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MACDONALD POSITIVITY VIA THE HARISH-CHANDRA D-MODULE

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ABSTRACT. Using the Harish-Chandra D-module, we give a proof of Haiman's theorem on the positivity of Macdonald polynomials. Ginzburg's work on the connection between this D-module and the isospectral commuting variety is fundamental to this approach.

1. INTRODUCTION

The (transformed) Macdonald polynomials $\tilde{H}_\mu(z;q,t)$ are symmetric functions with coefficients that are rational functions of two parameters q and t. They have remarkable specialisations to important families of symmetric functions including Hall-Littlewood polynomials, Jack polynomials and Schur functions.

Expanding the Macdonald polynomials in terms of Schur functions,

$$\tilde{H}_\mu(z;q,t) = \sum_\lambda \tilde{K}_{\lambda,\mu}(q,t)s_\lambda(z),$$

Macdonald conjectured that the coefficients $\tilde{K}_{\lambda,\mu}(q,t)$ belong to $\mathbb{N}[q,t]$. In a wonderful paper, [7], Haiman confirmed this conjecture by proving the $n!$ theorem. This showed the existence of a vector bundle $\tilde{\mathcal{P}}$ on $\mathsf{Hilb}^n\mathbb{C}^2$, the Hilbert scheme of points on the plane, with many remarkable properties. In particular, the fibres of $\tilde{\mathcal{P}}$ at the torus fixed points of $\mathsf{Hilb}^n\mathbb{C}^2$ are bigraded representations of $\mathfrak{S}_n$ encoding the Macdonald polynomials. Haiman's proof of the $n!$ theorem is a remarkable blend of sophisticated algebraic geometry and subtle combinatorics.

In this note we give a different proof of Macdonald positivity using recent work of Ginzburg, [4]. This proof again displays a vector bundle on $\mathsf{Hilb}^n\mathbb{C}^2$ whose fibres at torus fixed points carry the Macdonald polynomials. The bundle is constructed from a degeneration of the Harish-Chandra D-module on the Grothendieck-Springer resolution of type A_{n-1}; to describe its fibres requires only standard constructions from D-module theory and the Springer correspondence. It should be noted that in [4] Ginzburg showed that this bundle is isomorphic to $\tilde{\mathcal{P}}$ *if* one assumes Haiman's results. We do not know if it is possible to give a new proof of the $n!$ theorem along similar lines.

Following Haiman's pioneering work there have been two recent proofs of generalisations of Macdonald positivity, [1] and [5]. These are of a different flavour to this note.

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2. Positivity

Let V be an n-dimensional complex vector space, $G = GL(V)$ with Lie algebra $\mathfrak{g} = \mathfrak{gl}(V)$, and set $\mathfrak{t}$ to be the subalgebra of $\mathfrak{g}$ consisting of diagonal matrices. Let $B \leq G$ be the Borel subgroup of upper triangular matrices, with Lie algebra $\mathfrak{b}$. The Weyl group, $W = \mathfrak{S}_n$, acts on $\mathfrak{t}$. We will identify $\mathfrak{g}$ and $\mathfrak{t}$ with $\mathfrak{g}^*$ and $\mathfrak{t}^*$ via the trace pairing.

Let $\kappa : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ be the commutator. The commuting variety, $\mathfrak{C}$, is the scheme-theoretic fibre $\kappa^{-1}(0)$. Set $\mathfrak{T} = \mathfrak{t} \times \mathfrak{t}$. Simultaneous conjugation provides an action of G on $\mathfrak{C}$ such that the algebraic geometric quotient $\mathfrak{C}/G$ is isomorphic to $\mathfrak{T}/W$, see [2, Theorem 1.3]. Let $\mathfrak{X} = [\mathfrak{C} \times_{\mathfrak{T}/W} \mathfrak{T}]_{\text{red}}$, the reduced *isospectral commuting variety*, and let $\mathfrak{X}_{\text{norm}}$ be its normalisation with morphism $\psi : \mathfrak{X}_{\text{norm}} \longrightarrow \mathfrak{X}$. There is a projection morphism $p_{\mathfrak{C}} : \mathfrak{X} \longrightarrow \mathfrak{C}$ and an induced morphism on the normalisations $p : \mathfrak{X}_{\text{norm}} \longrightarrow \mathfrak{C}_{\text{norm}}$.

