On the probabilistic well-posedness of the nonlinear Schrödinger equations with non-algebraic nonlinearities

Citation for published version:
Oh, T, Okamoto, M & Pocovnicu, O 2019, 'On the probabilistic well-posedness of the nonlinear Schrödinger equations with non-algebraic nonlinearities', *Discrete and Continuous Dynamical Systems - Series A*, vol. 39, no. 6, pp. 3479-3520. https://doi.org/10.3934/dcds.2019144

Digital Object Identifier (DOI):
10.3934/dcds.2019144

Link:
Link to publication record in Edinburgh Research Explorer

Document Version:
Peer reviewed version

Published In:
Discrete and Continuous Dynamical Systems - Series A

ON THE PROBABILISTIC WELL-POSEDNESS OF THE NONLINEAR SCHRÖDINGER EQUATIONS WITH NON-ALGEBRAIC NONLINEARITIES

TADAHIRO OH, MAMORU OKAMOTO, AND OANA POCOVNICU

Abstract. We consider the Cauchy problem for the nonlinear Schrödinger equations (NLS) with non-algebraic nonlinearities on the Euclidean space. In particular, we study the energy-critical NLS on $\mathbb{R}^d$, $d = 5, 6$, and energy-critical NLS without gauge invariance and prove that they are almost surely locally well-posed with respect to randomized initial data below the energy space. We also study the long time behavior of solutions to these equations: (i) we prove almost sure global well-posedness of the (standard) energy-critical NLS on $\mathbb{R}^d$, $d = 5, 6$, in the defocusing case, and (ii) we present a probabilistic construction of finite time blowup solutions to the energy-critical NLS without gauge invariance below the energy space.

Contents

2010 *Mathematics Subject Classification.* 35Q55.
Key words and phrases. nonlinear Schrödinger equation; almost sure local well-posedness; almost sure global well-posedness; finite time blowup.

1. Introduction

1.1. **Nonlinear Schrödinger equations.** We consider the Cauchy problem for the following energy-critical nonlinear Schrödinger equation (NLS) on $\mathbb{R}^d$, $d = 5, 6$:

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^{\frac{4}{d-2}} u \\ u|_{t=0} = \phi, \end{cases} \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^d. \tag{1.1}$$

This equation enjoys the following dilation symmetry:

$$u(t,x) \longmapsto u_\mu(t,x) := \mu^{\frac{d-2}{2}} u(\mu^2 t, \mu x)$$

for $\mu > 0$. This dilation symmetry preserves the $\dot{H}^1$-norm of the initial data ϕ, thus inducing the scaling critical Sobolev regularity $s_{\text{crit}} = 1$. Moreover, the energy (= Hamiltonian) of a solution u remains invariant under this dilation symmetry. For this reason, we refer to (1.1) as energy-critical and $\dot{H}^1(\mathbb{R}^d)$ as the energy space.

The Cauchy problem (1.1) in a general dimension has been at the core of the study of dispersive equations for several decades and has been studied extensively. In particular, for $d \geq 5$, it is known that (1.1) is (i) locally well-posed in the energy space [13] and (ii) globally well-posed in the defocusing case [55] and also in the focusing case under some assumption on the (kinetic) energy [34]. On the other hand, (1.1) is known to be ill-posed in $H^s(\mathbb{R}^d)$, $s < s_{\text{crit}} = 1$, in the sense of norm inflation [15]; there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ of (smooth) solutions to (1.1) and $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $\|u_n(0)\|_{H^s} < \frac{1}{n}$ but $\|u_n(t_n)\|_{H^s} > n$ with $t_n < \frac{1}{n}$. This in particular shows that the solution map to (1.1) can *not* be extended to be a continuous map on $H^s(\mathbb{R}^d)$, $s < 1$, thus violating one of the important criteria for well-posedness.

Despite the ill-posedness below the energy space, one may still hope to construct unique local-in-time solutions in a probabilistic manner, thus establishing almost sure local well-posedness in some suitable sense; see [5] for a general review on this topic. Such an approach first appeared in the work by McKean [40] and Bourgain [7] in the study of invariant Gibbs measures for the cubic NLS on $\mathbb{T}^d$, $d = 1, 2$. In particular, they established almost sure local well-posedness with respect to particular random initial data.[1] This random initial data in [40, 7] can be viewed as a randomization of the Fourier coefficients of a particular function (basically the antiderivative of the Dirac delta function) via the multiplication by independent Gaussian random variables. Such randomization of the Fourier series is classical and well studied [48, 32]. In [10], Burq-Tzvetkov elaborated this idea further. In particular, in the context of the cubic nonlinear wave equation (NLW) on a three dimensional compact Riemannian manifold, they considered a randomization via the Fourier series expansion as above for *any* rough initial condition below the scaling critical Sobolev regularity and established almost sure local well-posedness with respect to the randomization. Such randomization via the Fourier series expansion is natural on compact domains and more generally in situations where the associated elliptic operators have discrete spectra [54, 20, 17].

Our main focus is to study NLS (1.1) on the Euclidean space $\mathbb{R}^d$. In this setting, the randomization via the Fourier series expansion does not quite work as the frequency space $\mathbb{R}^d_\xi$ is

[1] These local-in-time solutions were then extended globally in time by invariance of the Gibbs measures. In the following, however, we do not use any invariant measure.

not discrete. We instead consider a randomization associated to the Wiener decomposition $\mathbb{R}^d_\xi = \bigcup_{n\in\mathbb{Z}^d}(n + (-\frac{1}{2}, \frac{1}{2}]^d)$. See [59, 38, 2, 3, 26]. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ satisfy

$$\operatorname{supp}\psi \subset [-1,1]^d \qquad \text{and} \qquad \sum_{n\in\mathbb{Z}^d} \psi(\xi - n) = 1 \quad \text{for any } \xi \in \mathbb{R}^d.$$

Then, given a function ϕ on $\mathbb{R}^d$, we have

$$\phi = \sum_{n\in\mathbb{Z}^d} \psi(D-n)\phi.$$

This replaces the role of the Fourier series expansion on compact domains. We then define the Wiener randomization of ϕ by

$$\phi^\omega := \sum_{n\in\mathbb{Z}^d} g_n(\omega)\psi(D-n)\phi, \tag{1.2}$$

where $\{g_n\}$ is a sequence of independent mean zero complex-valued random variables on a probability space $(\Omega, \mathcal{F}, P)$. In the following, we assume that the real and imaginary parts of g_n are independent and endowed with probability distributions $\mu_n^{(1)}$ and $\mu_n^{(2)}$, satisfying the following exponential moment bound:

$$\int_{\mathbb{R}} e^{\kappa x} d\mu_n^{(j)}(x) \le e^{c\kappa^2}$$

for all $\kappa \in \mathbb{R}$, $n \in \mathbb{Z}^d$, $j = 1, 2$. This condition is satisfied by the standard complex-valued Gaussian random variables and the standard Bernoulli random variables.

On the one hand, the randomization does not improve differentiability just like the randomization via the Fourier series expansion [10, 1]. On the other hand, it improves integrability as for the classical random Fourier series [48, 32]. From this point of view, the randomization makes the problem *subcritical* in some sense, at least for local-in-time problems.

In the following, we study the Cauchy problem (1.1) with random initial data given by the Wiener randomization ϕ^ω of a given function $\phi \in H^s(\mathbb{R}^d)$, $d = 5, 6$. In view of the deterministic well-posedness result for $s \ge 1$, we only consider $s < s_{\text{crit}} = 1$.

Theorem 1.1. *Let $d = 5, 6$ and $1 - \frac{1}{d} < s < 1$. Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in* (1.2). *Then, the Cauchy problem* (1.1) *is almost surely locally well-posed with respect to the random initial data ϕ^ω.*

More precisely, there exist $C, c, \gamma > 0$ such that for each $0 < T \ll 1$, there exists $\Omega_T \subset \Omega$ with $P(\Omega_T^c) \le C \exp\Big(-\frac{c}{T^\gamma \|\phi\|_{H^s}^2}\Big)$ such that for each $\omega \in \Omega_T$, there exists a unique solution $u = u^\omega \in C([-T,T]; H^s(\mathbb{R}^d))$ to (1.1) *with $u|_{t=0} = \phi^\omega$ in the class*

$$S(t)\phi^\omega + X_T^1 \subset S(t)\phi^\omega + C([-T,T]; H^1(\mathbb{R}^d)) \subset C([-T,T]; H^s(\mathbb{R}^d)),$$

where $S(t) = e^{it\Delta}$ and X_T^1 is defined in Section 3 below.

Almost sure local well-posedness with respect to the Wiener randomization has been studied in the context of the cubic NLS and the quintic NLS on $\mathbb{R}^d$ [2, 3, 9] which are energy-critical in dimensions 4 and 3, respectively. Note that when $d = 5, 6$, the energy-critical nonlinearity $|u|^{\frac{4}{d-2}}u$ is no longer algebraic, presenting a new difficulty in applying the argument in [2, 3, 9].

Let $z(t) = z^\omega(t) := S(t)\phi^\omega$ denote the random linear solution with ϕ^ω as initial data. If u is a solution to (1.1), then the residual term $v := u - z$ satisfies the following perturbed NLS:

$$\begin{cases} i\partial_t + \Delta v = \mathcal{N}(v + z^\omega) \\ v|_{t=0} = 0, \end{cases} \tag{1.3}$$

where $\mathcal{N}(u) = \pm|u|^{\frac{4}{d-2}}u$. In terms of the Duhamel formulation, (1.3) reads as

$$v(t) = -i\int_0^t S(t-t')\mathcal{N}(v + z^\omega)(t')dt'. \tag{1.4}$$

Then, the main objective is to solve the fixed point problem (1.4).[2] In fact, the first and third authors (with Bényi) [2, 3] studied this problem for the residual term v in the context of the cubic NLS on $\mathbb{R}^d$ by carrying out case-by-case analysis and estimating terms of the form $v\overline{v}v$, $v\overline{v}z$, $v\overline{z}z$, etc. In [9], Brereton carried out similar analysis for the quintic NLS on $\mathbb{R}^d$. Such case-by-case analysis is possible only for algebraic, i.e. smooth, nonlinearities and thus is not applicable to our problem at hand. In this paper, we adjust the analysis from [3] in order to handle non-algebraic nonlinearities. Moreover, our analysis in this paper is simpler than that in [2, 3] in the sense that we avoid thorough case-by-case analysis. There is, however, a price to pay: (i) While our approach for non-algebraic nonlinearities in this paper can be applied to the energy-critical cubic NLS on $\mathbb{R}^4$, this would yield a worse regularity range $s \in (\frac{3}{4}, 1)$ than the regularity range $s \in (\frac{3}{5}, 1)$ obtained in [3]. This is due to the fact that we adjust our calculation to a non-smooth nonlinearity. (ii) The constants in the nonlinear estimates in Section 4 depend on the local existence time $T > 0$ (see Proposition 4.1 below). In particular, Theorem 1.1 is not accompanied by almost sure small data global well-posedness and scattering. This is in sharp contrast with the situation for the cubic nonlinearity considered in [3].

Our main tools for proving Theorem 1.1 are similar to those in [3]; the Fourier restriction norm method adapted to the spaces V^p of functions of bounded p-variation and their preduals U^p, the bilinear refinement of the Strichartz estimate, and the probabilistic Strichartz estimates thanks to the gain of integrability via the Wiener randomization. In order to avoid the use of fractional derivatives, we focus on the energy-critical NLS and solve the fixed point problem (1.4) in X^1_T at the critical regularity (for the residual term) by performing a precise computation. Namely, it is important that we use this refined version of the Fourier restriction norm method, since if we were to use the usual $X^{\sigma,b}$-spaces introduced in [6], then we would need to study the problem at the subcritical regularity $\sigma = 1 + \varepsilon$ as in [2], creating a further difficulty. Moreover, in proving almost sure global well-posedness of (1.1), it is essential that we only use the X^σ_T-norm, $\sigma \le 1$, for the residual part v. See Theorem 1.5 below.

[2]In the field of stochastic parabolic PDEs, this change of viewpoint and solving the fixed point problem for the residual term v is called the Da Prato-Debussche trick [18, 19]. In the context of deterministic dispersive PDEs with random initial data, this goes back to the work by McKean [40] and Bourgain [7], which precedes [18, 19].

Next, we consider the following energy-critical NLS without gauge invariance on $\mathbb{R}^d$, $d = 5, 6$:

$$\begin{cases} i\partial_t u + \Delta u = \lambda |u|^{\frac{d+2}{d-2}} \\ u|_{t=0} = \phi, \end{cases} \tag{1.5}$$

where $\lambda \in \mathbb{C} \setminus \{0\}$. As in the case of the standard NLS (1.1), one can prove local well-posedness of (1.5) in $H^s(\mathbb{R}^d)$, $s \geq 1$, via the Strichartz estimates. On the other hand, Ikeda-Inui [28] showed that (1.5) is ill-posed in $H^s(\mathbb{R}^d)$ with $s < 1$. More precisely, they proved non-existence of solutions for rough initial data, satisfying a certain condition. This ill-posedness result by non-existence is much stronger than the norm inflation proved for the standard NLS (1.1). The non-existence result in [28] studies a rough initial condition and exhibits a pathological behavior in a direct manner, while the norm inflation result in [15] is proved by studying the behavior of a sequence of smooth solutions; in particular it does not say anything about rough solutions.

Theorem 1.2. *Let $d = 5, 6$ and $1 - \frac{1}{d} < s < 1$. Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in* (1.2)*. Then, the Cauchy problem* (1.5) *is almost surely locally well-posed with respect to the random initial data ϕ^ω in the sense of Theorem 1.1.*

Theorem 1.2, in particular, states that upon the randomization, we can avoid these pathological initial data constructed in [28] for which no solution exists. Compare this with the "standard" almost sure local well-posedness results such as Theorem 1.1 above, where the only known obstruction to well-posedness below a threshold regularity is discontinuity of the solution map.[3] In this sense, Theorem 1.2 provides a more striking role of randomization, overcoming the non-existence result below the scaling critical regularity, and it seems that Theorem 1.2 is the first such result.

The proof of Theorem 1.2 follows the same lines as that of Theorem 1.1. When $d = 6$, the nonlinearity $|u|^2 = u\overline{u}$ in (1.5) is algebraic. Hence, one may also perform case-by-case analysis as in [3]. We, however, do not pursue this direction since our purpose is to present a unified approach to the problem.

Next, let us state an almost sure local well-posedness result with slightly more general initial data. Fix $\phi \in H^s(\mathbb{R}^d) \setminus H^1(\mathbb{R}^d)$. Then, we consider the following Cauchy problem for given $v_0 \in H^1(\mathbb{R}^d)$:

$$\begin{cases} i\partial_t u + \Delta u = \mathcal{N}(u) \\ u|_{t=0} = v_0 + \phi^\omega, \end{cases} \tag{1.6}$$

where $\mathcal{N}(u) = \pm|u|^{\frac{4}{d-2}}u$ or $\lambda|u|^{\frac{d+2}{d-2}}$ and ϕ^ω is the Wiener randomization of ϕ. Then, as a corollary to (the proof of) Theorems 1.1 and 1.2, we have the following proposition.

Proposition 1.3. *Let $d = 5, 6$ and $1 - \frac{1}{d} < s < 1$. Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in* (1.2)*. Then, given $v_0 \in H^1(\mathbb{R}^d)$, the Cauchy problem* (1.6) *is almost surely locally well-posed with respect to the Wiener randomization ϕ^ω, where the (random) local existence time $T = T_\omega$ is assumed to be sufficiently small, depending on the*

[3] Namely, the pathological behavior of the standard NLS (1.1) below the scaling critical regularity $s_{\text{crit}} = 1$ is about the solution map (stability under perturbation) and is not about individual solutions (such as existence). On the contrary, in the case of (1.5), there are individual initial data, each of which is responsible for the pathological behavior (non-existence of solutions).

deterministic part v_0 of the initial data. Moreover, the following blowup alternative holds; let $T^* = T^*(\omega, v_0)$ be the forward maximal time of existence. Then, either

$$T^* = \infty \qquad \textit{or} \qquad \lim_{T \to T^*} \|u - S(t)\phi^\omega\|_{L_t^{q_d}([0,T); W_x^{1,r_d})} = \infty, \tag{1.7}$$

where (q_d, r_d) is a particular admissible pair given by

$$(q_d, r_d) := \big(\tfrac{2d}{d-2}, \tfrac{2d^2}{d^2-2d+4}\big). \tag{1.8}$$

Namely, this is an almost sure local well-posedness result with the initial data of the form: "a *fixed* smooth deterministic function + a rough random perturbation". See, for example, [45]. The proof of Proposition 1.3 is based on studying the equation for the residual term $v = u - z^\omega$ as above:

$$\begin{cases} i\partial_t v + \Delta v = \mathcal{N}(v + z^\omega) \\ v|_{t=0} = v_0 \in H^1(\mathbb{R}^d), \end{cases} \tag{1.9}$$

where we now have a non-zero initial condition. For this fixed point problem, the critical nature of the problem appears through the deterministic initial condition v_0. In particular, the local existence time $T = T(v_0)$ depends on the profile of the (deterministic) initial data v_0. We point out that the good set of probability 1 on which almost sure local well-posedness holds does *not* depend on the choice of $v_0 \in H^1(\mathbb{R}^d)$.

In the next two subsections, we state results on the long time behavior of solutions to (1.1) and (1.5), using Proposition 1.3. In particular, we prove almost sure global well-posedness of the defocusing energy-critical NLS (1.1) below the energy space (Theorem 1.5). As for NLS (1.5) without gauge invariance, we use Proposition 1.3 to construct finite time blowup solutions below the critical regularity in a probabilistic manner (Theorem 1.7).

Remark 1.4. When $s < 1$, the solution map

$$\Phi : u_0 \in H^s(\mathbb{R}^d) \longmapsto u \in C([-T, T]; H^s(\mathbb{R}^d))$$

is not continuous for (1.1) and is not even well defined for (1.5); see [15, 28]. Once we view $z^\omega = S(t)\phi^\omega$ as a probabilistically pre-defined data, we can factorize the solution map for (1.6) as

$$u_0 = v_0 + \phi^\omega \in H^s(\mathbb{R}^d) \longmapsto (v_0, z^\omega) \longmapsto v \in C([-T_\omega, T_\omega]; H^1(\mathbb{R}^d)),$$

where the first map can be viewed as a universal lift map and the second map is the solution map Ψ to (1.9), which is in fact continuous in $(v_0, z^\omega) \in H^1(\mathbb{R}^d) \times S^s([0,T])$, where $S^s([0,T]) \subset C([0,T]; H^s(\mathbb{R}^d))$ is the intersection of suitable space-time function spaces. See (4.3) below for example. We also point out that under this factorization, it is clear that the probabilistic component appears only in the first step while the second step is entirely deterministic.

One can go further and introduce more probabilistically pre-defined objects in order to improve the regularity threshold. In the context of the cubic NLS on $\mathbb{R}^3$ [4], the first and third authors (with Bényi) decomposed u as $u = z_1^\omega + z_3^\omega + v$, where $z_1^\omega = S(t)\phi^\omega$ and $z_3^\omega = -i\int_0^t S(t-t')|z_1|^2 z_1(t')dt'$, thus leading to the following factorization:

$$u_0 = v_0 + \phi^\omega \in H^s(\mathbb{R}^3) \longmapsto (v_0, z_1^\omega, z_3^\omega) \longmapsto v \in C([-T_\omega, T_\omega]; H^1(\mathbb{R}^3)).$$

The introduction of the higher order pre-defined object z_3 allowed us to lower the regularity threshold from the previous work [3]. See [4] for a further discussion. For NLS with non-algebraic nonlinearities such as (1.1) and (1.5), it is not clear how to introduce a further decomposition at this point. This is due to the non-smoothness of the nonlinearities. If one has an algebraic (or analytic) nonlinearity, then a Picard iteration yields analytic dependence (at least for smooth data), thus enabling us to write a solution as a power series in terms of initial data, at least in theory. See [14, 43]. On the other hand, if a nonlinearity is non-smooth, then a Picard iteration does not yield analytic dependence, which makes it hard to find a higher order term.

More recently, the first author (with Tzvetkov and Wang) proved invariance of the white noise for the (renormalized) cubic fourth order NLS on the circle [46]. In this work, we introduced an infinite sequence $\{z_{2j-1}\}_{j\in\mathbb{N}}$ of pre-defined objects of order $2j-1$ (depending only on the random initial data) and wrote $u = \sum_{j=1}^\infty z_{2j-1} + v$, thus considering the following factorization:

$$u_0^\omega \in H^s(\mathbb{T}) \longmapsto (z_1^\omega, z_3^\omega, z_5^\omega, \dots) \longmapsto v \in C(\mathbb{R}; H^s(\mathbb{T})),$$

for $s < -\frac{1}{2}$, where u_0^ω is the Gaussian white noise on the circle. We conclude this remark by pointing out an analogy of this factorization of the ill-posed solution map to that in the rough path theory [21] and more recent studies on stochastic parabolic PDEs [23, 25].

