$$0 = g_{t,12} - \left(f_{t,12}^A + b^A\right), \qquad \tau_{t,12} \text{ free, } t = 1, \tag{10a}$$

$$0 = g_{t,12} - f_{t,12}^A, \qquad \tau_{t,12} \text{ free, } t = 2, \tag{10b}$$

$$0 = h^S - \left(b^A + b^B\right), \qquad \omega^S \text{ free.} \tag{10c}$$

By relaxing the binary requirement of variables v_{t1}^A, v_{t2}^B, y_{t1}^A and y_{t2}^B, the KKT conditions for each of the above optimization problems are as follows.

- Producers A at node 1.

$$0 \le -\pi_{t1} + \delta_{t1}^A \perp s_{t1}^A \ge 0 \tag{11a}$$

$$0 \le \gamma^A + \left(\lambda_{t,max}^A - \lambda_{t,min}^A\right) - \delta_{t1}^A \perp q_{t1}^A \ge 0 \tag{11b}$$

$$0 \le -\pi_{t2} + (\tau_{12}^{Reg} + \tau_{t,12}) + \delta_{t1}^A \perp f_{t,12}^A \ge 0 \tag{11c}$$

$$0 \le -\pi_{t+1,2} + (\tau_{12}^{Reg} + \tau_{t,12}) + \omega^S + \delta_{t1}^A \perp b^A \ge 0 \tag{11d}$$

$$0 \le \rho^A + \left(\xi_{t,max}^A - \xi_{t,min}^A\right) - \delta_{t1}^A \perp E_{t1}^A \ge 0 \tag{11e}$$

$$0 \le v_{t1}^A q_{max}^A - q_{t1}^A \perp \lambda_{t,max}^A \ge 0 \tag{11f}$$

$$0 \le q_{t1}^A - v_{t1}^A q_{min}^A \perp \lambda_{t,min}^A \ge 0 \tag{11g}$$

$$0 \le y_{t1}^A E_{max}^A - E_{t1}^A \perp \xi_{t,max}^A \ge 0 \tag{11h}$$

$$0 \le E_{t1}^A - y_{t1}^A E_{min}^A \perp \xi_{t,min}^A \ge 0 \tag{11i}$$

$$0 \le y_{t1}^A - v_{t1}^A \perp l_{t1}^A \ge 0 \tag{11j}$$

$$0 \le \lambda_{t,max}^A q_{max}^A - \lambda_{t,min}^A q_{min}^A - l_{t1}^A \perp v_{t1}^A \ge 0 \tag{11k}$$

$$0 \le \xi_{t,max}^A E_{max}^A - \xi_{t,min}^A E_{min}^A + l_{t1}^A \perp y_{t1}^A \ge 0 \tag{11l}$$

$$0 \le 1 - v_{t1}^A \perp \eta_{t1}^A \ge 0, \tag{11m}$$

$$0 \le 1 - y_{t1}^A \perp \beta_{t1}^A \ge 0, \tag{11n}$$

$$0 = s_{t1}^A - q_{t1}^A - E_{t1}^A + f_{t,12}^A + b^A, \qquad \delta_{t1}^A \text{ free.} \tag{11o}$$

- Producers B at node 2.

$$0 \le -\pi_{t2} + \delta^B_{t2} \perp s^B_{t2} \ge 0, \tag{12a}$$
$$0 \le \gamma^B + \left(\lambda^B_{t,max} - \lambda^B_{t,min}\right) - \delta^B_{t2} \perp q^B_{t2} \ge 0, \tag{12b}$$
$$0 \le -\pi_{t+1,2} + \omega^S + \delta^B_{t2} \perp b^B \ge 0, \tag{12c}$$
$$0 \le \rho^B + \left(\xi^B_{t,max} - \xi^B_{t,min}\right) - \delta^B_{t2} \perp E^B_{t2} \ge 0, \tag{12d}$$
$$0 \le v^B_{t2} q^B_{max} - q^B_{t2} \perp \lambda^B_{t,max} \ge 0, \tag{12e}$$
$$0 \le q^B_{t2} - v^B_{t2} q^B_{min} \perp \lambda^B_{t,min} \ge 0, \tag{12f}$$
$$0 \le y^B_{t2} E^B_{max} - E^B_{t2} \perp \xi^B_{t,max} \ge 0, \tag{12g}$$
$$0 \le E^B_{t2} - y^B_{t2} E^B_{min} \perp \xi^B_{t,min} \ge 0, \tag{12h}$$
$$0 \le y^B_{t2} - v^B_{t2} \perp l^B_{t2} \ge 0 \tag{12i}$$
$$0 \le \lambda^B_{t,max} q^B_{max} - \lambda^B_{t,min} q^B_{min} - l^B_{t2} \perp v^B_{t1} \ge 0 \tag{12j}$$
$$0 \le \xi^B_{t,max} E^B_{max} - \xi^B_{t,min} E^B_{min} + l^B_{t2} \perp y^B_{t1} \ge 0 \tag{12k}$$
$$0 \le 1 - v^B_{t2} \perp \eta^B_{t2} \ge 0, \tag{12l}$$
$$0 \le 1 - y^B_{t2} \perp \beta^B_{t2} \ge 0, \tag{12m}$$
$$0 = s^B_{t2} + b^B - q^B_{t2} - E^B_{t2}, \qquad \delta^B_{t2} \text{ free.} \tag{12n}$$