There is an action of G on $\mathfrak{X}$ induced from $\mathfrak{C}$, of $\mathbb{C}^* \times \mathbb{C}^*$ by dilation in both sets of variables, and of W from the diagonal action on $\mathfrak{T}$. All these lift to $\mathfrak{X}_{\text{norm}}$.

Let $\tilde{\mathfrak{g}} = G \times_B \mathfrak{b}$ be the Grothendieck-Springer resolution. It admits morphisms $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ and $\nu : \tilde{\mathfrak{g}} \to \mathfrak{t}$ defined by $(g, x) \mapsto gxg^{-1}$, respectively $(g, x) \mapsto x \bmod [\mathfrak{b}, \mathfrak{b}]$. Let $\mathcal{M} = \int_{\mu \times \nu} \mathcal{O}_{\tilde{\mathfrak{g}}}$, the *Harish-Chandra $D_{\mathfrak{g} \times \mathfrak{t}}$-module.* It is holonomic.

Theorem 1. [4, Theorem 1.3.3, Theorem 1.3.4, Theorem 1.5.2]

(1) *There is a filtration on $\mathcal{M}$, the Hodge filtration, such that* $\operatorname{gr} \mathcal{M} \cong \psi_* \mathcal{O}_{\mathfrak{X}_{norm}}$.
(2) *$\mathfrak{X}_{norm}$ is Cohen-Macaulay and Gorenstein.*
(3) *Set $\mathcal{R} = p_* \mathcal{O}_{\mathfrak{X}_{norm}}$. Over the smooth locus of $\mathfrak{C}$, $\mathcal{R}$ is a $G \times W \times \mathbb{C}^* \times \mathbb{C}^*$-equivariant vector bundle whose fibres carry the regular representation of W.*

Let $\mathcal{S} = \{(X, Y, v) \in \mathfrak{g} \times \mathfrak{g} \times V : [X, Y] = 0,\ \mathbb{C}\langle X, Y\rangle v = V\}$. The action of G on $\mathcal{S}$ is free, and its quotient is $\mathsf{Hilb}^n\mathbb{C}^2$, the Hilbert scheme of n points on the plane. The $\mathbb{C}^* \times \mathbb{C}^*$-action on $\mathsf{Hilb}^n\mathbb{C}^2$ has a finite number of fixed points, I_μ, labelled by partitions of n, see for instance [7, §3.2].

The projection morphism from $\mathcal{S}$ to $\mathfrak{g} \times \mathfrak{g}$ has image $\mathfrak{C}^\circ$, the set of pairs $(X, Y) \in \mathfrak{C}$ that have a cyclic vector. This makes $\mathcal{S}$ a torsor over $\mathfrak{C}^\circ$.

Since $\mathfrak{C}^\circ$ is smooth we may define an open set $\mathfrak{X}^\circ = p^{-1}(\mathfrak{C}^\circ)$ in $\mathfrak{X}_{\text{norm}}$ and then set $\mathfrak{W} = (\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S})/G$. We have the following diagram, see [4, (8.2.1)]

$$\begin{array}{ccccccc}
\mathfrak{X}^\circ & \overset{\tilde{\delta}}{\twoheadleftarrow} & \mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S} & \xrightarrow[\cong]{h} & \mathfrak{W} \times_{\mathsf{Hilb}^n\mathbb{C}^2} \mathcal{S} & \overset{\tilde{\rho}}{\twoheadrightarrow} & \mathfrak{W} \\
& \searrow^{p} & & \searrow^{\tilde{p}} \quad \swarrow^{\tilde{\eta}} & & & \swarrow^{\eta} \\
& & \mathfrak{C}^\circ & \underset{\delta}{\twoheadleftarrow} \ \mathcal{S} \ \underset{\rho}{\twoheadrightarrow} & \mathsf{Hilb}^n\mathbb{C}^2 & &
\end{array}$$

Set $\mathcal{P} = (\rho_* \delta^* (\mathcal{R}|_{\mathfrak{C}^\circ}))^G$. By [4, Corollary 8.1.3] this is a $W \times \mathbb{C}^* \times \mathbb{C}^*$-equivariant vector bundle on $\mathsf{Hilb}^n\mathbb{C}^2$ whose fibres carry the regular representation of W. It is shown in [4, §8.2] that $\mathfrak{W}$ is isomorphic to the relative spectrum of $\mathcal{P}$, so $\mathcal{P} \cong \eta_* \mathcal{O}_{\mathfrak{W}}$.