1.2. **Almost sure global well-posedness of the defocusing energy-critical NLS below the energy space.** In this subsection, we consider the energy-critical NLS (1.1) in the defocusing case (i.e. with the + sign). Let us first recall the known related result in this direction. In [3], the first and third authors (with Bényi) studied the global-in-time behavior of solutions to the defocusing energy-critical cubic NLS (1.1) on $\mathbb{R}^4$. By implementing the probabilistic perturbation theory, we proved conditional almost sure global well-posedness of the defocusing energy-critical cubic NLS on $\mathbb{R}^4$, assuming the following energy bound on the residual part $v = u - z$:

Energy bound: Given any $T, \varepsilon > 0$, there exists $R = R(T, \varepsilon)$ and $\Omega_{T,\varepsilon} \subset \Omega$ such that

(i) $P(\Omega_{T,\varepsilon}^c) < \varepsilon$, and
(ii) If $v = v^\omega$ is the solution to (1.3) for $\omega \in \Omega_{T,\varepsilon}$, then the following *a priori* energy estimate holds:

$$\|v(t)\|_{L^\infty([0,T];H^1(\mathbb{R}^d))} \le R(T, \varepsilon). \tag{1.10}$$

The main ingredient in this conditional almost sure global well-posedness result in [3] is a perturbation lemma (see Lemma 7.4 below). Assuming the energy bound (1.10) above, we iteratively applied the perturbation lemma in the probabilistic setting to show that a solution can be extended to a time depending only on the H^1-norm of the residual part v. Such a perturbative approach was previously used by Tao-Vişan-Zhang [53] and Killip-Vişan with the first and third authors [35]. The main novelty in [3] was an application of such a technique in the probabilistic setting, allowing us to study the long time behavior of solutions when there is no invariant measure available for the problem.[4]

[4]It is worthwhile to mention that the conditional almost sure global well-posedness in [3] and Theorem 1.5 below exploit certain "invariance" property of the distribution of the linear solution $S(t)\phi^\omega$; the distribution of $S(t)\phi^\omega$ on an interval $[t_0, t_0 + \tau*]$ (measured in a suitable space-time norm) depends only on the length

This probabilistic perturbation method can be easily adapted to other critical equations. In [49, 44], by establishing the energy bound (1.10), we implemented the probabilistic perturbation theory in the context of the defocusing energy-critical NLW on $\mathbb{R}^d$, $d = 3, 4, 5$, and proved almost sure global well-posedness.

For our problem at hand, Proposition 1.3 (more precisely Lemma 6.2 below) allows us to repeat the argument in [3]. Furthermore, we show that the energy bound (1.10) holds true for $d = 5, 6$ in the defocusing case and hence we prove the following almost sure global well-posedness of the defocusing energy-critical NLS (1.1).

Theorem 1.5. *Let $d = 5, 6$ and set $s_* = s_*(d)$ by*

$$\text{(i) } s_* = \frac{63}{68} \quad \textit{when } d = 5 \qquad \textit{and} \qquad \text{(ii) } s_* = \frac{20}{23} \quad \textit{when } d = 6.$$

Given $\phi \in H^s(\mathbb{R}^d)$, $s_* < s < 1$, let ϕ^ω be its Wiener randomization defined in (1.2)*. Then, the defocusing energy-critical NLS* (1.1) *on $\mathbb{R}^d$ is almost surely globally well-posed with respect to the random initial data ϕ^ω.*

More precisely, there exists a set $\Sigma \subset \Omega$ with $P(\Sigma) = 1$ such that, for each $\omega \in \Sigma$, there exists a (unique) global-in-time solution u to (1.1) *with $u|_{t=0} = \phi^\omega$ in the class*

$$S(t)\phi^\omega + C(\mathbb{R}; H^1(\mathbb{R}^d)) \subset C(\mathbb{R}; H^s(\mathbb{R}^d)).$$

Theorem 1.5 establishes the first almost sure global well-posedness result for the defocusing energy-critical NLS (1.1) below the energy space (without the radial assumption). In a recent preprint [36], Killip-Murphy-Vişan studied the defocusing energy-critical cubic NLS with randomized initial data when $d = 4$. In particular, under the radial assumption, they proved almost sure global well-posedness and scattering below the energy space by implementing a double bootstrap argument intertwining the energy and Morawetz estimates.

Our main goal in Theorem 1.5 is to simply prove almost sure global well-posedness (without scattering) by establishing the energy bound (1.10). In particular, As mentioned above, the main difficulty in proving Theorem 1.5 is to establish the a priori energy bound (1.10). For this purpose, let us recall the following conservation laws for (1.1):

$$\text{Mass: } M(u)(t) = \int_{\mathbb{R}^d} |u(t,x)|^2 dx,$$

$$\text{Energy: } E(u)(t) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t,x)|^2 dx + \frac{d-2}{2d} \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2d}{d-2}} dx.$$

The main task is to control the growth of the energy $E(v)$ for the residual part $v = u - z$ by estimating the time derivative of $E(v)$. We first point out that while $M(v)$ is not conserved, one can easily establish a global-in-time bound on $M(v)$. See Lemma 7.1.

By a direct computation with (1.3), we have

$$\begin{aligned} \partial_t E(v) &= \operatorname{Re} i \int \big\{ |v+z|^{\frac{4}{d-2}}(v+z) - |v|^{\frac{4}{d-2}} v \big\} \Delta \overline{v} dx \\ &\quad - \operatorname{Re} i \int |v+z|^{\frac{4}{d-2}}(v+z)|v|^{\frac{4}{d-2}} \overline{v} dx \\ &=: \mathrm{I} + \mathrm{II}. \end{aligned} \tag{1.11}$$

τ_* of the interval. In [17], similar invariance of the distribution of the random linear solution played an essential role in proving almost sure global well-posedness.

We need to estimate $\partial_t E(v)$ by $E(v)$ and various norms of the random linear solution $z = S(t)\phi^\omega$. Moreover, we are allowed to use at most one power of $E(v)$ in order to close a Gronwall-type argument. Note that the energy $E(v)$ consists of two parts. On the one hand, while the kinetic part controls the derivative of v, its homogeneity (= degree) is low and hence can not be used to control a nonlinear term of a high degree (in v). On the other hand, the potential part has a higher homogeneity but it can not be used to control any derivative. Hence, we need to combine the kinetic and potential parts of the energy in an intricate manner.

The main contribution to I in (1.11) is given by a term of the form:

$$\int |v|^{\frac{4}{d-2}} |\nabla v \cdot \nabla z| dx \lesssim \int |\nabla v|^2 dx + \big\| |v|^{\frac{4}{d-2}} \nabla z \big\|_{L^2_x}^2. \tag{1.12}$$

In order to estimate the second term on the right-hand side, we integrate in time and perform multilinear space-time analysis. More precisely, we divide the second term on the right-hand side of (1.12) into a θ-power and a $(1-\theta)$-power for some $\theta = \theta(s) \in (0,1)$ and estimate them in different manners. As for the θ-power, we apply the refinement of the bilinear Strichartz estimate (Lemma 3.6), substitute the Duhamel formula for v (yielding a higher order term in v), and control the resulting contribution (by ignoring the derivative on v) by the potential part of the energy. We then use the $(1-\theta)$-power to absorb the derivative on v from the θ-power and control the resulting contribution by the kinetic part of the energy and the mass. See Propositions 7.2 and 7.3.

When $d = 6$, the main contribution to the second term II in (1.11) is given by $\int |v|^3 |z| dx$, which can be controlled by (the potential part of) the energy $E(v)$. On the other hand, when $d = 5$, the main contribution to the second term II in (1.11) is given by $\int |v|^{\frac{11}{3}} |z| dx$, which we can not control by the energy $E(v)$. In order to overcome this problem, we use the following modified energy when $d = 5$:

$$\mathcal{E}(v) = \frac{1}{2} \int |\nabla v|^2 dx + \frac{3}{10} \int |v + z|^{\frac{10}{3}} dx. \tag{1.13}$$

The use of this modified energy $\mathcal{E}(v)$ eliminates the contribution II in (1.11) at the expense of introducing Δz in I. It turns out, however, the worst term is still given by the second term on the right-hand side of (1.12) and hence there is no loss in using the modified energy $\mathcal{E}(v)$.

Lastly, we point out the following. On the one hand, the regularity for almost sure local well-posedness in Theorem 1.1 is worse when $d = 6$. On the other hand, the regularity for almost sure global well-posedness in Theorem 1.5 is worse when $d = 5$:

$$\frac{20}{23} \approx 0.8696 < \frac{63}{68} \approx 0.9265.$$

This is due to the fact that the main contribution (1.12) to the energy estimate comes with a higher order term in v when $d = 5$. In fact, when $d = 4$, our argument completely breaks down. In this case, the left-hand side of (1.12) becomes

$$\int |v|^2 |\nabla v \cdot \nabla z| dx \lesssim \int |\nabla v|^2 dx + \int |v|^4 |\nabla z|^2 dx.$$

Recalling that the potential energy is given by $\frac{1}{4} \int |v|^4 dx$, it is easy to see that we can not pass a part of the derivative on z to $|v|^4$ in the second term on the right-hand side and hence it is not possible to bound it by $\big(E(v)\big)^\alpha$, $\alpha \le 1$, since $z \notin W^{1,p}(\mathbb{R}^4)$ for any p, almost surely. For this problem, some other space-time control such as the (interaction) Morawetz estimate[5] is required.[6]

Remark 1.6. In [39], Lührmann-Mendelson used a modified energy with the potential part given by $\frac{1}{p+1}\int |v+z|^{p+1}dx$ in studying the defocusing energy-subcritical NLW on $\mathbb{R}^3$ $(3 < p < 5)$:

$$\partial_t^2 u - \Delta u + |u|^{p-1}u = 0$$

with randomized initial data below the scaling critical regularity. In particular, they adapted the technique from [44] and proved almost sure global well-posedness in $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ for $\frac{p-1}{p+1} < s < 1$ by establishing an energy estimate for the modified energy. We point out, however, that the use of the modified energy for NLW in [39] is not necessary. On the contrary, it provides a worse regularity restriction than the same argument with the standard energy for NLW. In fact, Sun-Xia [51] independently studied the same problem[7] with the standard energy and proved almost sure global well-posedness with a better regularity threshold: $\frac{p-3}{p-1} < s < 1$, which interpolates the almost sure global well-posedness results by Burq-Tzvetkov ($p = 3$) in [11] and the first and third authors ($p = 5$) in [44].

While our use of the modified energy $\mathcal{E}(v)$ in (1.13) removes the issue with the time derivative of the potential part of the energy (i.e. II in (1.11)), it does not worsen the regularity threshold in the sense that the worst term is still given by (1.12).

1.3. **Probabilistic construction of finite time blowup solutions below the critical regularity.**

In this subsection, we focus on NLS (1.5) without gauge invariance. As compared to the standard NLS (1.1) with the gauge invariant nonlinearity, the equation (1.5) is less understood, in particular due to lack of structures such as conservation laws.

In recent years, starting with the work by Ikeda-Wakasugi [30], there has been some development in the construction of finite time blowup solutions for (1.5), including the case of small initial data. See also [41, 42, 29]. While there are some variations, the criteria for finite time blowup solutions are very different from those for the standard NLS (1.1) and they are given in terms of a condition on the sign of the product of the real part (and the imaginary part, respectively) of the coefficient $\lambda \in \mathbb{C} \setminus \{0\}$ in (1.5) and the imaginary part (and the real part, respectively) of (the spatial integral of) an initial condition. We now recall the result of particular interest due to Ikeda-Inui [28, Theorem 2.3 and Remark 2.1].

Given $v_0 \in H^1(\mathbb{R}^d)$, consider NLS (1.5) without gauge invariance equipped with an initial condition of the form $\phi = \alpha v_0$, $\alpha \ge 0$. Moreover, assume that v_0 satisfies

$$(\operatorname{Im}\lambda)(\operatorname{Re} v_0)(x) \ge \mathbf{1}_{|x|\le 1}|x|^{-k} \quad \text{for all } x \in \mathbb{R}^d, \tag{1.14}$$

$$\text{or} \quad -(\operatorname{Re}\lambda)(\operatorname{Im} v_0)(x) \ge \mathbf{1}_{|x|\le 1}|x|^{-k} \quad \text{for all } x \in \mathbb{R}^d \tag{1.15}$$

for some positive $k < \frac{d}{2} - 1$. Then, there exists $\alpha_0 = \alpha_0(d, k, |\lambda|) > 0$ such that, for any $\alpha > \alpha_0$, the solution $u = u(\alpha)$ to (1.5) with $u|_{t=0} = \alpha v_0$ blows up forward in finite time.

[5] See [53] for the interaction Morawetz estimate for NLS with a perturbation.

[6] In a recent preprint [36], Killip-Murphy-Vişan proved almost sure global well-posedness and scattering below the energy space for the defocusing energy-critical cubic NLS on $\mathbb{R}^4$ in the radial setting, where the Morawetz estimate (among other tools available in the radial setting) played an important role.

[7] While the main result in [51] is stated on the three-dimensional torus $\mathbb{T}^3$, the same result holds on $\mathbb{R}^3$ by the same proof.

If we denote $T^*(\alpha) > 0$ to be the forward maximal time of existence, then the following estimate holds:

$$T^*(\alpha) \le C\alpha^{-\frac{1}{\kappa}} \tag{1.16}$$

for all $\alpha > \alpha_0$, where $\kappa = \frac{d-2}{4} - \frac{k}{2}$. Moreover, we have

$$\lim_{T \to T^*} \|u\|_{L^{q_d}([0,T); W_x^{1,r_d})} = \infty,$$

where (q_d, r_d) is as in (1.8). A similar statement holds for the negative time direction if we replace (1.14) and (1.15) by $-(\operatorname{Im}\lambda)(\operatorname{Re} v_0)(x) \ge \mathbf{1}_{|x|\le 1}|x|^{-k}$ and $(\operatorname{Re}\lambda)(\operatorname{Im} v_0)(x) \ge \mathbf{1}_{|x|\le 1}|x|^{-k}$, respectively.

In the following, we fix v_0 satisfying (1.14) or (1.15) and consider (1.5) with $u|_{t=0} = \alpha v_0 + \varepsilon \phi^\omega$, where ϕ^ω is the Wiener randomization of some fixed $\phi \in H^s(\mathbb{R}^d) \setminus H^1(\mathbb{R}^d)$, $s < 1$, $d = 5, 6$. Namely, we study stability of the finite time blowup solution constructed in [28] under a rough perturbation in a probabilistic manner.

Theorem 1.7. *Let $d = 5, 6$, $1 - \frac{1}{d} < s < 1$, and $k < \frac{d}{2} - 1$. Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in* (1.2)*. Fix $v_0 \in H^1(\mathbb{R}^d)$, satisfying* (1.14) *or* (1.15)*. Then, for each $R > 0$ and $\varepsilon > 0$, there exists $\Omega_{R,\varepsilon} \subset \Omega$ with*

$$P(\Omega_{R,\varepsilon}^c) \le C \exp\bigg(- c \frac{R^2}{\varepsilon^2 \|\phi\|_{L^2}^2} \bigg)$$

and $\alpha_0 = \alpha_0(d, k, |\lambda|, R, \varepsilon) > 0$ such that for each $\omega \in \Omega_{R,\varepsilon}$ and any $\alpha > \alpha_0$, the solution $u = u^\omega$ to (1.5) *with initial data*

$$u|_{t=0} = \alpha v_0 + \varepsilon \phi^\omega$$

blows up forward in finite time with the forward maximal time $T^*(\alpha)$ of existence satisfying (1.16)*, where the implicit constant depends only on $R > 0$. Moreover, we have*

$$\lim_{T \to T^*} \|u - \varepsilon z^\omega\|_{L^{q_d}([0,T); W_x^{1,r_d})} = \infty, \tag{1.17}$$

where $z^\omega = S(t)\phi^\omega$.

This result in particular allows us to construct finite time blowup solutions *below* the critical regularity $s_{\text{crit}} = 1$. Moreover, it can be viewed as a *probabilistic stability* result of the finite time blowup solutions in $H^1(\mathbb{R}^d)$ constructed in [28] under random and rough perturbations. Note that $P(\Omega_{R,\varepsilon}) \to 1$ as $\varepsilon \to 0$.

The proof of Theorem 1.7 is a straightforward combination of Proposition 1.3 and the finite time blowup result in [28]. More precisely, we prove Theorem 1.7 by writing $u = \varepsilon z + v$ and considering the equation for the residual term v:

$$\begin{cases} i\partial_t v + \Delta v = \lambda |v + \varepsilon z^\omega|^{\frac{d+2}{d-2}}, \\ v|_{t=0} = \alpha v_0, \end{cases} \tag{1.18}$$

where $z^\omega = S(t)\phi^\omega$ as before. In view of Proposition 1.3, the equation (1.18) is almost surely locally well-posed with a blowup alternative (1.7). This allows us to show that the solution v is a weak solution in the sense of Definition 8.1 and hence to carry out the analysis in [28] with a small modification coming from the random perturbation term. One crucial point to note is that once we reduce our analysis to the weak formulation in (8.1), we only require space-time integrability of the random perturbation z^ω and its differentiability plays no role. This enables us to prove Theorem 1.7.

We now give a brief outline of this article. In Sections 2 and 3, we recall probabilistic and deterministic lemmas along with the definitions of the basic function spaces. We then prove the crucial nonlinear estimates in Section 4, and present the proof of the almost sure local well-posedness (Theorems 1.1 and 1.2) in Section 5. In Section 6, we prove a variant of almost sure local well-posedness (Proposition 1.3). In Section 7, we establish the crucial energy bound (1.10) and present the proof of almost sure global well-posedness of the defocusing energy-critical NLS (1.1) (Theorem 1.5). In Section 8, we use Proposition 1.3 to construct finite time blowup solutions below the critical regularity in a probabilistic manner.

In view of the time reversibility of the equations, we only consider positive times in the following. Moreover, in the local-in-time theory, the defocusing/focusing nature of (1.1) does not play any role, so we assume that it is defocusing (with the +-sign in (1.1)). Similarly, we simply set $\lambda = 1$ in (1.5).

2. Probabilistic lemmas

In this section, we state the probabilistic lemmas used in this paper. See [2, 44] for their proofs. The first lemma states that the Wiener randomization almost surely preserves the differentiability of a given function.

Lemma 2.1. *Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in* (1.2)*. Then, there exist $C, c > 0$ such that*

$$P\big(\|\phi^\omega\|_{H^s} > \lambda\big) \le C \exp\bigg(-c\frac{\lambda^2}{\|\phi\|_{H^s}^2}\bigg)$$

for all $\lambda > 0$.

In fact, one can also show that there is almost surely no smoothing upon randomization in terms of differentiability (see, for example, Lemma B.1 in [10]). We, however, do not need such a non-smoothing result in the following.

Next, we state the probabilistic Strichartz estimates. Before doing so, we first recall the usual Strichartz estimates on $\mathbb{R}^d$ for readers' convenience. We say that a pair (q, r) is admissible if $2 \le q, r \le \infty$, $(q, r, d) \ne (2, \infty, 2)$, and

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}. \tag{2.1}$$

Then, the following Strichartz estimates are known to hold. See [50, 56, 22, 33].

Lemma 2.2. *Let (q, r) be admissible. Then, we have*

$$\|S(t)\phi\|_{L^q_t L^r_x} \lesssim \|\phi\|_{L^2}.$$

As a corollary, we obtain

$$\|S(t)\phi\|_{L^p_{t,x}} \lesssim \big\||\nabla|^{\frac{d}{2}-\frac{d+2}{p}}\phi\big\|_{L^2}. \tag{2.2}$$

for $p \ge \frac{2(d+2)}{d}$, which follows from Sobolev's inequality and Lemma 2.2.

The following lemma shows an improvement of the Strichartz estimates upon the randomization of initial data. The improvement appears in the form of integrability and not differentiability. Note that such a gain of integrability is classical in the context of random Fourier series [48]. The first estimate (2.3) follows from Minkowski's integral inequality along with Bernstein's inequality. As for the L^∞_T-estimate (2.4), see [44] for the proof (in the context of the wave equation).

Lemma 2.3. *Given ϕ on $\mathbb{R}^d$, let ϕ^ω be its Wiener randomization defined in* (1.2). *Then, given finite $q, r \geq 2$, there exist $C, c > 0$ such that*

$$P\Big(\|S(t)\phi^\omega\|_{L^q_t L^r_x([0,T)\times\mathbb{R}^d)} > \lambda \Big) \leq C \exp\bigg(-c \frac{\lambda^2}{T^{\frac{2}{q}} \|\phi\|^2_{H^s}} \bigg) \tag{2.3}$$

for all $T > 0$ and $\lambda > 0$ with (i) *$s = 0$ if $r < \infty$ and* (ii) *$s > 0$ if $r = \infty$. Moreover, when $q = \infty$, given $2 \leq r \leq \infty$, there exist $C, c > 0$ such that*

$$P\Big(\|S(t)\phi^\omega\|_{L^\infty_t L^r_x([0,T)\times\mathbb{R}^d)} > \lambda \Big) \leq C(1+T) \exp\bigg(-c \frac{\lambda^2}{\|\phi\|^2_{H^s}} \bigg) \tag{2.4}$$

for all $\lambda > 0$ with $s > 0$.

3. Function spaces and their basic properties

In this section, we go over the basic definitions and properties of the functions spaces used for the Fourier restriction norm method adapted to the space of functions of bounded p-variation and its pre-dual, introduced and developed by Tataru, Koch, and their collaborators [37, 24, 27]. We refer readers to Hadac-Herr-Koch [24] and Herr-Tataru-Tzvetkov [27] for proofs of the basic properties. See also [3].

Let $\mathcal{Z}$ be the set of finite partitions $-\infty < t_0 < t_1 < \cdots < t_K \leq \infty$ of the real line. By convention, we set $u(t_K) := 0$ if $t_K = \infty$.

Definition 3.1. Let $1 \leq p < \infty$. We define a U^p-atom to be a step function $a : \mathbb{R} \to L^2(\mathbb{R}^d)$ of the form

$$a = \sum_{k=1}^{K} \phi_{k-1} \chi_{[t_{k-1}, t_k)},$$

where $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset L^2(\mathbb{R}^d)$ with $\sum_{k=0}^{K-1} \|\phi_k\|^p_{L^2} = 1$. Furthermore, we define the atomic space $U^p = U^p(\mathbb{R}; L^2(\mathbb{R}^d))$ by

$$U^p := \bigg\{ u : \mathbb{R} \to L^2(\mathbb{R}^d) : u = \sum_{j=1}^\infty \lambda_j a_j \text{ for } U^p\text{-atoms } a_j,\ \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}; \mathbb{C}) \bigg\}$$

with the norm

$$\|u\|_{U^p} := \inf \bigg\{ \sum_{j=1}^\infty |\lambda_j| : u = \sum_{j=1}^\infty \lambda_j a_j \text{ for } U^p\text{-atoms } a_j,\ \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}; \mathbb{C}) \bigg\},$$

where the infimum is taken over all possible representations for u.

Definition 3.2. Let $1 \leq p < \infty$.