- The TSO:

$$0 \le -\tau^{Reg}_{12} - \tau_{t,12} + \gamma^{TSO} + \epsilon_{t,12} \perp g_{t,12} \ge 0, \tag{13a}$$
$$0 \le \bar{g}_{12} - g_{t,12} \perp \epsilon_{t,12} \ge 0. \tag{13b}$$

- The storage operator:

$$0 \le -\omega^S - \omega^{Reg} + \gamma^S + \theta^S \perp h^S \ge 0, \tag{14a}$$
$$0 \le \sigma^S - h^S \perp \theta^S \ge 0. \tag{14b}$$

We carried out several tests for this example with demand function given by $D_{tn}(\pi_{tn}) = a_{tn} - b_{tn}\pi_{tn}$. The values of the other parameters used for these tests are as follows.

$$\begin{array}{rrrr}
\gamma^A = 7, & \gamma^B = 7, & \gamma^{TSO} = 1, & \gamma^S = 0.5; \\
a_{11} = 20, & a_{21} = 40, & a_{12} = 30, & a_{22} = 80; \\
b_{11} = 0.1, & b_{21} = 0.1, & b_{12} = 0.1, & b_{22} = 0.1; \\
q^A_{max} = 60, & q^A_{min} = 5, & q^B_{max} = 40, & q^B_{min} = 5; \\
E^A_{max} = 25, & E^A_{min} = 5, & E^B_{max} = 25, & E^B_{min} = 5; \\
\bar{g}_{12} = 30, & \tau^{Reg} = 0.5. & &
\end{array}$$

It can be seen from these parameters that the demand ($D_{tn}(\pi_{tn}) = a_{tn} - b_{tn}\pi_{tn}$) during the second time period is very high at node 2 ($a_{22} = 80$). After supplying up to 40 units of electricity to node 1, producer A can supply up to 20 units of electricity, which coupled with the production capacity of existing generators of B will still not be enough to satisfy the demand at a low cost[1]. Therefore, either production capacity expansion and/or storage may be needed for relatively low prices. In the following subsection, we report on numerical experiments whose output was used to analyse profitability and to value storage. We look at both cases where the producers can and cannot expand their production capacities along with storage activity.

4.2 Computational Results

It is important to point out that our computational experiments here consist of solving the BC-MLCP which includes of the KKT equations (11), (12), (13) (14) together with the market clearing conditions (9) and (10) after re-imposing the binary conditions on variables v_{t1}^A, v_{t2}^B, y_{t1}^A and y_{t2}^B. This BC-MLCP is solved using our previously developed algorithm (see (Fomeni et al. 2015)). It may or may not be the case that the players have an incentive to deviate from the solution obtained in this manner. Thus we can only say in general, without more specific assumptions, that a stationary point of the problem is sought.

We tested two scenarios to analyze the profitability of the storage operator. In the first test, we assume that there is no option for the producers to expand their generation capacities. In this test, it can easily be seen from the parameters chosen above that without storage, the price of electricity will be very high during the second time period as the demand is high ($a_{22} = 80$). Thus, storage is an obvious option to help stabilize the price of electricity. The results of this test are reported in Tables 4 and 5. We also consider various values for the capacity of storage because the impact and the profits of the storage operator vary with respect to its capacity σ^S. We report results for three values of the capacity of storage: $\sigma^S = 15$, $\sigma^S = 18$ and $\sigma^S = 30$. The results show that storage is indeed used at different levels for each of the values of σ^S. Looking at the resulting prices in Table 4, we can see that for lower values of σ^S, the prices of electricity during the second time period (π_{21} and π_{22}) can be very high, reflecting the fact that ($\sigma^S = 15$) there are not enough resources to meet the demand. However when the capacity of storage is enough to support the generation and transmission ($\sigma^S = 18, 30$), the prices are lower. Conversely, the results in Table 5 showing the profits of each player indicates that if the storage operator uses a large storage capacity, it may not make any profit at all.

