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Citation for published version:

Gess, B & Maurelli, M 2019, 'Well-posedness by noise for scalar conservation laws', *Communications in partial differential equations*, vol. 43, no. 12, pp. 1702-1736.
<https://doi.org/10.1080/03605302.2018.1535604>

Digital Object Identifier (DOI):

[10.1080/03605302.2018.1535604](https://doi.org/10.1080/03605302.2018.1535604)

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Early version, also known as pre-print

Published In:

Communications in partial differential equations

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WELL-POSEDNESS BY NOISE FOR SCALAR CONSERVATION LAWS

BENJAMIN GESS AND MARIO MAURELLI

ABSTRACT. We consider stochastic scalar conservation laws with spatially inhomogeneous flux. The regularity of the flux function with respect to its spatial variable is assumed to be low, so that entropy solutions are not necessarily unique in the corresponding deterministic scalar conservation law. We prove that perturbing the system by noise leads to well-posedness.

1. INTRODUCTION

The question of regularization and well-posedness by noise for SPDE has attracted considerable interest in recent years. One of the driving hopes in this field is to obtain the well-posedness by noise for nonlinear PDE arising in fluid dynamics, for which the deterministic counterpart does not or is not known to allow unique solutions. Despite considerable effort, only partial results in this direction could be obtained so far, cf. e.g. [17, 23, 27, 28] and the references therein. One of the prominent works in this direction is [24] in which the well-posedness by noise for (linear) transport equations with irregular drift has been shown. More precisely, while weak solutions to

$$(1.1) \quad \partial_t u(t, x) + b(x) \cdot \nabla u(t, x) = 0 \quad \text{on } \mathbb{R}^d$$

are not necessarily unique if $\operatorname{div} b \notin L^\infty(\mathbb{R}^d)$ (cf. [1, 19]), it has been shown in [24] that weak solutions to

$$(1.2) \quad du(t, x) + b(x) \cdot \nabla u(t, x) dt + \nabla u(t, x) \circ dW_t = 0 \quad \text{on } \mathbb{R}^d$$

are unique, provided $b \in C_b^\alpha(\mathbb{R}^d)$ for some $\alpha \in (0, 1)$, $\operatorname{div} b \in L^p(\mathbb{R}^d)$ for some $p > 2$ and W_t denotes a standard d -dimensional Wiener process. As pointed out in [24] their result yields *the first concrete example of a partial differential equation related to fluid dynamics that may lack uniqueness without noise, but is well-posed with a suitable noise* (cf. [24, p.3, 1.1 ff.]). On the other hand, as observed in [23, 24], in the nonlinear setting ($d = 1$ for simplicity)

$$(1.3) \quad \partial_t u(t, x) + \partial_x u^2(t, x) = 0 \quad \text{on } \mathbb{R}$$

the same type of noise seems to be of little use, since the stochastically perturbed equation

$$(1.4) \quad du(t, x) + \partial_x u^2(t, x) dt + \partial_x u(t, x) \circ dW_t = 0 \quad \text{on } \mathbb{R}$$

reduces to the deterministic case (1.3) via the transformation $v(t, x) := u(t, x - W_t)$. That is, if u is a solution to (1.4) then v is a solution to (1.3) and vice versa. In particular, shocks and non-uniqueness of weak solutions still appear in (1.4). Hence, no well-posedness by noise, nor regularization by noise seems to be present

Date: January 20, 2017.

2000 Mathematics Subject Classification. H6015, 35R60, 35L65.

Key words and phrases. Stochastic scalar conservation laws, well-posedness by noise.

M.M. acknowledges funding from the European Research Council under the European Union's Seventh Framework Program (FP7/2007-2013) / ERC grant agreement nr. 258237.

in this case and it was concluded in [24]: *The generalization to nonlinear transport equations, where b depends on u itself, would be a major next step for applications to fluid dynamics but it turns out to be a difficult problem* (cf. [24, p.6, 1.11 ff.]).

The purpose of this work is to shed more light on the effect of linear multiplicative noise on (nonlinear) scalar conservation laws. In contrast to the above observation, we show that a similar effect of well-posedness by noise as obtained in [24] for (1.1) can be observed for (nonlinear) scalar conservation laws. More precisely, we consider scalar conservation laws with irregular flux of the type

$$(1.5) \quad \partial_t u(t, x) + b(x, u(t, x)) \cdot \nabla u(t, x) = 0 \quad \text{on } \mathbb{R}^d,$$

with a possibly irregular b . In particular, this includes the special case of inhomogeneous Burgers' equations $b(x, u) = 2b(x)u$. The model example

$$(1.6) \quad b(x, u) = 2\text{sgn}(x)(\sqrt{|x|} \wedge K)u$$

for some $K > 0$, $u_0(\cdot) = 1_{[0,1]}(\cdot)$, $d = 1$ shows that entropy solutions to (1.5) are not necessarily unique. Indeed, fix some time $T > 0$ and choose $K > T/2 + 1$ for simplicity. Then, there are several entropy solutions to (1.5), including the following two particular ones

$$u^1(t, x) := \begin{cases} 1 & \text{if } 0 \leq x \leq (\frac{t}{2} + 1)^2 \\ 0 & \text{otherwise} \end{cases}, \quad u^2(t, x) := \begin{cases} 1 & \text{if } -(\frac{t}{2})^2 \leq x \leq (\frac{t}{2} + 1)^2 \\ 0 & \text{otherwise} \end{cases},$$

on $[0, T] \times \mathbb{R}$. In contrast, we prove that entropy solutions to the stochastically perturbed scalar conservation law

$$(1.7) \quad du(t, x) + b(x, u(t, x)) \cdot \nabla u(t, x) dt + \nabla u(t, x) \circ dW_t = 0 \quad \text{on } \mathbb{R}^d,$$

are unique, assuming $b \in L_{\xi, loc}^\infty(L_x^\infty) \cap L_{\xi, loc}^1(W_{x, loc}^{1,1})$ and $\text{div } b \in L_{\xi, loc}^1(L_x^1) \cap L_x^p(L_{\xi, loc}^\infty)$ for some $p > d$. Note that (1.6) satisfies these assumptions. Hence, this demonstrates that linear multiplicative noise has a similar regularizing effect in the case of (nonlinear) scalar conservation laws with irregular flux as it was obtained in the linear case in [24]. To the authors' knowledge, this provides the first example of a nonlinear scalar conservation law that becomes well-posed by the inclusion of noise.

Scalar conservation laws with irregular flux in divergence form have been used in several models, including models of traffic flow, flow in porous media and sedimentation processes (cf. [13]). In the present work, we choose to consider the non-divergence form in order to allow comparison to the results obtained in [9, 24]. We expect that related arguments can be also applied to the corresponding divergence type equations, as it was demonstrated in the linear case in [9], although nontrivial differences with the non-divergence case may arise. This will be treated in a subsequent work. The respective study of conservation laws with irregular flux has attracted considerable interest in recent years, see [3–5, 13, 14] among many more. Due to the spatial irregularity of the flux, entropy solutions to (1.5) are typically non-unique and several selection criteria to select a unique entropy solution have been introduced, corresponding to different physical phenomena and relative approximation procedures. Therefore, the study of selection methods for (1.5) is of high interest. The well-posedness result for

$$(1.8) \quad du(t, x) + b(x, u(t, x)) \cdot \nabla u(t, x) dt + \sigma \nabla u(t, x) \circ dW_t = 0 \quad \text{on } \mathbb{R}^d,$$

with $\sigma > 0$ obtained in this paper opens the way to study selection principles by vanishing noise $\sigma \rightarrow 0$. In the case of linear transport equations with irregular drift such vanishing noise selection methods have been analyzed in [6, 16] and it should be noted that in general the vanishing viscosity selection does not coincide with the vanishing noise selection. In analogy to linear stochastic transport equations (1.2),

stochastic scalar conservation laws (1.7) model the evolution of passive scalars in turbulent fluids, so-called Kraichnan models.

The literature on regularization (i.e. improvement of regularity) and well-posedness (i.e. existence, uniqueness and possibly stability) by noise is vast and giving a complete survey at this point would exceed the purpose of this paper. Therefore, we will restrict to those that seem most relevant for the content of this work and refer to [23] for a more complete account of the literature. Concerning the case of transport equations with irregular drift (1.1), we mention the works [9, 22, 24, 26] and the references therein. In particular, we would like to emphasize the work [7] which provides a purely analytic approach to the effect of well-posedness by noise for (1.2), since the proof has served as an inspiration for some of the steps of the proof presented in this paper. A regularization by noise effect for (1.2) has been first obtained in [22] where it has been shown that solutions to (1.2) are smooth if the initial condition is, assuming that b satisfies certain integrability conditions, slightly more restrictive than the Ladyzhenskaya-Prodi-Serrin condition. A PDE-based approach and a generalization of these results to drifts b satisfying the Ladyzhenskaya-Prodi-Serrin condition and to divergence-type equations has been given in [9]. A path-by-path approach to well-posedness by noise has been introduced in [11] and was used in [10] for transport equations. Another approach based on Malliavin calculus has been introduced in [39] and developed in a series of papers, cf. e.g. [40] on transport equations.

In some (typically nonlinear) situations, the spatial dependence of the noise coefficients has proven to be crucial in order to obtain well-posedness by noise results. More precisely, in [25] well-posedness by spatially dependent linear transport noise for point vortex dynamics informally related to stochastic 2D-Euler equations has been shown. In [17] it has been shown that the same type of noise can prevent the collapse of point charges in Vlasov-Poisson equations.

More recently, regularizing effects of nonlinear noise in the setting of (nonlinear) scalar conservation laws has been observed in [31] and in the setting of fully nonlinear PDE in [29]. Well-posedness of stochastic scalar conservation laws with random flux has been considered in [30, 34, 35, 38].

We next present the idea and an outline of the proof. Our treatment of (1.7) is based on the kinetic formulation of (stochastic) scalar conservation laws as introduced in [36]. For a function $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ we introduce the kinetic function $\chi(t, x, \xi) : [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$(1.9) \quad \chi(t, x, \xi) = \chi(u(t, x), \xi) := 1_{\xi < u(t, x)} - 1_{\xi < 0}.$$

In the case of a smooth spatial inhomogeneity b and smooth driving signal W , u is an entropy solution to (1.7) iff χ solves the following equation, in the sense of distributions,

$$(1.10) \quad \partial_t \chi = -b(x, \xi) \cdot \nabla \chi - \nabla \chi \cdot \dot{W}_t + \partial_\xi m,$$

where m is a nonnegative bounded random measure on $[0, T] \times \mathbb{R}^d \times \mathbb{R}$ and the derivatives are intended with respect to x unless differently specified. In the general case of (1.7), we take (1.9), (1.10) as the definition of an entropy solution to (1.7), where now the term $\nabla \chi \cdot \dot{W}_t$ should be interpreted as a Stratonovich integral, or more precisely,

$$(1.11) \quad \begin{aligned} \partial_t \chi &= -b(x, \xi) \cdot \nabla \chi - \nabla \chi \circ dW_t + \partial_\xi m \\ &= -b(x, \xi) \cdot \nabla \chi - \nabla \chi \cdot dW_t + \frac{1}{2} \Delta \chi + \partial_\xi m, \end{aligned}$$

see Definition 2.8 below for details. As in the deterministic case, the notion of a generalized kinetic solution is convenient in the construction of an entropy solution since, roughly speaking, the class of generalized kinetic solutions is stable under weak limits. Roughly speaking, a function f is said to be a generalized kinetic solution to (1.7) if f solves (1.10) for some nonnegative measure m and $|f| = \text{sgn}(\xi)f \leq 1$, $\partial_\xi f = 2\delta_0 - \nu$ for some nonnegative measure ν . The key difference to an entropy solution is that f is not assumed to be of the form of a kinetic function (1.9) for some function u .

The main difficulty then lies in proving that generalized kinetic solutions are in fact entropy solutions, which boils down to proving $|f| = 1$ a.e.. In order to prove this we aim to estimate the difference $|f| - f^2$ based on (1.11). The proof now consists of two steps. In the first step, an (in)equality for $|f| - f^2$ is derived based on renormalization techniques (cf. [1, 19]) using the assumption $b \in L^1_{\xi, \text{loc}}(W_{x, \text{loc}}^{1,1})$. Informally, this leads to the equality

$$\partial_t(|f| - f^2) + b(x, \xi) \cdot \nabla(|f| - f^2) + \nabla(|f| - f^2) \circ dW_t = (\text{sgn}(\xi) - 2f)\partial_\xi m.$$

Passing to the Itô formulation and taking the expectation, we informally “gain a Laplacian” similarly to [8]. The main difficulty at this point that is due to the nonlinearity of (1.7) is the additional singular term $\partial_\xi m$. To handle this term, in the second step, we integrate in both ω and ξ (in the Itô formulation), which informally yields

$$\begin{aligned} & \partial_t \int \mathbb{E}(|f| - f^2) d\xi + \int b(x, \xi) \cdot \nabla \mathbb{E}(|f| - f^2) d\xi + \frac{1}{2} \Delta \int \mathbb{E}(|f| - f^2) d\xi \\ &= \mathbb{E} \int \varphi(\text{sgn}(\xi) - 2f) \partial_\xi m d\xi. \end{aligned}$$

Since $\partial_\xi f = 2\delta_0 - \nu \leq 2\delta_0$ this implies

$$\partial_t \int \mathbb{E}(|f| - f^2) d\xi + \int b(x, \xi) \cdot \nabla \mathbb{E}(|f| - f^2) d\xi + \frac{1}{2} \Delta \int \mathbb{E}(|f| - f^2) d\xi \leq 0$$

This is a linear *parabolic* PDE in $\int \mathbb{E}(|f| - f^2) d\xi$ but, in contrast to the linear case, it is not closed, since it involves both $\int \mathbb{E}(|f| - f^2) d\xi$ and $\int b(x, \xi) \cdot \nabla \mathbb{E}(|f| - f^2) d\xi$. The rigorous analysis is carried out by passing to the distributional form. The problem that the above PDE is non-closed then relates to finding a nonnegative test function φ , independent of ξ , that satisfies for every ξ ,

$$\partial_t \varphi + \text{div}(b(x, \xi)\varphi) + \Delta \varphi \leq C,$$

for some constant $C > 0$. In the analysis of this PDE, we rely on the boundedness assumption on b and the integrability assumption on $\text{div}b$.

1.1. Notation. We let (Ω, \mathcal{A}, P) be a measurable space, $(\mathcal{F}_t)_t$ be a normal filtration on (Ω, \mathcal{A}, P) and $W = (W_t)_t$ be a d -dimensional Brownian motion on Ω with respect to the filtration $(\mathcal{F}_t)_t$. For a σ -finite measure space (E, \mathcal{E}, μ) , we say that a function $f : E \rightarrow \mathbb{R}$ is strictly measurable, resp. μ -measurable (or measurable, when μ is fixed) if, for every Borel subset A of \mathbb{R} , $f^{-1}(A)$ is in \mathcal{E} , resp. $\bar{\mathcal{E}}^\mu$, the completion of \mathcal{E} with the μ -null sets. Given a Banach space V , we define $L^0(E; V) = L^0(E, \mathcal{E}, \mu; V)$ in two cases:

- (1) if $V = U^*$ is the dual space of a separable Banach space U , $L^0(E; V)$ is defined as the space of classes of equivalence, under the relation “ $f = g$ μ -a.e.”, of weakly- $*$ measurable functions $f : E \rightarrow V$, i.e., for every φ in U , $x \mapsto \langle f(x), \varphi \rangle_{V, U}$ is measurable. This applies to the case of $V = \mathcal{M}(D)$, the space of finite signed measure over a domain D of \mathbb{R}^n , $L^\infty(D)$, $L^p(D)$ for $1 < p < \infty$;

- (2) if V is separable, $L^0(E; V)$ is defined as the space of classes of equivalence, under the relation “ $f = g$ μ -a.e.”, of weakly measurable functions $f : E \rightarrow V$, i.e., for every φ in V^* , $x \mapsto \langle f(x), \varphi \rangle_{V, V^*}$ is measurable. This applies to the case of $V = C_0(D)$, the space of continuous bounded function on a domain D of \mathbb{R}^n vanishing at infinity, $L^1(D)$, $L^p(D)$ for $1 < p < \infty$.

Similarly, one can define $L^p(E; V)$. When $V = \mathbb{R}$ we simply write $L^0(E)$, $L^p(E)$ (the usual L^p spaces). For a metric, locally compact, σ -compact space S , the space $\mathcal{M}(S)$ denotes the space of finite signed Borel measures on S , $\mathcal{M}_+(S)$ the subset of finite nonnegative Borel measures. More details on these spaces and on measurability issues are given in the appendix. When not otherwise stated, the spaces Ω , resp. $[0, T] \times \Omega$ are considered endowed with the σ -algebrae \mathcal{A} , resp. $\mathcal{B}([0, T]) \otimes \mathcal{A}$; \mathcal{P} denotes the predictable σ -algebra on $[0, T] \times \Omega$, progressive measurability is measurability with respect to \mathcal{P} . The concepts of entropy solutions, kinetic solutions, generalized kinetic solutions, kinetic measures have always to be understood in the sense of equivalence classes, although we will often consider them as functions when this does not create confusion. In cases where we need to work with representatives this will be indicated, although we will often use the same symbol for the class and the representative.

