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DEFORMATION QUANTISATION FOR (-2) -SHIFTED SYMPLECTIC STRUCTURES

J.P.PRIDHAM

ABSTRACT. We formulate a notion of E_{-1} quantisation of (-2) -shifted Poisson structures on derived algebraic stacks, depending on a flat right connection on the structure sheaf, as solutions of a quantum master equation. We then parametrise E_{-1} quantisations of (-2) -shifted symplectic structures by constructing a map to power series in de Rham cohomology. For a large class of examples, we show that these quantisations give rise to classes in Borel–Moore homology which are closely related to Borisov–Joyce invariants.

INTRODUCTION

Shifted symplectic structures in derived algebraic geometry were introduced in [PTVV], and shifted Poisson structures, together with the correspondence between shifted symplectic structures and non-degenerate shifted Poisson structures, in [Pri3, CPT⁺]. For $n \geq 1$, deformation quantisation of n -shifted Poisson structures is an immediate consequence of formality of the E_{n+1} operad, as observed in [CPT⁺]. For $n = 0$ and $n = -1$, deformation quantisation of n -shifted Poisson structures is more subtle, but was investigated and largely established in [Pri4, Pri2, Pri1]; we now look at the case $n = -2$.

By deformation quantisation of a derived scheme or stack X over R , we should mean some form of non-commutative formal deformation of the structure sheaf over $R[[\hbar]]$, for \hbar an element of homological degree 0. In particular, this excludes the red shift quantisations proposed in [CPT⁺]. Meanwhile the structures enhancing fundamental classes in [BBD⁺, BJ] are naturally defined over $R((\hbar))$, and should be recovered by localising deformation quantisations away from $\hbar = 0$ (cf. [Pri1, §4.2] for the $n = -1$ case).

For $n \geq -1$, an n -shifted quantisation is a (Beilinson–Drinfeld) BD_{n+1} -algebra. For $n \geq 0$, this is a filtered, almost commutative E_{n+1} -algebra deforming the P_{n+1} -algebra given by the Poisson structure. The category of modules over such an algebra is an n -tuply monoidal linear category, so $n = 0$ just gives a linear category. The case $n = -1$ concerns BD_0 -algebras, which are filtered, almost commutative Batalin–Vilkovisky (BV)-algebras. These are just objects in a category, and a (-2) -shifted quantisation will just be an element of an object. Since the hierarchy of BD_{n+1} -algebras has petered out by $n = -2$, we make use of the observation that for $n \geq -1$, n -shifted quantisations are parametrised by Maurer–Cartan elements of a natural BD_{n+2} -algebra (given by differential operators or Hochschild complexes) deforming the P_{n+2} -algebra of shifted polyvectors.

For $n = -2$, we thus consider the BD_0 -algebra given by the Hodge filtration on the right de Rham complex of \mathcal{O}_X associated to a flat right connection on \mathcal{O}_X , and formulate (Definition 1.16) deformation quantisations of (-2) -shifted Poisson structures as Maurer–Cartan elements of an associated BV -algebra, i.e. as solutions of the quantum

master equation. On inverting \hbar , this gives a Laurent series of cohomology classes in a right de Rham complex.

Our main results are Propositions 1.37 and 1.45, and their global analogues Proposition 3.1 and §refArtinsn. Proposition 1.37 parametrises E_{-1} -quantisations in terms of first-order quantisations and power series in de Rham cohomology; in particular, it shows that the only obstruction to quantising a (-2) -shifted Poisson structure is first order. The strategy of proof is adapted from [Pri1], involving a notion of compatibility between (-2) -shifted Poisson structures and de Rham power series. Proposition 1.45 then shows that if there exist any flat right connections on \mathcal{O}_X , then there is an essentially unique flat right connection admitting first-order quantisations of a given non-degenerate (-2) -shifted Poisson structure.

Right de Rham cohomology of the dualising complex is just Borel–Moore homology. However, for (-2) -shifted symplectic derived schemes, the dualising complex is seldom a line bundle, so generating fundamental classes from our quantisations is not just a matter of choosing orientation data. Instead, in §2 we look at toy models in the form of dg manifolds with strict shifted symplectic structures, where we are able to equate the right de Rham cohomology groups of \mathcal{O}_X and ω_X given orientation data. Over \mathbb{C} , we then show (Corollary 2.13) that for our quantisations S the images in Steenrod homology of the associated classes $[\exp(S)]$ are given by

$$\hbar^{(\dim X)/2}[X]_{BJ} \cdot (1 + \hbar\mathbb{C}[[\hbar]]) \subset H_{\dim X}^{\text{St}}(\pi^0 X(\mathbb{C})_{\text{an}}, \mathbb{C}[[\hbar]]),$$

where $\dim X$ is the virtual dimension of X , $[X]_{BJ}$ is the Borisov–Joyce virtual fundamental class $[X_{dm}]_{\text{virt}}$ of [BJ, Corollary 3.19], and $\pi^0 X(\mathbb{C})_{\text{an}}$ is the space given by the analytic topology on the \mathbb{C} -points of X .

It is important to note, however, that the quantisation itself is a richer structure than a cohomology class, because of restrictions in terms of the Hodge filtration. In particular, shifted Poisson structures can be recovered directly from our E_{-1} quantisations, and the space of homotopy classes of quantisations does not have an abelian structure.

I would like to thank Dominic Joyce for helpful comments on Borisov–Joyce invariants.

0.1. Notation. We denote the underlying graded module of a cochain complex (resp. chain complex) by $M^\#$ (resp. $M_\#$).

Given a differential graded associative algebra (DGAA) A , and A -modules M, N in cochain complexes, we write $\underline{\text{Hom}}_A(M, N)$ for the cochain complex given by

$$\underline{\text{Hom}}_A(M, N)^i = \text{Hom}(M^\#, N^\#[i]),$$

with differential $\delta f = \delta_N \circ f \pm f \circ \delta_M$.

CONTENTS

Introduction	1
0.1. Notation	2
1. Compatible quantisations on derived affine schemes	3
1.1. Quantised (-2) -shifted polyvectors	3
1.2. (-2) -shifted quantisations	6
1.3. Generalised pre-symplectic structures	9
1.4. Compatibility of quantisations and symplectic structures	11
1.5. Comparing quantisations and generalised symplectic structures	14

2.	Interpretation of the quantisation	18
2.1.	Right de Rham cohomology and Borel–Moore homology	18
2.2.	Calculation of virtual fundamental classes	23
3.	Global quantisations	27
3.1.	Étale functoriality and derived Deligne–Mumford stacks	27
3.2.	Derived Artin stacks	28
	References	28

1. COMPATIBLE QUANTISATIONS ON DERIVED AFFINE SCHEMES

Let R be a graded-commutative differential algebra (CDGA) over \mathbb{Q} , and fix a CDGA A over R . We will denote the differentials on A and R by δ .

1.1. Quantised (-2) -shifted polyvectors.

1.1.1. *Polyvectors.* The following is adapted from [Pri3, Definition 1.1], with the introduction of a dummy variable \hbar of cohomological degree 0 to assist comparison with quantisation constructions.

Definition 1.1. Define the complex of (-2) -shifted polyvector fields (or strictly speaking, multiderivations) on A by

$$\widehat{\text{Pol}}(A/R, -2) := \prod_{p \geq 0} \hbar^{p-1} \underline{\text{Hom}}_A(\Omega_A^p, A)[p].$$

with graded-commutative multiplication $(a, b) \mapsto ab$ on $\widehat{\text{Pol}}(A, -2)$ following the usual conventions for symmetric powers, so for $\pi \in \hbar^p \underline{\text{Hom}}_A(\Omega_A^p, A)$, $\nu \in \hbar^q \underline{\text{Hom}}_A(\Omega_A^q, A)$ we have

$$(\pi \cdot \nu)(adf_1 \wedge \dots \wedge df_{p+q}) = \frac{1}{(p+q)!} \sum_{\sigma \in S_{p+q}} \pm a \pi(df_{\sigma(1)} \wedge \dots \wedge df_{\sigma(p)}) \nu(df_{\sigma(p+1)} \wedge \dots \wedge df_{\sigma(p+q)}).$$

The Lie bracket on $\text{Hom}_A(\Omega_{A/R}^1, A)$ then extends to give a bracket (the Schouten–Nijenhuis bracket)

$$[-, -]: \widehat{\text{Pol}}(A/R, -2) \times \widehat{\text{Pol}}(A/R, -2) \rightarrow \widehat{\text{Pol}}(A/R, -2)[1],$$

determined by the property that it is a bi-derivation with respect to the multiplication operation.

Thus $\widehat{\text{Pol}}(A/R, -2)$ has the natural structure of a P_0 -algebra, and in particular $\widehat{\text{Pol}}(A/R, -2)[-1]$ is a differential graded Lie algebra (DGLA) over R .

Note that the differential δ on $\widehat{\text{Pol}}(A/R, -2)$ can be written as $[\delta, -]$, where $\delta \in \widehat{\text{Pol}}(A/R, -2)^1$ is the element defined by the derivation δ on A .

Definition 1.2. Define a decreasing filtration F on $\widehat{\text{Pol}}(A/R, -2)$ by

$$F^i \widehat{\text{Pol}}(A/R, -2) := \prod_{j \geq i} \hbar^{j-1} \underline{\text{Hom}}_A(\Omega_A^j, A)[j];$$

this has the properties that $\widehat{\text{Pol}}(A/R, -2) = \varprojlim_i \widehat{\text{Pol}}(A/R, -2)/F^i$, with $[F^i, F^j] \subset F^{i+j-1}$, $\delta F^i \subset F^i$, and $F^i F^j \subset \hbar^{-1} F^{i+j}$.

Observe that this filtration makes $F^2 \widehat{\text{Pol}}(A/R, -2)[-1]$ into a pro-nilpotent DGLA.

Definition 1.3. Define the tangent DGLA of polyvectors by

$$T\widehat{\text{Pol}}(A/R, -2) := \widehat{\text{Pol}}(A/R, -2) \oplus \widehat{\text{Pol}}(A/R, -2)\hbar\epsilon,$$

for ϵ of degree 0 with $\epsilon^2 = 0$. The Lie bracket is given by $[u + v\epsilon, x + y\epsilon] = [u, x] + [u, y]\epsilon + [v, x]\epsilon$.

Definition 1.4. Given a Maurer–Cartan element $\pi \in \text{MC}(F^2\widehat{\text{Pol}}(A/R, -2)[-1])$, define

$$T_\pi\widehat{\text{Pol}}(A/R, -2) := \prod_{p \geq 0} \hbar^p \underline{\text{Hom}}_A(\Omega_A^p, A)[p],$$

with derivation $\delta + [\pi, -]$ (necessarily square-zero by the Maurer–Cartan conditions). The product on polyvectors makes this a CDGA, and it inherits the filtration F from $\widehat{\text{Pol}}$.

Given $\pi \in \text{MC}(F^2\widehat{\text{Pol}}(A/R, -2)/F^p)$, we define $T_\pi\widehat{\text{Pol}}(A/R, -2)/F^p$ similarly. This is a CDGA because $F^i \cdot F^j \subset F^{i+j}$.

Regarding $T_\pi\widehat{\text{Pol}}(A/R, -2)[-1]$ as an abelian DGLA, observe that $\text{MC}(T_\pi\widehat{\text{Pol}}(A/R, -2)[-1])$ is just the fibre of

$$\text{MC}(T\widehat{\text{Pol}}(A/R, -2)[-1]) \rightarrow \text{MC}(\widehat{\text{Pol}}(A/R, -2)[-1])$$

over π . Evaluation at $\hbar = 1$ gives an isomorphism from $T\widehat{\text{Pol}}(A/R, -2)[-1]$ to the DGLA $\widehat{\text{Pol}}(A/R, -2)[-1] \otimes_{\mathbb{Q}} \mathbb{Q}[\epsilon]$ of [Pri3, §1.1.1], and the map σ of [Pri3, Definition 1.11] then becomes:

Definition 1.5. Define

$$\sigma = -\partial_{\hbar^{-1}}: \widehat{\text{Pol}}(A/R, -2) \rightarrow T\widehat{\text{Pol}}(A/R, -2)$$

by $\alpha \mapsto \alpha + \epsilon\hbar^2 \frac{\partial \alpha}{\partial \hbar}$. Note that this is a morphism of filtered DGLAs, so gives a map

$$\text{MC}(F^2\widehat{\text{Pol}}(A/R, -2)[-1]) \rightarrow \text{MC}(F^2T\widehat{\text{Pol}}(A/R, -2)[-1]),$$

with $\sigma(\pi) \in Z^1(F^2T_\pi\widehat{\text{Pol}}(A/R, -2)[-1])$.

1.1.2. Right connections and de Rham complexes.

Definition 1.6. We define a homotopy right \mathcal{D} -module structure (or flat right connection) on A over R to be a sequence of maps $\nabla_{p+1}: \underline{\text{Hom}}_A(\Omega_{A/R}^p, A)^\# \rightarrow A^\#[1-p]$ for $p \geq 1$, satisfying the following conditions:

- (1) For $a \in A$ and $\xi \in \underline{\text{Hom}}_A(\Omega_A^1, A)$, we have $\nabla_2(a\xi) = a\nabla_2(\xi) - \xi(da)$;
- (2) For $p \geq 2$, the maps ∇_{p+1} are A -linear;
- (3) The operations $(\nabla_2 - \text{id}, \nabla_3, \nabla_4, \dots)$ define an L_∞ -morphism from the DGLA $\underline{\text{Hom}}_A(\Omega_A^1, A)$ to the DGLA $(A \oplus \underline{\text{Hom}}_A(\Omega_A^1, A))^{\text{opp}}$ of first-order differential operators with bracket given by negating the commutator.

Remarks 1.7. The final condition in Definition 1.6 is equivalent to saying that ∇ is an L_∞ -derivation from the DGLA $\underline{\text{Hom}}_A(\Omega_A^1, A)$ to the R -module A given the $\underline{\text{Hom}}_A(\Omega_A^1, A)$ -module structure $\xi * a := -\xi(da)$. If we interchange the order of duals and tensor products (permissible if Ω_A^1 is a perfect A -module), then our flat right connections correspond to right $(A, \underline{\text{Hom}}_A(\Omega_A^1, A))$ -module structures on A , in the sense of [Vit, Definition 44], for the natural Lie–Rinehart algebra structure on $(A, \underline{\text{Hom}}_A(\Omega_A^1, A))$.

Definition 1.8. Given a flat right connection ∇ on A , we define the right de Rham complex $\mathrm{DR}^r(A, \nabla)$ associated to ∇ , and its increasing filtration F , by

$$F_i \mathrm{DR}^r(A, \nabla) := \bigoplus_{p \leq i} \underline{\mathrm{Hom}}_A(\Omega_A^p, A)[p],$$

equipped with differential $D^\nabla = \sum_{k \geq 1} D_k^\nabla$ given (for $\pi \in \underline{\mathrm{Hom}}_A(\Omega_A^p, A)[p]$, $\omega \in \Omega_A^{p+1-k}$) by

$$D_k^\nabla(\pi)(\omega) := \begin{cases} \nabla_k(\pi \lrcorner \omega) & k > 2, \\ \nabla_2(\pi \lrcorner \omega) + (-1)^{\deg \pi} \pi(d\omega) & k = 2, \\ \delta\pi(\omega) & k = 1, \end{cases}$$

where d is the de Rham differential and δ is induced by the differential δ on A .

As in [Vit, Corollary 50], the condition that ∇ be an L_∞ -derivation is equivalent to saying that the operator $D^\nabla = \sum_k D_k^\nabla$ satisfies $D^\nabla \circ D^\nabla = 0$.

Definition 1.9. We adapt [Kra, Definition 7] by defining a filtered BV_∞ -algebra B over R to be a graded-commutative unital R -algebra equipped with an increasing filtration F and a square-zero R -linear operator \square of degree 1 satisfying the conditions

- (1) $1 \in F_0 B$ and $F_r \cdot F_s \subset F_{r+s}$;
- (2) $\square(1) = 0$ and $\square(F_r) \subset F_r$;
- (3) for $a_i \in F_{r_i}$ and $b \in F_s$, the iterated graded commutators satisfy

$$[a_1, [a_2, \dots, [a_k, \square] \dots]](b) \in F_{s-k+\sum r_i}.$$

In particular, the conditions for the BV operator \square are satisfied if it admits a locally finite decomposition $\square = \sum_{k \geq 1} \square_k$, for \square_k a differential operator of order $\leq k$, such that $\square_k(1) = 0$ and $\sum_{i+j=k} [\square_i, \square_j] = 0$, with $\square_k: F_r B \rightarrow F_{r+1-k} B$. Such decompositions without a filtration correspond to BV_∞ -algebras in the sense of [Vit, Definition 52] and [BL, Definition 3.11].

Definition 1.10. Following [Kra, Proposition 2], the operations

$$[a_1, \dots, a_k]_{\square, k} := [\dots [\square, a_1], \dots, a_k](1)$$

define an L_∞ -algebra structure on the complex $B[-1]$ for any filtered BV_∞ -algebra B with differential \square .