One question therefore is: what is an "optimal" storage capacity that should be selected from both the player and the system perspective? In an attempt to answer this question, we experimented with various values of the storage capacity σ^S ranging from 0 to 20 and the results of the profits of Producers A, B and the storage operator are shown in the left hand side of Figure 4. On the right-hand side of Figure 4 is the behavior of the prices at each node during the second (on-peak) time period. Figure 4 shows that in the absence of sufficient storage resources, the prices during the second time period are quite high, thus allowing both producers to make high profits. However, when there is enough storage resources, both these profits and the prices go down. The most interesting part of this analysis is the profit made by the storage operator which has a concave shape, suggesting that there is a storage capacity that maximizes the profit made by the storage operator. One can see in Figure 4 that the value of the optimal storage capacity which provides the maximum profit is 10. However, the corresponding price of electricity during the second time period is quite

[1]Producer A: $20MW$, producer B: $40MW$. Demand: $80 - 0.1\pi_{22}$, so would need a price of $200\$/MWh$ to meet the demand.

[2]The variables $E_t^A, E_t^B, y_t^A, y_t^B$ are not reported here as they do not appear in the model solved for this part of our experiments

Table 4: Producer A and B cannot expand their generation capacities[2]

	Var.	$\sigma^S = 15$		$\sigma^S = 18$		$\sigma^S = 30$	
		t=1	t =2	t=1	t=2	t=1	t=2
Producer A	s^A	19.3	37.55	19.3	39.05	19.3	39.25
	q^A	23.5	60	26.5	60	26.9	60
	f^A	4.2	22.45	0.00	20.95	0	20.75
	b^A	0	NA	7.2	NA	7.6	NA
	λ^A_{max}	0	17.5	0	2.5	0	0.5
	λ^A_{min}	0	0	0	0	0	0
	v^A	1	1	1	1	1	1
	η^A	0	0	0	0	0	0
	δ^A	7	24.5	7	9.5	7	7.5
Producer B	s^B	25	40	29.2	40	29.2	40
	q^B	40	40	40	40	40	40
	b^B	15	NA	10.8	NA	10.8	NA
	λ^B_{max}	1	18.5	1	3.5	1	1.5
	λ^B_{min}	0	0	0	0	0	0
	v^B	1	1	1	1	1	1
	η^B	0	0	0	0	0	0
	δ^B	8	25.5	8	10.5	8	8.5
TSO	g_{12}	4.2	22.45	7.2	20.95	7.6	20.75
	ϵ_{12}	0	0	0	0	0	0
Sto. Ope.	h^S_t	15	NA	18	NA	18.4	NA
	θ^S_t	17	NA	2	NA	0	NA
Market	π_{t1}	7	24.5	7	9.5	7	7.5
	π_{t2}	8	25.5	8	10.5	8	8.5
	τ_{12}	0.5	0.5	0.5	0.5	0.5	0.5
	ω^S	17.5	NA	2.5	NA	0.5	NA

Table 5: Profit of each player when Producer A and B cannot expand their generation capacities

	Prod. A	Prod. B	TSO	Sto. Oper
$\sigma^S = 15$	1050	780	0	255
$\sigma^S = 18$	150	180	0	36
$\sigma^S = 30$	30	100	0	0

high. Therefore, if one is concerned about a capacity that stabilizes the price of electricity during the high-demand time period, a better value of the storage capacity is $\sigma^S = 18$, which guarantees profit for the storage operator and for which the prices are around 8.

We conducted a second set of experiments to analyze the profitability and the impact of storage in the network if both Producers A and B have the possibility to expand their generation capacities. Instead of paying for storage, the producers could decide to simply expand and generate the extra energy by themselves in order to meet the demand ($D_{tn}(\pi_{tn}) = a_{tn} - b_{tn}\pi_{tn}$) at low price and possibly make more profits. For this set of experiments, we considered the same parameters as above and experimented with different