The variables t, ω, x, ξ denote elements resp. in $[0, T], \Omega, \mathbb{R}^d, \mathbb{R}$. We often use the short notation $L_x^p, L_{t, \omega, x}^p, \mathcal{M}_x, \dots$ for the spaces $L^p(\mathbb{R}^d), L^p([0, T] \times \Omega \times \mathbb{R}^d), \dots$ and $L_{\xi, [-R, R]}^p$ for the space $L^p([-R, R])$. We also use the notation $b \in L_x^p(L_{\xi, loc}^\infty), b \in L_{\xi, loc}^1(W_{x, loc}^{1,1}), \dots$ to state that $b \in L_x^p(L_{\xi, [-R, R]}^\infty)$ for every $R > 0$, $b \in L_{\xi, [-R, R]}^1(W_{x, B_R}^{1,1})$ for every $R > 0, \dots$. The symbols $\nabla, \operatorname{div}, \Delta$, if not differently specified, are referred to derivatives in x , while derivatives in t and ξ are denoted by ∂_t, ∂_ξ . As usual in probability theory, φ_t denotes the evaluation at time t , that is, $\varphi_t = \varphi(t)$. The symbol $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L_{x, \xi}^2$, unless differently specified. For example, $\langle \cdot, \cdot \rangle_{t, x, \xi}$ denotes the scalar product in $L_{t, x, \xi}^2$. Sometimes, for a measure m on $[0, T] \times \mathbb{R}^d \times \mathbb{R}$, we use the notation $\langle m, \varphi \rangle dt$ for $\int \varphi(t, x, \xi) m(dt, dx, d\xi)$. The convolution operator is denoted by $*_{var}$, where var stands for the variable (usually x or ξ or both) for which the convolution is performed. The function ρ denotes a smooth nonnegative compactly supported even function on \mathbb{R}^d such that $\int_{\mathbb{R}^d} \rho(x) dx = 1$, and $\rho^\epsilon := \epsilon^{-d} \rho(\epsilon^{-1} \cdot)$. Similarly $\bar{\rho}$ denotes a smooth nonnegative compactly supported even function on \mathbb{R} such that $\int_{\mathbb{R}} \bar{\rho}(\xi) d\xi = 1$, and $\bar{\rho}^\delta = \delta^{-1} \bar{\rho}(\delta^{-1} \cdot)$. In statements and proofs, the letter C denotes a generic positive constant, which can change from line to line and can depend on d (dimension) and p (integrability exponent assumed for $\operatorname{div} b$). In accordance to (1.9) we use the notation $\chi(\xi, u) = 1_{\xi < u} - 1_{\xi < 0}$.

Throughout all the paper, we assume $b \in L_{loc}^1(\mathbb{R}^d \times \mathbb{R})$ and $\operatorname{div}(b) \in L_{loc}^1(\mathbb{R}^d \times \mathbb{R})$. When we use the kinetic formulation, we write b for $b(x, \xi)$.

1.2. Organization of the paper. In Section 2 we introduce the notions of entropy, kinetic and generalized kinetic solutions to (1.7), prove a flow-transformation result linking (1.7) to a scalar conservation law with random coefficients and prove the existence of generalized entropy solutions based on stable L^p -estimates. Some subtle measurability properties are postponed to the Appendix 5. The results and definitions in Section 2 are applicable under mild assumptions on b and, in particular, apply without change to the non-perturbed case. In Section 3 it is shown that generalized entropy solutions are entropy solutions and their uniqueness is deduced using certain parabolic PDE estimates given in Section 4.

2. DEFINITIONS AND THE EXISTENCE OF GENERALIZED KINETIC SOLUTIONS

In this section we give some general definitions and results, which hold also without noise. In the case of a smooth vector field b , there exists a unique entropy solution. In the general case, even the existence of an entropy solution may not hold in general. However, one can get the existence of a so-called generalized kinetic solution.

We start defining the concept of an entropy solution.

Definition 2.1. A (stochastic) bounded kinetic measure is a map $m : \Omega \rightarrow \mathcal{M}([0, T] \times \mathbb{R}^d \times \mathbb{R})$, weakly- $*$ measurable, satisfying the following properties:

- (i) $m \in L^\infty(\Omega; \mathcal{M}([0, T] \times \mathbb{R}^d \times \mathbb{R}))$;
- (ii) m is a.s. non-negative and supported on $[0, T] \times \mathbb{R}^d \times [-R, R]$ for some $R > 0$ independent of ω ;
- (iii) for every $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d \times \mathbb{R})$, the process $(t, \omega) \mapsto \int_{[0, t] \times \mathbb{R}^d \times \mathbb{R}} \varphi dm$ is progressively measurable.

Here and in what follows, we can extend definitions and formulations to test functions φ which are not necessarily compactly supported in the ξ variable, because of the assumption that m is supported on $[0, T] \times \mathbb{R}^d \times [-R, R]$.

Definition 2.2. Let $b \in L^1_{loc}(\mathbb{R}^{d+1})$ with $\operatorname{div}(b)$ in $L^1_{loc}(\mathbb{R}^{d+1})$ and let $u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$. An entropy solution to (1.7) is a measurable function $u : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$, such that $\chi(t, \omega, x, \xi) = \chi(u(t, \omega, x), \xi) = 1_{\xi < u(t, \omega, x)} - 1_{\xi < 0}$ satisfies the following properties:

- (i) $\chi \in L^\infty([0, T] \times \Omega; L^1(\mathbb{R}^d \times \mathbb{R}))$ and is supported on $[0, T] \times \Omega \times \mathbb{R}^d \times [-R, R]$ for some $R > 0$;
- (ii) χ is a weakly- $*$ progressively measurable $L^\infty_{x, \xi}$ -valued process;
- (iii) there exists a bounded kinetic measure m such that, for every test-function $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d \times \mathbb{R})$, it holds, for a.e. (t, ω) ,

$$(2.1) \quad \begin{aligned} \langle \chi_t, \varphi_t \rangle &= \langle \chi_0, \varphi_0 \rangle + \int_0^t \langle \chi, \partial_t \varphi + \operatorname{div}(b(x, \xi)\varphi) \rangle dr + \int_0^t \langle \chi, \nabla \varphi \rangle dW \\ &\quad + \frac{1}{2} \int_0^t \langle \chi, \Delta \varphi \rangle dr - \int_{[0, t] \times \mathbb{R}^d \times \mathbb{R}} \partial_\xi \varphi dm, \end{aligned}$$

$$\text{with } \chi_0(x, \xi) = 1_{\xi < u_0(x)} - 1_{\xi < 0}.$$

The function χ is called a kinetic solution.

The well-known definitions of entropy solutions, kinetic solutions and kinetic measures in the case of deterministic scalar conservation laws are recovered in the above definitions by removing the ω dependence, the progressive measurability assumptions as well as the second order term and stochastic integral in (2.1).

Remark 2.3. (i) For every kinetic solution χ and test function $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d \times \mathbb{R})$, $(t, \omega) \mapsto \langle \chi_t, \varphi_t \rangle$ is a semimartingale admitting a càdlàg version. More precisely, it admits a version which is the sum of a continuous martingale and a process with BV paths. Indeed, for every φ and every representative of m , for a.e. ω , the function $t \mapsto \int_{[0, t] \times \mathbb{R}^d \times \mathbb{R}} \partial_\xi \varphi dm$ is of finite variation. In particular, the processes $\int_{[0, t] \times \mathbb{R}^d \times \mathbb{R}} \partial_\xi \varphi dm$ and $\int_{[0, t] \times \mathbb{R}^d \times \mathbb{R}} \partial_\xi \varphi dm$ are progressively measurable and resp. càdlàg, càglàd.

(ii) More general, let $\varphi : [0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$ be a measurable bounded function such that: 1) for every (x, ξ, z) , $(t, \omega) \mapsto \varphi(t, \omega, x, \xi, z)$ is progressively measurable; 2) for a.e. ω , $(t, x, \xi, z) \mapsto \varphi(t, \omega, x, \xi, z)$ is continuous. Then, for every

representative of m , the maps $(t, z, \omega) \mapsto \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \varphi(r, \omega, x, \xi, z) m(r, x, \xi) dr dx d\xi$, $(t, z, \omega) \mapsto \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \varphi(r, \omega, x, \xi, z) m(r, x, \xi) dr dx d\xi$ are measurable and: 1) for each z fixed, progressively measurable in (t, ω) ; 2) for a.e. ω , with zero set independent of z , and each z , càdlàg, resp. càglàd, in t ; 3) for a.e. ω , with zero set independent of t , and each t fixed, continuous in z . This fact is a consequence of Remark 5.5 (ii) below, applied for z fixed, and of the continuity of the integral with respect to z , which follows from the dominated convergence theorem.

Remark 2.4. By equation (2.1), for every test function $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d \times \mathbb{R})$, the quadratic covariance between $\langle \chi, \nabla \varphi \rangle$ and W is $[\langle \chi, \nabla \varphi \rangle, W]_t = \int_0^t \langle \chi, \Delta \varphi \rangle dr$ for a.e. (t, ω) . Note that by abuse of notation we here use $\langle \chi, \nabla \varphi \rangle$ to also denote its càdlàg version. Therefore, the Stratonovich integral $\int_0^t \langle \chi, \nabla \varphi \rangle \circ dW$ makes sense and equation (2.1) can be rewritten as

$$\begin{aligned} \langle \chi_t, \varphi_t \rangle &= \langle \chi_0, \varphi_0 \rangle + \int_0^t \langle \chi, \partial_t \varphi + \operatorname{div}(b\varphi) \rangle dr + \int_0^t \langle \chi, \nabla \varphi \rangle \circ dW \\ &\quad - \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \partial_\xi \varphi dm. \end{aligned}$$

In particular, we see here that equation (2.1) is of hyperbolic type.

Remark 2.5. By the definition of χ we have immediately that, for every $1 \leq p < \infty$, for a.e. (t, ω, x) ,

$$(2.2) \quad \begin{aligned} u(t, \omega, x) &= \int_{\mathbb{R}} \chi(t, \omega, x, \xi) d\xi, \\ \frac{1}{p} |u(t, \omega, x)|^p &= \int_{\mathbb{R}} |\xi|^{p-1} \operatorname{sgn}(\xi) \chi(t, \omega, x, \xi) d\xi. \end{aligned}$$

Therefore, the weakly-* progressive measurability of χ implies that of u and $|u|^p$. Conversely, if u is an L_x^∞ -valued weakly-* progressively measurable process, then (by Proposition 5.2 below) u is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable as a real-valued function of (t, ω, x) and since $(v, \xi) \mapsto 1_{\xi < v} - 1_{\xi < 0}$ is a strictly measurable function, by Remark 5.4, the function $(t, \omega, x, \xi) \mapsto \chi(t, \omega, x, \xi)$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$ -measurable, that is, χ is a $L_{x, \xi}^\infty$ -valued weakly-* progressively measurable process. By the formulae above and the fact that $\chi = 0$ for $|\xi| > R$, we also have that u is in $L^\infty([0, T] \times \Omega; L^\infty(\mathbb{R}^d)) \cap L^\infty([0, T] \times \Omega; L^1(\mathbb{R}^d))$. Hence, u is in $L^\infty([0, T] \times \Omega; L^p(\mathbb{R}^d))$ and χ is in $L^\infty([0, T] \times \Omega; L^p(\mathbb{R}^d \times \mathbb{R}))$ for every $p \in [1, \infty]$.

2.1. A flow transformation. Before giving the existence result, we recall the following transformation that links equation (1.7) to a scalar conservation law with random coefficients.

Proposition 2.6. *Let $b \in L_{loc}^1(\mathbb{R}^{d+1})$ with $\operatorname{div}(b)$ in $L_{loc}^1(\mathbb{R}^{d+1})$. A function u is an entropy solution to (1.7) iff the function $\tilde{u}(t, x) := u(t, x + W_t)$ is L_x^∞ -valued weakly-* progressively measurable and is a.s. an entropy solution to*

$$(2.3) \quad \partial_t \tilde{u}(t, x) + b(x + W_t, \tilde{u}(t, x)) \cdot \nabla \tilde{u}(t, x) = 0.$$

More precisely, $\chi = \chi(u)$ is a kinetic solution to (1.7) with kinetic measure m iff:

- (i) $\tilde{\chi}(t, x, \xi) := 1_{\xi < \tilde{u}(t, \omega, x)} - 1_{\xi < 0} = \chi(t, x + W_t, \xi)$ is $L_{x, \xi}^\infty$ -valued weakly-* progressively measurable and $\tilde{m}(t, x, \xi) = m(t, x + W_t, \xi)$ is weakly-* progressively measurable, that is, for every $\psi \in C_c^\infty([0, T] \times \mathbb{R}^d \times \mathbb{R})$, the process $(t, \omega) \mapsto \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \psi d\tilde{m}$ is progressively measurable.

(ii) For a.e. ω , $\tilde{\chi}^\omega$ is a kinetic solution to (2.3) with kinetic measure \tilde{m}^ω . In particular, in the sense of distributions,

$$(2.4) \quad \partial_t \tilde{\chi} + b(x + W_t, \xi) \cdot \nabla \tilde{\chi} = \partial_\xi \tilde{m}.$$

Proof. Step 1: Progressive measurability.

Progressive measurability of $\tilde{\chi}$ can be deduced from progressive measurability of χ and vice versa. Indeed, for every φ in $C_c^\infty(\mathbb{R}^d \times \mathbb{R})$, $\langle \tilde{\chi}, \tilde{\varphi} \rangle = \langle \chi, \tilde{\varphi}(x - W_t, \xi) \rangle$ is progressively measurable, by Remark 5.5. A similar reasoning applies to \tilde{m} : For every $\tilde{\varphi} \in C_c^\infty([0, T] \times \mathbb{R}^d \times \mathbb{R})$, the process $(t, \omega) \mapsto \int_{[0, t] \times \mathbb{R}^d \times \mathbb{R}} \tilde{\varphi} d\tilde{m} = \int_{[0, t] \times \mathbb{R}^d \times \mathbb{R}} \varphi(r, x - W_r, \xi) dm$ is progressively measurable, again by Remark 5.5.

Step 2: Equation (2.1) implies (2.4).

Since for any (deterministic) test function $\tilde{\varphi}$, $\langle \tilde{\chi}, \tilde{\varphi} \rangle = \langle \chi, \tilde{\varphi}(x - W_t, \xi) \rangle$, the statement would follow if we could take $\tilde{\varphi}(x - W_t)$ as a test function. Unfortunately, this is not possible, since $\tilde{\varphi}(x + W_t)$ is not deterministic. Therefore, we use a regularization procedure: We consider χ^ϵ , a regularization of χ with respect to x and ξ . Then, for fixed x and ξ , we multiply χ^ϵ by $\tilde{\varphi}(x - W_t)$ using Itô's formula, integrate in x and ξ and pass to the limit $\epsilon \rightarrow 0$.

