Tangent measures of elliptic harmonic measure and applications

TANGENT MEASURES OF ELLIPTIC MEASURE AND APPLICATIONS

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ABSTRACT. Tangent measure and blow-up methods, are powerful tools for understanding the relationship between the infinitesimal structure of the boundary of a domain and the behavior of its harmonic measure. We introduce a method for studying tangent measures of elliptic measures in arbitrary domains associated with (possibly non-symmetric) elliptic operators in divergence form whose coefficients have vanishing mean oscillation at the boundary. In this setting, we show the following for domains $\Omega \subset \mathbb{R}^{n+1}, n \geq 2$:

(1) We extend the results of Kenig, Preiss, and Toro [KPT09] by showing mutual absolute continuity of interior and exterior elliptic measures for *any* domains implies the tangent measures are a.e. flat and the elliptic measures have dimension n.
(2) We generalize the work of Kenig and Toro [KT06] and show that VMO equivalence of doubling interior and exterior elliptic measures for general domains implies the tangent measures are always supported on the zero sets of elliptic polynomials.
(3) In a uniform domain that satisfies the capacity density condition and whose boundary is locally finite and has a.e. positive lower n-Hausdorff density, we show that if the elliptic measure is absolutely continuous with respect to n-Hausdorff measure then the boundary is rectifiable. This generalizes the work of Akman, Badger, Hofmann, and Martell [ABHM17].

CONTENTS

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1. Introduction

1.1. **Background.** In this paper, we study how the relationships between the elliptic measures of two complementary domains in $\mathbb{R}^{n+1}$, for $n \geq 2$, dictate the geometry of their common boundaries. We shall denote those domains by Ω^+ and Ω^- and the respective elliptic measures by ω^+ and ω^-. In [BCGJ89], Bishop, Carleson, Garnett and Jones showed that, for disjoint simply connected planar domains with mutually absolutely continuous harmonic measures, the boundary had tangents on a set of positive measure. In [KPT09], Kenig, Preiss, and Toro showed that if $\Omega^\pm$ are both nontangentially accesible (or NTA) domains in $\mathbb{R}^{n+1}$ (with $n \geq 2$) and the interior and exterior harmonic measures are mutually absolutely continuous, then at every point of the common boundary except for a set of harmonic measure zero, $\partial\Omega^+$ looks flatter and flatter as we zoom in. We will not define NTA but refer the reader to its inception in [JK82]. Recently, the authors of the current paper, along with Tolsa [AMT16], as well as with Tolsa and Volberg [AMTV16], showed that additionally the boundary is n-rectifiable in the sense that, off a set of harmonic measure zero, the boundary is a union of Lipschitz images of $\mathbb{R}^{n+1}$, and in fact Ω^+ and Ω^- need not be NTA but just connected.

These are, however, almost everywhere phenomena, so it is interesting to ask what assumptions we need on $\omega^{\pm}$ to guarantee some nice limiting behavior of our blowups at *every* point. In [KT06], Kenig and Toro showed that if Ω^+ is 2-sided NTA and $\log \frac{d\omega^-}{d\omega^+} \in \mathrm{VMO}(d\omega^+)$, then as we zoom in on any point of the boundary for a particular sequence of scales, $\partial\Omega^+$ begins to look more and more like the zero set of a harmonic polynomial (see Section 7 for the definition of VMO). In [Bad11], Badger further showed that these harmonic polynomials are always homogeneous, and later in [Bad13] investigated the topological properties of sets where the boundary is approximated by zero sets of harmonic polynomials in this way.

To explain these results in more detail, we need to discuss what we mean by "blowups" and what it means for these to look like not necessarily one object but any one of a class of objects as we zoom in on harmonic measure. There are two ways we can consider this. Firstly, we can look at the Hausdorff convergence of rescaled copies of the support of a measure as we zoom in. To do this, we follow the framework of Badger amd Lewis [BL15].

Definition 1.1. Let $A \subset \mathbb{R}^{n+1}$ be a set. For $x \in A$, $r > 0$, and $\mathscr{S}$ a collection of sets, define

$$\Theta_A^{\mathscr{S}}(x,r) = \inf_{S\in\mathscr{S}} \max\left\{ \sum_{a\in A\cap B(x,r)} \frac{\mathrm{dist}(a, x+S)}{r}, \sum_{z\in(x+S)\cap B(x,r)} \frac{\mathrm{dist}(z,A)}{r} \right\}.$$

We say $x \in A$ is a $\mathscr{S}$ *point* of A if $\lim_{r\to 0}\Theta_A^{\mathscr{S}}(x,r) = 0$. We say A is *locally bilaterally well approximated by* $\mathscr{S}$ (or simply $LBWA(\mathscr{S})$) if for all $\varepsilon > 0$ and all compact sets $K \subset A$, there is $r_{\varepsilon,K} > 0$ such that $\Theta_A^{\mathscr{S}}(x,r) < \varepsilon$ for all $x \in K$ and $0 < r < r_{\varepsilon,K}$.

Thus, for $x \in A$ to be a $\mathscr{S}$ point means that, as we zoom in on A at the point x, the set A resembles more and more like an element of $\mathscr{S}$ (though that element may change as we zoom in).

Secondly, we can look at the weak convergence of rescaled copies of the measure itself. To do this, we follow the framework of Preiss in [Pre87]. For $a \in \mathbb{R}^{n+1}$ and $r > 0$, set

$$T_{a,r}(x) = \frac{x-a}{r}.$$

Note that $T_{a,r}(B(a,r)) = B(0,1)$. Given a Radon measure μ, the notation $T_{a,r}[\mu]$ is the image measure of μ by $T_{a,r}$, that is,

$$T_{a,r}[\mu](A) = \mu(rA+a), \qquad A \subset \mathbb{R}^{n+1}.$$

Here and later, for a function f and a measure μ, we write $f[\mu]$ to denote the push-forward measure measure $f[\mu](A) = \mu(f^{-1}(A))$.

Definition 1.2. We say that ν is a *tangent measure* of μ at a point $a \in \mathbb{R}^{n+1}$ if ν is a non-zero Radon measure on $\mathbb{R}^{n+1}$ and there are sequences $c_i > 0$ and $r_i \downarrow 0$ so that $c_i T_{a,r_i}[\mu]$ converges weakly to ν as $i \to \infty$ and write $\nu \in \mathrm{Tan}(\mu, a)$.

That is, ν is a tangent measure of μ at a point ξ if, as we zoom in on μ at ξ for a sequence of scales, the rescaled μ converges weakly to ν.

The collections of measures and sets that we will consider are associated to zero sets of harmonic functions. Let H denote the set of harmonic functions vanishing at the origin, $P(k)$ denote the set of harmonic polynomials h of degree k such that $h(0) = 0$ and $F(k)$ the set of homogeneous polynomials of degree k. For $h \in H$, we define

$$\Sigma_h = \{h = 0\}, \quad \Omega_h = \{h > 0\},$$

and

$$\mathscr{H} = \{\omega_h : h \in H\}, \mathscr{P}(k) = \{\omega_h : h \in P(k)\}, \mathscr{F}(k) = \{\omega_h : h \in F(k)\},$$

where

$$\omega_h = -\nu_{\Omega_h} \cdot \nabla h d\sigma_{\Sigma_h}.$$

Also set

$$\mathscr{P}_\Sigma(k) = \{\Sigma_h : h \in P(1) \cup \cdots \cup P(k)\}, \quad \mathscr{F}_\Sigma(k) = \{\Sigma_h : h \in F(k)\}$$

and

$$\mathcal{H}_\Sigma = \{\Sigma_h : h \in H\}.$$

Here $\nu_{\Omega_h}(x)$ stands for the measure theoretic unit outward normal of Ω_h at $x \in \partial^*\Omega_h$, the reduced boundary of Ω_h. Since h is a harmonic function and thus, real analytic, which implies that Σ_h is an n-dimensional real analytic variety, Ω_h is a set of locally finite perimeter and one can prove that $\mathcal{H}^n(\partial\Omega_h \setminus \partial^*\Omega_h) = 0$, where $\mathcal{H}^n$ stands for the n-Hausdorff measure. Notice now that $\nu_{\Omega_h}(x)$ is defined at $\mathcal{H}^n$-almost every point of Σ_h and σ_{Σ_h} is the usual surface measure. For a detailed proof of this see [AMT16, p. 21].

In the rest of the paper we will be dealing with unbounded domains, i.e., open and connected sets in $\mathbb{R}^{n+1}$ with $n \geq 2$.

We summarize the best results to date. We first mention a result by the authors, Tolsa, and Volberg.

Theorem 1.3 ([AMT16], [AMTV16]). *Let $\Omega^\pm \subset \mathbb{R}^{n+1}$ be two disjoint domains and $\omega^\pm = \omega_{\Omega^\pm}^{x_\pm}$ for some $x_\pm \in \Omega^\pm$. If $\omega^\pm$ are mutually absolutely continuous on E, then for $\omega^\pm$-a.e. $\xi \in E$, $\mathrm{Tan}(\omega^\pm, \xi) \subset \mathscr{F}(1)$ and $\omega^+|_E$ can be covered up to a set of ω^+-measure zero by n-dimensional Lipschitz graphs. Furthermore, if $\partial\Omega^\pm$ are CDC, then $\lim_{r\to 0} \Theta_{\partial\Omega^+}^{\mathscr{F}_\Sigma(1)}(\xi, r) = 0$ for ω^+-a.e. $\xi \in E$.*

This was originally shown by Bishop, Carleson, Garnett, and Jones for simply connected planar domains [BCGJ89]. Later, Kenig, Preiss and Toro showed that, under the same assumptions, provided that the domain is also 2-sided locally NTA, it holds that $\dim \omega^+ = n$ (but not that ω^+ is rectifiable).

Below we summarize the results so far in the situation when Ω is 2-sided NTA and the interior and exterior harmonic measures are VMO equivalent, which brings together results and techniques from Badger [Bad11, Bad13] and Kenig and Toro [KT06].

Theorem 1.4. *Let $\Omega^+ \subset \mathbb{R}^{n+1}$ and $\Omega^- = \mathrm{ext}(\Omega^+)$ be NTA domains, and let $\omega^\pm$ be the harmonic measure in $\Omega^\pm$ with pole $x^\pm \in \Omega^\pm$. Assume that ω^+ and ω^- are mutually absolutely continuous and $f := \frac{d\omega^-}{d\omega^+}$ satisfies $\log f \in \mathrm{VMO}(d\omega^+)$. Then, there exists $d \in \mathbb{N}$ (depending on n and the NTA constants) such that the boundary $\partial\Omega^+$ is $LBWA(\mathscr{P}_\Sigma(d))$ and may be decomposed into sets $\Gamma_1, ..., \Gamma_d$ satisfying the following.*

(1) For $1 \le k \le d$, $\Gamma_k = \{\xi \in \partial\Omega^+ : \mathrm{Tan}(\omega^+, \xi) \subset \mathscr{F}(k)\}$.
(2) $\Gamma_1 \cup \cdots \cup \Gamma_d = \partial\Omega^+$.
(3) $\lim_{r\to 0} \Theta^{\mathscr{F}_\Sigma(k)}_{\partial\Omega^+}(\xi, r) = 0$ for $\xi \in \Gamma_k$.

The work of [BET17] studies the geometric structure of the set as well as the tangent measure structure using the conclusions of the results above. We refer to their work for more details.

1.2. **Blowups of elliptic measures.** In this paper, our objective is to recreate some parts of these results to a class of elliptic measures. Admittedly, there are more results that could be generalized to this setting, like Tsirelson's theorem (using the method of Tolsa and Volberg [TV16]), but we content ourselves with the present results to convey the flexibility of the method.

Let $\Omega \subset \mathbb{R}^{n+1}$ be open and $A = A(\cdot) = (a_{ij}(\cdot))_{1\le i,j\le n+1}$ be a matrix with real measurable coefficients in Ω. We say that A is a *uniformly elliptic matrix* in Ω with constant $\Lambda \ge 1$ and write $A \in \mathscr{A}$, if it satisfies the following conditions:

$$\Lambda^{-1}|\xi|^2 \le \langle A(x)\xi, \xi\rangle, \qquad \text{for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^{n+1}, \tag{1.1}$$

$$\langle A(x)\xi, \eta\rangle \le \Lambda|\xi||\eta|, \qquad \text{for a.e. } x \in \Omega \text{ and for all } \xi, \eta \in \mathbb{R}^{n+1}. \tag{1.2}$$

Notice that the matrix is *possibly non-symmetric,* and has variable coefficients. If $A \in \mathscr{A}$, we define a *uniformly elliptic operator* associated with A by

$$L_A = -\mathrm{div}(A(\cdot)\nabla).$$

We will let $\omega_\Omega^{A,x}$ denote the L_A-*harmonic measure* in Ω with pole at x (see Section 11 in [HKM06] for the definition), which we also call *elliptic* *measure*. It is clear that the transpose matrix of A, which we denote A^T, is also uniformly elliptic in Ω. Finally, a function $u : \Omega \to \mathbb{R}$ that satisfies the equation $L_A u = 0$ in the weak sense is called L_A-*harmonic*. We will denote $\mathscr{C}$ the subclass of $\mathscr{A}$ consisting of matrices with constant entries.

To make sense of tangent measures of an elliptic measure at a point ξ in its support, we need to assume that the coefficients A do not oscillate too much there on small scales.

Definition 1.5. Let $\Omega \subset \mathbb{R}^{n+1}$ and let L_A be an elliptic operator on Ω. For a compact set $K \subset \partial\Omega$, we will say that the coefficients of L_A have *vanishing mean oscillation on* K with respect to Ω (or just $L_A \in \mathrm{VMO}(\Omega, K)$) if

$$\lim_{r\to 0} \sup_{\xi \in K} \frac{1}{r^{n+1}} \inf_{C\in\mathscr{C}} \int_{B(\xi,r)\cap\Omega} |A(x) - C| dx = 0. \tag{1.3}$$

We also say the coefficients of L_A have VMO *at* $\xi \in \partial\Omega$ if

$$\lim_{r\to 0} \frac{1}{r^{n+1}} \inf_{C\in\mathscr{C}} \int_{B(\xi,r)\cap\Omega} |A(x) - C| dx = 0. \tag{1.4}$$

Much like the harmonic case, the tangent measures we will obtain are supported on zero sets of elliptic polynomials associated with an elliptic operator with constant coefficients. For a constant coefficient matrix A with real entries, we will denote by H_A the set of L_A-harmonic functions u vanishing at zero, i.e. those functions u for which

$$\int A\nabla u \nabla \varphi dx = 0 \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^{n+1}) \text{ and } u(0) = 0.$$

We also let $P_A(k)$ denote the set of L_A-harmonic polynomials of degree k vanishing at the origin, and $F_A(k) \subset P_A(k)$ the subset of homogeneous L_A-harmonic polynomials of degree k. When $A = I$, we will simply write $F(k)$, $P(k)$ and H in place of $F_A(k)$, $P_A(k)$ and H_A.

For $h \in H_A$, we will write

$$d\omega_h^A = -\nu_{\Omega_h} \cdot A\nabla h \, d\sigma_{\Sigma_h},$$

where σ_S stands for the surface measure on a surface S and ν is the outward normal vector at $x \in \partial^*\Omega_h$, the reduced boundary of Ω_h. Once more, we used that h is real analytic since A has constant coefficients and $L_A h = 0$ (see e.g. Proposition 11.3 in [Mi13]). Again, when A is the identity, we will drop the superscripts and, for example, write ω_h in place of ω_h^A. For $\mathscr{S} \subset \mathscr{C}$, we write

$$\mathscr{H}_{\mathscr{S}} = \{\omega_h^A : h \in H_A,\ A \in \mathscr{S}\}, \quad \mathscr{P}_{\mathscr{S}}(k) = \{\omega_h^A : h \in P_A(k),\ A \in \mathscr{S}\},$$

$$\mathscr{F}_{\mathscr{S}}(k) = \{\omega_h^A : h \in F_A(k),\ A \in \mathscr{S}\},$$

$$\mathscr{H}_A = \mathscr{H}_{\{A\}}, \quad \mathscr{P}_A = \mathscr{P}_{\{A\}}, \quad \mathscr{F}_A = \mathscr{F}_{\{A\}},$$

and define $\mathscr{H}_{\mathscr{S},\Sigma}$, $\mathscr{P}_{\mathscr{S},\Sigma}$, and $\mathscr{F}_{\mathscr{S},\Sigma}$ as we did before. Observe that $\mathscr{F}_{\mathscr{C}}(1) = \mathscr{F}_A(1) = \mathscr{F}(1)$ for any $A \in \mathscr{C}$.

Our results also recover some LBWA properties implied in previous results if we consider domains satisfying the Capacity Density Condition (CDC), whose complements also satisfy the CDC (see Definition 4.3 below) and whose associated elliptic measures are doubling. Examples of domains satisfying these conditions are NTA domains or, by [Mar79, Theorem 3.1], any uniform domain Ω for which there is $s > n-1$ such that $\mathcal{H}^s_\infty(B(\xi,r)\cap\partial\Omega)/r^s \geq c > 0$ for all $\xi \in \partial\Omega$ and $r > 0$ is a CDC domain.

Our first result extends the work of [KPT09] to the elliptic case, and for domains beyond NTA. First, recall the dimension of a measure μ is

For a Borel measure μ in $\mathbb{R}^{n+1}$, we define the *Hausdorff dimension of* μ is defined by

$$\dim(\mu) = \inf\{\dim(Z) : \quad \mu(\mathbb{R}^{n+1}\setminus Z) = 0\}.$$

In practice, it is easier to compute this dimension as follows. Define *lower* and *upper pointwise dimension* at a point $x \in \operatorname{supp}\mu$ to be

$$\underline{d}_\mu(x) = \liminf_{r\to 0}\frac{\log\mu(B(x,r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}.$$

The common value $\underline{d}_\mu(x) = \overline{d}_\mu(x) = d_\mu(x)$, if it exists, we call it *pointwise dimension* of μ at $x \in \operatorname{supp}\mu$. It is shown in [BW06, Proposition 3] that

$$\dim(\mu) = \operatorname{ess\,sup}\{\underline{d}_\mu(x) : x \in \Lambda\}.$$

Theorem I. Let $\Omega^\pm \subset \mathbb{R}^{n+1}$ be two disjoint domains and let L_A be a uniformly elliptic operator on $\Omega^+ \cup \Omega^-$. Let also $\omega^\pm = \omega^{L_A,x_\pm}_{\Omega^\pm}$ for some $x_\pm \in \Omega^\pm$ be the L_A-harmonic measures in the respective domains and L_A be in $\mathrm{VMO}(\Omega^+\cup\Omega^-,\xi)$ at ω^+-almost every $\xi \in E \subset \partial\Omega^+\cap\partial\Omega^-$ with respect to either $\Omega^\pm$. If $\omega^\pm$ are mutually absolutely continuous on E, then for $\omega^\pm$-a.e. $\xi \in E$, $\operatorname{Tan}(\omega^\pm,\xi) \subset \mathscr{F}(1)$ and $\dim\omega^\pm|_E = n$. Furthermore, if $\partial\Omega^\pm$ are CDC, then $\lim_{r\to 0}\Theta^{\mathscr{F}_\Sigma(1)}_{\partial\Omega^+}(\xi,r) = 0$ for ω^+-a.e. $\xi \in E$.

Kenig, Preiss, and Toro originally showed this if $\Omega^\pm$ were both NTA domains, and the dimension was computed by estimating the Hausdorff dimension directly from above and then using the monotonicity formula of Alt, Cafarelli, and Friedman [ACF84] to estimate it from below. The latter is not available for L-harmonic functions when L satisfies the VMO condition above. For this reason, we use instead the fact that the tangent measures are all flat, which forces $\omega^\pm$ to decay like a planar n-dimensional Hausdorff measure on small scales.

Assuming a VMO condition on the interior and exterior elliptic measures, we can also obtain the results of [KT06] and [Bad11] for elliptic measures on domains that do not have to be NTA. We first state a pointwise version of these.

Theorem II. Let Ω^+ be a domain in $\mathbb{R}^{n+1}$, $\Omega^- := \text{ext}(\Omega^+)$ be its exterior, and let L_A be a uniformly elliptic operator in $\Omega^+ \cup \Omega^-$. Denote $\omega^\pm$ the L_A-harmonic measures of $\Omega^\pm$ with poles at some points $x^\pm \in \Omega^\pm$, and assume that $\omega^\pm$ are mutually absolutely continuous with $f = \frac{d\omega^-}{d\omega^+}$. If for a fixed $\xi \in \partial\Omega^+ \cap \partial\Omega^+$ it holds that $L_A \in \text{VMO}(\Omega^+ \cup \Omega^-, \xi)$,

$$\lim_{r\to 0} \left(\fint_{B(\xi,r)} f d\omega^+ \right) \exp\left(-\fint_{B(\xi,r)} \log f d\omega^+ \right) = 1, \tag{1.5}$$

and $\text{Tan}(\omega^+, \xi) \neq \varnothing$, then $\text{Tan}(\omega^+, \xi) \subset \mathscr{F}_{\mathscr{C}}(k)$ for some k and

$$\limsup_{r\to 0} \frac{\omega^+(B(\xi, 2r))}{\omega^+(B(\xi, r))} < \infty. \tag{1.6}$$

If $\Omega^\pm$ have the CDC, then additionally

$$\lim_{r\to 0} \Theta_{\partial\Omega^+}^{\mathscr{F}_{\mathscr{C}},\Sigma(k)}(\xi, r) = 0.$$

It is well known that $\text{Tan}(\omega^+, \xi) \neq \varnothing$ whenever ω^+ satisfies the pointwise doubling condition (1.6). In our situation, however, we do not assume that, but we get it for free since $\mathscr{F}_{\mathscr{C}}(k)$ is compact (see [Bad11, Lemma 4.10] for the harmonic case and Theorem 3.4 below).

One might have guessed that a pointwise version of Theorem 1.4 would have assumed instead that

$$\lim_{r\to 0} \fint_{B(\xi,r)} \left| f - \fint_{B(\xi,r)} \log f d\omega^+ \right| d\omega^+ = 0,$$

but we were not able to show that this implied Theorem II. However, under certain conditions they are equivalent. We will discuss this matter in depth in Section 7 below.

Next, we state a global version.

Theorem III. Let $\Omega^\pm \subset \mathbb{R}^{n+1}$ be two disjoint domains in $\mathbb{R}^{n+1}$ with common boundary, and let L_A be a uniformly elliptic operator in $\Omega^+ \cup \Omega^-$ such that $L_A \in \text{VMO}(\Omega^+ \cup \Omega^-, \xi)$ at every $\xi \in \partial\Omega^+ \cap \partial\Omega^-$. Denote $\omega^\pm$ the L_A-harmonic measures of $\Omega^\pm$ with poles at some points $x^\pm \in \Omega^\pm$. If ω^+ is C-doubling, $\omega^\pm$ are mutually absolutely continuous, and $\log f = \log \frac{d\omega^-}{d\omega^+} \in \text{VMO}(d\omega^+)$, then there is d depending on n and the doubling constant so that, for every compact subset $K \subseteq \partial\Omega^+$,

$$\lim_{r\to 0} \sup_{\xi \in K} d_1(T_{\xi,r}[\omega^+], \mathscr{P}_{\mathscr{C}}(d)) = 0. \tag{1.7}$$

If additionally $\Omega^{\pm}$ are CDC domains, then for any compact set $K \subseteq \partial\Omega$,

$$\lim_{r\to 0} \sup_{\xi\in K} \Theta_{\partial\Omega^+}^{\mathscr{P}_{\mathscr{C},\Sigma}(d)}(\xi, r) = 0.$$

That is, $\partial\Omega^+ \in LBWA(\mathscr{P}_{\mathscr{C},\Sigma}(d))$.

See Section 3 for the definition of $d_1(\cdot, \mathscr{P}_{\mathscr{C}}(d))$, which is essentially a distance between measures and the set $\mathscr{P}_{\mathscr{C}}(d)$.

The proof of Theorem II involves some useful lemmas about tangent measures that may be of independent interest. Specifically, we refer the reader to Lemma 3.10.

Over the course of working on this manuscript, we also resolved a question left open in [Bad11] (see the discussion on page 861 of [Bad11]).

Proposition I. The d-cone $\mathscr{P}(k)$ has compact basis for each $k \in (0, n]$.