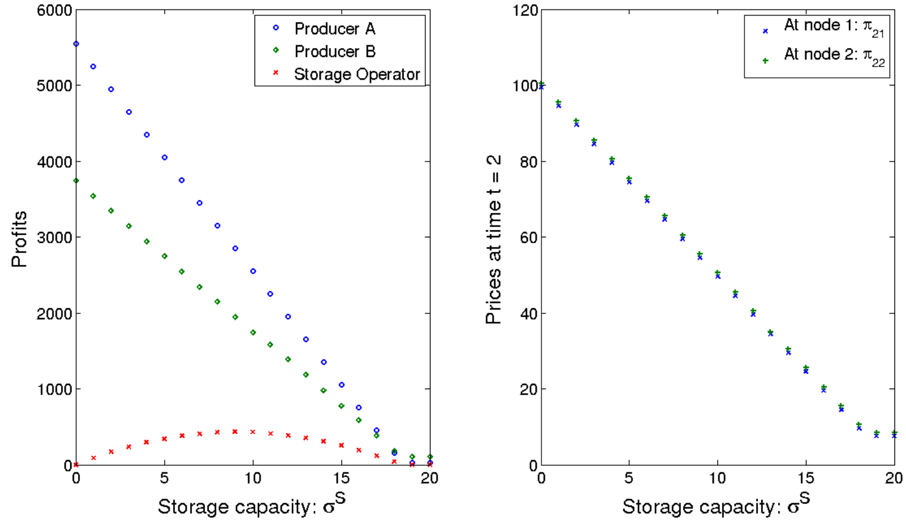

Figure 4: Profits and prices vs storage capacity

values for the unit costs of the expanded production units ρ^A and ρ^B. Thus, the unit cost of generating electricity for each producer is 7, the unit cost of storage is 0.5 and the unit cost of transporting electricity from node 1 to node 2 is 1. This means that, it would cost Producer B at least 7.5 to generate and store electricity, and it would cost Producer A at least 8 to sell electricity to node 2. Therefore we have investigated three possible value of ρ^A and ρ^B, namely $\rho^A = \rho^B = 7.4$, $\rho^A = \rho^B = 7.6$ and $\rho^A = \rho^B = 8.1$. These three values show us what will happen if expanding the generation capacities of the producers is respectively:

- less expensive than using storage,
- more expensive than using storage but less than the cost endured by Producer A for selling directly to node 2,
- more expensive than the above two points.

We present the results of this analysis in Tables 6 and 7. It can be seen in Table 6 that storage is consistently needed to support the supply of electricity and to keep the prices steady. One interesting thing to notice in Table 6 is that when $\rho^A = \rho^B = 7.4$, the new generator of Producer B is being used to generate 5 units of electricity at time $t = 1$ while the existing generator is not used to full capacity even though it is cheaper. This is simply because of the minimum output requirement of the new generator. The results in Table 7 suggest that the storage operator never makes a profit. However, further experiments shows that storage can be profitable if the appropriate storage capacity is chosen as also shown by the previous analysis. To support this, we have analyzed the profits of each player for different values of the storage capacity (σ^S) ranging from 0 to 20 when the costs of expansion (ρ^A and ρ^B) are set to 8.1. The results of this analysis are shown in Table 8. This table shows that for small values of $\sigma^S \leq 10$ there is a profit for the storage operator.

Table 6: Producers A and B can expand their generation capacity and $\sigma^S = 30$

	Var.	$\rho^A = \rho^B = 7.4$		$\rho^A = \rho^B = 7.6$		$\rho^A = \rho^B = 8.1$	
		t=1	t =2	t=1	t=2	t=1	t=2
Producer A	s^A	19.3	39.3	19.3	39.3	19.3	39.25
	q^A	19.3	39.3	19.3	39.3	26.9	60
	f^A	0.00	0.00	0.00	0.00	7.6	20.75
	b^A	0.00	N/A	0.00	N/A	0.00	N/A
	E^A	0.00	0.00	0.00	0.00	0.00	0.00
	v^A	1	1	1	1	1	1
	y^A	0	0	0	0	0	0
	δ^A	7	7	7	7	7	7.5
Producer B	s^B	29.3	65	29.29	63.53	21.6	40
	q^B	38.55	40	40	40	40	40
	b^B	14.25	N/A	15.71	N/A	18.4	N/A
	E^B	5	25	5.0	23.53	0.00	0
	v^B	1	1	1	1	1	1
	y^B	1	1	1	1	0	0
	δ^B	7	7.5	7.1	7.6	8	8.5
TSO	g_{12}	0.00	0.00	0.00	0.00	7.60	20.75
	ϵ_{12}	0.00	0.00	0.00	0.00	0.00	0.00
Sto. Ope.	h_t^S	14.25	N/A	15.71	N/A	18.4	N/A
	θ_t^S	0.00	N/A	0.00	N/A	0.00	N/A
Market	π_{t1}	7	7	7	7	7	7.5
	π_{t2}	7	7.5	7.1	7.6	8	8.5
	τ_{12}	0.5	0.5	0.5	0.1	0.5	0.5
	ω^S	0.5	N/A	0.5	N/A	0.5	N/A