We consider a regularization of χ in both x and ξ , i.e. $\chi_t^\epsilon(x, \xi) := \langle \chi_t, \rho_\epsilon(x - \cdot) \bar{\rho}_\epsilon(\xi - \cdot) \rangle$. For every (x, ξ) , we have the following equation, outside a null set possibly depending on (x, ξ) and ϵ ,

$$\begin{aligned} \chi^\epsilon(t, x, \xi) &= \chi^\epsilon(0, x, \xi) + \int_0^t \langle \chi, \operatorname{div}(b(\cdot, \cdot) \rho_\epsilon(x - \cdot) \bar{\rho}_\epsilon(\xi - \cdot)) \rangle dr \\ &\quad + \int_0^t \langle \chi, \nabla \rho_\epsilon(x - \cdot) \bar{\rho}_\epsilon(\xi - \cdot) \rangle dW + \frac{1}{2} \int_0^t \langle \chi, \Delta \rho_\epsilon(x - \cdot) \bar{\rho}_\epsilon(\xi - \cdot) \rangle dr \\ &\quad - \int_{[0, t]} \langle m, \partial_\xi \rho_\epsilon(x - \cdot) \bar{\rho}_\epsilon(\xi - \cdot) \rangle dr. \end{aligned}$$

We multiply χ^ϵ by $\tilde{\varphi}(t, x - W_t, \xi)$ and use Itô's formula for càdlàg processes (see, for example, [41, Theorem 33]), applied to $f(x, y) = xy$. Note that no jump term appears here because the function f is bilinear and thus, with the notation of [41, Theorem 33], $f(x_s, y_s) - f(x_{s-}, y_s) - \nabla f(x_{s-}, y_s) \cdot \Delta x_s = 0$. Hence, we get, outside a null set as above,

$$\begin{aligned} &\chi^\epsilon(t, x, \xi) \tilde{\varphi}(t, x - W_t, \xi) \\ &= \chi^\epsilon(0, x, \xi) \tilde{\varphi}(0, x, \xi) + \int_0^t \chi^\epsilon(r, x, \xi) \partial_t \tilde{\varphi}(r, x - W, \xi) dr \\ &\quad + \int_0^t \langle \chi, \operatorname{div}(b(\cdot, \cdot) \rho_\epsilon(x - \cdot) \bar{\rho}_\epsilon(\xi - \cdot)) \rangle \tilde{\varphi}(r, x - W, \xi) dr \\ &\quad + \int_0^t \langle \chi, \nabla \rho_\epsilon(x - \cdot) \bar{\rho}_\epsilon(\xi - \cdot) \rangle \tilde{\varphi}(r, x - W, \xi) dW \\ &\quad + \frac{1}{2} \int_0^t \langle \chi, \Delta \rho_\epsilon(x - \cdot) \bar{\rho}_\epsilon(\xi - \cdot) \rangle \tilde{\varphi}(r, x - W, \xi) dr \\ &\quad - \int_{[0, t]} \langle m, \rho_\epsilon(x - \cdot) \partial_\xi \bar{\rho}_\epsilon(\xi - \cdot) \rangle \tilde{\varphi}(r, x - W, \xi) dr \\ &\quad - \int_0^t \chi^\epsilon(r, x, \xi) \nabla \tilde{\varphi}(r, x - W, \xi) dW + \frac{1}{2} \int_0^t \chi^\epsilon(r, x, \xi) \Delta \tilde{\varphi}(r, x - W, \xi) dr \\ &\quad + \int_0^t \langle \chi, \nabla \rho_\epsilon(x - \cdot) \bar{\rho}_\epsilon(\xi - \cdot) \rangle \cdot \nabla \tilde{\varphi}(r, x - W, \xi) dr. \end{aligned}$$

By the stochastic Fubini theorem (see for example [42], Exercise 5.17) and Remark 2.3, all the addends have measurable versions in (t, ω, x, ξ) , for which the equality above is true for a.e. (t, ω, x, ξ) and we can integrate in x and in ξ and exchange the order of integration. Therefore, integrating the above formula in (x, ξ) and bringing the convolution on φ , we get, with $\varphi(t, x, \xi) = \tilde{\varphi}(t, x - W_t, \xi)$,

$$\begin{aligned} \langle \chi_t, \varphi_t^\epsilon \rangle &= \langle \chi_0, \varphi_0^\epsilon \rangle + \int_0^t \langle \chi, \partial_t \varphi^\epsilon \rangle dr + \int_0^t \langle \chi, \operatorname{div}(b\varphi^\epsilon) \rangle dr \\ &\quad + \int_0^t \langle \chi, \nabla \varphi^\epsilon \rangle dW + \frac{1}{2} \int_0^t \langle \chi, \Delta \varphi^\epsilon \rangle dr - \int_{[0,t]} \langle m, \partial_\xi \varphi^\epsilon \rangle dr \\ &\quad - \int_0^t \langle \chi, \nabla \varphi^\epsilon \rangle dW + \frac{1}{2} \int_0^t \langle \chi, \Delta \varphi^\epsilon \rangle dr - \int_0^t \langle \chi, \Delta \varphi^\epsilon \rangle dr \\ &= \langle \chi_0, \varphi_0^\epsilon \rangle + \int_0^t \langle \chi, \partial_t \varphi^\epsilon \rangle dr + \int_0^t \langle \chi, \operatorname{div}(b\varphi^\epsilon) \rangle dr - \int_{[0,t]} \langle m, \partial_\xi \varphi^\epsilon \rangle dr. \end{aligned}$$

Finally, we let ϵ go to 0 and use the change of variable $\tilde{x} = x - W_t$, to obtain

$$(2.5) \quad \langle \tilde{\chi}_t, \tilde{\varphi}_t \rangle = \langle \tilde{\chi}_0, \tilde{\varphi}_0 \rangle + \int_0^t \langle \tilde{\chi}, \partial_t \tilde{\varphi} \rangle dr + \int_0^t \langle \tilde{\chi}, \operatorname{div}(\tilde{b}\tilde{\varphi}) \rangle dr - \int_{[0,t]} \langle \tilde{m}, \partial_\xi \tilde{\varphi} \rangle dr.$$

This formula is valid for every $\tilde{\varphi}$ smooth test function (with compact support), on a full measure set in (t, ω) which can depend on $\tilde{\varphi}$. To make this set independent of $\tilde{\varphi}$, we use a density argument. Let D be a countable dense set in $C_c^\infty(\mathbb{R}^d \times \mathbb{R})$ and let F be a full measure set in (t, ω) such that $\tilde{m}(\omega)$ is a bounded measure and (2.5) holds for every (t, ω) in F and for every $\tilde{\varphi}$ in D . For a given test function $\tilde{\varphi}$, take a sequence $(\tilde{\varphi}_n)_n$ in D converging to $\tilde{\varphi}$ in C_b^2 ; passing to the limit in (2.5) for $\tilde{\varphi}_n$ (using that $\tilde{\chi}$ is bounded for every (t, ω)), we get (2.5) for $\tilde{\varphi}$ for every (t, ω) in F . The proof of the first part is complete.

Step 3: Equation (2.4) and weak-* progressive measurability imply (2.1).

Since the strategy is similar to that of the first part, we will only sketch it. We regularize $\tilde{\chi}$ by convolving it with an approximate identity, obtaining $\tilde{\chi}^\epsilon$. It follows from the progressive measurability hypothesis that $\tilde{\chi}^\epsilon$ is an Itô process, therefore, for every test function φ , we can multiply it by $\varphi(t, x + W_t, \xi)$ and apply Itô's formula. By Fubini's theorem, the stochastic Fubini theorem and Remark 2.3 we can integrate in x and in ξ and exchange the order of integration, then we bring the convolution on φ , let ϵ go to 0 and change variable to get finally (2.1). \square

2.2. The case of smooth coefficients. In this section we consider the case of a smooth coefficient $b \in C_c^\infty(\mathbb{R}^{d+1})$ and a smooth initial condition $u_0 \in C_c^\infty(\mathbb{R}^d)$ and derive stable a priori bounds.

Proposition 2.7. *Let $u_0 \in C_c^\infty(\mathbb{R}^d)$ and $b \in C_c^\infty(\mathbb{R}^{d+1})$. Then there is a unique entropy solution u to (1.7). Moreover, we have*

$$(2.6) \quad \operatorname{ess\,sup}_{\omega \in \Omega} \sup_{t \in [0, T]} \|u(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty}$$

and, for every $p \geq 1$ finite,

$$(2.7) \quad \begin{aligned} \operatorname{ess\,sup}_{\omega \in \Omega} \sup_{t \in [0, T]} \|u(t)\|_{L^p}^p + p(p-1) \int_0^T \int \int |\xi|^{p-2} m d\xi dx dr \\ \leq \|u_0\|_{L^p}^p + p \|u_0\|_{L_x^\infty}^{p-1} \|\operatorname{div} b\|_{L^1([0, T] \times \mathbb{R}^d \times [-\|u_0\|_{L_x^\infty}, \|u_0\|_{L_x^\infty})}. \end{aligned}$$

Moreover, χ and m are supported a.s. on $[0, T] \times \mathbb{R}^d \times [-\|u_0\|_{L^\infty}, \|u_0\|_{L^\infty}]$.

Proof. Step 1: We start with the equation

$$(2.8) \quad \partial_t v + g(t, x, v) \cdot \nabla v = 0$$

for some $g \in C([0, T]; C_b^3(\mathbb{R}^{d+1}))$, i.e. three times continuously differentiable with bounded derivatives, and initial condition $u_0 \in C_c^\infty(\mathbb{R}^d)$.

As a consequence of [33, Corollary 2.4 and Theorem 2.5], there exists a unique entropy solution $v = v^g \in L^\infty([0, T]; L^1(\mathbb{R}^d))$, v^g is in $C([0, T]; L^1(\mathbb{R}^d))$ and the map $C([0, T]; C_b^3(\mathbb{R}^{d+1})) \ni g \mapsto v^g \in C([0, T]; L^1(\mathbb{R}^d))$ is locally Lipschitz continuous. Note that, denoting by $\chi = \chi^g$ the associated kinetic solution, $\|\chi_t - \chi_s\|_{L_{x,\xi}^1} = \|v_t - v_s\|_{L_x^1}$ and $\|\chi_t^{g^1} - \chi_t^{g^2}\|_{L_{x,\xi}^1} = \|v_t^{g^1} - v_t^{g^2}\|_{L_x^1}$. Consequently, χ is in $C([0, T]; L^1(\mathbb{R}^d \times \mathbb{R}))$ and the map $C([0, T]; C_b^3(\mathbb{R}^{d+1})) \ni g \mapsto \chi^g \in C([0, T]; L^1(\mathbb{R}^d \times \mathbb{R}))$ is locally Lipschitz continuous. As a consequence, the maps $[0, T] \times C([0, T]; C_b^3(\mathbb{R}^{d+1})) \ni (t, g) \mapsto v_t^g \in L^1(\mathbb{R}^d)$ and $[0, T] \times C([0, T]; C_b^3(\mathbb{R}^{d+1})) \ni (t, g) \mapsto \chi_t^g \in L^1(\mathbb{R}^d \times \mathbb{R})$ are continuous.

Following [33, Corollary 2.4 and Theorem 2.5] the entropy solution $v = v^g$ can be constructed by first approximating g by a smooth g^δ and then considering a vanishing viscosity approximation. That is, v can be obtained as an a.e. limit of the solutions $v^{\varepsilon, \delta}$ to

$$\partial_t v^{\varepsilon, \delta} + g^\delta(t, x, v^{\varepsilon, \delta}) \cdot \nabla v^{\varepsilon, \delta} = \varepsilon \Delta v^{\varepsilon, \delta}.$$

The maximum principle applied to these equations yields $\|v^{\varepsilon, \delta}\|_{L_{t,x}^\infty} \leq \|u_0\|_{L_x^\infty}$. Passing to the limit, we obtain the bound

$$(2.9) \quad \|v\|_{L_{t,x}^\infty} \leq \|u_0\|_{L_x^\infty}.$$

This implies that, for every t and for a.e. x , $\chi(t, x, \cdot) = \chi^g(t, x, \cdot)$ is supported on $[-\|u_0\|_{L_x^\infty}, \|u_0\|_{L_x^\infty}]$.

The existence of a kinetic measure m^g associated to χ^g can be derived as in [15, Section 2.2] extended to the time dependent and non conservative case, that is, for every φ compactly supported we have

$$(2.10) \quad \int_{[0, T] \times \mathbb{R}^d \times \mathbb{R}} \partial_\xi \varphi dm^g = -\langle \chi_T^g, \varphi_T \rangle + \langle \chi_0^g, \varphi_0 \rangle + \int_0^T \langle \chi^g, \partial_t \varphi + \operatorname{div}(g(r, x, \xi) \varphi) \rangle dr.$$

Therefore, m^g is uniquely determined, is supported on $[0, T] \times \mathbb{R}^d \times [-\|u_0\|_{L_x^\infty}, \|u_0\|_{L_x^\infty}]$ and (2.10) is satisfied for all smooth φ compactly supported in x .

In order to obtain the estimate (2.7), we consider the test functions given by $(\operatorname{sgn}(\xi)|\xi|^{p-1})^\varepsilon \psi_{1/\varepsilon}(x)$, where $\psi_{1/\varepsilon}$ is an increasing sequence of smooth functions, $[0, 1]$ -valued, with values 1 on $B_{1/\varepsilon}$, 0 on $B_{2/\varepsilon}^c$ and such that $|\nabla \psi_{1/\varepsilon}(x)| \leq 2\varepsilon$ for every x and $(\operatorname{sgn}(\xi)|\xi|^{p-1})^\varepsilon := \operatorname{sgn}(\cdot) \cdot |\cdot|^{p-1} *_{\xi} \bar{\rho}_\varepsilon$. In particular, $\operatorname{sgn}(\xi)(\operatorname{sgn}(\xi)|\xi|^{p-1})^\varepsilon$ is a sequence of nonnegative functions converging pointwise on $\mathbb{R} \setminus \{0\}$ to $|\xi|^{p-1}$. Moreover, in the case $p > 1$, $\partial_\xi (\operatorname{sgn}(\xi)|\xi|^{p-1})^\varepsilon$ is a sequence of nonnegative functions converging pointwise on \mathbb{R} to $(p-1)|\xi|^{p-2}$, with the convention $|0|^{p-2} = +\infty$ for $p < 2$ and $|0|^0 = 1$ for $p = 2$. Due to (2.10), we have

$$\begin{aligned} & \langle \chi_t^g, (\operatorname{sgn}(\xi)|\xi|^{p-1})^\varepsilon \psi_{1/\varepsilon} \rangle + \int_{[0, t] \times \mathbb{R}^d \times \mathbb{R}} \frac{d}{d\xi} (\operatorname{sgn}(\xi)|\xi|^{p-1})^\varepsilon \psi_{1/\varepsilon} dm^g \\ &= \langle \chi_0^g, (\operatorname{sgn}(\xi)|\xi|^{p-1})^\varepsilon \psi_{1/\varepsilon} \rangle + \int_0^t \langle \chi^g, \operatorname{div} g(r, x, \xi) (\operatorname{sgn}(\xi)|\xi|^{p-1})^\varepsilon \psi_{1/\varepsilon} \rangle dr \\ &+ \int_0^t \langle \chi^g, g(r, x, \xi) (\operatorname{sgn}(\xi)|\xi|^{p-1})^\varepsilon \cdot \nabla \psi_{1/\varepsilon} \rangle dr. \end{aligned}$$

In the case $p > 1$, taking the \liminf for $\epsilon \rightarrow 0$ and recalling that $\chi_t^g(\operatorname{sgn}(\xi)|\xi|^{p-1})^\epsilon = |\chi_t^g| \operatorname{sgn}(\xi) |\xi|^{p-1}$, applying Fatou's lemma for the second term on the left hand side and the dominated convergence theorem for the remaining terms, we get

$$\begin{aligned} & \langle |\chi_t^g|, |\xi|^{p-1} \rangle + (p-1) \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} |\xi|^{p-2} dm^g \\ & \leq \langle |\chi_0^g|, |\xi|^{p-1} \rangle + \int_0^t \langle |\chi_r^g|, |\operatorname{div}g| |\xi|^{p-1} \rangle dr. \end{aligned}$$

Recalling (2.2) and (2.9) we obtain

$$\begin{aligned} & \|v^g\|_{L^p}^p + p(p-1) \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} |\xi|^{p-2} dm^g \\ (2.11) \quad & \leq \|u_0\|_{L^p}^p + p \int_0^t \int \int (\operatorname{div}g) |\xi|^{p-1} 1_{|\xi| \leq \|u_0\|_{L_x^\infty}} dx d\xi dr \\ & \leq \|u_0\|_{L^p}^p + p \|u_0\|_{L_x^\infty}^{p-1} \|\operatorname{div}g\|_{L^1([0,T] \times \mathbb{R}^d \times [-\|u_0\|_{L_x^\infty}, \|u_0\|_{L_x^\infty})} \end{aligned}$$

In particular, taking $p = 2$, we see that $\|m\|_{\mathcal{M}_{t,x,\xi}}$ is bounded in terms of u_0 and $\|\operatorname{div}g\|_{L^1([0,T] \times \mathbb{R}^d \times [-\|u_0\|_{L_x^\infty}, \|u_0\|_{L_x^\infty})}$. In the case $p = 1$, proceeding as before we get

$$\begin{aligned} & \|v^g\|_{L^1} + \liminf_{\epsilon \rightarrow 0} \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \frac{d}{d\xi} (\operatorname{sgn}(\xi))^\epsilon \psi_{1/\epsilon}(x) dm^g \\ & \leq \|u_0\|_{L_x^1} + \|\operatorname{div}g\|_{L^1([0,T] \times \mathbb{R}^d \times [-\|u_0\|_{L_x^\infty}, \|u_0\|_{L_x^\infty})}. \end{aligned}$$

In particular, recalling again (2.2), this gives the global $L_{x,\xi}^1$ bound

$$(2.12) \quad \sup_{t \in [0,T]} \|\chi^g(t)\|_{L_{x,\xi}^1} \leq \|u_0\|_{L_x^1} + \|\operatorname{div}g\|_{L^1([0,T] \times \mathbb{R}^d \times [-\|u_0\|_{L_x^\infty}, \|u_0\|_{L_x^\infty})}.$$

Step 2: We apply the previous results to $g = \tilde{b}^\omega = b(x + W_t(\omega), u)$ and, by Proposition 2.6, get the existence of an entropy solution that, by a change of variables, satisfies the estimates. The technical details are not difficult but not immediate, since we have to pass from a process with values in a space of functions of x to a measurable function of (t, ω, x) :

1) The map $(t, \omega) \mapsto \bar{u}(t, \omega) = v_t^{\tilde{b}^\omega}$ is measurable bounded from $\mathcal{B}([0, T]) \otimes \mathcal{F}_T$ to $\mathcal{B}(L_x^1)$, for every T , since it is the composition of the measurable map $(t, \omega) \mapsto (t, \tilde{b}^\omega)$ from $\mathcal{B}([0, T]) \otimes \mathcal{F}_T$ to $\mathcal{B}([0, T]) \otimes \mathcal{B}(C([0, T]; C_b^3(\mathbb{R}^{d+1})))$ and the continuous map $(t, g^\omega) \mapsto v_t^g$. Since the L_x^1 -valued process \bar{u} admits a time-continuous version, \bar{u} is actually measurable bounded from \mathcal{P} (the predictable σ -algebra) to $\mathcal{B}(L_x^1)$; in particular, it is weakly measurable with respect to \mathcal{P} . Therefore, by Proposition 5.2, there exists \tilde{u} in $L^1([0, T] \times \Omega \times \mathbb{R}^d, \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d))$ version of \bar{u} (in the sense that, for a.e. (t, ω) , $\bar{u}(t, \omega) = \tilde{u}(t, \omega)$). By the L^∞ bounds, \tilde{u} is in $L_{t,\omega,x}^\infty$. Similarly the map $(t, \omega) \mapsto \bar{\chi}(t, \omega) = \chi_t^{\tilde{b}^\omega}$ is measurable bounded from $\mathcal{B}([0, T]) \otimes \mathcal{F}_T$ to $\mathcal{B}(L_{x,\xi}^1)$, for every T and there exists $\tilde{\chi}$ in $L^1([0, T] \times \Omega \times \mathbb{R}^d, \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d))$ representing $\bar{\chi}$; clearly $\tilde{\chi} = 1_{\bar{u} < \xi} - 1_{0 < \xi}$ and $\tilde{\chi}$ is supported on $[-\|u_0\|_{L_x^\infty}, \|u_0\|_{L_x^\infty}]$.