The *transformed Macdonald polynomials* $\tilde{H}_\mu(z;q,t)$ are two parameter symmetric functions attached to partitions μ. They may be characterised by the following conditions in the ring of symmetric functions over the base field $\mathbb{Q}(q,t)$, [8, Definition 3.5.2].

(Mi) $\tilde{H}_\mu[(1-q)Z;q,t] \in \mathbb{Q}(q,t)\{s_\lambda(z) : \lambda \geq \mu\}$

(Mii) $\tilde{H}_\mu[(1-t)Z;q,t] \in \mathbb{Q}(q,t)\{s_\lambda(z) : \lambda \geq \mu^t\}$

(Miii) $\tilde{H}_\mu[1;q,t] = 1$.

Here $s_\lambda(z)$ is the Schur function attached to the partition λ, $\geq$ is the dominance ordering on partiations, and the $[\cdot\cdot]$ denotes plethystic substitution, see [8, §3.3].

The following theorem gives another proof of Macdonald positivity. This was proved first by Haiman in [7], and subsequently in [1] and [5]. We do not assert here that $\mathcal{P}$ is the Procesi bundle, although that does follow from the work of Haiman and Ginzburg, see [4, Corollary 8.2.5]. Recall the Frobenius characteristic is the unique linear map from the representation ring of $\mathfrak{S}_n$ to symmetric functions, sending the irreducible representation λ to the Schur function $s_\lambda(z)$, see [8, §3.2].

Theorem 2. *Let $\mathcal{P}(I_\mu)$ be the fibre of $\mathcal{P}$ above $I_\mu \in \mathsf{Hilb}^n\mathbb{C}^2$, which by the above carries a $W \times \mathbb{C}^* \times \mathbb{C}^*$-action. The Frobenius characteristic $F_{\mathcal{P}(I_\mu)}(z;q,t)$ equals $\tilde{H}_\mu(z;q,t)$.*

The proof of this will occupy the rest of this note. It proceeds in a similar way to the tactic of Haiman's own proof, using however basic facts about D-modules.

Any function in $\mathcal{O}(\mathfrak{T})$ pulls back to a regular function on $\mathfrak{X}_{\mathrm{norm}}$, and by construction these functions are invariant under the action of G. Thus the functions in $\mathcal{O}(\mathfrak{T})$ give rise to functions on $\mathfrak{W}$ and hence an action on $\mathcal{P}$. Let $y_1, \ldots, y_n$ be a basis of linear functionals on $\mathfrak{t} \times \{0\} \subset \mathfrak{T}$.

Claim 1. *The elements $y_1, \ldots, y_n$ are a regular sequence at any point in $\mathfrak{W}$ at which they vanish.*

Proof. Let $I = (y_1, \ldots, y_n)$ be the ideal of $\mathcal{O}_{\mathfrak{W}}$ generated by the y_i's. Thanks to [4, Proposition 3.2.4] $\mathfrak{W}$ is Cohen-Macaulay. Hence it is enough to show that $\operatorname{codim} I = n$. This follows just as in [7, Proposition 3.3.3], for instance. □

In [4, Proposition 3.2.4] it is shown that $\mathfrak{W} \cong [\mathsf{Hilb}^n\mathbb{C}^2 \times_{\mathfrak{T}/W} \mathfrak{T}]_{\text{red, norm}}$. Since the support of $I_\mu \in \mathsf{Hilb}^n\mathbb{C}^2$ is concentrated at the origin of $\mathfrak{T}/W$, there is a unique point $(I_\mu, 0) \in [\mathsf{Hilb}^n\mathbb{C}^2 \times_{\mathfrak{T}/W} \mathfrak{T}]_{\text{red}}$ lying above $I_\mu \in \mathsf{Hilb}^n\mathbb{C}^2$ and we let $\mathcal{J}_\mu$ be the corresponding maximal ideal sheaf. Let $A = \mathcal{O}_{[\mathsf{Hilb}^n\mathbb{C}^2 \times_{\mathfrak{T}/W} \mathfrak{T}]_{\text{red}}}$ and $B = \mathcal{O}_{\mathfrak{W}}$. We now know that $(y_1, \ldots, y_n)$ is a regular sequence in $({}_AB)_{\mathcal{J}_\mu}$. It follows that $({}_AB)_{\mathcal{J}_\mu}/(y_1, \ldots, y_n)({}_AB)_{\mathcal{J}_\mu}$ admits a Koszul resolution, and hence by [8, Proposition 3.3.1] that we have an equality of Frobenius characteristics