(i) We define $V^p = V^p(\mathbb{R}; L^2(\mathbb{R}^d))$ to be the space of functions $u : \mathbb{R} \to L^2(\mathbb{R}^d)$ of bounded p-variation with the standard p-variation norm

$$\|u\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \bigg(\sum_{k=1}^K \|u(t_k) - u(t_{k-1})\|_{L^2}^p \bigg)^{\frac{1}{p}}.$$

By convention, we impose that the limits $\lim_{t\to\pm\infty} u(t)$ exist in $L^2(\mathbb{R}^d)$.
(ii) Let V^p_{rc} be the closed subspace of V^p of all right-continuous functions $u \in V^p$ with $\lim_{t\to-\infty} u(t) = 0$.

Recall the following inclusion relation; for $1 \le p < q < \infty$,

$$U^p \hookrightarrow V^p_{\text{rc}} \hookrightarrow U^q \hookrightarrow L^\infty(\mathbb{R}; L^2(\mathbb{R}^d)). \tag{3.1}$$

The space V^p is the classical space of functions of bounded p-variation and the space U^p appears as the pre-dual of $V^{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Their duality relation and the atomic structure of the U^p-space turned out to be very effective in studying dispersive PDEs in critical settings.

Next, we define the U^p- and V^p-spaces adapted to the Schrödinger flow.

Definition 3.3. Let $1 \le p < \infty$. We define $U^p_\Delta := S(t)U^p$ and ($V^p_\Delta := S(t)V^p$, respectively) to be the space of all functions $u : \mathbb{R} \to L^2(\mathbb{R}^d)$ such that $t \to S(-t)u(t)$ is in U^p (and in V^p, respectively) with the norms

$$\|u\|_{U^p_\Delta} := \|S(-t)u\|_{U^p} \qquad \text{and} \qquad \|u\|_{V^p_\Delta} := \|S(-t)u\|_{V^p}.$$

The closed subspace $V^p_{\text{rc},\Delta}$ is defined in an analogous manner.

Next, we define the dyadically defined versions of U^p_Δ and V^p_Δ. We use the convention that capital letters denote dyadic numbers, e.g., $N = 2^n$ for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Fix a nonnegative even function $\varphi \in C_0^\infty((-2,2); [0,1])$ with $\varphi(r) = 1$ for $|r| \le 1$. Then, we set $\varphi_N(r) := \varphi(r/N) - \varphi(2r/N)$ for $N \ge 2$ and $\varphi_1(r) := \varphi(r)$. Given $N \in 2^{\mathbb{N}_0}$, let $\mathbf{P}_N$ denote the Littlewood-Paley projection operator with the Fourier multiplier $\varphi_N(|\xi|)$, i.e. $\mathbf{P}_N f := \mathcal{F}^{-1}[\varphi_N(|\xi|)\widehat{f}(\xi)]$. We also define $\mathbf{P}_{\le N} := \sum_{1\le M\le N} \mathbf{P}_M$ and $\mathbf{P}_{>N} := \text{Id} - \mathbf{P}_{\le N}$.

Definition 3.4. (i) Let $s \in \mathbb{R}$. We define $X^s(\mathbb{R})$ to be the closure of $C(\mathbb{R}; H^s(\mathbb{R}^d)) \cap U^2_\Delta$ with respect to the X^s-norm defined by

$$\|u\|_{X^s(\mathbb{R})} := \bigg(\sum_{\substack{N\ge1\\\text{dyadic}}} N^{2s} \|\mathbf{P}_N u\|^2_{U^2_\Delta L^2} \bigg)^{\frac{1}{2}}.$$

(ii) Let $s \in \mathbb{R}$. We define $Y^s(\mathbb{R})$ to be the space of all functions $u \in C(\mathbb{R}; H^s(\mathbb{R}^d))$ such that the map $t \mapsto \mathbf{P}_N u$ lies in $V^2_{\text{rc},\Delta} H^s$ for any $N \in 2^{\mathbb{N}_0}$ and $\|u\|_{Y^s(\mathbb{R})} < \infty$, where the Y^s-norm is defined by

$$\|u\|_{Y^s(\mathbb{R})} := \bigg(\sum_{\substack{N\ge1\\\text{dyadic}}} N^{2s} \|\mathbf{P}_N u\|^2_{V^2_\Delta L^2} \bigg)^{\frac{1}{2}}.$$

The transference principle ([24, Proposition 2.19]) and the interpolation lemma [24, Proposition 2.20] applied on the Strichartz estimates (Lemma 2.2 and (2.2)) imply the following estimate for the Y^0-space.

Lemma 3.5. *Let $d \geq 1$. Then, given any admissible pair (q, r) with $q > 2$ and $p \geq \frac{2(d+2)}{d}$, we have*

$$\|u\|_{L_t^q L_x^r} \lesssim \|u\|_{Y^0},$$
$$\|u\|_{L_{t,x}^p} \lesssim \big\||\nabla|^{\frac{d}{2}-\frac{d+2}{p}} u\big\|_{Y^0}.$$

Similarly, the bilinear refinement of the Strichartz estimate [8, 47, 16] implies the following bilinear estimate.

Lemma 3.6. *Let $N_1, N_2 \in 2^{\mathbb{N}_0}$ with $N_1 \leq N_2$. Then, given any $\varepsilon > 0$, we have*

$$\|\mathbf{P}_{N_1} u_1 \mathbf{P}_{N_2} u_2\|_{L_{t,x}^2} \lesssim N_1^{\frac{d-2}{2}} \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}-\varepsilon} \|\mathbf{P}_{N_1} u_1\|_{Y^0} \|\mathbf{P}_{N_2} u_2\|_{Y^0}$$

for all $u_1, u_2 \in Y^0$.

For our analysis, we need to introduce the local-in-time versions of the spaces defined above.

Definition 3.7. Let $\mathcal{B}$ be a Banach space consisting of continuous H-valued functions (in $t \in \mathbb{R}$) for some Hilbert space H. We define the corresponding restriction space $\mathcal{B}(I)$ to a given time interval $I \subset \mathbb{R}$ as

$$\mathcal{B}(I) := \big\{u \in C(I; H) : \text{ there exists } v \in \mathcal{B} \text{ such that } \ v|_I = u\big\}.$$

We endow $\mathcal{B}(I)$ with the norm

$$\|u\|_{\mathcal{B}(I)} := \inf\big\{\|v\|_{\mathcal{B}} : \ v|_I = u\big\},$$

where the infimum is taken over all possible extensions v of u onto the real line. When $I = [0, T)$, we simply set $\mathcal{B}_T := \mathcal{B}(I) = \mathcal{B}([0, T))$.

Recall that the space $\mathcal{B}(I)$ is a Banach space. As a consequence of (3.1), we have the following inclusion relation; for any interval $I \subset \mathbb{R}$, we have

$$X^s(I) \hookrightarrow Y^s(I) \hookrightarrow \langle \nabla \rangle^{-s} V_\Delta^2(I) \cap C(I; H^s(\mathbb{R}^d)).$$

We conclude this section by stating the linear estimates. Given $a \in \mathbb{R}$, we define the integral operator $\mathcal{I}_a$ on $L^1_{\text{loc}}([a, \infty); L^2(\mathbb{R}^d))$ by

$$\mathcal{I}_a[F](t) := \int_a^t S(t - t')F(t')dt' \tag{3.2}$$

for $t \geq a$ and $\mathcal{I}_a[F](t) = 0$ otherwise. When $a = 0$, we simply set $\mathcal{I} = \mathcal{I}_a$. Given an interval $I = [a, b)$, we set the dual norm $N^s(I)$ controlling the nonhomogeneous term on I by

$$\|F\|_{N^s(I)} = \big\|\mathcal{I}_a[F]\big\|_{X^s(I)}.$$

Then, we have the following linear estimates.

Lemma 3.8. *Let $s \in \mathbb{R}$ and $T \in (0, \infty]$. Then, the following linear estimates hold:*

$$\|S(t)\phi\|_{X_T^s} \leq \|\phi\|_{H^s},$$
$$\|F\|_{N_T^s} \leq \sup_{\substack{w \in Y_T^{-s} \\ \|w\|_{Y_T^{-s}} = 1}} \left|\int_0^T \langle F(t), w(t)\rangle_{L_x^2} dt\right|$$

for any $\phi \in H^s(\mathbb{R}^d)$ *and* $F \in L^1([0,T); H^s(\mathbb{R}^d))$.

The first estimate is immediate from the definition of the space X^s_T. The second estimate basically follows from the duality relation between U^2 and V^2 ([24, Proposition 2.10, Remark 2.11]). See also Proposition 2.11 in [27].

4. Nonlinear estimates

As in Section 1, let $z(t) = z^\omega(t) = S(t)\phi^\omega$ denote the linear solution with the randomized initial data ϕ^ω in (1.2). If u is a solution to (1.1), then the residual term $v = u - z$ satisfies the perturbed NLS (1.3). In this section, we establish relevant nonlinear estimates in solving the fixed point problem (1.4) for the residual term v.

Given $d = 5, 6$, fix an admissible pair:

$$(q_d, r_d) := \bigg(\frac{2d}{d-2}, \frac{2d^2}{d^2-2d+4}\bigg) = \begin{cases} \big(\frac{10}{3}, \frac{50}{19}\big), & d=5,\\ \big(3, \frac{18}{7}\big), & d=6. \end{cases} \tag{4.1}$$

Note that

$$\frac{d+2}{d-2} q_d' = q_d,$$

where q_d' denotes the Hölder conjugate of q_d. By Sobolev's inequality, we have

$$W^{1,r_d}(\mathbb{R}^d) \hookrightarrow L^{\rho_d}(\mathbb{R}^d), \qquad \rho_d := \frac{2d^2}{(d-2)^2} = \begin{cases} \frac{50}{9}, & d=5,\\ \frac{9}{2}, & d=6. \end{cases} \tag{4.2}$$

Before we state the main probabilistic nonlinear estimates, let us define the set of indices:

$$\mathfrak{S}_\delta := \bigg\{ \Big(\frac{q_d}{1-\delta q_d}, r_d\Big), \Big(\frac{q_d}{1-\delta q_d}, \frac{d+2}{d-2} r_d'\Big), \Big(\frac{q_d}{1-\delta q_d}, \rho_d\Big), \Big(\frac{4}{1-4\delta}, 4\Big), \Big(4, \frac{4+2\delta}{\delta}\Big) \bigg\}$$

for small $\delta > 0$. Given an interval $I \subset \mathbb{R}$ and $\delta > 0$, we define $S^s(I) = S^s(I;\delta)$ by[8]

$$\|u\|_{S^s(I)} := \max\big\{ \|\langle\nabla\rangle^s u\|_{L^q_t L^r_x(I\times\mathbb{R}^d)} : (q,r) \in \mathfrak{S}_\delta \big\}. \tag{4.3}$$

Furthermore, given $M > 0$ and an interval I, define the set $E_M(I) \subset \Omega$ by

$$E_M(I) := \big\{ \omega \in \Omega : \|\phi^\omega\|_{H^s} + \|S(t)\phi^\omega\|_{S^s(I)} \le M \big\}. \tag{4.4}$$

When $I = [0,T)$, we simply write $E_{M,T} = E_M([0,T))$.

Proposition 4.1. *Let* $d = 5, 6$, $1 - \frac{1}{d} < s < 1$, *and*

$$\mathcal{N}(u) = |u|^{\frac{4}{d-2}} u \qquad \text{or} \qquad \mathcal{N}(u) = |u|^{\frac{d+2}{d-2}}.$$

Given $\phi \in H^s(\mathbb{R}^d)$, *let* ϕ^ω *be its Wiener randomization defined in* (1.2) *and* $z = S(t)\phi^\omega$. *Then, there exist sufficiently small* $\delta = \delta(d,s) > 0$ *and* $\theta = \theta(d,s) > 0$ *such that*

$$\|\mathcal{N}(v+z)\|_{N^1_T} \le C_1 \Big\{ \|v\|^{\frac{d+2}{d-2}}_{Y^1_T} + T^\theta M^{\frac{d+2}{d-2}} \Big\}, \tag{4.5}$$

$$\|\mathcal{N}(v_1+z) - \mathcal{N}(v_2+z)\|_{N^1_T} \le C_2 \Big\{ \|v_1\|^{\frac{4}{d-2}}_{Y^1_T} + \|v_2\|^{\frac{4}{d-2}}_{Y^1_T} + T^\theta M^{\frac{4}{d-2}} \Big\} \|v_1 - v_2\|_{Y^1_T}, \tag{4.6}$$

---

[8]As we see below, we fix $\delta = \delta(d,s) > 0$ and hence we suppress the dependence on δ for simplicity of the presentation. A similar comment applies to $E_M(I)$ and $\widetilde{E}_M(I)$ defined in (4.4) and (6.2).

for any $T>0$, $v, v_1, v_2 \in Y^1_T$, and $\omega \in E_{M,T}$.

Note that we have

$$\|u\|_{X^1_T} \sim \|u\|_{X^0_T} + \|\nabla u\|_{X^0_T}. \tag{4.7}$$

It is crucial that we handle a regular gradient ∇ rather than $\langle\nabla\rangle$ for our purpose. We also point out that once we fix the set $E_{M,T}$, the nonlinear estimates are entirely *deterministic*.

Proof. **Part 1:** We first prove (4.5). In view of (4.7), Lemma 3.8 and Definition 3.7 of the time restriction norm, it suffices to show[9]

$$\bigg|\int_{[0,T]\times\mathbb{R}^d} \mathcal{N}(v+z)\cdot w dx dt\bigg| \lesssim \|v\|_{Y^1}^{\frac{d+2}{d-2}} + T^\theta M^{\frac{d+2}{d-2}}, \tag{4.8}$$

$$\bigg|\int_{[0,T]\times\mathbb{R}^d} \nabla\mathcal{N}(v+z)\cdot w dx dt\bigg| \lesssim \|v\|_{Y^1}^{\frac{d+2}{d-2}} + T^\theta M^{\frac{d+2}{d-2}}, \tag{4.9}$$

for all $w\in Y^0$ with $\|w\|_{Y^0}=1$ and any $\omega \in E_{M,T}$,.

Let us first consider (4.8). Hölder's inequality and the embedding $W^{\frac{4}{d+2}, r_d}(\mathbb{R}^d) \hookrightarrow L^{\frac{d+2}{d-2}r_d'}(\mathbb{R}^d)$ yield

$$\begin{aligned}
\text{LHS of (4.8)} &\lesssim \big\| |v+z|^{\frac{d+2}{d-2}}\big\|_{L^{q_d'}_T L^{r_d'}_x} \|w\|_{L^{q_d}_T L^{r_d}_x} \lesssim \|v+z\|^{\frac{d+2}{d-2}}_{L^{q_d}_T L_x^{\frac{d+2}{d-2}r_d'}}\\
&\lesssim \|v\|_{Y^1}^{\frac{d+2}{d-2}} + \|z\|^{\frac{d+2}{d-2}}_{L^{q_d}_T L_x^{\frac{d+2}{d-2}r_d'}}\\
&\lesssim \|v\|_{Y^1}^{\frac{d+2}{d-2}} + (T^\delta M)^{\frac{d+2}{d-2}}
\end{aligned} \tag{4.10}$$

for any $\omega\in E_{M,T}$, where we used

$$\|z\|_{L^{q_d}_T L_x^{\frac{d+2}{d-2}r_d'}} \le T^\delta \|z\|_{L_T^{\frac{q_d}{1-\delta q_d}} L_x^{\frac{d+2}{d-2}r_d'}} \le T^\delta M.$$

Next, we consider (4.9). The contribution from $\mathbf{P}_{\le 1} w$ can be estimated in an analogous manner to the computation above. Hence, without loss of generality, we assume $w = \mathbf{P}_{>1} w$ in the following.

We first prove (4.9) for $\mathcal{N}(u) = |u|^{\frac{d+2}{d-2}}$. With

$$\nabla(|f|^\alpha) = \alpha |f|^{\alpha-2}\operatorname{Re}(f\nabla\overline{f}), \tag{4.11}$$

the estimate (4.9) is reduced to showing

$$\bigg|\int_{[0,T]\times\mathbb{R}^d} (\nabla w_1)(v+z)|v+z|^{\frac{6-d}{d-2}} w dx dt\bigg| \lesssim \|v\|_{Y^1}^{\frac{d+2}{d-2}} + T^\theta M^{\frac{d+2}{d-2}} \tag{4.12}$$

for $w_1 = \overline{v}$ or $\overline{z}$. A small but important observation is that a derivative does not fall on the third factor with the absolute value. In the following, we preform analysis on the relative sizes of the frequencies of the first two factors.

[9] Strictly speaking, we need to work with a truncated nonlinearity as in [3] so that Lemma 3.8 is applicable. This modification, however, is standard and we omit details. See [3] for the details.

• **Case 1:** $w_1 = \overline{v}$. In this case, from Lemma 3.5 with (4.2) and (4.4), we have

$$\begin{aligned}
\text{LHS of (4.12)} & \lesssim \|\nabla v\|_{L^{q_d}_T L^{r_d}_x} \|v+z\|_{L^{q_d}_T L^{\rho_d}_x} \big\| |v+z|^{\frac{6-d}{d-2}} \big\|_{L_T^{\frac{2d}{6-d}} L_x^{\frac{2d^2}{(6-d)(d-2)}}} \|w\|_{L^{q_d}_T L^{r_d}_x} \\
& \lesssim \|v\|_{Y^1} \|v+z\|^{\frac{4}{d-2}}_{L^{q_d}_T L^{\rho_d}_x} \lesssim \|v\|_{Y^1} \big\{ \|v\|_{L^{q_d}_T W^{1,r_d}_x} + \|z\|_{L^{q_d}_T L^{\rho_d}_x} \big\}^{\frac{4}{d-2}} \\
& \lesssim \|v\|_{Y^1} \Big\{ \|v\|^{\frac{4}{d-2}}_{Y^1} + (T^\delta M)^{\frac{4}{d-2}} \Big\}
\end{aligned} \tag{4.13}$$

for any $\omega \in E_{M,T}$. Then, (4.12) follows from Young's inequality.

• **Case 2:** $w_1 = \overline{z}$. Using the Littlewood-Paley decomposition, we have

$$\text{LHS of (4.12)} \lesssim \sum_{N_1, N_2 \in 2^{\mathbb{N}_0}} \bigg| \int_{[0,T]\times\mathbb{R}^d} N_1 \mathbf{P}_{N_1} \overline{z}\, \mathbf{P}_{N_2}(v+z) |v+z|^{\frac{6-d}{d-2}} w dx dt \bigg|.$$

<u>Subcase 2.a:</u> We first consider the contribution from $N_2 \gtrsim N_1^{\frac{1}{d-1}}$. Note that we have

$$\|z\|_{L^{q_d}_T (W^{s,r_d}_x \cap L^{\rho_d}_x)} \le T^\delta \|z\|_{L_T^{\frac{q_d}{1-\delta q_d}} (W^{s,r_d}_x \cap L^{\rho_d}_x)} \le T^\delta M$$

on $E_{M,T}$. Then, proceeding as in Case 1 with Lemma 3.5, (4.2), and (4.4), we have

$$\begin{aligned}
\text{LHS of (4.12)} & \lesssim \sum_{\substack{N_1, N_2 \in 2^{\mathbb{N}_0} \\ N_2 \gtrsim N_1^{\frac{1}{d-1}}}} N_1 \|\mathbf{P}_{N_1} z\|_{L^{q_d}_T L^{\rho_d}_x} \|\mathbf{P}_{N_2}(v+z)\|_{L^{q_d}_T L^{r_d}_x} \\
& \qquad \times \big\| |v+z|^{\frac{6-d}{d-2}} \big\|_{L_T^{\frac{2d}{6-d}} L_x^{\frac{2d^2}{(6-d)(d-2)}}} \|w\|_{L^{q_d}_T L^{r_d}_x} \\
& \lesssim \sum_{\substack{N_1, N_2 \in 2^{\mathbb{N}_0} \\ N_2 \gtrsim N_1^{\frac{1}{d-1}}}} N_1^{-s+1} N_2^{-s} \|\mathbf{P}_{N_1} z\|_{L^{q_d}_T W^{s,\rho_d}_x} \big\{ \|\mathbf{P}_{N_2} v\|_{L^{q_d}_T W^{s,r_d}_x} + \|\mathbf{P}_{N_2} z\|_{L^{q_d}_T W^{s,r_d}_x} \big\} \\
& \qquad \times \big\{ \|v\|_{L^{q_d}_T W^{1,r_d}_x} + \|z\|_{L^{q_d}_T L^{\rho_d}_x} \big\}^{\frac{6-d}{d-2}} \|w\|_{L^{q_d}_T L^{r_d}_x} \\
& \lesssim \sum_{\substack{N_1, N_2 \in 2^{\mathbb{N}_0} \\ N_2 \gtrsim N_1^{\frac{1}{d-1}}}} N_1^{-s+1} N_2^{-s} T^\delta M \Big\{ \|v\|^{\frac{4}{d-2}}_{Y^1} + (T^\delta M)^{\frac{4}{d-2}} \Big\} \\
& \lesssim \|v\|^{\frac{d+2}{d-2}}_{Y^1} + (T^\delta M)^{\frac{d+2}{d-2}}
\end{aligned} \tag{4.14}$$

for any $\omega \in E_{M,T}$, provided that $s > 1 - \frac{1}{d}$.