2) Calling $u(t, \omega, x) = \tilde{u}(t, \omega, x - W_t(\omega))$, $\chi(t, \omega, x, \xi) = \tilde{\chi}(t, \omega, x - W_t(\omega), \xi)$, then u and χ are also $L_{x,\xi}^\infty$ -valued weakly-* progressively measurable. Indeed, for every φ in $C_c^\infty(\mathbb{R}^d \times \mathbb{R})$, $\langle u, \varphi \rangle = \langle \tilde{u}, \varphi(\cdot + W_t, \cdot) \rangle$ is progressively measurable, by Remark 5.5.

3) The map $\omega \mapsto \bar{m}(\omega) = m^{\tilde{b}^\omega}$ is bounded as an $\mathcal{M}_{t,x,\xi}$ -valued function, nonnegative and supported on $[0, T] \times \mathbb{R}^d \times [-\|u_0\|_{L_x^\infty}, \|u_0\|_{L_x^\infty}]$. Moreover, for every ψ in $C_c^\infty([0, T] \times \mathbb{R}^d \times \mathbb{R})$, calling φ a primitive function of ψ , then $\int_{[0,T] \times \mathbb{R}^d \times \mathbb{R}} \psi d\bar{m} = \int \partial_\xi \varphi d\bar{m}$ is \mathcal{F}_T -measurable for every T by equation (2.4) and admits a continuous version, hence is progressively measurable. Calling $m(t, \omega, x, \xi) = \bar{m}(t, \omega, x -$

$W_t(\omega, \xi)$ (more precisely, the image measure of \bar{m} under $(t, \omega, x, \xi) \mapsto (t, \omega, x + W_t(\omega, \xi))$), then $\int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \psi dm = \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \psi(t, x - W_t, \xi) dm$ is progressively measurable, by Remark 5.5. Therefore m is a kinetic measure.

4) By Proposition 2.6, u is an entropy solution of (1.7), with kinetic function χ and kinetic measure m .

5) Changing variable $x' = x - W_t$ in (2.11) and in (2.12), we get the estimates (2.6) and (2.7). \square

2.3. Existence of generalized kinetic solutions. We introduce the notion of a generalized kinetic solution.

Definition 2.8. Let $f_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d \times \mathbb{R})$. A generalized kinetic solution to (1.7) is a measurable function $f : [0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ with the following properties:

- (i) $f \in L^\infty([0, T] \times \Omega; L^1(\mathbb{R}^d \times \mathbb{R}))$ and is supported on $[0, T] \times \Omega \times \mathbb{R}^d \times [-R, R]$ for some $R > 0$;
- (ii) f is a weakly-* progressively measurable $L_{x, \xi}^\infty$ -valued process;
- (iii) there exists a kinetic bounded measure m on $[0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}$ such that, for every $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d \times \mathbb{R})$, it holds a.s.,

$$(2.13) \quad \langle f_t, \varphi_t \rangle = \langle f_0, \varphi_0 \rangle + \int_0^t \langle f, \partial_t \varphi + \operatorname{div}(b(x, \xi)\varphi) \rangle dr + \int_0^t \langle f, \nabla \varphi \rangle dW \\ + \frac{1}{2} \int_0^t \langle f, \Delta \varphi \rangle dr - \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \partial_\xi \varphi dm.$$

- (iv) there exists a kinetic bounded measure ν on $[0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}$, which moreover is in $L^\infty([0, T] \times \Omega, \mathcal{M}(\mathbb{R}^d \times \mathbb{R}))$, such that, for every $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d \times \mathbb{R})$, it holds for a.e. (t, ω) ,

$$(2.14) \quad |f(t, x, \xi)| = \operatorname{sgn}(\xi) f(t, x, \xi) \leq 1, \quad \text{for a.e. } (x, \xi), \\ \langle f_t, -\partial_\xi \varphi_t \rangle = \int \varphi(t, x, 0) dx - \int_{\mathbb{R}^d \times \mathbb{R}} \varphi_t d\nu_t.$$

A formal, short-hand notation for (2.13) and (2.14) is

$$\partial_t f + b(x, \xi) \cdot \nabla f + \nabla f \circ dW_t = \partial_\xi m,$$

and

$$|f|(t, x, \xi) = \operatorname{sgn}(\xi) f(t, x, \xi) \leq 1, \\ \frac{\partial f}{\partial \xi} = \delta(\xi) - \nu(t, x, \xi).$$

Remark 2.9. Kinetic solutions are a particular type of generalized kinetic solutions. Indeed, if $f_0(x, \xi) := \chi(u_0(x), \xi)$ and χ is a kinetic solution to (1.7), with associated kinetic measure m , then χ is also a generalized solution with kinetic measure m and $\nu = \delta_{\xi=u(t, \omega, x)}$.

The following theorem asserts the existence of a generalized kinetic solution.

Theorem 2.10. Let $b \in L_{loc}^1(\mathbb{R}^{d+1})$ with $\operatorname{div}(b)$ in $L_{\xi, loc}^1(L_x^1)$ and $u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$. Then there exists a generalized kinetic solution f to (1.7) starting from $f_0(x, \xi) := \chi(u_0(x), \xi)$.

Proof. Step 1: Approximation of f and convergence.

We introduce smooth approximations $b^\varepsilon \in C_c^\infty(\mathbb{R}^{d+1})$ of b with $b^\varepsilon \rightarrow b$ in $L_{x,\xi,loc}^1$ and $\operatorname{div} b^\varepsilon \rightarrow \operatorname{div} b$ in $L_{\xi,loc}^1(L_x^1)$ and $u_0^\varepsilon \in C_c^\infty(\mathbb{R}^d)$ of u_0 such that $\|u_0^\varepsilon\|_{L^p} \leq \|u_0\|_{L^p}$ for all $p \geq 1$ and $u_0^\varepsilon \rightarrow u_0$ in L_x^1 . We consider the corresponding unique entropy solution u^ε (see Proposition (2.7)) to

$$\partial_t u^\varepsilon(t, x) + b^\varepsilon(x, u^\varepsilon(t, x)) \cdot \nabla u^\varepsilon(t, x) + \nabla u^\varepsilon(t, x) \circ dW_t = 0,$$

that is $\chi^\varepsilon = \chi(u^\varepsilon)$ solves

$$(2.15) \quad \partial_t \chi^\varepsilon + b^\varepsilon(x, \xi) \cdot \nabla \chi^\varepsilon + \nabla \chi^\varepsilon \circ dW_t = \partial_\xi m^\varepsilon.$$

Since $|\chi^\varepsilon| \leq 1$, the sequence χ^ε converges weakly-*, up to taking a subsequence, to a limit f in $L^\infty([0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R})$. Since for every φ in $L_{x,\xi}^1$, $\langle \chi^\varepsilon, \varphi \rangle$ is progressively measurable, also $\langle f, \varphi \rangle$ is progressively measurable.

Step 2: Bounds and support of f .

Using Proposition 2.7 and $\operatorname{div}(b) \in L_{\xi,loc}^1(L_x^1)$, we obtain that $(\chi^\varepsilon)^+ = \chi^\varepsilon \vee 0$ is uniformly bounded in $L_{t,\omega}^\infty(L_{x,\xi}^1)$. Therefore, identifying $\chi^\varepsilon(x, \xi)$ with $\chi^\varepsilon(x, \xi) dx d\xi$, $(\chi^\varepsilon)^+$ is uniformly bounded in $L_{t,\omega}^\infty(\mathcal{M}_{x,\xi,+})$. By Theorem 5.3, up to the selection of a subsequence, $(\chi^\varepsilon)^+$ converges weakly-* in $L_{t,\omega}^\infty(\mathcal{M}_{x,\xi,+})$ to an element $g^+ \in L_{t,\omega}^\infty(\mathcal{M}_{x,\xi,+})$. Similarly $(\chi^\varepsilon)^- = (-\chi^\varepsilon) \vee 0$ converges weakly-*, up to the selection of a subsequence, to an element $g^- \in L_{t,\omega}^\infty(\mathcal{M}_{x,\xi,+})$. Moreover, we can take the same subsequence for the weakly-* convergence of χ^ε in $L_{t,\omega,x,\xi}^\infty$ and of $(\chi^\varepsilon)^+$ and $(\chi^\varepsilon)^-$ in $L_{t,\omega}^\infty(\mathcal{M}_{x,\xi,+})$. By a density argument, we see that $g := g^+ - g^- = f$. In particular,

$$\begin{aligned} \|f\|_{L_{t,\omega}^\infty(L_{x,\xi}^1)} &= \|g\|_{L_{t,\omega}^\infty(\mathcal{M}_{x,\xi})} \leq \|g^+\|_{L_{t,\omega}^\infty(\mathcal{M}_{x,\xi})} + \|g^-\|_{L_{t,\omega}^\infty(\mathcal{M}_{x,\xi})} \\ &\leq 2 \sup_\varepsilon \|\chi^\varepsilon\|_{L_{t,\omega}^\infty(L_{x,\xi}^1)}. \end{aligned}$$

For the support property of f , again Proposition 2.7 ensures that the functions χ^ε are concentrated a.s. on $[0, T] \times \mathbb{R}^d \times [-\|u_0\|_\infty, \|u_0\|_\infty]$. Therefore, $\mathbb{E}[F \langle \chi^\varepsilon, \varphi \rangle_{t,x,\xi}] = 0$ for every φ in $L^1([0, T] \times \mathbb{R}^d \times \mathbb{R})$ with support outside $[0, T] \times \mathbb{R}^d \times [-\|u_0\|_\infty, \|u_0\|_\infty]$ and every F in L_ω^1 . Passing to the limit in this equality, we conclude that f is concentrated a.s. on $[0, T] \times \mathbb{R}^d \times [-\|u_0\|_\infty, \|u_0\|_\infty]$.

Step 3: Convergence of m^ε .

By Proposition 2.7 (applied with $p = 2$), m^ε is a bounded sequence in the space $L^\infty(\Omega; \mathcal{M}_+([0, T] \times \mathbb{R}^d \times \mathbb{R}))$, therefore, by Theorem 5.3, it converges weakly-*, up to subsequences, to a limit m in $L^\infty(\Omega; \mathcal{M}_+([0, T] \times \mathbb{R}^d \times \mathbb{R}))$. The weakly-* progressive measurability, in the sense of Definition 2.1, of m is verified as for f . The support property of m follows from Proposition 2.7 as for f , replacing $L^1([0, T] \times \mathbb{R}^d \times \mathbb{R})$ with $C_0([0, T] \times \mathbb{R}^d \times \mathbb{R})$.

Step 4: Equation (2.13).

Equation (2.13) is obtained passing to the limit in (2.15) for φ in $C_c^\infty([0, T] \times \mathbb{R}^d \times \mathbb{R})$, exploiting the linearity of the equation and using that $b \cdot \nabla \varphi$ and $\varphi \operatorname{div} b$ are in $L_{t,\omega,x,\xi}^1$. More precisely, we multiply (2.15) by a measurable bounded function $G = G(t, \omega)$, we integrate in t and ω and we pass to the limit, thanks to the weak-* convergence of χ^ε and m^ε . By arbitrariness of G we get (2.13).

Step 5: Properties (2.14).

The bound $\|f\|_{L_{t,\omega,x,\xi}^\infty} \leq 1$ follows from the same bound for χ^ε . The property $|f| = \operatorname{sgn}(\xi) f$ follows from passing to the limit in $\mathbb{E}[\langle \chi^\varepsilon, \operatorname{sgn}(\xi) G \rangle_{t,x,\xi}] \geq 0$ for every

G nonnegative function in $L^1([0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R})$. Further, we have for a.e. (t, ω) ,

$$(2.16) \quad \langle \chi_t^\varepsilon, -\partial_\xi \varphi_t \rangle = \int_{\mathbb{R}^d} \varphi(t, x, 0) dx - \int_{\mathbb{R}^d \times \mathbb{R}} \varphi_t d\nu_t^\varepsilon,$$

where $d\nu^\varepsilon = \delta_{\xi=u^\varepsilon(t,x)} dx dt$. In particular, ν^ε is a bounded sequence in $L^\infty(\Omega \times [0, T]; \mathcal{M}(\mathbb{R}^d \times \mathbb{R}))$ of kinetic measures. Proceeding as for m^ε , we get that ν^ε converges weakly-*, up to subsequences, to a bounded kinetic measure ν . Passing to the limit in (2.16) in a way similar to the proof of equation (2.13), we obtain (2.14). \square

Remark 2.11. For any generalized kinetic solution f , which by definition is in $L^\infty(\Omega \times [0, T]; L^1(\mathbb{R}^d \times \mathbb{R})) \cap L^\infty(\Omega \times [0, T]; L^\infty(\mathbb{R}^d \times \mathbb{R}))$, we have by interpolation $f \in L^\infty(\Omega \times [0, T]; L^p(\mathbb{R}^d \times \mathbb{R}))$ for every $1 \leq p \leq \infty$. Moreover the global $L^1_{x,\xi}$ bound allows to consider also bounded test functions, independent of ξ , which are in $L^\infty([0, T]; W^{2,\infty}(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d; W^{1,\infty}([0, T]))$.

The following lemma will be useful in the next section.

Lemma 2.12. *Let f be a generalized kinetic solution to equation (1.7). For every test function ψ in $C_c^\infty(\mathbb{R}^d \times \mathbb{R})$, there exist measurable functions $f(\psi)^+$, $f(\psi)^-$ on $[0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}$, versions of $f *_{x,\xi} \psi$ (that is, for every (x, ξ) , $f(\psi)^+(x, \xi)$ and $f(\psi)^-(x, \xi)$ coincide with $f *_{x,\xi} \psi$ on a full-measure set in $[0, T] \times \Omega$, possibly depending on (x, ξ) and ψ), with the following properties:*

- (i) for every (x, ξ) , $f(\psi)^+(x, \xi)$, $f(\psi)^-(x, \xi)$ are progressively measurable processes;
- (ii) for a.e. ω it holds: for every (x, ξ) , $f(\psi)^+(x, \xi)$ is càdlàg, $f(\psi)^-(x, \xi)$ is càglàd;
- (iii) for a.e. ω it holds: for every t , $f(\psi)^+$ is $C^1_{x,\xi}$ and $\nabla_{x,\xi} f(\psi)^+ = f(\nabla_{x,\xi} \psi)^+$ and similarly for $f(\psi)^-$.

The above lemma is similar to Remark 2.3 but with the additional property (iii). The existence of such versions is needed when dealing with terms of the form $\int \partial_\xi f(\psi)^+ dm$, since for these both the precise version in time and the differentiability in ξ are needed.

Proof of Lemma 2.12. We call $\varphi^{x,\xi}(y, \zeta) = \psi(x-y, \xi-\zeta)$. We know that, for every (x, ξ) , it holds for a.e. (t, ω) ,

$$(2.17) \quad \begin{aligned} \langle f_t, \varphi_t^{x,\xi} \rangle &= \langle f_0, \varphi_0^{x,\xi} \rangle + \int_0^t \langle f, \partial_t \varphi^{x,\xi} + \operatorname{div}_y(b\varphi^{x,\xi}) \rangle dr + \int_0^t \langle f, \nabla_y \varphi^{x,\xi} \rangle dW \\ &+ \frac{1}{2} \int_0^t \langle f, \Delta_y \varphi^{x,\xi} \rangle dr - \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \partial_\zeta \varphi^{x,\xi} dm. \end{aligned}$$

For the integrals $\int_0^t \langle f, \partial_t \varphi^{x,\xi} + \operatorname{div}_y(b\varphi^{x,\xi}) \rangle dr$, $\frac{1}{2} \int_0^t \langle f, \Delta_y \varphi^{x,\xi} \rangle dr$, there exist resp. versions $A(t, \omega, x, \xi)$, $B(t, \omega, x, \xi)$ which satisfy the first and the third property above and are continuous (a.s.) in (t, x, ξ) . Such a version $C(t, \omega, x, \xi)$ exists also for the stochastic integral $-\int_0^t \langle f, \nabla_y \varphi^{x,\xi} \rangle dW$, by stochastic Fubini theorem (see again [42], Exercise 5.17). Finally, for $-\int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \partial_\zeta \varphi^{x,\xi} dm$, by Remark 2.3 there exist versions $D^+(t, \omega, x, \xi)$, $D^-(t, \omega, x, \xi)$ which satisfy the first and the third property above and are resp. càdlàg, càglàd for fixed (x, ξ) . Therefore $f(\psi)^+ = \langle f_0, \varphi_0^{x,\xi} \rangle + A + B + C + D^+$ and $f(\psi)^- = \langle f_0, \varphi_0^{x,\xi} \rangle + A + B + C + D^-$ are measurable versions of $f *_{x,\xi} \psi$ with the desired properties. \square

From now on, when this does not create confusion, the first three integrals in formula (2.17) will denote their continuous versions. The càdlàg version D^+ of the last integral will be denoted still by $-\int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \partial_\zeta \varphi^{x,\xi} dm$, while the càglàd version D^- by $-\int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \partial_\zeta \varphi^{x,\xi} dm$, coherently with the continuity property in t of the integral on $[0, t]$.

