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Effects of refractory period on stochastic resetting

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Abstract.

We consider a stochastic process undergoing resetting after which a random refractory period is imposed. In this period the process is quiescent and remains at the resetting position. Using a first-renewal approach, we compute exactly the stationary position distribution and analyse the emergence of a delta peak at the resetting position. In the case of a power-law distribution for the refractory period we find slow relaxation. We generalise our results to the case when the resetting period and the refractory period are correlated, by computing the Laplace transform of the survival probability of the process and the mean first passage time, i.e., the mean time to completion of a task. We also compute exactly the joint distribution of the active and absorption time to a fixed target.

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1. Introduction

Resetting a stochastic process may improve the time to complete a task considerably. For example if the task at hand is to locate a fixed target, a purely diffusive process has infinite mean first passage time (MFPT) to the target. However, when a resetting process is introduced the MFPT is rendered finite [1]. The idea of resetting a random process to improve efficiency or throughput extends to diffusion processes with random target position [2] and to arbitrary dimensions [3], non-diffusive processes such as Lévy flights [4] and active run and tumble particles [5], to computer algorithms [6, 7, 8], Michaelis-Menten type reaction schemes [9, 10, 11] and kinetic proofreading mechanisms [12, 13, 14]. The central idea is that resetting may eliminate a long tail of completion times in the absence of resetting.

However, in most contexts resetting does not come without a cost. For example, ‘teleporting’ a searcher back to where he started must take some time and effort [15, 16]. In a different context, neuron and nerve cells undergo resetting events involving the firing of action potentials after which the cell is in a quiescent state in which subsequent stimuli are ineffective. Such a *refractory* period may be absolute, in which case a subsequent action potential cannot occur, or relative in which case a subsequent action potential is inhibited but not impossible. Refractory periods may be psychological as well as physiological in nature. For example, large movements in the stock exchange are often interpreted as resetting events after which there is often a slow-down period in which activity abates [17].

A recent series of papers [9, 10, 11] has considered a Generalised Michaelis-Menten chemical reaction scheme in which there are three stages: i) unbound enzyme and substrate; ii) enzyme bound to substrate; iii) the catalytic production of a product. The cycle goes from i) to ii) when the enzyme binds with a certain rate k_{on} and reverts to i) when the enzyme unbinds with rate k_{off} ; the cycle goes from ii) to iii) through a complicated possible series of reactions that result in a waiting time distribution $f(t)$ for the production of the product.

Remarkably, as noted in [9, 10, 11], this reaction scheme maps precisely onto a general resetting process (the unbinding which occurs with rate k_{off}) which then incurs a refractory period corresponding to the binding time controlled by rate k_{on} . The waiting distribution $f(t)$ characterises the stochastic process which is reset. The works [9, 10] (see also [18]) have derived important results for the mean time for the production of a product and the Laplace transform of the distribution of the production time. Moreover, it has been shown that interesting universality emerges when the resetting process is chosen to optimise the time to completion of the process [11].

In this note we study and extend further the effects of the presence of a non-zero refractory period on other properties of a generic process undergoing stochastic resetting, going beyond the first-passage time distribution. More precisely, We consider a stochastic process $x(t)$ in continuous time t under resetting, wherein immediately after a resetting event we impose a refractory period: during a refractory period, the particle ‘sleeps’ (no movement or activity, i.e., a quiescent state). The duration of the refractory period, τ , is taken to be a random variable, drawn independently after each reset event from a distribution $W(\tau)$. The distribution $W(\tau)$ may or may not have finite mean. At the end of the random refractory period, the process resumes its random motion till the next reset event. The evolution is shown schematically in Fig. 1, where the reset position X_r is taken to be the origin $x = 0$ for simplicity. We shall initially consider the case where the resetting is Poissonian i.e. occurs with fixed rate but later generalise to arbitrary joint distribution for the resetting period and the subsequent refractory period.

We consider three novel effects of a refractory period on resetting. (i) We first derive in Section 2.1 the stationary state for the case of Poissonian reset (where resetting occurs with constant rate r), but an arbitrary distribution $W(\tau)$ for τ , the refractory period

after the reset. We show how the stationary state contains a delta peak at the resetting position corresponding to the process in the quiescent state during its refractory period. In section 2.2 we study the long-time relaxation of this delta peak (ii) In section 3 We then go on to consider correlations between the refractory period and the period of activity before the reset that preceded the refractory period. For example, it would be natural if after a long period of activity before the reset the refractory period were longer than after a short period of activity. We show that the stationary state depends on these correlations in a *weak* way, i.e., it depends only on the marginal distribution of the first resetting time (which may depend on the correlations). However the survival probability of the process in the presence of an absorbing target is *strongly* affected by the correlation and we compute explicitly the Laplace transform of the survival probability, from which we obtain the mean first passage time under the reset. (iii) Finally, in Section 4, for a process with Poissonian resetting and independent refractory period, we compute exactly the joint distribution of the survival time and the active time till the target is found, i.e. the time during which the process is not in the quiescent state. This quantity is of interest since in an optimisation problem it may be appropriate to consider the active time as distinct from time spent in the quiescent state.