<u>Subcase 2.b:</u> Next, we estimate the contribution from $N_2 \ll N_1^{\frac{1}{d-1}}$. Noting that $\big(\frac{4d}{(6-d)(d-2)}, \frac{d^2}{d^2-4d+6}\big)$ is an admissible pair, Hölder's inequality and Lemma 3.5 yield

$$\|w\|_{L_T^{\frac{d}{d-3}} L_x^{\frac{d^2}{d^2-4d+6}}} \le T^{\frac{d}{4}-1} \|w\|_{L_T^{\frac{4d}{(6-d)(d-2)}} L_x^{\frac{d^2}{d^2-4d+6}}} \lesssim T^{\frac{d}{4}-1} \|w\|_{Y^0}. \tag{4.15}$$

Then, by applying Lemma 3.6 with Lemmas 3.5 and 3.8 and (4.4), we obtain

$$\begin{aligned}
\text{LHS of (4.12)} &\lesssim \sum_{\substack{N_1, N_2 \in 2^{\mathbb{N}_0} \\ N_2 \ll N_1^{\frac{1}{d-1}}}} N_1 \|\mathbf{P}_{N_1} z \mathbf{P}_{N_2}(v+z)\|_{L^2_{T,x}} \\
&\qquad \times \big\| |v+z|^{\frac{6-d}{d-2}} \big\|_{L_T^{\frac{2d}{6-d}} L_x^{\frac{2d^2}{(6-d)(d-2)}}} \|w\|_{L_T^{\frac{d}{d-3}} L_x^{\frac{d^2}{d^2-4d+6}}} \\
&\lesssim T^{\frac{d}{4}-1} \sum_{\substack{N_1, N_2 \in 2^{\mathbb{N}_0} \\ N_2 \ll N_1^{\frac{1}{d-1}}}} N_1 N_2^{\frac{d}{2}-1} \bigg(\frac{N_2}{N_1}\bigg)^{\frac{1}{2}-\varepsilon} \|\mathbf{P}_{N_1} z\|_{Y^0_T} \|\mathbf{P}_{N_2}(v+z)\|_{Y^0_T} \\
&\qquad \times \big\{ \|v\|_{L_T^{q_d} W_x^{1,r_d}} + \|z\|_{L_T^{q_d} L_x^{\rho_d}} \big\}^{\frac{6-d}{d-2}} \|w\|_{Y^0} \\
&\lesssim T^{\frac{d}{4}-1} \sum_{\substack{N_1, N_2 \in 2^{\mathbb{N}_0} \\ N_2 \ll N_1^{\frac{1}{d-1}}}} N_1^{-s+\frac{1}{2}+\varepsilon} N_2^{-s+\frac{d-1}{2}-\varepsilon} M \\
&\qquad \times (\|v\|_{Y^s} + M) \Big\{ \|v\|_{Y^1}^{\frac{6-d}{d-2}} + (T^\delta M)^{\frac{6-d}{d-2}} \Big\} \\
&\lesssim T^{\theta'} M \Big\{ \|v\|_{Y^1}^{\frac{4}{d-2}} + M^{\frac{4}{d-2}} \Big\} \\
&\lesssim \|v\|_{Y^1}^{\frac{d+2}{d-2}} + T^\theta M^{\frac{d+2}{d-2}}
\end{aligned} \tag{4.16}$$

for any $\omega \in E_{M,T}$, provided that $s > 1 - \frac{1}{d}$. This proves (4.5) for $\mathcal{N}(u) = |u|^{\frac{d+2}{d-2}}$.

We now prove (4.9) for $\mathcal{N}(u) = |u|^{\frac{4}{d-2}} u$. In this case, we have[10]

$$\nabla(|f|^{\alpha-1} f) = (\alpha - 1)|f|^{\alpha-2} \tfrac{f}{|f|} \operatorname{Re}(f \nabla \overline{f}) + |f|^{\alpha-1} \nabla f. \tag{4.17}$$

Noting that $\big| |f|^{\alpha-3} f \big| = |f|^{\alpha-2}$, we can estimate the first term in (4.17) using (4.12). It remains to estimate the contribution from the second term in (4.17). Namely, we prove

$$\bigg| \int_{[0,T] \times \mathbb{R}^d} (\nabla w_1) |v+z|^{\frac{4}{d-2}} w \, dx dt \bigg| \lesssim \|v\|_{Y^1}^{\frac{d+2}{d-2}} + T^\theta M^{\frac{d+2}{d-2}} \tag{4.18}$$

for $w_1 = v$ or z. When $w_1 = v$, (4.18) follows from Case 1 above. Hence, we assume that $w_1 = z$ in the following. By writing $(\nabla z)|v+z|^{\frac{4}{d-2}} = (\nabla z)|v+z| \cdot |v+z|^{\frac{6-d}{d-2}}$, it follows from

[10] Here, we assumed that $\partial\{x \in \mathbb{R}^d : f(x) = 0\}$ has measure 0. This assumption can be verified for smooth truncated $\mathbf{P}_{\leq N} z$ and smooth v_N. Then, we can establish the desired estimates for smooth $\mathbf{P}_{\leq N} z$ and v_N and take a limit as $N \to \infty$.

Lemma 3.5 and (4.15) with (4.4) that

$$
\begin{aligned}
\text{LHS of (4.18)} &\lesssim \|(\nabla z)(v+z)\|_{L^2_{T,x}} \Big\| |v+z|^{\frac{6-d}{d-2}} \Big\|_{L_T^{\frac{2d}{6-d}} L_x^{\frac{2d^2}{(6-d)(d-2)}}} \|w\|_{L_T^{\frac{d}{d-3}} L_x^{\frac{d^2}{d^2-4d+6}}} \\
&\lesssim \sum_{N_1, N_2 \in 2^{\mathbb{N}_0}} N_1 \|\mathbf{P}_{N_1} z \mathbf{P}_{N_2}(v+z)\|_{L^2_{T,x}} \\
&\qquad \times \Big\| |v+z|^{\frac{6-d}{d-2}} \Big\|_{L_T^{\frac{2d}{6-d}} L_x^{\frac{2d^2}{(6-d)(d-2)}}} \|w\|_{L_T^{\frac{d}{d-3}} L_x^{\frac{d^2}{d^2-4d+6}}} \\
&\lesssim T^{\frac{d}{4}-1} \Big\{ \|v\|_{Y^1}^{\frac{6-d}{d-2}} + (T^\delta M)^{\frac{6-d}{d-2}} \Big\} \sum_{N_1, N_2 \in 2^{\mathbb{N}_0}} N_1 \|\mathbf{P}_{N_1} z \mathbf{P}_{N_2}(v+z)\|_{L^2_{T,x}}
\end{aligned} \tag{4.19}
$$

for any $\omega \in E_{M,T}$. When $N_2 \ll N_1^{\frac{1}{d-1}}$, we can apply Lemma 3.6 as in Subcase 2.b and establish (4.18).

Let us consider the remaining case $N_2 \gtrsim N_1^{\frac{1}{d-1}}$. As in Subcase 2.a, we have

$$
\begin{aligned}
\sum_{\substack{N_1, N_2 \in 2^{\mathbb{N}_0} \\ N_2 \gtrsim N_1^{\frac{1}{d-1}}}} N_1 \|\mathbf{P}_{N_1} z \mathbf{P}_{N_2} z\|_{L^2_{T,x}} &\lesssim \sum_{\substack{N_1, N_2 \in 2^{\mathbb{N}_0} \\ N_2 \gtrsim N_1^{\frac{1}{d-1}}}} N_1^{-s+1} N_2^{-s} \|\mathbf{P}_{N_1} z\|_{L^4_T W^{s,4}_x} \|\mathbf{P}_{N_2} z\|_{L^4_T W^{s,4}_x} \\
&\lesssim (T^\delta M)^2
\end{aligned}
$$

for any $\omega \in E_{M,T}$, provided that $s > 1 - \frac{1}{d}$. Similarly, it follows from Sobolev's inequality (with sufficiently small $\delta > 0$ such that $\frac{1-s}{d} \geq \frac{1}{2} - \frac{1}{2+\delta}$) and (4.4) that

$$
\begin{aligned}
\sum_{\substack{N_1, N_2 \in 2^{\mathbb{N}_0} \\ N_2 \gtrsim N_1^{\frac{1}{d-1}}}} N_1 \|\mathbf{P}_{N_1} z \mathbf{P}_{N_2} v\|_{L^2_{T,x}} &\lesssim \sum_{\substack{N_1, N_2 \in 2^{\mathbb{N}_0} \\ N_2 \gtrsim N_1^{\frac{1}{d-1}}}} N_1^{-s+1} N_2^{-s} \|\mathbf{P}_{N_1} z\|_{L^4_T W^{s, \frac{4+2\delta}{\delta}}_x} \|\mathbf{P}_{N_2} v\|_{L^4_T W^{s,2+\delta}_x} \\
&\lesssim T^{\frac{1}{4}} \|v\|_{Y^1} M
\end{aligned}
$$

for any $\omega \in E_{M,T}$, provided that $s > 1 - \frac{1}{d}$. This proves (4.5) for $\mathcal{N}(u) = |u|^{\frac{4}{d-2}} u$.

Part 2: Next, we prove the difference estimates (4.6). Our main goal is to prove

$$
\begin{aligned}
&\bigg| \int_{[0,T]\times\mathbb{R}^d} \big\{ \mathcal{N}(v_1+z) - \mathcal{N}(v_2+z) \big\} w dx dt \bigg| \\
&\qquad \lesssim \Big\{ \|v_1\|_{Y^1}^{\frac{4}{d-2}} + \|v_2\|_{Y^1}^{\frac{4}{d-2}} + T^\theta M^{\frac{4}{d-2}} \Big\} \|v_1 - v_2\|_{Y^1},
\end{aligned} \tag{4.20}
$$

and

$$
\begin{aligned}
&\bigg| \int_{[0,T]\times\mathbb{R}^d} \big\{ \nabla\mathcal{N}(v_1+z) - \nabla\mathcal{N}(v_2+z) \big\} w dx dt \bigg| \\
&\qquad \lesssim \Big\{ \|v_1\|_{Y^1}^{\frac{4}{d-2}} + \|v_2\|_{Y^1}^{\frac{4}{d-2}} + T^\theta M^{\frac{4}{d-2}} \Big\} \|v_1 - v_2\|_{Y^1}
\end{aligned} \tag{4.21}
$$

for all $w \in Y^0$ with $\|w\|_{Y^0} = 1$. In the following, we only consider (4.21) and discuss how to apply the computations in Part 1. The first difference estimate (4.20) follows in a similar, but simpler manner.

• **Case 3:** $\mathcal{N}(u) = |u|^{\frac{d+2}{d-2}}$. Let $F(\zeta) = F(\zeta, \overline{\zeta}) = |\zeta|^{\frac{6-d}{d-2}}\zeta$. Then, we have

$$\partial_\zeta F = \tfrac{2+d}{2d-4}|\zeta|^{\frac{6-d}{d-2}} \qquad \text{and} \qquad \partial_{\overline{\zeta}} F = \tfrac{6-d}{2d-4}|\zeta|^{\frac{6-d}{d-2}}\tfrac{\zeta^2}{|\zeta|^2}. \tag{4.22}$$

By Fundamental Theorem of Calculus, we have

$$\begin{aligned} F(v_1 + z) - F(v_2 + z) = \int_0^1 & \partial_\zeta F(v_2 + z + \theta(v_1 - v_2))(v_1 - v_2) \\ & + \partial_{\overline{\zeta}} F(v_2 + z + \theta(v_1 - v_2))(\overline{v_1 - v_2})d\theta. \end{aligned} \tag{4.23}$$

Then, from (4.11) and (4.23), we have

$$\begin{aligned} \nabla(|v_1 + z|^{\frac{d+2}{d-2}}) & - \nabla(|v_2 + z|^{\frac{d+2}{d-2}}) \\ & = \tfrac{d+2}{d-2}\operatorname{Re}\big\{F(v_1 + z)\nabla(\overline{v_1 + z}) - F(v_2 + z)\nabla(\overline{v_2 + z})\big\} \\ & = \tfrac{d+2}{d-2}\operatorname{Re}\bigg\{F(v_1 + z)\nabla(\overline{v_1 - v_2}) \\ & \quad + \int_0^1 \partial_\zeta F(v_2 + z + \theta(v_1 - v_2))(v_1 - v_2)d\theta \cdot \nabla(\overline{v_2 + z}) \\ & \quad + \int_0^1 \partial_{\overline{\zeta}} F(v_2 + z + \theta(v_1 - v_2))(\overline{v_1 - v_2})d\theta \cdot \nabla(\overline{v_2 + z})\bigg\}. \end{aligned} \tag{4.24}$$

The contribution to (4.21) from the first term on the right-hand side of (4.24) can be estimated as in (4.12). As for the second term on the right-hand side of (4.24), the estimate (4.21) is reduced to

$$\begin{aligned} \int_0^1 \bigg| \int_{[0,T]\times\mathbb{R}^d} (\nabla w_1)(v_1 - v_2) \cdot \big|v_2 + z + \theta(v_1 - v_2)\big|^{\frac{6-d}{d-2}} w dx dt \bigg| d\theta \\ \lesssim \Big\{\|v_1\|_{Y^1}^{\frac{4}{d-2}} + \|v_2\|_{Y^1}^{\frac{4}{d-2}} + T^\theta M^{\frac{4}{d-2}}\Big\}\|v_1 - v_2\|_{Y^1} \end{aligned}$$

for $w_1 = \overline{v}_2$ or $\overline{z}$, which once again follows from (4.12) in Part 1. In view of (4.22), we have $|\partial_{\overline{\zeta}} F| \sim |\zeta|^{\frac{6-d}{d-2}}$. Hence, the third term on the right-hand side of (4.24) can be estimated in a similar manner.

• **Case 4:** $\mathcal{N}(u) = |u|^{\frac{4}{d-2}}u$. In view of (4.17), there are two contributions to

$$\nabla\mathcal{N}(v_1 + z) - \nabla\mathcal{N}(v_2 + z).$$

Let $G(\zeta) = G(\zeta, \overline{\zeta}) = |\zeta|^{\frac{8-2d}{d-2}}\zeta^2$. Then, we have

$$\partial_\zeta G = \tfrac{d}{d-2}|\zeta|^{\frac{6-d}{d-2}}\tfrac{\zeta}{|\zeta|} \qquad \text{and} \qquad \partial_{\overline{\zeta}} G = \tfrac{4-d}{d-2}|\zeta|^{\frac{6-d}{d-2}}\tfrac{\zeta^3}{|\zeta|^3}. \tag{4.25}$$

Next, let $H(z) = H(\zeta, \overline{\zeta}) = |\zeta|^{\frac{4}{d-2}}$. Then, we have

$$\partial_\zeta H = \tfrac{2}{d-2}|\zeta|^{\frac{6-d}{d-2}}\tfrac{\overline{\zeta}}{|\zeta|} \qquad \text{and} \qquad \partial_{\overline{\zeta}} H = \tfrac{2}{d-2}|\zeta|^{\frac{6-d}{d-2}}\tfrac{\zeta}{|\zeta|}. \tag{4.26}$$

Then, from (4.17), (4.25), and (4.26), we have

$$\begin{aligned} \nabla\mathcal{N}(v_1 + z) & - \nabla\mathcal{N}(v_2 + z) \\ & = \tfrac{4}{d-2}\operatorname{Re}\big\{G(v_1 + z)\nabla(\overline{v_1 + z}) - G(v_2 + z)\nabla(\overline{v_2 + z})\big\} \\ & \quad + H(v_1 + z)\nabla(v_1 + z) - H(v_2 + z)\nabla(v_2 + z). \end{aligned}$$

Noting that

$$|\partial_\zeta G| \sim |\partial_{\overline{\zeta}} G| \sim |\partial_\zeta H| \sim |\partial_{\overline{\zeta}} H| \sim |\zeta|^{\frac{6-d}{d-2}},$$

we can use (4.23) with G and H replacing F and repeat the computation in Part 1 to establish (4.21). This completes the proof of Proposition 4.1. □

5. Proof of Theorems 1.1 and 1.2

We present the proof of Theorems 1.1 and 1.2. Namely, we solve the following fixed point problem:

$$v = -i\mathcal{I}[\mathcal{N}(v+z)],$$

where

$$\mathcal{N}(u) = |u|^{\frac{4}{d-2}}u \qquad \text{or} \qquad \mathcal{N}(u) = |u|^{\frac{d+2}{d-2}}.$$

Let $\eta > 0$ be sufficiently small such that

$$2C_1\eta^{\frac{4}{d-2}} \le 1 \qquad \text{and} \qquad 3C_2\eta^{\frac{4}{d-2}} \le \tfrac{1}{2},$$

where C_1 and C_2 are the constants in (4.5) and (4.6). Given $M > 0$, we set

$$T := \min\Big\{ \big(\tfrac{\eta}{M}\big)^{\frac{d+2}{d-2}}, \big(\tfrac{\eta}{M}\big)^{\frac{4}{d-2}} \Big\}^{\frac{1}{\theta}}. \tag{5.1}$$

Then, it follows from Proposition 4.1 with $X^1_T \hookrightarrow Y^1_T$ that for each $\omega \in E_{M,T}$, the mapping $v \mapsto -i\mathcal{I}[\mathcal{N}(v+z)]$ is a contraction on the ball $B_\eta \subset X^1_T$ defined by

$$B_\eta := \{v \in X^1_T : \|v\|_{X^1_T} \le \eta\}.$$

Moreover, it follows from Lemmas 2.1 and 2.3 with (5.1) imply the following tail estimate:

$$\begin{aligned} P(\Omega \setminus E_{M,T}) &\le C\exp\bigg(-c\frac{M^2}{\|\phi\|^2_{H^s}}\bigg) + C\exp\bigg(-c\frac{M^2}{T^\gamma\|\phi\|^2_{H^s}}\bigg) \\ &\le C\exp\bigg(-\frac{c}{T^\gamma\|\phi\|^2_{H^s}}\bigg) \end{aligned}$$

for some $\gamma > 0$. This proves almost sure local well-posedness of (1.1) and (1.5).

6. A variant of almost sure local well-posedness

In this section, we briefly discuss the proof of Proposition 1.3. In particular, we consider the perturbed NLS (1.9) with a non-zero initial condition v_0. This will be useful in proving Theorems 1.5 and 1.7. As in [3], we consider the following Cauchy problem for NLS with a perturbation:

$$\begin{cases} i\partial_t v + \Delta v = \mathcal{N}(v+f), \\ v|_{t=0} = v_0 \in H^1(\mathbb{R}^d), \end{cases} \tag{6.1}$$

where f is a given deterministic function, satisfying certain regularity conditions. This allows us to separate the probabilistic and deterministic components of the argument in a clear manner.

First, note that, since our initial condition is not 0, the Y^1_T-norm of the solution v does not tend to 0 even when $T \to 0$. Hence, we need to use an auxiliary norm that tends to 0 as $T \to 0$. As a corollary to (the proof of) Proposition 4.1, we obtain the following nonlinear estimates, which are stated for a general time interval $I \subset \mathbb{R}$. Note that all the terms on the right-hand side in the first estimate (6.3) have (i) two factors of the $L^{q_d}_t(I; W^{1,r_d}_x)$-norm of v (which is weaker than the $X^1(I)$-norm) or (ii) a factor of $|I|^\theta$, which can be made small by shrinking the interval I.

In the following, let (q_d, r_d) be the admissible pair defined in (4.1). Given $\delta > 0$, $M > 0$, and an interval I, define $\widetilde{E}_M(I)$ by

$$\widetilde{E}_M(I) := \big\{ f \in Y^s(I) \cap S^s(I) : \|f\|_{Y^s}(I) + \|f\|_{S^s(I)} \le M \big\}, \tag{6.2}$$

where $S^s(I) = S^s(I; \delta)$ is as in (4.3). When $I = [0, T)$, we simply write $\widetilde{E}_{M,T} = \widetilde{E}_M([0, T))$.

Corollary 6.1. *Let $d = 5, 6$, $1 - \frac{1}{d} < s < 1$, and*

$$\mathcal{N}(u) = |u|^{\frac{4}{d-2}} u \qquad or \qquad \mathcal{N}(u) = |u|^{\frac{d+2}{d-2}}.$$

Then, there exist sufficiently small $\delta = \delta(d, s) > 0$ and $\theta = \theta(d, s) > 0$ such that

$$\|[\mathcal{N}(v + f)\|_{N^1(I)} \lesssim \|v\|^{\frac{d+2}{d-2}}_{L_t^{q_d}(I;W_x^{1,r_d})} + |I|^\theta M^{\frac{d+2}{d-2}} + |I|^\theta M \|v\|^{\frac{6-d}{d-2}}_{L_t^{q_d}(I;W_x^{1,r_d})} \|v\|_{Y^1(I)}, \tag{6.3}$$

$$\begin{aligned} &\|\mathcal{N}(v_1 + f) - \mathcal{N}(v_2 + f)\|_{N^1(I)} \\ &\qquad \lesssim \Big\{ \|v_1\|^{\frac{4}{d-2}}_{L_t^{q_d}(I;W_x^{1,r_d})} + \|v_2\|^{\frac{4}{d-2}}_{L_t^{q_d}(I;W_x^{1,r_d})} + |I|^\theta M^{\frac{4}{d-2}} \Big\} \|v_1 - v_2\|_{Y^1(I)}, \end{aligned} \tag{6.4}$$

for any interval $I \subset \mathbb{R}$, $v, v_1, v_2 \in Y^1(I)$, and $f \in \widetilde{E}_M(I)$.

Proof. This corollary follows from the proof of Proposition 4.1 simply by not applying the Strichartz estimates (Lemma 3.5). In particular, a small modification to (4.10), (4.13), and (4.14) yields (6.3) for the corresponding cases, where the left-hand side is controlled by the first two terms on the right-hand side of (6.3). In (4.16) and (4.19), the subcritical nature of the perturbation f allows us to gain a small power of $|I|$ through (4.15). Hence, we obtain (6.3), where the left-hand side is controlled by the last two terms on the right-hand side of (6.3). The difference estimate (6.4) also follows from a similar modification. ☐

By following the proof of Proposition 6.3 in [3], we obtain the following almost sure local well-posedness of the perturbed NLS (6.1) with non-zero initial data. Proposition 1.3 in Section 1 then follows from this lemma with Lemmas 2.1 and 2.3 by setting $f = z^\omega = S(t)\phi^\omega$.