Remark 2.13. Consider $f^{\epsilon,\delta} = f *_{x,\xi} (\rho_\epsilon \bar{\rho}_\delta)$, where $\rho = \rho(x)$, $\bar{\rho} = \bar{\rho}(\xi)$ are two C_c^∞ even functions and $\rho_\epsilon(x) = \epsilon^{-d} \rho(\epsilon^{-1}x)$, $\bar{\rho}_\delta(\xi) = \delta^{-1} \bar{\rho}(\delta^{-1}\xi)$. We call $f^{\epsilon,\delta,+}$, $f^{\epsilon,\delta,-}$ the versions of $f^{\epsilon,\delta}$ as in the previous Lemma. Note that, by construction, for a.e. ω , it holds for every (t, x, ξ) , with the above convention on the integrals,

$$\begin{aligned} f_t^{\epsilon,\delta,+}(x, \xi) &= f_0(x, \xi) + \int_0^t \frac{1}{2} \Delta f^{\epsilon,\delta}(x, \xi) dr - \int_0^t (b \cdot \nabla f)^{\epsilon,\delta}(x, \xi) dr \\ &\quad - \int_0^t \nabla f^{\epsilon,\delta}(x, \xi) dW_r + \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \rho^\epsilon(x-y) (\bar{\rho}^\delta)'(\xi-\zeta) m(r, y, \zeta) dy d\zeta dr, \end{aligned}$$

where $(b \cdot \nabla f)^{\epsilon,\delta} = \nabla(\rho^\epsilon \bar{\rho}^\delta) *_{x,\xi} (bf) + (\rho^\epsilon \bar{\rho}^\delta) *_{x,\xi} (\operatorname{div}(b)f)$. The integrands $\int_{\mathbb{R}^d \times \mathbb{R}} \rho^\epsilon(x-y) \bar{\rho}^\delta(\xi-\zeta) m(\cdot, y, \zeta) dy d\zeta$, $\int_{\mathbb{R}^d \times \mathbb{R}} \rho^\epsilon(x-y) (\bar{\rho}^\delta)'(\xi-\zeta) m(\cdot, y, \zeta) dy d\zeta$ will be denoted resp. by $m^{\epsilon,\delta}(r, x, \xi)$ and $\partial_\xi m^{\epsilon,\delta}(\cdot, x, \xi)$ and they are measures on $[0, T]$ parametrized by (ω, x, ξ) . Moreover, for every fixed representative of m and every test function ψ , the function $(t, \omega, x, \xi) \mapsto \int_{[0,t]} f(\psi)^+(x, \xi) \partial_\xi m^{\epsilon,\delta}(r, x, \xi) dr$ is measurable, càdlàg in t and continuous in (x, ξ) for a.e. ω , and, for a.e. ω , it holds that for every $t \geq 0$,

$$\begin{aligned} &\int_{\mathbb{R}^d \times \mathbb{R}} \int_{[0,t]} f(\psi)^+(r, x, \xi) \partial_\xi m^{\epsilon,\delta}(r, x, \xi) dr dx d\xi \\ &= \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \int_{\mathbb{R}^d \times \mathbb{R}} f(\psi)^+(r, x, \xi) \rho^\epsilon(x-y) (\bar{\rho}^\delta)'(\xi-\zeta) dx d\xi m(r, y, \zeta) dy d\zeta dr \\ &= - \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \partial_\xi f(\psi)^+(r, x, \xi) m^{\epsilon,\delta}(r, x, \xi) dx d\xi dr. \end{aligned}$$

Indeed, the measurability follows from Remark 5.5 below applied at (x, ξ) fixed and from the continuity property of the integral with respect to (x, ξ) . The above equality follows from Fubini's theorem, Lemma 2.12 and the càdlàg property of the integrals. An analogous property holds replacing $f(\psi)^+$ with $f(\psi)^+ \varphi$ or $f(\psi)^- \varphi$ for regular test functions φ .

3. WELL-POSEDNESS OF ENTROPY SOLUTIONS

In this section we prove the well-posedness by noise result, namely the existence, uniqueness and stability of entropy solutions:

Theorem 3.1. *Assume that $b \in L_{\xi,loc}^\infty(L_x^\infty) \cap L_{\xi,loc}^1(W_{x,loc}^{1,1})$ and that $\operatorname{div} b \in L_{\xi,loc}^1(L_x^1) \cap L_x^p(L_{\xi,loc}^\infty)$ for some $p > d$, $p \leq \infty$. For every initial datum u_0 in $(L^1 \cap L^\infty)(\mathbb{R}^d)$, there exists a unique entropy solution u to (1.7). Moreover, for every initial data u_0^1, u_0^2 in $(L^1 \cap L^\infty)(\mathbb{R}^d)$, the two corresponding entropy solutions u^1, u^2 satisfy*

$$\mathbb{E} \int |u_t^1 - u_t^2| dx \leq C \int |u_0^1 - u_0^2| dx,$$

for a.e. $t \in [0, T]$ and some constant $C > 0$, depending only on T , $\|b\|_{L_{\xi,[-M,M]}^\infty(L_x^\infty)}$ and $\|\operatorname{div} b\|_{L_x^p(L_{\xi,[-M,M]}^\infty)}$, where $M = \max\{\|u_0^1\|_{L_x^\infty}, \|u_0^2\|_{L_x^\infty}\}$.

Remark 3.2. As it will be clear from the proof, the result can be generalized to fluxes with $b(x, u)$ replaced by

$$\sum_{k=1}^N b_k(x, u),$$

where b_k are vector fields satisfying the assumptions of Theorem 3.1 with integrability exponents $p_k > d$ (i.e. $\operatorname{div} b_k \in L_x^{p_k}(L_{\xi, \operatorname{loc}}^{\infty})$) which can be different one from each others.

The proof of Theorem 3.1 follows from the following two preliminary results, the key estimate being the following

Lemma 3.3. *Assume that $b \in L_{\xi, \operatorname{loc}}^{\infty}(L_x^{\infty}) \cap L_{\xi, \operatorname{loc}}^1(W_{x, \operatorname{loc}}^{1,1})$ and that $\operatorname{div} b \in L_x^p(L_{\xi, \operatorname{loc}}^{\infty})$ for some $p > d$, $p \leq \infty$. Let f be a generalized kinetic solution to (1.7), supported on $[0, T] \times \Omega \times \mathbb{R}^d \times [-R, R]$ for some $R \geq 0$. Then,*

$$(3.1) \quad \mathbb{E} \int_{\mathbb{R}^d \times \mathbb{R}} (|f_t| - f_t^2) d\xi dx \leq C \int_{\mathbb{R}^d \times \mathbb{R}} (|f_0| - f_0^2) d\xi dx,$$

for a.e. $t \in [0, T]$ and some constant $C > 0$, depending only on T , $\|b\|_{L_{\xi, [-R, R]}^{\infty}(L_x^{\infty})}$ and $\|\operatorname{div} b\|_{L_x^p(L_{\xi, [-R, R]}^{\infty})}$.

Note that $|f_t| - f_t^2 \geq 0$ for any generalized kinetic solution, since $|f| \leq 1$ by definition. When the initial datum f_0 is the kinetic function of some u_0 , that is, if $f_0(x, \xi) = \chi(u_0)(x, \xi)$, then Lemma 3.3 implies that f takes values in $\{0, \pm 1\}$. In this case f is a true kinetic function:

Proposition 3.4. *Assume that b satisfies the assumptions of Lemma 3.3 and let f be a generalized kinetic solution to (1.7) starting from $f_0 = \chi(u_0)$, for some u_0 in $(L^1 \cap L^{\infty})(\mathbb{R}^d)$. Then there exists an entropy solution u to (1.7) such that $f(x, \xi, t) = \chi(\xi, u(x, t))$ a.e. in (t, ω, x, ξ) .*

Lemma 3.3 and Proposition 3.4, together with Theorem 2.10, imply the well-posedness result Theorem 3.1:

Proof of Theorem 3.1. Concerning the existence of an entropy solution, Theorem 2.10 yields the existence of a generalized kinetic solution f to (1.7). Proposition 3.4 then implies the existence of an entropy solution to (1.7).

For stability, let $\chi^i = \chi(u^i, \xi)$ be the kinetic functions associated to u^i , $i = 1, 2$. Note that $|\chi^1 - \chi^2|^2 = |\chi^1 - \chi^2| = 1_{u^1 \leq \xi < u^2} + 1_{u^2 \leq \xi < u^1}$ for a.e. ξ and, in particular, $\int |\chi_t^1 - \chi_t^2|^2 d\xi = |u_t^1 - u_t^2|$. Therefore, the statement is equivalent to

$$(3.2) \quad \mathbb{E} \int |\chi_t^1 - \chi_t^2|^2 dx d\xi \leq C \int |\chi_0^1 - \chi_0^2|^2 dx d\xi.$$

Now consider $f := \frac{1}{2}(\chi^1 + \chi^2)$. Then f is a generalized kinetic solution, with associated Young measure $\nu = \delta_0 - \frac{1}{2}(\delta_{\xi=u^1} + \delta_{\xi=u^2})$. Moreover,

$$\begin{aligned} |f| - f^2 &= \frac{1}{2} \operatorname{sgn}(\xi)(\chi^1 + \chi^2) - \frac{1}{4}((\chi^1)^2 + (\chi^2)^2 + 2\chi^1\chi^2) \\ &= \frac{1}{2}(|\chi^1| + |\chi^2|) - \frac{1}{4}(|\chi^1| + |\chi^2| + 2\chi^1\chi^2) \\ &= \frac{1}{4}|\chi^1 - \chi^2|^2. \end{aligned}$$

Therefore, applying Lemma 3.3 implies (3.2). Uniqueness follows from stability, thus, the proof is complete. \square

In order to prove Lemma 3.3, we will use the equations (more precisely, certain inequalities) satisfied by $|f|$ and f^2 . We recall that, since f satisfies a transport-type equation, for any function β regular enough, informally $\beta(f)$ also satisfies a transport-type equation. This property is known as renormalization. When coming to a rigorous proof, however, problems can appear from the drift term, when b is not regular enough and from the kinetic measure term m . The Sobolev assumption on b , as in the theory of DiPerna-Lions [19] and Ambrosio [1], ensures that the drift term behaves nicely. The presence of the kinetic measure m does not allow to write an equation for $|f|$ and f^2 themselves but is enough for the following inequality.

Lemma 3.5. *Assume that $b \in L_{\xi,loc}^1(W_{x,loc}^{1,1})$. Let f be a generalized kinetic solution to (1.7). Then, for every nonnegative test function φ in $C_c^\infty([0, T] \times \mathbb{R}^d)$ independent of ξ , it holds for a.e. (t, ω) ,*

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}} (|f_t| - f_t^2) \varphi_t \, dx d\xi &\leq \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}} [\partial_t \varphi + \frac{1}{2} \Delta \varphi + \operatorname{div}(b\varphi)] (|f| - f^2) \, dx d\xi dr \\ &\quad + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}} \nabla \varphi (|f| - f^2) \, dx d\xi dW_r. \end{aligned}$$

Proof. Step 1: We start with the equation for $|f|$. Since, by (2.14), $|f| = f \operatorname{sgn}(\xi)$, we aim to use $\operatorname{sgn}(\xi)$ as a test function in (2.13). To do so, we regularize sgn via $\operatorname{sgn} *_{\xi} \bar{\rho}^\delta =: \operatorname{sgn}^\delta$. Note that $\partial_\xi \operatorname{sgn}^\delta = 2\bar{\rho}^\delta$. For technical reasons that will become clear in the second step, we write an equation for $\int_{\mathbb{R}^d \times \mathbb{R}} f^{\epsilon, \delta} \operatorname{sgn}^\delta(\xi) \varphi \, dx d\xi$, where $f^{\epsilon, \delta} = f *_{x, \xi} (\rho^\epsilon \bar{\rho}^\delta)$, that is, we take $(\operatorname{sgn}^\delta(\xi) \varphi) *_{x, \xi} (\rho^\epsilon \bar{\rho}^\delta)$ as a test function in (2.13). Moreover, again in (2.13) we take the càdlàg version of the integral and thus get

$$\begin{aligned} &\int_{\mathbb{R}^d \times \mathbb{R}} f_t^{\epsilon, \delta, +} \operatorname{sgn}^\delta(\xi) \varphi \, dx d\xi \\ &= \int_{\mathbb{R}^d \times \mathbb{R}} f_0^{\epsilon, \delta} \operatorname{sgn}^\delta(\xi) \varphi_0 \, dx d\xi + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}} f^{\epsilon, \delta} \operatorname{sgn}^\delta(\xi) (\partial_t \varphi + \frac{1}{2} \Delta \varphi) \, dx d\xi dr \\ (3.3) \quad &+ \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}} f b \cdot \nabla (\operatorname{sgn}^\delta(\xi) \varphi)^{\epsilon, \delta} \, dx d\xi dr + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}} f \operatorname{div} b (\operatorname{sgn}^\delta(\xi) \varphi)^{\epsilon, \delta} \, dx d\xi dr \\ &- \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}} f^{\epsilon, \delta} \operatorname{sgn}^\delta(\xi) \nabla \varphi \, dx d\xi dW_r - 2 \int_{[0, t] \times \mathbb{R}^d \times \mathbb{R}} \varphi \bar{\rho}^\delta(\xi) m^{\epsilon, \delta} \, dx d\xi dr, \end{aligned}$$

where $f^{\epsilon, \delta, +}$ is the càdlàg version of $f^{\epsilon, \delta}$ (see Remark 2.13).

Step 2: For f^2 , we would like to take $f\varphi$ as a test function in (2.13). Since f is not regular, we regularize it in both x and ξ . More precisely we take $f^{\epsilon, \delta, +}$, $f^{\epsilon, \delta, -}$ resp. càdlàg, càglàd versions of $f^{\epsilon, \delta}$, as in Lemma 2.12 and Remark 2.13. Itô's formula for càdlàg processes (cf. [41, Theorem 33]) yields

$$\begin{aligned} &(f_t^{\epsilon, \delta, +}(x, \xi))^2 \varphi_t(x) - (f_0^{\epsilon, \delta}(x, \xi))^2 \varphi_0(x) \\ &= \int_0^t f_r^{\epsilon, \delta, +}(x, \xi)^2 \partial_t \varphi_r(x) \, dr + \int_{[0, t]} (f_r^{\epsilon, \delta, +}(x, \xi) + f_r^{\epsilon, \delta, -}(x, \xi)) \varphi_r(x) \, df_r^{\epsilon, \delta} \\ &\quad + \int_0^t \varphi_r(x) \, d[f^{\epsilon, \delta}]_r \\ &= \int_0^t f_r^{\epsilon, \delta}(x, \xi)^2 \partial_t \varphi_r(x) \, dr + \int_0^t f_r^{\epsilon, \delta}(x, \xi) \Delta f_r^{\epsilon, \delta}(x, \xi) \varphi_r(x) \, dr \\ &\quad - \int_0^t 2 f_r^{\epsilon, \delta}(x, \xi) (b \cdot \nabla f_r)^{\epsilon, \delta}(x, \xi) \varphi_r(x) \, dr - \int_0^t 2 f_r^{\epsilon, \delta}(x, \xi) \nabla f_r^{\epsilon, \delta}(x, \xi) \varphi_r(x) \, dW_r \end{aligned}$$

$$\begin{aligned}
& + \int_{[0,t]} (f_r^{\epsilon,\delta,+}(x,\xi) + f_r^{\epsilon,\delta,-}(x,\xi)) \varphi_r(x) \partial_\xi m^{\epsilon,\delta}(r,x,\xi) dx d\xi dr \\
& + \int_0^t |\nabla f_r^{\epsilon,\delta}(x,\xi)|^2 \varphi_r(x) dr.
\end{aligned}$$