2. Renewal equation for the probability distribution

For simplicity we begin with the established case of Poissonian resetting and refractory period probability density $W(\tau)$ (see [9, 10, 11] although, as noted above, the notation is slightly different). Between resets we follow a general stochastic process $x(t)$. We denote by $G_0(x, t)$ the time-dependent probability density for this process to have reached x after time t (in the absence of resetting) where we have suppressed the initial condition which we take to be coincident with the resetting position $X_r = 0$ (taken here to be at the origin). We shall refer to $G_0(x, t)$ as the propagator of the process.

We begin by writing a renewal equation for the probability density which uses the first reset to renew the process. Related first-renewal equations have been previously used in the case of the Generalised Michaelis-Menten reaction scheme [9, 10, 11] and non-Poissonian reset [19]. In other contexts last-renewal equations have been employed [1, 20, 4, 5]. Recently, a unified renewal approach has been developed [21, 22, 23].

First note that for a Poissonian resetting process (constant rate r) the probability of no resets occurring up until time t is given by e^{-rt} and the probability density for the first reset to occur at time t_1 is re^{-rt_1} . Then we may write down an equation for the probability density for the process to be at x at time t .

$$\begin{aligned}
 P(x, t) &= e^{-rt} G_0(x, t) \\
 &+ r \int_0^t dt_1 e^{-rt_1} \int_0^{t-t_1} d\tau_2 W(\tau_2) P(x, t - t_1 - \tau_2) \\
 &+ r \int_0^t dt_1 e^{-rt_1} \int_{t-t_1}^{\infty} d\tau_2 W(\tau_2) \delta(x) .
 \end{aligned} \tag{1}$$

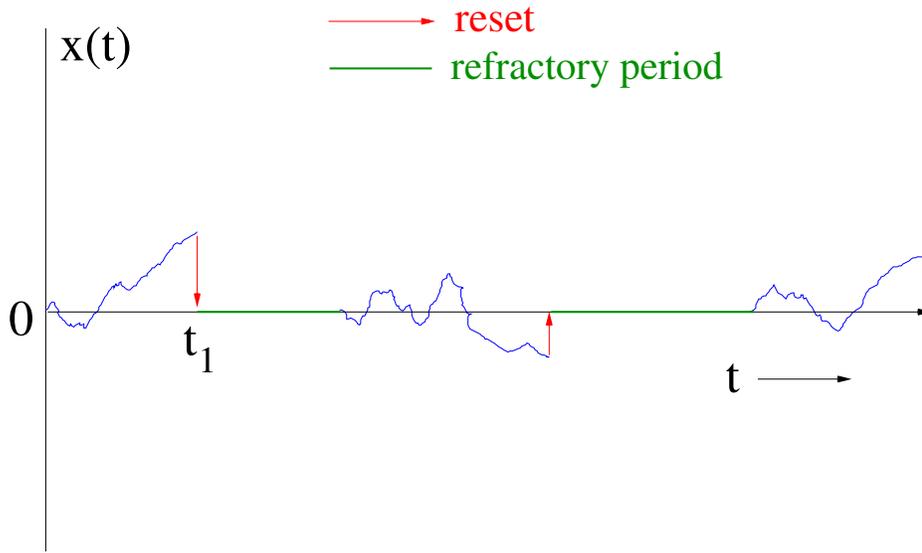


Figure 1. Schematic evolution of a stochastic process $x(t)$ as a function of time t , starting from $x(0) = 0$. The red arrows denote the reset (to the origin) events. The first reset happens at time t_1 . Immediately following a reset, the particle spends a random refractory period during which the particle does not move (shown by the thick green lines). The process resumes its dynamics only at the end of the refractory period.

We note here that, since our refractory period occurs after a reset, the initial condition is different from that considered in the series of papers [9, 10, 11] where the process starts off with a refractory period.