It follows from Definition 1.9 that these L_∞ operations satisfy

$$[F_{i_1}, \dots, F_{i_k}]_{\square, k} \subset F_{i_1 + \dots + i_k + 1 - k}.$$

Lemma 1.11. *The operator $D^\nabla = \sum_{k \geq 1} D_k^\nabla$ defines a filtered BV_∞ -algebra structure on the filtered graded-commutative algebra $\mathrm{DR}^r(A, \nabla)$. On the associated graded complex $\mathrm{gr}^F \mathrm{DR}^r(A, \nabla)$, the induced L_∞ bracket $[-]_{\nabla, k}$ of weight $1 - k$ is trivial for $k \geq 3$ and corresponds to the Schouten–Nijenhuis bracket for $k = 2$.*

Proof. The argument of [Vit, Proposition 53] adapts to our slightly different setting to show that the operators D_k^∇ define a BV_∞ -algebra structure. It follows directly from the definitions that $F_i \cdot F_j \subset F_{i+j}$ and that $D_k^\nabla(F_i) \subset F_{i+1-k}$.

Now, differential operators of order less than k do not contribute to the $(k-1)$ -fold commutator $[-]_{\nabla, k}$, so $[-]_{\nabla, k} = \sum_{j \geq k} [-]_{\nabla, j, k}$, which is of weights at most $(1-k)$ with respect to the filtration. Observe that the leading term of $[-]_{\nabla, k}$ is $[-]_{\nabla, k, k}$. The

calculation of [Vit, Proposition 53] shows that this structure corresponds to the L_∞ -structure on polyvectors induced by the L_∞ -structure on $\underline{\mathbf{Hom}}_A(\Omega_{A/R}^1, A)$, which is just the Schouten–Nijenhuis bracket. \square

1.1.3. Quantised polyvectors.

Definition 1.12. Given a flat right connection ∇ on A , define the complex of quantised (-2) -shifted polyvector fields on A by

$$Q\widehat{\mathbf{P}ol}(A, \nabla, -2) := \prod_j \hbar^{j-1} F_j \mathbf{DR}^r(A, \nabla).$$

It follows from Lemma 1.11 that for $a_i, b \in \hbar Q\widehat{\mathbf{P}ol}(A, \nabla, -2)$, the iterated graded commutators satisfy

$$[a_1, [a_2, \dots, [a_k, D^\nabla] \dots]](b) \in \hbar^{k+1} Q\widehat{\mathbf{P}ol}(A, \nabla, -2),$$

and that $\hbar Q\widehat{\mathbf{P}ol}(A, \nabla, -2)$ is closed under multiplication. Thus multiplication and the operator D^∇ make $\hbar Q\widehat{\mathbf{P}ol}(A, \nabla, -2)$ into a filtered BV_∞ -algebra with respect to the \hbar -adic filtration. Moreover, the induced L_∞ -algebra structure from Definition 1.10 extends naturally to an $R[[\hbar]]$ -linear L_∞ -algebra structure on $Q\widehat{\mathbf{P}ol}(A, \nabla, -2)[-1]$.

Definition 1.13. Define a decreasing filtration \tilde{F} on $Q\widehat{\mathbf{P}ol}(A, \nabla, -2)$ by

$$\tilde{F}^i Q\widehat{\mathbf{P}ol}(A, \nabla, -2) := \prod_{j \geq i} \hbar^{j-1} F_j \mathbf{DR}^r(A, \nabla).$$

This filtration has the properties that $Q\widehat{\mathbf{P}ol}(A, \nabla, -2) = \varprojlim_i Q\widehat{\mathbf{P}ol}(A, \nabla, -2)/\tilde{F}^i$, with multiplication in \mathbf{DR}^r giving us a commutative product

$$\tilde{F}^i Q\widehat{\mathbf{P}ol}(A, \nabla, -2) \times \tilde{F}^j Q\widehat{\mathbf{P}ol}(A, \nabla, -2) \rightarrow \hbar^{-1} \tilde{F}^{i+j} Q\widehat{\mathbf{P}ol}(A, \nabla, -2).$$

The operators D_k^∇ satisfy $D_k^\nabla(\tilde{F}^i) \subset \hbar^{k-1} \tilde{F}^{i+1-k} \subset \tilde{F}^i$ and

$$[\tilde{F}^{i_1}, \dots, \tilde{F}^{i_m}]_{\nabla, m} \subset \tilde{F}^{i_1 + \dots + i_m + 1 - m}.$$

1.2. (-2) -shifted quantisations.

1.2.1. The space of quantisations.

Definition 1.14. Given an L_∞ -algebra L , the Maurer–Cartan set is defined by

$$\mathbf{MC}(L) := \{\omega \in L^1 \mid \sum_{n \geq 1} \frac{[\omega, \dots, \omega]_n}{n!} = 0 \in L^2\},$$

where $[-]_1$ is the differential.

Following [Hin], define the Maurer–Cartan space $\underline{\mathbf{MC}}(L)$ (a simplicial set) of L by

$$\underline{\mathbf{MC}}(L)_n := \mathbf{MC}(L \otimes_{\mathbb{Q}} \Omega^\bullet(\Delta^n)),$$

with the obvious simplicial operations, where

$$\Omega^\bullet(\Delta^n) = \mathbb{Q}[t_0, t_1, \dots, t_n, \delta t_0, \delta t_1, \dots, \delta t_n] / (\sum t_i - 1, \sum \delta t_i)$$

is the commutative dg algebra of de Rham polynomial forms on the n -simplex, with the t_i of degree 0.

Definition 1.15. We now define another decreasing filtration G on $Q\widehat{\text{Pol}}(A, \nabla, -2)$ by setting

$$G^i Q\widehat{\text{Pol}}(A, \nabla, -2) := \hbar^i Q\widehat{\text{Pol}}(A, \nabla, -2).$$

We then set $G^i \tilde{F}^p := G^i \cap \tilde{F}^p$.

Note that $G^i \subset \tilde{F}^i$, and beware that $G^i \tilde{F}^p$ is not the same as $\hbar^i \tilde{F}^p$ in general, since

$$\begin{aligned} G^i \tilde{F}^p Q\widehat{\text{Pol}}(A, \nabla, -2) &= \prod_{j \geq p} \hbar^{j-1} F_{j-i} \text{DR}^r(A, \nabla), \\ \hbar^i \tilde{F}^p Q\widehat{\text{Pol}}(A, \nabla, -2) &= \prod_{j \geq p+i} \hbar^{j-1} F_{j-i} \text{DR}^r(A, \nabla). \end{aligned}$$

We will also consider the convolution $G * \tilde{F}$, given by $(G * \tilde{F})^p := \sum_{i+j=p} G^i \cap \tilde{F}^j$; explicitly,

$$(G * \tilde{F})^p Q\widehat{\text{Pol}}(A, \nabla, -2) = \prod_{j < p} \hbar^{j-1} F_{2j-p} \text{DR}^r(A, \nabla) \times \prod_{j \geq p} \hbar^{j-1} F_j \text{DR}^r(A, \nabla).$$

In particular, $(G * \tilde{F})^2 Q\widehat{\text{Pol}}(A, \nabla, -2) = A \oplus \tilde{F}^2 Q\widehat{\text{Pol}}(A, \nabla, -2)$.

Definition 1.16. Define the space $Q\mathcal{P}(A, \nabla, -2)$ of E_{-1} quantisations of (A, ∇) over R to be given by the simplicial set

$$Q\mathcal{P}(A, \nabla, -2) := \varprojlim_i \underline{\text{MC}}(\tilde{F}^2 Q\widehat{\text{Pol}}(A, \nabla, -2)[-1]/\tilde{F}^{i+2}).$$

Also write

$$Q\mathcal{P}(A, \nabla, -2)/G^k := \varprojlim_i \underline{\text{MC}}(\tilde{F}^2 Q\widehat{\text{Pol}}(A, \nabla, -2)[-1]/(\tilde{F}^{i+2} + G^k)),$$

so $Q\mathcal{P}(A, \nabla, -2) = \varprojlim_k Q\mathcal{P}(A, \nabla, -2)/G^k$.

1.2.2. The quantum master equation. By [BL, Theorem 3.7], there is an L_∞ -isomorphism from $\text{DR}^r(A, \nabla)[-1]$ with the L_∞ -structure $[-]_\nabla$ of Definition 1.10 to the complex $\text{DR}^r(A, \nabla)[-1]$ with abelian L_∞ structure. Applied to the pro-nilpotent L_∞ -algebra $\hbar \text{DR}^r(A, \nabla)[[\hbar]][-1]$, this gives an isomorphism

$$\begin{aligned} \varprojlim_r \underline{\text{MC}}(\hbar(\text{DR}^r(A, \nabla)[\hbar]/\hbar^r)[-1]; [-]_\nabla) &\rightarrow \varprojlim_r \underline{\text{MC}}(\hbar(\text{DR}^r(A, \nabla)[\hbar]/\hbar^r)[-1]; D^\nabla, 0, 0, \dots) \\ S &\mapsto e^S - 1. \end{aligned}$$

In particular, for $S \in \text{DR}^r(A, \nabla)^0$, [BL, Remark 3.6] shows that the expression $\sum_n [S, \dots, S]_{n, \nabla} / n!$ can be rewritten as $e^{-S} D_S^\nabla(e^S)$, so the Maurer–Cartan equation $\sum_n [S, \dots, S]_{n, \nabla} / n! = 0$ is equivalent to the quantum master equation $D^\nabla(e^S) = 0$.

Since the target L_∞ -algebra is abelian, $\varprojlim_r \underline{\text{MC}}(\hbar(\text{DR}^r(A, \nabla)[\hbar]/\hbar^r)[-1]; D^\nabla, 0, 0, \dots)$ should be thought of as the space of 0-cocycles in the right de Rham complex $\hbar \text{DR}^r(A, \nabla)[[\hbar]]$. Its homotopy groups are given by

$$\pi_i \varprojlim_r \underline{\text{MC}}(\hbar(\text{DR}^r(A, \nabla)[\hbar]/\hbar^r)[-1]; D^\nabla, 0, 0, \dots) \cong \hbar H^{-i}(\text{DR}^r(A, \nabla)[[\hbar]]).$$

A smaller, but weakly equivalent space can be constructed by truncating the complex $\text{DR}^r(A, \nabla)$ in non-positive degrees, and applying the inverse of the Dold–Kan normalisation functor to obtain a simplicial abelian group.

Our complex $Q\widehat{\text{Pol}}(A, \nabla, -2)$ is not itself a BV_∞ -algebra, but it is an L_∞ -subalgebra of $(\hbar(\text{DR}^r(A, \nabla)[\hbar/\hbar^r])[-1]; [-]_\nabla)$. Therefore sending S to $e^S - 1$ gives natural maps

$$\begin{aligned} Q\mathcal{P}(A, \nabla, -2) &\rightarrow \varprojlim_r \underline{\text{MC}}(\hbar(\text{DR}^r(A, \nabla)[\hbar/\hbar^r])[-1]; D^\nabla, 0, 0, \dots), \\ \pi_i Q\mathcal{P}(A, \nabla, -2) &\rightarrow \hbar H^{-i}(\text{DR}^r(A, \nabla))[[\hbar]] \end{aligned}$$

from quantisations to power series in right de Rham cohomology. This will lead to comparisons with other constructions in §2.

1.2.3. The centre of a quantisation.

Definition 1.17. Define the filtered tangent L_∞ -algebra of quantised polyvectors by

$$\begin{aligned} TQ\widehat{\text{Pol}}(A, \nabla, -2) &:= Q\widehat{\text{Pol}}(A, \nabla, -2) \oplus \prod_{p \geq 0} \hbar^p F_p \text{DR}^r(A, \nabla)\epsilon, \\ \tilde{F}^j TQ\widehat{\text{Pol}}(A, \nabla, -2) &:= \tilde{F}^j Q\widehat{\text{Pol}}(A, \nabla, -2) \oplus \prod_{p \geq j} \hbar^p F_p \text{DR}^r(A, \nabla)(M)\epsilon, \end{aligned}$$

for ϵ of degree 0 with $\epsilon^2 = 0$. The L_∞ operations are given by $[u_1 + v_1\epsilon, \dots, u_n + v_n\epsilon]_n = [u_1, \dots, u_n]_n + \sum_{i=1}^n [u_1, \dots, u_{i-1}, v_i, u_{i+1}, \dots, u_n]\epsilon$.

Definition 1.18. Given a Maurer–Cartan element $S \in \text{MC}(\tilde{F}^2 Q\widehat{\text{Pol}}(A, \nabla, -2)[-1])$, define the centre of (A, ∇, S) by

$$T_S Q\widehat{\text{Pol}}(A, \nabla, -2) := \prod_{p \geq 0} \hbar^p F_p \text{DR}^r(A, \nabla),$$

with differential $D_S^\nabla = e^{-S} \circ D^\nabla \circ e^S$ (necessarily square-zero).

This inherits a commutative multiplication from $\text{DR}^r(A, \nabla)$, and it has a filtration

$$\tilde{F}^i T_S Q\widehat{\text{Pol}}(A, \nabla, -2) := \prod_{p \geq i} \hbar^p F_p \text{DR}^r(A, \nabla),$$

with $\tilde{F}^i \cdot \tilde{F}^j \subset \tilde{F}^{i+j}$.

Given $S \in \text{MC}(F^2 Q\widehat{\text{Pol}}(A, \nabla, -2)[-1]/\tilde{F}^p)$, we define $T_S Q\widehat{\text{Pol}}(A, \nabla, -2)/\tilde{F}^p$ similarly.

Observe that regarding $T_S Q\widehat{\text{Pol}}(A, \nabla, -2)$ as an abelian L_∞ -algebra, the space

$$T_S Q\mathcal{P}(A, \nabla, -2)/\tilde{F}^p := \underline{\text{MC}}(\tilde{F}^2 T_S Q\widehat{\text{Pol}}(A, \nabla, -2)[-1]/\tilde{F}^p; D_S^\nabla, 0, 0, \dots)$$

is just the fibre of

$$\underline{\text{MC}}(\tilde{F}^2 TQ\widehat{\text{Pol}}(A, \nabla, -2)[-1]/\tilde{F}^p) \rightarrow \underline{\text{MC}}(\tilde{F}^2 Q\widehat{\text{Pol}}(A, \nabla, -2)[-1]/\tilde{F}^p)$$

over S .

Similarly to Definition 1.15, there is a filtration G on $TQ\widehat{\text{Pol}}(A, \nabla, -2)$ and $T_S Q\widehat{\text{Pol}}(A, \nabla, -2)$ given by powers of \hbar . This filtration makes $T_S Q\widehat{\text{Pol}}(A, \nabla, -2)$ into a filtered BV_∞ -algebra in the sense of Definition 1.9, and $T_S Q\widehat{\text{Pol}}(A, \nabla, -2)/\tilde{F}^p$ is then also a filtered BV_∞ -algebra (with respect to the filtration G) since \tilde{F}^p is an ideal.

Since $\mathrm{gr}_G^i \tilde{F}^{p-i} \widehat{Q\mathcal{P}ol} = \prod_{j \geq p-i} \hbar^{j-1} \mathrm{gr}_{j-i}^F \mathrm{DR}^r(A, \nabla)$, the associated gradeds of the filtration G admit maps

$$\begin{aligned} \mathrm{gr}_G^i \tilde{F}^p \widehat{Q\mathcal{P}ol}(A, \nabla, -2) &\rightarrow \prod_{j \geq p} \hbar^{j-1} \underline{\mathrm{Hom}}_A(\Omega_{A/R}^{j-i}, A)[j-i] \\ \mathrm{gr}_G^i \tilde{F}^p T_S \widehat{Q\mathcal{P}ol}(A, \nabla, -2) &\rightarrow \prod_{j \geq p} \hbar^j \underline{\mathrm{Hom}}_A(\Omega_{A/R}^{j-i}, A)[j-i]. \end{aligned}$$

which are isomorphisms when A is semi-smooth (in particular whenever A is cofibrant as a CDGA over R).

For the filtrations F of Definitions 1.2 and 1.4, we may rewrite these maps as

$$\begin{aligned} \mathrm{gr}_G^i \tilde{F}^p \widehat{Q\mathcal{P}ol}(A, \nabla, -2) &\rightarrow \hbar^i F^{p-i} \widehat{\mathcal{P}ol}(A, -2), \\ \mathrm{gr}_G^i \tilde{F}^p T_S \widehat{Q\mathcal{P}ol}(A, \nabla, -2) &\rightarrow \hbar^i F^{p-i} T_{\pi_S} \widehat{\mathcal{P}ol}(A, -2), \end{aligned}$$

where $\pi_S \in \mathrm{MC}(F^2 \widehat{\mathcal{P}ol}(A, \nabla, -2))$ denotes the image of S under the map $\mathrm{gr}_G^0 \tilde{F}^2 \widehat{Q\mathcal{P}ol}(A, \nabla, -2) \rightarrow F^2 \widehat{\mathcal{P}ol}(A, -2)$.