$$F_{({}_AB)_{\mathcal{J}_\mu}}([1-q]Z;q,t) = F_{({}_AB)_{\mathcal{J}_\mu}/(y_1,\ldots,y_n)({}_AB)_{\mathcal{J}_\mu}}(z;q,t).$$

Since $\eta : \mathfrak{W} \longrightarrow \mathsf{Hilb}^n\mathbb{C}^2$ factors through $[\mathsf{Hilb}^n\mathbb{C}^2 \times_{\mathfrak{T}/W} \mathfrak{T}]_{\text{red}}$, the stalk $\mathcal{P}_\mu$ of $\mathcal{P}$ at I_μ equals $({}_AB)_{\mathcal{J}_\mu}$. By freeness $F_{\mathcal{P}_\mu}(z;q,t) = F_{\mathcal{P}(I_\mu)}(z;q,t)p_\mu(q,t)$ where $p_\mu(q,t) \in \mathbb{Q}(q,t)$ is the bigraded Poincaré series for the local ring of $\mathsf{Hilb}^n\mathbb{C}^2$ at the point I_μ. It follows that

$$F_{\mathcal{P}(I_\mu)}([1-q]Z;q,t) = F_{\mathcal{P}_\mu}([1-q]Z;q,t)p_\mu(q,t) = F_{({}_AB)_{\mathcal{J}_\mu}/(y_1,\ldots,y_n)({}_AB)_{\mathcal{J}_\mu}}(z;q,t)p_\mu(q,t).$$

Therefore to check (Mi), we need only show that

$$F_{({}_AB)_{\mathcal{J}_\mu}/(y_1,\dots,y_n)({}_AB)_{\mathcal{J}_\mu}}(z;q,t) \in \mathbb{Q}(q,t)\{s_\lambda(z) : \lambda \geq \mu\}.$$

By [6, Proposition 5.3] this is implied by the following.

Claim 2. *The λ isotypic component of $({}_AB)_{\mathcal{J}_\mu}/(y_1,\dots,y_n)({}_AB)_{\mathcal{J}_\mu}$ is zero unless $\lambda \geq \mu$.*

Proof. Since $\mathfrak{C}^\circ$ belongs to smooth locus of $\mathfrak{C}$, the restriction of $p : \mathfrak{X}_{\text{norm}} \longrightarrow \mathfrak{C}_{\text{norm}}$ to $\mathfrak{X}^\circ$ factors through $\mathfrak{X}$, that is $p|_{\mathfrak{X}^\circ} = (p_{\mathfrak{C}} \circ \psi)|_{\mathfrak{X}^\circ}$. It follows that

$$\mathcal{R}|_{\mathfrak{C}^\circ} = p_*\left(\mathcal{O}_{\mathfrak{X}_{\text{norm}}}|_{\mathfrak{X}^\circ}\right) = (p_{\mathfrak{C}})_*\left((\operatorname{gr}\mathcal{M})|_{p_{\mathfrak{C}}^{-1}(\mathfrak{C}^\circ)}\right).$$

Now let (X_μ, Y_μ) be an element in the principal nilpotent pair orbit corresponding to μ, see [3, (0.1)]. We deduce that the stalk of $\mathcal{R}$ above (X_μ, Y_μ) equals $(\operatorname{gr}\mathcal{M})_{K_\mu}$ where K_μ is the maximal ideal of $(X_\mu, Y_\mu, 0, 0)$, the unique point in $\mathfrak{X}$ lying over (X_μ, Y_μ).