Lemma 6.2. *Assume the hypotheses of Corollary 6.1. Given $M > 0$, let $\widetilde{E}_M(\cdot)$ be as in (6.2) and let $\theta > 0$ be as in Corollary 6.1. Then, there exists small $\eta_0 = \eta_0(\|v_0\|_{H^1}, M) > 0$ such that if*

$$\|S(t - t_0)v_0\|_{L_t^{q_d}(I;W_x^{1,r_d})} \le \eta \qquad and \qquad |I| \le \eta^{\frac{2}{\theta}}$$

for some $\eta \le \eta_0$ and some time interval $I = [t_0, t_1] \subset \mathbb{R}$, then for any $f \in \widetilde{E}_M(I)$, there exists a unique solution $v \in X^1(I) \cap C(I; H^1(\mathbb{R}^d))$ to (1.9) with $v|_{t=t_0} = v_0$, satisfying

$$\begin{aligned} \|v\|_{L_t^{q_d}(I;W_x^{1,r_d})} &\le 2\eta, \\ \|v - S(t - t_0)v_0\|_{X^1(I)} &\lesssim \eta. \end{aligned}$$

Proof. As mentioned above, one can prove Lemma 6.2 by following the proof of Proposition 6.3 in [3]. More precisely, by applying Corollary 6.1 and choosing

$$\eta_0 \ll \widetilde{R}^{-\frac{d+2}{d-2}}$$

with $\widetilde{R} := \max(\|v_0\|_{H^1}, M)$, a straightforward computation shows that the map Γ defined by

$$\Gamma v(t) := S(t-t_0)v_0 - i\int_{t_0}^t S(t-t')\mathcal{N}(v+f)(t')dt'$$

is a contraction on

$$B_{R,M,\eta} = \big\{v \in X^1(I)\cap C(I;H^1) : \|v\|_{X^1(I)} \le 2\widetilde{R},\ \|v\|_{L_t^{q_d}(I;W_x^{1,r_d})} \le 2\eta\big\},$$

provided that $f \in \widetilde{E}_M(I)$. □

Lastly, note that Lemma 6.2 yields the following blowup alternative. Suppose that there exists $M(t)$ such that $f \in \widetilde{E}_{M(t)}([0,t))$ for each $t>0$. Then, given $v_0 \in H^1(\mathbb{R}^d)$, let v be the solution to the perturbed NLS (6.1) with $v|_{t=0} = v_0$ on a forward maximal time interval $[0,T^*)$ of existence. Then, either $T^* = \infty$ or

$$\lim_{T\to T^*} \|v\|_{L_t^{q_d}([0,T);W_x^{1,r_d})} = \infty. \tag{6.5}$$

In view of Lemma 6.2, this blowup alternative follows from a standard argument as in [12]. In fact, suppose $T^* < \infty$ and

$$A^* := \lim_{T\to T^*} \|v\|_{L_t^{q_d}([0,T);W_x^{1,r_d})} < \infty.$$

Then, we will derive a contradiction in the following.

Without loss of generality, assume that $M(t)$ is non-decreasing and set

$$M^* := \sup_{t\in[0,T^*+1]} M(t) < \infty. \tag{6.6}$$

Partition the interval $[0,T^*]$ as

$$[0,T^*] = \bigcup_{j=0}^{J} I_j \cap [0,T^*]$$

where $I_j = [t_j, t_{j+1}]$ with $t_0 = 0$ and $t_{J+1} = T^*$. From (6.3) in Corollary 6.1 with Lemma 3.8, we have

$$\begin{aligned}\|v\|_{X^1(I_j)} &\le \|v(t_j)\|_{H^1} + \|\mathcal{N}(v+z)\|_{N^1(I_j)}\\ &\le \|v(t_j)\|_{H^1} + C(T^*,A^*,M^*) + |I_j|^\theta M^*(A^*)^{\frac{6-d}{d-2}}\|v\|_{X^1(I_j)}.\end{aligned}$$

Hence by imposing that the lengths of the subintervals I_j are sufficiently small, depending only on A^* and M^*, we obtain

$$\sup_{t\in I_j}\|v(t)\|_{H^1} \lesssim \|v\|_{X^1(I_j)} \lesssim \|v(t_j)\|_{H^1} + C(T^*,A^*,M^*), \tag{6.7}$$

where the implicit constants are independent of $j = 0,1,\dots,J$. By iteratively applying the estimate (6.7), we obtain

$$R^* := \sup_{t\in[0,T^*]}\|v(t)\|_{H^1} \le C(T^*,A^*,M^*) < \infty. \tag{6.8}$$

Then, combining (6.7) and (6.8), we obtain

$$\|v\|_{X^1(I_j)} \le C(T^*,A^*,M^*) < \infty \tag{6.9}$$

uniformly in $j = 0,1,\dots,J$.

Given $\widetilde{\eta} > 0$ (to be chosen later), we refine the partition and assume that

$$\|v\|_{L_t^{q_d}(I_j; W_x^{1,r_d})} < \widetilde{\eta}. \tag{6.10}$$

Fix $\eta_0 = \eta_0(R^*, M^*) > 0$, where η_0 is as in Lemma 6.2 and R^* and M^* are as in (6.8) and (6.6). Then, by taking the $L_t^{q_d}(I_j; W_x^{1,r_d})$-norm of the Duhamel formulation:

$$S(t - t_j)v(t_j) = v(t) + i\int_{t_j}^{t} S(t - t')\mathcal{N}(v + f)dt',$$

applying Corollary 6.1 with (6.9) and the smallness condition (6.10), and taking $\widetilde{\eta} = \widetilde{\eta}(\eta_0) = \widetilde{\eta}(R^*, M^*) > 0$ and $|I_j| = |I_j|(T^*, A^*, M^*, \eta_0)$ sufficiently small, we have

$$\begin{aligned}\|S(t - t_j)v(t_j)\|_{L_t^{q_d}(I_j; W_x^{1,r_d})} &\leq \widetilde{\eta} + C\widetilde{\eta}^{\frac{d+2}{d-2}} + C(T^*, A^*, M^*)|I_j|^{\theta} \\ &\leq \tfrac{1}{2}\eta_0.\end{aligned}$$

In particular, with $j = J$, this implies that there exists some $\varepsilon > 0$ such that

$$\|S(t - t_J)v(t_J)\|_{L_t^{q_d}([t_J, T^* + \varepsilon]; W_x^{1,r_d})} \leq \eta_0.$$

By further imposing that $|I_J| \leq \frac{1}{2}\eta_0^{\frac{2}{\theta}}$, we conclude from Lemma 6.2 that the solution v can be extended to $[0, T^* + \varepsilon]$ for some $\varepsilon > 0$, which is a contradiction to the assumption $T^* < \infty$. Therefore, if $T^* < \infty$, then we must have (6.5).

Remark 6.3. Suppose $T^* < \infty$. Then, it follows from the argument above with Lemma 6.2 and the subadditivity of the X^1-norm over disjoint intervals (Lemma A.4 in [3]) that $v \in X^1([0, T^* - \delta))$ for any $\delta > 0$. If $T^* = \infty$, we have $v \in X^1([0, T))$ for any finite $T > 0$.

7. Almost sure global well-posedness of the defocusing energy-critical NLS below the energy space

In this section, we present the proof of Theorem 1.5. Namely, we prove almost sure global well-posedness of the defocusing energy-critical NLS on $\mathbb{R}^d$, $d = 5, 6$:

$$\begin{cases} i\partial_t u + \Delta u = |u|^{\frac{4}{d-2}}u, \\ u|_{t=0} = \phi^\omega, \end{cases} \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^d. \tag{7.1}$$

where ϕ^ω is the Wiener randomization of a given function $\phi \in H^s(\mathbb{R}^d)$ for some $s < 1$. As in Section 6, we consider the following Cauchy problem for the defocusing NLS with a deterministic perturbation:

$$\begin{cases} i\partial_t v + \Delta v = |v + f|^{\frac{4}{d-2}}(v + f) \\ v|_{t=0} = 0. \end{cases} \tag{7.2}$$

Under a suitable regularity assumption on f, Lemma 6.2 guarantees local existence of solutions to (7.2). In the following, we assume

(i) f is a linear solution $f = S(t)\psi$ for some deterministic initial condition ψ,
(ii) f satisfies certain space-time integrability conditions.

Under these assumptions, we first establish crucial energy estimates (Proposition 7.2 for $d = 6$ and Proposition 7.3 for $d = 5$) for a solution v to the perturbed NLS (7.2). This is the main new ingredient in this paper as compared to [3]. Once we have these energy estimates, we can proceed as in [3] and hence we only sketch the argument. Fix an interval $[0, T)$. Given $t_0 \in [0, T)$, we iteratively apply the perturbation lemma (Lemma 7.4) on short time intervals $I_j = [t_j, t_{j+1}]$ and approximate a solution v to the perturbed NLS (7.2) by the global solution w to the original NLS (7.1) with $w|_{t=t_0} = v(t_0)$. This allows us to show that the solution v to the perturbed NLS (7.2) exists on $[t_0, t_0 + \tau]$, where τ is independent of $t_0 \in [0, T)$ (Proposition 7.5). By iterating this "good" local well-posedness, we can extend the solution v to the entire interval $[0, T]$. Since the choice of $T > 0$ was arbitrary, this shows that the perturbed NLS (7.2) is globally well-posed. In Subsection 7.3, we verify that the conditions imposed on f for long time existence are satisfied with a large probability by setting $f(t) = z(t) = S(t)\phi^\omega$. This yields Theorem 1.5.

7.1. **Energy estimate for the perturbed NLS.** First, we discuss the following a priori[11] control on the mass.

Lemma 7.1. *Let v be a solution to (7.2) with $f = S(t)\psi$. Then, we have*

$$\int |v(t)|^2 dx \lesssim \int |\psi|^2 dx, \tag{7.3}$$

where the implicit constant is independent of $t \in \mathbb{R}$.

Proof. Note that $u = v + f$ satisfies (7.1). Hence, by the mass conservation for (7.1), we have

$$\int |\psi|^2 dx = \int |v(t) + f(t)|^2 dx = \int |v(t)|^2 dx + 2\,\mathrm{Re} \int v(t)\overline{f(t)} dx + \int |f(t)|^2 dx.$$

By the unitarity of the linear solution operator, we obtain

$$\int |v(t)|^2 dx = -2\,\mathrm{Re} \int v(t)\overline{f(t)} dx \le \frac{1}{2} \int |v(t)|^2 dx + 2 \int |f(t)|^2 dx.$$

By invoking the unitarity of the linear solution operator once again, we obtain (7.3). □

Next, we establish an energy estimate when $d = 6$. Recall the following conserved energy for NLS (7.1):

$$E(u) = \frac{1}{2} \int |\nabla u|^2 dx + \frac{1}{3} \int |u|^3 dx.$$

In the following, we estimate the growth of $E(v)$ for a solution v to the perturbed NLS (7.2).

Proposition 7.2. *Let $d = 6$ and $s > \frac{20}{23}$. Then, the following energy estimate holds for a solution v to the perturbed NLS (7.2) with $f = S(t)\psi$:*

$$\begin{aligned} \partial_t E(v)(t) \lesssim \big(1 + \|f(t)\|_{L^\infty_x}\big) E(v)(t) + \|f(t)\|^6_{L^6_x} \\ + \|f(t)\nabla f(t)\|^2_{L^2_x} + \|v(t)\nabla f(t)\|^2_{L^2_x}. \end{aligned} \tag{7.4}$$

[11]In Lemma 7.1 and Propositions 7.2 and 7.3, we prove a priori estimates for a smooth solution v with smooth ψ and hence f. By the standard argument via the local theory, one can show that these a priori estimates also hold for rough solutions as long as they exist.

In particular, given $T > 0$, we have

$$\sup_{t\in[0,T]} E(v)(t) \le C\big(T, \|f\|_{A^s(T)}\big) \tag{7.5}$$

for any solution $v \in C([0,T]; H^1(\mathbb{R}^6))$ to the perturbed NLS (7.2) with $f = S(t)\psi$, where the $A^s(T)$-norm is defined by

$$\|f\|_{A^s(T)} := \max\Big(\|\langle\nabla\rangle^{s-} f\|_{L^\infty_{T,x}}, \|f\|_{L^6_{T,x}}, \|f\|_{L^4_T W^{s,4}_x}, \|f\|_{L^4_T L^3_x}, \|\psi\|_{L^2_x}, \|f\|_{Y^s_T}\Big).$$

Proof. We first prove (7.4). Since we work for fixed t, we suppress the t-dependence in the following. Noting that $\partial_t(|v|^3) = 3|v|\operatorname{Re}(\overline{v}\partial_t v)$, we have

$$\begin{aligned}
\partial_t E(v) &= \underbrace{-\operatorname{Re} i\int \Delta v\Delta\overline{v}dx}_{=0} + \operatorname{Re} i\int |v+f|(v+f)\Delta\overline{v}dx\\
&\quad + \operatorname{Re} i\int \Delta v|v|\overline{v}dx - \operatorname{Re} i\int |v+f|(v+f)|v|\overline{v}dx\\
&= \operatorname{Re} i\int \big\{|v+f|(v+f) - |v|v\big\}\Delta\overline{v}dx - \operatorname{Re} i\int |v+f|(v+f)|v|\overline{v}dx\\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned} \tag{7.6}$$

By Young's inequality, we have

$$\begin{aligned}
\mathrm{II} &= \underbrace{-\operatorname{Re} i\int |v+f||v|^3dx}_{=0} - \operatorname{Re} i\int |v+f|\cdot f\cdot |v|\overline{v}dx\\
&\lesssim (1 + \|f\|_{L^\infty_x})\int |v|^3 dx + \|f\|^6_{L^6_x}\\
&\lesssim (1 + \|f\|_{L^\infty_x})E(v) + \|f\|^6_{L^6_x}.
\end{aligned} \tag{7.7}$$

Integrating by parts, we have

$$\mathrm{I} = -\operatorname{Re} i\int \nabla\big\{|v+f|(v+f) - |v|v\big\}\cdot\nabla\overline{v}dx. \tag{7.8}$$

Then, from (4.17), (4.25), and (4.26), we have

$$\begin{aligned}
&\nabla\mathcal{N}(v+f) - \nabla\mathcal{N}(v)\\
&\qquad = \operatorname{Re}\big\{G(v+f)\nabla(\overline{v+f}) - G(v)\nabla\overline{v}\big\} + H(v+f)\nabla(v+f) - H(v)\nabla v\\
&\qquad = \operatorname{Re}\big\{G(v+f)\nabla\overline{f}\big\} + \operatorname{Re}\big\{(G(v+f) - G(v))\nabla\overline{v}\big\}\\
&\qquad\quad + H(v+f)\nabla f + (H(v+f) - H(v))\nabla v,
\end{aligned} \tag{7.9}$$

where $G(\zeta) = \frac{\zeta^2}{|\zeta|}$ and $H(\zeta) = |\zeta|$ are as in (4.25) and (4.26) (with $d = 6$), respectively. Let us denote by I_j, $j = 1, \dots, 4$, the contribution to I in (7.8) from the jth term on the right-hand side of (7.9).

Proceeding as in (4.23), we have

$$\begin{aligned}
G(v+f) - G(v) &= \int_0^1 \partial_\zeta G(v+\theta f)\cdot f + \partial_{\overline{\zeta}} G(v+\theta f)\cdot\overline{f}d\theta,\\
H(v+f) - H(v) &= \int_0^1 \partial_\zeta H(v+\theta f)\cdot f + \partial_{\overline{\zeta}} H(v+\theta f)\cdot\overline{f}d\theta.
\end{aligned}$$

Then, it follows from (4.25) and (4.26) that

$$\|G(v+f)-G(v)\|_{L^\infty_x}+\|H(v+f)-H(v)\|_{L^\infty_x}\lesssim\|f\|_{L^\infty_x}. \tag{7.10}$$

Hence, from (7.8), (7.9), and (7.10), we have

$$|\mathrm{I}_2+\mathrm{I}_4|\lesssim\|f\|_{L^\infty_x}\|\nabla v\|^2_{L^2_x}\lesssim\|f\|_{L^\infty_x}E(v). \tag{7.11}$$

Note that $|G(\zeta)|=|H(\zeta)|=|\zeta|$. Then, integrating by parts (in x), we have

$$\begin{aligned}|\mathrm{I}_1+\mathrm{I}_3|&\lesssim\|\nabla v\|^2_{L^2_x}+\|(v+f)\nabla f\|^2_{L^2_x}\\&\lesssim E(v)+\|f\nabla f\|^2_{L^2_x}+\|v\nabla f\|^2_{L^2_x}.\end{aligned} \tag{7.12}$$

Hence, (7.4) follows from (7.6), (7.7), (7.11), and (7.12).

Next, we discuss the second estimate (7.5). By solving the differential inequality (7.4) with $v|_{t=0}=0$ in a crude manner, we obtain

$$\begin{aligned}E(v)(\tau)&\le C\int_0^\tau e^{C(1+\|f\|_{L^\infty_{T,x}})(\tau-t)}\Big\{\|f(t)\|^6_{L^6_x}+\|f(t)\nabla f(t)\|^2_{L^2_x}+\|v(t)\nabla f(t)\|^2_{L^2_x}\Big\}dt\\&\le Ce^{C(1+\|f\|_{L^\infty_{T,x}})T}\Big\{\|f\|^6_{L^6_{\tau,x}}+\|f\nabla f\|^2_{L^2_{\tau,x}}+\|v\nabla f\|^2_{L^2_{\tau,x}}\Big\}\end{aligned} \tag{7.13}$$

for any $\tau\in[0,T]$. The estimate (7.13) is by no means sharp. It, however, suffices for our purpose.

We can estimate $\|f\nabla f\|_{L^2_{\tau,x}}$ as in the proof of Proposition 4.1. Namely, by writing

$$\|f\nabla f\|_{L^2_{\tau,x}}\le\sum_{N_1,N_2\in2^{\mathbb{N}_0}}N_2\|\mathbf{P}_{N_1}f\mathbf{P}_{N_2}f\|_{L^2_{\tau,x}}, \tag{7.14}$$

we separate the estimate into two cases (i) $N_1\gtrsim N_2^{\frac15}$ and (ii) $N_1\ll N_2^{\frac15}$. Then, we can estimate the contribution from (i) by $\|f\|^2_{L^4_\tau W^{s,4}_x}$ for $s>\frac56$, while we can apply Lemma 3.6 and estimate the contribution from (ii) by $\|f\|^2_{Y^s_\tau}$ for $s>\frac56$. Hence, we obtain

$$\|f\nabla f\|^2_{L^2_{\tau,x}}\lesssim\|f\|^4_{L^4_\tau W^{s,4}_x}+\|f\|^4_{Y^s_\tau}, \tag{7.15}$$

provided that $s>\frac56$.

Next, we consider $\|v\nabla f\|_{L^2_\tau L^2_x}$. By writing

$$\|v\nabla f\|_{L^2_{\tau,x}}\le\sum_{N_1,N_2\in2^{\mathbb{N}_0}}N_2\|\mathbf{P}_{N_1}v\mathbf{P}_{N_2}f\|_{L^2_{\tau,x}},$$

we divide the argument into the following two cases:

$$\text{(i) } N_1\gtrsim N_2^\gamma\qquad\text{and}\qquad\text{(ii) } N_1\ll N_2^\gamma$$

for some $\gamma > 0$ (to be chosen later). We first estimate the contribution from (i) $N_1 \gtrsim N_2^\gamma$. By interpolation and Lemma 7.1, we have

$$\begin{aligned}\sum_{\substack{N_1, N_2 \in 2^{\mathbb{N}_0}\\ N_1 \gtrsim N_2^\gamma}} N_2 \|\mathbf{P}_{N_1} v \mathbf{P}_{N_2} f\|_{L^2_{\tau,x}} &\lesssim \sum_{\substack{N_1, N_2 \in 2^{\mathbb{N}_0}\\ N_1 \gtrsim N_2^\gamma}} N_1^{1-} N_2^{1-\gamma+} \|\mathbf{P}_{N_1} v \mathbf{P}_{N_2} f\|_{L^2_{\tau,x}}\\
&\lesssim \sum_{\substack{N_1, N_2 \in 2^{\mathbb{N}_0}\\ N_1 \gtrsim N_2^\gamma}} \|\mathbf{P}_{N_1} \langle \nabla \rangle^{1-} v \mathbf{P}_{N_2} \langle \nabla \rangle^{s-} f\|_{L^2_{\tau,x}}\\
&\le C(T) \Big\{ \sup_{t \in [0,\tau]} \big(E(v)(t)\big)^{\frac{1}{2}-} \|\psi\|_{L^2_x}^{0+} + \|\psi\|_{L^2_x} \Big\} \|\langle \nabla \rangle^{s-} f\|_{L^\infty_{\tau,x}} \end{aligned} \tag{7.16}$$

for any $\tau \in [0, T]$, provided that

$$s > 1 - \gamma. \tag{7.17}$$

We now turn our attention to (ii) $N_1 \ll N_2^\gamma$. Recall that $(q, r) = (2, 3)$ is admissible. Hence, by Lemma 3.6, the Duhamel formula (with $v|_{t=0} = 0$), the linear estimate (Lemma 3.8) and the Strichartz estimates (Lemma 3.5), we have

$$\begin{aligned}\|\mathbf{P}_{N_1} v \mathbf{P}_{N_2} f\|_{L^2_{\tau,x}} &\lesssim N_1^{\frac{5}{2}-} N_2^{-\frac{1}{2}+} \|\mathbf{P}_{N_1} v\|_{Y^0_\tau} \|\mathbf{P}_{N_2} f\|_{Y^0_\tau}\\
&\lesssim N_1^{\frac{5}{2}-} N_2^{-\frac{1}{2}+} \bigg\| \mathbf{P}_{N_1} \int_0^t S(t-t') |v+f|(v+f)(t') dt' \bigg\|_{Y^0_\tau} \|\mathbf{P}_{N_2} f\|_{Y^0_\tau}\\
&\lesssim N_1^{\frac{5}{2}-} N_2^{-\frac{1}{2}+} \big(\|v\|^2_{L^4_\tau L^3_x} + \|f\|^2_{L^4_\tau L^3_x} \big) \|\mathbf{P}_{N_2} f\|_{Y^0_\tau} \end{aligned} \tag{7.18}$$

Fix $\theta \in (0, 1)$ (to be chosen later). We apply (7.18) only to the θ-power of the factor in

$$\sum_{\substack{N_1, N_2 \in 2^{\mathbb{N}_0}\\ N_1 \ll N_2^\gamma}} N_2 \|\mathbf{P}_{N_1} v \mathbf{P}_{N_2} f\|_{L^2_{\tau,x}}.$$

Then, with (7.18), we have

$$\begin{aligned}&\sum_{\substack{N_1, N_2 \in 2^{\mathbb{N}_0}\\ N_1 \ll N_2^\gamma}} N_2 \|\mathbf{P}_{N_1} v \mathbf{P}_{N_2} f\|_{L^2_{\tau,x}}\\
&\quad\lesssim \sum_{\substack{N_1, N_2 \in 2^{\mathbb{N}_0}\\ N_1 \ll N_2^\gamma}} N_2^{1-\frac{\theta}{2}+} \big(\|v\|^2_{L^4_\tau L^3_x} + \|f\|^2_{L^4_\tau L^3_x} \big)^\theta \|\mathbf{P}_{N_2} f\|^\theta_{Y^0_\tau} \|N_1^{\frac{5}{2}\frac{\theta}{1-\theta}-} \mathbf{P}_{N_1} v \mathbf{P}_{N_2} f\|^{1-\theta}_{L^2_{\tau,x}}\end{aligned}$$

By interpolation,

$$\begin{aligned}&\lesssim \sum_{\substack{N_1, N_2 \in 2^{\mathbb{N}_0}\\ N_1 \ll N_2^\gamma}} \big(\|v\|^2_{L^4_\tau L^3_x} + \|f\|^2_{L^4_\tau L^3_x} \big)^\theta \|\mathbf{P}_{N_2} f\|^\theta_{Y^s_\tau} \|\mathbf{P}_{N_1} v\|^{1-\frac{7}{2}\theta+}_{L^2_{\tau,x}}\\
&\qquad\qquad \times \|\mathbf{P}_{N_1} \langle \nabla \rangle v\|^{\frac{5}{2}\theta-}_{L^2_{\tau,x}} \|\mathbf{P}_{N_2} \langle \nabla \rangle^{s-} f\|^{1-\theta}_{L^\infty_{\tau,x}},\end{aligned}$$

provided that

$$1 - \frac{\theta}{2} < s. \tag{7.19}$$

Summing over N_1 and N_2 and applying Lemma 7.1, we obtain

$$\|v\nabla f\|_{L^2_{\tau,x}}^2 \le C(T, \|f\|_{A^s(T)})\Big\{1 + \sup_{t\in[0,\tau]} \big(E(v)(t)\big)^{1-}\Big\} \tag{7.20}$$

for any $\tau \in [0, T]$, provided that

$$\frac{4}{3}\theta + \frac{5}{2}\theta < 1. \tag{7.21}$$

Optimizing (7.17), (7.19), and (7.21), we obtain

$$s > \frac{20}{23}$$

with $\theta = \frac{6}{23}-$ and $\gamma = 1 - s+$.