Since the cohomology groups of $T_{\pi_S} \widehat{\mathcal{P}ol}(A, -2)$ are shifted Poisson cohomology, we will refer to the cohomology groups of $T_S \widehat{Q\mathcal{P}ol}(A, \nabla, -2)$ as quantised Poisson cohomology.

We write $T_S Q^{tw} \mathcal{P}(A, \nabla, -2) := \underline{\mathrm{MC}}((G * \tilde{F})^2 T_S \widehat{Q\mathcal{P}ol}(A, \nabla, -2))$.

Definition 1.19. Say that an E_{-1} quantisation $S = \sum_{j \geq 2} S_j \hbar^j$ is non-degenerate if the map

$$S_2^\sharp: \Omega_A^1 \rightarrow \underline{\mathrm{Hom}}_A(\Omega_A^1, A)[2]$$

is a quasi-isomorphism and Ω_A^1 is perfect.

Definition 1.20. Define the tangent spaces of quantisations and of twisted quantisations by

$$\begin{aligned} TQ\mathcal{P}(A, \nabla, -2) &:= \varprojlim_i \underline{\mathrm{MC}}(\tilde{F}^2 TQ\widehat{\mathcal{P}ol}(A, \nabla, -2) / \tilde{F}^{i+2}), \\ TQ^{tw}\mathcal{P}(A, \nabla, -2) &:= \varprojlim_i \underline{\mathrm{MC}}((G * \tilde{F})^2 TQ\widehat{\mathcal{P}ol}(A, \nabla, -2) / \tilde{F}^{i+2}), \end{aligned}$$

with $TQ\mathcal{P}(A, \nabla, -2)/G^k$, $TQ^{tw}\mathcal{P}(A, \nabla, -2)/G^k$ defined similarly.

These are simplicial sets over $Q\mathcal{P}(A, \nabla, -2)$ (resp. $Q^{tw}\mathcal{P}(A, \nabla, -2)$, $Q\mathcal{P}(A, \nabla, -2)/G^k$, $Q^{tw}\mathcal{P}(A, \nabla, -2)/G^k$), fibred in simplicial abelian groups.

Definition 1.21. Define the canonical tangent vector

$$\sigma = -\partial_{\hbar^{-1}}: Q\widehat{\mathcal{P}ol}(A, \nabla, -2) \rightarrow TQ\widehat{\mathcal{P}ol}(A, \nabla, -2)$$

by $\alpha \mapsto \alpha + \epsilon \hbar^2 \frac{\partial \alpha}{\partial \hbar}$. Note that this is a morphism of filtered DGLAs, so gives a map $\sigma: Q\mathcal{P}(A, \nabla, -2) \rightarrow TQ\mathcal{P}(A, \nabla, -2)$, with $\sigma(S) \in S + \epsilon Z^1(\tilde{F}^2 T_S Q\widehat{\mathcal{P}ol}(A, \nabla, -2))$ for $S \in Q\mathcal{P}(A, \nabla, -2)_0$.

1.3. Generalised pre-symplectic structures.

Definition 1.22. Define the (left) de Rham complex $\mathrm{DR}(A/R)$ to be the product total complex of the bicomplex

$$A \xrightarrow{d} \Omega_{A/R}^1 \xrightarrow{d} \Omega_{A/R}^2 \xrightarrow{d} \dots,$$

so the total differential is $d \pm \delta$.

We define the Hodge filtration F on $\mathrm{DR}(A/R)$ by setting $F^p \mathrm{DR}(A/R) \subset \mathrm{DR}(A/R)$ to consist of terms $\Omega_{A/R}^i$ with $i \geq p$. In particular, $F^p \mathrm{DR}(A/R) = \mathrm{DR}(A/R)$ for $p \leq 0$.

Definition 1.23. When A is a semi-smooth CDGA over R , recall that a (-2) -shifted pre-symplectic structure ω on A/R is an element

$$\omega \in Z^0 F^2 \mathrm{DR}(A/R).$$

In [PTVV], shifted pre-symplectic structures are referred to as closed 2-forms.

A (-2) -shifted pre-symplectic structure ω is called symplectic if $\omega_2 \in Z^{-2} \Omega_{A/R}^2$ induces a quasi-isomorphism

$$\omega_2^\sharp: \mathrm{Hom}_A(\Omega_{A/R}^1, A) \rightarrow \Omega_{A/R}^1[-2],$$

and $\Omega_{A/R}^1$ is perfect as an A -module.

We now recall a construction from [Pri1, Definition 1.26] which allows us to formulate compatibility between quantisations and a generalisation of pre-symplectic structures.

Definition 1.24. Write $A^{\otimes \bullet+1}$ for the cosimplicial CDGA $n \mapsto A^{n+1}$ given by the Čech nerve, with I the kernel of the diagonal map $A^{\otimes \bullet+1} \rightarrow A$. This has a filtration F given by powers $F^p := (I)^p$ of I , and we define the filtered cosimplicial CDGA $\hat{A}^{\otimes \bullet+1}$ to be the completion

$$\begin{aligned} \hat{A}^{\otimes \bullet+1} &:= \varprojlim_q A^{\otimes \bullet+1} / F^q, \\ F^p \hat{A}^{\otimes \bullet+1} &:= \varprojlim_q F^p / F^q. \end{aligned}$$

We then take the Dold–Kan conormalisation $N\hat{A}^{\bullet+1}$, which becomes a filtered bi-DGAA via the Alexander–Whitney cup product. Explicitly, $N^n \hat{A}^{\bullet+1}$ is the intersection of the kernels of all the big diagonals $\hat{A}^{n+1} \rightarrow \hat{A}^n$, and the cup product is given by

$$(a_0 \otimes \dots \otimes a_m) \smile (b_0 \otimes \dots \otimes b_n) = a_0 \otimes \dots \otimes a_{m-1} \otimes (a_m b_0) \otimes b_1 \otimes \dots \otimes b_n.$$

We then define $\mathrm{DR}'(A/R)$ to be the product total complex

$$\mathrm{DR}'(A/R) := \mathrm{Tot}^\Pi N\hat{A}^{\bullet+1}$$

regarded as a filtered DGAA over R , with $F^p \mathrm{DR}'(A/R) := \mathrm{Tot}^\Pi N F^p \hat{A}^{\bullet+1}$.

The following is standard (for instance [Pri1, Lemma 1.27]):

Lemma 1.25. *There is a filtered quasi-isomorphism $\mathrm{DR}'(A/R) \rightarrow \mathrm{DR}(A/R)$, given by $N^n \hat{A}^{\bullet+1} \rightarrow N^n \hat{A}^{\bullet+1} / F^{n+1} \cong (\Omega_{A/R}^1)^{\otimes A^n} \rightarrow \Omega_{A/R}^n$.*

Definition 1.26. Define a decreasing filtration \tilde{F} on $\mathrm{DR}'(A/R)[[\hbar]]$ by

$$\tilde{F}^p \mathrm{DR}'(A/R) := \prod_{i \geq 0} \hbar^i F^{p-i} \mathrm{DR}'(A/R),$$

where we adopt the convention that $F^j \mathrm{DR}' = \mathrm{DR}'$ for all $j \leq 0$.

Define further filtrations $G, G * \tilde{F}$ by $G^k \mathrm{DR}'(A/R)[[\hbar]] = \hbar^k \mathrm{DR}'(A/R)[[\hbar]]$, and $(G * \tilde{F})^p := \sum_{i+j=p} G^i \cap \tilde{F}^j$, so

$$(G * \tilde{F})^p = \prod_{i \geq 0} \hbar^i F^{p-2i} \mathrm{DR}'(A/R).$$

This makes $(\mathrm{DR}'(A/R)[[\hbar]], G * \tilde{F})$ into a filtered DGAA, since $\tilde{F}^p \tilde{F}^q \subset \tilde{F}^{p+q}$ and similarly for G .

Definition 1.27. Define a generalised (-2) -shifted pre-symplectic structure on a cofibrant (or just semi-smooth) CDGA A/R to be an element

$$\omega \in Z^0((G * \tilde{F})^2 \mathrm{DR}'(A/R)[[\hbar]]) = Z^0(F^2 \mathrm{DR}'(A/R)) \times \hbar Z^0 \mathrm{DR}'(A/R)[[\hbar]].$$

Call this symplectic if $\Omega_{A/R}^1$ is perfect as an A -module and the leading term $\omega_0 \in Z^0 F^2 \mathrm{DR}'(A/R)$ induces a quasi-isomorphism

$$[\omega_0]^\sharp: \mathrm{Hom}_A(\Omega_{A/R}^1, A) \rightarrow \Omega_{A/R}^1[-2],$$

for $[\omega_0] \in Z^{-2} \Omega_{A/R}^2$ the image of ω_0 modulo F^3 .

Definition 1.28. Define the space of generalised (-2) -shifted pre-symplectic structures on A/R to be the simplicial set

$$\mathrm{GPreSp}(A/R, -2) := \varprojlim_i \underline{\mathrm{MC}}((G * \tilde{F})^2 \mathrm{DR}'(A/R)[[\hbar]][-1]/\tilde{F}^{i+2}),$$

where we regard the cochain complex $\mathrm{DR}'(A/R)$ as a DGLA with trivial bracket.

Also write $\mathrm{GPreSp}(A/R, -2)/\hbar^k := \varprojlim_i \underline{\mathrm{MC}}(((G * \tilde{F})^2 \mathrm{DR}'(A/R)[[\hbar]]/(G^k + \tilde{F}^{i+2})))$, so $\mathrm{GPreSp}(A/R, -2) = \varprojlim_k \mathrm{GPreSp}(A/R, -2)/\hbar^k$. Write $\mathrm{PreSp} = \mathrm{GPreSp}/\hbar$.

Set $\mathrm{GSp}(A/R, -2) \subset \mathrm{GPreSp}(A/R, -2)$ to consist of the symplectic structures — this is a union of path-components.

Note that $\mathrm{GPreSp}(A/R, -2)$ is canonically weakly equivalent to the Dold–Kan de-normalisation of the good truncation complex $\tau^{\leq 0}((G * \tilde{F})^2 \mathrm{DR}(A/R)[[\hbar]])$ (and similarly for the various quotients we consider), but the description in terms of $\underline{\mathrm{MC}}$ will simplify comparisons. In particular, we have

$$\pi_i \mathrm{GPreSp}(A/R, -2) \cong \mathrm{H}^{-i}(F^2 \mathrm{DR}(A/R)) \times \hbar \mathrm{H}^{-i}(\mathrm{DR}(A/R)[[\hbar]]).$$

1.4. Compatibility of quantisations and symplectic structures. We will now develop the notion of compatibility between a (truncated) generalised (-2) -shifted pre-symplectic structure and a (truncated) E_{-1} quantisation. The case $k = 1$ recovers the notion of compatibility between (-2) -shifted pre-symplectic and Poisson structures from [Pri3]. From now on we fix a semi-smooth CDGA A over R .

Proposition 1.29. *Given $S \in ((G * \tilde{F})^2 \widehat{Q\mathrm{Pol}}(A, \nabla, -2)/G^k)^1$, there is a chain map*

$$\mu(-, S): \mathrm{DR}'(A/R)[[\hbar]]/G^k \rightarrow T_S \widehat{Q\mathrm{Pol}}(A, \nabla, -1)/G^k$$

*of graded associative $R[[\hbar]]/\hbar^k$ -modules, respecting the filtrations $(G * \tilde{F})$; this is induced by the maps*

$$a_0 \otimes a_1 \otimes \dots \otimes a_n \mapsto a_0 (D_S^\nabla - \delta)(a_1 (D_S^\nabla - \delta)(\dots (D_S^\nabla - \delta)(a_n) \dots))$$

on $A^{\otimes \bullet+1}$.

*Given $\rho \in ((G * \tilde{F})^p \widehat{Q\mathrm{Pol}}(A, \nabla, -2)/G^k)^r$, there is then an $R[[\hbar]]/\hbar^k$ -linear derivation*

$$\nu(-, S, \rho): (\mathrm{DR}'(A/R)[[\hbar]]/\hbar^k, (G * \tilde{F})^\bullet) \rightarrow (T_S \widehat{Q\mathrm{Pol}}(A, \nabla, -2)[r]/G^k, (G * \tilde{F})^{\bullet+p}),$$

which is characterised by the expression

$$\mu(\omega, S + \epsilon\rho) = \mu(\omega, S) + \epsilon\nu(\omega, S, \rho),$$

where $\epsilon^2 = 0$.

Proof. We adapt the proof of [Pri1, Lemma 1.32]. It suffices to prove this for the limit over all k , as S and ρ always lift to $(G * \tilde{F})^2 \widehat{Q\text{Pol}}(A, \nabla, -2)$ and the maps are $R[[\hbar]]$ -linear. First, let $T = T_0 \widehat{Q\text{Pol}}(A, \nabla, -2)[\hbar^{-1}]$, with filtrations \tilde{F} given by powers of \hbar and $G^i T := \hbar^i T_0 \widehat{Q\text{Pol}}(A, \nabla, -2)$. We then consider the convolution filtration $G * \tilde{F}$, which is given explicitly by $(G * \tilde{F})^p T = \prod_k \hbar^k F_{2k-p} \text{DR}^r(A, \nabla)$.

The filtration $G * \tilde{F}$ induces a filtration on the ring $\mathcal{D}_{R[[\hbar]]}(T)$ of graded $R[[\hbar]]$ -linear differential operators on T , which we also denote by $G * \tilde{F}$, given by saying that

$$(G * \tilde{F})^i \underline{\text{End}}_{R[[\hbar]]}(T) = \{f \in \underline{\text{End}}_{R[[\hbar]]}(T) : f((G * \tilde{F})^p T) \subset (G * \tilde{F})^{i+p} T \forall p\}.$$

We then let B be the completion of $\mathcal{D}_{R[[\hbar]]}(T)$ with respect to the filtration $G * \tilde{F}$, and define a filtration Fil on $\mathcal{D}_{R[[\hbar]]}(T)$ by first setting

$$\text{Fil}^i(\mathcal{D}/(G * \tilde{F})^j) = \sum_r (G * \tilde{F})^r (\mathcal{D}_{\leq 2r-i}/(G * \tilde{F})^j),$$

where $\mathcal{D}_{\leq r}$ denotes differential operators of order $\leq r$, then by letting $\text{Fil}^i(B) = \varprojlim_j \text{Fil}^i(\mathcal{D}/(G * \tilde{F})^j)$.

Now, we have $D^\nabla = \sum_{k \geq 1} D_k^\nabla$, where the operator D_k^∇ has order $\leq k$, preserves \tilde{F} and shifts the index of G by $k-1$. Thus $D_k^\nabla \in (G * \tilde{F})^{k-1} \mathcal{D}_{\leq k}$, so $D_k^\nabla \in \text{Fil}^0 B$ for $k \geq 2$ and hence $D^\nabla - \delta \in \text{Fil}^0 B$.

Moreover, $\tilde{F}^p \widehat{Q\text{Pol}}(A, \nabla, -2) = \tilde{F}^{p-1} G^{-1} T$, so in particular $S \in \tilde{F}^1 G^{-1} T \subset (G * \tilde{F})^0 T$. If we regard S as a differential operator on T of order 0, this also gives $S \in \text{Fil}^0 B$. Since

$$D_S^\nabla = e^{-S} D^\nabla e^S = e^{-S} (D^\nabla - \delta) e^S + (\delta + \delta S),$$

it follows that $D_S^\nabla - \delta \in \text{Fil}^0 B$.

Since the associated graded ring $\text{gr}_{\text{Fil}}^\bullet B$ is commutative, we may therefore appeal to [Pri1, Lemma 1.31], which gives a filtered map $\mu'(-, S): (\text{DR}'(A/R), F) \rightarrow (B, \text{Fil})$, from which we obtain our filtered morphisms

$$\begin{aligned} \mu(-, S): (\text{DR}'(A/R), F^\bullet) &\rightarrow (T, G * \tilde{F}^\bullet) \\ \nu(-, S, \rho): (\text{DR}'(A/R), F^\bullet) &\rightarrow (T[r], (G * \tilde{F})^{\bullet+p}) \end{aligned}$$

by evaluating the operators at 1.

We next show that the images of these maps lie in the submodule $G^0 T = T_0 \widehat{Q\text{Pol}}(A, \nabla, -1)$ of T . Given $a \in A$, we may rewrite the expression for $[D_S^\nabla, a]$ as the sum of iterated commutators $\sum_n (-\text{ad}_a)(-\text{ad}_S)^n (D^\nabla)/n!$, and hence as $\sum_n (-\text{ad}_a)(-\text{ad}_S)^n (D_{\geq n+1}^\nabla)/n!$, the lower order operators being annihilated. Since $S \in G^{-1} B$ and $D_k^\nabla \in G^{k-1} B$, this means that $[D_S^\nabla, a] \in G^0 B$, and hence $[D_S^\nabla - \delta, a] \in G^0 B$. Thus $\mu'(F^1(A \otimes A), S) \subset G^0 B$; since $F^1(A \otimes A)$ topologically generates $\text{DR}'(A/R)$ under multiplication, $\mu'(\text{DR}'(A/R), S)$ is thus contained in the subalgebra $G^0 B$ of B , and the result follows by evaluation at 1. The statements for ν are an immediate consequence.