See Section 3 for the definition of compact bases. A consequence of this result is that we can improve on the following result of Badger.

Theorem 1.6 ([Bad11] Theorem 1.1)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be an NTA domain with harmonic measure ω and let $\xi \in \partial\Omega$. If $\mathrm{Tan}(\omega, \xi) \subset \mathscr{P}(d)$, then $\mathrm{Tan}(\omega, \xi) \subset \mathscr{F}(k)$ for some $k \leq d$.*

In the proof of this result, Badger relied on the NTA assumption to conclude that $\mathrm{Tan}(\omega, \xi)$ was compact. By using Proposition I (whose proof is rather short), the compactness of $\mathscr{F}(k)$ (to which much of the proof of Theorem 1.6 is dedicated), and a connectivity theorem of Preiss, we can improve this by showing that, to get the same conclusion, no a priori information about the geometry of ω is needed: it need not have been a harmonic measure, let alone one for an NTA domain:

Proposition II. Let ω be a Radon measure in $\mathbb{R}^{n+1}$ and $\xi \in \mathbb{R}^{n+1}$ such that $\mathrm{Tan}(\omega, \xi) \subset \mathscr{P}(k)$ for some integer k. If $\mathrm{Tan}(\omega, \xi) \cap \mathscr{F}(k) \neq \varnothing$ for some integer k, then $\mathrm{Tan}(\omega, \xi) \subset \mathscr{F}(k)$.

1.3. **Rectifiability and elliptic measure for uniform domains.** The blow-up arguments we use also have an application to studying the relationship between rectifiability and harmonic measure, a subject in which there have been a flurry of results in the last few years. For simply connected planar domains, the problem of when harmonic measure is absolutely continuous with respect to $\mathcal{H}^1$ is classical. Bishop and Jones showed in [BJ90] that, if Ω is simply connected, $\omega_\Omega^x \ll \mathcal{H}^1$ on the subset of any Lipschitz curve intersecting $\partial\Omega$. Conversely, Pommerenke showed in [Pom86] that if $\omega_\Omega \ll \mathcal{H}^1$ on a subset $E \subset \partial\Omega$, then that set can be covered by Lipschitz graphs up to a set of harmonic measure zero. In fact, a much earlier result of the

Riesz brothers says that any Jordan domain has harmonic measure and $\mathcal{H}^1$ mutually absolutely continuous if and only if the boundary is rectifiable (see [RR16] or [GM08, Chapter VI.1]).

In higher dimensions, the problem is more delicate. There are some examples of simply connected domains $\Omega \subset \mathbb{R}^{n+1}$ with n-rectifiable boundaries of finite $\mathcal{H}^n$-measure so that either $\omega_\Omega \not\ll \mathcal{H}^n$ or $\mathcal{H}^n \not\ll \omega_\Omega$ (see [Wu86], [Zie74]). David and Jerison showed in [DJ90] that mutual absolute continuity occured for NTA domains with Ahlfors-David regular boundaries. Building on that, Badger showed in [Bad12] that $\mathcal{H}^n \ll \omega_\Omega$ if Ω was an NTA domain whose boundary simply had locally finite $\mathcal{H}^n$ measure, although we showed with Tolsa that the converse relation $\omega_\Omega \ll \mathcal{H}^n$ could be false for such domains [AMT17].

However, in [AHM$^+$16], along with Hofmann, Martell, Mayboroda, Tolsa, and Volberg, we showed that for *any* domain $\Omega \subset \mathbb{R}^{n+1}$ and $E \subset \partial\Omega$ with $\omega_\Omega(E) > 0$ and $\mathcal{H}^n(E) < \infty$, if $\omega_\Omega \ll \mathcal{H}^n$ on E, then E may be covered up to ω_Ω measure zero by Lipschitz graphs. By a theorem of Wolff, harmonic measure in the plane lies on a set of σ-finite $\mathcal{H}^1$-measure, and so the assumption that $\mathcal{H}^1(E) < \infty$ is unnecessary in this case (although very necessary in higher dimensions due to the existence of Wolff snowflakes). With Akman, we developed a converse for domains $\Omega \subset \mathbb{R}^{n+1}$ with *big complements*, meaning

$$\mathcal{H}^n_\infty(B(\xi, r)\backslash\Omega) \geq cr^n \text{ for all } \xi \in \partial\Omega \text{ and } 0 < r < \operatorname{diam}\partial\Omega. \tag{1.8}$$

We showed that, for such domains, $\omega_\Omega \ll \mathcal{H}^n$ on the subset of any n-dimensional Lipschitz graph [AAM16], and hence, for these domains, we know that absolute continuity is equivalent to rectifiability of harmonic measure (versus rectifiability of the boundary).

There are fewer positive results concerning absolute continuity and rectifiability of *elliptic* measures. Even in the case of the half plane, without some extra assumptions on the behavior of the elliptic coefficients, elliptic measure can be singular [CFK81, Swe92, Wu94], and some sort of Dini condition on the coefficients near the boundary is needed [FJK84, FKP91]. In [KP01], for example, Kenig and Pipher considered the following condition.

Definition 1.7. Let $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. We will say that an elliptic operator $L = -\operatorname{div} A\nabla$ satisfies the *Kenig-Pipher condition (or KP-condition)* if $A = (a_{ij}(x))$ is a uniformly elliptic real matrix that has distributional derivatives such that

$$\varepsilon^L_\Omega(z) := \sup\{\delta(x)|\nabla a_{ij}(x)|^2 : x \in B(z, \delta(z))/2, \;\; 1 \leq i, j \leq n+1\} \tag{1.9}$$

is a Carleson measure in Ω, by which we mean that for all $x \in \partial\Omega$ and $r \in (0, \operatorname{diam} \partial\Omega)$,

$$\int_{B(x,r)\cap\Omega} \varepsilon_\Omega^L(z)dz \leq Cr^n.$$

In [KP01], they showed that for Lipschitz domains in $\mathbb{R}^{n+1}$, elliptic operators satisfying the KP-*condition* gave rise to elliptic measures which were A_∞-equivalent to surface measure. In fact, it was proved in [HMT16] that the same result can be obtained under the following more general assumptions on the coefficients:

$$(\widetilde{KP}) = \begin{cases} \nabla a_{ij} \in \mathrm{Lip}_{loc}(\Omega), \\ \|\delta_\Omega |\nabla a_{ij}|\|_{L^\infty(\Omega)} < \infty, \\ \delta(x)|\nabla a_{ij}(x)|^2 \text{ is a Carleson measure,} \end{cases} , \tag{1.10}$$

for $1 \leq i, j \leq n+1$. Akman, Badger, Hofmann, and Martell observed in [ABHM17, Section 3.2] that, using the same arguments in [DJ90], this result can be extended to NTA domains with Ahlfors-David regular boundaries. They used this fact to show that, on a *uniform domain* Ω (see Definition 9.1 below) with Ahlfors-David regular boundary, if L_A is a *symmetric* elliptic operator satisfying a local L^1 version of (1.9), i.e., $A \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and $\sup\{|\nabla a_{ij}(x)| : x \in B(z, \delta(z))/2, \quad 1 \leq i, j \leq n+1\}|$ is a Carleson measure with Carleson constant depending on the ball, then $\mathcal{H}^n \ll \omega_\Omega^L$ implies n-rectifiability of the boundary.

Using our blowup arguments, we can obtain the following improvement.

Theorem IV. Let $\Omega \subset \mathbb{R}^{n+1}$ be a uniform CDC domain so that $\mathcal{H}^n|_{\partial\Omega}$ is locally finite. Let $\omega_\Omega^{L_A}$ be the L_A-harmonic measure associated to a (possibly non-symmetric) elliptic operator satisfying (1.1) and (1.2). Let $E \subseteq \partial\Omega$ be a set with $\mathcal{H}^n(E) > 0$ such that $\mathcal{H}^n \ll \omega_\Omega^{L_A}$ on E and for $\mathcal{H}^n$-a.e. $\xi \in E$,

$$\theta_{\partial\Omega,*}^n(\xi, r) := \liminf_{r\to 0} \frac{\mathcal{H}^n(B(\xi, r) \cap \partial\Omega)}{(2r)^n} > 0$$

and A has vanishing mean oscillation at ξ. Then E is n-rectifiable.

Surprisingly, to get this improvement requires a very different set of techniques than originally considered in [ABHM17]. Let us point out that the argument therein uses the symmetry hypothesis on the coefficients in a significant way and does not seem easy to extend to the non-symmetric case unless one additionally assumes that $\mathcal{H}^n \ll \omega_\Omega^{L_{A^T}}$.

Having VMO coefficients $\mathcal{H}^n$-a.e. on $\partial\Omega$ is natural as it is implied by the Carleson condition considered in [ABHM17] and [KP01] by the following proposition:

Proposition III. Let $\Omega \subset \mathbb{R}^{n+1}$ be a uniform domain and suppose that A is an elliptic matrix satisfying (1.1) and (1.2) such that $A \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and, for some ball B_0 centered on $\partial\Omega$,

$$\int_{B_0} \delta(x)|\nabla a_{ij}(x)|^2 dx < \infty. \tag{1.11}$$

Then $L_A \in \mathrm{VMO}(\Omega, \xi)$ for $\mathcal{H}^n$-a.e. $\xi \in B_0 \cap \partial\Omega$.

Discussion of related results. Near the completion of this work, we learned that Toro and Zhao simultaneously had proved that $\mathcal{H}^n \ll \omega_\Omega$ implies rectifiability of the boundary if $\Omega \subseteq \mathbb{R}^{n+}$ is a uniform domain with Ahlfors-David n-regular boundary and the elliptic coefficients are in $W^{1,1}(\Omega)$ [TZ17]. They also exploit the vanishing oscillation of the coefficients at almost every boundary point (which they show is implied by the $W^{1,1}$ condition) in the context of uniform domains, though their proof is distinct by their use of pseudo-tangents and stopping-time arguments.

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2. Notation

We will write $a \lesssim b$ if there is $C > 0$ so that $a \leq Cb$ and $a \lesssim_t b$ if the constant C depends on the parameter t. We write $a \approx b$ to mean $a \lesssim b \lesssim a$ and define $a \approx_t b$ similarly.

3. Tangent Measures

3.1. Cones and Compactness.

Given two Radon measure μ and σ, we set

$$F_B(\mu, \sigma) = \sup_f \int f \, d(\mu - \sigma),$$

where the supremum is taken over all the 1-Lipschitz functions supported on B. For $r>0$, we write

$$F_r(\mu,\nu)=F_{B(0,r)}, \qquad F_r(\mu)=F_r(\mu,0)=\int (r-|z|)_+ d\mu.$$

A set of Radon measures $\mathscr{M}$ is a *d-cone* if $cT_{0,r}[\mu]\in\mathscr{M}$ for all $\mu\in\mathscr{M}$, $c>0$ and $r>0$. We say a d-cone has *closed (resp. compact) basis* if its basis $\{\mu\in\mathscr{M} : F_1(\mu)=1\}$ is closed (resp. compact) with respect to the weak topology.

For a d-cone $\mathscr{M}$, $r>0$, and μ a Radon measure with $0<F_r(\mu)<\infty$, we define the *distance* between μ and $\mathscr{M}$ as

$$d_r(\mu,\mathscr{M})=\inf\left\{F_r\left(\frac{\mu}{F_r(\mu)},\nu\right) : \nu\in\mathscr{M}, F_r(\nu)=1\right\}.$$

Lemma 3.1 ([KPT09] Section 2)**.** *Let μ be a Radon measure in $\mathbb{R}^{n+1}$ and $\mathscr{M}$ a d-cone. For $\xi\in\mathbb{R}^{n+1}$ and $r>0$,*

(1) $T_{\xi,r}[\mu](B(0,s))=\mu(B(\xi,sr))$,
(2) $\int f\,dT_{\xi,r}[\mu]=\int f\circ T_{\xi,r}\,d\mu$,
(3) $F_{B(\xi,r)}(\mu)=rF_1(T_{\xi,r}[\mu])$,
(4) $F_{B(\xi,r)}(\mu,\nu)=rF_1(T_{\xi,r}[\mu],T_{\xi,r}[\nu])$,
(5) $\mu_i\to\mu$ weakly if and only if $F_r(\mu_i,\mu)\to 0$ for all $r>0$,
(6) $d_r(\mu,\mathscr{M})\le 1$,
(7) $d_r(\mu,\mathscr{M})=d_1(T_{0,r}[\mu],\mathscr{M})$,
(8) if $\mu_i\to\mu$ weakly and $F_r(\mu)>0$, then $d_r(\mu_i,\mathscr{M})\to d_r(\mu,\mathscr{M})$.

Lemma 3.2 ([KPT09] Remark 2.13)**.** *A d-cone $\mathscr{M}$ of Radon measures in $\mathbb{R}^{n+1}$ has a closed basis if and only if it is a relatively closed subset of the non-zero Radon measures in $\mathbb{R}^{n+1}$.*

Proof. One direction is obvious, so suppose $\mathscr{M}$ has closed basis and $\mu_i\in\mathscr{M}$ converges weakly to some non-zero Radon measure μ. Then $F_r(\mu)>0$ for some $r>0$. The set $\{\nu\in\mathscr{M} : F_1(\nu)=1\}$ is closed by assumption, and since $\mathscr{M}$ is a d-cone, the set $\{\nu\in\mathscr{F} : F_r(\nu)=1\}$ is also closed. Hence, since $\mu_i/F_r(\mu_i)\to\mu/F_r(\mu)$, we know $\mu/F_r(\mu)\in\mathscr{M}$, and thus $\mu\in\mathscr{M}$. □

Lemma 3.3. *If μ is a nonzero Radon measure and $\mathscr{M}$ is a d-cone with closed basis, then $\mu\in\mathscr{M}$ if and only if $d_r(\mu,\mathscr{M})=0$ for all $r>0$ for which $F_r(\mu)>0$.*

Proof. Suppose $d_r(\mu,\mathscr{M})=0$ for all $r>0$ for which $F_r(\mu)>0$. For $j\in\mathbb{N}$ large enough, we can find a sequence $\mu_{j,k}\in\mathscr{M}$ such that

$$F_j(\mu_{j,k})=1 \text{ and } \lim_{k\to\infty}F_j\left(\frac{\mu}{F_j(\mu)},\mu_{j,k}\right)=0. \tag{3.1}$$

In particular, we can pass to a subsequence so that $\mu_{j,k}$ converges weakly in $B(0,j)$ to a measure μ_j supported in $B(0,j)$ with $F_j(\mu_j)=1$. In view of (3.1), the latter implies that $\mu=F_j(\mu)\,\mu_j$ in $B(0,j)$, and thus,

$$F_j(\mu)\,\mu_j \rightharpoonup \mu.$$

Since, $\mu_{j,k}\rightharpoonup\mu_j$ and $F_j(\mu)\neq 0$ for j large, we can pick k_j so that

$$F_j(\mu_{j,k_j},\mu_j)<\frac{1}{jF_j(\mu)}.$$

In particular, for any $r>0$ and $j>r$,

$$\begin{aligned}F_r\left(\mu_{j,k_j}F_j(\mu),\mu\right) &\leq F_r\left(\mu_{j,k_j}F_j(\mu),\mu_jF_j(\mu)\right)+F_r\left(\mu_jF_j(\mu),\mu\right)\\ &\leq F_j\left(\mu_{j,k_j}F_j(\mu),\mu_jF_j(\mu)\right)+F_r\left(\mu_jF_j(\mu),\mu\right)\\ &<\frac{1}{j}+F_r\left(\mu_jF_j(\mu),\mu\right)\to 0.\end{aligned}$$

Thus, $\mu_{j,k_j}F_j(\mu)\rightharpoonup\mu$. By Lemma 3.2, $\mathscr{M}$ is closed, and since we have $\mu_{j,k_j}F_j(\mu)\in\mathscr{M}$ for all j, this implies $\mu\in\mathscr{M}$. The other implication is trivial. $\square$

Theorem 3.4 ([Pre87] Corollary 2.7)**.** *Let μ be a Radon measure on $\mathbb{R}^{n+1}$, and $\xi\in\operatorname{supp}\mu$. Then $\operatorname{Tan}(\mu,\xi)$ has compact basis if and only if*

$$\limsup_{r\to 0}\frac{\mu(B(\xi,2r))}{\mu(B(\xi,r))}<\infty. \tag{3.2}$$

In this case, for any $\nu\in\operatorname{Tan}(\mu,\xi)$, it holds that $0\in\operatorname{supp}\nu$ and

$$\frac{\nu(B(0,2r))}{\nu(B(0,r))}\leq\limsup_{\rho\to 0}\frac{\mu(B(\xi,2\rho))}{\mu(B(\xi,\rho))},\ \textit{for all } r>0.$$

Lemma 3.5 ([Mat95] Theorem 14.3)**.** *Let μ be a Radon measure on $\mathbb{R}^{n+1}$. If $\xi\in\mathbb{R}^{n+1}$ and (3.2) holds, then every sequence $r_i\downarrow 0$ contains a subsequence such that*

$$\frac{T_{\xi,r_j}[\mu]}{\mu(B(\xi,r_j))}\rightharpoonup\nu, \tag{3.3}$$

for some measure $\nu\in\operatorname{Tan}(\mu,\xi)$.

Having tangent measures that arise as limits of the form (3.3) is very convenient, but this limit does not always converge weakly to something. This may happen if μ is not pointwise doubling at the point a. However, all tangent measures are at least dilations of tangent measures arising in this way.

Lemma 3.6 ([Mat95] Remark 14.4(1))**.** *Let μ be a nonzero Radon measure, $\xi \in \operatorname{supp}\mu$, and $\nu \in \operatorname{Tan}(\mu,\xi)$. Then there are $\rho_j \downarrow 0$ and $\rho, c > 0$ so that*

$$\frac{T_{\xi,\rho_j}[\mu]}{\mu(B(\xi,\rho_j))} \rightharpoonup c\,T_{0,\rho}[\nu] \quad \text{and} \quad c\,T_{0,\rho}[\nu](\mathbb{B}) > 0.$$

Proposition 3.7 ([Pre87] Proposition 2.2)**.** *Let $\mathscr{M}$ be a d-cone. Then $\mathscr{M}$ has compact basis if and only if for every $\lambda > 1$ there is $\tau > 1$ such that*

$$F_{\tau r}(\Psi) \leq \lambda F_r(\Psi) \text{ for every } \Psi \in \mathscr{M} \text{ and } r > 0. \tag{3.4}$$

In this case, $0 \in \operatorname{supp}\Psi$ for all $\Psi \in \mathscr{M}$.

Theorem 3.8. [Mat95, Theorem 14.16] *Let μ be a Radon measure on $\mathbb{R}^{n+1}$. For μ-almost every $x \in \mathbb{R}^{n+1}$, if $\nu \in \operatorname{Tan}(\mu,x)$, the following hold:*

(1) $T_{y,r}[\nu] \in \operatorname{Tan}(\mu,x)$ for all $y \in \operatorname{supp}\nu$ and $r > 0$.
(2) $\operatorname{Tan}(\nu,y) \subset \operatorname{Tan}(\mu,x)$ for all $y \in \operatorname{supp}\nu$.

Lemma 3.9. [Bad11, Lemma 2.6] *Let μ be a non-zero Radon measure on $\mathbb{R}^{n+1}$ and $x \in \operatorname{supp}(\mu)$. If $\nu \in \operatorname{Tan}(\mu,x)$, then $\operatorname{Tan}(\nu,0) \subset \operatorname{Tan}(\mu,x)$.*

3.2. **Connectivity of cones.** The main tool from [KPT09] and [Bad11] is the following "connectivity" lemma, which was originally shown in [KPT09, Corollary 2.16] under the assumption that $\mathscr{M}$ had compact basis. For our purposes, we need to remove this assumption.

Lemma 3.10. *Let $\mathscr{F}$ and $\mathscr{M}$ be d-cones and assume $\mathscr{F}$ has compact basis. Furthermore, suppose that there is $\varepsilon_0 > 0$ such that for $\mu \in \mathscr{M}$, if there is $r_0 > 0$ so that $d_r(\mu,\mathscr{F}) \leq \varepsilon$ for all $r \geq r_0$, then $\mu \in \mathscr{F}$. For a Radon measure η and $x \in \operatorname{supp}\eta$, if $\operatorname{Tan}(\eta,x) \subset \mathscr{M}$ and $\operatorname{Tan}(\eta,x) \cap \mathscr{F} \neq \varnothing$, then $\operatorname{Tan}(\eta,x) \subset \mathscr{F}$.*

We will first require some lemmas.

Lemma 3.11. *Let $\mathscr{F}$ be a d-cone with compact basis. There is $\beta > 0$ depending only on $\mathscr{F}$ so that the following holds. Suppose ω is a Radon measure in $\mathbb{R}^{n+1}$, $\xi \in \operatorname{supp}\omega$, $\operatorname{Tan}(\omega,\xi) \cap \mathscr{F} \neq \varnothing$ and*

$$\limsup_{r\to 0} d_{r_0}(T_{\xi,r}[\omega],\mathscr{F}) \geq \varepsilon_0 > 0, \text{ for some } r_0 > 0.$$

Then for $\varepsilon < \varepsilon_0$ small enough, we may find $\mu \in \operatorname{Tan}(\omega,\xi)\backslash\mathscr{F}$ so that

(1) $d_{r_0}(\mu,\mathscr{F}) = \varepsilon$,
(2) $d_r(\mu,\mathscr{F}) \leq \varepsilon$ for all $r > r_0$, and
(3) $\mu(B(0,r)) \leq r^\beta\, \mu(B(0,4r_0))$ for all $r \geq r_0$.

This is an adaptation of the proof of [KPT09, Corollary 2.16], but with some extra care.