This formula is valid for each (x, ξ) for a.e. (t, ω) , where the exceptional set may depend on (x, ξ) . However, by Remark 2.13, for a fixed representative of m , the integral with the kinetic measure m is measurable and càdlàg in t for (ω, x, ξ) fixed and continuous in (x, ξ) for (t, ω) fixed. Also the other integrals have versions that are continuous in (t, x, ξ) for ω fixed and, in particular, are measurable in (t, ω, x, ξ) . For such versions, for a.e. ω , the above equality above holds for every (t, x, ξ) . The idea at this point is to first integrate in x and ξ , then to use integration by parts to bring the derivatives onto φ and thereby to get an equation for $f^{\epsilon,\delta,+}$ which is similar to the one satisfied by f itself, plus a remainder. Indeed, integrating in x and ξ , using Remark 2.13, Fubini's theorem and the stochastic Fubini theorem, we obtain the following equality, valid for every t and for every ω in a full-measure set independent of t ,

$$\begin{aligned}
& \int_{\mathbb{R}^d \times \mathbb{R}} (f_t^{\epsilon,\delta,+}(x,\xi))^2 \varphi_t(x) dx d\xi - \int_{\mathbb{R}^d \times \mathbb{R}} (f_0^{\epsilon,\delta}(x,\xi))^2 \varphi_0(x) dx d\xi \\
& = \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}} f_r^{\epsilon,\delta}(x,\xi)^2 \left(\partial_t \varphi_r(x) + \frac{1}{2} \Delta \varphi_r(x) + \operatorname{div}(b(x,\xi) \varphi_r(x)) \right) dx d\xi dr \\
& + 2 \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}} \int_{\mathbb{R}^d \times \mathbb{R}} f_r^{\epsilon,\delta}(x,\xi) f_r(y,\zeta) \bar{\rho}_\delta(\xi - \zeta) \left(\nabla \rho_\epsilon(x-y) \cdot (b(x,\xi) - b(y,\zeta)) \right. \\
& \quad \left. + \rho_\epsilon(x-y) \operatorname{div}_y b(y,\zeta) \right) \varphi_r(x) dy d\zeta dx d\xi dr \\
& - \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} (\partial_\xi f_r^{\epsilon,\delta,+} + \partial_\xi f_r^{\epsilon,\delta,-})(x,\xi) m^{\epsilon,\delta}(r,x,\xi) \varphi_r(x) dx d\xi dr \\
& - \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}} f_r^{\epsilon,\delta}(x,\xi)^2 \nabla \varphi_r(x) dx d\xi dW_r.
\end{aligned}$$

For the third addend, note that, for every (x, ξ) , it holds for a.e. (t, ω) , $\partial_\xi f^{\epsilon,\delta}(x, \xi) = \bar{\rho}_\delta(\xi) - \nu *_\xi \bar{\rho}_\delta(\xi)$ (the convolution being in the ξ direction) and so $\partial_\xi f^{\epsilon,\delta}(x, \xi) \geq \bar{\rho}_\delta(\xi)$. Therefore, by the càdlàg/càglàd properties of $\partial_\xi f^{\epsilon,\delta,+}$ and $\partial_\xi f^{\epsilon,\delta,-}$, for a.e. ω , it holds for every (t, x, ξ) , $\partial_\xi f^{\epsilon,\delta,+}(x, \xi) \geq \bar{\rho}_\delta(\xi)$ and $\partial_\xi f^{\epsilon,\delta,-}(x, \xi) \geq \bar{\rho}_\delta(\xi)$. So we obtain

$$\begin{aligned}
& - \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} (\partial_\xi f_r^{\epsilon,\delta,+} + \partial_\xi f_r^{\epsilon,\delta,-})(x,\xi) m^{\epsilon,\delta}(r,x,\xi) \varphi_r(x) dx d\xi dr \\
& \leq -2 \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \bar{\rho}_\delta(\xi) m^{\epsilon,\delta}(r,x,\xi) \varphi_r(x) dx d\xi dr.
\end{aligned}$$

Here we see the reason for the additional regularization of $f^{\epsilon,\delta}$ in the first step: in this way the right hand side of the above inequality is equal to the last term in formula (3.3). In conclusion we get, for a.e. ω (on a full-measure set independent of t), for every t ,

$$\begin{aligned}
& \int_{\mathbb{R}^d \times \mathbb{R}} (f_t^{\epsilon,\delta,+} \operatorname{sgn}^\delta(\xi) - (f_t^{\epsilon,\delta,+})^2) \varphi_t dx d\xi - \int_{\mathbb{R}^d \times \mathbb{R}} (f_0^{\epsilon,\delta} \operatorname{sgn}^\delta(\xi) - (f_0^{\epsilon,\delta})^2) \varphi_0 dx d\xi \\
& \leq \int_0^t \int f^{\epsilon,\delta} \operatorname{sgn}^\delta(\xi) (\partial_t \varphi_r + \frac{1}{2} \Delta \varphi_r) dx d\xi dr \\
& + \int_0^t \int f b \cdot \nabla (\operatorname{sgn}^\delta(\xi) \varphi)^{\epsilon,\delta} dx d\xi dr + \int_0^t \int f \operatorname{div} b (\operatorname{sgn}^\delta(\xi) \varphi)^{\epsilon,\delta} dx d\xi dr
\end{aligned}$$

$$\begin{aligned}
(3.4) \quad & - \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}} (f^{\epsilon, \delta})^2 (\partial_t \varphi + \frac{1}{2} \Delta \varphi + \operatorname{div}(b\varphi)) \, dx d\xi dr \\
& - \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}} (f^{\epsilon, \delta} \operatorname{sgn}^\delta(\xi) - (f^{\epsilon, \delta})^2) \nabla \varphi \, dx d\xi dW_r \\
& - 2 \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}} \int_{\mathbb{R}^d \times \mathbb{R}} f_r^{\epsilon, \delta}(x, \xi) f_r(y, \zeta) \bar{\rho}_\delta(\xi - \zeta) (\nabla \rho_\epsilon(x - y) \cdot (b(x, \xi) - b(y, \zeta)) \\
& \quad + \rho_\epsilon(x - y) \operatorname{div}_y b(y, \zeta)) \varphi_r(x) \, dy d\zeta \, dx d\xi dr.
\end{aligned}$$

Step 3: The last addend in the right hand side above is the commutator error, which by Lemma 3.6 below goes to zero in $L_{t, \omega}^2$ letting first $\delta \rightarrow 0$ and then $\epsilon \rightarrow 0$. Therefore, taking the $L_{t, \omega}^2$ -limit in (3.4) first for $\delta \rightarrow 0$ then for $\epsilon \rightarrow 0$, we obtain the statement. \square

We used the following commutator lemma:

Lemma 3.6. *Assume that $b \in L_{\xi, \text{loc}}^1(W_{x, \text{loc}}^{1,1})$. Then it holds, for every finite $m \geq 1$,*

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \mathbb{E} \int_0^T & \left| \int_{\mathbb{R}^d \times \mathbb{R}} \int_{\mathbb{R}^d \times \mathbb{R}} f_r^{\epsilon, \delta}(x, \xi) f_r(y, \zeta) \bar{\rho}_\delta(\xi - \zeta) (\nabla \rho_\epsilon(x - y) \cdot (b(x, \xi) - b(y, \zeta)) \right. \\
& \quad \left. + \rho_\epsilon(x - y) \operatorname{div}_y b(y, \zeta)) \varphi_r(x) \, dy d\zeta \, dx d\xi \right|^m dr \\
& = 0.
\end{aligned}$$

Proof. The proof is obtained by adapting the classical commutator lemma (see for example [2, 19]) to this anisotropic regularization in x and ξ , which was also used in [12]. Since b is weakly differentiable in the x -variable, we have for a.e. (x, y, ξ)

$$b(x, \xi) - b(y, \xi) = \int_0^1 D_x b(y + a(x - y), \xi) (x - y) \, da.$$

This formula can be obtained by approximation of b in $L_{\xi, \text{loc}}^1(W_{x, \text{loc}}^{1,1})$ with regular b^n . By the change of variable $z = (x - y)/\epsilon$, $\eta = (\xi - \zeta)/\delta$, we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^d \times \mathbb{R}} \int_{\mathbb{R}^d \times \mathbb{R}} f_r^{\epsilon, \delta}(x, \xi) f_r(y, \zeta) \bar{\rho}_\delta(\xi - \zeta) (\nabla \rho_\epsilon(x - y) \cdot (b(x, \xi) - b(y, \zeta)) \\
& \quad + \rho_\epsilon(x - y) \operatorname{div}_y b(y, \zeta)) \varphi_r(x) \, dy d\zeta \, dx d\xi \\
& = \int_0^1 \int \bar{\rho}(\eta) \nabla \rho(z) \cdot \int f_r^{\epsilon, \delta}(x, \xi) f_r(x - \epsilon z, \xi - \delta \eta) \cdot D_x b(x - \epsilon z, \xi) z \varphi_r(x) \, dx d\xi dz d\eta da \\
& + \int \bar{\rho}(\eta) \rho(z) \int f_r^{\epsilon, \delta}(x, \xi) f_r(x - \epsilon z, \xi - \delta \eta) \operatorname{div} b(x - \epsilon z, \xi - \delta \eta) \varphi_r(x) \, dx d\xi dz d\eta \\
& + \frac{1}{\epsilon} \int \bar{\rho}(\eta) \nabla \rho(z) \cdot \int f_r^{\epsilon, \delta}(x, \xi) f_r(x - \epsilon z, \xi - \delta \eta) (b(x, \xi - \delta \eta) - b(x, \xi)) \varphi_r(x) \, dx d\xi dz d\eta \\
(3.5) \quad & =: A + B + C
\end{aligned}$$

Here and in the following we can suppose without loss of generality that all the integrals range over a compact set independent of ϵ , δ , r and ω , since the test functions φ , ρ , $\bar{\rho}$ are compactly supported and $f_r(x, \xi)$ and $f_r^{\epsilon, \delta}(x, \xi)$ are compactly supported in ξ uniformly in ϵ , δ , r and ω . We start with the first integral A on the right hand side of (3.5). We first take the $L_{t, \omega}^m$ -limit as $\delta \rightarrow 0$ and we find that A

converges to

$$(3.6) \quad \int_0^1 \int \nabla \rho(z) \cdot \int f_r^\epsilon(x, \xi) f_r(x - \epsilon z, \xi) \cdot D_x b(x - a\epsilon z, \xi) z \varphi_r(x) \, dx d\xi dz da,$$

where $f^\epsilon(x, \xi) = f(\cdot, \xi) * \rho^\epsilon(x)$. Since the proof of this fact is standard and relies arguments similar to but simpler than the proof of the limit as $\epsilon \rightarrow 0$, we omit it. Now we take the $L_{t,\omega}^m$ -limit of s(3.6) as $\epsilon \rightarrow 0$. First we fix z, a, r and ω . For the inner integral, we have

$$\begin{aligned} & \int f_r^\epsilon(x, \xi) f_r(x - \epsilon z, \xi) D_x b(x - a\epsilon z, \xi) z \varphi_r(x) \, dx d\xi - \int f_r(x, \xi)^2 \cdot D_x b(x, \xi) z \varphi_r(x) \, dx d\xi \\ &= \int f_r^\epsilon(x, \xi) f_r(x - \epsilon z, \xi) (D_x b(x - a\epsilon z, \xi) - D_x b(x - \epsilon z, \xi)) z \varphi_r(x) \, dx d\xi \\ & \quad + \int f_r^\epsilon(x, \xi) (f_r(x - \epsilon z, \xi) D_x b(x - \epsilon z, \xi) - f_r(x, \xi) D_x b(x, \xi)) z \varphi_r(x) \, dx d\xi \\ & \quad + \int (f_r^\epsilon(x, \xi) - f_r(x, \xi)) f_r(x, \xi) D_x b(x, \xi) z \varphi_r(x) \, dx d\xi \end{aligned}$$

The first addend on the right hand side above goes to 0 for $\epsilon \rightarrow 0$, since both $D_x b(x - a\epsilon z, \xi)$ and $D_x b(x - \epsilon z, \xi)$ tend to $D_x b(x, \xi)$ in $L_{x,\xi}^1$ by continuity of translation and $f^\epsilon(x, \xi) f(x - \epsilon z, \xi) \varphi(x)$ is bounded in $L_{x,\xi}^\infty$ uniformly in ϵ . The second addend also goes to 0 for $\epsilon \rightarrow 0$, since $f(x - \epsilon z, \xi) D_x b(x - a\epsilon z, \xi)$ tends to $f(x, \xi) D_x b(x, \xi)$ in $L_{x,\xi}^1$ by continuity of translation and $f^\epsilon(x, \xi) \varphi(x)$ is bounded in $L_{x,\xi}^\infty$ uniformly in ϵ . Finally, the third addend goes to 0 by dominated convergence, since $f^\epsilon(x, \xi) - f(x, \xi)$ tends to 0 for a.e. (x, ξ) and the integrand is bounded by $C|D_x b|(x, \xi)$, for some $C > 0$. Therefore, for fixed z, η, r and a , the inner integral in the first addend of (3.5) converges to $\int f_r^2(x, \xi) D_x b(x, \xi) z \varphi_r(x) \, dx d\xi$. Since, moreover, the inner integral is bounded uniformly in z, a, r and ω , dominated convergence implies for A

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \left(\lim_{\delta \rightarrow 0} \left(\int_0^1 \int \bar{\rho}(\eta) \nabla \rho(z) \cdot \int f_r^{\epsilon, \delta}(x, \xi) f_r(x - \epsilon z, \xi - \delta \eta) \cdot \right. \right. \\ & \quad \left. \left. \cdot D_x b(x - a\epsilon z, \xi) z \varphi_r(x) \, dx d\xi dz d\eta da \right) \right) \\ &= \int_0^1 \int \nabla \rho(z) \cdot \int f_r(x, \xi)^2 \cdot D_x b(x, \xi) z \varphi_r(x) \, dx d\xi dz da \\ &= - \int f_r(x, \xi)^2 \operatorname{div} b(x, \xi) \varphi_r(x) \, dx d\xi, \end{aligned}$$

where we have used that $\int \partial_i \rho(z) z_j \, dz = -\delta_{ij}$ and the limits are taken in $L_{t,\omega}^m$. Similarly, for the second integral B on the right hand side of (3.5) we have

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \left(\lim_{\delta \rightarrow 0} \left(\int \bar{\rho}(\eta) \rho(z) \int f_r^{\epsilon, \delta}(x, \xi) f_r(x - \epsilon z, \xi - \delta \eta) \cdot \right. \right. \\ & \quad \left. \left. \cdot \operatorname{div} b(x - \epsilon z, \xi - \delta \eta) \varphi_r(x) \, dx d\xi dz d\eta \right) \right) \\ &= \int f_r(x, \xi)^2 \operatorname{div} b(x, \xi) \varphi_r(x) \, dx d\xi, \end{aligned}$$

where again the limits are taken in $L_{t,\omega}^m$. For the third integral C, again with similar reasoning but now taking only the limit $\delta \rightarrow 0$, we get

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \mathbb{E} \int_0^T \left| \frac{1}{\epsilon} \int \bar{\rho}(\eta) \nabla \rho(z) \cdot \right. \\ & \quad \left. \cdot \int f_r^{\epsilon, \delta}(x, \xi) f(x - \epsilon z, \xi - \delta \eta) (b(x, \xi - \delta \eta) - b(x, \xi)) \varphi(x) \, dx d\xi dz d\eta \right|^m dr = 0. \end{aligned}$$

Putting together these limits we obtain the desired statement. \square

We are ready to prove the key Lemma 3.3.

Proof of Lemma 3.3. By Lemma 3.5 we have, for every nonnegative test function φ in $C_c^\infty([0, T] \times \mathbb{R}^d)$ independent of ξ , for a.e. t (with the exceptional set possibly depending on φ),

$$(3.7) \quad \begin{aligned} \mathbb{E} \int_{\mathbb{R}^d \times \mathbb{R}} (|f_t| - f_t^2) \varphi_t \, dx d\xi &\leq \int_{\mathbb{R}^d \times \mathbb{R}} (|f_0| - f_0^2) \varphi_0 \, dx d\xi \\ &+ \mathbb{E} \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}} [\partial_t \varphi + \frac{1}{2} \Delta \varphi + \operatorname{div}(b(x, \xi) \varphi)] (|f| - f^2) \, dx d\xi dr, \end{aligned}$$

where we used that $\int_0^t \int_{\mathbb{R}^d \times \mathbb{R}} \nabla \varphi (|f| - f^2) \, dx d\xi dW_r$ is an L^2 martingale with zero mean, since $\nabla \varphi (|f| - f^2)$ is bounded and compactly supported.