Finally, to see that $\mu(-, S)$ is a chain map, we may appeal to [Pri1, Lemma 1.33], which gives the expression

$$D_S^\nabla \mu'(\omega, S) = \mu'((d + \delta)\omega, S) + \nu'(\omega, S, \frac{1}{2}(D_S^\nabla)^2),$$

the final term vanishing because D_S^∇ is square-zero. That μ is a chain map then follows by evaluation at $1 \in T$. \square

Definition 1.30. We say that a generalised (-2) -shifted pre-symplectic structure ω and an E_{-1} quantisation S of a flat right connection (A, ∇) are compatible (or a compatible pair) if

$$[\mu(\omega, S)] = [-\partial_{\hbar^{-1}}(S)] \in H^0((G * \tilde{F})^2 T_S \widehat{QPol}(A, \nabla, -2)),$$

where $\sigma = -\partial_{\hbar^{-1}}$ is the canonical tangent vector of Definition 1.21.

Definition 1.31. Given a simplicial set Z , an abelian group object A in simplicial sets over Z , a space X over Z and a morphism $s: X \rightarrow A$ over Z , define the homotopy vanishing locus of s over Z to be the homotopy limit of the diagram

$$X \begin{array}{c} \xrightarrow{s} \\ \rightrightarrows \\ \xrightarrow{0} \end{array} A \longrightarrow Z.$$

Definition 1.32. Define the space $QComp(A, \nabla, -2)$ of compatible quantised (-2) -shifted pairs to be the homotopy vanishing locus of

$$(\mu - \sigma): GPreSp(A/R, -2) \times Q\mathcal{P}(A, \nabla, -2) \rightarrow TQ^{tw}\mathcal{P}(A, \nabla, -2)$$

over $Q^{tw}\mathcal{P}(A, \nabla, -2)$.

We define a cofiltration on this space by setting $QComp(A, \nabla, -2)/G^k$ to be the homotopy vanishing locus of

$$(\mu - \sigma): (GPreSp(A/R, -2)/\hbar^k) \times (Q\mathcal{P}(A, \nabla, -2)/G^k) \rightarrow TQ^{tw}\mathcal{P}(A, \nabla, -2)/G^k$$

over $Q^{tw}\mathcal{P}(A, \nabla, -2)/G^k$.

When $k = 1$, note that this recovers the notion of compatible (-2) -shifted pairs from [Pri3], because as in the proof of Proposition 1.29, we have

$$\begin{aligned} \mu'(da, S) &= -\delta a + \sum_{n \geq 0} [[\dots [D^\nabla, \overbrace{S, \dots, S}^n], a]/n! \\ &\equiv \sum_{n \geq 1} [[\dots [D_{n+1}^\nabla, S], \dots, S], a]/n! \pmod{G^1}, \\ &= \sum_{n \geq 1} [S, S, \dots, S, a]_{\nabla, n+1}/n! \end{aligned}$$

which by Lemma 1.11 is just the Schouten–Nijenhuis bracket $[S, a]_{\nabla, 2}$, the higher brackets vanishing.

Definition 1.33. Define $QComp(A, \nabla, -2)^{\text{nondeg}} \subset QComp(A, \nabla, -2)$ to consist of compatible quantised pairs (ω, Δ) with Δ non-degenerate. This is a union of path-components, and by [Pri3, Lemma 1.22] has a natural map

$$QComp(A, \nabla, -2)^{\text{nondeg}} \rightarrow GSp(A/R, -1)$$

as well as the canonical map

$$QComp(A, \nabla, -2)^{\text{nondeg}} \rightarrow Q\mathcal{P}(A, \nabla, -2)^{\text{nondeg}}.$$

Proposition 1.34. For any flat right connection (A, ∇) , the canonical map

$$QComp(A, \nabla, -2)^{\text{nondeg}} \rightarrow Q\mathcal{P}(A, \nabla, -2)^{\text{nondeg}}$$

is a weak equivalence. In particular, there is a morphism

$$Q\mathcal{P}(A, \nabla, -2)^{\text{nondeg}} \rightarrow GSp(A/R, -2)$$

in the homotopy category of simplicial sets.

Proof. We adapt the proof of [Pri3, Proposition 1.26] and [Pri1, Proposition 1.38]. For any $S \in Q\mathcal{P}(A, \nabla, -2)$, the homotopy fibre of $Q\text{Comp}(A/R, -2)^{\text{nondeg}}$ over S is just the homotopy fibre of

$$\mu(-, S): G\text{PreSp}(A/R, -2) \rightarrow T_S Q^{tw}\mathcal{P}(A, \nabla, -2)$$

over $-\partial_{\hbar^{-1}}(S)$.

The map $\mu(-, S): \text{DR}'(A/R)[[\hbar]] \rightarrow T_S Q\widehat{\text{Pol}}(A, \nabla, -2)$ is a morphism of complete $(G * \tilde{F})$ -filtered $R[[\hbar]]$ -modules. Since the morphism is $R[[\hbar]]$ -linear, it maps $G^k(G * \tilde{F})^p \text{DR}'(A/R)[[\hbar]]$ to $G^k(G * \tilde{F})^p T_S Q\widehat{\text{Pol}}(A, \nabla, -2)$. Non-degeneracy of S_2 modulo F_1 implies that $\mu(-, S)$ induces quasi-isomorphisms

$$\hbar^k \Omega^{p-2k}[2k-p] \rightarrow \hbar^{p-k} \underline{\text{Hom}}_A(\Omega_{A/R}^{p-2k}, A)[p-2k]$$

on the associated gradeds $\text{gr}_G^k \text{gr}_{(G*\tilde{F})}^p$. We therefore have a quasi-isomorphism of bifiltered complexes, so we have isomorphisms on homotopy groups:

$$\begin{aligned} \pi_j G\text{PreSp}(A/R, -2) &\rightarrow \pi_j T_S Q^{tw}\mathcal{P}(A, \nabla, -2) \\ \text{H}^{-j}((G * \tilde{F})^2 \text{DR}(A/R)[[\hbar]]) &\rightarrow \text{H}^{-j}((G * \tilde{F})^2 T_S Q\widehat{\text{Pol}}(A, \nabla, -2)). \quad \square \end{aligned}$$

1.5. Comparing quantisations and generalised symplectic structures.

Definition 1.35. Given a compatible pair $(\omega, \pi) \in \text{Comp}(A, -2) = Q\text{Comp}(A, \nabla, -2)/G^1$, and $k \geq 0$, define the complex $N(\omega, \pi, k)$ to be the cocone of the map

$$\text{gr}_G^k(G * \tilde{F})^2(\text{DR}'(A/R)[[\hbar]] \oplus \text{gr}_G^k \tilde{F}^2 Q\widehat{\text{Pol}}(A, \nabla, -2)) \rightarrow \text{gr}_G^k(G * \tilde{F})^2 T_\pi Q\widehat{\text{Pol}}(A, \nabla, -2)$$

given by combining

$$\begin{aligned} \text{gr}_G^k \mu(-, \pi): \text{gr}_G^k(G * \tilde{F})^2 \text{DR}'(A/R)[[\hbar]] &\rightarrow \text{gr}_G^k(G * \tilde{F})^2 T_\pi Q\widehat{\text{Pol}}(A, \nabla, -2) \\ \hbar^k F^{2-2k} \text{DR}(A/R) &\rightarrow \prod_{i \geq (2-2k), 0} \hbar^{i+k} \underline{\text{Hom}}_A(\Omega_{A/R}^i, A)[i] \end{aligned}$$

with the maps

$$\begin{aligned} \text{gr}_G^k \nu(\omega, \pi) + \partial_{\hbar^{-1}}: (\text{gr}_G^k \tilde{F}^2 Q\widehat{\text{Pol}}(A, \nabla, -2), \delta_\pi) &\rightarrow \text{gr}_G^k(G * \tilde{F})^2 T_\pi Q\widehat{\text{Pol}}(A, \nabla, -2) \\ \prod_{i \geq (2-k), 0} \hbar^{i+k-1} \underline{\text{Hom}}_A(\Omega_{A/R}^i, A)[i] &\rightarrow \prod_{i \geq (2-2k), 0} \hbar^{i+k} \underline{\text{Hom}}_A(\Omega_{A/R}^i, A)[i], \end{aligned}$$

where

$$\nu(\omega, \pi)(b) := \nu(\omega, \pi, b).$$

It follows from the proof of Proposition 1.34 that the maps $\text{gr}_G^k \mu(-, \pi)$ are all F -filtered quasi-isomorphisms when π is non-degenerate, so the projection maps $N(\omega, \pi, k) \rightarrow \text{gr}_G^k \tilde{F}^2 Q\widehat{\text{Pol}}(A, \nabla, -2)$ are also quasi-isomorphisms. The behaviour of the other projection is more subtle for low k , but it behaves well thereafter, the proof of [Pri1, Lemma 1.40] adapting verbatim to give:

Lemma 1.36. *The projection maps*

$$N(\omega, \pi, k) \rightarrow \hbar^k \text{DR}(A/R)$$

are F -filtered quasi-isomorphisms for all $k \geq 2$.

1.5.1. *The comparison.*

Proposition 1.37. *The maps*

$$\begin{aligned} Q\mathcal{P}(A, \nabla, -2)^{\text{nondeg}}/G^k &\rightarrow (Q\mathcal{P}(A, \nabla, -2)^{\text{nondeg}}/G^2) \times_{(G\text{Sp}(A, -2)/G^2)}^{\hbar} (G\text{Sp}(A, -2)/G^k) \\ &\simeq (Q\mathcal{P}(A, \nabla, -2)^{\text{nondeg}}/G^2) \times \prod_{i=2}^{k-1} \underline{\text{MC}}(\hbar^i \text{DR}(A/R)[-1]) \end{aligned}$$

coming from Proposition 1.34 are weak equivalences for all $k \geq 2$.

Proof. Proposition 1.34 gives equivalences between $Q\mathcal{P}^{\text{nondeg}}$ and $Q\text{Comp}^{\text{nondeg}}$. Fix $(\omega, \pi) \in \text{Comp}(A, -2)$ and denote homotopy fibres by subscripts. Arguing as in the proof of [Pri1, Proposition 1.41], Lemma 1.36 shows that for $k \geq 2$, the right-hand map is a weak equivalence in the commutative diagram

$$\begin{array}{ccccc} (Q\text{Comp}(A, \nabla, -2)/G^{k+1})_{(\omega, \pi)} & \longrightarrow & (Q\text{Comp}(A, \nabla, -2)/G^k)_{(\omega, \pi)} & \longrightarrow & \underline{\text{MC}}(N(\omega, \pi, k)) \\ \downarrow & & \downarrow & & \downarrow \\ (G\text{PreSp}(A, -2)/G^{k+1})_{\omega} & \longrightarrow & (G\text{PreSp}(A, -2)/G^k)_{\omega} & \longrightarrow & \underline{\text{MC}}(\hbar^k F^{2-2k} \text{DR}(A/R)) \end{array}$$

of homotopy fibre sequences, so

$$(Q\text{Comp}(A, \nabla, -2)/G^k) \times_{G\text{PreSp}(A, -2)/G^k}^{\hbar} G\text{PreSp}(A, -2)/G^{k+1},$$

and the result follows by induction. \square

Remark 1.38. Taking the limit over all k , Proposition 1.37 gives an equivalence

$$Q\mathcal{P}(A, \nabla, -2)^{\text{nondeg}} \simeq (Q\mathcal{P}(A, \nabla, -2)^{\text{nondeg}}/G^2) \times \prod_{i \geq 2} \underline{\text{MC}}(\hbar^i \text{DR}(A/R)[-1]);$$

in particular, this means that there is a canonical map

$$(Q\mathcal{P}(A, \nabla, -2)^{\text{nondeg}}/G^2) \rightarrow Q\mathcal{P}(A, \nabla, -2)^{\text{nondeg}},$$

corresponding to the distinguished point $0 \in \underline{\text{MC}}(\hbar^2 \text{DR}(A/R)[-2][[\hbar]])$.

Thus to quantise a non-degenerate (-2) -shifted Poisson structure $\pi_{\hbar} = \sum_{j \geq 2} \hbar^{j-1} \pi_j$ (or equivalently, by [Pri3, Corollary 1.38], a (-2) -shifted symplectic structure), it suffices to lift the power series π_{\hbar} to a Maurer–Cartan element of the L_{∞} -algebra $\prod_{j \geq 2} \hbar^{j-1} (F_j \text{DR}^r(A, \nabla)/F_{j-2})$.

Even in the degenerate case, the proof of Proposition 1.37 gives a sufficient first-order criterion for quantisations to exist:

$$Q\text{Comp}(A, \nabla, -2) \simeq (Q\text{Comp}(A, \nabla, -2)/G^2) \times \prod_{i \geq 2} \underline{\text{MC}}(\hbar^i \text{DR}(A/R)[-1]).$$

For the (-1) -shifted and 0 -shifted quantisations considered in [Pri1, Pri4], there was a notion of self-duality for quantisations, and restricting to these allowed us to show that even the first-order obstruction to quantising non-degenerate Poisson structures vanishes. The following lemma and example show that the same is not true for our notion of (-2) -shifted quantisations, and that we need an additional condition on compatibility of the Poisson structure and the connection.

Lemma 1.39. *Given a (-2) -shifted Poisson structure $\pi_{\hbar} = \sum_{j \geq 2} \hbar^{j-1} \pi_j$, the obstruction $\text{ob}_{\nabla}(\pi_{\hbar})$ to lifting π_{\hbar} to $Q\mathcal{P}(A, \nabla, -2)/G^2$ (resp. $Q^{tw}\mathcal{P}(A, \nabla, -2)/G^2$) is given by the class*

$$\left[\sum_{n \geq 1} [\pi_{\hbar}, \dots, \pi_{\hbar}]_{\nabla_{n+1, n}/n!} \right]$$

in $H^1(F^1 T_{\pi} \widehat{\text{Pol}}(A/R, -2))$ (resp. $H^1(T_{\pi} \widehat{\text{Pol}}(A/R, -2))$).

Proof. Just observe that π_{\hbar} naturally defines an element of

$$(G * \tilde{F})^2 Q\widehat{\text{Pol}}(A/R, -2)/G^2 = F_0 \text{DR}^r(A, \nabla) \times \prod_{j \geq 0} \hbar^{j-1} \text{DR}^r(A, \nabla)/F_{j-2},$$

and that the Maurer-Cartan expression is given by

$$\sum_{n \geq 1} [\pi_{\hbar}, \dots, \pi_{\hbar}]_{\nabla, n}/n! = \sum_{n \geq 1} [\pi_{\hbar}, \dots, \pi_{\hbar}]_{\nabla_{\geq n, n}/n!}.$$

Since π is a Poisson structure, we know that

$$\sum_{n \geq 1} [\pi_{\hbar}, \dots, \pi_{\hbar}]_{\nabla_{n, n}/n!} = 0,$$

and we also know that the terms $[\pi_{\hbar}, \dots, \pi_{\hbar}]_{\nabla_{\geq n+2, n}}$ lie in $G^2 Q\widehat{\text{Pol}}$, leaving only the terms $[\pi_{\hbar}, \dots, \pi_{\hbar}]_{\nabla_{n+1, n}}$ to contribute to the obstruction. \square

Example 1.40. Consider the shifted cotangent bundle $T^*\mathbb{G}_m[2]$, whose ring of functions is given by $A := R[x, x^{-1}, \xi]$ for x of degree 0 and ξ of cochain degree -2 . Consider the (-2) -shifted symplectic structure $x dx d\xi$, with associated non-degenerate (-2) -shifted Poisson structure $\pi_{\hbar} := \hbar x^{-1} \partial_{\xi} \partial_x$. Now consider the right \mathcal{D} -module structure on A given by the connection $\nabla_2(a \partial_x + b \partial_{\xi}) = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial \xi}$ (induced by the isomorphism $f \mapsto f dx d\xi$ from A to the natural right \mathcal{D} -module $\Omega_A^2[-2]$).

The obstruction of Lemma 1.39 is given by $[D_2^{\nabla}(\pi_{\hbar})] = [\hbar x^{-2} \partial_{\xi}] \in H^1(T_{\pi} \widehat{\text{Pol}}(A/R, -2))$. Under the isomorphism $\mu(-, \pi_{\hbar})$, this corresponds to the element $[x^{-1} dx] \in H^1(\text{DR}(A/R) \cong H^1(\mathbb{G}_m/R))$, which is non-zero, so in this case the homotopy fibres of $Q\mathcal{P}(A, \nabla, -2)/G^2 \rightarrow \mathcal{P}(A, -2)$ over π_{\hbar} are empty.