Proof. Without loss of generality, we will assume $r_0 = 1$. Let $c_j > 0$ and $r_j \downarrow 0$ be such that $c_j T_{\xi,r_j}[\omega] \to \nu \in \mathscr{F}$. Since $\mathscr{F}$ is compact, by Proposition 3.7, $0 \in \operatorname{supp}\nu$ and so $\nu(\mathbb{B}) > 0$. Thus, by Lemma 3.1 (5), $c_j T_{\xi,r_j}[\omega](\mathbb{B}) > 0$ for j large. By Lemma 3.1 (8), we have that, given $\varepsilon > 0$, for j large enough,

$$d_1(T_{\xi,r_j}[\omega], \mathscr{F}) = d_1(c_j T_{\xi,r_j}[\omega], \mathscr{F}) < \varepsilon. \tag{3.5}$$

Note that $0 \in \operatorname{supp} T_{\xi,r_j}[\omega]$ since $\xi \in \operatorname{supp}\omega$, and so there is no accidental dividing by zero in the definition of d_1. By assumption, there is also $s_j \downarrow 0$ so that

$$d_1(T_{\xi,s_j}[\omega], \mathscr{F}) > \varepsilon. \tag{3.6}$$

We can assume $s_j < r_j$ by passing to a subsequence. Then by (3.5) and (3.6), let $\rho_j \in (s_j, r_j)$ be the maximal number such that

$$d_1(T_{\xi,\rho_j}[\omega], \mathscr{F}) = \varepsilon. \tag{3.7}$$

Then, by the maximality of ρ_j,

$$\sup_{t\in[\rho_j,r_j]} d_1(T_{\xi,t}[\omega], \mathscr{F}) \leq \varepsilon. \tag{3.8}$$

We claim $\rho_j/r_j \to 0$. If not, then since $\rho_j/r_j \leq 1$, we may pass to a subsequence so that $\rho_j/r_j \to t \in (0,1)$, and so

$$c_j T_{\xi,\rho_j}[\omega] = T_{0,\rho_j/r_j}\left[c_j T_{\xi,r_j}[\omega]\right] \to T_{0,t}[\nu] \in \mathscr{F},$$

which contradicts (3.7). Thus, $\rho_j/r_j \to 0$, and so (3.8) implies that for $\alpha \geq 1$, if j is large enough, we have $1 \leq \alpha < r_j/\rho_j$. If $\omega_j = T_{\xi,\rho_j}[\omega]$, then by Lemma 3.1 (7), it holds

$$d_\alpha(\omega_j, \mathscr{F}) = d_\alpha(T_{\xi,\rho_j}[\omega], \mathscr{F}) = d_1(T_{\xi,\alpha\rho_j}[\omega], \mathscr{F}) \overset{(3.8)}{\leq} \varepsilon, \tag{3.9}$$

which by (3.7) implies that

$$d_1(\omega_j, \mathscr{F}) = \varepsilon > 0 \quad \text{and} \quad \limsup_{j\to\infty} d_r(\omega_j, \mathscr{F}) \leq \varepsilon \text{ for } r > 1. \tag{3.10}$$

For $r \geq 1$, let $\mu_{j,r} \in \mathscr{F}$ be such that $F_{\tau r}(\mu_{j,r}) = 1$ and

$$F_{\tau r}\left(\frac{\omega_j}{F_{\tau r}(\omega_j)}, \mu_{j,r}\right) < \frac{3}{2} d_{\tau r}(\omega_j, \mathscr{F}).$$

By (3.10), for j large enough,

$$F_r\left(\frac{\omega_j}{F_{\tau r}(\omega_j)}, \mu_{j,r}\right) \leq F_{\tau r}\left(\frac{\omega_j}{F_{\tau r}(\omega_j)}, \mu_{j,r}\right) < \frac{3}{2} d_{\tau r}(\omega_j, \mathscr{F}) < 2\varepsilon. \tag{3.11}$$

Since $\mathscr{F}$ has compact basis, by Proposition 3.7 with $\lambda = 2$, there is $\tau > 1$ depending only on $\mathscr{F}$ so that (3.4) holds for $\mathscr{M} = \mathscr{F}$. Thus, if $\varepsilon < 1/8$, by the triangle inequality for F_r and (3.11),

$$\frac{F_r(\omega_j)}{F_{\tau r}(\omega_j)} \geq F_r(\mu_{j,r}) - 2\varepsilon \geq \frac{1}{2} F_{\tau r}(\mu_{j,r}) - 2\varepsilon = \frac{1}{2} - 2\varepsilon > \frac{1}{4}. \tag{3.12}$$

Hence, for any $r \geq 1$,

$$F_{\tau r}(\omega_j) \leq 4 F_r(\omega_j).$$

Set $\mu_j = \omega_j / F_1(\omega_j)$. Then iterating the above inequality and letting $j \to \infty$, we get that for all $\ell \in \mathbb{N}$,

$$\limsup_{j\to\infty} F_{\tau^\ell}(\mu_j) \leq 4^\ell.$$

This implies that we can pass to a subsequence so that μ_j converges weakly to a measure $\mu \in \mathrm{Tan}(\omega, \xi)$. In particular, for $r \geq 1$, since $F_1(\mu_j) = 1$, we may compute

$$d_1(\mu, \mathscr{F}) = \lim_{j\to\infty} d_1(\mu_j, \mathscr{F}) = \lim_{j\to\infty} d_1(\omega_j, \mathscr{F}) \overset{(3.10)}{=} \varepsilon,$$

$$d_r(\mu, \mathscr{F}) = \lim_{j\to\infty} d_r(\mu_j, \mathscr{F}) = \lim_{j\to\infty} d_r(\omega_j, \mathscr{F}) \overset{(3.10)}{\leq} \varepsilon,$$

and

$$\tau^\ell \mu(B(0,\tau^\ell)) \leq F_{2\tau^\ell}(\mu) \leq 4^\ell F_2(\mu) \text{ for all } \ell \in \mathbb{N}. \tag{3.13}$$

Since $\tau > 1$, for any $r \geq 1$, there exists $\ell > 0$ such that $\tau^{\ell-1} < r \leq \tau^\ell$. If $\tau \in (1,4)$, (3.13) implies

$$\tau^\ell\, \mu(B(0,\tau^\ell)) \leq \tau^\alpha r^\alpha \mu(\overline{B(0,2)}),$$

where $\alpha = \frac{1}{\log_4 \tau} \in (1,\infty)$ and we used that $4^\ell = \tau^{\ell\alpha}$. Therefore,

$$\mu(B(0,r)) \leq \tau^{\alpha-\ell}\, r^\alpha\, \mu(\overline{B(0,2)}),$$

and notice that $\tau^{\alpha-\ell} \leq 1$ whenever $\tau^\ell \geq 4$, i.e., the constant is independent of τ. In the case that $1 \leq r \leq \tau^\ell < 4$, we simply use that $B(0,r) \subset B(0,4)$ to conclude that

$$\mu(B(0,r)) \leq \mu(B(0,4)).$$

If $\tau \geq 4$, then (3.13) trivially gives

$$\tau^\ell \mu(B(0,\tau^\ell)) \leq 4^\ell \mu(\overline{B(0,2)}) \leq \tau^\ell\, \mu(\overline{B(0,2)}),$$

which can only be true if $r \leq \tau^\ell \leq 2$. Thus, $B(0,r) \subset B(0,2)$ and (3) readily follows. $\square$

Corollary 3.12. *Let $\mathscr{F}$ be a d-cone with compact basis. There is $\beta > 0$ so that the following holds. Suppose μ is a Radon measure in $\mathbb{R}^{n+1}$ so that*

(1) $\mathrm{Tan}(\mu, \xi) \cap \mathscr{F} \neq \varnothing$ *and*

(2) $\operatorname{Tan}(\mu,\xi)\backslash\mathscr{F}\neq\varnothing$.

Then there is $r_0>0$ so that for any $\varepsilon>0$ sufficiently small, the conclusion of Lemma 3.11 holds.

Proof. Let $\nu\in\operatorname{Tan}(\mu,\xi)\backslash\mathscr{F}$. By Lemma 3.3, there exists $r_0>0$ so that $F_{r_0}(\nu)>0$ and $d_{r_0}(\nu,\mathscr{F})>0$. Let $c_j>0$ and $r_j\downarrow 0$ be so that $c_jT_{\xi,r_j}[\mu]\to\nu$. Then, for j large enough, $d_{r_0}(T_{\xi,r_j}[\mu],\mathscr{F})>d_{r_0}(\nu,\mathscr{F})/2>0$. The corollary now follows from Lemma 3.11 with $\varepsilon_0=d_{r_0}(\nu,\mathscr{F})/2$. □

Proof of Lemma 3.10. If $\operatorname{Tan}(\eta,x)\backslash\mathscr{F}\neq\varnothing$, then, by Corollary 3.12, we may find $\mu\in\operatorname{Tan}(\eta,x)\setminus\mathscr{F}$ and $\varepsilon,r_0>0$ so that $d_{r_0}(\mu,\mathscr{F})=\varepsilon$ and $d_r(\mu,\mathscr{F})\leq\varepsilon$ for all $r>r_0$. By assumption, this implies $\mu\in\mathscr{F}$, which is a contradiction. Thus, $\operatorname{Tan}(\eta,x)\subset\mathscr{F}$. □

4. Elliptic Measures

4.1. **Uniformly elliptic operators in divergence form.** Let A be a real matrix with measurable coefficients that satisfies (1.1) and (1.2). We consider the second order elliptic operator $L=-\operatorname{div}A\nabla$ and we say that a function $u\in W^{1,2}_{loc}(\Omega)$ is a *weak solution* of the equation $Lu=0$ in Ω (or just *L-harmonic*) if

$$\int A\nabla u\cdot\nabla\varphi=0,\quad\text{for all }\varphi\in C_0^\infty(\Omega).\tag{4.1}$$

We also say that $u\in W^{1,2}_{loc}(\Omega)$ is a *supersolution* (resp. *subsolution*) for L in Ω or just *L-superharmonic* (resp. *L-subharmonic*) if $\int A\nabla u\nabla\varphi\geq 0$ (resp. $\int A\nabla u\nabla\varphi\leq 0$) for all non-negative $\varphi\in C_0^\infty(\Omega)$.

In this section, we assume $n\geq 2$.

4.2. **Regularity of the domain and Dirichlet problem.** We say that a point $x_0\in\partial\Omega$ is *Sobolev L-regular* if, for each function $\varphi\in W^{1,2}(\Omega)\cap C(\overline{\Omega})$, the L-harmonic function h in Ω with $h-\varphi\in W_0^{1,2}(\Omega)$ satisfies

$$\lim_{x\to x_0}h(x)=\varphi(x_0).$$

Theorem 4.1 (Theorem 6.27 in [HKM06])**.** *If for $x_0\in\partial\Omega$ it holds that*

$$\int_0^1\frac{\operatorname{cap}(B(x_0,r)\cap\Omega^c,B(x_0,2r))}{\operatorname{cap}(B(x_0,r),B(x_0,2r))}\,\frac{dr}{r}=+\infty,$$

then x_0 is Sobolev L-regular. Here $\operatorname{cap}(\cdot,\cdot)$ stands for the variational 2–capacity of the condenser $(\cdot,\cdot)$ (see e.g. [HKM06, p. 27]*).*

We say that a point $x_0 \in \partial\Omega$ is *Wiener regular* if, for each function $f \in C(\partial\Omega;\mathbb{R})$, the L-harmonic function H_f constructed by the Perron's method satisfies

$$\lim_{x\to x_0} H_f(x) = f(x_0).$$

See [HKM06, Chapter 9].

Lemma 4.2 (Theorem 9.20 in [HKM06])**.** *Suppose that $x_0 \in \partial\Omega$. If x_0 is Sobolev L-regular then it is also Wiener regular.*

The aforementioned result from [HKM06] is only stated for Ω bounded but in fact it holds for unbounded domains, since the only part of the proof that requires the domain to be bounded is the existence of a unique solution of the Dirichlet problem with Sobolev Dirichlet data in bounded domains. This is true though in the unbounded case as well. See e.g. on p. 11 in [AGMT17] where this is shown. Moreover, ∞ is also a Wiener regular point for each unbounded $\Omega \subset \mathbb{R}^{n+1}$, if $n \geq 2$ (see e.g. Theorem 9.22 in [HKM06]).

We say that Ω is Sobolev L-regular (resp. Wiener regular) if all the points in $\partial\Omega$ are Sobolev L-regular (resp. Wiener regular).

Definition 4.3. A domain $\Omega \subset \mathbb{R}^{n+1}$ is called *regular* if every point of $\partial\Omega$ is regular (i.e., if the classical Dirichlet problem is solvable in Ω for the elliptic operator $\mathcal{L}$), where $\partial\Omega$ denotes the boundary of Ω. For $K \subset \partial\Omega$, we say that Ω has the *capacity density condition (CDC)* if, for all $x \in \partial\Omega$ and $0 < r < \operatorname{diam}\partial\Omega$,

$$\operatorname{cap}(B(x,r)\cap\Omega^c, B(x,2r)) \gtrsim r^{n-1}.$$

Note that if $n \geq 2$, by Wiener's criterion, domains satisfying the CDC are both Wiener regular and L-Sobolev regular.

Let $\Omega \subset \mathbb{R}^{n+1}$ be Wiener regular and $x \in \Omega$. If $f \in C(\partial\Omega)$, then the map $f \mapsto \overline{H}_f(x)$ is a bounded linear functional on $C(\partial\Omega)$. Therefore, by Riesz representation theorem and the maximum principle, there exists a probability measure ω^x on $\partial\Omega$ (associated to L and the point $x \in \Omega$) defined on Borel subsets of $\partial\Omega$ so that

$$\overline{H}_f(x) = \int_{\partial\Omega} f\, d\omega^x, \quad \text{for all } x \in \Omega.$$

We call ω^x the *elliptic measure* or *L-harmonic measure* associated to L and x.

4.3. **Green function and PDE estimates.**

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open, connected set so that $\partial\Omega$ is Sobolev L-regular. There exists a Green function $G : \Omega \times \Omega \setminus \{(x,y) : x = y\} \to \mathbb{R}$ associated with L which satisfies the following. For $0 < a < 1$, there are are positive constants C and c depending on a, n and Λ such that for all $x, y \in \Omega$ with $x \neq y$, it holds:*

$$0 \leq G(x,y) \leq C\,|x-y|^{1-n}$$

$$G(x,y) \geq c\,|x-y|^{1-n} \quad \text{if } |x-y| \leq a\,\delta_\Omega(x),$$

$$G(x,\cdot) \in C(\overline{\Omega} \setminus \{x\}) \cap W^{1,2}_{loc}(\Omega \setminus \{x\}) \quad \text{and} \quad G(x,\cdot)|_{\partial\Omega} \equiv 0,$$

$$G(x,y) = G^T(y,x),$$

where G^T is the Green function associated with the operator L_{A^T}, and for every $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$,

$$\int_{\partial\Omega} \varphi\, d\omega^x - \varphi(x) = -\int_\Omega A^T(y)\nabla_y G(x,y) \cdot \nabla\varphi(y)\, dy, \quad \text{for a.e. } x \in \Omega. \tag{4.2}$$

In the statement of (4.2), one should understand that the integral on right hand side is absolutely convergent for a.e. $x \in \Omega$ and a proof of it can be found in Lemma 2.6 in [AGMT17]. The rest were proved in [GW82] and [HK07].

The lemma below is frequently called Bourgain's Lemma, as he proved a similar estimate for harmonic measure in [Bou87].

Lemma 4.5 ([HKM06, Lemma 11.21])**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be any domain satisfying the CDC condition, $x_0 \in \partial\Omega$, and $r > 0$ so that $\Omega \backslash B(x_0, 2r) \neq \varnothing$. Then*

$$\omega_\Omega^{L,x}(B(x_0,2r)) \geq c > 0 \quad \text{for all } x \in \Omega \cap B(x_0,r), \tag{4.3}$$

where c depends on d and the constant in the CDC.

Lemma 4.6. *For $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, and the assumptions of Lemma 4.4, if B is centered on $\partial\Omega$, then*

$$G(x,y)\, r_B^{n-1} \inf_{z \in 2B} \omega^{L,z}(4B) \lesssim \omega^{L,y}(4B) \text{ for } x \in B \cap \Omega \text{ and } y \in \Omega \backslash 2B. \tag{4.4}$$

In particular, for a CDC domain, we have

$$G(x,y) r_B^{n-1} \lesssim \omega^{L,y}(4B) \text{ for } x \in B \cap \Omega \text{ and } y \in \Omega \backslash 2B.$$

Proof. This was originally shown for harmonic measure in [AHM+16], but we cover the details here.

By Bourgain's estimate, $\omega^{L,y}(4B) \gtrsim 1$ for $y \in 2B \cap \Omega$, and so for $y \in \Omega \backslash 2B$ and $x \in B \cap \Omega$

$$\inf_{z\in 2B} \omega^{L,z}(4B) G(x,y) r_B^{n-1} \lesssim \frac{\inf_{z\in 2B} \omega^{L,z}(4B)}{|x-y|^{n-1}} r_B^{n-1} \lesssim \inf_{z\in 2B} \omega^{L,z}(4B)$$

and since $G(x,\cdot)$ vanishes on $\partial\Omega$, we thus have that, for some constant $C > 0$,

$$\limsup_{y\to\xi} C\omega^{L,y}(4B) - \inf_{z\in 2B} \omega^{L,z}(4B) G(x,y) r_B^{n-1} \geq 0 \quad \text{for all } \xi \in \partial(\Omega\backslash 2B)$$

and so (4.4) follows from the maximum principle [HKM06, Theorem 11.9]. □

By an iteration argument using Lemma 4.5, one can obtain the following lemma.

Lemma 4.7. *Let $\Omega \subsetneq \mathbb{R}^{n+1}$ be open with the CDC. Let $x \in \partial\Omega$ and $0 < r < \operatorname{diam}\Omega$. Let u be a non-negative L-harmonic function in $B(x,4r)\cap\Omega$ and continuous in $B(x,4r)\cap\overline{\Omega}$ so that $u \equiv 0$ in $\partial\Omega\cap B(x,4r)$. Then extending u by 0 in $B(x,4r)\setminus\overline{\Omega}$, there exists a constant $\alpha > 0$ such that*

$$u(y) \leq C\left(\frac{\delta_\Omega(y)}{r}\right)^\alpha \sup_{B(x,2r)} u \quad \text{for all } y \in B(x,r), \tag{4.5}$$

where C and α depend on n, Λ and the CDC constant, and $\delta_\Omega(y) = \operatorname{dist}(y,\Omega^c)$. In particular, u is α-Hölder continuous in $B(x,r)$.

The following lemma is standard but we provide a proof for the sake of completeness.

Lemma 4.8. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and assume that A is an elliptic matrix and $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a bi-Lipschitz map. Set*

$$\widetilde{A} := |\det D_\Phi| D_{\Phi^{-1}}(A\circ\Phi) D_{\Phi^{-1}}^T.$$

Then u is a weak solution of $L_A u = 0$ in $\Phi(\Omega)$ if and only if $\tilde{u} = u\circ\Phi$ is a weak solution of $L_{\widetilde{A}}\tilde{u} = 0$ in Ω.

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ and $\varphi = \psi \circ \Phi$. Then by change of variables and the chain rule

$$\begin{aligned}\int_{\Phi(\Omega)} A\nabla u \cdot \nabla\psi &= \int_\Omega (A\circ\Phi)\nabla u\circ\Phi \cdot \nabla\psi\circ\Phi \,|\det D_\Phi| \\ &= \int_\Omega (A\circ\Phi)D_{\Phi^{-1}}^T\nabla(u\circ\Phi)\cdot D_{\Phi^{-1}}^T\nabla(\psi\circ\Phi)\,|\det D_\Phi| \\ &= \int_\Omega |\det D_\Phi| D_{\Phi^{-1}}(A\circ\Phi)D_{\Phi^{-1}}^T\nabla(u\circ\Phi)\cdot\nabla(\psi\circ\Phi) \\ &= \int_\Omega \widetilde{A}\nabla\widetilde{u}\cdot\nabla\varphi.\end{aligned}$$

The lemma readily follows. $\square$

We will usually apply the above lemma when $\Phi(x) = Sx$ for some matrix S, in which case

$$\tilde{A} = (\det S)S^{-1}(A\circ S)(S^{-1})^T. \tag{4.6}$$

Lemma 4.9. *With the same assumptions as Lemma 4.8, and assuming Ω is a Wiener regular domain, we have that for any set $E \subset \Phi(\partial\Omega) = \partial\Phi(\Omega)$ and $x \in \Omega$,*

$$\omega_{\Phi(\Omega)}^{L_A,\Phi(x)}(E) = \omega_\Omega^{L_{\widetilde{A}},x}(\Phi^{-1}(E)). \tag{4.7}$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$. Since the function

$$v(x) = \int \varphi\, d\omega_{\Phi(\Omega)}^{L,x}$$

is L_A-harmonic for $x \in \Phi(\Omega)$, by the previous lemma we know that the function

$$\widetilde{v}(x) = \int \varphi\, d\omega_{\Phi(\Omega)}^{L,\Phi(x)}$$

is $L_{\widetilde{A}}$-harmonic for $x \in \Omega$. If $\xi \in \partial\Omega$, then as $x \to \xi$ in Ω, $\Phi(x) \to \Phi(\xi)$ in $\Phi(\Omega)$, and so

$$\widetilde{v}(x) = \int \varphi\, d\omega_{\Phi(\Omega)}^{L,\Phi(x)} \to \varphi(\Phi(\xi)).$$

Thus, $\widetilde{v}$ is the $L_{\widetilde{A}}$-harmonic extension of $(\varphi\circ\Phi)|_{\partial\Omega}$ to Ω, and so

$$\int_{\partial\Phi(\Omega)} \varphi\, d\omega_{\Phi(\Omega)}^{L_A,\Phi(x)} = \int_{\partial\Omega} \varphi\circ\Phi\, d\omega_\Omega^{L_{\widetilde{A}},x}, \quad \text{for all } x \in \Omega.$$

Since this holds for all such φ, we get that for any set $E \subset \partial\Phi(\Omega) = \Phi(\partial\Omega)$,

$$\omega_{\Phi(\Omega)}^{L_A,\Phi(x)}(E) = \omega_\Omega^{L_{\widetilde{A}},x}(\Phi^{-1}(E)),$$

which gives the lemma.

$\square$

The following lemma will help us relate measures generated by elliptic polynomials to just measures generated by harmonic polynomials. In particular, if A is an elliptic matrix with constant and real coefficients, by the change of variables described below (which is just a linear transformation), if h is a harmonic polynomial solution in an open set Ω and $S = \sqrt{A_s}$ (where A_s is the symmetric part of A), then $\widetilde{h} = h \circ S^{-1}$ is a polynomial solution of $-\operatorname{div} A\nabla u = 0$ in $S(\Omega)$. So, there is a bijection between the set of harmonic polynomials and the set of polynomials solutions of $-\operatorname{div} A\nabla u = 0$ in $S(\Omega)$ (for a fixed constant elliptic matrix A). Recall also that p is a harmonic polynomial in an open set if and only if it is a harmonic polynomial in $\mathbb{R}^{n+1}$. So, if A is as above, there is an abundance of non-trivial polynomial solutions of $-\operatorname{div} A\nabla u = 0$ in any open subset of $\mathbb{R}^{n+1}$ (including $\mathbb{R}^{n+1}$ itself). In fact, Theorem 2 in [AP12] states that for such L_A, for any $k \in \mathbb{N}$, there exists a polynomial solution of $L_A h = 0$ of degree k.

Lemma 4.10. *Let A be an elliptic constant matrix, $A_s = (A + A^T)/2$, and $S = \sqrt{A_s}$. Let $h \in H_A$ and $\widetilde{h} = h \circ S$. Then $\tilde{A} = (\det S)I$, $\widetilde{h} \in H$ and*

$$\omega_{\widetilde{h}} = (\det S)^{-1} S^{-1}[\omega_h^A]. \tag{4.8}$$

Proof. Note that since L_A has constant coefficients, then $L_{A_s} = L_A$ by the fact that for $u \in C^2$

$$\begin{aligned} L_A u = \sum_{i,j} a_{ij}\partial_i\partial_j u &= \frac{1}{2}\sum_{i,j} a_{ij}\partial_i\partial_j u + \frac{1}{2}\sum_{i,j} a_{ij}\partial_j\partial_i u \\ &= \sum_{i,j} \frac{(a_{ij} + a_{ji})}{2}\partial_i\partial_j u = L_{A_s} u. \end{aligned}$$

Thus, if h is an L_A-harmonic function, it is also an L_{A_s}-harmonic function. Moreover, for any $\psi \in C_c^\infty(\mathbb{R}^{n+1})$

$$\int \psi \, d\omega_h^{A_s} = \int_{\Omega_h} h \, L_{A_s}(\psi) = \int_{\Omega_h} h \, L_A(\psi) = \int \psi \, d\omega_h^A.$$

In fact, without loss of generality, we may assume that $A = A_s$.

Recall now that since A_s is a symmetric, positive definite and invertible matrix with constant real entries, then it has a unique real symmetric positive definite square root $S = \sqrt{A_s}$ which is also invertible. Hence, by Lemma 4.8 and (4.6) with $A = A_s$, we have that $\tilde{A} = (\det S)I$ and $\widetilde{h}$ is $L_{(\det S)I}$-harmonic, and thus just harmonic.

Let now $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ and $\psi \circ S = \varphi$. By Green's formula and the fact that S is also symmetric, we have that

$$
\begin{aligned}
(\det S) \int \varphi \, d\omega_{\widetilde{h}} &= (\det S) \int_{\Omega_{\widetilde{h}}} \widetilde{h} \, \Delta\varphi = -(\det S) \int_{\Omega_{\widetilde{h}}} \nabla\widetilde{h} \cdot \nabla\varphi \\
&= -(\det S) \int_{\Omega_{\widetilde{h}}} S^T \nabla h \circ S \cdot S^T \nabla\psi \circ S \\
&= -\int_{S^{-1}(\Omega_h)} S S^T \nabla h \circ S \cdot \nabla\psi \circ S \\
&= -\int_{\Omega_h} A_s \nabla h \cdot \nabla\psi = \int_{\Omega_h} h \, L_{A_s}(\psi) \\
&= \int_{\Omega_h} h \, L_A(\psi) = \int \psi \, d\omega_h^A = \int \varphi \, dS^{-1}[\omega_h^A].
\end{aligned}
$$

□

Let us recall some simple facts from linear algebra which help us understand how the geometry of Ω is affected by the linear transformation above. Note that S is orthogonally diagonalizable since it is symmetric, which means that it represents a linear transformation with scaling in mutually perpendicular directions. Hence S^{-1} is a special bi-Lipschitz change of variables that takes balls to ellipsoids, where eigenvectors determine directions of semi-axes, eigenvalues determine lengths of semi-axes and its maximum eccentricity is given by $\sqrt{(\lambda_{\max}/\lambda_{\min})}$ (where $\lambda_{\max}$ are $\lambda_{\min}$ are the maximal and minimal eigenvalues of S^{-1}), which is in turn bounded below by $\sqrt{\Lambda}^{-1}$ and above by $\sqrt{\Lambda}$. In particular, $S^{-1}(\partial\Omega) = \partial(S^{-1}(\Omega))$, $\Lambda^{-1/2} \leq \|S^{-1}\| \leq \Lambda^{1/2}$, i.e., S^{-1} distorts distances by at most a constant depending on ellipticity.