The first term on the right hand side (rhs) of (1) is the contribution from trajectories in which there is no resetting, which occurs with probability e^{-rt} ; the second term integrates over the first resetting time t_1 , and the subsequent refractory period with duration τ_2 which finishes before t leaving a period $t - t_1 - \tau_2$ for the particle to pass from the resetting site to x ; the third contribution is from first resets at t_1 with refractory periods τ_2 which last until beyond time t , leaving the particle at the resetting position (here the origin).

We now take the Laplace transform of (1) which is defined as

$$\tilde{P}(x, s) = \int_0^\infty dt e^{-st} P(x, t). \quad (2)$$

The Laplace transform of the first term yields $\tilde{G}_0(x, r+s)$ which is the Laplace transform (with Laplace variable $r+s$) of the propagator. The Laplace transform of the second term yields from the convolution theorem $\frac{r}{r+s} \tilde{W}(s) \tilde{P}(x, s)$, where $\tilde{W}(s)$ is the Laplace transform of the refractory period distribution, and the Laplace transform of the third term yields

$$\delta(x)r \int_0^\infty dt e^{-st} \int_0^t dt_1 e^{-rt_1} \int_{t-t_1}^\infty d\tau_2 W(\tau_2)$$

$$\begin{aligned}
&= \delta(x)r \int_0^\infty dt_1 e^{-rt_1} \int_0^\infty d\tau_2 W(\tau_2) \int_{t_1}^{\tau_2+t_1} dt e^{-st} \\
&= \delta(x) \frac{r}{s(r+s)} \left[1 - \tilde{W}(s) \right] .
\end{aligned} \tag{3}$$

Putting these results together yields

$$\tilde{P}(x, s) = \frac{1}{r+s-r\tilde{W}(s)} \left[(r+s)\tilde{G}_0(x, r+s) + \delta(x) \frac{r}{s} \left[1 - \tilde{W}(s) \right] \right] . \tag{4}$$

As a check, we may integrate Eq. (4) over x , using $\int_{-\infty}^\infty \tilde{G}_0(x, s) dx = 1/s$. We then recover $\int_{-\infty}^\infty \tilde{P}(x, s) dx = 1/s$ as expected, since $P(x, t)$ is normalised to unity. Another useful check concerns the the case of vanishing refractory period $W(\tau) = \delta(\tau)$. Then $\tilde{W}(s) = 1$ and we find

$$\tilde{P}_{\tau=0}(x, s) \rightarrow \frac{(r+s)}{s} \tilde{G}_0(x, r+s) . \tag{5}$$

For instance, for a simple diffusing particle on a line with diffusion constant D and starting at the origin $x = 0$, the bare propagator is $G_0(x, t) = e^{-x^2/4Dt}/\sqrt{4\pi Dt}$. The Laplace transform with respect to t gives, $\tilde{G}_0(x, s) = e^{-\sqrt{s/D}|x|}/\sqrt{4Ds}$. Hence, in this case, Eq. (5) gives

$$\tilde{P}_{\tau=0}(x, s) = \frac{1}{s} \sqrt{\frac{r+s}{4D}} e^{-\sqrt{\frac{r+s}{D}}|x|} \tag{6}$$

which is completely consistent with the full time-dependent solution in the absence of a refractory period [3, 20].

2.1. Stationary State

The stationary state $P^{\text{st}}(x)$ is easy to extract as the coefficient of $1/s$ in the small s expansion of (4). In the case of vanishing refractory period (5) we recover $P^{\text{st}}(x) = r\tilde{G}_0(x, r)$ [1]. In the general case, assuming $W(\tau)$ has finite mean $\langle \tau \rangle$, we have for small s

$$\tilde{W}(s) \simeq 1 - s\langle \tau \rangle . \tag{7}$$

Then we obtain the stationary distribution

$$P^{\text{st}}(x) = \frac{r}{1+r\langle \tau \rangle} \left[\tilde{G}_0(x, r) + \delta(x) \langle \tau \rangle \right] . \tag{8}$$

Thus the probability distribution is a superposition of the distribution with zero refractory period (which is given by $r\tilde{G}_0(x, r)$) and a delta function at the resetting position with relative weight equal to $r\langle \tau \rangle$ i.e. the ratio of the mean refractory period to the mean period between resets. We shall see in Section 3 that the form of (8) is robust to the introduction of correlations between refractory period and resetting period.

On the other hand in the case where the mean of $W(\tau)$ diverges we find that the stationary state $P^{\text{st}}(x) \rightarrow \delta(x)$, i.e., the stationary state is dominated by the divergent mean refractory period.