Corollary 1.41. *If the scheme $\text{Spec } H^0 A$ is connected and $R = H^0 R$, then whenever the obstruction of Lemma 1.39 vanishes, the twisted quantisation of a given non-degenerate (-2) -shifted Poisson structure is essentially unique up to addition by $\hbar R[[\hbar]]$.*

Proof. Since scalars R are untouched by all the operations, the additive group $\hbar R[[\hbar]]$ acts on the space $Q^{tw}\mathcal{P}(A, \nabla, -2)$ by addition: if S is a twisted quantisation then so is $S + r(\hbar)$ for $r(\hbar) \in \hbar R[[\hbar]]$. Also note that the pair $(\omega + \hbar^2 r'(\hbar), S + r(\hbar))$ are compatible whenever $(\omega, S) \in Q\text{Comp}(A, \nabla, -2)$.

The hypotheses imply that $H^0 \text{DR}(A/R) = R$, with $H^{<0} \text{DR}(A/R) = 0$, and hence also $H^{\leq 0} T_{\pi} \widehat{\text{Pol}}(A/R, -2) \cong R$ for non-degenerate Poisson structures π , via the isomorphism $\mu(-, \pi)$. When the obstruction of Lemma 1.39 vanishes, Proposition 1.37 ensures that the space of twisted quantisations of π is non-empty, and comparison of tangent spaces for the tower $\{Q^{tw}\mathcal{P}(A, \nabla, -2)/G^k\}_k$ shows that $Q^{tw}\mathcal{P}(A, \nabla, -2)_{\pi}$ must be an $\hbar R[[\hbar]]$ -torsor under the action above. \square

1.5.2. *Compatibility of connections and symplectic structures.* We now show that varying the right \mathcal{D} -module structure allows us to eliminate the obstruction of Lemma 1.39, and that there is a unique choice which does so.

Lemma 1.42. *Given $\alpha = \sum_{p \geq 1} \alpha_p \in Z^1(F^1\mathrm{DR}(A))$ and a flat right connection ∇ on A in the sense of Definition 1.6, there is a right \mathcal{D} -module structure ∇^α given by*

$$\nabla_{p+1}^\alpha = \nabla_{p+1} + \lrcorner \alpha_p: \underline{\mathrm{Hom}}_A(\Omega_{A/R}^p, A)^\# \rightarrow A^\#[1-p].$$

Proof. The only non-trivial condition to check is that $\{\nabla_{p+1}^\alpha\}_{p \geq 1}$ defines an L_∞ -derivation for the opposite module structure, or equivalently that $D^{\nabla^\alpha} \circ D^{\nabla^\alpha} = 0$. Now, observe that

$$D^{\nabla^\alpha}(\pi) = D^\nabla(\pi) + \pi \lrcorner \alpha,$$

so

$$\begin{aligned} D^{\nabla^\alpha} \circ D^{\nabla^\alpha} &= D^\nabla \circ D^\nabla + (\lrcorner \alpha) \circ D^\nabla + D^\nabla \circ (\lrcorner \alpha) + \lrcorner(\alpha \wedge \alpha) \\ &= (\lrcorner \alpha) \circ D^\nabla + D^\nabla \circ (\lrcorner \alpha). \end{aligned}$$

For $\pi \in \underline{\mathrm{Hom}}_A(\Omega_A^p, A)[p]$ and $\omega \in \Omega_A^q$, we have $D^\nabla(\pi)(\omega) = \nabla(\pi \lrcorner \omega) \pm \pi(d\omega) + (\delta\pi)(\omega)$, so

$$\begin{aligned} ((\lrcorner \alpha) \circ D^\nabla(\pi))(\omega) &= \mp D^\nabla(\pi)(\alpha \wedge \omega) \\ &= \mp \nabla(\pi \lrcorner(\alpha \wedge \omega)) \mp \pi(d(\alpha \wedge \omega)) \mp \delta(\pi(\alpha \wedge \omega)) \pm \pi(\delta(\alpha \wedge \omega)). \\ (D^\nabla \circ (\lrcorner \alpha))(\pi)(\omega) &= \pm D^\nabla(\pi \lrcorner \alpha)(\omega) \\ &= \pm \nabla(\pi \lrcorner(\alpha \wedge \omega)) \pm \pi(\alpha \wedge d\omega) \pm \delta(\pi(\alpha \wedge \omega)) \mp \pi(\alpha \wedge \delta\omega). \end{aligned}$$

Cancelling terms, this gives

$$D^{\nabla^\alpha}(D^{\nabla^\alpha}\pi)(\omega) = \mp \pi(d\alpha \wedge \omega) \pm \pi(\delta\alpha \wedge \omega),$$

but $d\alpha \mp \delta\alpha = 0$ because $\alpha \in Z^1(F^1\mathrm{DR}(A))$. \square

Lemma 1.43. *Given a (-2) -shifted Poisson structure $\pi_\hbar = \sum_{j \geq 2} \hbar^{j-1} \pi_j$, a flat right connection ∇ on A , and an element $\alpha = \sum_{p \geq 1} \alpha_p \in Z^1(F^1\mathrm{DR}(A))$, the difference*

$$\mathrm{ob}_{\nabla^\alpha}(\pi_\hbar) - \mathrm{ob}_\nabla(\pi_\hbar) \in H^1(F^1T_\pi \widehat{\mathrm{Pol}}(A/R, -2))$$

between the obstructions to lifting π_\hbar to $Q\mathcal{P}(A, \nabla^\alpha, -2)/G^2$ or to $Q\mathcal{P}(A, \nabla, -2)/G^2$ (cf. Lemma 1.39) is

$$\mu(\alpha, \pi_\hbar),$$

for the compatibility map $\mu(-, \pi_\hbar): \mathrm{DR}(A) \rightarrow T_{\pi_\hbar} \mathcal{P}(A, -2)$ of [Pri3, Definition 1.16], a multiplicative map given on generators by $\mu(\mathrm{adf}, \pi_\hbar) := \pi_\hbar \lrcorner(\mathrm{adf})$.

Proof. Since $\mathrm{ob}_\nabla(\pi_\hbar) = \sum_{n \geq 1} [\pi_\hbar, \dots, \pi_\hbar]_{\nabla_{n+1}, n}/n!$, we have

$$\mathrm{ob}_{\nabla^\alpha}(\pi_\hbar) - \mathrm{ob}_\nabla(\pi_\hbar) = \sum_{n \geq 1} [\pi_\hbar, \dots, \pi_\hbar]_{\lrcorner \alpha_n, n}/n!,$$

where $[v_1, \dots, v_n]_{\lrcorner \alpha_n, n} = [\dots [\lrcorner \alpha_n, v_1], \dots, v_n](1)$.

Now, for the insertion operator i , we have $[\lrcorner \alpha, v] = \lrcorner i_v(\alpha_n)$, so

$$[\pi_\hbar, \dots, \pi_\hbar]_{\lrcorner \alpha_n, n}/n! = 1 \lrcorner (i_{\pi_\hbar}^{\circ n}(\alpha_n))/n!,$$

which is just $\mu(\alpha_n, \pi_\hbar)$. \square

Definition 1.44. We define the space of flat right connections on A over R to be the simplicial set given in level n by the set of flat right connections on $A \otimes_{\mathbb{Q}} \Omega^{\bullet}(\Delta^n)$ over $R \otimes_{\mathbb{Q}} \Omega^{\bullet}(\Delta^n)$, with the obvious simplicial operations.

Proposition 1.45. *Given a non-degenerate (-2) -shifted Poisson structure $\pi_{\hbar} = \sum_{j \geq 2} \hbar^{j-1} \pi_j$ on A over R , there is essentially at most one pair (∇, S) where ∇ is a flat right connection on A and S is a first-order deformation quantisation of π_{\hbar} relative to ∇ .*

Explicitly, the space of pairs (∇, S) , for S in the homotopy fibre of $Q\mathcal{P}(A, \nabla, -2)/G^2 \rightarrow \mathcal{P}(A, -2)$ over π_{\hbar} , is either empty or contractible, depending on whether any flat right connections on A exist.

Proof. If there do not exist flat right connections on A , then the space is empty. Otherwise, choose a connection ∇^0 . Lemma 1.42 gives a morphism $\alpha \mapsto \nabla^{0, \alpha}$ from $\underline{\text{MC}}(F^1\text{DR}(A))$ to the space of flat right connections. It follows from the non-degeneracy hypothesis that each Ω_A^p is perfect as an A -module, so the map from Ω_A^p to its double dual is a quasi-isomorphism. Obstruction calculus as in [Pri3, §1.4] then shows that $\underline{\text{MC}}(F^1\text{DR}(A))$ is weakly equivalent to the space of flat right connections.

By Lemma 1.43, the space of pairs (∇, S) as above for varying π is given by the homotopy fibre of

$$\begin{aligned} \underline{\text{MC}}(F^1\text{DR}(A)) \times \mathcal{P}(A, -2)^{\text{nondeg}} &\rightarrow \underline{\text{MC}}(F^1T_{\pi}\widehat{\text{Pol}}(A/R, -2)) \\ (\alpha, \pi) &\mapsto \text{ob}_{\nabla^0}(\pi) + \mu(\alpha, \pi) \end{aligned}$$

over 0. Since π is non-degenerate, the map $\mu(-, \pi)$ is a filtered quasi-isomorphism, so the natural map from this space of pairs down to $\mathcal{P}(A, -2)^{\text{nondeg}}$ is a weak equivalence. In other words, π admits an essentially unique first-order quantisation compatible with an essentially unique flat right connection. \square

Combining Proposition 1.45 with Corollary 1.41, we have:

Corollary 1.46. *Take a non-degenerate (-2) -shifted Poisson structure $\pi_{\hbar} = \sum_{j \geq 2} \hbar^{j-1} \pi_j$ on A over R , with $\text{Spec } H^0 A$ is connected and $R = H^0 R$. If A admits any flat right connections, then pairs (∇, S) , with ∇ a flat right connection on A and $S \in Q^{\text{tw}}\mathcal{P}(A, \nabla, -2)$ a quantisation of π_{\hbar} , are essentially unique up to addition by $(0, \hbar R[[\hbar]])$.*

2. INTERPRETATION OF THE QUANTISATION

2.1. Right de Rham cohomology and Borel–Moore homology.

2.1.1. *Comparison with Borel–Moore homology.* There is a dualising complex ω_A on the derived scheme $\text{Spec } A$, given for A semi-smooth by $\omega_A = \underline{\text{Hom}}_{A_0}(A, \Omega_{A_0}^n)[n]$ when A_0 has dimension n . Since ω_A is a right \mathcal{D} -module in the sense of [GR], by [Pri1, Example 4.1] it is a right \mathcal{D} -module in our sense. When the base ring R is \mathbb{C} , the right de Rham complex $\text{DR}^r(\omega_A)$ can be identified, via Serre duality, the Riemann–Hilbert correspondence, and Verdier duality, with Borel–Moore homology of $\text{Spec } H_0 A$ with complex coefficients, as follows.

The following generalises Definition 1.8:

Definition 2.1. Given a morphism $f: X \rightarrow S$ of derived schemes and a right $\mathcal{D}_{X/S}$ -module \mathcal{E} on X , define the hypersheaf $\mathbf{LDR}_{X/S}^r(\mathcal{E})$ on X by

$$\mathbf{LDR}_{X/S}^r(\mathcal{E}) := \mathcal{E} \otimes_{\mathcal{D}_{X/S}}^{\mathbf{L}} \mathcal{O}_X.$$

Lemma 2.2. For a quasi-compact quasi-separated derived scheme X locally of finite presentation over \mathbb{C} , with underived truncation $\pi^0 X$, we have

$$\mathbf{H}_d^{BM}(\pi^0 X(\mathbb{C}), \mathbb{C}) \simeq \mathbb{H}^{-d}(X, \mathbf{LDR}^r(\omega_X)).$$

Proof (sketch). We use the characterisation of Borel–Moore homology $\mathbf{H}_d^{BM}(\pi^0 X(\mathbb{C}), \mathbb{C})$ as cohomology $\mathbf{H}^{-d}(\pi^0 X(\mathbb{C})_{\text{an}}, \mathbb{D}_{\pi^0 X(\mathbb{C})})$ of the \mathbb{C} -dualising complex $\mathbb{D}_{\pi^0 X(\mathbb{C})}$ on the analytic site of $\pi^0 X$. Assume for simplicity that X admits a derived closed immersion $i: X \rightarrow X^0$ with X^0 smooth. For the associated closed immersion $\bar{i}: \pi^0 X \rightarrow X^0$, it suffices to show that the complexes $\mathbf{R}\bar{i}_* \mathbb{D}_{\pi^0 X(\mathbb{C})} \simeq \mathbf{R}\bar{i}_* \bar{i}^! \mathbb{D}_{X^0(\mathbb{C})}$ (on the analytic site of X^0) and $\mathbf{R}i_* \mathbf{DR}^r(\omega_X) \simeq \mathbf{R}i_*(\omega_X \otimes_{\mathcal{D}_X}^{\mathbf{L}} \mathcal{O}_X)$ (on the Zariski site of X^0) have isomorphic hypercohomology.

By [Sai, 6.1.2], the complex $\mathbf{R}\bar{i}_* \bar{i}^! \mathbb{D}_{X^0(\mathbb{C})}$ is given by the right de Rham complex

$$(\mathbf{R}\Gamma_{[\pi^0 X(\mathbb{C})]} \omega_{X^0}^{\text{an}}) \otimes_{\mathcal{D}_{X^0}^{\text{an}}}^{\mathbf{L}} \mathcal{O}_{X^0},$$

where $\Gamma_{[\pi^0 X(\mathbb{C})]} M = \varinjlim \mathcal{H}om_{\mathcal{O}_{X^0}^{\text{an}}}(\mathcal{O}_{X^0}^{\text{an}}/\mathcal{I}_{\pi^0 X}^{\text{an}}, M)$. Meanwhile, we can write

$$\omega_X \otimes_{\mathcal{D}_X}^{\mathbf{L}} \mathcal{O}_X \simeq \omega_X \otimes_{\mathcal{D}_X}^{\mathbf{L}} \mathbf{Diff}(i^{-1} \mathcal{O}_{X^0}, \mathcal{O}_X) \otimes_{i^{-1} \mathcal{D}_{X^0}}^{\mathbf{L}} i^{-1} \mathcal{O}_{X^0},$$

and $\omega_X \otimes_{\mathcal{D}_X}^{\mathbf{L}} \mathbf{Diff}(i^{-1} \mathcal{O}_{X^0}, \mathcal{O}_X) \simeq \omega_X \otimes_{\mathcal{D}_{X/X^0}}^{\mathbf{L}} \mathcal{O}_X$. It thus suffices to show that the complex $\mathbf{R}\Gamma_{[\pi^0 X(\mathbb{C})]} \omega_{X^0}^{\text{an}}$ of analytic right \mathcal{D} -modules on X^0 is the analytification of the complex $i_*(\omega_X \otimes_{\mathcal{D}_{X/X^0}}^{\mathbf{L}} \mathcal{O}_X)$ of algebraic \mathcal{D} -modules.

Now, since $\omega_X \simeq \mathbf{R}\mathcal{H}om_{i^{-1} \mathcal{O}_{X^0}}(\mathcal{O}_X, i^{-1} \omega_{X^0})$, we have

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_{X^0}}(i_*(\omega_X \otimes_{\mathcal{D}_{X/X^0}}^{\mathbf{L}} \mathcal{O}_X), \mathcal{O}_{X^0}) \simeq i_* \mathbf{R}\mathcal{H}om_{\mathcal{D}_{X/X^0}}(\mathcal{O}_X \otimes_{i^{-1} \mathcal{O}_{X^0}} i^{-1} \omega_{X^0}, \mathcal{O}_X).$$

Since i is a derived closed immersion, the left de Rham complex $\mathbf{DR}(\mathcal{O}_X/\mathcal{O}_{X^0})$ is quasi-isomorphic to the completion of \mathcal{O}_X along $\pi^0 X$, so the last expression above is just the completion of $\omega_{X^0}^*$ with respect to the ideal $\mathcal{I}_{\pi^0 X}$. Meanwhile, the $\mathcal{O}_{X^0}^{\text{an}}$ -linear dual of $\mathbf{R}\Gamma_{[\pi^0 X(\mathbb{C})]} \omega_{X^0}^{\text{an}}$ is the completion of $(\omega_{X^0}^{\text{an}})^*$ with respect to the analytic ideal $\mathcal{I}_{\pi^0 X}^{\text{an}}$, permitting the desired comparison along the lines of [Har]. \square

Remark 2.3. More generally, to right any right \mathcal{D} -module we may associate a complex $\mathbf{LDR}^r(\mathcal{E})^{\text{an}}$ of sheaves on $\pi^0 X(\mathbb{C})^{\text{an}}$. This respects the six functor formalism via the comparison in [Pri1, Example 4.1] between our ind-coherent right \mathcal{D} -modules and those of [GR], and we can also rewrite $\mathbf{R}\Gamma(X, \mathbf{LDR}^r(\omega_X))$ as $f_{\text{dR}*} f_{\text{dR}}^! R$ for $f: X \rightarrow \text{Spec } R$.

2.1.2. Reduction to Gorenstein derived schemes. As in §1.2.2, since the space $Q\mathcal{P}(A, \nabla, -2)$ of E_{-1} quantisations from Definition 1.16 consists of solutions of the quantum master equation, any quantisation $S \in Q\mathcal{P}(A, \nabla, -2)$ gives rise to a 0-cocycle e^S in the right de Rham complex $\mathbf{DR}^r(A, \nabla)[[\hbar]]$.