4.4. **The main blow-up lemma.** We now introduce the main tool of this paper, which is a variant of previous blow-up arguments, first introduced by Kenig and Toro for NTA domains [KT06], then extended to CDC domains in [AMT16]. Both these cases applied to harmonic measure, but it can be extended to elliptic measures with a VMO condition on the coefficients.

Lemma 4.11. *Let $\Omega^+ \subset \mathbb{R}^{n+1}$ be a CDC domain, $K \subset \partial\Omega^+$ a compact set, $\xi_j \in K$ a sequence of points, and $L = -\operatorname{div} A\nabla$ be a uniformly elliptic operator in Ω^+ such that*

$$
\limsup_{r\to 0} \sup_{\xi\in K} \frac{1}{r^{n+1}} \inf_{C\in\mathscr{C}} \int_{B(\xi,r)\cap\Omega^+} |A(x) - C| dx = 0. \tag{4.9}
$$

Let ω^+ be the elliptic measure for Ω^+. and$c_j \geq 0$, and $r_j \to 0$ such that $\omega_j^+ = c_j T_{\xi_j, r_j}[\omega^+] \rightharpoonup \omega_\infty^+$ for some nonzero measure ω_∞^+. Let $\Omega_j^+ = T_{\xi_j, r_j}(\Omega^+)$. Then there is a subsequence and a closed set $\Sigma \subset \mathbb{R}^{n+1}$ such that

(a) For all $R > 0$ sufficiently large, $B(0,R) \cap \partial\Omega_j^+ \neq \emptyset$ and $\partial\Omega_j^+ \cap \overline{B(0,R)} \to \Sigma \cap \overline{B(0,R)}$ in the Hausdorff metric.

(b) $\Sigma^c = \Omega_\infty^+ \cup \Omega_\infty^-$ where Ω_∞^+ is a nonempty open set and Ω_∞^- is also open but possibly empty. Further, they satisfy that for any ball B with $\overline{B} \subset \Omega_\infty^\pm$, a neighborhood of $\overline{B}$ is contained in $\Omega_j^\pm$ for all j large enough.

(c) $\operatorname{supp} \omega_\infty^+ \subset \Sigma$.

(d) Let $u^+(x) = G_{\Omega^+}(x, x^+)$ on Ω^+ and $u^+(x) = 0$ on $(\Omega^+)^c$. Set

$$u_j^+(x) = c_j\, u^+(x r_j + \xi_j)\, r_j^{n-1}.$$

Then u_j^+ converges locally uniformly in $\mathbb{R}^{n+1}$ and in $W^{1,2}_{\text{loc}}(\mathbb{R}^{n+1})$ to a nonzero function u_∞^+ which is continuous in $\mathbb{R}^{n+1}$, vanishes in $(\Omega_\infty^+)^c$ and satisfies

$$u_\infty^+(y) \lesssim \omega_\infty^+(\overline{B}(x, 4r))\, r^{1-n}, \tag{4.10}$$

for $x \in \Sigma$, $r > 0$, and $y \in B(x,r) \cap \Omega_\infty^+$. Moreover, there is A_0^+ a constant elliptic matrix so that if $L_0^+ = -\operatorname{div} A_0^+ \nabla$, then

$$\int \varphi\, d\omega_\infty^+ = \int_{\mathbb{R}^{n+1}} u_\infty^+\, L_0^+ \varphi, \quad \text{for any } \varphi \in C_c^\infty(\mathbb{R}^{n+1}). \tag{4.11}$$

Suppose now that $\Omega^- = \mathbb{R}^{n+1} \backslash \overline{\Omega^+}$, so that $\partial\Omega^+ = \partial\Omega^-$ and Ω^- is also connected and has the CDC. Define analogously ω_j^-, u^-, u_j^-, and u_∞^-. Assume that A is uniformly elliptic in $\Omega^+ \cup \Omega^-$, (4.9) holds for $\Omega^+ \cup \Omega^-$ in place of Ω^+ and ω_j^- converges weakly to $\omega_\infty^- = c\omega_\infty^+$ for some number $c \in (0, \infty)$. Then $\Omega_\infty^- \neq \varnothing$ and for a suitable subsequence, (d) holds for u_j^-, u_∞^-, and Ω_∞^-. Furthermore, if we set $u_\infty = u_\infty^+ - c^{-1} u_\infty^-$, then

(e) u_∞ extends to a continuous function on $\mathbb{R}^{n+1}$ which satisfies $L_0 u_\infty = 0$ in $\mathbb{R}^{n+1}$.

(f) $\Sigma = \{u_\infty = 0\}$, with $u_\infty > 0$ on Ω_∞^+ and $u_\infty < 0$ on Ω_∞^-. Further, Σ is a real analytic variety of dimension n.

(g) $d\omega_\infty^+ = -\frac{\partial u_\infty}{\partial \nu_{A_0}}\, d\sigma_{\partial\Omega_\infty^+}$, where σ_S stands for the surface measure on a surface S and $\frac{\partial}{\partial \nu_{A_0}} = \nu \cdot A_0 \nabla$ is the outward co-normal derivative.

Proof. The proof of this lemma can be found in [AMT16] for harmonic measure for the case that $K = \{\xi\}$ (i.e. so that (1.4) holds). The proof for general K is essentially the same in this setting with minor changes. Here we shall only record the required modifications (some of which are quite substantial) for the $K = \{\xi\}$ case in order for the same proof to work for any elliptic measure as well. In this case, $\xi_j = \xi$ for all j. We set

$$A_j(x) := A(r_j x + \xi), \;\; u_j^{\pm}(x) := c_j r_j^{n-1} u^{\pm}(r_j x + \xi)$$

and

$$\varphi_j(x) := \varphi\left(\frac{x-\xi}{r_j}\right).$$

Without loss of generality we can only work with u^+ since the results for u^- can be proved analogously.

Notice now that for j large enough, the pole $x^+ \notin \operatorname{supp}(\varphi_j)$. In fact, for any ball B centered at the boundary of Ω_j, we can find $j_0 \in \mathbb{N}$, such that for all $j \geq j_0$, $x^+ \notin T_{\xi,r_j}(B)$. Moreover, for $x \in B \cap \Omega_j$ and j large enough,

$$\begin{aligned} u_j^+(x) &= c_j\, r_j^{n-1}\, u^+(r_j x + \xi) \\ &\overset{(4.4)}{\lesssim} c_j\, r_j^{n-1} (r_j r_B)^{1-n}\, \omega^+(4 r_j B + \xi) = r_B^{1-n}\, \omega_j^+(4B). \end{aligned} \tag{4.12}$$

Proof of (b): We only need to prove the existence of $B \subset \Omega_j^+$ for large $j \in \mathbb{N}$. Suppose there is no such ball. Let φ be any continuous compactly supported non-negative function for which $\int \varphi\, d\omega_\infty^+ \neq 0$, and let $M > 0$ be so that $\operatorname{supp} \varphi \subset B(0,M)$. Thus, there must be $x_0 \in B(0,M) \cap \operatorname{supp} \omega_\infty^+$. We set

$$\delta_j := \sup\{\operatorname{dist}(x, (\Omega_j^+)^c) : x \in B(0,2M)\},$$

which goes to zero by assumption. For $x \in B(0,2M)$ and $j \in \mathbb{N}$, let $\zeta_j(x) \in (\Omega_j^+)^c$ be closest to x so that $|x - \zeta_j(x)| \leq \delta_j \leq 2M$ (the second inequality holds because $0 \in \partial\Omega_j^+$). It also holds that for all $x \in B(0,2M)$, $|x - x_0| \leq |x| + |x_0| < 3M$.

Notice now that for any j big enough, u_j^+ is a solution in $B(0,2M) \cap \Omega_j^+$ and a subsolution in $B(0,2M)$. Moreover, if $x \in \Omega_j^+$, then $\zeta_j(x) \in \partial\Omega_j^+$. Thus, for j large, by Cauchy-Scwharz, Caccioppoli's inequality in $B(0,M)$ (which also holds for subsolutions) and the fact that u_j^+ and φ are supported in Ω_j^+ and $B(0, M)$ respectively,

$$0 < \int \varphi \, d\omega_j^+ = \int_{\Omega_j^+} A_j \nabla u_j^+ \cdot \nabla \varphi \lesssim_{\lambda,\Lambda,n,M} \|\nabla \varphi\|_\infty \left(\int_{B(0,2M)} |u_j^+|^2 \right)^{1/2}$$

$$\overset{(4.5)}{\lesssim} \left(\int_{\Omega_j^+ \cap B(0,2M)} \left(\sup_{B(\zeta_j(x),2M)} u_j^+ \right)^2 \left(\frac{x - \zeta_j(x)}{2M} \right)^{2\alpha} dx \right)^{1/2}$$

$$\overset{(4.12)}{\lesssim} \left(\int_{\Omega_j^+ \cap B(0,2M)} \left[\omega_j^+ (B(\zeta_j(x), 8M)) \, (2M)^{1-n} \right]^2 dx \right)^{1/2} \left(\frac{\delta_j}{2M} \right)^\alpha$$

$$\lesssim (2M)^{\frac{n+1}{2}} \omega_j^+ (B(x_0, 13M)) \, (2M)^{1-n} \left(\frac{\delta_j}{2M} \right)^\alpha,$$

and thus

$$0 < \int \varphi \, d\omega_\infty^+ \lesssim_{\lambda,\Lambda,n,M,\varphi} \left(\limsup_{j \to \infty} \omega_j^+ (B(x_0, 13M)) \right) \lim_j \delta_j^\alpha$$
$$\leq \omega_\infty^+ (\overline{B}(x_0, 13M)) \cdot 0 = 0,$$

which is a contradiction. Thus, there is $B \subset \Omega_j$ for all large j (after passing to a subsequence).

Proof of (d): Arguing as in [AMT16], there exists u_∞^+ which is continuous in $\mathbb{R}^{n+1}$ and vanishes on $(\Omega_\infty^+)^c$ such that (after passing to a subsequence) $u_j^+ \to u_\infty^+$ uniformly on compact sets of $\mathbb{R}^{n+1}$. Moreover, it is not hard to see that $u_j^+ \in W^{1,2}(B)$ for large j. Indeed, by (4.12), it is clear that

$$\|u_j^+\|_{L^2(B)}^2 \lesssim r_B^{3-n} \, [\omega_j^+ (4B)]^2, \tag{4.13}$$

while by Caccioppoli's inequality and (4.12),

$$\int_B |\nabla u_j^+|^2 \lesssim r_B^{-2} \int_B |u_j^+|^2 \lesssim r_B^{-2} [r_B^{1-n} \, \omega_j^+ (4B)]^2 r_B^{n+1} = r_B^{1-n} \, [\omega_j^+ (4B)]^2. \tag{4.14}$$

In view of (4.13) and (4.14) we have that

$$\limsup_{j \to \infty} \|u_j^+\|_{W^{1,2}(B)} \lesssim r_B^{\frac{1-n}{2}} (1 + r_B) \, \limsup_{j \to \infty} \omega_j^+ (4B)$$
$$\leq r_B^{\frac{1-n}{2}} (1 + r_B) \, \omega_\infty^+ (\overline{4B}) < \infty.$$

Therefore, by [HKM06, Theorem 1.32], $u_\infty^+ \in W^{1,2}_{\text{loc}}(\mathbb{R}^{n+1})$ and there exists a further subsequence of u_j^+ that converges weakly to u_∞^+ in $W^{1,2}_{\text{loc}}(\mathbb{R}^{n+1})$.

Notice that

$$- \int_{\Omega_j^+} A_j \nabla u_j^+ \cdot \nabla \varphi = \int \varphi \, d\omega_j^+.$$

Indeed, by a change of variables, and letting $\varphi_j = \varphi \circ T_{\xi,r_j}$ and $\varphi_j = \varphi \circ T_{\xi,r_j}$

$$\begin{aligned}\int \varphi\, d\omega_j^+ = c_j \int \varphi_j\, d\omega^+ &= \int_{\Omega^+} A\nabla u^+ \cdot \nabla \varphi_j \\ &= c_j r_j^n \int_{\Omega_j^+} A(r_j x+\xi)\nabla u^+(r_j x+\xi)\cdot \nabla\varphi(x)dx \\ &= \int_{\Omega_j^+} A_j \nabla u_j^+ \cdot \nabla \varphi.\end{aligned}$$

Let $C_{j,k}$ be a constant elliptic matrix so that

$$\lim_j (kr_j)^{-1-n} \int_{B(\xi,kr_j)\cap\Omega^+} |A - C_{j,k}| = 0.$$

By a diagonalization argument and compactness, we may pass to a subsequence so that for each k, $C_{j,k}$ converges to a uniformly elliptic matrix C_k with constant coefficients. It is not hard to check that we must in fact have that $C_k = A_0^+$ for some fixed matrix A_0^+ (using the fact that $\inf \delta_j > 0$). Thus, we have

$$\lim_j (Mr_j)^{-1-n} \int_{B(\xi,Mr_j)\cap\Omega^+} |A - A_0^+| = 0 \quad \text{for all } M \geq 1. \tag{4.15}$$

To see the ellipticity of A_0^+ is pretty easy but we show the details for completeness. Note that since A is uniformly elliptic for a.e. $x\in\Omega^+$, then for $\xi \in \mathbb{R}^{n+1}$,

$$\Lambda^{-1}|\xi|^2 \leq A(x)\xi\cdot\xi = (A(x) - A_0^+)\,\xi\cdot\xi + A_0^+\,\xi\cdot\xi.$$

Then, if we take averages over $B(\xi, Mr_j)\cap\Omega$, use the existence of corkscrew balls in Ω_j for large j proved in (b) and then take limits as $j\to\infty$, by (4.15) we have

$$\Lambda^{-1}|\xi|^2 \leq A_0^+\,\xi\cdot\xi.$$

The upper bound follows by a similar argument and the proof is omitted.

We will now estimate the difference

$$\int_{\Omega_j^+} A_j \nabla u_j^+ \cdot \nabla\varphi - \int_{\Omega_\infty^+} A_0^+ \nabla u_\infty^+ \cdot \nabla \varphi, \tag{4.16}$$

for sufficiently large j.

To this end, let $\operatorname{supp}(\varphi) \subset B(0, M)$. Note that

$$|(4.16)| \leq \left| \int_{\Omega_j^+} (A(r_j x + \xi) - A_0^+) \nabla u_j^+ \cdot \nabla \varphi \right| \\ + \left| \int_{B(0,M)} (\nabla u_j^+ \mathbb{1}_{\Omega_j} - \nabla u_\infty^+ \mathbb{1}_{\Omega_\infty}) \cdot A_0^{+,T} \nabla \varphi \right| \leq I_1 + I_2.$$

Note that $u_j^+, u_\infty^+ \in W^{1,2}(\mathbb{R}^{n+1})$, $u_j^+ > 0$ only in Ω_j^+, and $u_\infty^+ > 0$ only in Ω_∞^+. Since the extension of the gradient of a function $f \in W_0^{1,2}(\Omega)$ by zero to $\mathbb{R}^{n+1}$ (where Ω is any domain) is the same as the gradient of the extension of f by zero[1], we have that in $W^{1,2}(B(0, M))$,

$$\nabla u_j^+ \mathbb{1}_{\Omega_j^+} = \nabla (u_j^+ \mathbb{1}_{\Omega_j^+}) = \nabla u_j^+ \rightharpoonup \nabla u_\infty^+ = \nabla (u_\infty^+ \mathbb{1}_{\Omega_\infty^+}) = \nabla u_\infty^+ \mathbb{1}_{\Omega_\infty^+}.$$

so we have that $I_2 \to 0$. On the other hand, since A and $A_0^+ \in L^\infty(\Omega)$,

$$I_1 \leq ||\nabla u_j^+||_{L^2(B(0,M))} ||\nabla \varphi||_\infty \left(\int_{B(0,M) \cap \Omega_j^+} |A(r_j x + \xi) - A_0^+|^2 dx \right)^{1/2} \\ \overset{(4.14)}{\lesssim}_\Lambda M^{\frac{1-n}{2}} \omega_\infty^+(\overline{B(0, 4M)}) \left(\frac{1}{r_j^{1+n}} \int_{B(0, M r_j) \cap \Omega^+} |A(x) - A_0^+| dx \right)^{1/2} \\ \overset{(4.15)}{\to} 0 \, .$$

Thus, combining the above estimates and taking $j \to \infty$, we infer that

$$- \int_{\Omega_\infty^+} A_0^+ \nabla u_\infty^+ \cdot \nabla \varphi = \int \varphi \, d\omega_\infty^+.$$

In particular, u_∞^+ is a continuous weak solution of

$$L_0^+ w = - \operatorname{div} A_0^+ \nabla w = 0 \text{ in } \Omega_\infty^+.$$

Since L_0^+ is a second order elliptic operator with constant coefficients, u_∞^+ is real analytic in Ω_∞^+. Thus, by definition of u_∞^+ and since the gradient of its extension by zero is the extension by zero of the gradient, we have that

$$\int_{\Omega_\infty^+} A_0^+ \nabla u_\infty^+ \cdot \nabla \varphi = \int_{\mathbb{R}^{n+1}} A_0^+ \nabla u_\infty^+ \cdot \nabla \varphi.$$

[1] See Proposition 9.18 in [Bre11]. It is stated for C^1-domains, but the direction we need holds for general Ω.

We now use the divergence theorem along with the fact that $\operatorname{supp}(\nabla\varphi) \subset B(0.M)$ and obtain (writing $L_0^{+,T} = L_{A_0^{+,T}}$)

$$\int \varphi \, d\omega_\infty^+ = -\int_{\mathbb{R}^{n+1}} \operatorname{div}[u_\infty^+ \, A_0^{+,T} \nabla\varphi] + \int_{\mathbb{R}^{n+1}} u_\infty^+ \, L_0^{+,T} \varphi$$
$$= -0 + \int_{\mathbb{R}^{n+1}} u_\infty^+ \, L_0^{+,T} \varphi,$$

which finishes the proof of (d). The rest of the proof is almost identical since one only uses that u_∞ real analytic in $\mathbb{R}^{n+1}$ and Liouville's theorem for positive solutions of uniformly elliptic equations (see e.g. Corollary 6.11 in [HKM06]).

On may argue similarly in the case of u_j^-. Notice that in this case, we will obtain a constant coefficient uniformly elliptic matrix A_0 such that

$$\lim_j (Mr_j)^{-1-n} \int_{B(\xi, Mr_j) \cap (\Omega^+ \cup \Omega^-)} |A - A_0| = 0 \quad \text{for all } M \geq 1. \tag{4.17}$$

□

Now we prove a slightly weaker version of this result in the next two lemmas. Again, this is based on the details in the proof of [AMTV16, Lemma 5.3], but with some adjustments for elliptic measure.

Lemma 4.12. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain. Let $\xi_j \in \partial\Omega$ and $L = -\operatorname{div} A\nabla$ be a uniformly elliptic operator in Ω such that (1.3) holds with $K = \{\xi_j\}$ and, if $\omega = \omega_\Omega^{L_A, x_0}$ is its L_A-harmonic measure with pole at $x_0 \in \Omega$, there is $r_j \to 0$ and $c_j > 0$ so that*

$$\omega_j := c_j T_{\xi_j, r_j}[\omega] \to \omega_\infty$$
$$\liminf_j \frac{|\Omega \cap B(\xi_j, r_j)|}{r_j^{n+1}} > 0, \tag{4.18}$$

and

$$\omega^z(B(\xi_j, 2r_j)) \gtrsim 1 \text{ for all } j \text{ and } z \in B(\xi_j, r_j) \cap \Omega. \tag{4.19}$$

Then there is a subsequence such that the following hold: If $u(x) = G_\Omega(x, x_0)$ on Ω and $u(x) = 0$ on Ω^c, and

$$u_j(x) = c_j \, u(xr_j + \xi_j) \, r_j^{n-1},$$

then u_j converges in $L^2_{loc}(\frac{1}{2}\mathbb{B})$ to a nonzero function u_∞ which is L_{A_0}-harmonic in $\{x : u_\infty > 0\} \cap \frac{1}{2}\mathbb{B}$, for constant uniformly elliptic matrix A_0, and such that

$$||u_\infty||_{L^2(\frac{1}{2}\mathbb{B})} \lesssim \omega_\infty(\overline{B(0,2)}), \tag{4.20}$$

and for any $\varphi \in C_c^\infty(\frac{1}{2}\mathbb{B})$*,*

$$\int \varphi \, d\omega_\infty = \int_{\mathbb{R}^{n+1}} u_\infty \, L_{A_0}\varphi. \tag{4.21}$$

If $\xi = \xi_j$ *and* A *is continuous at* ξ*, then* A_0 *is just the value of* A *at* ξ*.*

Proof. Recall that we denote $\mathbb{B} = B(0,1)$. Again, to simplify notation, we'll just prove the case when $\xi_j = \xi \in \partial\Omega$.

By (4.19), without loss of generality, we can scale the c_j so that

$$\omega_\infty\big(\tfrac{1}{4}\mathbb{B}\big) = 1. \tag{4.22}$$

Let $\Omega_j = T_{\xi,r_j}(\Omega)$. By (4.19) and (4.4),

$$\omega(B(\xi,2r_j)) \gtrsim r_j^{n-1}\, u(x) \quad \text{for all } x \in B(\xi,r_j) \cap \Omega_1, \tag{4.23}$$

and so,

$$\omega_j(2\mathbb{B}) \gtrsim \; u_j(x) \quad \text{for all } x \in \mathbb{B} \cap \Omega_1^j, \tag{4.24}$$

By Caccioppoli's inequality for L-subharmonic functions and the uniform boundedness of u in $\mathbb{B}$, we deduce that, for $i = 1, 2$,

$$\begin{aligned}\limsup_{j\to\infty} \|\nabla u_j\|_{L^2(\frac{1}{2}\mathbb{B})} &\lesssim \limsup_{j\to\infty} \|u_j\|_{L^2(\mathbb{B})} \\ &\lesssim \limsup_{j\to\infty} \omega_j(2\mathbb{B}) \leq \omega_\infty(\overline{2\mathbb{B}}).\end{aligned}$$

By the Rellich-Kondrachov theorem, the unit ball of the Sobolev space $W^{1,2}(\frac{1}{2}\mathbb{B})$ is relatively compact in $L^2(\frac{1}{2}\mathbb{B})$, and thus there exists a subsequence of the functions u_j which converges *strongly* in $L^2(\frac{1}{2}\mathbb{B})$ to another function $u_\infty \in L^2(\frac{1}{2}\mathbb{B})$. This and the above inequality imply (4.20).