Finally, putting (7.13), (7.15), (7.16), and (7.20) together with $v|_{t=0} = 0$, we obtain

$$\sup_{t\in[0,\tau]} E(v)(t) \le C(T, \|f\|_{A^s(T)})\Big\{1 + \sup_{t\in[0,\tau]} \big(E(v)(t)\big)^{1-}\Big\}$$

for any $\tau \in [0, T]$. Then, (7.5) follows from the standard continuity argument. □

We conclude this subsection by establishing an energy estimate when $d = 5$. As mentioned in Section 1, we study the growth of the following modified energy:

$$\mathcal{E}(v) = \frac{1}{2}\int |\nabla v|^2 dx + \frac{3}{10}\int |v + f|^{\frac{10}{3}} dx$$

for a solution v to the perturbed NLS (7.2).

Proposition 7.3. *Let $d = 5$ and $s > \frac{63}{68}$. Then, the following energy estimate holds: given $T > 0$, we have*

$$\sup_{t\in[0,T]} \mathcal{E}(v)(t) \le C\big(T, \|f\|_{B^s(T)}\big) \tag{7.22}$$

for any solution $v \in C([0, T]; H^1(\mathbb{R}^5))$ to the perturbed NLS (7.2) with $f = S(t)\psi$, where the $B^s(T)$-norm is defined by

$$\|f\|_{B^s(T)} := \max_{\substack{p=\frac{5}{2},3,4\\ q=2,\frac{10}{3}}} \Big(\|\langle\nabla\rangle^{s-} f\|_{L^\infty_{T,x}}, \|\langle\nabla\rangle^{s} f\|_{L^p_{T,x}}, \|f\|_{L_T^{\frac{14}{3}} L_x^{\frac{10}{3}}}, \|f\|_{L^\infty_T L^q_x}, \|f\|_{Y^s_T}\Big).$$

The proof of Proposition 7.3 is similar to that of Proposition 7.2 but is more complicated due to the (higher) fractional power of the nonlinearity.

Proof. Proceeding as in (7.6) with $\partial_t\big(|v+f|^{\frac{10}{3}}\big) = \frac{10}{3}|v+f|^{\frac{4}{3}}\operatorname{Re}\big((\overline{v+f})\partial_t(v+f)\big)$, we have

$$\begin{aligned}
\partial_t \mathcal{E}(v) &= \operatorname{Re} i \int |v+f|^{\frac{4}{3}}(v+f)\Delta\overline{v} dx \\
&\quad + \operatorname{Re} i \int (\Delta v + \Delta f)|v+f|^{\frac{4}{3}}(\overline{v+f}) dx - \underbrace{\operatorname{Re} i \int |v+f|^{\frac{14}{3}} dx}_{=0} \\
&= \operatorname{Re} i \int \nabla\big(|v+f|^{\frac{4}{3}}(v+f)\big)\cdot\nabla\overline{f} dx
\end{aligned}$$

With (4.17),

$$
\begin{aligned}
&= \frac{4}{3}\operatorname{Re} i \int \frac{v+f}{|v+f|^{\frac{2}{3}}} \operatorname{Re}\big((v+f)\nabla(\overline{v+f})\big)\cdot \nabla \overline{f} dx \\
&\quad + \operatorname{Re} i \int |v+f|^{\frac{4}{3}} \nabla(v+f)\cdot \nabla \overline{f} dx \\
&= \frac{5}{3}\operatorname{Re} i \int |v+f|^{\frac{4}{3}} \nabla v \cdot \nabla \overline{f} dx + \frac{2}{3}\operatorname{Re} i \int \frac{(v+f)^2}{|v+f|^{\frac{2}{3}}} \nabla \overline{v}\cdot \nabla \overline{f} dx \\
&\quad + \frac{2}{3}\operatorname{Re} i \int \frac{(v+f)^2}{|v+f|^{\frac{2}{3}}} \nabla \overline{f}\cdot \nabla \overline{f} dx \\
&\lesssim \|\nabla v\|_{L^2_x}^2 + \big\||v+f|^{\frac{4}{3}}\nabla f\big\|_{L^2_x}^2 + \big\||v+f|^{\frac{4}{3}}\nabla f\cdot \nabla f\big\|_{L^1_x} \\
&\lesssim \mathcal{E}(v) + \big\||v+f|^{\frac{4}{3}}\nabla f\big\|_{L^2_x}^2 + \big\||v+f|^{\frac{4}{3}}\nabla f\cdot \nabla f\big\|_{L^1_x}.
\end{aligned} \tag{7.23}
$$

By solving the differential inequality (7.23) with $v|_{t=0} = 0$ in a crude manner, we obtain

$$
\begin{aligned}
\mathcal{E}(v)(\tau) &\lesssim \int_0^\tau e^{C(\tau-t)}\Big\{\big\||v+f|^{\frac{4}{3}}\nabla f\big\|_{L^2_x}^2 + \big\||v+f|^{\frac{4}{3}}\nabla f\cdot\nabla f\big\|_{L^1_x}\Big\}dt \\
&\le e^{CT}\Big\{\big\||v+f|^{\frac{4}{3}}\nabla f\big\|_{L^2_{\tau,x}}^2 + \big\||v+f|^{\frac{4}{3}}\nabla f\cdot\nabla f\big\|_{L^1_{\tau,x}}\Big\} \\
&=: e^{CT}\big\{\,\mathrm{I} + \mathrm{II}\big\}
\end{aligned} \tag{7.24}
$$

for any $\tau \in [0, T]$,

We first consider I. By Hölder's inequality, we have

$$
\begin{aligned}
\big\||v+f|^{\frac{4}{3}}\nabla f\big\|_{L^2_{\tau,x}} &\lesssim \big\||f|^{\frac{4}{3}}\nabla f\big\|_{L^2_{\tau,x}} + \big\||v|^{\frac{4}{3}}\nabla f\big\|_{L^2_{\tau,x}} \\
&\lesssim \|f\|_{L^\infty_{\tau,x}}^{\frac{1}{3}}\|f\nabla f\|_{L^2_{\tau,x}} + \|v\|^{\frac{1}{3}}_{L^\infty_\tau L_x^{\frac{10}{3}}}\|v\nabla f\|_{L^2_\tau L_x^{\frac{5}{2}}}.
\end{aligned} \tag{7.25}
$$

Arguing as in (7.14), we have

$$
\|f\nabla f\|_{L^2_{\tau,x}}^2 \lesssim \|f\|_{L^4_\tau W^{s,4}_x}^4 + \|f\|_{Y^s_\tau}^4, \tag{7.26}
$$

provided that $s > \frac{4}{5}$. On the other hand, by the dyadic decomposition, we have

$$
\begin{aligned}
\|v\|^{\frac{1}{3}}_{L^\infty_\tau L_x^{\frac{10}{3}}}\|v\nabla f\|_{L^2_\tau L_x^{\frac{5}{2}}} &\lesssim \Big\{\sup_{t\in[0,\tau]}\big(\mathcal{E}(v)(t)\big)^{\frac{1}{10}} + \|f\|^{\frac{1}{3}}_{L^\infty_\tau L_x^{\frac{10}{3}}}\Big\} \\
&\quad \times \sum_{N_1, N_2 \in 2^{\mathbb{N}_0}} N_2\|\mathbf{P}_{N_1} v \mathbf{P}_{N_2} f\|_{L^2_\tau L_x^{\frac{5}{2}}}.
\end{aligned} \tag{7.27}
$$

Then, by interpolation, we have[12]

$$\begin{aligned}
N_2\|\mathbf{P}_{N_1}v\mathbf{P}_{N_2}f\|_{L^2_\tau L^{\frac{5}{2}}_x} &\le N_2\|\mathbf{P}_{N_1}v\mathbf{P}_{N_2}f\|^{\frac{1}{2}}_{L^2_{\tau,x}}\|\mathbf{P}_{N_1}v\mathbf{P}_{N_2}f\|^{\frac{1}{2}}_{L^2_\tau L^{\frac{10}{3}}_x}\\
&\le N_2\|\mathbf{P}_{N_1}v\mathbf{P}_{N_2}f\|^{\frac{1}{2}}_{L^2_{\tau,x}}\|\mathbf{P}_{N_1}v\|^{\frac{1}{2}}_{L^\infty_\tau L^{\frac{10}{3}}_x}\|\mathbf{P}_{N_2}f\|^{\frac{1}{2}}_{L^2_\tau L^\infty_x}\\
&\lesssim \Big\{\sup_{t\in[0,\tau]}\big(\mathcal{E}(v)(t)\big)^{\frac{3}{20}} + \|\mathbf{P}_{N_1}f\|^{\frac{1}{2}}_{L^\infty_\tau L^{\frac{10}{3}}_x}\Big\}\\
&\qquad \times N_2\|\mathbf{P}_{N_1}v\mathbf{P}_{N_2}f\|^{\frac{1}{2}}_{L^2_{\tau,x}}\|\mathbf{P}_{N_2}f\|^{\frac{1}{2}}_{L^2_\tau L^\infty_x}.
\end{aligned}\tag{7.28}$$

We now divide the argument into the following two cases:

$$\text{(i) } N_1 \gtrsim N_2^\gamma \qquad \text{and} \qquad \text{(ii) } N_1 \ll N_2^\gamma$$

for some $\gamma \in (0,1)$ (to be chosen later). We first estimate the contribution from (i) $N_1 \gtrsim N_2^\gamma$. By interpolation and Lemma 7.1, we have

$$\begin{aligned}
N_2\|\mathbf{P}_{N_1}v\mathbf{P}_{N_2}f\|^{\frac{1}{2}}_{L^2_{\tau,x}}\|\mathbf{P}_{N_2}f\|^{\frac{1}{2}}_{L^2_\tau L^\infty_x} &\lesssim N_1^{\frac{1}{2}-}N_2^{1-\frac{1}{2}\gamma+}\|\mathbf{P}_{N_1}v\mathbf{P}_{N_2}f\|^{\frac{1}{2}}_{L^2_{\tau,x}}\|\mathbf{P}_{N_2}f\|^{\frac{1}{2}}_{L^2_\tau L^\infty_x}\\
&\lesssim \|\mathbf{P}_{N_1}\langle\nabla\rangle^{1-}v\mathbf{P}_{N_2}\langle\nabla\rangle^{s-}f\|^{\frac{1}{2}}_{L^2_{\tau,x}}\|\langle\nabla\rangle^{s-}f\|^{\frac{1}{2}}_{L^2_\tau L^\infty_x}\\
&\le C(T)\Big\{\sup_{t\in[0,\tau]}\big(\mathcal{E}(v)(t)\big)^{\frac{1}{4}-}\|\psi\|^{0+}_{L^2_x} + \|\psi\|^{\frac{1}{2}}_{L^2_x}\Big\}\|\langle\nabla\rangle^{s-}f\|_{L^\infty_{\tau,x}}
\end{aligned}\tag{7.29}$$

for any $\tau \in [0,T]$, provided that

$$s > 1 - \tfrac{1}{2}\gamma. \tag{7.30}$$

Next, we consider (ii) $N_1 \ll N_2^\gamma$. Recall that $(q,r) = \big(2, \frac{10}{3}\big)$ is admissible. Then, proceeding as in (7.18) with Lemma 3.6, the Duhamel formula (with $v|_{t=0} = 0$), the linear estimate (Lemma 3.8) and the Strichartz estimates (Lemma 3.5), we have

$$\begin{aligned}
\|\mathbf{P}_{N_1}v\mathbf{P}_{N_2}f\|_{L^2_{\tau,x}} &\lesssim N_1^{2-}N_2^{-\frac{1}{2}+}\|\mathbf{P}_{N_1}v\|_{Y^0_\tau}\|\mathbf{P}_{N_2}f\|_{Y^0_\tau}\\
&\lesssim N_1^{2-}N_2^{-\frac{1}{2}+}\big(\|v\|^{\frac{7}{3}}_{L^{\frac{14}{3}}_\tau L^{\frac{10}{3}}_x} + \|f\|^{\frac{7}{3}}_{L^{\frac{14}{3}}_\tau L^{\frac{10}{3}}_x}\big)\|\mathbf{P}_{N_2}f\|_{Y^0_\tau}
\end{aligned}\tag{7.31}$$

As in the proof of Proposition 7.2, we apply (7.31) only to the θ-power for some $\theta \in (0,1)$. With (7.31), we have

$$\begin{aligned}
&N_2\|\mathbf{P}_{N_1}v\mathbf{P}_{N_2}f\|^{\frac{1}{2}}_{L^2_\tau L^2_x}\|\mathbf{P}_{N_2}f\|^{\frac{1}{2}}_{L^2_\tau L^\infty_x}\\
&\quad\lesssim N_2^{1-\frac{\theta}{4}+}\big(\|v\|^{\frac{7}{3}}_{L^{\frac{14}{3}}_\tau L^{\frac{10}{3}}_x} + \|f\|^{\frac{7}{3}}_{L^{\frac{14}{3}}_\tau L^{\frac{10}{3}}_x}\big)^{\frac{1}{2}\theta}\|\mathbf{P}_{N_2}f\|^{\frac{1}{2}\theta}_{Y^0_\tau}\\
&\qquad\qquad \times \|N_1^{\frac{2\theta}{1-\theta}-}\mathbf{P}_{N_1}v\mathbf{P}_{N_2}f\|^{\frac{1}{2}(1-\theta)}_{L^2_{\tau,x}}\|\mathbf{P}_{N_2}f\|^{\frac{1}{2}}_{L^2_\tau L^\infty_x}
\end{aligned}$$

[12]In the following, we drop the summation over N_1 and N_2 for conciseness of the presentation. Note that we can simply sum over N_1 and N_2 at the end by losing an ε-amount of derivative. Similar comments apply to other dyadic summations.

By interpolation and Lemma 7.1,

$$\lesssim C(T)\big(\|v\|^{\frac{7}{3}}_{L_\tau^{\frac{14}{3}} L_x^{\frac{10}{3}}} + \|f\|^{\frac{7}{3}}_{L_\tau^{\frac{14}{3}} L_x^{\frac{10}{3}}}\big)^{\frac{1}{2}\theta} \|\mathbf{P}_{N_2} f\|^{\frac{1}{2}\theta}_{Y^s_\tau} \|\mathbf{P}_{N_1} f\|^{\frac{1-3\theta}{2}}_{L^2_{\tau,x}} \times \|\mathbf{P}_{N_1}\langle\nabla\rangle v\|^{\theta}_{L^\infty_\tau L^2_x} \|\mathbf{P}_{N_2}\langle\nabla\rangle^{s-} f\|^{1-\theta}_{L^\infty_{\tau,x}} \tag{7.32}$$

for any $\tau \in [0, T]$, provided that

$$1 - \frac{\theta}{4} < s. \tag{7.33}$$

Hence, from (7.25), (7.26), (7.27), (7.28), (7.29), and (7.32), we obtain

$$\mathrm{I} = \big\||v + f|^{\frac{4}{3}} \nabla f\big\|^2_{L^2_{\tau,x}} \le C(T, \|f\|_{B^s(T)}) \sup_{t\in[0,\tau]} \Big\{1 + \big(\mathcal{E}(v)(t)\big)^{1-}\Big\} \tag{7.34}$$

for any $\tau \in [0, T]$, provided that

$$\frac{1}{2} + \frac{7}{10}\theta + \theta < 1.$$

In particular, by choosing $\theta = \frac{5}{17}-$ and $\gamma = \frac{1}{2}\theta$, it follows from (7.30) and (7.33) that the estimate (7.34) holds for

$$s > \frac{63}{68} \approx 0.9265. \tag{7.35}$$

Next, we estimate II in (7.23). By symmetry, we have

$$\begin{aligned}
\mathrm{II} &= \big\||v + f|^{\frac{4}{3}} \nabla f \cdot \nabla f\big\|_{L^1_{\tau,x}} \lesssim \sum_{\substack{N_2, N_3 \in 2^{\mathbb{N}_0} \\ N_2 \ge N_3}} N_2 N_3 \big\||v + f|^{\frac{4}{3}} \mathbf{P}_{N_2} f \cdot \mathbf{P}_{N_3} f\big\|_{L^1_{\tau,x}} \\
&\lesssim \sum_{\substack{N_2, N_3 \in 2^{\mathbb{N}_0} \\ N_2 \ge N_3}} N_2 N_3 \big\||v|^{\frac{4}{3}} \mathbf{P}_{N_2} f \cdot \mathbf{P}_{N_3} f\big\|_{L^1_{\tau,x}} \\
&\qquad + \sum_{\substack{N_2, N_3 \in 2^{\mathbb{N}_0} \\ N_2 \ge N_3}} N_2 N_3 \|f\|^{\frac{1}{3}}_{L^2_{\tau,x}} \|f \mathbf{P}_{N_2} f\|_{L^2_{\tau,x}} \|\mathbf{P}_{N_3} f\|_{L^3_{\tau,x}} \\
&=: \mathrm{II}_1 + \mathrm{II}_2.
\end{aligned}$$

We first estimate II_2. By the dyadic decomposition, we have

$$\begin{aligned}
\mathrm{II}_2 &= \sum_{\substack{N_1 N_2, N_3 \in 2^{\mathbb{N}_0} \\ N_2 \ge N_3}} N_2 N_3 \|f\|^{\frac{1}{3}}_{L^2_{\tau,x}} \|\mathbf{P}_{N_1} f \mathbf{P}_{N_2} f\|_{L^2_{\tau,x}} \|\mathbf{P}_{N_3} f\|_{L^3_{\tau,x}} \\
&\le \sum_{\substack{N_1 N_2, N_3 \in 2^{\mathbb{N}_0} \\ N_2 \ge N_3}} N_2^{2-2s} \|f\|^{\frac{1}{3}}_{L^2_{\tau,x}} \|\mathbf{P}_{N_1} f \mathbf{P}_{N_2} \langle\nabla\rangle^s f\|_{L^2_{\tau,x}} \|\mathbf{P}_{N_3} \langle\nabla\rangle^s f\|_{L^3_{\tau,x}} \\
&\le C(T, \|f\|_{B^s(T)}) \sum_{N_1 N_2 \in 2^{\mathbb{N}_0}} N_2^{2-2s+} \|\mathbf{P}_{N_1} f \mathbf{P}_{N_2} \langle\nabla\rangle^s f\|_{L^2_{\tau,x}}
\end{aligned}$$

for any $\tau \in [0, T]$. If $N_1 \gtrsim N_2^{\gamma}$ for some $\gamma \in (0, 1)$, then we have

$$\begin{aligned}
N_2^{2-2s+} \|\mathbf{P}_{N_1} f \mathbf{P}_{N_2} \langle\nabla\rangle^s f\|_{L^2_{\tau,x}} &\lesssim N_1^{0-} N_2^{2-2s-\gamma s+} \|\mathbf{P}_{N_1} \langle\nabla\rangle^s f \mathbf{P}_{N_2} \langle\nabla\rangle^s f\|_{L^2_{\tau,x}} \\
&\lesssim \|f\|^2_{L^4_\tau W^{s,4}_x},
\end{aligned} \tag{7.36}$$

provided that $2-2s-\gamma s<0$, namely

$$s>\frac{2}{2+\gamma}. \tag{7.37}$$

If $N_1 \ll N_2^\gamma$, then by applying Lemma 3.6, we have

$$\begin{aligned} N_2^{2-2s+}\|\mathbf{P}_{N_1} f \mathbf{P}_{N_2}\langle\nabla\rangle^s f\|_{L^2_{\tau,x}} &\lesssim N_1^{2-s-} N_2^{\frac{3}{2}-2s+}\|\mathbf{P}_{N_1} f\|_{Y^s_\tau}\|\mathbf{P}_{N_2} f\|_{Y^s_\tau} \\ &\ll N_2^{\frac{3}{2}-2s+\gamma(2-s)+}\|\mathbf{P}_{N_1} f\|_{Y^s_\tau}\|\mathbf{P}_{N_2} f\|_{Y^s_\tau} \\ &\ll \|f\|^2_{Y^s_\tau}, \end{aligned} \tag{7.38}$$

provided that $\frac{3}{2}-2s+\gamma(2-s)<0$, namely

$$s>\frac{3+4\gamma}{4+2\gamma}. \tag{7.39}$$

It follows from (7.36) and (7.38) with (7.37) and (7.39) that

$$\mathrm{II}_2 \le C(T, \|f\|_{B^s(T)}) \tag{7.40}$$

for any $\tau \in [0,T]$, provided that

$$s>\frac{8}{9}\approx 0.8889. \tag{7.41}$$

Finally, we estimate II_1. By Hölder's inequality, we have

$$\mathrm{II}_1 \lesssim \sum_{\substack{N_2,N_3\in 2^{\mathbb{N}_0}\\ N_2\ge N_3}} N_2^{2-2s}\|v\|^{\frac{1}{3}}_{L^{\frac{10}{3}}_{\tau,x}}\|v\mathbf{P}_{N_2}\langle\nabla\rangle^{s-} f\|_{L^2_{\tau,x}}\|\mathbf{P}_{N_3}\langle\nabla\rangle^s f\|_{L^{\frac{5}{2}}_{\tau,x}}. \tag{7.42}$$

In the following, we estimate

$$\|v\mathbf{P}_{N_2}\langle\nabla\rangle^{s-} f\|_{L^2_{\tau,x}} \lesssim \sum_{N_1\in 2^{\mathbb{N}_0}} \|\mathbf{P}_{N_1} v\mathbf{P}_{N_2}\langle\nabla\rangle^{s-} f\|_{L^2_{\tau,x}}.$$

If $N_1 \gtrsim N_2^\gamma$ for some $\gamma\in(0,1)$, then

$$\begin{aligned} N_2^{2-2s+}\|v\mathbf{P}_{N_2}\langle\nabla\rangle^{s-} f\|_{L^2_{\tau,x}} &\lesssim N_1^{1-} N_2^{2-2s-\gamma+}\|v\mathbf{P}_{N_2}\langle\nabla\rangle^{s-} f\|_{L^2_{\tau,x}} \\ &\lesssim C(T)\|\langle\nabla\rangle v\|_{L^\infty_\tau L^2_x}\|\langle\nabla\rangle^{s-} f\|_{L^\infty_{\tau,x}} \end{aligned} \tag{7.43}$$

for any $\tau\in[0,T]$, provided that $2-2s<\gamma<1$.