If we write $\dim A$ for the virtual dimension of $\mathbf{R}\text{Spec } A$ over R , then a morphism $(A, \nabla) \rightarrow \omega_A[-\dim A]$ of right \mathcal{D} -modules would then give us a class of degree $\dim A$ in Borel–Moore homology associated to e^S . However, (-2) -shifted symplectic derived schemes are seldom Gorenstein, so ω_A will not be a line bundle in the cases which interest us. Instead, we now establish some fairly general circumstances in which the right de

Rham complex $\mathrm{DR}^r(A, \nabla)$ is quasi-isomorphic to a shift of $\mathrm{DR}^r(\omega_A)$. Beware that for the derived schemes we consider, the structure sheaf is unbounded, so not ind-coherent, ruling out comparisons with the right crystals of [GR].

Example 2.4. Consider the shifted cotangent space $T^*[-2]\mathbb{A}^1$ of the affine line, corresponding to the CDGA $A := R[x, \xi]$ with $\deg x = 0$, $\deg \xi = -2$. This has a natural (-2) -shifted symplectic structure $dx d\xi$, and corresponding non-degenerate (-2) -shifted Poisson structure π determined by the equation $[x, \xi]_\pi = 1$. The essentially unique right connection ∇ compatible with π is given by $\nabla(a\partial_x + b\partial_\xi) = -\frac{\partial a}{\partial x} - \frac{\partial b}{\partial \xi}$, and then contraction with $dx \wedge d\xi$ defines an isomorphism $\mathrm{DR}^r(A, \nabla) \rightarrow \mathrm{DR}(A)$, which in turn is quasi-isomorphic to the base ring R .

On the other hand, $\omega_A \cong (R[x, \xi, \xi^{-1}]/R[x, \xi])dx \wedge d\xi[1]$, and then $\mathrm{DR}^r(\omega_A) \cong (\mathrm{DR}(A)[\xi^{-1}]/\mathrm{DR}(A))[1]$, which is quasi-isomorphic to $R[2]$ via the element $\xi^{-1}d\xi$, so we do have $\mathrm{DR}^r(A, \nabla) \simeq \mathrm{DR}^r(\omega_A)[-2]$, which tallies well with $T^*[-2]\mathbb{A}^1$ having virtual dimension 2.

However, comparison of the Hodge filtrations shows that this quasi-isomorphism does not come from a right \mathcal{D} -module morphism $A \rightarrow \omega_A[-2]$, since the generator $\partial_x \partial_\xi$ of $\mathrm{DR}^r(A, \nabla)$ lies in $F_2 \setminus F_1$, while the generator $(\xi^{-1}d\xi \wedge dx)\partial_x$ of $\mathrm{DR}^r(\omega_A)$ lies in $F_1 \setminus F_0$.

Under the isomorphism $\mathrm{DR}^r(A, \nabla) \simeq \mathrm{DR}^r(\omega_A)[-2]$, the quantisation corresponding to the constant power series $dx d\xi$ is then simply $\hbar\pi = \hbar\partial_x \partial_\xi$, giving virtual fundamental class $\exp(\hbar\partial_x \partial_\xi) = 1 + \hbar\partial_x \partial_\xi \sim \hbar\partial_x \partial_\xi \in H^0\mathrm{DR}^r(A, \nabla)[[\hbar]]$. This corresponds to the class $\hbar \in H^0\mathrm{DR}(A)[[\hbar]]$ under the isomorphism above, and hence to

$$\hbar[\mathbb{A}^1] \in H_2^{BM}(\mathbb{A}^1, \mathbb{C}[[\hbar]])$$

when $R = \mathbb{C}$, via Lemma 2.2. In general for X smooth over \mathbb{C} , a similar argument (cf. Proposition 2.10) gives the quantisation class of a shifted cotangent complex as

$$[T^*[-2]X] = \hbar^{\dim X}[X] \in H_2^{BM}_{\dim X}(X, \mathbb{C}[[\hbar]]).$$

The reduction in this example of the right de Rham cohomology of a derived scheme to the right de Rham cohomology of a line bundle on a Gorenstein scheme generalises to the following lemma.

Lemma 2.5. *Take a dg scheme $X = (X^0, \mathcal{O}_X)$ over R in the sense of [CFK], with X^0 smooth and \mathcal{O}_X freely generated as a sheaf of graded-commutative algebras over $\mathcal{O}_{X,0} = \mathcal{O}_{X^0}$ by finite rank vector bundles \mathcal{E}, \mathcal{F} in homological degrees 1 and 2 respectively. Let $\mathcal{O}_Y \subset \mathcal{O}_X$ be the dg $\mathcal{O}_{X,0}$ -subalgebra generated by \mathcal{E} . Then for any flat right connection ∇ on \mathcal{O}_X , there exists a flat right connection ∇ on the line bundle $\det \mathcal{F}^* \otimes \mathcal{O}_Y$ on Y and a quasi-isomorphism*

$$\mathrm{DR}_{X/R}^r(\mathcal{O}_X, \nabla) \rightarrow \mathrm{DR}_{Y/R}^r(\det \mathcal{F}^* \otimes_{\mathcal{O}_{X^0}} \mathcal{O}_Y, \nabla)[- \mathrm{rk}(\mathcal{F})].$$

Proof. Ignoring differentials, the expression of \mathcal{O}_X as an \mathcal{O}_Y -algebra generated by \mathcal{F} gives a decomposition of graded sheaves

$$\Omega_{X/R, \#}^1 \otimes_{\mathcal{O}_{X, \#}} \mathcal{O}_{Y, \#} \cong \Omega_{Y/R, \#}^1 \oplus (\mathcal{F}[2] \otimes_{\mathcal{O}_{X^0}} \mathcal{O}_{Y, \#}).$$

This in turn gives us morphisms

$$\theta_p: \underline{\mathrm{Hom}}_A(\Omega_Y^p, \det \mathcal{F}^* \otimes_{\mathcal{O}_{X^0}} \mathcal{O}_Y)_{\#}[-2 \mathrm{rk}(\mathcal{F})] \rightarrow \underline{\mathrm{Hom}}_A(\Omega_X^{p+\mathrm{rk}(\mathcal{F})}, \mathcal{O}_Y)_{\#}.$$

Now consider the quotient \mathcal{Q} of $\mathrm{DR}^r(\mathcal{O}_X, \nabla)$ by the smallest subcomplex containing $(\mathcal{F})\mathrm{DR}^r(\mathcal{O}_X, \nabla)$, where (\mathcal{F}) denotes the ideal of $\mathcal{O}_{X, \#}$ generated by \mathcal{F} . It follows

immediately from definition 1.8 that the contraction map

$$\mathrm{DR}(\mathcal{O}_X) \otimes \mathrm{DR}^r(\mathcal{O}_X, \nabla) \xrightarrow{\lrcorner} \mathrm{DR}^r(\mathcal{O}_X, \nabla)$$

is a chain map, so \mathcal{Q} is the quotient of $\mathrm{DR}^r(\mathcal{O}_X, \nabla)/(\mathcal{F})$ by the relations $(\delta f)\pi \simeq df \lrcorner \pi$ for $f \in \mathcal{F}$.

Thus the maps

$$\theta: \bigoplus_{p \geq 0} \underline{\mathrm{Hom}}_A(\Omega_Y^p, \det \mathcal{F}^* \otimes_{\mathcal{O}_{X^0}} \mathcal{O}_Y)_{\#}[p - \mathrm{rk}(\mathcal{F})] \rightarrow \mathcal{Q}_{\#}$$

are isomorphisms, and consideration of local co-ordinates shows that the projection $\mathrm{DR}^r(\mathcal{O}_X, \nabla) \rightarrow \mathcal{Q}$ is a quasi-isomorphism. The differential induced on \mathcal{Q} by D^{∇} transfers to a differential $(D^{\nabla})'$ on $\bigoplus_{p \geq 0} \underline{\mathrm{Hom}}_A(\Omega_Y^p, \det \mathcal{F}^* \otimes_{\mathcal{O}_{X^0}} \mathcal{O}_Y)_{\#}[p - \mathrm{rk}(\mathcal{F})]$, and since the contraction map

$$\mathrm{DR}(\mathcal{O}_Y) \otimes \mathcal{Q} \xrightarrow{\lrcorner} \mathcal{Q}$$

is a chain map, it follows that $(D^{\nabla})' = D^{\nabla'}$ for some flat right connection ∇' on $\det \mathcal{F}^* \otimes_{\mathcal{O}_{X^0}} \mathcal{O}_Y$. \square

Our main interest in Lemma 2.5 is that whenever we have an isomorphism between $\det \mathcal{F}^*[-\mathrm{rk}(\mathcal{F})] \otimes_{\mathcal{O}_{X^0}} \mathcal{O}_Y$ and a shift of the dualising complex ω_Y on Y , respecting their natural right connections, the lemma gives a quasi-isomorphism from $\mathrm{DR}^r(\mathcal{O}_X, \nabla)$ to a shift of the Borel–Moore homology complex of $\pi^0 Y = \pi^0 X$, via Lemma 2.2.

Remark 2.6. Beware that a dg manifold (in the sense of [CFK]) generated in degrees $[0, -2]$ need not take the form in Lemma 2.5, since there could be an algebraic obstruction to the existence of a section of the map $\mathcal{O}_X^{-2} \rightarrow \mathcal{O}_X^{-2}/\Lambda^2 \mathcal{O}_X^{-1}$. However, in the C^∞ setting of [Pri5] this would not be an issue, and all of our results carry over straightforwardly to that setting. It is also worth noting that the expressions we will obtain for virtual fundamental classes in Proposition 2.11 below do not depend on the choice of morphism $\mathcal{F} \rightarrow \mathcal{O}_{X,\#}$ in Lemma 2.5.

2.1.3. *Poisson structures and dualising complexes.* We now consider fairly general cases in which we can use Lemma 2.5 to construct maps from $\mathrm{DR}^r(\mathcal{O}_X, \nabla)$ to right de Rham cohomology of a dualising complex, and hence to Borel–Moore homology.

Definition 2.7. Say that a (-2) -shifted Poisson structure π on a CDGA A is *strict* if $\pi = \pi_2$. In particular, this makes $A[-2]$ a DGLA rather than just an L_∞ -algebra.

Say that a (-2) -shifted Poisson structure π is *strictly non-degenerate* if the map $\pi_2^{\flat}: \Omega_A^1 \rightarrow \underline{\mathrm{Hom}}_A(\Omega_A^1, A)[2]$ is an isomorphism (not just a quasi-isomorphism).

Lemma 2.8. *Let X be a dg scheme as in Lemma 2.5, so X^0 is smooth and $\mathcal{O}_{X,\#} \cong \mathcal{O}_{X^0}[\mathcal{E}[1] \oplus \mathcal{F}[2]]$. Then strictly non-degenerate strict (-2) -shifted Poisson structures π on X correspond to the following data:*

- (1) an isomorphism $\alpha: \mathcal{F} \cong \mathcal{T}_{X^0}$ to the tangent sheaf of X^0 ,
- (2) a (not necessarily flat) left connection $\nabla_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_{X^0}} \Omega_{X^0}^1$ on \mathcal{E} ,
- (3) a non-degenerate inner product $Q: \mathrm{Symm}_{\mathcal{O}_{X^0}}^2 \mathcal{E} \rightarrow \mathcal{O}_{X^0}$ compatible with $\nabla_{\mathcal{E}}$,

with the differential δ on \mathcal{O}_X determined as follows by an element $\phi \in \Gamma(X^0, \mathcal{E})$ with $dQ(\phi, \phi) = 0$

- (1) the map $\delta: \mathcal{E} \rightarrow \mathcal{O}_X$ is given by $Q(\phi, -)$,
- (2) the map $\delta: \mathcal{F} \rightarrow \mathcal{E}$ is given by $\delta f = -\alpha(f) \lrcorner \nabla_{\mathcal{E}}(\phi)$.

Proof. Putting such a Poisson structure on X is equivalent to defining a shifted Lie bracket $[-, -]: \mathcal{O}_X^{\otimes 2} \rightarrow \mathcal{O}_X[2]$ which respects the differential δ , is a biderivation with respect to the multiplication on \mathcal{O}_X , and satisfies a strict non-degeneracy condition. This is determined in terms of the data above by

- (1) $[f, a] := \alpha(f) \lrcorner da \in \mathcal{O}_{X^0}$ for $f \in \mathcal{F}$, $a \in \mathcal{O}_{X^0}$,
- (2) $[e_1, e_2] := Q(e_1, e_2) \in \mathcal{O}_{X^0}$ for $e_1, e_2 \in \mathcal{E}$,
- (3) $[f, e] := \alpha(f) \lrcorner \nabla_{\mathcal{E}}(e) \in \mathcal{E}$ for $f \in \mathcal{F}$, $e \in \mathcal{E}$,
- (4) $[f_1, f_2] := \alpha^{-1}([\alpha f_1, \alpha f_2]) + (f_1 \wedge f_2) \lrcorner Q(\kappa) \in \mathcal{F} \oplus \Lambda^2 \mathcal{E}$ for $f_1, f_2 \in \mathcal{F}$, where $\kappa \in \text{End}(\mathcal{E}) \otimes \Omega_{X^0}^2$ is the curvature of $\nabla_{\mathcal{E}}$ (necessarily antisymmetric under Q) and $Q(\kappa) \in \Lambda^2 \mathcal{E} \otimes \Omega_{X^0}^2$ its image under the isomorphism $Q: \mathcal{E}^* \rightarrow \mathcal{E}$;

the various conditions follow from the Jacobi identity.

The differential is then just given by $\delta = [\phi, -]$, which also determines ϕ in terms of δ , because Q is non-degenerate. The condition $dQ(\phi, \phi) = 0$ equivalent to saying that $[\phi, \phi]$ is central, and hence that $\delta^2 = 0$. \square

Proposition 2.9. *Take a dg scheme $X = (X^0; \mathcal{O}_{X^0}[\mathcal{E}[1] \oplus \mathcal{F}[2]], \delta)$ equipped with a strictly non-degenerate strict (-2) -shifted Poisson structure π as in Lemma 2.8, such that the determinant bundle $(\det \mathcal{E}, \nabla_{\mathcal{E}})$ is trivial as a line bundle with connection on X^0 .*

Then there exists an essentially unique right connection ∇ on \mathcal{O}_X satisfying the conditions of Proposition 1.45, and for $Y = (X^0; \mathcal{O}_{X^0}[\mathcal{E}[1]], \delta)$ there is a quasi-isomorphism

$$\text{DR}_{X/R}^r(\mathcal{O}_X, \nabla) \rightarrow \text{DR}_{Y/R}^r(\omega_Y)[- \dim X]$$

induced by Lemma 2.5.

Proof. Lemma 2.8 gives an isomorphism $\alpha: \mathcal{F} \cong \mathcal{T}_{X^0}$, which has dual $\Omega_{X^0}^1$, so the quasi-isomorphism of Lemma 2.5 is a map

$$\text{DR}_{X/R}^r(\mathcal{O}_X, \nabla) \rightarrow \text{DR}_{Y/R}^r(\Omega_{X^0}^{\dim X^0} \otimes_{\mathcal{O}_{X^0}} \mathcal{O}_Y, \bar{\nabla})[- \dim X^0].$$

Meanwhile, the dualising complex ω_{X^0} on X^0 is $\Omega_{X^0}^{\dim X^0}[\dim X^0]$, and the dualising complex ω_Y on Y is given by $\underline{\text{Hom}}_{\mathcal{O}_{X^0}}(\mathcal{O}_Y, \omega_X)$, since $Y \rightarrow X^0$ is a derived closed immersion. Since \mathcal{O}_Y is the exterior algebra of \mathcal{E} over \mathcal{O}_{X^0} , its \mathcal{O}_{X^0} -linear dual is $\det \mathcal{E}^* \otimes_{\mathcal{O}_{X^0}} \mathcal{O}_Y[-\text{rk}(\mathcal{E})]$. As $\det \mathcal{E} \cong \mathcal{O}_{X^0}$, this reduces to give an isomorphism

$$\omega_Y \cong \Omega_{X^0}^{\dim X^0} \otimes_{\mathcal{O}_{X^0}} \mathcal{O}_Y[\dim X^0 - \text{rk}(\mathcal{E})]$$

of line bundles, and since $\dim X = 2 \dim X^0 - \text{rk}(\mathcal{E})$, it remains to show that their respective right connections agree.

The right connection ∇ on \mathcal{O}_X from Proposition 1.45 is determined by the condition $\nabla \pi = 0$. Since $\nabla(\alpha \lrcorner \pi) = d\alpha \lrcorner \pi \mp \alpha \lrcorner \nabla(\pi)$ for all $\alpha \in \Omega_X^1$, we then define $\nabla: \mathcal{T}_X \rightarrow \mathcal{O}_X$ by

$$\begin{aligned} \nabla(v) &= \nabla((\pi^b)^{-1}(v) \lrcorner \pi) \\ &= d((\pi^b)^{-1}(v)) \lrcorner \pi, \end{aligned}$$

for $\pi^b: \Omega_X^1 \rightarrow \mathcal{T}_X[2]$ the isomorphism given by contraction. Since this is flat and satisfies $\nabla \pi = 0$, it must be the essentially unique connection of Proposition 1.45, up to coherent homotopy.