The idea at this point is to use duality, that is, we would like to take a test function φ , independent of ξ , nonnegative and sufficiently regular, with $\varphi_T > 0$, such that, for every ξ in a bounded interval $[-R, R]$,

$$(3.8) \quad \partial_t \varphi + \frac{1}{2} \Delta \varphi + \operatorname{div}(b(x, \xi) \varphi) \leq C.$$

Then we could conclude by Gronwall's inequality. To do so, the strategy is to take φ as a nonnegative solution to

$$\partial_t \varphi + \frac{1}{2} \Delta \varphi + F(x) \varphi = 0, \quad \varphi(t_{fin}, x) = 1,$$

where $F(x) = \|\operatorname{div} b(x, \cdot)\|_{L_{\xi, B_R}^\infty}$ (measurable function), with t_{fin} a given time and R such that the support of f is in $[0, T] \times \Omega \times \mathbb{R}^d \times [-R, R]$, and then to use a bound on the transport term $b \cdot \nabla \varphi$ to obtain (3.8). For technical reasons, we take, for $\epsilon, t_{fin} > 0$ fixed, φ^ϵ to be a solution on $[0, t_{fin}]$ to

$$(3.9) \quad \partial_t \varphi^\epsilon + \frac{1}{2} \Delta \varphi^\epsilon + F^\epsilon \varphi^\epsilon = 0, \quad \varphi^\epsilon(t_{fin}, x) = \psi_{1/\epsilon}(x),$$

where $\psi_{1/\epsilon}$ is a C_c^∞ nonnegative function, with values in $[0, 1]$, equal to 1 on $B_{1/\epsilon}(0)$ and uniformly bounded (in ϵ) in the $W^{1, \infty}(\mathbb{R}^d)$ norm; F^ϵ is a compactly supported regularization of F , converging to F a.e. and in L^p , if $p < \infty$, or a.e. and with uniform L^∞ bound, if $p = \infty$. We extend φ^ϵ to the whole interval $[0, T]$ by taking $\varphi^\epsilon(t, x) = \psi_{1/\epsilon}(x)$ for $t \in [t_{fin}, T]$. By Remark 4.1 below, φ^ϵ is nonnegative and in $L_t^\infty(W_x^{2, \infty}) \cap L_x^\infty(W_t^{1, \infty})$ for every $\epsilon > 0$. Therefore, reasoning as in Remark 2.11, φ^ϵ can be used as test function in (3.7). Consequently, we have, for a.e. $t \leq t_{fin}$,

with the exceptional set $N^{\epsilon, t_{fin}}$ possibly depending on ϵ and t_{fin} ,

$$\begin{aligned}
& \mathbb{E} \int \varphi_t^\epsilon (|f_t| - f_t^2) dx d\xi \\
& \leq \int_{\mathbb{R}^d \times \mathbb{R}} \varphi_0^\epsilon (|f_0| - f_0^2) dx d\xi \\
& \quad + \int_0^t \mathbb{E} \int [\partial_t \varphi^\epsilon + \frac{1}{2} \Delta \varphi^\epsilon + F^\epsilon \varphi^\epsilon] (|f| - f^2) dx d\xi dr \\
& \quad + \int_0^t \mathbb{E} \int [b \cdot \nabla \varphi^\epsilon + \operatorname{div}(b) \varphi^\epsilon - F \varphi^\epsilon] (|f| - f^2) dx d\xi dr \\
(3.10) \quad & \quad + \int_0^t \mathbb{E} \int [F \varphi^\epsilon - F^\epsilon \varphi^\epsilon] (|f| - f^2) dx d\xi dr \\
& \leq \int_0^t \mathbb{E} \int (b \cdot \nabla \varphi^\epsilon) (|f| - f^2) dx d\xi dr \\
& \quad + \int_0^t \mathbb{E} \int (F - F^\epsilon) \varphi^\epsilon (|f| - f^2) dx d\xi dr,
\end{aligned}$$

where we have used that $|f| - f^2 \geq 0$ and that f is supported on $[0, T] \times \Omega \times \mathbb{R}^d \times [-R, R]$.

Before passing to the limit $\epsilon \rightarrow 0$, we aim to replace t by t_{fin} in the above inequality. This is not immediate, since the function $t \mapsto \mathbb{E}[|f_t| - f_t^2]$ is not known to be (even weakly) continuous. To overcome this difficulty, we fix a version of the map $[0, T] \rightarrow L^1(\mathbb{R}^d \times \mathbb{R})$ given by $t \mapsto \mathbb{E}[|f_t| - f_t^2]$ and we use Lusin's theorem for separable Banach space-valued functions, see for example [37]: for every $\delta > 0$, there exists a measurable set $A_\delta \subseteq [0, T]$ with Lebesgue measure $|A_\delta| \geq T - \delta$, such that $t \mapsto \mathbb{E}[|f_t| - f_t^2]$ is continuous on A_δ as an $L^1(\mathbb{R}^d \times \mathbb{R})$ -valued map. We can also assume that A_δ has no points which are isolated from the left, where we say that t_0 is isolated from the left in A_δ if $(t_0 - \eta, t_0) \cap A_\delta = \emptyset$ for some $\eta > 0$. Indeed, the set of points of A_δ which are isolated from the left is at most countable and thus has zero Lebesgue measure. Therefore, for $t_{fin} \in A_\delta$, we can find a sequence $t_n \leq t_{fin}$ in $A_\delta \setminus N^{\epsilon, t_{fin}}$ converging to t_{fin} (as $n \rightarrow \infty$) and such that (3.10) holds for t_n and $\mathbb{E}[|f_{t_n}| - f_{t_n}^2] \rightarrow \mathbb{E}[|f_{t_{fin}}| - f_{t_{fin}}^2]$ in $L^1(\mathbb{R}^d \times \mathbb{R})$. Moreover, by Remark 4.1 φ^ϵ is in $L_x^\infty(W_t^{1, \infty})$ and so the map $[0, T] \rightarrow L^\infty(\mathbb{R}^d \times \mathbb{R})$ given by $t \mapsto \varphi_t^\epsilon$ is continuous. Hence, by Hölder's inequality,

$$\mathbb{E} \int \varphi_{t_n}^\epsilon (|f_{t_n}| - f_{t_n}^2) dx d\xi \rightarrow \mathbb{E} \int \varphi_{t_{fin}}^\epsilon (|f_{t_{fin}}| - f_{t_{fin}}^2) dx d\xi.$$

Since the right hand side of (3.10) is continuous in time, we can pass to the limit in (3.10) for $t_n \rightarrow t_{fin}$ and obtain (3.10) for $t_{fin} \in A_\delta$. Since this is true for any $\delta > 0$, we obtain (3.10) for a.e. $t = t_{fin}$.

Now we let ϵ go to 0. By Lemma 4.3, applied to the backward PDE (3.9), and the uniform bound on F^ϵ in L^p , we have a uniform (in ϵ) bound on $\|\varphi^\epsilon\|_{L_t^\infty(W_x^{1, \infty})}$. Therefore, we can bound the first addend of the right hand side in (3.10) by

$$\begin{aligned}
& \limsup_{\epsilon \rightarrow 0} \int_0^{t_{fin}} \mathbb{E} \int (b \cdot \nabla \varphi^\epsilon) (|f| - f^2) d\xi dx \\
& \leq \|b\|_{L_{\xi, [-R, R]}^\infty(L_x^\infty)} \sup_\epsilon \|\varphi^\epsilon\|_{L_t^\infty(W_x^{1, \infty})} \int_0^{t_{fin}} \mathbb{E} \int (|f| - f^2) dx d\xi dr \\
(3.11) \quad & \leq C \int_0^{t_{fin}} \mathbb{E} \int (|f| - f^2) dx d\xi dr.
\end{aligned}$$

Concerning the second addend in (3.10), in the case $p < \infty$, $F - F^\epsilon$ converges to 0 in L_x^p and thus in $L^p([0, T] \times \Omega \times \mathbb{R}^d \times [-R, R])$, φ^ϵ is uniformly bounded in $L_{t,x}^\infty$ and $(|f| - f^2)$ is in $L^{p'}([0, T] \times \Omega \times \mathbb{R}^d \times [-R, R])$ by Remark 2.11. Therefore, by Hölder's inequality,

$$\limsup_{\epsilon \rightarrow 0} \int_0^{t_{fin}} \mathbb{E} \int (F - F^\epsilon) \varphi^\epsilon (|f| - f^2) dx d\xi dr = 0.$$

In the case $p = \infty$ we get the same result but exploiting the dominated convergence theorem and using that $\varphi^\epsilon (F - F^\epsilon)$ converges to 0 a.e. and is uniformly bounded and that $(|f| - f^2)$ is in $L^1([0, T] \times \Omega \times \mathbb{R}^d \times [-R, R])$. Finally, concerning the initial condition, using again the uniform bound from Lemma 4.3 we get $\int_{\mathbb{R}^d \times \mathbb{R}} \varphi_0^\epsilon (|f_0| - f_0^2) dx d\xi \leq C \int_{\mathbb{R}^d \times \mathbb{R}} (|f_0| - f_0^2) dx d\xi$. Putting all together we have, for a.e. $t_{fin} > 0$,

$$\begin{aligned} \mathbb{E} \int (|f_{t_{fin}}| - f_{t_{fin}}^2) d\xi dx &\leq C \int_{\mathbb{R}^d \times \mathbb{R}} (|f_0| - f_0^2) dx d\xi \\ &\quad + C \int_0^{t_{fin}} \mathbb{E} \int (|f| - f^2) d\xi dx dr. \end{aligned}$$

We conclude by Gronwall's lemma for discontinuous functions (cf. [20, Theorem 5.1 in the Appendix]) that, for a.e. $t \in [0, T]$,

$$\mathbb{E} \int (|f_t| - f_t^2) d\xi dx \leq C \int_{\mathbb{R}^d \times \mathbb{R}} (|f_0| - f_0^2) dx d\xi,$$

where C is a constant that depends only on the bound (3.11) and on the a priori estimates in Lemma 4.3, applied to the backward PDE (3.9). Therefore, it depends only on T , $\|b\|_{L_{\xi, [-R, R]}^\infty(L_x^\infty)}$ and $\|\operatorname{div} b\|_{L_x^p(L_{\xi, [-R, R]}^\infty)}$. The proof is complete. \square

Finally we prove Proposition 3.4.

Proof of Proposition 3.4. Since f_0 takes values in $\{0, \pm 1\}$, we have $|f_0| - f_0^2 = 0$. Therefore, Lemma 3.3 implies $f^2 - |f| = 0$ a.s. (recall $|f| \leq 1$ by definition) and thus f takes values in $\{0, \pm 1\}$ for a.e. (t, ω, x, ξ) . We then define $u(t, \omega, x) := \int_{\mathbb{R}} f(t, \omega, x, \xi) d\xi$. Note that u is well-defined since f is compactly supported in ξ and measurable by Fubini's theorem.

Now we claim that, for every $h > 0$, for a.e. (t, ω, x, ξ) ,

$$(3.12) \quad (f(t, \omega, x, \xi) - f(t, \omega, x, \xi + h))(1_{-\infty < \xi < -h} + 1_{h < \xi < +\infty}) \geq 0,$$

$$(3.13) \quad f(t, \omega, x, \xi) - f(t, \omega, x, \xi + h) + 1 \geq 0$$

and we use these inequalities to conclude. Since the pushforward of the Lebesgue measure via the map $(\xi, h) \mapsto (\xi, \xi + h)$ is equivalent to the Lebesgue measure, the two inequalities above imply, for a.e. $(t, \omega, x, \xi, \eta)$,

$$(3.14) \quad (f(t, \omega, x, \xi) - f(t, \omega, x, \eta))(1_{\xi < \eta < 0} + 1_{0 < \xi < \eta}) \geq 0,$$

$$(3.15) \quad (f(t, \omega, x, \xi) - f(t, \omega, x, \eta) + 1)1_{\xi < \eta} \geq 0.$$

Now we fix a version of f and we consider, for fixed (t, ω, x) , the set $A = A(t, \omega, x) = \{\xi < 0 : (f(t, \omega, x, \xi) - f(t, \omega, x, \eta)) \operatorname{sgn}(\xi - \eta) \leq 0 \text{ for a.e. } \eta < 0\}$. By Fubini's theorem, (3.14) implies that, for a.e. (t, ω, x) , $A(t, \omega, x)$ is a full-measure set on $(-\infty, 0)$. Moreover, for any (t, ω, x) , f is non-increasing on $A(t, \omega, x)$. Indeed, if this would not be true, we could find $\xi < \eta$ in A with $f(t, \omega, x, \xi) - f(t, \omega, x, \eta) < 0$. Thus, since $f(t, \omega, x, \xi) - f(t, \omega, x, \zeta) \geq 0$ for a.e. $\zeta > \xi$ we obtain $f(t, \omega, x, \zeta) - f(t, \omega, x, \eta) < 0$ for a.e. $\zeta \in (\xi, \eta)$, in contradiction to $\eta \in A$. Similarly, for a.e. (t, ω, x) , $B(t, \omega, x) = \{\xi > 0 : (f(t, \omega, x, \xi) - f(t, \omega, x, \eta)) \operatorname{sgn}(\xi - \eta) \leq 0 \text{ for a.e. } \eta > 0\}$ is a full-measure set on $(0, +\infty)$ on which f is non-increasing. Since f is compactly supported in ξ and takes values a.e. in $\{0, \pm 1\}$, we conclude for a.e. (t, ω, x) ,

$f = -1_{\{a < \xi < 0\}} + 1_{\{0 < \xi < b\}}$ for some $a \leq 0 \leq b$ (depending on (t, ω, x)) on the full-measure set $A(t, \omega, x) \cup B(t, \omega, x)$. By (3.15) this yields that either $f = -1_{\{a < \xi < 0\}}$ a.e. or $f = 1_{\{0 < \xi < b\}}$ a.e. and thus $f = \chi(u)$ a.e.. Progressive measurability of u follows from the respective property of $f = \chi(u)$, by Remark 2.5.

It remains to prove the claim above, that is, (3.12) and (3.13). To prove (3.12) we take a nonnegative test function ψ in $C_c^\infty([0, T] \times \mathbb{R}^d \times \mathbb{R})$ with support contained in $(-\infty, -h)$ and we call φ the function such that $\psi = -\partial_\xi \varphi$ and that $\varphi(-a) = 0$ for a large enough; φ is a nonpositive nonincreasing function, constant on $[-h, +\infty)$. We then have by (2.14), for a.e. ω ,

$$\begin{aligned} & \int (f(t, x, \xi) - f(t, x, \xi + h)) \psi(t, x, \xi) \, dx d\xi dt \\ &= \int f(t, x, \xi) (\psi(t, x, \xi) - \psi(t, x, \xi - h)) \, dx d\xi dt \\ &= \int (\varphi(t, x, 0) - \varphi(t, x, -h)) \, dx dt - \int (\varphi(t, x, \xi) - \varphi(t, x, \xi - h)) \nu(dx, d\xi, dt) \\ &= - \int (\varphi(t, x, \xi) - \varphi(t, x, \xi - h)) \nu(dx, d\xi, dt) \geq 0. \end{aligned}$$

This proves that $f(t, \omega, x, \xi) - f(t, \omega, x, \xi + h) \geq 0$ on $\{-\infty < \xi < -h\}$; similarly for $\{h < \xi < +\infty\}$. For (3.13), we take a nonnegative test function ψ in $C_c^\infty([0, T] \times \mathbb{R}^d \times \mathbb{R})$ and we call φ the nonpositive, nonincreasing function such that $\psi = -\partial_\xi \varphi$ and that $\varphi(-a) = 0$ for a large enough. Again we have by (2.14)

$$\begin{aligned} & \int (f(t, x, \xi) - f(t, x, \xi + h) + 1) \psi(t, x, \xi) \, dx d\xi dt \\ &= \int f(t, x, \xi) (\psi(t, x, \xi) - \psi(t, x, \xi - h)) \, dx d\xi dt + \int (\int \psi(t, x, \xi) \, d\xi) \, dx dt \\ &= \int (\varphi(t, x, 0) - \varphi(t, x, -h)) \, dx dt - \int (\varphi(t, x, \xi) - \varphi(t, x, \xi - h)) \nu(dx, d\xi, dt) \\ &\quad + \int (\varphi(t, x, -R_1) - \varphi(t, x, R_1)) \, dx dt \end{aligned}$$

for some R_1 such that the support of ψ is contained in $[0, T] \times \mathbb{R}^d \times [-R_1, R_1]$. Now the monotonicity property of φ gives that $\varphi(t, x, \xi) - \varphi(t, x, \xi - h) \leq 0$ for every ξ , $\varphi(t, x, -R_1) - \varphi(t, x, -h) \geq 0$ and $\varphi(t, x, 0) - \varphi(t, x, R_1) \geq 0$. Therefore, the right hand side of the formula above is ≥ 0 . This proves (3.13) and concludes the proof of the claim. \square

Remark 3.7. The condition $\operatorname{div} b \in L_{\xi, \operatorname{loc}}^1(L_x^1) \cap L_x^p(L_{\xi, \operatorname{loc}}^\infty)$ can be relaxed to $\operatorname{div} b \in L_{x, \xi, \operatorname{loc}}^1$, $(\operatorname{div} b)_+ \in L_{\xi, \operatorname{loc}}^1(L_x^1) \cap L_x^p(L_{\xi, \operatorname{loc}}^\infty)$, since only the bound on the positive part of $\operatorname{div} b$ is required for the a priori estimates in the proof of the existence of generalized solutions as well as in the proof of uniqueness.

4. PDE

In this section we provide a priori estimates for a linear parabolic PDE on \mathbb{R}^d of the form

$$(4.1) \quad \partial_t \varphi = \frac{1}{2} \Delta \varphi + g \varphi + h \varphi,$$

where $g \in L_x^p$ for some finite $p > d$ and $h \in L_x^\infty$. Since we are interested in a priori estimates in this section, we suppose that g, h and the initial datum φ_0 are smooth, nonnegative and compactly supported. The estimates can be applied also to the backward PDE, by a change of time. The methods used in this section are

essentially classical, see for example (among many other references) [32] for heat kernel estimates in L^p spaces and [21] for estimates on Kolmogorov-type PDEs.