If $N_1 \ll N_2^\gamma$, then by applying (7.31) to the θ-power $\|v\mathbf{P}_{N_2}\langle\nabla\rangle^{s-} f\|_{L^2_{\tau,x}}$ as before, we have

$$\begin{aligned} & N_2^{2-2s+}\|v\mathbf{P}_{N_2}\langle\nabla\rangle^{s-} f\|_{L^2_{\tau,x}} \\ &\quad \lesssim N_2^{2-2s-\frac{1}{2}\theta+}\Big(\|v\|^{\frac{7}{3}}_{L^{\frac{14}{3}}_\tau L^{\frac{10}{3}}_x} + \|f\|^{\frac{7}{3}}_{L^{\frac{14}{3}}_\tau L^{\frac{10}{3}}_x}\Big)^\theta \|\mathbf{P}_{N_2} f\|^\theta_{Y^s_\tau} \\ &\qquad \times \|N_1^{\frac{2\theta-}{1-\theta}}\mathbf{P}_{N_1} v\|^{1-\theta}_{L^2_{\tau,x}}\|\mathbf{P}_{N_2}\langle\nabla\rangle^{s-} f\|^{1-\theta}_{L^\infty_{\tau,x}} \end{aligned}$$

By interpolation and Lemma 7.1,

$$\begin{aligned}
&\leq C(T) N_2^{2-2s-\frac{1}{2}\theta+} \big(\|v\|^{\frac{7}{3}}_{L^{\frac{14}{3}}_\tau L^{\frac{10}{3}}_x} + \|f\|^{\frac{7}{3}}_{L^{\frac{14}{3}}_\tau L^{\frac{10}{3}}_x} \big)^\theta \|\mathbf{P}_{N_2} f\|^\theta_{Y^s_\tau} \|f\|^{1-3\theta+}_{L^\infty L^2_x} \\
&\qquad \times \|\mathbf{P}_{N_1} \langle \nabla \rangle v\|^{2\theta-}_{L^2_{\tau,x}} \|\mathbf{P}_{N_2} \langle \nabla \rangle^{s-} f\|^{1-\theta}_{L^\infty_{\tau,x}} \\
&\leq C(T, \|f\|_{B^s(T)}) \big(1 + \|v\|^{\frac{7}{3}\theta}_{L^\infty_\tau L^{\frac{10}{3}}_x} \big) \|\langle \nabla \rangle v\|^{2\theta-}_{L^\infty_\tau L^2_x}
\end{aligned} \tag{7.44}$$

for any $\tau \in [0, T]$, provided that

$$s > 1 - \frac{\theta}{4}. \tag{7.45}$$

Putting (7.42), (7.43), and (7.44) together, we obtain

$$\mathrm{II}_1 \leq C(T, \|f\|_{B^s(T)}) \sup_{t \in [0,\tau]} \Big\{ 1 + \big(\mathcal{E}(v)(t) \big)^{1-} \Big\} \tag{7.46}$$

by choosing $\theta \in (0, 1)$ such that $\frac{1}{10} + \frac{7}{10}\theta + \theta < 1$ with $\gamma = \frac{\theta}{2}$. In particular, by choosing $\theta = \frac{9}{17}-$, the regularity restriction (7.45) yields

$$s > \frac{59}{68} \approx 0.8676. \tag{7.47}$$

Therefore, it follows from (7.24), (7.34), (7.40), and (7.46) with (7.35), (7.41), and (7.47) that

$$\sup_{t \in [0,\tau]} \mathcal{E}(v)(t) \leq C(T, \|f\|_{B^s(T)}) \Big\{ 1 + \sup_{t \in [0,\tau]} \big(\mathcal{E}(v)(t) \big)^{1-} \Big\}$$

for any $\tau \in [0, T]$, provided that $s > \frac{63}{68}$. Therefore, (7.22) follows from the standard continuity argument. □

7.2. **Long time existence of solutions to the perturbed NLS.** Our main goal in this subsection is to prove long time existence of solutions to the perturbed NLS (7.2) under some regularity assumptions on the perturbation f (Proposition 7.5). The main ingredients are the energy estimates (Propositions 7.2 and 7.3) and the following perturbation lemma.

Lemma 7.4 (Perturbation lemma)**.** *Given $d = 5$ or 6, let (q_d, r_d) be the admissible pair in* (4.1)*. Let I be a compact interval with $|I| \leq 1$. Suppose that $v \in C(I; H^1(\mathbb{R}^d))$ satisfies the following perturbed NLS:*

$$i\partial_t v + \Delta v = |v|^{\frac{4}{d-2}} v + e,$$

satisfying

$$\|v\|_{L^{q_d}_t (I; W^{1,r_d}_x (\mathbb{R}^d))} + \|v\|_{L^\infty(I; H^1(\mathbb{R}^d))} \leq R$$

for some $R \geq 1$. Then, there exists $\varepsilon_0 = \varepsilon_0(R) > 0$ such that if we have

$$\|w_0 - v(t_0)\|_{H^1(\mathbb{R}^d)} + \|e\|_{N^1(I)} \leq \varepsilon$$

for some $w_0 \in H^1(\mathbb{R}^d)$, some $t_0 \in I$, and some $\varepsilon < \varepsilon_0$, then there exists a solution $w \in X^1(I) \cap C(I; H^1(\mathbb{R}^d))$ to the defocusing NLS (7.1) *with $w(t_0) = w_0$ such that*

$$\begin{aligned}
\|w\|_{X^1(I)} + \|v\|_{X^1(I)} &\leq C(R), \\
\|w - v\|_{X^1(I)} &\leq C(R)\varepsilon,
\end{aligned}$$

where $C(R)$ is a non-decreasing function of R.

See [16, 52, 53] for perturbation and stability results on the usual Strichartz and Lebesgue spaces. For perturbation lemmas involving the critical X^1-norm, see [31, 3]. The proof of Lemma 7.4 follows from a straightforward modification of the proof of Lemma 7.1 in [3] and hence we omit details.

We now state a long time existence result for the perturbed NLS (7.2). Fix $d = 5$ or 6 and let $s \in (s_*, 1)$, where s_* is as in Theorem 1.5. Then, let $\delta = \delta(d, s) > 0$ be as in Corollary 6.1. Given $T > 0$, suppose that $f \in \widetilde{E}_{M,T}$ for some $M > 0$, where $\widetilde{E}_{M,T}$ is as in (6.2). Namely, we have

$$\|f\|_{Y^s([0,T))} + \|f\|_{S^s([0,T))} \le M. \tag{7.48}$$

Then, Lemma 6.2 guarantees existence of a solution $v \in C([0, \tau_0]; H^1(\mathbb{T}^d)) \cap X^1([0, \tau_0])$ to the perturbed NLS (7.2), at least for a short time $\tau_0 > 0$. Furthermore, assume that there exists $K > 0$ such that

$$\text{(i) } \|f\|_{A^s(T)} \le K \text{ when } d = 6 \qquad \text{and} \qquad \text{(ii) } \|f\|_{B^s(T)} \le K \text{ when } d = 5, \tag{7.49}$$

where $A^s(T)$ and $B^s(T)$ are as in Propositions 7.2 and 7.3. Then, it follows from Lemma 7.1 and Propositions 7.2 and 7.3 that there exists $R = R(K, T) > 0$ such that

$$\|v\|_{L^\infty([0,T];H^1(\mathbb{R}^d))} \le R \tag{7.50}$$

for a solution v to (7.2).

Under these assumptions, by iteratively applying Lemma 7.4, we obtain the following long time existence result for the perturbed NLS (7.2) on $[0, T]$.

Proposition 7.5. *Let $d = 5, 6$ and $s \in (s_*, 1)$, where s_* is as in Theorem 1.5. Given $T > 0$, assume that the hypotheses* (7.48) *and* (7.49) *hold. Then, there exists $\tau = \tau(R, M, T, s) > 0$ such that, given any $t_0 \in [0, T)$, the solution v to* (7.2) *exists on $[t_0, t_0 + \tau] \cap [0, T]$. In particular, the energy estimate* (7.50) *guarantees existence of v on the entire interval* $[0, T]$.

Proposition 7.5 follows from a straightforward modification of the proof of Proposition 7.2 in [3]. Hence, we omit the details of the proof but we briefly describe the main idea in the following. Given $t_0 \in [0, T)$, the main idea is to approximate a solution v to the perturbed NLS (7.2) by the global solution w to the original NLS (7.1) with $w|_{t=t_0} = v(t_0)$ on $[t_0, t_0 + \tau]$, where $\tau = \tau(R, M, T, s) > 0$ is independent of $t_0 \in [0, T)$. We achieve this goal by iteratively applying the perturbation lemma (Lemma 7.4) on short time intervals. This is possible thanks to (i) the a priori control (7.48) and (7.50) on f and the H^1-norm of $v(t)$, respectively, on $[0, T]$ and (ii) the following space-time control on the global solution w to (7.1) due to Vişan [55]:

$$\|w\|_{L^{\frac{2(d+2)}{d-2}}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \le C(\|v(t_0)\|_{H^1}) = C(R).$$

See the proof of Proposition 7.2 in [3] for details. In the following, we point out the difference between the assumptions in Proposition 7.5 above and those in Proposition 7.2 in [3]. The assumption in [3] would read as "$\|f\|_{S^s(I)} \le |I|^\beta$ for any interval $I \subset [0, T]$" in our context. Note that we are making a weaker assumption on the S^s-norm in (7.48). This is possible thanks to the appearance of the factor $|I|^\theta$ in the nonlinear estimate (6.3) in Corollary 6.1.

Namely, in this paper, we already exploited the subcritical nature of the perturbation and created the factor $|I|^\theta$ in (6.3). Compare this with Lemma 6.2 in [3].

7.3. **Proof of Theorem 1.5.** In this subsection, we present the proof of Theorem 1.5. By Borel-Cantelli lemma, it suffices to prove the following "almost" almost sure global existence result. See [17, 3].

Proposition 7.6. *Let $d = 5, 6$ and $s \in (s_*, 1)$, where s_* is as in Theorem 1.5. Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in* (1.2)*. Then, given any $T, \varepsilon > 0$, there exists a set $\widetilde{\Omega}_{T,\varepsilon} \subset \Omega$ such that*

(i) *$P(\widetilde{\Omega}^c_{T,\varepsilon}) < \varepsilon$,*
(ii) *For each $\omega \in \widetilde{\Omega}_{T,\varepsilon}$, there exists a (unique) solution u to* (1.1) *on $[0, T]$ with $u|_{t=0} = \phi^\omega$.*

The proof of Proposition 7.6 is analogous to that of Proposition 8.1 in [3]. The main difference appears in the definitions of Ω_2 and Ω_3 below, incorporating the energy estimate (7.50) and the simplified assumption (7.48).

Proof. Fix $T, \varepsilon > 0$. Set $M = M(\varepsilon, \|\phi\|_{H^s})$ by

$$M \sim \|\phi\|_{H^s} \Big(\log \frac{1}{\varepsilon}\Big)^{\frac{1}{2}}.$$

Without loss of generality, we assume that $\varepsilon > 0$ is sufficiently small such that $M = M(\varepsilon, \|\phi\|_{H^s}) \geq 1$. Defining $\Omega_1 = \Omega_1(\varepsilon)$ by

$$\Omega_1 := \big\{\omega \in \Omega : \|\phi^\omega\|_{H^s} \leq M\big\},$$

it follows from Lemma 2.1 that

$$P(\Omega_1^c) < \frac{\varepsilon}{3}. \tag{7.51}$$

Given $K > 0$, define $\Omega_2 = \Omega_2(T, K)$ by

$$\Omega_2 := \big\{\omega \in \Omega : \|S(t)\phi^\omega\|_{F^s(T)} \leq K\big\},$$

where $F^s(T) = A^s(T)$ when $d = 6$ and $= B^s(T)$ when $d = 5$. Then, by Lemmas 2.1 and 2.3, we can choose $K = K(T, \varepsilon, \|\phi\|_{H^s}) \gg 1$ such that

$$P(\Omega_2^c) < \frac{\varepsilon}{3}. \tag{7.52}$$

Hence, the energy estimate (7.50) holds with some $R = R(K, T) = R(T, \varepsilon) > 0$.

Now, let $\tau = \tau(R, M, T, s)$ be as in Proposition 7.5. Let $\delta = \delta(d, s) > 0$ be as in Corollary 6.1 and set $q = \frac{4}{1-4\delta}$. With $I_j = [j\tau_*, (j+1)\tau_*]$ for some $\tau_* \leq \tau$ (to be chosen later), we partition the interval $[0, T]$ as

$$[0, T] = \bigcup_{j=0}^{[\frac{T}{\tau_*}]} I_j \cap [0, T]$$

and define Ω_3 by

$$\Omega_3 := \Big\{\omega \in \Omega : \|S(t)\phi^\omega\|_{S^s(I_j)} \leq M,\ j = 0, \dots, \big[\tfrac{T}{\tau_*}\big]\Big\}.$$

Then, by Lemma 2.3 and taking $\tau_* = \tau_*(T, \varepsilon, \|\phi\|_{H^s}) > 0$ sufficiently small, we have

$$\begin{aligned} P(\Omega_3^c) &\le \sum_{j=0}^{[\frac{T}{\tau_*}]} P\Big(\|S(t)\phi^\omega\|_{S^s(I_j)} > M \Big) \le C\frac{T}{\tau_*} \exp\bigg(- c\frac{M^2}{\tau_*^{\frac{2}{q}} \|\phi\|_{H^s}^2} \bigg) \\ &\le C\frac{T}{\tau_*} \cdot \tau_* \exp\bigg(- c\frac{M^2}{2\tau_*^{\frac{2}{q}} \|\phi\|_{H^s}^2} \bigg) \le CT \exp\bigg(- \frac{c}{2\tau_*^{\frac{2}{q}} \|\phi\|_{H^s}^2} \bigg) \\ &< \frac{\varepsilon}{3}. \end{aligned} \tag{7.53}$$

Finally, set $\widetilde{\Omega}_{T,\varepsilon} := \Omega_1 \cap \Omega_2 \cap \Omega_3$. Then, from (7.51), (7.52), and (7.53), we conclude that

$$P(\widetilde{\Omega}_{T,\varepsilon}^c) < \varepsilon.$$

Moreover, for $\omega \in \widetilde{\Omega}_{T,\varepsilon}$, we can iteratively apply Proposition 7.5 and construct the solution $v = v^\omega$ to (1.3) on each $[j\tau_*, (j+1)\tau_*]$, $j = 0, \dots, [\frac{T}{\tau_*}] - 1$, and $\big[[\frac{T}{\tau_*}]\tau_*, T\big]$. This completes the proof of Proposition 7.6. □

8. Probabilistic construction of finite time blowup solutions below the energy space

In this section, we present the proof of Theorem 1.7. We first recall the following definition of a weak solution to (1.18). See [30].

Definition 8.1. We say that v is a weak solution to (1.18) on $[0, T)$ if v belongs to $L^{\frac{d+2}{d-2}}_{\text{loc}}([0, T) \times \mathbb{R}^d)$ and satisfies

$$\int_{[0,T]\times\mathbb{R}^d} v \cdot (-i\partial_t \psi + \Delta\psi)\, dxdt = i\alpha \int_{\mathbb{R}^d} v_0 \cdot \psi(0)\, dx + \lambda \int_{[0,T]\times\mathbb{R}^d} |v + \varepsilon z|^{\frac{d+2}{d-2}} \cdot \psi\, dxdt \tag{8.1}$$

for any test function[13] $\psi \in C_c^\infty([0, T) \times \mathbb{R}^d)$.

Fix $v_0 \in H^1(\mathbb{R}^d)$. Then, for any $\alpha > 0$ and $\varepsilon > 0$, Proposition 1.3 establishes almost sure local well-posedness of the following Duhamel formulation:

$$v(t) = \alpha S(t) v_0 - i\lambda \int_0^t S(t - t')|v + \varepsilon z^\omega|^{\frac{d+2}{d-2}}(t')dt'. \tag{8.2}$$

The following lemma shows that the solution v to (8.2) is indeed a weak solution to (8.1).

Lemma 8.2. *Let $d = 5, 6$ and $1 - \frac{1}{d} < s < 1$. Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in* (1.2) *and let $z^\omega = S(t)\phi^\omega$. Then, given any $v_0 \in H^1(\mathbb{R}^d)$, $\alpha > 0$, $\varepsilon > 0$, and $T > 0$, any local-in-time solution $v \in C([0, T); H^1(\mathbb{R}^d)) \cap X^1([0, T))$ to the Duhamel formulation* (8.2) *is almost surely a weak solution on $[0, T)$ in the sense of Definition 8.1.*

[13] By convention, our test function ψ has compact support but does not have to vanish at $t = 0$. The same comment applies to the test function $\eta = \eta(t)$ below.

We first present the proof of Theorem 1.7, assuming Lemma 8.2. We prove Lemma 8.2 at this end of this section. Note that while Proposition 1.3 guarantees the existence of the solution v to (8.2) at least for some small $T_\omega > 0$, Lemma 8.2 *assumes* its existence on $[0, T)$ for some given $T > 0$.

In the following, we only consider (1.5) with $\lambda = 1$ and assume that v_0 satisfies (1.15). The proof of Theorem 1.7 is based on the so-called test function method [57, 58] and we closely follow the argument in [28]. We first define two test functions $\eta = \eta(t) \in C_c^\infty([0,\infty); [0,1])$ and $\theta = \theta(x) \in C_c^\infty(\mathbb{R}^d; [0,1])$ such that

$$\eta(t) = \begin{cases} 1 & \text{for } 0 \le t < \frac{1}{2}, \\ 0 & \text{for } t \ge 1, \end{cases} \qquad \text{and} \qquad \theta(x) = \begin{cases} 1 & \text{for } 0 \le |x| < \frac{1}{2}, \\ 0 & \text{for } |x| \ge 1. \end{cases}$$

We also define the scaled test functions $\eta_{_T}$ and $\theta_{_T}$ by $\eta_{_T}(t) := \eta\big(\frac{t}{T}\big)$ and $\theta_{_T}(x) := \big(\frac{x}{\sqrt{T}}\big)$. Finally, we set $\psi_{_T}(t,x) := \eta_{_T}(t)\theta_{_T}(x)$.