By the same diagonalization argument as in the proof of the previous lemma (although using (4.18) instead of $\inf \delta_j > 0$ that we used in the previous lemma), we can pass to a subsequence so that, for some uniformly elliptic matrix A_0 with constant coefficients,

$$\lim_j (Mr_j)^{-1-n} \int_{B(\xi,Mr_j)\cap\Omega} |A(x) - A_0| = 0, \quad \text{for all } M \geq 1. \tag{4.25}$$

It easy to check that

$$\int \varphi \, d\omega_j = \int A_j \nabla u_j \cdot \nabla \varphi \, dx,$$

for any C^∞ function φ compactly supported in $\frac{1}{2}\mathbb{B}$. Then passing to a limit, it follows that

$$\int \varphi \, d\omega_\infty = \int A_0 \nabla u_\infty \cdot \nabla \varphi \, dx, \;\; \text{for any } \varphi \in C_c^\infty(\tfrac{1}{2}\mathbb{B}). \tag{4.26}$$

$\square$

Theorem 4.13. *Let $\Omega^{\pm} \subset \mathbb{R}^{n+1}$ be disjoint domains. Let $\xi_j \in \partial\Omega^+ \cap \partial\Omega^-$ and $L = -\operatorname{div} A\nabla$ be a uniformly elliptic operator in $\Omega^+ \cup \Omega^-$ such that such that (1.3) holds with $K = \{\xi_j\}$ with respect to both $\Omega^+ \cup \Omega^-$. If $\omega^{\pm} = \omega_{\Omega^{\pm}}^{L_A, x^{\pm}}$ is the L_A-harmonic measure with pole at $x^{\pm} \in \Omega^{\pm}$, and if there is $r_j \to 0$ and $c_j > 0$ so that*

$$\omega_j^+ := c_j T_{\xi_j, r_j}[\omega^+] \to \omega_\infty$$

and

$$\omega_j^- := c_j T_{\xi_j, r_j}[\omega^-] \to c\,\omega_\infty$$

for some constant $c > 0$, then there is a subsequence such that the following hold. If $u^{\pm}(x) = G_{\Omega^{\pm}}(x, x^{\pm})$ on $\Omega^{\pm}$, $u(x) = 0$ on $(\Omega^{\pm})^c$ and

$$u_j^{\pm}(x) = c_j\, u^{\pm}(x r_j + \xi_j)\, r_j^{n-1},$$

then $u_j := u_j^+ - c^{-1} u_j^-$ converges in $L^2(\frac{1}{2}\mathbb{B})$ to a nonzero function u_∞, which is L_{A_0}-harmonic in $\frac{1}{2}\mathbb{B}$ for some constant uniformly elliptic matrix A_0, and moreover,

$$\tfrac{1}{2}\mathbb{B} \cap \operatorname{supp} \omega_\infty = \{u_\infty = 0\} \cap \tfrac{1}{2}\mathbb{B} \tag{4.27}$$

and (4.20) and (4.21) hold. If $\xi_j = \xi$ and A is continuous at ξ, then A_0 is just the value of A at ξ.

By applying this result to the sequences $c_j T_{\xi_j, a r_j}[\omega^{\pm}]$ for all $a > 0$, we see that u_∞ extends to a L_{A_0}-harmonic function on $\mathbb{R}^{n+1}$ so that for $r > 0$,

$$||u_\infty||_{L^2(B(0,r))} \lesssim r^{1-n}\, \omega_\infty(\overline{B(0,4r)}), \tag{4.28}$$

and for any $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$,

$$\int \varphi\, d\omega_\infty = \int_{\mathbb{R}^{n+1}} u_\infty\, L_{A_0} \varphi. \tag{4.29}$$

Proof. The proof is mostly the same as the proof of [AMTV16, Lemma 5.3], but we provided some of the details here to show the differences. Again, we assume $\xi_j = \xi$. Note that since Ω^+ and Ω^- are disjoint, we may assume without loss of generality that

$$|B(\xi, r_j/8) \backslash \Omega^+| \geq \frac{|B(\xi, r_j/8)|}{2}$$

and so Bourgain's estimate implies

$$\omega^{+,z}(B(\xi, 2r_j)) \gtrsim 1 \text{ for all } z \in B(\xi, r_j)).$$

Hence, the conclusions of Lemma 4.12 apply to $\omega = \omega^+$, $\Omega = \Omega^+$ and $u = u^+$. In particular, (4.24) in our scenario is

$$\omega_j^+(2\mathbb{B}) \gtrsim u_j^+(x) \qquad \text{for all } x \in \mathbb{B} \cap \Omega_1^j. \tag{4.30}$$

Again, by rescaling, we can assume that $\omega_\infty(\frac{1}{4}\mathbb{B}) = 1$.

Observe now that for any non-negative $\varphi \in C_c^\infty(\frac{1}{2}\mathbb{B})$ with $\phi = 1$ in $\frac{1}{4}\mathbb{B}$, by Cauchy-Schwartz and Caccioppoli's inequality (since $u_j^\pm$ is positive and L_{A_j}-harmonic in $\mathbb{B} \cap \Omega_j^\pm$ and zero in $\mathbb{B} \setminus \Omega_j^\pm$) we have that

$$
\begin{aligned}
1 =&\omega_\infty(\frac{1}{4}\mathbb{B}) \leq \int \varphi \, d\omega_\infty = \int A_0 \nabla u_\infty^+ \cdot \nabla\varphi \, dx \\
&= \lim_j \int_{\Omega_j^+} A_j \nabla u_j^+ \cdot \nabla \varphi \, dx \\
&\leq \|A\|_{L^\infty} \|\nabla \varphi\|_{L^\infty(\mathbb{B})} \lim_j \int_{\Omega_j^+ \cap \frac{1}{2}\mathbb{B}} |\nabla u_j^+| \\
&\lesssim \|A\|_{L^\infty} \|\nabla \varphi\|_{L^\infty(\mathbb{B})} \lim_j \left(\int_{\Omega_j^+ \cap \mathbb{B}} |u_j^+|^2 \right)^{1/2} \\
&\lesssim \lim_j \left(\int_{\mathbb{B} \cap \Omega_j^+ \cap \{u_j^+ > t\}} |u_j^+|^2 \, dx + \int_{\mathbb{B} \cap \Omega_j^+ \cap \{u_j^+ \leq t\}} |u_j^+|^2 \, dx \right)^{1/2} \\
&\lesssim \liminf_j \left(|\{x \in \mathbb{B} \cap \Omega_j^+ : u_j^+ > t\}|^{1/2} \cdot ||u_j^+||_{L^\infty(\mathbb{B} \cap \Omega_j^+)} \right) + t \\
&\overset{(4.32)}{\lesssim} \liminf_j \left(|\{x \in \mathbb{B} \cap \Omega_j^+ : u_j^+ > t\}|^{1/2} \, \omega_\infty \left(\overline{2\mathbb{B}} \right) + t \right),
\end{aligned}
$$

and so, for t small enough,

$$
|\mathbb{B} \cap \Omega_j^+| \geq |\{x \in \mathbb{B} \cap \Omega_j^+ : u_j^+(x) > t\}| \gtrsim \omega_\infty(\overline{2\mathbb{B}})^{-2}.
$$

In particular,

$$
|B(\xi, r_j) \backslash \Omega^-| \geq |B(\xi, r_j) \cap \Omega^+| \gtrsim r_j^{n+1} \omega_\infty(\overline{2\mathbb{B}})^{-2}. \tag{4.31}
$$

Thus, by the same arguments as earlier in proving (4.24), we have that for j large,

$$
\omega_j^-(B(\xi, 2r_j)) \gtrsim u_j^-(x) \, \omega_\infty(\overline{2\mathbb{B}})^{-2}, \qquad \text{for all } x \in B(\xi, r_j) \cap \Omega^-. \tag{4.32}
$$

Thus, we can apply Lemma 4.12 and can pass to a subsequence so that u_j^- converges in $L^2(\frac{1}{2}\mathbb{B})$ to a function u_∞^-. Hence, $u_j^+ - c^{-1} u_j^- \to u_\infty^+ - c^{-1} u_\infty^- =: u_\infty$ and

$$
c \int \varphi \, d\omega_\infty = \int L_{A_0^*} \varphi \, u_\infty^- \, dx, \quad \text{for any } \phi \in C_c^\infty(\tfrac{1}{2}\mathbb{B}). \tag{4.33}
$$

In particular, we can show that u_∞ is L_{A_0}-harmonic in $\frac{1}{2}\mathbb{B}$, and the rest of the proof is exactly as in [AMTV16] starting from equation (5.15).

$\square$

5. Harmonic Polynomial Measures

5.1. **Preliminaries.** In this section, we review and collect some lemmas that will help us work with the quantities ω_h^A.

Lemma 5.1. *Let $h \in H_A$ and $r > 0$. Then*

$$T_{0,r}[\omega_h^A] = r^{n-1}\omega^A_{h\circ T_{0,r}^{-1}} \tag{5.1}$$

and

$$F_r(\omega_h^A) = r^n F_1(\omega^A_{h\circ T_{0,r}^{-1}}). \tag{5.2}$$

Proof. By Lemma 4.10, it suffices to prove this in the case that $h \in H$. Note that if h is a harmonic function and $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$, then

$$\begin{aligned}\int \varphi\, dT_{0,r}[\omega_h] &= \int \varphi \circ T_{0,r}\, d\omega_h \\ &= \int h\,\Delta(\varphi\circ T_{0,r})\,dx = r^{-2}\int h\,\Delta\varphi\circ T_{0,r}\,dx \\ &= r^{n-1}\int h\circ T_{0,r}^{-1}\,\Delta\varphi\,dx = r^{n-1}\int \varphi\, d\omega_{h\circ T_{0,r}^{-1}},\end{aligned}$$

and so (5.1) follows. Moreover, by Lemma 3.1 (3),

$$F_r(\omega_h) = rF_1(T_{0,r}[\omega_h]) \overset{(5.1)}{=} r^n F_1(\omega_{h\circ T_{0,r}^{-1}}). \tag{5.3}$$

□

Lemma 5.2. *Let $h \in F_A(k)$ and $r > 0$. Then*

$$F_r(\omega_h^A) = r^{n+k}F_1(\omega_h^A). \tag{5.4}$$

Proof. Note that since h is homogeneous of degree k,

$$h \circ T_{0,r}^{-1}(x) = h(rx) = r^k h(x),$$

and thus, by (5.2),

$$F_r(\omega_h^A) = r^n F_1(\omega^A_{h\circ T_{0,r}^{-1}}) = r^n F_1(\omega^A_{r^k h}) = r^{n+k}F_1(\omega_h^A).$$

□

The following is an immediate consequence of Lemma 5.1

Lemma 5.3 (Lemma 4.1 [Bad11])**.** *$\mathscr{F}_A(k)$, $\mathscr{P}_A(k)$, and $\mathscr{H}_A$ are d-cones. Hence, so are $\mathscr{F}_{\mathscr{S}}(k)$, $\mathscr{P}_{\mathscr{S}}(k)$, and $\mathscr{H}_{\mathscr{S}}$ for any $\mathscr{S} \subset \mathscr{C}$.*

Lemma 5.4. *Let $A_j \in \mathscr{C}$ converge to a matrix $A \in \mathscr{C}$ and let $h_j \in H_{A_j}$ converge uniformly on compact subsets to some $h \in H_A$. Then $\omega_{h_j}^{A_j} \to \omega_h^A$ weakly.*

Proof. First we will deal with the case that $A_j = A = I$ for all j.

We first claim that, since h and h_j are harmonic, $\mathbb{1}_{\Omega_{h_j}} \to \mathbb{1}_{\Omega_h}$ a.e.. Indeed, if $\mathbb{1}_{\Omega_h}(x) = 1$, then $h(x) > 0$, and by uniform convergence, $h_j(x) > 0$ for all large j, and so $\mathbb{1}_{\Omega_{h_j}}(x) = 1$ for all large j; similarly, if $\mathbb{1}_{\Omega}(x) = 0$, then either $x \in \partial\Omega_h$ (which has measure zero) or $h_j(x) < 0$ for all large j, in which case $\mathbb{1}_{\Omega_{h_j}}(x) = 0$ for all large j. Thus, $\mathbb{1}_{\Omega_{h_j}} \to \mathbb{1}_{\Omega_h}$ pointwise everywhere in $(\partial\Omega_h)^c$ and thus a.e. in $\mathbb{R}^{n+1}$. In particular, $h_j\mathbb{1}_{\Omega_j} \to h\mathbb{1}_{\Omega}$ a.e.. Hence, for $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$, by the dominated convergence theorem,

$$\lim_{j\to\infty} \int \varphi \, d\omega_{h_j} = \lim_{j\to\infty} \int_{\Omega_{h_j}} h_j \, \Delta\varphi = \int_{\Omega_h} h \, \Delta\varphi = \int \varphi \, d\omega_h,$$

which implies $\omega_{h_j} \rightharpoonup \omega_h$ as $j \to \infty$.

Now we handle the general case. Let $A_{j,s} = (A_j + A_j^T)/2$, and $S_j = \sqrt{A_{j,s}}$, and define A_s and S similarly. Let $\tilde{A}_j$ and $\tilde{A}$ be defined as in (4.6), and let $\tilde{h} = h \circ S$ and $\tilde{h}_j = h_j \circ S_j$. Since $\sqrt{\cdot}$ is continuous on the set of real symmetric matrices, $\tilde{h}_j \to \tilde{h}$ uniformly on compact subsets and both are harmonic. Thus, $\omega_{\tilde{h}_j} \rightharpoonup \omega_{\tilde{h}}$, and so

$$\lim_{j\to\infty} \omega^A_{h_j} \overset{(4.8)}{=} \lim_{j\to\infty} (\det S_j) S_j[\omega_{\tilde{h}_j}] = (\det S) S[\omega_{\tilde{h}}] \overset{(4.8)}{=} \omega^A_h.$$

$\square$

Lemma 5.5. *If $A \in \mathscr{C}$ and $h \in P_A(k)$ for some $k \in \mathbb{N}$, then*

$$||h||_{L^\infty(\mathbb{B})} \lesssim_{k,\Lambda} F_1(\omega^A_h). \tag{5.5}$$

Proof. Suppose instead that there exist $A_j \in \mathscr{C}$ and $h_j \in P_{A_j}(k)$ for which $||h_j||_{L^\infty(\mathbb{B})} > jF_1(\omega^{A_j}_{h_j})$. Without loss of generality, we may assume $||h_j||_{L^\infty(\mathbb{B})} = 1$, and thus $F_1(\omega^{A_j}_{h_j}) \to 0$. Using Cauchy estimates (see e.g. Proposition 11.3 [Mi13]), $\{h_j\}_{j=1}^\infty$ forms a normal family in $\mathbb{B}$ and thus, we can pass to a subsequence so that h_j converges uniformly on compact subsets of $\mathbb{B}$ and so that A_j converges to some $A \in \mathscr{C}$. Since all h_j are polynomials of order k, we know that the coefficients of h_j converge, which, in turn, implies that h_j converges to some function $h \in \mathscr{P}_{\mathscr{C}}(k)$ uniformly on compact subsets of $\mathbb{R}^{n+1}$. By Lemma 5.4, $\omega^{A_j}_{h_j} \to \omega^A_h$. In particular,

$$F_1(\omega^A_h) = \lim_{j\to\infty} F_1(\omega^{A_j}_{h_j}) = 0.$$

Thus, $\omega(B(0,r)) = 0$ for all $r < 1$, and so $0 \notin \operatorname{supp} \omega_h$. We will now show that in fact $0 \in \operatorname{supp} \omega^A_h$ in order to get a contradiction.

First, by Lemma 4.10, we can assume without loss of generality that $A = I$ and $\omega_h^A = \omega_h$. Secondly, notice that as $h_j \in \mathscr{P}_{\mathscr{C}}(k)$, $h \in \mathscr{P}(k)$ and so $h(0) = 0$. By Lojasiewicz's structure theorem for real analytic varieties (see e.g. [KP02, Theorem 6.3.3, p.168]), if U is a small enough neighborhood of a point $0 \in \Sigma_h$, we have that

$$U \cap \Sigma_h = V^n \cup V^{n-1} \cup \cdots \cup V^0,$$

where V^0 is either the empty set or the singleton $\{0\}$ and for each $i \in \{1, \ldots, n\}$, we may write V^i as a finite, disjoint union $V^i = \bigcup_{j=1}^{N_k} \Gamma_j^i$, of i-dimensional real analytic submanifolds. Further, for each $1 \leq i \leq n-1$,

$$U \cap \overline{V^i} \supset V^{i-1} \cup \cdots \cup V^0.$$

Moreover, for $1 \leq k \leq n$ and $1 \leq j \leq N_k$, $U \cap \partial \Gamma_j^i$ is a union of sets of the form Γ_m^ℓ, for $1 \leq \ell < i$ and $1 \leq m \leq N_\ell$ and possibly V^0.

By the main result in [CNV15], $\dim\{\nabla h = 0\} \leq n-1$, and thus $V^n \cap \{\nabla h = 0\}$ is a closed set of relatively empty interior in V^n, so in particular,

$$\overline{V^n \backslash \{\nabla h = 0\}} \cap U = \overline{V^n} \cap U = \Sigma_h \cap U \ni 0.$$

For $\zeta \in U \cap V^n \backslash \{\nabla h = 0\}$, the derivative of h at ζ tangent to V^n is always zero, as h is zero on V^n, which forces ∇h to be perpendicular to V^n. Since the normal derivative is nonzero,

$$U \cap V^n \backslash \{\nabla h = 0\} \subset \left\{ \zeta \in U \cap V^n : \frac{\partial h}{\partial \nu} \neq 0 \right\} \subset U \cap V^n \cap \operatorname{supp} \omega_h.$$

Thus, $0 \in U \cap \overline{V^n \backslash \{\nabla h = 0\}} \subset \operatorname{supp} \omega_h$, which gives us the contradiction and concludes the proof.

□

5.2. **Proof of Proposition I.** Proposition I will follow from the following more general result.

Lemma 5.6. *Let $\mathscr{S} \subset \mathscr{C}$ be closed (hence compact). Then $P_{\mathscr{S}}(k)$ and $\mathscr{F}_{\mathscr{S}}(k)$ have compact basis.*

Proof. Let $h_j \in P_{A_j}(k)$ with $A_j \in \mathscr{S}$ and assume $\mathscr{F}(\omega_{h_j}^{A_j}) = 1$. Then by (5.5) and Cauchy estimates, we can bound each coefficient of the polynomials h_j uniformly, and then pass to a subsequence so that $A_j \to A \in \mathscr{S}$ and h_j converges on compact subsets of $\mathbb{R}^{n+1}$ to a function $h \in P_A(k) \subset P_{\mathscr{S}}(k)$. By Lemma 5.4, we have that $\omega_{h_j} \to \omega_h$, which implies that $\mathscr{P}_{\mathscr{S}}(k)$ has compact basis. The proof for $\mathscr{F}_{\mathscr{S}}(k)$ is similar. □

As a corollary, we show the following stronger version of (5.5).

Corollary 5.7. *For $h \in P_{\mathscr{C}}(k)$ and $r > 0$,*

$$||h||_{L^\infty(r\mathbb{B})} \approx_k r^{-n} F_r(\omega_h). \tag{5.6}$$

Proof. Let $h \in P_{\mathscr{C}}(k)$ and $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ be such that $\mathbb{1}_{\frac{1}{2}\mathbb{B}} \leq \varphi \leq \mathbb{1}_{\mathbb{B}}$. Since $\mathscr{P}_{\mathscr{C}}(k)$ has compact basis by Lemma 5.6, we can estimate

$$\begin{aligned} F_1(\omega_h) &\overset{(3.4)}{\lesssim} F_{1/2}(\omega_h) \leq \int \varphi d\omega_h = \int_{\Omega_h} h\, \Delta\varphi \leq ||\Delta\varphi||_\infty \int_{\mathbb{B}} |h| \\ &\lesssim ||h||_{L^\infty(\mathbb{B})} \overset{(5.5)}{\lesssim} F_1(\omega_h). \end{aligned}$$

For $r \neq 1$, by the previous inequalities we have

$$F_r(\omega_h) \overset{(5.2)}{=} r^n F_1(\omega_{h\circ T_{0,r}^{-1}}) \approx r^n ||h \circ T_{0,r}^{-1}||_{L^\infty(\mathbb{B})} \approx r^n ||h||_{L^\infty(r\mathbb{B})}.$$

□

5.3. **Proof of Proposition II.**

Lemma 5.8. *Let $h \in H_A$, $A \in \mathscr{C}$, and*

$$h(x) = \sum_{j=m}^\infty \sum_{|\alpha|=j} \frac{D^\alpha h(0)}{\alpha!} x^\alpha = \sum_{j=m}^\infty h_j(x)$$

be its Taylor series (where $m > 0$ and $h_m \neq 0$) which converges uniformly to h on compact subsets of $\mathbb{R}^{n+1}$. Then $\mathrm{Tan}(\omega_h^A, 0) = \{c\,\omega_{h_m}^A : c > 0\}$.

Proof. For notational convenience, we will just consider the case $A = I$, the general case is identical. Note that as $r \to 0$, $r^{-m} h \circ T_{0,r}^{-1} \to h_m$ uniformly on compact subsets of $\mathbb{R}^{n+1}$. Indeed, fix $R > 0$. Then the series

$$r^{-m} \sum_{j=m}^\infty \sum_{|\alpha|=j} \frac{D^\alpha h(0)}{\alpha!} (rx)^\alpha = \sum_{j=m}^\infty \sum_{|\alpha|=j} \frac{D^\alpha h(0)}{\alpha!} x^\alpha r^{|\alpha|-m}$$

converges uniformly to $r^{-m} h \circ T_{0,r}^{-1}$ on compact subsets of $B(0,R)$, provided r is small enough. In fact, by Cauchy estimates,

$$|D^\alpha h(0)| \lesssim_n |\alpha|^{|\alpha|},$$

and since there exists a constant $C > 1$ such that $\frac{k^k}{k!} \lesssim C^k$, then, for $x \in B(0,R)$ and $r \in (0, \frac{1}{CR})$, we have that

$$\begin{aligned} \left| r^{-m} h \circ T_{0,r}^{-1}(x) - h_m(x) \right| &\leq \sum_{j=m+1}^\infty \sum_{|\alpha|=j} \left| \frac{D^\alpha h(0)}{\alpha!} \right| R^{|\alpha|} r^{|\alpha|-m} \\ &\lesssim_{n,m} \sum_{j=m+1}^\infty C^j R^j r^{j-m} \lesssim r^{-m}(CRr)^{m+1} = (CR)^{m+1} r \xrightarrow{r\downarrow 0} 0. \end{aligned}$$

Let now

$$\nu_r := r^{-m-n+1} T_{0,r}[\omega_h] \overset{(5.1)}{=} r^{-m}\, \omega_{h\circ T_{0,r}^{-1}} = \omega_{r^{-m}h\circ T_{0,r}^{-1}}.$$

By Lemma 5.4, $\nu_r \rightharpoonup \omega_{h_m} \in \mathscr{F}(m)$. In particular, every tangent measure of ω_h at zero must be a multiple of this one. $\square$

We now state an interesting consequence of these results: that if a portion of tangent measures of an arbitrary Radon measure are in $\mathscr{P}(k)$, then in fact they are all in $\mathscr{F}(k)$ (that is, we did not have to assume the original measure was special like harmonic measure).

Lemma 5.9. *Let ω be a Radon measure, $\xi \in \operatorname{supp}\omega$, and k be the minimal integer such that $\operatorname{Tan}(\omega,\xi)\cap\mathscr{P}(k) \neq \varnothing$, then $\operatorname{Tan}(\omega,\xi)\cap\mathscr{P}(k) \subset \mathscr{F}(k)$.*

We follow the proof in [Bad11, Lemma 5.9], which originally supposed that ω was harmonic measure for an NTA domain.

Proof. If $k = 1$, then $\mathscr{P}(1) = \mathscr{F}(1)$. Now suppose $k > 1$ and there is $h \in P(k)$ non-homogeneous such that $\omega_h \in \operatorname{Tan}(\omega,\xi)\cap\mathscr{P}(k)$. Since $h \in \mathscr{P}(k)$, we may write

$$h(x) = \sum_{j=m}^{k}\sum_{|\alpha|=j}\frac{D^\alpha h(0)}{\alpha!}x^\alpha = \sum_{j=m}^{k} h_m(x),$$

where $m < k$ since $h \in \mathscr{P}(k)$ is not homogeneous. By Lemma 5.8, $\operatorname{Tan}(\omega_h,0) = \{c\omega_{h_m} : c > 0\} \subset \mathscr{F}(m)$, and since $\operatorname{Tan}(\omega_h,0) \subset \operatorname{Tan}(\omega,\xi)$ by Lemma 3.9, $\operatorname{Tan}(\omega,\xi)\cap\mathscr{F}(m) \neq \varnothing$, contradicting the minimality of k. Thus, $\operatorname{Tan}(\omega,\xi)\cap\mathscr{P}(k) \subset \mathscr{F}(k)$. $\square$

We will also need the following result.