In order to check that the right connection $\bar{\nabla}$ induced by ∇ on ω_Y from Lemma 2.5 and the isomorphism above is the canonical right connection ∇_{ω_Y} , it suffices to restrict

to local co-ordinates. For $n := \dim X^0$, without loss of generality we may replace X^0 with an étale neighbourhood U^0 admitting an étale map $U^0 \rightarrow \mathbb{A}^n$ such that $\mathcal{E}|_{U^0}$ is free. This gives co-ordinates x_1, \dots, x_n in \mathcal{O}_{U^0} , with the dx_i a basis for $\Omega_{U^0}^1$, and dual basis ∂_{x_i} of \mathcal{T}_{U^0} . For the isomorphism $\alpha: \mathcal{F} \rightarrow \mathcal{T}_{X^0}$ of Lemma 2.8, set $\xi_i := \alpha^{-1}(\partial_{x_i}) \in \mathcal{F}|_{U^0}$. Let $r = \text{rk}(\mathcal{E})$, and choose a basis e_1, \dots, e_r for $\mathcal{E}|_{U^0}$ such that $e_1 \wedge \dots \wedge e_r$ maps to 1 under the given isomorphism $\det \mathcal{E} \cong \mathcal{O}_{X^0}$.

For $U := (U^0, \mathcal{O}_X|_{U^0})$, these co-ordinates give rise to a basis $\{dx_i, de_j, d\xi_i\}_{1 \leq i \leq n, 1 \leq j \leq r}$ of Ω_U^1 , and we denote the dual basis of \mathcal{T}_U by $\{\partial_{x_i}, \partial_{e_j}, \partial_{\xi_i}\}$. Consider the right connection ∇' on \mathcal{O}_X which sends each of these basis vectors to 0. Expanding out Lemma 2.8 then gives an expression for $\pi - \sum_i \partial_{x_i} \partial_{\xi_i}$ as an \mathcal{O}_{U^0} -linear combination of terms

$$\partial_{e_j} \partial_{e_l}, e_l \partial_{e_j} \partial_{\xi_i}, e_j e_l \partial_{\xi_i} \partial_{\xi_k}.$$

Since the operators ∂_{ξ_i} and ∂_{e_j} annihilate \mathcal{O}_{X^0} , while ∂_{ξ_i} annihilates e_j , only the coefficients c_{jli} of $e_l \partial_{e_j} \partial_{\xi_i}$ contribute to give

$$\nabla' \pi = \sum_{j^i} c_{j^i} \partial_{\xi_i}.$$

Now, $\nabla_{\mathcal{E}}(e_j) = \sum_{l,i} e_l c_{lji} dx_i$, so $\nabla' \pi$ is determined by the trace of the connection $\nabla_{\mathcal{E}}$, which is 0 because $(\det \mathcal{E}, \nabla_{\mathcal{E}})$ is a bundle with connection. Thus $\nabla = \nabla'$.

In these co-ordinates, Lemma 2.5 kills the ξ_i and any polyvectors not divisible by $\prod_i \partial_{\xi_i}$. Pulling out a factor of the globally defined element $\prod_i \partial_{x_i} \partial_{\xi_i}$ (corresponding to the isomorphism $\alpha: \mathcal{F} \rightarrow \mathcal{T}_{X^0}$), we see that

$$\bar{\nabla}: \Omega_{U^0}^n \otimes_{\mathcal{O}_{U^0}} \mathcal{T}_Y \rightarrow \Omega_{U^0}^n \otimes_{\mathcal{O}_{U^0}} \mathcal{O}_Y$$

is determined by the property that the elements $dx_1 \wedge \dots \wedge dx_n \otimes \partial_{x_i}$ and $dx_1 \wedge \dots \wedge dx_n \otimes \partial_{e_j}$ all lie in its kernel.

Under the isomorphism $\Omega_{X^0}^n \otimes_{\mathcal{O}_{X^0}} \mathcal{O}_Y \cong \omega_Y[n-r]$ above, an element $\beta \in \Omega_{U^0}^n$ corresponds to the morphism $\beta \cdot P$ from \mathcal{O}_Y to $\Omega_{U^0}^n$ given by composing the projection $\mathcal{O}_Y \rightarrow \det \mathcal{E}[r]$ with the isomorphism $\det \mathcal{E} \cong \mathcal{O}_{X^0}$ and multiplication by β . The canonical right connection ∇_{ω_Y} on $\omega_Y = \underline{\text{Hom}}_{\mathcal{O}_{X^0}}(\mathcal{O}_Y, \Omega_{X^0}^n)[n]$ is given by combining the right \mathcal{D}_{X^0} -module structure $\nabla_{\omega_{X^0}}$ on $\Omega_{X^0}^n$ with the left \mathcal{D}_Y -module structure ∇_Y on \mathcal{O}_Y . Since $\partial_{e_j}(\det \mathcal{E})[r] \subset \Lambda^{r-1}(\mathcal{E})[r-1] \subset \mathcal{O}_Y$, it follows that $\nabla_{\omega_Y}(\beta \cdot P \partial_{e_j}) = 0$. Since $P(e_1 \wedge \dots \wedge e_r) = 1$ and $\partial_{x_i}(e_j) = 0$, we also have $\nabla_{\omega_Y}(\beta \cdot P \partial_{x_i}) = P \cdot \nabla_{\omega_X}(\beta \partial_{x_i})$. In particular, these vanish when $\beta = dx_1 \wedge \dots \wedge dx_n$, so $\bar{\nabla} = \nabla_{\omega_Y}$, as required. \square

2.2. Calculation of virtual fundamental classes. In the fairly general cases described above, we now construct virtual fundamental classes from our quantisations, and compare them with the Borisov–Joyce virtual fundamental classes.

Proposition 2.10. *In the setting of Proposition 2.9, the image under the map*

$$\text{DR}_{X/R}^r(\mathcal{O}_X, \nabla)[[\hbar]] \rightarrow \text{DR}_{Y/R}^r(\omega_Y)[- \dim X][[\hbar]]$$

of the 0-cocycle $\exp(S)$ associated to any twisted quantisation

$$S \in \Gamma(X^0, Q^{tw} \mathcal{P}(\mathcal{O}_X, \nabla, -2))$$

of $\pi_{\hbar} \in \Gamma(X^0, \mathcal{P}(\mathcal{O}_X, \nabla, -2))$ is given by an element of

$$\hbar^{\dim X^0} u \exp(\hbar Q + \nabla_{\mathcal{E}} + \hbar^{-1} Q(\kappa) + \hbar^{-1} \nabla_{\mathcal{E}}(\phi)) \cdot (1 + \hbar R[[\hbar]]) \in Z^{-\dim X} \text{DR}_{Y/R}^r(\omega_Y)[[\hbar]],$$

for $u \in \Gamma(X^0, \det \mathcal{E}^*)$ the orientation coming from the isomorphism $\det \mathcal{E} \cong \mathcal{O}_{X^0}$, and all other notation as in Lemma 2.8.

The images of these cocycles in $H^{-\dim X} \mathrm{DR}_{X^0/R}^r(\omega_{X^0})[[\hbar]] = H^{\mathrm{rk}(\mathcal{E})} \mathrm{DR}(X^0/R)[[\hbar]]$ are given by the cohomology classes

$$[\exp(S)] \mapsto \begin{cases} \frac{\hbar^{\dim X/2} [u(Q(\kappa)^{\mathrm{rk}(\mathcal{E})/2})]}{(\mathrm{rk}(\mathcal{E})/2)!} \cdot (1 + \hbar R[[\hbar]]) & 2 \mid \dim(X) \\ 0 & 2 \nmid \dim X. \end{cases}$$

Proof. First observe that since the right connection ∇ constructed in the proof of Proposition 2.9 has no higher terms (a consequence of the strict Poisson structure), Lemma 1.11 ensures that the associated L_∞ structure $\{[-, -]_{\nabla, n}\}_n$ on the complex of polyvectors is just the Schouten–Nijenhuis bracket, giving us a natural filtered DGLA isomorphism

$$Q\widehat{\mathrm{Pol}}(\mathcal{O}_X, \nabla, -2)[-1] \simeq \widehat{\mathrm{Pol}}(\mathcal{O}_X, \nabla, -2)[-1][[\hbar]].$$

In particular, inclusion of constants gives us a natural map

$$\mathcal{P}(\mathcal{O}_X, \nabla, -2) \rightarrow Q^{tw}\mathcal{P}(\mathcal{O}_X, \nabla, -2),$$

so $\pi_{\hbar} = \hbar\pi$ is a natural quantisation of itself.

By Corollary 1.46, any other twisted quantisation S lies in $\hbar\pi + \hbar R[[\hbar]]$, so $\exp(S) \in \exp(\hbar\pi)(1 + \hbar R[[\hbar]])$, and from now on we restrict to the case $S = \hbar\pi$.

In order to proceed further, we pass to local co-ordinates x_i, ξ_i, e_j as in the proof of Proposition 2.9, and observe that for $I = \prod_i \partial_{x_i} \partial_{\xi_i}$, we have

$$\begin{aligned} \prod_{j \in J} \partial_{\xi_j} \exp(\hbar \sum_{i=1}^n \partial_{x_i} \partial_{\xi_i}) &= \prod_{j \in J} \partial_{\xi_j} \prod_{i=1}^n (1 + \hbar \partial_{x_i} \partial_{\xi_i}) \\ &= \prod_{j \in J} \partial_{\xi_j} \prod_{i \notin J} (1 + \hbar \partial_{x_i} \partial_{\xi_i}) \\ &= \left(\prod_{j \in J} dx_j \prod_{i \notin J} (dx_i dx_j + \hbar) \right) \lrcorner I \\ &= \hbar^n \left(\prod_{j \in J} \hbar^{-1} dx_j \prod_{i \notin J} (\hbar^{-1} dx_i dx_j + 1) \right) \lrcorner I \\ &= \left(\prod_{j \in J} \hbar^{-1} dx_j \right) \exp(\hbar^{-1} \sum_{i=1}^n dx_i d\xi_i) \lrcorner \hbar^n I. \end{aligned}$$

Locally, π admits an expression

$$\pi = \sum_i \partial_{x_i} \partial_{\xi_i} + \sum_{jl} q_{jl} \partial_{e_j} \partial_{e_l} + \sum_{ijl} c_{jli} e_l \partial_{e_j} \partial_{\xi_i} + \sum_{ikjl} \lambda_{jl}^{ik} e_j e_l \partial_{\xi_i} \partial_{\xi_k};$$

expanding and contracting then gives

$$\exp(\hbar\pi) = \left(\exp(\hbar \sum_{jl} q_{jl} \partial_{e_j} \partial_{e_l} + \sum_{ijl} c_{jli} e_l \partial_{e_j} dx_i + \hbar^{-1} \sum_{ikjl} \lambda_{jl}^{ik} e_j e_l dx_i dx_k) \exp(\hbar^{-1} \sum_i dx_i d\xi_i) \right) \lrcorner \hbar^n I.$$

The quasi-isomorphism from Lemma 2.5 kills all terms in the image of $(d\xi_i + \delta\xi_i) \lrcorner -$, giving

$$\exp(\hbar^{-1} \sum_i dx_i d\xi_i) \lrcorner \beta \mapsto \exp(\hbar^{-1} \sum_i \delta\xi_i dx_i) \lrcorner \beta = \exp(\hbar^{-1} \nabla_{\mathcal{E}}(\phi)) \lrcorner \beta$$

by Lemma 2.8. The rest of the expression then simplifies in terms of that lemma to give

$$\exp(\hbar\pi) \mapsto \exp(\hbar Q + \nabla_{\mathcal{E}} + \hbar^{-1} Q(\kappa) + \hbar^{-1} \nabla_{\mathcal{E}}(\phi)) \hbar^n$$

in $\mathrm{DR}_{Y/R}^r(\Omega_{X^0}^n \otimes_{\mathcal{O}_{X^0}} \mathcal{O}_Y, \bar{\nabla})[-n]$, once we pull out the factor of I , and multiplication by the orientation $u \in \Gamma(X^0, \det \mathcal{E}^*)$ gives the first required expression.

We now consider the image under the natural map $\mathrm{DR}^r(\omega_Y) \rightarrow \mathrm{DR}^r(\omega_{X^0})$. This map destroys all terms involving \mathcal{T}_{Y/X^0} (corresponding in co-ordinates to the vectors ∂_{e_j}), so the image of $\exp(\hbar\pi)$ becomes simply $\exp(\hbar^{-1}Q(\kappa) + \hbar^{-1}\nabla_{\mathcal{E}}(\phi))\hbar^n$. Now observe that because $\det \mathcal{E}$ is trivial as a module with connection, the projection $P: \mathcal{O}_Y \otimes \Omega_{X^0}^* \rightarrow \Omega_{X^0}^*[r]$ from the proof of Proposition 2.9 is a chain map sending $\nabla_{\mathcal{E}}$ to d , even though $\nabla_{\mathcal{E}}$ is not closed. Since P kills $\Lambda^{<r} \mathcal{E}$, we have

$$P(\hbar^n \exp(\hbar^{-1}Q(\kappa) + \hbar^{-1}\nabla_{\mathcal{E}}(\phi))) = \hbar^n \sum_{2i+j=r} \hbar^{-i-j} u \left(\frac{Q(\kappa)^i \nabla_{\mathcal{E}}(\phi)^j}{i!j!} \right),$$

but terms with $j \neq 0$ are coboundaries with respect to $\nabla_{\mathcal{E}}$, so we get

$$[P \exp(\hbar\pi)] = \begin{cases} \hbar^{n-(r/2)} [u(Q(\kappa)^{r/2})]/(r/2)! & 2 \mid r \\ 0 & 2 \nmid r. \end{cases}$$

□

Proposition 2.11. *Take a connected dg scheme $X = (X^0; \mathcal{O}_{X^0}[\mathcal{E}[1] \oplus \mathcal{F}[2]], \delta)$ over \mathbb{C} equipped with a strictly non-degenerate strict (-2) -shifted Poisson structure π as in Lemma 2.8, such that the determinant bundle $(\det \mathcal{E}, \nabla_{\mathcal{E}})$ is trivial as a line bundle with connection on X^0 . For any special orthogonal real \mathcal{C}^∞ -bundle \mathcal{V} on the analytic site of $X^0(\mathbb{C})$ with an isomorphism between $\mathcal{V} \otimes_{\mathbb{R}} \mathbb{C}$ and the smooth sections of \mathcal{E} , the images under the map*

$$\mathrm{H}^0 \mathrm{DR}_{X/R}^r(\mathcal{O}_X, \nabla)[\hbar] \rightarrow \mathrm{H}^{-\dim X} \mathrm{DR}_{X^0/R}^r(\omega_{X^0})[\hbar] \xrightarrow{\frown[X^0]} \mathrm{H}_{\dim X}^{BM}(X^0, \mathbb{C}[\hbar])$$

(from Proposition 2.9) of the classes $[\exp(S)]$ associated to twisted quantisations S of π_{\hbar} are given by the cohomology classes

$$[\exp(S)] \mapsto \hbar^{(\dim X)/2} e(\mathcal{V}) \frown [X^0] \cdot (1 + \hbar R[\hbar]),$$

where e denotes the Euler class of a vector bundle with trivial determinant.

Proof (sketch). The case $2 \nmid \dim X$ is immediate, since $\mathrm{H}^r(\mathrm{BSO}_r(\mathbb{R}), \mathbb{Z})$ is torsion for r odd (see for instance [Bro, Theorem 1.5]), so the complex Euler class of any special orthogonal bundle is 0. From now on, assume $2 \mid \dim X$.

Next, observe that the cohomology class $[u(Q(\kappa)^{r/2})]/(r/2)!$ from Proposition 2.10 depends only on the special orthogonal bundle \mathcal{E} on X^0 , since a different choice of special orthogonal connection $\nabla'_{\mathcal{E}} = \nabla_{\mathcal{E}} + \gamma$ has curvature $\kappa + \frac{1}{2}(\nabla_{\mathcal{E}} + \nabla'_{\mathcal{E}})\gamma$, and the map P as in the proof of Proposition 2.10 sends terms involving $\nabla_{\mathcal{E}}$ or $\nabla'_{\mathcal{E}}$ to coboundaries.