That is because storage is indeed needed but not providing enough capacity and therefore the price of using storage is high due to congestion. When the value of σ^S is between 11 and 16, there is no profit for the storage operator. However, we can note that the amount of storage used is the same (10.76) for all these values. This suggests that the expanded production units are needed to meet the demand, since the available storage capacity is not able to fully support the transmission. Thus, when σ^S is around 18, it changes the dynamic and the storage re-become profitable, simply because the producer are no longer using the expensive (new) generators. This can also be seen in Table 6, when $\rho^A = \rho^B = 8.1$ the amount of power stored is 18.4 and none of the producers are using the new generators.

Table 7: Profit of each player when Producer A and B can expand their generation capacities

	Prod. A	Prod. B	TSO	Sto. Oper
$\rho^A = \rho^B = 7.4$	0	20.50	0	0.00
$\rho^A = \rho^B = 7.6$	0	25.50	0	0.00
$\rho^A = \rho^B = 8.1$	30.00	100.00	0	0.00

Table 8: Different storage capacities with their associated profits when $\rho^A = \rho^B = 8.1$

	Profits								
σ^S	A	B	TSO	Storage	π_{11}	π_{21}	π_{12}	π_{22}	h^S
0	6	44	0	0	7	7.1	7	8.1	0
1	6	44	0	0.6	7	7.1	7	8.1	1
2	6	44	0	1.2	7	7.1	7	8.1	2
3	6	44	0	1.8	7	7.1	7	8.1	3
4	6	44	0	2.4	7	7.1	7	8.1	4
5	6	44	0	3	7	7.1	7	8.1	5
6	6	44	0	3.6	7	7.1	7	8.1	6
7	6	44	0	4.2	7	7.1	7	8.1	7
8	6	44	0	4.8	7	7.1	7	8.1	8
9	6	44	0	5.4	7	7.1	7	8.1	9
10	6	44	0	6	7	7.1	7	8.1	10
11	6	68	0	0	7	7.1	7.6	8.1	10.76
12	6	68	0	0	7	7.1	7.6	8.1	10.76
13	6	68	0	0	7	7.1	7.6	8.1	10.76
14	6	68	0	0	7	7.1	7.6	8.1	10.76
15	6	68	0	0	7	7.1	7.6	8.1	10.76
16	6	68	0	0	7	7.1	7.6	8.1	10.76
17	450	380	0	119	7	14.5	8	15.5	17
18	150	180	0	36	7	9.5	8	10.5	18
19	30	100	0	0	7	7.5	8	8.5	18.4
20	30	100	0	0	7	7.5	8	8.5	18.4

5 Conclusion

In this paper we presented a study of the equilibria in traffic network and power system network with storage in the presence of logic constraints. These constraints are modeled as binary variables which are added to standard complementarity-based equilibrium models. For example in the traffic equilibrium, we showed how binary variables can be introduced to model varying degrees of equity in the flow of traffic. While in power system networks with storage, we used binary variables to model physically-based non-convexities such as the minimum power outputs of production units.

With regards to the traffic equilibrium problem, we have observed that the equilibrium solution provided by complementarity models alone may suggest that the flows along two or more paths of the same OD pairs are largely disproportionate. We have therefore shown that logic constraints can be added to these models to control the flows on different paths without increasing the travel costs of the vehicles. Our computational results showed that the addition of these logic constraints to traffic equilibrium models can achieve more equitable equilibrium.

We also studied the equilibrium problem of power markets with the addition of a storage operator. We have analyzed some benefits of including a storage operator in a power system network with the role of storing electricity during the off-peak demand periods and releases it during the peak demand periods to be sold by the producers. Our analyses showed that the presence of a storage operator in the market is beneficial in terms of stabilizing the price of electricity during on-peak demand periods and in terms of supporting the transmission in meeting the demand satisfaction at all time. The analysis also suggests that it can be profitable for a storage investor to operate as a service provider. However, it is important to point out that the example for the power network application is purely illustrative as it consists of only two nodes, two time periods, one storage unit, two producers, and a linear representation of the network. A more realistic example would not only be larger but also account for power losses as well as the levels of charge and discharge of storage during each time period. Nevertheless the positive results reported in this paper are motivation for carrying out a more detailed study.

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