Let $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}\times\mathfrak{t}$ be the inclusion that sends X to $(X, 0)$. Define

$$T^*(\mathfrak{g}) = \mathfrak{g}\times\mathfrak{g}^* \xleftarrow{\rho_\pi} \mathfrak{g}\times_{\mathfrak{g}\times\mathfrak{t}} T^*(\mathfrak{g}\times\mathfrak{t}) = \mathfrak{g}\times\mathfrak{g}^*\times\mathfrak{t}^* \xrightarrow{\varpi_\pi} T^*(\mathfrak{g}\times\mathfrak{t}) = \mathfrak{g}\times\mathfrak{g}^*\times\mathfrak{t}\times\mathfrak{t}^*$$

by $\rho_\pi(X,Y,w) = (X,Y)$ and $\varpi_\pi(X,Y,w) = (X,Y,0,w)$. We set $T^*_{\mathfrak{g}}(\mathfrak{g}\times\mathfrak{t}) = \rho_\pi^{-1}(T^*_{\mathfrak{g}}(\mathfrak{g})) = \mathfrak{g}\times\{0\}\times\mathfrak{t}^*$. The characteristic variety of $\mathcal{M}$ is $\operatorname{Ch}(\mathcal{M}) = [\mathfrak{X}]$, [4, Corollary 2.4.1]. Now

$$\begin{aligned}\varpi_\pi^{-1}(\mathfrak{X})\cap T^*_{\mathfrak{g}}(\mathfrak{g}\times\mathfrak{t}) &= \{(X,Y,w) : [X,Y]=0, X \text{ nilpotent, e-vals}(Y)=w\}\cap\{(X,0,w)\} \\ &= \{(X,0,0) : X \text{ nilpotent}\} \subset \mathfrak{g}\times\{0\}\times\{0\} = \mathfrak{g}\times_{\mathfrak{g}\times\mathfrak{t}} T^*_{\mathfrak{g}\times\mathfrak{t}}(\mathfrak{g}\times\mathfrak{t}).\end{aligned}$$

Thus π is non-characteristic with respect to $\mathcal{M}$. In particular we deduce from [10, Theorem 4.7] that $\operatorname{Ch}(\pi^*\mathcal{M}) = \rho_\pi\omega_\pi^{-1}(\operatorname{Ch}(\mathcal{M})) = \{(X,Y) : [X,Y]=0, X \text{ nilpotent}\} \subset \mathfrak{C}$. In fact, the $y_1,\dots,y_n$ form a regular sequence for $\operatorname{gr}\mathcal{M}$ by [4, Proposition 9.1.3], so multiplication by each y_i on $\operatorname{gr}\mathcal{M}/(y_1,\dots y_{i-1})\operatorname{gr}\mathcal{M}$ is injective, and iterating the proof of Step 1 of [10, Theorem 4.7] shows that $(\rho_\pi)_*\varpi_\pi^*(\operatorname{gr}\mathcal{M})$ is isomorphic to $\operatorname{gr}\pi^*\mathcal{M}$.

The support of $(\rho_\pi)_*\varpi_\pi^*(\operatorname{gr}\mathcal{M})$ is $\{(X,Y) : [X,Y]=0, X \text{ nilpotent}\}$. Since $\mathcal{M}$ is holonomic this space is lagrangian in $T^*(\mathfrak{g})$, a union of conormal bundles $\bigcup_\lambda \overline{T^*_{\mathcal{O}_\lambda}(\mathfrak{g})}$, where $\mathcal{O}_\lambda$ denotes the nilpotent orbit in $\mathfrak{g}$ of type λ. The D-module $\mathcal{M}$ carries a W-action, [9, §5] and this induces the W-action that is inherited by $\mathcal{R}$ in the statement of Theorem 1(3). The λ-isotypic component of the stalk of $\mathcal{R}|_{\mathfrak{C}^\circ}/(y_1,\dots,y_n)\mathcal{R}|_{\mathfrak{C}^\circ}$ at (X_μ, Y_μ) is non-zero if and only if (X_μ, Y_μ) is in the support of the λ-isotypic component of $(\rho_\pi)_*\varpi_\pi^*(\operatorname{gr}\mathcal{M})$.