Lemma 5.10 ([Bad11] Lemma 4.7)**.** *Suppose $h \in P(m)$ for some m. There exist $\varepsilon = \varepsilon(n,m,k) > 0$ and $r_0 > 0$ so that if $d_r(\omega_h,\mathscr{F}(k)) < \varepsilon$ for all $r \geq r_0$, then $m = k$.*

Proof of Proposition II. Suppose $\operatorname{Tan}(\omega,\xi) \subset \mathscr{P}(k)$. Let m be the minimal integer for which $\operatorname{Tan}(\omega,\xi)\cap\mathscr{P}(m) \neq \varnothing$, so $m \leq k$. Then, by Lemma 5.9, $\operatorname{Tan}(\omega,\xi)\cap\mathscr{P}(m) \subset \mathscr{F}(m)$. In particular, $\operatorname{Tan}(\omega,\xi)\cap\mathscr{F}(m) \neq \varnothing$. Since, by Proposition I, $\mathscr{P}(k)$ has compact basis, we can use Lemma 5.10 and Lemma 3.10 to conclude $\operatorname{Tan}(\omega,\xi) \subset \mathscr{F}(m)$. $\square$

6. Proof of Theorem I

Lemma 6.1. *Let $\mathscr{S} \subset \mathscr{C}$ be closed and $\omega = \omega_\Omega^{A,x}$ be an L_A-harmonic measure where $A \in \mathscr{A}$ and $L_A \in \mathrm{VMO}(\Omega,\xi)$ at $\xi \in \operatorname{supp}\omega$. Also assume we have $\operatorname{Tan}(\omega,\xi) \subset \mathscr{H}_{\mathscr{S}}$. Let k be the smallest integer for which $\operatorname{Tan}(\omega,\xi) \cap \mathscr{F}_{\mathscr{S}}(k) \neq \varnothing$. Then $\operatorname{Tan}(\omega,\xi) \subset \mathscr{F}_{\mathscr{S}}(k)$. In particular,*

$$\lim_{r\to 0} \frac{\log \omega(B(\xi,r))}{\log r} = n+k-1, \tag{6.1}$$

i.e., the pointwise dimension of harmonic measure at the point ξ is $n+k-1$.

Proof. If $\operatorname{Tan}(\omega,\xi) \not\subset \mathscr{F}_{\mathscr{S}}(k)$, then by Corollary 3.12, there is $r_0 > 0$ so that for any $\varepsilon > 0$ small we may find $\nu \in \operatorname{Tan}(\omega,\xi)\backslash\mathscr{F}_{\mathscr{S}}(k)$ so that $d_{r_0}(\nu,\mathscr{F}_{\mathscr{S}}(k)) = \varepsilon$ and $d_r(\nu,\mathscr{F}_{\mathscr{S}}(k)) \leq \varepsilon$ for all $r \geq r_0$. Without loss of generality, we can assume $r_0 = 1$. For each $r > 1$, choose $\mu_r \in \mathscr{F}_{\mathscr{S}}(k)$ such that $F_r(\mu_r) = 1$ and

$$F_r\left(\frac{\nu}{F_r(\nu)}, \mu_r\right) < 2\varepsilon.$$

Then for $r \geq 1$,

$$\begin{aligned}\frac{F_r(\nu)}{F_{2r}(\nu)} &= \int (r-|x|)_+ d\frac{\nu}{F_{2r}(\nu)} < 2\varepsilon + \int (r-|x|)_+ d\mu_{2r} = 2\varepsilon + F_r(\mu_{2r}) \\ &\overset{(5.4)}{=} 2\varepsilon + 2^{-n-k}F_{2r}(\mu_{2r}) = 2\varepsilon + 2^{-n-k} = 2^{-n-k+\beta},\end{aligned}$$

for some $\beta > 0$ that goes to zero as $\varepsilon \to 0$. Similarly,

$$\frac{F_r(\nu)}{F_{2r}(\nu)} \geq 2^{-n-k-\beta}.$$

Hence, for $\ell \in \mathbb{N}$,

$$2^{\ell(n+k-\beta)} \leq \frac{F_{2^\ell r}(\nu)}{F_r(\nu)} \leq 2^{\ell(n+k+\beta)}. \tag{6.2}$$

Note that $\nu = \omega_h^A$ for some $h \in \mathscr{H}_A$ by Lemma 4.13 and $A \in \mathscr{S}$, and so

$$\begin{aligned}||h||_{L^\infty(2^\ell \mathbb{B})} &\overset{(4.28)}{\lesssim} 2^{\ell(1-n)}\omega_h(B(0,2^{\ell+1})) \leq 2^{-\ell n-1}F_{2^{\ell+2}}(\omega_h) \\ &\overset{(6.2)}{\leq} 2^{\ell(k+\beta)-1}F_{2^2}(\omega_h). \end{aligned} \tag{6.3}$$

Let α be a multi-index of length $|\alpha| > k$. Then we can pick $\varepsilon > 0$ small enough so that β is so small that $|\alpha| - k - \beta > 0$ holds. Thus, by Cauchy estimates,

$$|\partial^\alpha h(0)| \lesssim_\alpha 2^{-\ell|\alpha|}||h||_{L^\infty(2^\ell\mathbb{B})} \overset{(6.3)}{\lesssim} 2^{-\ell(|\alpha|-k-\beta)}F_{2^2}(\omega_h) \to 0$$

as $\ell \to \infty$, and so $h \in \mathscr{P}_A(k)$.

Suppose $h = \sum_{j=1}^{k} h_j$. If $\omega_h \notin \mathscr{F}_A(k)$, then there exists $j < k$ such that $h_j \neq 0$, and by Lemma 5.8, we infer that $\mathrm{Tan}(\omega_h^A, 0)$ contains an element of $\mathscr{F}_A(j)$. Since $\omega_h^A \in \mathrm{Tan}(\omega, \xi)$, we know that $\mathrm{Tan}(\omega_h^A, 0) \subset \mathrm{Tan}(\omega, \xi)$ by Lemma 3.9 and thus, $\mathrm{Tan}(\omega,\xi) \cap \mathscr{F}_A(j) \neq \varnothing$. Hence $\mathrm{Tan}(\omega,\xi) \cap \mathscr{F}_{\mathscr{S}}(j) \neq \varnothing$, contradicting the minimality of k. This proves $\mathrm{Tan}(\omega,\xi) \subset \mathscr{F}_{\mathscr{S}}(k)$.

For the final equality, note that $\mathrm{Tan}(\omega,\xi) \subset \mathscr{F}_{\mathscr{S}}(k)$ and so $\mathrm{Tan}(\omega,\xi)$ has compact basis. In particular, by Lemma 3.11,

$$\lim_{r\to 0} d_1(T_{\xi,r}[\omega], \mathscr{F}_{\mathscr{S}}(k)) = 0.$$

Thus, for $\varepsilon > 0$, there is $r_0 > 0$ such that for each $r \leq r_0$, there exists $\mu_r \in \mathscr{F}_{\mathscr{S}}(k)$ so that $F_1(\mu_r) = 1$ and

$$F_1\left(\frac{T_{\xi,r}[\omega]}{F_1(T_{\xi,r}[\omega])}, \mu_r\right) < \varepsilon.$$

Setting $\nu_r = r^{-1} T_{\xi,r}^{-1}[\mu_r]$, this gives $F_r(\nu_r) = 1$ and

$$F_r\left(\frac{\omega}{F_r(\omega)}, \nu_r\right) < \varepsilon.$$

By the same arguments as earlier, we can show that there exists $\gamma > 0$, which goes to zero as $\varepsilon \to 0$, so that for all $\ell \geq 0$ and $r < 2^{-\ell-1} r_0$,

$$2^{\ell(n+k-\gamma)} \leq \frac{F_{2^\ell r}(\omega)}{F_r(\omega)} \leq 2^{\ell(n+k+\gamma)}. \tag{6.4}$$

Hence, if we set $d = n + k - 1$, we get

$$\begin{aligned}
\omega(B(\xi, 2^\ell r)) &= T_{\xi,r}[\omega](B(0,2^\ell)) \leq 2^{-\ell} F_{2^{\ell+1}}(T_{\xi,r}[\omega]) \\
&\leq 2^{(\ell+1)(n+k+\gamma)-\ell} F_1(T_{\xi,r}[\omega]) \\
&\leq 2^{\ell(d+\gamma)+n+k+\gamma} T_{\xi,r}[\omega](B(0,1)) \\
&= 2^{\ell(d+\gamma)+n+k+\gamma} \omega(B(\xi,r)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\omega(B(\xi,r)) &= T_{\xi,r}[\omega](B(0,1)) \leq F_2(T_{\xi,r}[\omega]) \\
&\leq 2^{-(\ell-1)(n+k-\gamma)} F_{2^\ell}(T_{\xi,r}[\omega]) \\
&\leq 2^{-(\ell-1)(n+k-\gamma)+\ell} \omega(B(\xi, 2^\ell r)) \\
&= 2^{-\ell(d-\gamma)+n+k-\gamma} \omega(B(\xi, 2^\ell r)).
\end{aligned}$$

For $r < r_0/2$, let $\ell \in \mathbb{N}$ be so that $2^{-\ell-1} r_0 \leq r \leq 2^{-\ell} r_0$. Then

$$\begin{aligned}
\omega(B(\xi,r)) \leq \omega(B(\xi, 2^{-\ell} r_0)) &\leq 2^{-\ell(d-\gamma)+n+k-\gamma} \omega(B(\xi, r_0)) \\
&\leq 2^{1+(n+k-\gamma)} r^{d-\gamma} \omega(B(\xi, r_0)).
\end{aligned}$$

Hence, recalling that these logs are negative, we conclude

$$\liminf_{r\to 0}\frac{\log\omega(B(\xi,r))}{\log r}\geq\liminf_{r\to 0}\frac{\log\left(2^{1+(n+k-\gamma)}\omega(B(\xi,r_0))\right)}{\log r}+d-\gamma=d-\gamma.$$

A similar estimate gives

$$\limsup_{r\to 0}\frac{\log\omega(B(\xi,r))}{\log r}\leq d+\gamma.$$

If we let $\gamma\to 0$, then (6.1) follows.

□

Proof of Theorem I. We set

$$E^*=\left\{\xi\in E:\lim_{r\to 0}\frac{\omega^+(E\cap B(\xi,r))}{\omega^+(B(\xi,r))}=\lim_{r\to 0}\frac{\omega^-(E\cap B(\xi,r))}{\omega^-(B(\xi,r))}=1\right\}\text{ and}$$

$$E^{**}=\{\xi\in E^*:(1.4)\text{ holds}\}.$$

Notice that by [Mat95, Corollary 2.14 (1)] and because ω_1 and ω_2 are mutually absolutely continuous on E,

$$\omega^+(E\backslash E^{**})=\omega^-(E\backslash E^{**})=0.$$

Also, set

$$\begin{aligned}\Lambda_1=\Big\{\xi\in E^{**}:0<h(\xi):=\frac{d\omega^-}{d\omega^+}(\xi)&=\lim_{r\to 0}\frac{\omega^-(B(\xi,r))}{\omega^+(B(\xi,r))}\\&=\lim_{r\to 0}\frac{\omega^-(E\cap B(\xi,r))}{\omega^+(E\cap B(\xi,r))}<\infty\Big\}\end{aligned}$$

and

$$\Gamma=\left\{\xi\in\Lambda_1:\xi\text{ is a Lebesgue point for }h\text{ with respect to }\omega^+\right\}.$$

Again, by Lebesgue differentiation for measures (see [Mat95, Corollary 2.14 (2) and Remark 2.15 (3)]), Γ has full measure in E^{**} and hence in E.

Next, we record a lemma which was proven in [AMT16, Lemma 5.8] (which in turn is based on the work of [KPT09]) in the case of the harmonic functions in domains that satisfy the CDC condition, but its proof goes through unchanged for L-harmonic functions in general domains.

Lemma 6.2. *Let $\xi\in\Gamma$, $c_j\geq 0$, and $r_j\to 0$ be so that $\omega_j^+=c_jT_{\xi,r_j}[\omega^+]\to\omega_\infty$. Then $\omega_j^-=c_jT_{\xi,r_j}[\omega^-]\to h(\xi)\omega_\infty$.*

We define

$$\mathscr{F}:=\{c\mathcal{H}^n|_V:c>0,\quad V\text{ a }d\text{-dimensional plane containing the origin}\}.$$

It is not hard to show that $\mathscr{F}$ has compact basis.

Lemma 6.3. *For ω^+-a.e. $\xi \in \Gamma$,*

$$\operatorname{Tan}(\omega^+, \xi) \cap \mathscr{F} \neq \varnothing.$$

Proof. We can pick $\xi \in \Gamma$ so that $\operatorname{Tan}(\omega^+, \xi) \neq \varnothing$, let $\omega_\infty \in \operatorname{Tan}(\omega^+, \xi)$, so there is $c_j > 0$ and $r_j \downarrow 0$ so that $c_j T_{\xi,r_j}[\omega^+] \to \omega_\infty$. By Lemma 6.2, we also have $c_j T_{\xi,r_j}[\omega^-] \to h(\xi)\omega_\infty$. By Lemma 4.13, (4.27) holds.

In particular, $\frac{1}{2}\mathbb{B} \cap \operatorname{supp} \omega_\infty$ is a smooth real analytic variety, and arguing as in [AMTV16], for example, one deduces that

$$d\omega_\infty|_{\frac{1}{2}\mathbb{B}} = -c_n(\nu_{\Omega_\infty^+} \cdot A_0 \nabla u_\infty)\, d\mathcal{H}^n|_{\partial^*\Omega_\infty^+ \cap \frac{1}{2}\mathbb{B}},$$

where A_0 is the matrix from Lemma 4.13, $\partial^*\Omega_\infty^+$ is the reduced boundary of $\Omega_\infty^+ = \{u_\infty > 0\}$ and $\nu_{\Omega_\infty^+}$ is the measure theoretic outer unit normal. Hence, ω_∞ is absolutely continuous with respect to surface measure of $\partial\Omega_\infty^+$ in $\frac{1}{2}\mathbb{B}$. Thus, since the tangent measure at $\mathcal{H}^n$-almost every point of $\partial\Omega_\infty^+$ is contained in $\mathscr{F}$, we can take another tangent measure of ω_∞ that is in $\mathscr{F}$ and apply Theorem 3.8 to conclude the proof. □

By Lemmas 6.1 and 6.3, we also have that $\dim \omega^+|_E = n$. It remains to show that, if $\Omega^\pm$ both have the CDC, then $\lim_{r\to 0} \Theta^{\mathscr{F}}_{\partial\Omega^+}(\xi, r) = 0$ for ω^+-a.e. $\xi \in E$. But this follows almost immediately because, for almost every $\xi \in \Gamma$ and any $r_j \downarrow 0$, we may pass to a subsequence so that, by Lemma 4.11 (a) and (f), $\lim_{j\to\infty} \Theta^{\mathscr{F}}_{\partial\Omega^+}(\xi, r_j) = 0$. This finishes the proof.

□

7. BMO, VMO, and vanishing A_∞

In this section, we will prove some estimates relating the logarithm of a Radon-Nikodym derivative to the mutual absolute continuity continuity properties of two measures. We will apply them to the specific case of elliptic measure, but we will prove them for general measures.

Definition 7.1. Let μ be a Radon measure on a metric space X. We say that a function $f \in L^1_{loc}(\mu)$ is of *bounded mean oscillation* and write $f \in \mathrm{BMO}(\mu)$, if there exists a constant $C > 0$ such that

$$\sup_{r\in(0,\infty)} \sup_{x\in\operatorname{supp}\mu} \fint_{B(x,r)} |f - f_{B(x,r)}| d\mu \leq C, \tag{7.1}$$

where $f_A := \fint_A f\, d\mu := \mu(A)^{-1} \int_A f\, d\mu$, for any $A \subset X$ with $\mu(A) > 0$. We define the space of *vanishing mean oscillation* $\mathrm{VMO}(\mu)$ to be the closure in the $\mathrm{BMO}(\mu)$ norm of the set of bounded uniformly continuous functions defined on X. Equivalently, we say $f \in \mathrm{VMO}(\mu)$ if $f \in L^1_{loc}(\mu)$ and

$$\lim_{r\to 0} \sup_{x\in\operatorname{supp}\mu} \fint_{B(x,r)} |f - f_{B(x,r)}| d\mu = 0, \tag{7.2}$$

Definition 7.2. For two measures μ and ν on a metric space X, we will say $\nu \in A_\infty(\mu)$ if $\mu \ll \nu$ there is $K = K(\mu,\nu)$ so that for any ball B centered on the support of μ,

$$\fint_B \frac{d\nu}{d\mu} d\mu \exp\left(-\fint_B \log \frac{d\nu}{d\mu} d\mu\right) \leq K(\mu,\nu). \tag{7.3}$$

We will say $\nu \in A_\infty'(\mu)$ if there are $\varepsilon, \delta \in (0,1)$ so that for all $B \subseteq X$ and $E \subseteq B$,

$$\frac{\mu(E)}{\mu(B)} < \delta \text{ implies } \frac{\nu(E)}{\nu(B)} < \varepsilon. \tag{7.4}$$

We will say $\nu \in VA_\infty(\mu)$ (or *vanishing A_∞ with respect to μ*) if

$$\lim_{r\to 0} \sup_{\xi \in \operatorname{supp}\mu} \fint_B \frac{d\nu}{d\mu} d\mu \exp\left(-\fint_B \log \frac{d\nu}{d\mu} d\mu\right) = 1 \tag{7.5}$$

and $\nu \in VA_\infty'(\mu)$ if for all $r > 0$ there is $\varepsilon_r \in (0,1)$ so that $\lim_{r\to 0} \varepsilon_r = 0$ and $\delta_r > 0$ so that for all balls $B \subset X$ with $r_B < r$ and $E \subset B$,

$$\frac{\mu(E)}{\mu(B)} < \delta_r \quad \Rightarrow \quad \frac{\nu(E)}{\nu(B)} < \varepsilon_r. \tag{7.6}$$

In the case that $X = \mathbb{R}^{n+1}$ and μ is equal to the $(n+1)$-dimensional Lebesgue measure, A_∞ equivalence is the same as A_∞'-equivalence, and this is from Reimann and Rychener [RR75], although it was also shown later by Hruščev in [Hru84] and García-Cuerva and Rubio de Francia in [GaCRdF85].

We recall a notion introduced by Korey [Kor98].

Definition 7.3. A probability space (X,μ) is *halving* if every subset $E \subset X$ of positive measure has a subset $F \subset E$ so that $\mu(F) = \mu(E)/2$.

We will first focus on proving the following after a series of other lemmas.

Lemma 7.4. *Let (X,μ) be a metric measure space, $\nu \ll \mu$, and $f = \frac{d\nu}{d\mu}$.*

(1) If $\nu \in A_\infty'(\mu)$ and $\log f \in \mathrm{BMO}(\mu)$, then $\nu \in A_\infty(\mu)$. If X is also halving, then $\nu \in A_\infty(\mu)$ implies $\nu \in A_\infty'(\mu)$ and $\log f \in \mathrm{BMO}(\mu)$.

(2) If $\nu \in VA_\infty'(\mu)$ and $\log f \in \mathrm{VMO}(\mu)$, then $\nu \in VA_\infty(\mu)$. If X is also halving, then $\nu \in VA_\infty(\mu)$ implies $\nu \in VA_\infty'(\mu)$ and $\log f \in \mathrm{VMO}(\mu)$.

The first implication of the second half of (1) of the lemma follows from the following theorem.

Theorem 7.5 (Theorem 1, [Hru84])**.** *Suppose $\nu \ll \mu$, B is a ball centered on $\operatorname{supp}\mu$, and*

$$\fint_B \frac{d\nu}{d\mu} d\mu \exp\left(-\fint_B \log \frac{d\nu}{d\mu} d\mu\right) \leq C.$$

Then there are $\varepsilon, \delta > 0$ so that, for any $F \subset B \cap \operatorname{supp}\mu$,

$$\frac{\mu(F)}{\mu(B)} < \delta \text{ implies } \frac{\nu(F)}{\nu(B)} < \varepsilon. \tag{7.7}$$

Moreover, there is $\delta > 0$ so that

$$\frac{\mu(F)}{\mu(B)} < \delta \text{ implies } \frac{\nu(F)}{\nu(B)} < 2(C-1). \tag{7.8}$$

In particular, if $\nu \in A_\infty(\mu)$, then $\nu \in A_\infty'(\nu)$, and if $\nu \in VA_\infty(\mu)$, then $\nu \in VA_\infty'(\mu)$.

Proof. We follow the proof from [Hru84, Theorem 1], since he proves (7.7) but not (7.8). Let $\delta \in (0,1)$ to be chosen later, $F \subseteq B$ and suppose $\mu(F) = \delta\mu(B)$, we will pick δ later. Let $f = \frac{d\nu}{d\mu}$, $E = B \backslash F$, and set

$$t = \frac{\nu(E)}{\nu(F)}.$$

Let $g_B = \fint_B f d\mu$. Then

$$\log C \geq (\log f^{-1})_B + \log f_B = \frac{\mu(E)}{\mu(B)}(\log f^{-1})_E + \frac{\mu(F)}{\mu(B)}(\log f^{-1})_F + \log f_B. \tag{7.9}$$

By Jensen's inequality, for any set S

$$(\log f^{-1})_S = -(\log f)_S \geq -\log f_S$$

and applying this to $S = E, F$, we have

$$\begin{aligned}
\log C &\geq -\frac{\mu(E)}{\mu(B)} \log f_E - \frac{\mu(F)}{\mu(B)} \log f_F + \log f_B \\
&\geq -\frac{\mu(E)}{\mu(B)} \log f_E - \frac{\mu(F)}{\mu(B)} \log f_E + \frac{\mu(F)}{\mu(B)} \log \frac{\mu(F)}{\mu(E)} + \frac{\mu(F)}{\mu(B)} \log t \\
&\quad + \log f_B \\
&= -\log f_E + \frac{\mu(F)}{\mu(B)} \log \frac{\mu(F)}{\mu(E)} + \frac{\mu(F)}{\mu(B)} \log t + \log f_B.
\end{aligned}$$

Now observe that

$$-\log f_E = \log\left(\frac{\mu(E)}{\mu(B)} \frac{\mu(B)}{\nu(B)} \frac{\nu(B)}{\nu(E)}\right) = \log \frac{\mu(E)}{\mu(B)} - \log f_B + \log\left(1 + \frac{1}{t}\right)$$

and so we have

$$\begin{aligned}\log C &\geq \log \frac{\mu(E)}{\mu(B)} + \log\left(1+\frac{1}{t}\right) + \frac{\mu(F)}{\mu(B)}\log\frac{\mu(F)}{\mu(E)} + \frac{\mu(F)}{\mu(B)}\log t \\ &= \frac{\mu(F)}{\mu(B)}\log\frac{\mu(F)}{\mu(B)} + \frac{\mu(E)}{\mu(B)}\log\frac{\mu(E)}{\mu(B)} + \log(1+t) + \frac{\mu(E)}{\mu(B)}\log\frac{1}{t} \\ &= \underbrace{\delta\log\delta + (1-\delta)\log(1-\delta)}_{=:\phi(\delta)} + \log(1+t) + \frac{\mu(E)}{\mu(B)}\log\frac{1}{t}.\end{aligned}$$

Note that $\lim_{\delta\to 0}\phi(\delta) = 0$. Let $\alpha > 0$ and pick $\delta > 0$ so that $|\phi(\delta)| < \alpha \log C$. Then

$$(1+\alpha)\log C \geq \log(1+t) + \frac{\mu(E)}{\mu(B)}\log\frac{1}{t}. \tag{7.10}$$

We restrict δ further so that $\delta < \alpha$. If $t > 1$, then $\frac{\mu(E)}{\mu(B)}\log\frac{1}{t} \geq \log\frac{1}{t}$; otherwise, $\frac{\mu(E)}{\mu(B)}\log\frac{1}{t} \geq (1-\alpha)\log\frac{1}{t}$ since $\frac{\mu(E)}{\mu(B)} = 1-\delta > 1-\alpha$. Thus, in any case, we have

$$\frac{1+\alpha}{1-\alpha}\log C > \log\frac{1}{t}. \tag{7.11}$$

This implies $t \geq c = C^{-(1+\alpha)/(1-\alpha)}$, and so

$$\nu(F) = \frac{\nu(F)}{1+t} + \frac{t\nu(F)}{1+t} = \frac{\nu(F)+\nu(E)}{1+t} = \frac{\nu(B)}{1+t} \leq \frac{\nu(B)}{1+c}.$$

This proves (7.7) with $\varepsilon = (1+c)^{-1}$. To prove (7.8), we go back to (7.10) with the same bound on δ. Then, since $t \geq c$,

$$\begin{aligned}(1+\alpha)\log C &\geq \log(1+t) + \frac{\mu(E)}{\mu(B)}\log\frac{1}{t} = \log\left(1+\frac{1}{t}\right) + \frac{\mu(F)}{\mu(B)}\log t \\ &\geq \log\left(1+\frac{1}{t}\right) - \delta\frac{1+\alpha}{1-\alpha}\log C.\end{aligned}$$

Since $\delta < \alpha$, this implies

$$\begin{aligned}\log\left(1+\frac{1}{t}\right) &< \left(1+\alpha+\delta\frac{1+\alpha}{1-\alpha}\right)\log C = (1+\alpha)\left(1+\frac{\delta}{1-\alpha}\right)\log C \\ &< \frac{1+\alpha}{1-\alpha}\log C,\end{aligned}$$

and so

$$C^{(1+\alpha)/(1-\alpha)} - 1 > \frac{1}{t}.$$

We now pick α so that $C^{(1+\alpha)/(1-\alpha)} - 1 = 2(C-1)$, and we are done.