Remark 4.1. The existence of a nonnegative solution φ in $L_t^\infty(W_x^{2,\infty})$ to (4.1) in the case of smooth compactly supported coefficients and nonnegative initial datum is ensured, for example, by the representation formula

$$(4.2) \quad \varphi(t, x) = \mathbb{E}[\exp[\int_0^t (g(x + W_r - W_t) + h(x + W_r - W_t)) dr] \varphi_0(x - W_t)].$$

The equation also implies, again for smooth compactly supported data, that such a solution is in $L_x^\infty(W_t^{1,\infty})$.

We start by recalling the regularizing properties of the heat kernel, of easy (and classical) proof:

Lemma 4.2. *Let $p_t(x) = t^{-d/2} p_1(t^{-1/2}x)$ be the heat kernel on \mathbb{R}^d , i.e. $p_1(x) = (2\pi)^{-d/2} e^{-|x|^2/2}$. Then we have, for $m \in [1, \infty]$,*

$$\|p_t\|_{L_x^m} \leq C_{m,d} t^{-(d-d/m)/2} \quad \text{and} \quad \|\nabla p_t\|_{L_x^m} \leq C_{m,d} t^{-(1+d-d/m)/2}.$$

Proof. We only prove the second inequality, the proof of the first one being similar. The case $m = \infty$ is obvious, thus let $m \in [1, \infty)$. Note that $\nabla p_t(x) = t^{-(1+d)/2} \nabla p_1(t^{-1/2}x)$. By the change of variable $y = t^{-1/2}x$, we get

$$\int_{\mathbb{R}^d} |\nabla p_t(x)|^m dx = t^{-(1+d)m/2} t^{d/2} \int_{\mathbb{R}^d} |\nabla p_1(y)|^m dy = t^{-m(1+d-d/m)/2} \|\nabla p_1\|_{L_x^m}^m,$$

which is the desired estimates. \square

We write the PDE (4.1) using the variational formulation:

$$\varphi_t = p_t * \varphi_0 + \int_0^t p_{t-s} * (g\varphi_s) ds + \int_0^t p_{t-s} * (h\varphi_s) ds.$$

Lemma 4.3. *There exists a locally bounded function $c = c(T, \|g\|_{L_x^p}, \|h\|_{L_x^\infty})$ such that, for every φ_0 in C_c^∞ , it holds*

$$\|\varphi_t\|_{W_x^{1,\infty}} \leq \|\varphi_0\|_{W_x^{1,\infty}} c(T, \|g\|_{L_x^p}, \|h\|_{L_x^\infty}).$$

Proof. Here C denotes any positive constant, which can change from line to line, possibly depending on T , p and d . We start with the L_x^∞ estimate. Using Young's inequality for convolutions we get

$$\begin{aligned} \|\varphi_t\|_{L^\infty} &\leq \|p_t * \varphi_0\|_{L^\infty} + \int_0^t \|p_{t-s} * (g\varphi_s)\|_{L^\infty} ds + \int_0^t \|p_{t-s} * (h\varphi_s)\|_{L^\infty} ds \\ &\leq C\|\varphi_0\|_{L^\infty} + C \int_0^t (t-s)^{-d/2p} \|g\varphi_s\|_{L^p} ds + C \int_0^t \|h\varphi_s\|_{L^\infty} ds \\ &\leq C\|\varphi_0\|_{L^\infty} + C \int_0^t (t-s)^{-d/2p} \|g\|_{L^p} \|\varphi_s\|_{L^\infty} ds + C \int_0^t \|h\|_{L^\infty} \|\varphi_s\|_{L^\infty} ds. \end{aligned}$$

Since $p > d$, $(t-s)^{-d/2p}$ is locally in L^2 , Hölder's inequality yields

$$\|\varphi_t\|_{L^\infty} \leq C\|\varphi_0\|_{L^\infty} + C\|g\|_{L^p} \left(\int_0^t \|\varphi_s\|_{L^\infty}^2 ds \right)^{1/2} + C\|h\|_{L^\infty} \left(\int_0^t \|\varphi_s\|_{L^\infty}^2 ds \right)^{1/2}$$

and thus

$$\|\varphi_t\|_{L^\infty}^2 \leq C\|\varphi_0\|_{L^\infty}^2 + C\|g\|_{L^p}^2 \int_0^t \|\varphi_s\|_{L^\infty}^2 ds + C\|h\|_{L^\infty}^2 \int_0^t \|\varphi_s\|_{L^\infty}^2 ds.$$

Gronwall's inequality implies

$$(4.3) \quad \|\varphi_t\|_{L^\infty} \leq C\|\varphi_0\|_{L^\infty} \exp[C(\|g\|_{L^p}^2 + \|h\|_{L^\infty}^2)].$$

We continue with the L_x^∞ estimate for $\nabla\varphi_t$. Using again Young's inequality we get

$$\begin{aligned} & \|\nabla\varphi_t\|_{L^\infty} \\ & \leq \|p_t * \nabla\varphi_0\|_{L^\infty} + \int_0^t \|\nabla p_{t-s} * (g\varphi_s)\|_{L^\infty} ds + \int_0^t \|\nabla p_{t-s} * (h\varphi_s)\|_{L^\infty} ds \\ & \leq C\|\nabla\varphi_0\|_{L^\infty} + C \int_0^t (t-s)^{-(1+d/p)/2} \|g\varphi_s\|_{L^p} ds + C \int_0^t (t-s)^{-1/2} \|h\varphi_s\|_{L^\infty} ds \\ & \leq C\|\nabla\varphi_0\|_{L^\infty} + C \int_0^t (t-s)^{-(1+d/p)/2} \|g\|_{L^p} \|\varphi_s\|_{L^\infty} ds \\ & \quad + C \int_0^t (t-s)^{-1/2} \|h\|_{L^\infty} \|\varphi_s\|_{L^\infty} ds. \end{aligned}$$

Since $p > d$, $(t-s)^{-(1+d/p)/2}$ is locally integrable and we obtain, with (4.3),

$$\|\nabla\varphi_t\|_{L^\infty} \leq C\|\nabla\varphi_0\|_{L^\infty} + C(\|g\|_{L^p} + \|h\|_{L^\infty})\|\varphi_0\|_{L^\infty} \exp[C(\|g\|_{L^p}^2 + \|h\|_{L^\infty}^2)].$$

The proof is complete. \square

5. APPENDIX

In the following, let (E, \mathcal{E}, μ) be a σ -finite measure space. We say that a function $f : E \rightarrow \mathbb{R}$ is strictly measurable, resp. μ -measurable (or measurable, when μ is fixed) if, for every Borel subset A of \mathbb{R} , $f^{-1}(A)$ is in \mathcal{E} , resp. $\bar{\mathcal{E}}^\mu$, the completion of \mathcal{E} with the μ -null sets. It is easy to see that f is measurable if and only if it is the μ -a.e. limit of a sequence of simple measurable functions. Given a Banach space V and a function $f : E \rightarrow V$, we recall the following three definitions of (μ -)measurability of f :

- we say that f is strongly measurable if it is the μ -a.e. limit of a sequence of V -valued simple measurable functions (i.e. of the form $\sum_{i=1}^N v_i 1_{A_i}$ for A_i in \mathcal{E} and v_i in V);
- we say that f is weakly measurable if, for every φ in V^* , $x \mapsto \langle f(x), \varphi \rangle_{V, V^*}$ is μ -measurable;
- if $V = U^*$ is the dual space of a Banach space U , we say that f is weakly-* measurable if, for every φ in U , $x \mapsto \langle f(x), \varphi \rangle_{V, U}$ is μ -measurable;
- we say that f is Borel measurable if, for every open set A in V (endowed with the strong topology), $f^{-1}(A)$ is in $\bar{\mathcal{E}}^\mu$.

The following result holds:

Proposition 5.1. *Let V be a separable Banach space. Then the notions of strong measurability, weak measurability and Borel measurability coincide. They also coincide with the weak-* measurability if moreover V is reflexive (in particular if $V = \mathbb{R}$).*

Proof. The fact that strong measurability and weak measurability coincide when V is separable is the well-known Pettis measurability theorem, see for example Theorem 2 in [18], Chapter II. Moreover, it is true in general that any Borel measurable function f is also weakly measurable, since in this case, for every φ in V^* , $f \mapsto \langle f, \varphi \rangle$ is the composition of a continuous map and a Borel measurable map. It remains to prove that a strongly measurable function f is also Borel measurable. To prove this, by separability of V , it is enough to prove that, for every y in V , for every $R > 0$, $f^{-1}(B_R(y)) = \{x \in V \mid \|f(x) - y\| < R\}$ is in $\bar{\mathcal{E}}^\mu$, that

is, it is enough to prove that, for every y in V , the function $x \mapsto \|f(x) - y\|$ is μ -measurable. Now, if f_k are simple measurable functions approximating (μ -a.s.) f , then $\|f_k(x) - y\|$ are also simple measurable functions approximating $\|f(x) - y\|$, therefore $x \mapsto \|f(x) - y\|$ is μ -measurable. \square

As mentioned in the introduction, in the definition of L^p spaces we only consider two cases: (1) $V = U^*$ is the dual space of a separable Banach space, where we define $L^0(E; V)$ as the space of equivalent classes of weakly- $*$ measurable functions; (2) V is a separable Banach space, where we define $L^0(E; V)$ as the space of equivalent classes of weakly (or strongly or Borel) measurable functions. In both cases, for any function f in $L^0(E; V)$, the function $x \mapsto \|f(x)\|_V$ is measurable: in the case (1) because $\|f(x)\| = \sup_{\varphi \in D} |\langle f(x), \varphi \rangle|$ where D is a countable dense set of B_1^U (the unit centered ball in U); in the case (2) as shown in the previous proof. Therefore, it makes sense to define the spaces $L^p(E; V)$ for $1 \leq p \leq \infty$.

Proposition 5.2. *Let D be a domain of \mathbb{R}^n . For every $1 \leq p \leq \infty$, the space $L^p(E \times D, \mathcal{E} \otimes \mathcal{B}(D))$ is canonically embedded in $L^p(E; L^p(D))$ (whose functions are weakly measurable for $1 \leq p < \infty$, weakly- $*$ measurable for $p = \infty$). This embedding is a surjective isomorphism.*

Proof. The embedding result is easy to show using Fubini's theorem, we prove only the surjectivity. We start with the case $p < \infty$. To prove this, let F be an element (more precisely, a representative of an element) in $L^p(E; L^p(D))$. By Proposition 5.1, F is strongly measurable, i.e. there exists a sequence $(F_n)_n$ of simple functions in $L^p(E; L^p(D))$ which converges to F in $L^p(D)$ for a.e. x and, without loss of generality, in $L^p(E; L^p(D))$. We can write F_n as

$$F_n(x) = \sum_{k=1}^{N(n)} F_{n,k} 1_{A_{n,k}}(x)$$

for some measurable sets $A_{n,k}$ and some elements $F_{n,k}$ in $L^p(D)$. Now we define, for each n , the map $G_n : E_x \times E_y \rightarrow V$ by

$$G_n(x, y) = \sum_{k=1}^{N(n)} G_{n,k}(y) 1_{A_{n,k}}(x),$$

where $G_{n,k}$ is a representative of $F_{n,k}$. The function G_n is measurable in (x, y) and, since $\|G_n - G_m\|_{L^p(E \times D)} = \|F_n - F_m\|_{L^p(E; L^p(D))}$, the sequence $(G_n)_n$ is Cauchy in $L^p(E \times D)$, therefore it converges to some G in $L^p(E \times D)$. In particular $x \mapsto [y \mapsto G_n(x, y)]$ (where $[y \mapsto G_n(x, y)]$ is the equivalence class of $y \mapsto G_n(x, y)$) converges to $x \mapsto [y \mapsto G(x, y)]$ in $L^p(E; L^p(D))$; it follows that $x \mapsto [y \mapsto G(x, y)]$ coincides with F . Hence G is the desired representative in $L^p(E \times D)$ of F . This concludes the proof in the case $p < \infty$. The case $p = \infty$ can be reduced to the case $p < \infty$. Indeed, calling $(E_n)_n$ an increasing sequence of sets with finite measure and with $E_n \nearrow E$, any function f in $L^\infty(E; L^\infty(D))$, restricted to $L^\infty(E_n; L^\infty(B_R \cap D))$, is also a weakly measurable function in $L^2(E_n; L^2(B_R \cap D))$. Hence, it has a representative in $L^2(E_n \times (B_R \cap D))$ and thus in $L^2_{loc}(E \times D)$, by arbitrariness of R and n , and this representative is essentially bounded. \square

The following result is in [43], Theorem 2 (see also Theorem A.4):

Theorem 5.3. *Let S be a metric σ -compact locally compact space and, for any $R > 0$, denote by $L_R^\infty(E; \mathcal{M}_+(S))$ the subset of $L^\infty(E; \mathcal{M}(S))$ of nonnegative measure-valued functions g with $\|g\|_{L^\infty(E; \mathcal{M}(S))} \leq R$. Then $L_R^\infty(E; \mathcal{M}_+(S))$ is (embedded isomorphically in) a bounded sequentially weakly- $*$ closed subset of the*

dual space of $L^1(E; C_0(S))$. In particular, every sequence in $L_R^\infty(E; \mathcal{M}_+(S))$ admits a subsequence converging weakly- $*$ to an element of $L_R^\infty(E; \mathcal{M}_+(S))$.

We close with two remarks on the composition of measurable functions:

Remark 5.4. Let $f : E \rightarrow V$ be a Borel measurable map and let $h : V \rightarrow \mathbb{R}$ be a strictly measurable function, where V is endowed with its Borel σ -algebra. Then the composition $h(f)$ is Borel measurable.

Let (F, \mathcal{F}, ν) be another σ -finite measure space and let $g : F \rightarrow E$ be a measurable map such that the image measure of ν under g is μ . Then, for every element (equivalent class) f of $L^0(E, V)$ (whenever V is separable or is the dual of a separable space), the composition $f(g)$ is well defined as element (equivalence class) in $L^0(F, V)$. Moreover the L^p norm (when defined) is also preserved in this composition.

Remark 5.5. (i) Assume that V is the dual space of a separable space U . Let $f : E \rightarrow V$ be a weakly- $*$ measurable map and let $\varphi : E \rightarrow U$ be a (weakly or equivalently strongly) measurable map. Then the map $x \mapsto \langle f(x), \varphi(x) \rangle_{V,U}$ is measurable. Indeed, if φ_k are simple measurable functions approximating a.e. φ , then $x \mapsto \langle f(x), \varphi_k(x) \rangle_{V,U}$ are measurable functions approximating a.e. $x \mapsto \langle f(x), \varphi(x) \rangle_{V,U}$. In particular, if $f : [0, T] \times \Omega \rightarrow L_{x,\xi}^\infty$ is a weakly- $*$ progressively measurable function and $\varphi : [0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$ -measurable integrable function, so that $\varphi : [0, T] \times \Omega \rightarrow L_{x,\xi}^1$ is a progressively measurable function, then $(t, \omega) \mapsto \langle f(t, \omega), \varphi(t, \omega) \rangle_{x,\xi}$ is a progressively measurable function.

(ii) An analogous property holds for bounded kinetic measures m . In this case one can consider a more general class of test functions. Let $\varphi : [0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function (not an equivalence class) such that: 1) for every (x, ξ) , $(t, \omega) \mapsto \varphi(t, \omega, x, \xi)$ is progressively measurable; 2) for a.e. ω , the zero set being independent of (t, x, ξ) , and for every (x, ξ) , $t \mapsto \varphi(t, \omega, x, \xi)$ is càdlàg (or càglàd); 3) for a.e. ω , the zero set being independent of (t, x, ξ) and for every t , $(x, \xi) \mapsto \varphi(t, \omega, x, \xi)$ is continuous. Then, for every fixed representative of m^ω , the map

$$(t, \omega) \mapsto \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \varphi^\omega(r, x, \xi) m^\omega(r, x, \xi) dx d\xi dr$$

is progressively measurable and has a.e. càdlàg paths. Indeed, (in the càdlàg case, the càglàd case being similar), for every fixed t , $\int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \varphi^\omega m^\omega dx d\xi dr$ is the a.e. limit of $\int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \varphi_n^\omega m^\omega dx d\xi dr$, where $\varphi_n = \psi_n \varphi *_{t} 1_{[0,1/n]}$ and ψ_n is a regular function with $0 \leq \psi_n \leq 1$, $\psi_n = 1$ on $B_{1/n}$ and with support on $B_{2/n}$. Since $\omega \mapsto \varphi_n^\omega \in C_0([0, T] \times \mathbb{R}^d \times \mathbb{R})$ is $\mathcal{F}_{t+1/n}$ -measurable, $\int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \varphi_n^\omega m^\omega dx d\xi dr$ is $\mathcal{F}_{t+1/n}$ -measurable and thus $\int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \varphi^\omega m^\omega dx d\xi dr$ is \mathcal{F}_t -measurable. Moreover, for any fixed representative of m^ω , for a.e. ω , the map $t \mapsto \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \varphi^\omega m^\omega dx d\xi dr$ is càdlàg. Therefore, $(t, \omega) \mapsto \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \varphi^\omega m^\omega dx d\xi dr$ has the desired properties. The same result, replacing càdlàg by càglàd, holds for $\int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \varphi^\omega m^\omega dx d\xi dr$.

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