Given $T \ge 1$, let $v = v^\omega \in X^1([0,T))$ be a solution to the Duhamel formulation (8.2) on $[0, T)$. Define I and II by

$$\mathrm{I}(T) = \int_{[0,T)\times B_{\sqrt{T}}} |v + \varepsilon z^\omega|^p \cdot \psi_{_T}^\ell \, dxdt \qquad \text{and} \qquad \mathrm{II}(T) = \operatorname{Im} \int_{B_{\sqrt{T}}} v_0 \cdot \theta_{_T}^\ell \, dx, \tag{8.3}$$

where B_r denotes the ball of radius r centered at 0 in $\mathbb{R}^d$ and $\ell \in \mathbb{N}$ such that $\ell \ge 2p' + 1$. Here, $p' = \frac{d+2}{4}$ denotes the Hölder conjugate of $p = \frac{d+2}{d-2}$. By Lemma 8.2 and taking the real part of the weak formulation (8.1), we obtain

$$\begin{aligned} \mathrm{I}(T) - \alpha \mathrm{II}(T) &= \operatorname{Im} \int_{[0,T]\times\mathbb{R}^d} v \cdot \partial_t \psi_T^\ell \, dxdt + \operatorname{Re} \int_{[0,T]\times\mathbb{R}^d} v \cdot \Delta \psi_T^\ell \, dxdt \\ &=: \mathrm{III}_1(T) + \mathrm{III}_2(T). \end{aligned} \tag{8.4}$$

By $\ell - 1 \ge \frac{\ell}{p}$ and the triangle inequality, we have

$$\begin{aligned} \mathrm{III}_1(T) &\lesssim T^{-1} \int_{[0,T)\times B_{\sqrt{T}}} |v| \cdot \eta_{_T}^{\ell-1} \theta_{_T}^\ell \eta'\big(\tfrac{t}{T}\big) \, dxdt \\ &\lesssim T^{-1} \int_{[0,T)\times B_{\sqrt{T}}} |v + \varepsilon z^\omega| \cdot \psi_T^{\frac{\ell}{p}} \, dxdt + \varepsilon T^{-1} \int_{[0,T)\times B_{\sqrt{T}}} |z^\omega| \, dxdt \\ &\lesssim T\big(\mathrm{I}(T)\big)^{\frac{1}{p}} + \varepsilon T^{-1} \|z^\omega\|_{L^1_{t,x}([0,T)\times B_{\sqrt{T}})}. \end{aligned} \tag{8.5}$$

A similar computation with $\ell - 2 \ge \frac{\ell}{p}$ and the triangle inequality yields

$$\mathrm{III}_2(T) \lesssim T\big(\mathrm{I}(T)\big)^{\frac{1}{p}} + \varepsilon T^{-1} \|z^\omega\|_{L^1_{t,x}([0,T)\times B_{\sqrt{T}})}. \tag{8.6}$$

From (8.4), (8.5), and (8.6) with Young's inequality, we have

$$\begin{aligned} -\alpha \mathrm{II}(T) &\le -\mathrm{I}(T) + CT\big(\mathrm{I}(T)\big)^{\frac{1}{p}} + C\varepsilon T^{-1} \|z^\omega\|_{L^1_{t,x}([0,T)\times B_{\sqrt{T}})} \\ &\le -\mathrm{I}(T) + C'T^{p'} + \mathrm{I}(T) + C\varepsilon T^{-1} \|z^\omega\|_{L^1_{t,x}([0,T)\times B_{\sqrt{T}})} \\ &\le C'T^{\frac{d+2}{4}} + C\varepsilon T^{-1} \|z^\omega\|_{L^1_{t,x}([0,T)\times B_{\sqrt{T}})}. \end{aligned} \tag{8.7}$$

On the other hand, from (1.15) and (8.3) with a change of variables, we have

$$-\text{II}(T) \geq T^{\frac{d-k}{2}} L(T) := T^{\frac{d-k}{2}} \int_{B_{\frac{1}{\sqrt{T}}}} |x|^{-k} \theta^{\ell}\, dx. \tag{8.8}$$

Given $R > 0$, $T \geq 1$, and $\varepsilon > 0$, define the set $\Omega_{R,\varepsilon}$ by

$$\Omega_{R,\varepsilon} := \big\{ \omega \in \Omega : \|\phi^\omega\|_{L^2_x} \leq \varepsilon^{-1} R \big\}.$$

Then, it follows from Lemma 2.1 that

$$P(\Omega_{R,\varepsilon}^c) \leq C \exp\bigg(- c \frac{R^2}{\varepsilon^2 \|\phi\|_{L^2}^2} \bigg).$$

In particular, $P(\Omega_{R,\varepsilon}^c) \to 0$ as $R \to \infty$ or $\varepsilon \to 0$ (while keeping the other fixed).

Then, putting (8.7) and (8.8) together with $T \geq 1$, we obtain

$$\begin{aligned} \alpha &\leq C L^{-1}(T) \Big\{ T^{\frac{-d+2k+2}{4}} + \varepsilon T^{\frac{-d+k-2}{2}} \|z^\omega\|_{L^1_{t,x}([0,T)\times B_{\sqrt{T}})} \Big\} \\ &\leq C L^{-1}(T) \Big\{ T^{\frac{-d+2k+2}{4}} + \varepsilon T^{\frac{-d+2k+2}{4}} \|z^\omega\|_{L^\infty_t L^2_x([0,T)\times B_{\sqrt{T}})} \Big\} \\ &\leq C L^{-1}(T) T^{\frac{-d+2k+2}{4}} (1+R) \end{aligned} \tag{8.9}$$

for $\omega \in \Omega_{R,\varepsilon}$. In the following, we fix $R > 0$ and $\varepsilon > 0$ and work on $\Omega_{R,\varepsilon}$. Namely, the following argument holds uniformly in $\omega \in \Omega_{R,\varepsilon}$ and we suppress the dependence on ω.

Suppose that given $\alpha > 0$, the maximal existence time $T^*(\alpha) \geq 4$. Since $k < d$, we have $L(4) < \infty$. In particular, by setting $T = 4$ in (8.9), we obtain

$$\alpha \lesssim 1 + R.$$

This in turn implies that there exists $\alpha_0 = \alpha_0(R) > 0$ such that $T^*(\alpha) < 4$ for all $\alpha \geq \alpha_0$.

Fix $\alpha > \alpha_0$. Then, by noting that $L(T)$ defined in (8.8) is decreasing on $[0, \infty)$, we conclude from (8.9) that

$$\alpha \leq C L^{-1}(4) T^{\frac{-d+2k+2}{4}} (1+R) \leq C(R) T^{\frac{-d+2k+2}{4}}$$

for any $0 < T \leq T^*(\alpha) < 4$. Hence, we obtain the following upper bound on the maximal time of existence:

$$T^*(\alpha) \leq C'(R) \alpha^{\frac{4}{-d+2k+2}}.$$

Lastly, (1.17) follows from the blowup alternative (6.5). This proves Theorem 1.7.

We conclude this paper by presenting the proof of Lemma 8.2. While the proof is standard, we include it for completeness.

Proof of Lemma 8.2. Write the solution v to (8.2) on $[0, T)$ as

$$v(t) = \alpha S(t) v_0 - i\lambda \mathcal{I}[\mathcal{N}(v + \varepsilon z^\omega)](t),$$

where $\mathcal{I}$ is as in (3.2) and $\mathcal{N}(u) = |u|^{\frac{d+2}{d-2}}$. First, we show that the linear part $\alpha S(t) v_0$ satisfies

$$\int_{[0,T]\times\mathbb{R}^d} v \cdot (-i\partial_t \psi + \Delta\psi)\, dx dt = i\alpha \int_{\mathbb{R}^d} v_0 \cdot \psi(0)\, dx \tag{8.10}$$

for any test function $\psi \in C_c^\infty([0, T) \times \mathbb{R}^d)$.

Let $v_{0,n}$ be smooth functions converging to v_0 in $H^1(\mathbb{R}^d)$. Then, $\alpha S(t)v_{0,n}$, $n \in \mathbb{N}$, solves the linear Schrödinger equation: $i\partial_t v + \Delta v = 0$ and is smooth on $[0, T) \times \mathbb{R}^d$. Integrating by parts, we have

$$\int_{[0,T]\times\mathbb{R}^d} \alpha S(t)v_{0,n} \cdot (-i\partial_t \psi + \Delta\psi)\, dxdt = i\alpha \int_{\mathbb{R}^d} v_{0,n} \cdot \psi(0)\, dx. \tag{8.11}$$

By Hölder's inequality and the unitarity of $S(t)$ on $L^2(\mathbb{R}^d)$, we have

$$\bigg| \int_{[0,T]\times\mathbb{R}^d} \alpha(S(t)v_0 - S(t)v_{0,n})(-i\partial_t\psi + \Delta\psi)\, dxdt \bigg| \lesssim \|v_0 - v_{0,n}\|_{L^2} \big(\|\psi\|_{W^{1,1}_T L^2_x} + \|\psi\|_{L^1_T H^2_x} \big) \longrightarrow 0.$$

Similarly, the right-hand side of (8.11) converges to the right-hand side of (8.10) as $n \to \infty$. Hence, (8.10) holds.

Next, we consider the nonlinear part $-i\lambda\mathcal{I}(v + \varepsilon z^\omega)$. Let v_n be smooth functions on $[0, T) \times \mathbb{R}^d$ converging to v in $X^1([0, T))$. Then, by Proposition 4.1 with Lemmas 2.1 and 2.3, we have

$$\big\| \mathcal{I}[\mathcal{N}(v + \varepsilon z^\omega)] - \mathcal{I}[\mathcal{N}(v_n + \varepsilon z^\omega)] \big\|_{C_T H^1} \longrightarrow 0, \tag{8.12}$$

almost surely. Let $w_n = -i\lambda\mathcal{I}[\mathcal{N}(v_n + \varepsilon z^\omega)]$. Then, w_n is the smooth solution to the following inhomogeneous linear Schrödinger equation:

$$\begin{cases} i\partial_t w_n + \Delta w_n = \lambda |v_n + \varepsilon z|^{\frac{d+2}{d-2}} \\ w_n|_{t=0} = 0. \end{cases}$$

Then, proceeding as above with (8.12) and integrating by parts, we have

$$\begin{aligned} \int_{[0,T]\times\mathbb{R}^d} - i\lambda\mathcal{I}(v + \varepsilon z^\omega) \cdot (-i\partial_t\psi + \Delta\psi)\, dxdt &= \lim_{n\to\infty} \int_{[0,T]\times\mathbb{R}^d} w_n \cdot (-i\partial_t\psi + \Delta\psi)\, dxdt \\ &= \lim_{n\to\infty} \int_{[0,T]\times\mathbb{R}^d} (i\partial_t w_n + \Delta w_n) \cdot \psi\, dxdt \\ &= \lim_{n\to\infty} \lambda \int_{[0,T]\times\mathbb{R}^d} |v_n + \varepsilon z|^{\frac{d+2}{d-2}} \cdot \psi\, dxdt = \lambda \int_{[0,T]\times\mathbb{R}^d} |v + \varepsilon z|^{\frac{d+2}{d-2}} \cdot \psi\, dxdt \end{aligned} \tag{8.13}$$

for any test function $\psi \in C^\infty_c([0, T) \times \mathbb{R}^d)$. Hence, the weak formulation (8.1) follows from (8.10) and (8.13). This completes the proof of Lemma 8.2. □

Acknowledgments. T.O. was supported by the European Research Council (grant no. 637995 "ProbDynDispEq"). M.O. was supported by JSPS KAKENHI Grant number JP16K17624.

References

[1] A. Ayache, N. Tzvetkov, *L^p properties for Gaussian random series,* Trans. Amer. Math. Soc. **360** (2008), no. 8, 4425–4439.

[2] Á. Bényi, T. Oh, O. Pocovnicu, *Wiener randomization on unbounded domains and an application to almost sure well-posedness of NLS,* Excursions in harmonic analysis. Vol. 4, 3–25, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, Cham, 2015.

[3] Á. Bényi, T. Oh, O. Pocovnicu, On the probabilistic Cauchy theory of the cubic nonlinear Schrödinger equation on $\mathbb{R}^d$, $d \geq 3$, Trans. Amer. Math. Soc. Ser. B **2** (2015), 1–50.

[4] Á. Bényi, T. Oh, O. Pocovnicu, *Higher order expansions for the probabilistic local Cauchy theory of the cubic nonlinear Schrödinger equation on* $\mathbb{R}^3$, to appear in Trans. Amer. Math. Soc.

[5] Á. Bényi, T. Oh, O. Pocovnicu, *On the probabilistic Cauchy theory for nonlinear dispersive PDEs*, to appear in Landscapes of Time-Frequency Analysis, Appl. Numer. Harmon. Anal., Birkhäuser/Springer.

[6] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations*, Geom. Funct. Anal. **3** (1993), 107–156.

[7] J. Bourgain, *Invariant measures for the 2D-defocusing nonlinear Schrödinger equation,* Comm. Math. Phys. **176** (1996), no. 2, 421–445.

[8] J. Bourgain, *Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity,* Internat. Math. Res. Notices **1998**, no. 5, 253–283.

[9] J. Brereton, *Almost sure local well-posedness for the supercritical quintic NLS,* Tunisian J. Math. **1** (2019), no. 3, 427–453.

[10] N. Burq, N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations I: local theory,* Invent. Math. **173** (2008), no. 3, 449–475.

[11] N. Burq, N. Tzvetkov, *Probabilistic well-posedness for the cubic wave equation,* J. Eur. Math. Soc. (JEMS) **16** (2014), no. 1, 1–30.

[12] T. Cazenave, *Semilinear Schrödinger equations,* Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003. xiv+323 pp.

[13] T. Cazenave, F. Weissler, *Some remarks on the nonlinear Schrödinger equation in the critical case,* Nonlinear semigroups, partial differential equations and attractors (Washington, DC, 1987), 18–29, Lecture Notes in Math., 1394.

[14] M. Christ, *Power series solution of a nonlinear Schrödinger equation*, Mathematical aspects of nonlinear dispersive equations, 131–155, Ann. of Math. Stud., 163, Princeton Univ. Press, Princeton, NJ, 2007.

[15] M. Christ, J. Colliander, T. Tao, *Ill-posedness for nonlinear Schrödinger and wave equations*, arXiv:math/0311048 [math.AP].

[16] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in* $\mathbb{R}^3$, Ann. of Math. (2) **167** (2008), no. 3, 767–865.

[17] J. Colliander, T. Oh, *Almost sure well-posedness of the cubic nonlinear Schrödinger equation below* $L^2(\mathbb{T})$*,* Duke Math. J. **161** (2012), no. 3, 367–414.

[18] G. Da Prato, A. Debussche, *Two-dimensional Navier-Stokes equations driven by a space-time white noise*, J. Funct. Anal. **196** (2002), no. 1, 180–210.

[19] G. Da Prato, A. Debussche, *Strong solutions to the stochastic quantization equations*, Ann. Probab. **31** (2003), no. 4, 1900–1916.

[20] C. Deng, S. Cui, *Random-data Cauchy problem for the Navier-Stokes equations on* $\mathbb{T}^3$*,* J. Differential Equations **251** (2011), no. 4-5, 902–917.

[21] P. Friz, M. Hairer, *A course on rough paths. With an introduction to regularity structures,* Universitext. Springer, Cham, 2014. xiv+251 pp.

[22] J. Ginibre, G. Velo, *Smoothing properties and retarded estimates for some dispersive evolution equations*, Comm. Math. Phys. **144** (1992), no. 1, 163–188.

[23] M. Gubinelli, N. Perkowski, *Lectures on singular stochastic PDEs,* Ensaios Matemáticos [Mathematical Surveys], 29. Sociedade Brasileira de Matemática, Rio de Janeiro, 2015. 89 pp.

[24] M. Hadac, S. Herr, H. Koch, *Well-posedness and scattering for the KP-II equation in a critical space,* Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 3, 917–941. *Erratum to "Well-posedness and scattering for the KP-II equation in a critical space"*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), no. 3, 971–972.

[25] M. Hairer, *Introduction to regularity structures,* Braz. J. Probab. Stat. **29** (2015), no. 2, 175–210.

[26] H. Hirayama, M. Okamoto, *Random data Cauchy problem for the nonlinear Schrödinger equation with derivative nonlinearity*, Discrete Contin. Dyn. Syst. **36** (2016), no. 12, 6943–6974.

[27] S. Herr, D. Tataru, N. Tzvetkov, *Global well-posedness of the energy-critical nonlinear Schrödinger equation with small initial data in* $H^1(\mathbb{T}^3)$*,* Duke Math. J. **159** (2011), no. 2, 329–349.

[28] M. Ikeda, T. Inui, *Some non-existence results for the semilinear Schrödinger equation without gauge invariance,* J. Math. Anal. Appl. **425** (2015), no. 2, 758–773.

[29] M. Ikeda, T. Inui, *Small data blow-up of L^2 or H^1-solution for the semilinear Schrödinger equation without gauge invariance,* J. Evol. Equ. **15** (2015), no. 3, 571–581.

[30] M. Ikeda, Y. Wakasugi, *Small-data blow-up of L^2-solution for the nonlinear Schrödinger equation without gauge invariance,* Differential Integral Equations **26** (2013), no. 11-12, 1275–1285.

[31] A. Ionescu, B. Pausader, *Global well-posedness of the energy-critical defocusing NLS on* $\mathbb{R} \times \mathbb{T}^3$, Comm. Math. Phys. **312** (2012), no. 3, 781–831.

[32] J. P. Kahane, *Some Random Series of Functions,* Second edition. Cambridge Studies in Advanced Mathematics, 5. Cambridge University Press, Cambridge, 1985. xiv+305 pp.

[33] M. Keel, T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), no. 5, 955–980.

[34] R. Killip, M. Vişan, *The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher,* Amer. J. Math. **132** (2010), no. 2, 361–424.

[35] R. Killip, T. Oh, O. Pocovnicu, M. Vişan, *Global well-posedness of the Gross-Pitaevskii and cubic-quintic nonlinear Schrödinger equations with non-vanishing boundary conditions,* Math. Res. Lett. 19 (2012), no. 5, 969–986.

[36] R. Killip, J. Murphy, M. Visan, *Almost sure scattering for the energy-critical NLS with radial data below* $H^1(\mathbb{R}^4)$, Comm. Partial Differential Equations.

[37] H. Koch, D. Tataru, *A priori bounds for the 1D cubic NLS in negative Sobolev spaces,* Int. Math. Res. Not. **2007**, no. 16, Art. ID rnm053, 36 pp.

[38] J. Lührmann, D. Mendelson, *Random data Cauchy theory for nonlinear wave equations of power-type on* $\mathbb{R}^3$*,* Comm. Partial Differential Equations **39** (2014), no. 12, 2262–2283.

[39] J. Lührmann, D. Mendelson, *On the almost sure global well-posedness of energy sub-critical nonlinear wave equations on* $\mathbb{R}^3$, New York J. Math. **22** (2016), 209–227.

[40] H.P. McKean, *Statistical mechanics of nonlinear wave equations. IV. Cubic Schrödinger,* Comm. Math. Phys. **168** (1995), no. 3, 479–491. *Erratum: "Statistical mechanics of nonlinear wave equations. IV. Cubic Schrödinger",* Comm. Math. Phys. **173** (1995), no. 3, 675.

[41] T. Oh, *A blowup result for the periodic NLS without gauge invariance*, C. R. Math. Acad. Sci. Paris **350** (2012), no. 7-8, 389–392.

[42] T. Oh, *On nonlinear Schrödinger equations with almost periodic initial data,* SIAM J. Math. Anal. **47** (2015), no. 2, 1253–1270.

[43] T. Oh, *A remark on norm inflation with general initial data for the cubic nonlinear Schrödinger equations in negative Sobolev spaces*, Funkcial. Ekvac. 60 (2017) 259–277.

[44] T. Oh, O. Pocovnicu, *Probabilistic global well-posedness of the energy-critical defocusing quintic nonlinear wave equation on* $\mathbb{R}^3$, J. Math. Pures Appl. **105** (2016), 342–366.

[45] T. Oh, J. Quastel, *On Cameron-Martin theorem and almost sure global existence*, Proc. Edinb. Math. Soc. **59** (2016), 483–501.

[46] T. Oh, N. Tzvetkov, Y. Wang, *Solving the 4NLS with white noise initial data*, preprint.

[47] T. Ozawa, Y. Tsutsumi, *Space-time estimates for null gauge forms and nonlinear Schrödinger equations,* Differential Integral Equations **11** (1998), no. 2, 201–222.

[48] R.E.A.C. Paley, A. Zygmund, *On some series of functions (1), (2), (3)*, Proc. Camb. Philos. Soc. **26** (1930), 337–357, 458–474; **28** (1932), 190–205.

[49] O. Pocovnicu, *Almost sure global well-posedness for the energy-critical defocusing nonlinear wave equation on* $\mathbb{R}^d$*,* $d = 4$ *and* 5*,* J. Eur. Math. Soc. (JEMS) **19** (2017), 2321–2375.

[50] R. S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44** (1977), no. 3, 705–714.

[51] C. Sun, B. Xia, *Probabilistic well-posedness for supercritical wave equation on* $\mathbb{T}^3$*,* Illinois J. Math. 60 (2016), no. 2, 481–503.

[52] T. Tao, M. Vişan, *Stability of energy-critical nonlinear Schrödinger equations in high dimensions,* Electron. J. Differential Equations **2005** (2005), 28 pp.

[53] T. Tao, M. Vişan, X. Zhang, *The nonlinear Schrödinger equation with combined power-type nonlinearities,* Comm. Partial Differential Equations **32** (2007), no. 7-9, 1281–1343.

[54] L. Thomann, *Random data Cauchy problem for supercritical Schrödinger equations.* Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 6, 2385–2402.

[55] M. Vişan, *The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions,* Duke Math. J. **138** (2007), no. 2, 281–374.

[56] K. Yajima, *Existence of solutions for Schrödinger evolution equations*, Comm. Math. Phys. **110** (1987), no. 3, 415–426.

[57] Q.S. Zhang, *Blow-up results for nonlinear parabolic equations on manifolds,* Duke Math. J. **97** (1999) 515–539.

[58] Q.S. Zhang, *A blow-up result for a nonlinear wave equation with damping: the critical case,* C. R. Acad. Sci. Paris, Ser. I **333** (2001) 109–114.

[59] T. Zhang, D. Fang, *Random data Cauchy theory for the generalized incompressible Navier-Stokes equations*, J. Math. Fluid Mech. **14** (2012), no. 2, 311–324.

Tadahiro Oh, School of Mathematics, The University of Edinburgh, and The Maxwell Institute for the Mathematical Sciences, James Clerk Maxwell Building, The King's Buildings, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, United Kingdom

Email address: hiro.oh@ed.ac.uk

Mamoru Okamoto, Division of Mathematics and Physics, Faculty of Engineering, Shinshu University, 4-17-1 Wakasato, Nagano City 380-8553, Japan

Email address: m_okamoto@shinshu-u.ac.jp

Oana Pocovnicu, Department of Mathematics, Heriot-Watt University and The Maxwell Institute for the Mathematical Sciences, Edinburgh, EH14 4AS, United Kingdom

Email address: o.pocovnicu@hw.ac.uk