$\square$

Showing that VA_∞ implies the logarithm of the density is VMO was shown by Korey.

Theorem 7.6 (Theorem 4 and Section 3.5 [Kor98])**.** *There is a universal constant $c>0$ so that the following holds. Let (X,μ) be a halving probability space, and suppose that*

$$\left(\int_X \exp g d\mu\right) / \exp\left(\int_X g d\mu\right) \leq K. \tag{7.12}$$

Then

$$\int_X \left| g - \int_X g d\mu \right| d\mu \leq \log 2K \tag{7.13}$$

and as $K \to 1$,

$$\int_X \left| g - \int_X g d\mu \right| d\mu \leq c\sqrt{K-1} \tag{7.14}$$

Lemma 7.7. *let (X,μ) be a metric probability space and suppose $\nu \ll \mu$. Let $\varepsilon, \delta \in (0,1)$ be so that for any $E \subset X$,*

$$\mu(E) < \delta\mu(X) \quad \Rightarrow \quad \nu(E) < \varepsilon\nu(X). \tag{7.15}$$

Set $f = \frac{d\nu}{d\mu}$ and assume

$$⨍_X \left| \log f - ⨍_X \log f d\mu \right| d\mu < \eta. \tag{7.16}$$

Then

$$1 \leq ⨍_X f d\mu \exp\left(-⨍_X \log f d\mu\right) \leq \frac{e^{\eta/\delta}}{1-\varepsilon}. \tag{7.17}$$

Proof. Without loss of generality, we may assume $\mu(X) = \nu(X) = 1$. Let $\varepsilon > 0$ and pick δ so that (7.15) holds.

Let $c = \int_X \log f \, d\mu$ and

$$G = \{|\log f - c| < \rho := \eta\delta^{-1}\}, \quad F = G^c. \tag{7.18}$$

Then, by Chebysev's inequality and (7.16), we infer that $\mu(F) < \delta$, which, in turn, by (7.15), implies

$$\nu(F) < \varepsilon. \tag{7.19}$$

Moreover, on the set G,

$$\frac{\eta}{\delta} > |\log f - c|$$

and so

$$f \leq e^{c+\eta/\delta} \text{ on } G. \tag{7.20}$$

Then,

$$1=\frac{\nu(X)}{\mu(X)}=\int_X f\,d\mu \overset{(7.20)}{\leq} \left(\int_G e^{c+\eta/\delta}d\mu+\int_F f\,d\mu\right)$$
$$\leq e^{c+\eta/\delta}+\nu(F)\overset{(7.19)}{<} e^{c+\eta/\delta}+\varepsilon.$$

Thus,

$$(1-\varepsilon)\int_X f\,d\mu=1-\varepsilon<e^{c+\eta/\delta}$$

and so

$$\int_X f\,d\mu<\frac{e^{c+\eta/\delta}}{1-\varepsilon}.$$

This and Jensen's inequality imply

$$1\leq e^{-c}\int_X f\,d\mu<e^{-c}\frac{1}{1-\varepsilon}e^{c+\eta/\delta}=\frac{1}{1-\varepsilon}e^{\eta/\delta}. \tag{7.21}$$

□

Corollary 7.8. *let (X,μ) be a metric measure space. Set $f=\frac{d\nu}{d\mu}$ and assume that for some sequence of balls B_j in X,*

$$\lim_j \fint_{B_j}\left|\log f-\fint_{B_j}\log f\,d\mu\right|d\mu=0. \tag{7.22}$$

and for all $\varepsilon>0$ there is $\delta>0$ so that for j sufficiently large,

$$\frac{\mu(E)}{\mu(B_j)}<\delta \text{ implies } \frac{\nu(E)}{\nu(B_j)}<\varepsilon. \tag{7.23}$$

Then

$$\lim_{j\to\infty}\fint_{B_j} f\,d\mu\exp\left(-\fint_{B_j}\log f\,d\mu\right)=1. \tag{7.24}$$

In particular, if $\log f\in \mathrm{VMO}(d\mu)$ and $\nu\in VA_\infty'(\mu)$, then $\nu\in VA_\infty(\mu)$.

Proof. Let $\varepsilon,\eta>0$ and let $\delta>0$ be so that (7.23) holds for j large enough. Then (7.16) holds (with B_j in place of X and $\mu|_{B_j}$ in place of μ). Then (7.17) must hold. In particular,

$$\limsup_{j\to\infty}\fint_{B_j} f\,d\mu\exp\left(-\fint_{B_j}\log f\,d\mu\right)\leq\frac{e^{\eta/\delta}}{1-\varepsilon}.$$

As ε and δ did not depend on η, we can send $\eta\to 0$, and then $\varepsilon\to 0$ since δ now vanishes from the inequality, and then we obtain (7.24). □

Proof of Lemma 7.4. The second halves of (1) and (2) follow from Theorems 7.5 and 7.6. The first half of (1) follows from Lemma 7.7, and the first half of (2) is from Corollary 7.8. □

Lemma 7.9. *Let $\Omega \subset \mathbb{R}^{n+1}$ be any connected domain and $\omega = \omega_{\Omega}^{L_A,x}$ where $A \in \mathscr{A}(\Omega)$. Then ω is halving.*

Proof. Suppose there is $E \subset \partial\Omega$ with $\omega(E) > 0$ that is not halving. For $t \in \mathbb{R}$ and $v \in \mathbb{S}^{n-1}$, let $H_{t,v} = \{x \in \mathbb{R}^{n+1} : x \cdot v \geq t\}$. Then $t \mapsto \omega(H_{t,v} \cap E)$ is not continuous for any $v \in \mathbb{S}^n$, and so there is t_v so that $\omega(\partial H_{t_v,v} \cap E) > 0$. Let $V_v = \partial H_{t_v,v}$, which is an n-dimensional plane. Since $\mathbb{S}^n$ is uncountable, there is $\varepsilon > 0$ so that $\omega(V_v \cap E) > \varepsilon > 0$ for all v in some uncountable set $A \subset \mathbb{S}^n$. Let $A' \subset A$ be countable. Note that for any $u, v \in A'$ distinct, $V_u \cap V_v$ is an $(n-1)$-dimensional subspace. This implies $V_u \cap V_v$ has 2-capacity zero [HKM06, Theorem 2.27], hence is a polar set for ω [HKM06, Theorem 10.1] and polar sets have L_A-harmonic measure zero [HKM06, Theorem 11.15]. Thus, if we set

$$W_u := V_u \backslash \bigcup_{\substack{v \in A' \\ v \neq u}} V_v,$$

we have that $\omega(W_u \cap E) = \omega(V_u \cap E) \geq \varepsilon$ and W_u are mutually disjoint. But since A' is infinite, this implies $\omega(E) = \infty$, which is a contradiction. $\square$

Lemma 7.10. *Let $\Omega^+ \subset \mathbb{R}^{n+1}$ be a connected domain with connected complement $\Omega^- = \text{ext}(\Omega^+)$ and let L_A be a uniformly elliptic operator with real coefficients. If $\omega^\pm$ denote the L_A-harmonic measures of $\Omega^\pm$ with fixed poles $x^\pm \in \Omega^\pm$, then $\omega^- \in A_\infty(\omega^+)$ if and only if $\omega^- \in A'_\infty(\omega^+)$ and $\log \frac{d\omega^-}{d\omega^+} \in \text{BMO}(d\omega^+)$. Moreover, $\omega^- \in VA_\infty(\omega^+)$ if and only if $\omega^- \in VA'_\infty(\omega^+)$ and $\log \frac{d\omega^-}{d\omega^+} \in \text{VMO}(d\omega^+)$.*

Proof. This follows from Lemmas 7.4 and 7.9. $\square$

8. Proofs of Theorems II and III

Lemma 8.1. *Let $\omega^\pm$ be two halving Radon measures with equal supports and set $f = \log \frac{d\omega^-}{d\omega^+}$. Suppose there are $r_j \downarrow 0$ and $\xi_j \in \partial\Omega^+$ so that $\omega_j^+ = T_{\xi_j,r_j}[\omega^+]/\omega(B(\xi_j, r_j))$ converges weakly to some measure ω with $\omega(\mathbb{B}) > 0$. Further assume that for all $M > 0$*

$$\lim_j \fint_{B(\xi_j,Mr_j)} f\, d\omega^+ \exp\left(-\fint_{B(\xi_j,Mr_j)} \log f\, d\omega^+\right) = 1. \tag{8.1}$$

Then $\omega_j^- \rightharpoonup \omega$ as well.

The proof is similar to that of [KT06, Theorem 4.4], though using the techniques of the previous section, we no longer require the doubling assumption.

Proof. Let $B_j = B(\xi_j, r_j)$ and for a ball B set $c_B = ⨍_B \log f$. By assumption, for each $M > 0$,

$$e^{-c_{MB_j}} \frac{\omega^-(MB_j)}{\omega^+(MB_j)} \to 1 \text{ as } j \to \infty. \tag{8.2}$$

Let $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ with support in $B(0, M)$ for some $M > 0$ and let $\varphi_j = \varphi \circ T_{\xi_j, r_j}$. Then $\operatorname{supp} \varphi_j \subset MB_j$. Let $\varepsilon > 0$. By (8.2), for j large enough, we have that

$$0 \leq e^{-c_{B_j}} \frac{\omega^-(B_j)}{\omega^+(B_j)} - 1 < \varepsilon \quad \text{and} \quad 0 \leq e^{-c_{MB_j}} \frac{\omega^-(MB_j)}{\omega^+(MB_j)} - 1 < \varepsilon. \tag{8.3}$$

Let now $\eta = c\sqrt{1-\varepsilon}$, where c is the constant in (7.14). For j large enough, Theorem 7.6 and (8.2) imply

$$⨍_{B_j} |\log f - c_{B_j}| \, d\omega^+ < \eta \quad \text{and} \quad ⨍_{MB_j} |\log f - c_{MB_j}| \, d\omega^+ < \eta. \tag{8.4}$$

Note that ε is independent of η. For fixed $\delta > 0$ and for a ball B, we set

$$G_B = \{\xi \in B \cap \partial\Omega^+ : |\log f(\xi) - c_B| \leq \eta/\delta\}, \quad F_B = B \backslash G_B.$$

Then, Chebyshev's inequality and (8.4) imply

$$\omega^+(F_{B_j}) < \delta\, \omega^+(B_j) \quad \text{and} \quad \omega^+(F_{MB_j}) < \delta\, \omega^+(MB_j), \tag{8.5}$$

and for $\delta > 0$ small enough and j large enough, Theorem 7.5 and (8.2) imply

$$\omega^-(F_{B_j}) < \varepsilon\, \omega^-(B_j) \quad \text{and} \quad \omega^-(F_{MB_j}) < \varepsilon\, \omega^-(MB_j). \tag{8.6}$$

Let $C = 2\frac{\omega(\overline{M\mathbb{B}})}{\omega(\mathbb{B})}$. Since $\omega(\mathbb{B}) > 0$, we know

$$\limsup_{j\to\infty} \frac{\omega^+(MB_j)}{\omega^+(B_j)} = \limsup_{j\to\infty} \frac{\omega_j^+(M\mathbb{B})}{\omega_j^+(\mathbb{B})} \leq \frac{\omega(\overline{M\mathbb{B}})}{\omega(\mathbb{B})} = C/2,$$

and so for j large enough,

$$\omega^+(MB_j) \leq C\omega^+(B_j). \tag{8.7}$$

Also, note that for j large enough,

$$\begin{aligned} |c_{B_j} - c_{MB_j}| &= \left| ⨍_{B_j} (c_{B_j} - c_{MB_j}) \right| d\omega^+ \\ &\leq ⨍_{B_j} |c_{B_j} - \log f| \, d\omega^+ + ⨍_{B_j} |\log f - c_{MB_j}| \, d\omega^+ \\ &\overset{(8.4)}{<} \eta + \frac{\omega^+(MB_j)}{\omega^+(B_j)} ⨍_{MB_j} |\log f - c_{MB_j}| \, d\omega^+ \overset{(8.4)}{\underset{(8.7)}{<}} (1+C)\eta. \end{aligned} \tag{8.8}$$

Hence,

$$\begin{aligned}\omega^-(MB_j) &\overset{(8.3)}{\leq} \omega^+(MB_j)(1+\varepsilon)e^{c_{MB_j}} \overset{(8.7)}{\underset{(8.8)}{<}} C\omega^+(B_j)(1+\varepsilon)e^{c_{B_j}+(1+C)\eta} \\ &\overset{(8.3)}{\leq} C\omega^-(B_j)(1+\varepsilon)e^{(1+C)\eta} \leq 2Ce^{(1+C)}\omega^-(B_j) \lesssim_C \omega^-(B_j).\end{aligned} \tag{8.9}$$

Then

$$\begin{aligned}\int \varphi d\omega_j^- - \int \varphi d\omega_j^+ &= \frac{1}{\omega^-(B_j)}\int_{MB_j}\varphi_j d\omega^- - \frac{1}{\omega^+(B_j)}\int_{MB_j}\varphi_j d\omega^+ \\ &= \underbrace{\frac{1}{\omega^-(B_j)}\int_{MB_j\cap F_{MB_j}}\varphi_j f d\omega^+}_{=:I_1} \\ &\quad + \underbrace{\frac{1}{\omega^-(B_j)}\int_{MB_j\cap G_{MB_j}}\left(f-e^{c_{MB_j}}\right)\varphi_j d\omega^+}_{=:I_2} \\ &\quad - \underbrace{\frac{e^{c_{MB_j}}}{\omega^-(B_j)}\int_{MB_j\cap F_{MB_j}}\varphi_j d\omega^+}_{=:I_3} \\ &\quad + \underbrace{\frac{e^{c_{MB_j}}}{\omega^-(B_j)}\int_{MB_j}\varphi_j d\omega^+ - \frac{1}{\omega^+(B_j)}\int_{MB_j}\varphi_j d\omega^+}_{=:I_4} \\ &= I_1+I_2-I_3+I_4.\end{aligned}$$

We will estimate each of these terms separately, with the understanding that j is large enough (depending on M and η).

$$\begin{aligned}|I_1| &\leq \frac{||\varphi||_\infty}{\omega^-(B_j)}\int_{MB_j}\mathbb{1}_{F_{MB_j}} f d\omega^+ \\ &= \frac{||\varphi||_\infty \omega^-(F_{MB_j})}{\omega^-(B_j)} = \frac{\omega^-(MB_j)}{\omega^-(B_j)}\frac{||\varphi||_\infty\omega^-(F_{MB_j})}{\omega^-(MB_j)} \overset{(8.6)}{\underset{(8.9)}{\lesssim}}{}_{C,M,||\varphi||_\infty} \varepsilon.\end{aligned}$$

Next, for points in G_{MB_j},

$$e^{-\eta/\delta}e^{c_{MB_j}} \leq f \leq e^{\eta/\delta}e^{c_{MB_j}}$$

and so

$$e^{c_{MB_j}}(e^{-\eta/\delta}-1) \leq f - e^{c_{MB_j}} \leq e^{c_{MB_j}}(e^{\eta/\delta}-1).$$

Thus, for $\eta>0$ small enough (i.e., for j large enough), we can make

$$|f-e^{c_{MB_j}}| < \delta e^{c_{MB_j}} \quad \text{on } G_{MB_j}.$$

Therefore,

$$|I_2| \leq \frac{\delta e^{c_{MB_j}} ||\varphi||_\infty}{\omega^-(B_j)} \omega^+(G_{MB_j}) \leq \frac{\delta e^{c_{MB_j}} ||\varphi||_\infty}{\omega^-(B_j)} \omega^+(MB_j)$$
$$= e^{c_{MB_j}} \frac{\omega^+(MB_j)}{\omega^-(MB_j)} \frac{\delta ||\varphi||_\infty \omega^-(MB_j)}{\omega^-(B_j)} \overset{(8.9)}{\underset{(8.3)}{\lesssim}}_{||\varphi||_\infty, C, M} \delta.$$

$$|I_3| \leq \frac{e^{c_{MB_j}} ||\varphi||_\infty}{\omega^-(B_j)} \omega^+(F_{MB_j}) \overset{(8.5)}{<} \delta \frac{e^{c_{MB_j}} ||\varphi||_\infty}{\omega^-(B_j)} \omega^+(MB_j)$$
$$= \delta \frac{e^{c_{MB_j}} ||\varphi||_\infty \omega^-(MB_j)}{\omega^-(B_j)} \frac{\omega^+(MB_j)}{\omega^-(MB_j)} \overset{(8.3)}{\underset{(8.9)}{\lesssim}}_{C, M, ||\varphi||_\infty} \delta.$$

Finally,

$$|I_4| \leq \left(e^{c_{MB_j}} \frac{\omega^+(B_j)}{\omega^-(B_j)} - 1 \right) \frac{\omega^+(MB_j)}{\omega^+(B_j)} \fint_{MB_j} \varphi_j d\omega^+ \overset{(8.3)}{\underset{(8.7)}{\lesssim}}_{C, ||\varphi||_\infty, M} \varepsilon.$$

Since these estimates hold for all j large enough, we can conclude

$$\limsup_{j \to \infty} \left| \int \varphi d\omega_j^- - \int \varphi_j d\omega_j^+ \right| \lesssim_{C, M, ||\varphi||_\infty} \varepsilon + \delta.$$

Now send δ to zero since it only had to be small enough depending on ε. Finally, ε was arbitrarily chosen, which implies that the above limit is zero. Since this holds for all φ, we get that $\omega_j^\pm$ have the same weak limit.

□

Proof of Theorem II. Let $\omega \in \mathrm{Tan}(\omega^+, \xi)$. We claim that $\omega \in \mathscr{H}_{\mathscr{C}}$. By Lemma 3.6, $\omega = c T_{0,r}(\mu)$ for some constants $c, r > 0$ and some measure μ of the form $\mu = \lim_{j \to 0} T_{\xi, r_j}[\omega^+]/\omega^+(B(\xi, r_j))$ for some $r_j \downarrow 0$ where $\mu(\mathbb{B}) > 0$. By Lemma 8.1, $\mu = \lim_{j \to 0} T_{\xi, r_j}[\omega^-]/\omega^-(B(\xi, r_j))$ as well. By Lemma 4.13 (or Lemma 4.11(g) if $\Omega^\pm$ have the CDC), $\mu \in \mathscr{H}_{\mathscr{C}}$, and since $\mathscr{H}_{\mathscr{C}}$ is a d-cone by Lemma 5.3, we also have that $\omega \in \mathscr{H}_{\mathscr{C}}$, which proves the claim.

Hence, $\omega = \omega_u$ for some $u \in H_A$ and some $A \in \mathscr{C}$. By Lemma 5.8, for some $k > 0$,

$$\mathrm{Tan}(\omega_u, 0) = \{c \omega_{u_k} : c > 0\} \subset \mathscr{F}_A(k) \subset \mathscr{F}_{\mathscr{C}}(k),$$

and since $\mathrm{Tan}(\omega_u, 0) \subset \mathrm{Tan}(\omega^+, \xi)$ by Lemma 3.9, we now know that $\mathrm{Tan}(\omega^+, \xi) \cap \mathscr{F}_{\mathscr{C}}(k) \neq \varnothing$ as well. By Lemma 6.1, $\mathrm{Tan}(\omega^+, \xi) \subset \mathscr{F}_{\mathscr{C}}(k)$. The proof that $\Theta_{\partial \Omega^+}^{\mathscr{F}_{\Sigma, \mathscr{C}}(k)}(\xi, r) \to 0$ if $\Omega^\pm$ have the CDC is similar to the proof of Theorem I.

□

Proof of Theorem III. Let K be any compact subset of $\partial\Omega^+$. Suppose there was a sequence of radii $r_j \downarrow 0$ and $\xi_j \in K$ so that

$$d_1(T_{\xi_j,r_j}[\omega^+], \mathscr{P}_{\mathscr{C}}(d)) \geq \varepsilon > 0 \tag{8.10}$$

where d will be chosen later, but it will depend only on n and the doubling constant of ω^+.

Since ω^+ is doubling, we may pass to a subsequence so that $\omega_j^+ := T_{\xi_j,r_j}[\omega^+]/\omega^+(B(\xi_j,r_j))$ converges weakly to some measure ω.

If $f = \frac{d\omega^-}{d\omega^+}$ satisfies $\log f \in \mathrm{VMO}(\omega^-)$, then doubling also implies that $\omega^- \in VA_\infty'(\omega^+)$. Indeed, if ω^+ is doubling, then the John-Nirenberg theorem holds, and the VMO condition tells us that on small enough balls, f is a traditional A_p-weight (c.f. [Gar07, Chapter 6.2]). This easily implies $f d\omega^+ = d\omega^- \in VA_\infty'(\omega^+)$. Thus, by Corollary 7.8, we know $\omega^- \in VA_\infty(\omega^+)$ that (8.1) holds for every $M > 0$. By Lemma 8.1, $\omega_j^- \rightharpoonup \omega$ as well. Thus, we can pass to a subsequence so that the conclusions of Lemma 4.13 hold. In particular, $\omega = \omega_h$ for some L_0-harmonic function h, where L_0 is a uniformly elliptic operator with constant coefficients, and also, for any $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$, (4.21) holds.

Now we apply the same standard trick from [KT06]. Notice that since ω^+ is doubling, so is ω_h, which combined with Cauchy estimates, implies that there exists $\beta > 0$ such that for any $\ell \in \mathbb{N}$ and any multi-index α,

$$|\partial_\alpha h(0)| \lesssim 2^{-|\alpha|\ell} ||h||_{L^\infty(2^\ell\mathbb{B})} \overset{(4.28)}{\lesssim} 2^{\ell(-|\alpha|+1-n)}\omega_h(B(0,2^{\ell+1})) \tag{8.11}$$

$$\lesssim 2^{\ell(-|\alpha|+1-n+\beta)}\omega_h(B(0,2)). \tag{8.12}$$

Hence, if $|\alpha| > 1-n+\beta$, letting $\ell \to \infty$ gives $|\partial_\alpha h(0)| = 0$, which implies h is a polynomial of degree at most $1-n+\beta$. Setting $d = \lceil 1-n+\beta \rceil$ gives a contradiction to (8.10). The proof of (1.7) is similar to the proof of Theorem I, where we use instead Lemma 4.11 instead of 4.13. $\square$

9. Proof of Theorem IV

All elliptic operators in this section will be assumed to satisfy (1.1) and (1.2). We will require a few lemmas about elliptic measures in uniform domains as well as some new notation.

Definition 9.1. Let $\Omega \subseteq \mathbb{R}^{n+1}$.

- We say Ω satisfies the *corkscrew condition* if for some uniform constant $c > 0$ and every ball B centered on $\partial\Omega$ with $0 < r_B < \mathrm{diam}(\partial\Omega)$, there is a ball $B(x_B, cr_B) \subseteq \Omega \cap B$. The point x_B is called a *corkscrew point relative to* B.
- We say Ω satisfies the *Harnack chain condition* if there is a uniform constant C such that for every $\rho > 0$, $\Lambda \geq 1$, and every pair of

points $x, y \in \Omega$ with $\delta(x)$, $\delta(y) \geq \rho$ and $|x - y| < \Lambda\,\rho$, there is a chain of open balls $B_1, \ldots, B_N \subset \Omega$, $N \leq C(\Lambda)$, with $x \in B_1$, $y \in B_N$, $B_k \cap B_{k+1} \neq \emptyset$ and $C^{-1}\operatorname{diam}(B_k) \leq \operatorname{dist}(B_k, \partial\Omega) \leq C\operatorname{diam}(B_k)$. The chain of balls is called a *Harnack chain*.