We have a decomposition $\pi^*\mathcal{M} = \bigoplus_\lambda(\pi^*\mathcal{M})_\lambda$. We've seen above that the support of $\operatorname{gr}(\pi^*\mathcal{M})_\lambda$ equals the support of the λ-isotypic component of $(\rho_\pi)_*\varpi_\pi^*(\operatorname{gr}\mathcal{M})$. By [9, Proposition 4.8.1 and Theorem 5.3(3)], $(\pi^*\mathcal{M})_\lambda$ is supported on the closure of the nilpotent orbit $\mathcal{O}_\lambda$, and so $\operatorname{Ch}((\pi^*\mathcal{M})_\lambda) \subseteq \bigcup_{\nu\leq\lambda}\overline{T^*_{\mathcal{O}_\nu}\mathfrak{g}}$. Thus the λ-isotypic component of the stalk of $\mathcal{R}|_{\mathfrak{C}^\circ}/(y_1,\dots,y_n)\mathcal{R}|_{\mathfrak{C}^\circ}$ at (X_μ, Y_μ) is non-zero only if $(X_\mu, Y_\mu) \in \overline{T^*_{\mathcal{O}_\nu}(\mathfrak{g})}$ for some $\nu \leq \lambda$. But since $X_\mu \in \mathcal{O}_\mu$, this in turn requires that $\mu \leq \nu$. So we deduce that the stalk at (X_μ, Y_μ) of the λ-isotypic component of $\mathcal{R}|_{\mathfrak{C}^\circ}/(y_1,\dots,y_n)\mathcal{R}|_{\mathfrak{C}^\circ}$ is non-zero only if $\mu \leq \lambda$.

Given any $s \in \mathcal{S}$ we have by definition

$$(\mathcal{P}/(y_1,\ldots,y_n)\mathcal{P})_{\rho(s)} \otimes_{\mathcal{O}_{\mathsf{Hilb}^n\mathbb{C}^2,\rho(s)}} \mathcal{O}_{\mathcal{S},s} \cong (\mathcal{R}|_{\mathfrak{C}^\circ}/(y_1,\ldots,y_n)\mathcal{R}|_{\mathfrak{C}^\circ})_{\delta(s)} \otimes_{\mathcal{O}_{\mathfrak{C}^\circ,\delta(s)}} \mathcal{O}_{\mathcal{S},s}.$$

If $s \in \delta^{-1}(X_\mu, Y_\mu)$ then $\rho(s) = I_\mu$ and it follows that the λ-isotypic component of $\mathcal{P}_\mu/(y_1,\ldots,y_n)\mathcal{P}_\mu$ is non-zero only if $\mu \leq \lambda$. Since $({}_AB)_{\mathcal{J}_\mu}/(y_1,\ldots,y_n)({}_AB)_{\mathcal{J}_\mu} = \mathcal{P}_\mu/(y_1,\ldots,y_n)\mathcal{P}_\mu$, this proves our claim. □

To deal with (Mii) we argue similarly, reducing the calculations about $\mathcal{P}$ to ones on $\mathfrak{X}$. We need to factor out a basis $z_1,\ldots,z_n$ of $\mathfrak{t}^*$. To see this is a regular sequence observe first that there is an automorphism of $\mathfrak{X}$ induced by interchanging $\mathfrak{g}\times\mathfrak{t}$ with $\mathfrak{g}^*\times\mathfrak{t}^*$. This induces an automorphism of the normalisation $\mathfrak{X}_{\text{norm}}$ and we see that $z_1,\ldots,z_n$ is a regular sequence since $y_1,\ldots,y_n$ is. Now recall that $(Y_\mu, X_\mu) = (X_{\mu^t}, Y_{\mu^t})$. Thus we deduce that the λ-isotypic component of $\mathcal{P}_\mu/(z_1,\ldots,z_n)\mathcal{P}_\mu$ is non-zero only if $\mu^t \leq \lambda$. This implies (Mii).

Condition (Miii) states that the trivial representation appears in $\mathbb{C}^*\times\mathbb{C}^*$-bidegree $(0,0)$ and nowhere else. But since $\mathcal{P}_\mu$ carries the regular representation of W and the trivial isotypic component is spanned by the constant functions, this is immediate.

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