The construction $\mathcal{E} \mapsto [u(Q(\kappa)^{r/2})]/(r/2)!$ can be defined for any SO_r -bundle on a smooth scheme, or even on a smooth Artin stack, the universal case being BSO_r . Write $\lambda_r \in \mathrm{H}^r \mathrm{DR}(\mathrm{BSO}_r)$ for the de Rham cohomology class which gives rise to the classes $[u(Q(\kappa)^{r/2})]/(r/2)!$ by pullback. By GAGA and Riemann–Hilbert, when $R = \mathbb{C}$ the algebraic de Rham cohomology of the stack $\mathrm{BSO}_{r,\mathbb{C}}$ is naturally isomorphic to the complex Betti cohomology $\mathrm{H}^*(\mathrm{BSO}_r(\mathbb{C}), \mathbb{C})$ of the classifying space $\mathrm{BSO}_r(\mathbb{C})$ with complex coefficients, and since $\mathrm{SO}_r(\mathbb{R}) \rightarrow \mathrm{SO}_r(\mathbb{C})$ is a deformation retract, this in turn is isomorphic to $\mathrm{H}^*(\mathrm{BSO}_r(\mathbb{R}), \mathbb{C})$.

Since curvature is additive and exponentials send sums to products, the Whitney sum $\oplus: (\mathrm{SO}_2)^{r/2} \rightarrow \mathrm{SO}_r$ gives

$$\oplus^* \lambda_r = \lambda_2^{\otimes(r/2)} \in H^*(B\mathrm{SO}_r(\mathbb{C}), \mathbb{C})^{\otimes(r/2)}.$$

Now, with respect to the universal bundle \mathcal{C}^∞ bundle \mathcal{U} on $B\mathrm{SO}_r$, [Bro, Theorem 1.5] gives cohomology as $H^*(B\mathrm{SO}_r(\mathbb{R}), \mathbb{Q}) \cong \mathbb{Q}[p_1, \dots, p_{(r/2)-1}, X_r]$ for $p_i = (-1)^i c_{2i}(\mathcal{U} \otimes \mathbb{C}) \in H^{4i}(B\mathrm{SO}_r(\mathbb{R}), \mathbb{Q})$ (Chern classes) and $X_r \in H^r(B\mathrm{SO}_{2s}(\mathbb{R}), \mathbb{Q})$ (the Euler class), with $(X_r)^2 = p_{r/2}$. From the fundamental theorem of symmetric polynomials, we can deduce that the Whitney sum gives an injective map

$$\oplus^*: H^*(B\mathrm{SO}_r(\mathbb{R}), \mathbb{Q}) \rightarrow H^*((B\mathrm{SO}_2)^{r/2}, \mathbb{Q})^{S_{r/2}},$$

with X_r mapping to $X_2^{\otimes(r/2)}$ and p_i mapping to the i th elementary symmetric function in the variables $1^{\otimes j-1} \otimes p_1 \otimes 1^{(r/2)-j} = 1^{\otimes j-1} \otimes (X_2)^2 \otimes 1^{(r/2)-j}$.

Meanwhile, the class λ_2 is easy to describe. We have an isomorphism $\mathrm{SO}_{2,\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}}$ given by $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \mapsto \alpha + i\beta$. Thus any special orthogonal algebraic bundle of rank 2 takes the form $\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^*$, with determinant via $\mathcal{L} \otimes \mathcal{L}^* \cong \mathcal{O}_{X^0}$. In this case, we have curvature $\kappa(\mathcal{L}) \in \Omega^2 = \Omega^2 \otimes \mathcal{E}nd(\mathcal{L})$, with $\kappa(\mathcal{L} \oplus \mathcal{L}^*) = \kappa(\mathcal{L}) \oplus \kappa(\mathcal{L}^*)$ and $uQ(\kappa(\mathcal{L} \oplus \mathcal{L}^*)) = \kappa(\mathcal{L}) = c_1(\mathcal{L})$, the first Chern class. Then $c_2(\mathcal{E}) = -c_1(\mathcal{L})^2$, from which it follows that $(\lambda_2)^2 = p_1$, so we must have $\lambda_2 = X_2$.

The Whitney sum thus combines with the rank 2 calculation above to show that $\lambda_r = X_r$, the Euler class of the universal real \mathcal{C}^∞ bundle. On pulling back to X^0 , this gives $[u(Q(\kappa)^{r/2})]/(r/2)! = e(\mathcal{V})$, the final result then following by substitution in Proposition 2.10. \square

Remark 2.12. By Lemma 2.2 applied to Y , each quantisation S in Proposition 2.10 gives a Borel–Moore homology class

$$[\exp(S)] \in H_{\dim X}^{BM}(\pi^0 X(\mathbb{C}), \mathbb{C})[[\hbar]].$$

when $R = \mathbb{C}$. Proposition 2.11 describes its image in $H_{\dim X}^{BM}(X^0(\mathbb{C}), \mathbb{C})[[\hbar]]$; exactness of the sequence

$$H_{\dim X}^{BM}(\pi^0 X(\mathbb{C}), \mathbb{C}) \rightarrow H_{\dim X}^{BM}(X^0(\mathbb{C}), \mathbb{C}) \rightarrow H_{\dim X}^{BM}((X^0 \setminus \pi^0 X)(\mathbb{C}), \mathbb{C})$$

can be verified for this class by the observations that $\pi^0 X$ is the vanishing locus of the section ϕ from Lemma 2.8 and that non-vanishing sections kill Euler classes.

Corollary 2.13. *In the setting of Proposition 2.11 with $\pi^0 X$ proper, the classes $[\exp(S)] \in H_{\dim X}(\pi^0 X(\mathbb{C}), \mathbb{C})[[\hbar]]$ associated to twisted quantisations S of π_{\hbar} via the map of Proposition 2.9 are given by*

$$\hbar^{(\dim X)/2} [X]_{BJ} \cdot (1 + \hbar \mathbb{C}[[\hbar]]),$$

where $[X]_{BJ}$ is the Borisov–Joyce virtual fundamental class $[X_{dm}]_{\mathrm{virt}}$ of [BJ, Corollary 3.19].

Proof. Observe that the dual vector bundles to \mathcal{V} and \mathcal{E} satisfy the conditions of [BJ, Definition 3.6] (in terms of their notation, \mathcal{E} is given by algebraic sections of E^* , and \mathcal{V} is given by smooth sections of $(E^+)^*$); this relies on the observation that $\mathrm{Re} Q$ is positive definite on both \mathcal{V} and its $\mathrm{Re} Q$ -orthogonal complement $i\mathcal{V}$. Thus $X_{dm} := (X^0(\mathbb{C}), \mathcal{V}, \phi)$ determines a Kuranishi neighbourhood of the form in [BJ, 3.16], and $[X]_{BJ}$ is the associated class in Steenrod homology, or equivalently in ordinary homology as $\pi^0 X(\mathbb{C})$ is a Euclidean neighbourhood retract (cf. [BJ, Corollary 3.19]).

In the equivalence [Joy, Theorem 4.42] between bordism and derived bordism classes, X_{dm} corresponds to the class of the vanishing locus Z of a generic section of \mathcal{V} , and then $[X]_{BJ} = [Z]$ in Steenrod homology $H_{\dim X}^{\text{St}}(\pi^0 X(\mathbb{C}), \mathbb{Z}) = \varprojlim_i H_{\dim X}^{\text{St}}(U_i, \mathbb{Z})$, for a system of open neighbourhoods U_i with $\pi^0 X(\mathbb{C}) = \bigcap_i U_i$. Now, by [BT, Proposition 12.8], the class $[Z]$ is Poincaré dual to the Euler class, so $[Z \cap U_i] = e(\mathcal{V}) \frown [U_i] \in H_{\dim X}^{BM}(U_i, \mathbb{Z})$. The result now follows from Proposition 2.11 by taking limits. \square

3. GLOBAL QUANTISATIONS

The derived affine schemes considered in §1 are a fairly limited class of objects with which to work, so in this section we indicate how to formulate and study quantisations of (-2) -shifted symplectic structures on derived Deligne–Mumford stacks or on derived Artin stacks. The generalisation proceeds along much the same lines as [Pri1, §§3.1, 3.2] and [Pri3, §§3.1, 3.2], so we concentrate on those features which are specific to our setting.

3.1. Étale functoriality and derived Deligne–Mumford stacks. In §2, we saw some (-2) -shifted quantisations on non-affine schemes, defined in terms of strict functoriality, but the definitions and constructions of §1 adapt much more generally using derived étale functoriality.

The construction of [Pri3, Definitions 2.16 and Definition 2.18] uses homotopy étale descent to construct, for any strongly quasi-compact derived Deligne–Mumford n -stack \mathfrak{X} over R , a simplicial set $F(\mathfrak{X})$ associated to any ∞ -functor F on homotopy étale morphisms of R -CDGAs. As in [Pri3, §2.1.2], such ∞ -functors can be constructed from any construction F on fibrant cofibrant $[m]$ -diagrams of R -CDGAs satisfying [Pri3, Properties 2.5]. In order to construct $\widehat{Q\text{Pol}}(\mathfrak{X}, \nabla, -2)$, we need such a construction for right connections ∇ and for right de Rham complexes.

By [Pri3, Lemma 2.3], a construction exists for the tangent sheaf $\mathbf{R}\underline{\text{Hom}}_{\mathcal{O}_{\mathfrak{X}}}(\Omega_{\mathfrak{X}}^1, \mathcal{O}_X)$ satisfying [Pri3, Properties 2.5], while [Pri1, Definition 2.1] extends this construction to the DGAA $\mathcal{D}_{\mathfrak{X}/R}$ of differential operators. In particular, we have an ∞ -functor $A \mapsto \mathbf{R}F_1 \mathcal{D}_{A/R}$ of dg Atiyah algebras on the site of homotopy étale affines over \mathfrak{X} .

Moreover, the left \mathcal{D} -module structure of $\mathcal{O}_{\mathfrak{X}}$ induces a functorial decomposition $\mathbf{R}F_1 \mathcal{D}_{A/R} \cong \mathbf{R}F_0 \mathcal{D}_{A/R} \oplus \mathbf{R}\text{gr}_1^F \mathcal{D}_{A/R}$ of left $\mathbf{R}F_0 \mathcal{D}_{A/R}$ -modules which respects the commutator Lie bracket (of weight -1). In particular, the decomposition makes $\mathbf{R}\text{gr}_1^F \mathcal{D}_{A/R}$ a dg Lie–Rinehart algebra (or Lie algebroid) over $\mathbf{R}F_0 \mathcal{D}_{A/R}$, and these are resolutions of the tangent sheaf and structure sheaf respectively.

Constructing a flat right connection ∇ on $\mathcal{O}_{\mathfrak{X}}$ then amounts to constructing a functorial homotopy anti-involution on the Atiyah algebra $\mathbf{R}F_1 \mathcal{D}_{-/R}$, or equivalently on its universal enveloping algebra $\mathbf{R}\mathcal{D}_{-/R}$. Applying the description of [Pri2, Remarks 1.18 and 1.19] functorially, the potential obstruction to such a right connection existing lies in $H^2(F^1\text{DR}(\mathfrak{X}/R))$, while if the obstruction vanishes the space of flat right connections is a torsor for the additive group space $\underline{\text{MC}}(F^1\text{DR}(\mathfrak{X}/R))$.

Adapting the relevant Definitions along the lines of [Pri3, Definitions 2.16 and Definition 2.18], Propositions 1.37 and 1.45 then adapt with the results above to give:

Proposition 3.1. *Given a strongly quasi-compact derived Deligne–Mumford n -stack \mathfrak{X} locally of finite type over R , and a homotopy right connection ∇ on $\mathcal{O}_{\mathfrak{X}}$, the maps*

$$\begin{aligned} Q\mathcal{P}(\mathfrak{X}, \nabla, -2)^{\text{nondeg}}/G^k &\rightarrow (Q\mathcal{P}(\mathfrak{X}, \nabla, -2)^{\text{nondeg}}/G^2) \times_{(G\text{Sp}(\mathfrak{X}, -2)/G^2)}^h (G\text{Sp}(\mathfrak{X}, -2)/G^k) \\ &\simeq (Q\mathcal{P}(\mathfrak{X}, \nabla, -2)^{\text{nondeg}}/G^2) \times \prod_{i=2}^{k-1} \underline{\text{MC}}(\hbar^i \text{DR}(\mathfrak{X}/R)[-1]) \end{aligned}$$

coming from Proposition 1.34 are weak equivalences for all $k \geq 2$.

Moreover, for any non-degenerate (-2) -shifted Poisson structure π_{\hbar} on \mathfrak{X} , the space of pairs (∇, S) , for S in the homotopy fibre of $Q\mathcal{P}(\mathfrak{X}, \nabla, -2)/G^2 \rightarrow \mathcal{P}(\mathfrak{X}, -2)$ over π_{\hbar} , is either empty or contractible, depending on whether any flat right connections exist on $\mathcal{O}_{\mathfrak{X}}$; the potential obstruction lies in $H^2(F^1\text{DR}(\mathfrak{X}/R))$.

3.2. Derived Artin stacks. The stacky CDGAs (commutative bidifferential bigraded algebras) of [Pri3, §3.1] model formal completions of affine atlases over derived Artin n -stacks, giving formally étale resolutions of derived Artin n -stacks by affine objects. Polyvectors and differential operators satisfy formally étale functoriality as in [Pri3, §3.2] and [Pri1, §3.2], so the reasoning of §3.1 adapts, with Proposition 3.1 adapting verbatim for derived Artin n -stacks.

Note that the right de Rham complexes involved in formulating (-2) -shifted quantisations for derived Artin n -stacks are thus defined in terms of stacky CDGAs, giving rise to complexes of quantised (-2) -shifted polyvectors which are formal deformations of the complexes of polyvectors from [Pri3, §3.2].

REFERENCES

- [BBD⁺] C. Brav, V. Bussi, D. Dupont, D. Joyce, and B. Szendrői. Symmetries and stabilization for sheaves of vanishing cycles. *J. Singul.*, 11:85–151, 2015. arXiv:1211.3259 [math.AG]. With an appendix by Jörg Schürmann.
- [BJ] D. Borisov and D. Joyce. Virtual fundamental classes for moduli spaces of sheaves on Calabi–Yau four-folds. *Geom. Topol.*, to appear. 2015. arXiv: 1504.00690 [math.AG].
- [BL] C. Braun and A. Lazarev. Homotopy BV algebras in Poisson geometry. *Trans. Moscow Math. Soc.*, pages 217–227, 2013.
- [Bro] Edgar H. Brown, Jr. The cohomology of $B\text{SO}_n$ and BO_n with integer coefficients. *Proc. Amer. Math. Soc.*, 85(2):283–288, 1982.
- [BT] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York–Berlin, 1982.
- [CFK] Ionuț Ciocan-Fontanine and Mikhail Kapranov. Derived Quot schemes. *Ann. Sci. École Norm. Sup. (4)*, 34(3):403–440, 2001.
- [CPT⁺] D. Calaque, T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi. Shifted Poisson structures and deformation quantization. *J. Topol.*, 10(2):483–584, 2017. arXiv:1506.03699v4 [math.AG].
- [GR] Dennis Gaitsgory and Nick Rozenblyum. Notes on geometric Langlands: crystals and D -modules. arXiv:1111.2087, 2011.
- [Har] Robin Hartshorne. Algebraic de Rham cohomology. *Manuscripta Math.*, 7:125–140, 1972.
- [Hin] Vladimir Hinich. DG coalgebras as formal stacks. *J. Pure Appl. Algebra*, 162(2-3):209–250, 2001.
- [Joy] Dominic Joyce. An introduction to d-manifolds and derived differential geometry. In *Moduli spaces*, volume 411 of *London Math. Soc. Lecture Note Ser.*, pages 230–281. Cambridge Univ. Press, Cambridge, 2014.
- [Kra] Olga Kravchenko. Deformations of Batalin–Vilkovisky algebras. In *Poisson geometry (Warsaw, 1998)*, volume 51 of *Banach Center Publ.*, pages 131–139, Warsaw, 2000. Polish Acad. Sci.
- [Pri1] J. P. Pridham. Deformation quantisation for (-1) -shifted symplectic structures and vanishing cycles. arXiv:1508.07936v4 [math.AG], 2015.

- [Pri2] J. P. Pridham. Quantisation of derived Poisson structures. arXiv: 1708.00496v2 [math.AG], 2017.
- [Pri3] J. P. Pridham. Shifted Poisson and symplectic structures on derived N -stacks. *J. Topol.*, 10(1):178–210, 2017. arXiv:1504.01940v5 [math.AG].
- [Pri4] J. P. Pridham. Deformation quantisation for unshifted symplectic structures on derived Artin stacks. *Selecta Math. (N.S.)*, 24(4):3027–3059, 2018. arXiv: 1604.04458v4 [math.AG].
- [Pri5] J. P. Pridham. An outline of shifted Poisson structures and deformation quantisation in derived differential geometry. arXiv: 1804.07622v1 [math.AG], 2018.
- [PTVV] T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi. Shifted symplectic structures. *Publ. Math. Inst. Hautes Études Sci.*, 117:271–328, 2013. arXiv: 1111.3209v4 [math.AG].
- [Sai] Morihiko Saito. \mathcal{D} -modules on analytic spaces. *Publ. Res. Inst. Math. Sci.*, 27(2):291–332, 1991.
- [Vit] Luca Vitagliano. Representations of homotopy Lie-Rinehart algebras. *Math. Proc. Cambridge Philos. Soc.*, 158(1):155–191, 2015.