Definition 9.2. If Ω satisfies both the corkscrew and the Harnack chain conditions, then we say that Ω is a *uniform domain*.

Theorem 9.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a uniform domain with the CDC and u a nonnegative L_A-elliptic function vanishing on $2B \cap \partial\Omega$ where B is a ball with $r_B < \operatorname{diam}\partial\Omega$ and $A \in \mathscr{A}(\Omega)$. Then*

$$\sup_{x \in B\cap\Omega} u(x) \lesssim u(x_B). \tag{9.1}$$

This was originally shown in section 4 of [JK82] for NTA domains, but the proof only uses the Hölder continuity of u at the boundary and the fact that NTA domains are uniform, and so the proof of the above result is exactly the same.

Theorem 9.4. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a uniform domain with the CDC and L_A an elliptic operator satisfying* (1.1) *and* (1.2)*. Then, for all B centered on $\partial\Omega$,*

$$\omega^{L_A,x}(B) \approx r_B^{n-1} G_\Omega(x, x_B) \text{ for all } x \in \Omega\backslash 2B. \tag{9.2}$$

This follows from the work of Aikawa and Hirata [AH08]. Their proof is originally for harmonic measures, but an inspection of the proof shows that it carries through for elliptic measure as well.

Theorem 9.5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a uniform domain with the CDC. If L_A is an elliptic operator satisfying* (1.1) *and* (1.2)*, B is a ball centered on $\partial\Omega$ and $E \subset B \cap \partial\Omega$ is Borel, then*

$$\omega_\Omega^{L_A,x_B}(E) \approx \frac{\omega_\Omega^{L_A,x}(E)}{\omega_\Omega^{L_A,x}(B)}. \tag{9.3}$$

Again, this is [JK82, Lemma 4.11], and since the previous two lemmas are available, the proof is exactly the same for elliptic measures modulo the proof of [JK82, Lemma 4.10]. The latter can also be proved as in [JK82] to build a sub-uniform domain, and then showing as in [AAM16, Lemma 2.26] that the resulting domain is also CDC (all of this instead of a geometric localization theorem due to Jones, which only works for NTA domains).

Lemma 9.6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a uniform domain with the CDC and L_A an elliptic operator satisfying* (1.1) *and* (1.2)*, and also* (1.4) *at ξ. If $\xi \in \partial\Omega$ and $\omega_j = \omega^{L_A,x_0}(B(\xi, r_j))^{-1} T_{\xi,r_j}(\omega^{L_A,x_0})$ converges weakly to a tangent measure $\omega_\infty \in \operatorname{Tan}(\omega^{L_A,x_0}, \xi)$. Then there is a uniform domain Ω_∞ and a* *constant matrix $A_0 \in \mathscr{C}$ such that, for each $x \in \Omega_\infty$, $\omega_{\Omega_j}^x \rightharpoonup \omega_{\Omega_\infty}^x$ and, for all balls $B' \subset B$ centered on $\partial\Omega_\infty$, if x_B is a corkscrew point in $\Omega_\infty \cap B$,*

$$\omega_{\Omega_\infty}^{L_{A_0},x_B}(B') \approx \frac{\omega_\infty(B')}{\omega_\infty(B)}. \tag{9.4}$$

This was originally shown in [AM15] for harmonic measure. In our situation, the proof is much shorter, so we provide it here.

Proof. By Lemma 4.11, there is $A_0 \in \mathscr{C}$ so that we can pass to a subsequence so that $u_j(x) = c_j u(xr_j + \xi) r_j^{n-1}$ converges uniformly in $\mathbb{R}^{n+1}$ to a nonzero L_{A_0}-elliptic function u_∞ and also so that, if $\Omega_j = T_{\xi,r_j}(\Omega)$, then $\partial\Omega_j$ converges in the Hausdorff metric on compact subsets. Let $\Omega_\infty = \{u_\infty > 0\}$.

Claim: Ω_∞ is uniform. If $x, y \in \Omega_\infty$ with $\text{dist}(\{x,y\}, \partial\Omega) \geq \varepsilon|x-y|$, then they are contained in Ω_j and $\text{dist}(\{x,y\}, \partial\Omega_j) \geq \frac{\varepsilon}{2}|x-y|$ for sufficiently large j. Since the Ω_j are uniform, for each j we can find a Harnack chain of length $N = N(\varepsilon)$ contained in Ω_j. By passing to a subsequence, we can assume the length of this chain is constant and their centers and radii are converging, and hence the chain converges to a Harnack chain in Ω_∞ of length no more than N. A similar proof shows that Ω_∞ is a corkscrew domain. Hence, Ω_∞ is uniform.

Suppose $B' \subset \mathbb{B}$ are centered on $\partial\Omega_\infty$. Let

$$\omega_{\Omega_j}^{T_{\xi,r_j}(x)} = T_{\xi,r_j}[\omega^{L_A,x}].$$

If $x_j = T_{\xi,r_j}(x_0)$, then

$$\omega_{\Omega_j}^{x_B}(B') \approx \frac{\omega_{\Omega_j}^{x_j}(B')}{\omega_{\Omega_j}^{x_j}(B)} = \frac{\omega_{\Omega_j}^{x_j}(\mathbb{B})}{\omega_{\Omega_j}^{x_j}(B)} \frac{\omega_{\Omega_j}^{x_j}(B')}{\omega_{\Omega_j}^{x_j}(\mathbb{B})} = \frac{\omega_j(B')}{\omega_j(B)}.$$

Since ω_j and ω_{Ω_j} are doubling measures, we have

$$\omega_{\Omega_\infty}^{x_B}(B') \leq \liminf_{j\to\infty} \omega_{\Omega_j}^{x_B}(B') \lesssim \limsup_{j\to\infty} \frac{\omega_j(B')}{\omega_j(B)} \leq \frac{\omega_\infty(\overline{B'})}{\omega_\infty(B)} \lesssim \frac{\omega_\infty(B')}{\omega_\infty(B)}.$$

A similar estimate gives the reverse inequality, and hence proves (9.4). $\square$

We will use the following criterion for uniform rectifiability due to Hofmann, Martell, and Uriarte-Tuero. See Theorem 1.23, Equation 1.22, and Remark 1.25 in [HMUT14] (for a local version see Corollary 11.2 in [MT15]).

Theorem 9.7. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a uniform domain with n-regular boundary and let ω_Ω be the harmonic measure defined in Ω. Suppose there is $q > 1$*

so that, for any balls $B' \subset B$ centered on $\partial\Omega$, if $k_B = \frac{d\omega_\Omega^{x_B}}{d\mathcal{H}^n|_{\partial\Omega}}$, then

$$\left(\fint_{B'\cap\partial\Omega} k_B^q \, d\mathcal{H}^n\right)^{\frac{1}{q}} \lesssim \fint_{2B'\cap\partial\Omega} k_B \, d\mathcal{H}^n.$$

Then $\partial\Omega$ is uniformly rectifiable.

Recall that, by the main result of [AH08], harmonic measure is doubling in uniform domains satisfying the CDC, and thus, by (9.3), the right side of this inequality is comparable to $\fint_{B'\cap\partial\Omega} k_B \, d\mathcal{H}^n$ (that is, with B' instead of $2B'$), which we will use below.

Remark 9.8. This result still holds for constant coefficients. Indeed, it is easy to see that the A_∞ property is preserved under linear transformations that map balls to ellipsoids, as is the one in Lemma 4.10 (see the paragraph after the proof of this lemma), using that such weights are doubling.Thus, by Lemma 4.10 and the fact that being a uniformly rectifiable set, by its very definition, is invariant under bi-lipschitz maps, $\partial\Omega_\infty$ is uniformly rectifiable.

Recall that an Ahlfors n-regular set E is *uniformly rectifiable* if there are $c, L > 0$ so that, for every ball B centered on E with $r_B < \operatorname{diam} E$, there is an L-Lipschitz map $f : B(0, r_B) \cap \mathbb{R}^n \to \mathbb{R}^{n+1}$ so that

$$\mathcal{H}^n(f(B(0, r_B)) \cap E) \geq c r_B^n.$$

Now we prove Theorem IV. Let $\Omega \subset \mathbb{R}^{n+1}$ be a uniform CDC domain so that $\mathcal{H}^n|_{\partial\Omega}$ is locally finite. Let $\omega = \omega_\Omega^{L_A}$ be the L_A-harmonic measure associated to a (possibly non-symmetric) elliptic operator satisfying (1.1) and (1.2). Let $E \subseteq \partial\Omega$ be a set with $\mathcal{H}^n(E) > 0$ such that $\mathcal{H}^n \ll \omega_\Omega^{L_A}$ on E and for $\mathcal{H}^n$-a.e. $\xi \in E$,

$$\theta^n_{\partial\Omega,*}(\xi, r) := \liminf_{r\to 0} \frac{\mathcal{H}^n(B(\xi, r) \cap \partial\Omega)}{(2r)^n} > 0$$

and A has vanishing mean oscillation at ξ.

Assume $\mathcal{H}^n(E) > 0$ (otherwise the theorem is trivial). Then we may find a subset E' of full $\mathcal{H}^n$-measure where ω and $\mathcal{H}^n$ are mutually absolutely continuous (in particular, $\mathcal{H}^n = g\omega$ for some function g, so we pick $E' = \{x : g(x) > 0\}$). For $\mathcal{H}^n|_{\partial\Omega}$-a.e. $\xi \in E'$, we also have

$$0 < \theta^n_*(\mathcal{H}^n|_{\partial\Omega}, \xi) \leq \theta^{n,*}(\mathcal{H}^n|_{\partial\Omega}, \xi) < \infty. \tag{9.5}$$

The lower bound is by assumption, and the upper bound is from [Mat95, Theorem 6.2]. By [Mat95, Theorem 14.7], for $\mathcal{H}^n|_{\partial\Omega}$-a.e. $\xi \in E'$, $\operatorname{Tan}(\mathcal{H}^n|_{\partial\Omega}, \xi)$ consists of Ahlfors-David n-regular measures. By [Mat95, Lemma 14.5] and [Mat95, Lemma 14.6], for $\mathcal{H}^n|_{\partial\Omega}$-a.e. $\xi \in E'$,

$$\operatorname{Tan}(\mathcal{H}^n|_{\partial\Omega}, \xi) = \operatorname{Tan}(\mathcal{H}^n|_{E'}, \xi) = \operatorname{Tan}(\omega, \xi)$$

and $\operatorname{Tan}(\omega,\xi)$ consists only of Ahlfors-David n-regular measures. Let $E'' \subset E'$ be the set of points where this holds.

By the Besicovitch decomposition theorem, we can split E'' into two sets F_1 and F_2 where F_1 is n-rectifiable and F_2 is purely n-unrectifiable. Suppose $\mathcal{H}^n(F_2) > 0$. Let $\xi \in F_2$ be a point of density of F_2 with respect to $\mathcal{H}^n$.

Let $r_j \downarrow 0$ be so that $\omega_j := \omega^{L_A,x_0}(B(\xi,r_j))^{-1}T_{\xi,r_j}(\omega^{L_A,x_0})$ converges weakly to some Ahlfors-David n-regular measure $\omega_\infty \in \operatorname{Tan}(\omega,\xi)$. By Lemma 9.6, we may find a uniform domain Ω_∞ so that $\operatorname{supp}\omega_\infty = \partial\Omega_\infty$ and, for any balls $B' \subset B$ centered on $\partial\Omega$,

$$\omega_{\Omega_\infty}^{L_{A_0},x_B}(B') \approx \frac{\omega_\infty(B')}{\omega_\infty(B)} \approx \frac{r_{B'}^n}{r_B^n},$$

for some $A_0 \in \mathscr{C}$. If $\sigma = \mathcal{H}^n|_{\partial\Omega_\infty}$, then σ is Ahlfors-David n-regular and so if we set

$$k_B := \frac{d\omega_{\Omega_\infty}^{L_{A_0},x_B}}{d\sigma},$$

then we have that for σ-a.e. $x \in B \cap \partial\Omega$,

$$k_B(x) = \lim_{r\to 0} \frac{\omega_{\Omega_\infty}^{L_{A_0},x_B}(B(x,r))}{\sigma(B(x,r))} \approx \frac{r^n/r_B^n}{r^n} = r_B^{-n}.$$

Hence, if $B' \subset B$ is centered on $\partial\Omega$,

$$\left(\fint_{B'} k_B^2\, d\sigma\right)^{\frac{1}{2}} \approx r_B^{-n} \approx \fint_{B'} k_B\, d\sigma.$$

Thus, in light of Remark 9.8, $\partial\Omega_\infty$ is uniformly rectifiable. By the main result of [AHM+17], Ω_∞ is an NTA domain. In particular, we can find corkscrew balls $B_1 \subset \mathbb{B} \cap \Omega_\infty$ and $B_2 \subseteq \mathbb{B}\backslash\Omega_\infty$. We claim that, for all j sufficiently large, $\frac{1}{2}B_1 \subset \Omega_j \cap \mathbb{B}$ and $\frac{1}{2}B_2 \subset \mathbb{B}\backslash\Omega_j$. Indeed, if $\frac{1}{2}B_i \cap \partial\Omega_j \neq \emptyset$ for infinitely many j, then since ω_j is doubling, $\omega_j(\frac{2}{3}B_i) \sim \omega_j(\mathbb{B}) = 1$ for all j, and so $\omega_\infty(B_i) > 0$, in particular $\partial\Omega_\infty \cap B_i \neq \emptyset$, which is a contradiction. Thus, B_1 and B_2 do not intersect $\partial\Omega_j$ for sufficiently large j. They cannot both be in Ω_j for all large j, since otherwise, if they were both in Ω_j for infinitely many j, then in each such Ω_j, they are connected by a Harnack chain in Ω_j of bounded length; passing to a subsequence, this implies there is a Harnack chain connecting B_1 to B_2, and since $B_1 \subseteq \Omega_\infty$, the whole chain, including B_2, must be in Ω_∞, which is a contradiction. Thus, at least one of these balls is in Ω_j^c for all j large. By the proof of Lemma 9.6, $\Omega_\infty = \{u_\infty > 0\}$, and since $u_j \to u_\infty$ uniformly on compact subsets of Ω_∞ and $u_\infty > 0$ on B_1, we have that $B_1 \subset \Omega_j$ for j large, and so $B_2 \subset \Omega_j^c$ for j large. This proves the claim.

Now there is a small angle of direcitons around the vector parallel to the line between the centers of B_1 and B_2 where the orthogonal projection of $\partial\Omega_j \cap \mathbb{B}$ has Lebesgue measure comparable to 1. By the Besicovitch-Federer projection theorem, the purely unrectifiable bart of $\partial\Omega_j$ has zero Lebesgue measure projection in almost all of these directions, and so $\partial\Omega_j \cap \mathbb{B}$ contains an n-rectifiable set of $\mathcal{H}^n$-measure $\gtrsim 1$ (with constant depending on the sizes of B_1 and B_2). Thus,

$$\liminf_{j\to\infty} \frac{\mathcal{H}^n(B(\xi, r_j) \cap \partial\Omega \backslash F_2)}{\mathcal{H}^n(B(\xi, r_j) \cap \partial\Omega)} \gtrsim \liminf_{j\to\infty} \frac{r_j^n}{\mathcal{H}^n(B(\xi, r_j) \cap \partial\Omega)} \overset{(9.5)}{>} 0.$$

But this contradicts that ξ is a point of density for F_2. Therefore, $\mathcal{H}^n(F_2) = 0$, and we have now shown that $\mathcal{H}^n$-almost all of E' is rectifiable, and thus ω^{x_0}-almost all of E is contained in a countable union of Lipschitz graphs. This finishes the proof of Theorem IV.

10. Proof of Proposition III

Assume the conditions of the proposition. We recall the following result.

Theorem 10.1. [HS94, Theorem 1.3] *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded C-uniform*[2] *domain. If*

$$p \leq q \leq \frac{(n+1)p}{n+1-p(1-\delta)} \text{ and } p(1-\delta) < n+1,$$

then for all $u \in L^1_{\text{loc}}(\Omega)$ such that $\nabla u(x) d(x, \partial\Omega)^\delta \in L^p(\Omega)$,

$$\inf_{a\in\mathbb{R}} ||u(x) - a||_{L^q(\Omega)} \lesssim_{n,p,q,\delta,C} |\Omega|^{\frac{1-\delta}{n+1}+\frac{1}{q}-\frac{1}{p}} ||\nabla u \, \text{dist}(\cdot, \Omega^c)^\delta||_{L^p(\Omega)}. \quad (10.1)$$

(The explicit constant in (10.1) is written at the end of the proof on page 218 of [HS94].) We will use this in the case that $\delta = \frac{1}{2}$ and $p = q = 2$, so (10.1) becomes

$$\inf_{a\in\mathbb{R}} ||u(x) - a||_{L^2(\Omega)} \lesssim_{n,p,q,\delta,C} |\Omega|^{\frac{1}{2(n+1)}} ||\nabla u \, \text{dist}(\cdot, \Omega^c)^{\frac{1}{2}}||_{L^2(\Omega)}. \quad (10.2)$$

Lemma 10.2. *Suppose $E \subset \mathbb{R}^{n+1}$ is a closed set and $\varepsilon : E^c \to [0, \infty]$ is a function such that for some ball B_0 centered on E,*

$$\int_{E^c \cap B_0} \varepsilon(z) dz < \infty.$$

Then for $\mathcal{H}^n$-a.e. $x \in E \cap B_0$,

$$\lim_{r\to 0} r^{-n} \int_{E^c \cap B(x,r)} \varepsilon(z) dz = 0.$$

[2]In fact it holds for John domains.

Proof. Without loss of generality, we can assume $E \subset B_0$. Let $d\mu(z) = \varepsilon(z)\, dz|_{E^c}$. For $x \in E$ and $r > 0$, set

$$a(x,r) = \frac{\mu(B(x,r))}{r^n} = r^{-n} \int_{E^c \cap B(x,r)} \varepsilon(z) dz.$$

Suppose there is $F \subset E$ with $\mathcal{H}^n(F) > 0$ such that

$$\limsup_{r \to 0} a(x,r) > 0.$$

Then there is $t > 0$ and a compact set $G \subset F$ with $\mathcal{H}^n_\infty(G) > 0$ and

$$\limsup_{r \to 0} a(x,r) > t > 0 \quad \text{for all } x \in G.$$

For each $x \in G$, pick $r_{x,1} > 0$ so that $B(x, r_{x,1}) \subset B_0$ and $a(x, r_{x,1}) > t$. Let B_j^1 be a Besicovitch subcovering from $\mathscr{G}_1 := \{B(x, r_x^1) : x \in G\}$, that is, a countable collection of balls in $\mathscr{G}_1$ so that

$$\mathbb{1}_G \leq \sum_j \mathbb{1}_{B_j^1} \lesssim_n 1.$$

Since the B_j^1 come from $\mathscr{G}$, we have that for all j,

$$\frac{\mu(B_j^1)}{r_{B_j^1}^n} = a(x_{B_j^1}, r_{B_j^1}) > t.$$

Let

$$L_1 = \bigcup B_j^1 \backslash E.$$

Then since the B_j^1 have bounded overlap and come from $\mathscr{G}_1$,

$$\mu(L_1) = \int_{L_1} d\mu \gtrsim \int_{L_1} \sum_j \mathbb{1}_{B_j^1} d\mu = \sum_j \mu(B_j^1) > t \sum_j r_{B_j^1}^n \geq t\, \mathcal{H}^n_\infty(G).$$

Since $\mu(G) = 0$, there is $\delta_1 > 0$ so that if $G_{\delta_1} = \{x \in \mathbb{R}^n : \operatorname{dist}(x, G) < \delta_1\}$ and $L^1 = L_1 \backslash G_{\delta_1}$, then

$$\mu(L^1) > \frac{\mu(L_1)}{2} \geq \frac{t}{2} \mathcal{H}^n_\infty(G).$$

Now inductively, suppose we have constructed disjoint sets $L^1, ..., L^k \subset B_0$ where

$$\mu(L^j) \gtrsim t\mathcal{H}^n_\infty(G) \quad \text{for all } j = 1, 2, ..., k,$$

and there is $\delta_k > 0$ so that $L^1 \cup \cdots \cup L^k \cap G_{\delta_k} = \emptyset$.

For each $x \in G$, we may find $r_{x,k+1} \in (0, \delta_k)$ so that $B(x, r_{x,k+1}) \subset B_0$ and $a(x, r_{x,k+1}) > t$. Let $\{B_j^{k+1}\}$ be a Besicovitch subcovering of the collection $\mathscr{G}_{k+1} = \{B(x, r_{x,k+1}) : x \in G\}$, so

$$\mathbb{1}_G \leq \sum_j \mathbb{1}_{B_j^{k+1}} \lesssim_n \mathbb{1}_{L_{k+1}},$$

where $L_{k+1} = \bigcup_j B_j^{k+1}$. Since G has $\mu(G) = 0$, there is $\delta_{k+1} \in (0, \delta_k)$ so that $L^{k+1} = L_{k+1} \backslash G_{\delta_{k+1}}$ has,

$$\mu(L^{k+1}) \geq \frac{\mu(L_{k+1})}{2} = \frac{1}{2}\int \mathbb{1}_{L_{k+1}} d\mu \gtrsim \int \sum_j \mu(B_j^{k+1}) \geq t \sum_j r^n_{B_j^{k+1}}$$
$$\gtrsim t\, \mathcal{H}^n_\infty(G).$$

Also note that by our induction hypothesis

$$L^{k+1} \subset L_{k+1} \subset G_{\delta_k} \subset (L^1 \cup \cdots \cup L^k)^c.$$

Thus, by induction, we can come up with a sequence of disjoint sets $L^k \subset B_0$ so that $\mu(L^k) \gtrsim t\, \mathcal{H}^n_\infty(G)$ for all k, which contradicts the finiteness of μ since ε is locally integrable. $\square$

Now we finish the proof of Proposition III. By the previous lemma, for $\varepsilon(z) = |\nabla A(z)|^2 \operatorname{dist}(z, \Omega^c)$ and $E = \partial\Omega$, we have that for $\mathcal{H}^n$-a.e. $\xi \in B_0 \cap \partial\Omega$,

$$\lim_{r\to 0} r^{-n} \int_{B(\xi,r)\cap\Omega} |\nabla A|^2 \operatorname{dist}(z, \Omega^c)\, dz = 0. \tag{10.3}$$

Let $\xi \in B_0 \cap \partial\Omega$ be such a point. There is a universal constant M depending on the uniformity constants so that, for all $r > 0$, there is a MC-uniform domain Ω_r such that

$$\Omega \cap B(\xi, r) \subset \Omega_r \subset \Omega \cap B(\xi, Mr).$$

This follows from the proof of [HM14, Lemma 3.61]. See also [Azz16, Lemma 4.1] or [JK82, Lemma 6.3].

Hence, by Cauchy-Schwarz inequality,

$$
\begin{aligned}
\inf_C r^{-(n+1)} & \int_{B(\xi,r)\cap\Omega} |A-C| \\
&\lesssim \inf_C \left(r^{-(n+1)} \int_{B(\xi,r)\cap\Omega} |A-C|^2 \right)^{1/2} \\
&\leq \inf_C \left(r^{-(n+1)} \int_{\Omega_r} |A-C|^2 \right)^{1/2} \\
&\overset{(10.2)}{\lesssim} |\Omega_r|^{\frac{1}{2(n+1)}} \left(\frac{1}{r^{n+1}} \int_{\Omega_r} |\nabla A|^2 \operatorname{dist}(z,\Omega_r^c) dz \right)^{1/2} \\
&\lesssim \left(r^{-n} \int_{\Omega\cap B(\xi,Mr)} |\nabla A|^2 \operatorname{dist}(z,\Omega^c) dz \right)^{1/2} \to 0, \text{ as } r \to 0.
\end{aligned}
$$

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