2.2. Long-time relaxation

We now consider the long-time relaxation to the stationary state. In the absence of a refractory period, the relaxation to the stationary state of a stochastic process in presence of resetting was analysed in Ref. [20] for diffusion on a line, and an interesting dynamical phase transition was found. Here, we will focus on the relaxation of the new feature in $P(x, t)$ that is induced entirely by the presence of a refractory period. More specifically, we note from Eq. (4) that the delta peak at the resetting position $x = 0$, induced by the refractory period, is present at all times, and not just in the stationary state. In this section, we focus on the growth of this delta peak in (4). If one inverts the Laplace transform in Eq. 4, the delta peak $\delta(x)$ will have a prefactor which we denote $B(t)$. From Eq. (4), the Laplace transform of $B(t)$ reads

$$\tilde{B}(s) = \frac{r \left[1 - \tilde{W}(s) \right]}{s(r + s - r\tilde{W}(s))}. \quad (9)$$

The late time behaviour is determined by the singularities of $\tilde{B}(s)$ with largest real part.

To illustrate the case in which $W(\tau)$ has finite mean, let us consider an exponential distribution $W(\tau) = ae^{-a\tau}$ where $a = 1/\langle\tau\rangle$. Then after easy computation we obtain

$$\tilde{B}(s) = \frac{r}{s(s + a + r)} \quad (10)$$

and inversion of this expression yields

$$B(t) = \frac{r}{(a + r)} \left(1 - e^{-(a+r)t} \right). \quad (11)$$

Thus the relaxation time $\tau_{rel} = (a + r)^{-1}$ for the delta peak is given by half the harmonic mean of the mean resetting time and the exponential decay scale of the refractory period. We expect a similar relaxation scenario to hold whenever $W(\tau)$ has an exponential tail.

We next consider the case when the distribution of the refractory period $W(\tau)$ has a power law decay for large τ , but still with a finite first moment

$$W(\tau) \simeq \frac{A}{\tau^\gamma} \quad \gamma > 2, \quad (12)$$

where A is a constant and the exponent $\gamma > 2$ ensures that $\langle\tau\rangle$ is finite. In this case, the small s behaviour of the Laplace transform is given by (for a related analysis in the context of mass transport models, see Ref. [24])

$$\tilde{W}(s) = \sum_{k=0}^{n-1} (-1)^k \frac{\mu_k}{k!} s^k + b s^{\gamma-1} + \dots \quad (13)$$

where $\mu_0 = 1$, $\mu_1 = \langle\tau\rangle$, $b = A\Gamma(1 - \gamma)$ and $n = \text{int}(\gamma)$. We consider, for simplicity, the case for non-integer $\gamma > 2$. For integer $\gamma > 2$, there are additional logarithmic corrections [24]. Substituting this small s behaviour of $\tilde{W}(s)$ in Eq. (9), we get for small s

$$\tilde{B}(s) = \frac{r\langle\tau\rangle}{1 + r\langle\tau\rangle} \frac{1}{s} - \frac{br}{(1 + r\langle\tau\rangle)^2} s^{\gamma-3} + \text{analytic terms} \quad (14)$$

where we have kept the most dominant nonanalytic term in the small s expansion. The first term on the rhs, when Laplace inverted, will lead to the stationary value at late times. The next subleading non-analytic term, when inverted, gives rise to a temporal power-law decay to the stationary state. The other analytic terms correspond to faster than power-law decay at late times. Inverting explicitly the first two terms in Eq. (14) then leads to the late time result (for noninteger $\gamma > 2$)

$$B(t) \simeq \frac{r\langle\tau\rangle}{1+r\langle\tau\rangle} - \frac{1}{(\gamma-2)(\gamma-1)} \frac{Ar}{(1+r\langle\tau\rangle)^2} \frac{1}{t^{\gamma-2}}, \quad (15)$$

where A is defined in Eq. (12), and we have used $\mathcal{L}_s^{-1}[s^p] = t^{-1-p}/\Gamma(-p)$. Thus, the late time relaxation to the stationary value is algebraic with an exponent $\gamma - 2$. A similar analysis can be performed for the case when $\gamma > 2$ is an integer.

As an example of a case where the mean refractory period $\langle\tau\rangle$ diverges, we again consider $W(\tau)$ with an algebraic tail as in Eq. 12, but with exponent $1 < \gamma < 2$, i.e. The Laplace transform of $W(\tau)$ behaves for small s as in Eq. (13), except that now $n = \text{int}(\gamma) = 1$, hence the first two leading terms predict

$$\tilde{W}(s) \simeq 1 + b s^{\gamma-1} + \dots \quad \text{with } 1 < \gamma < 2 \quad (16)$$

where the constant $b = A\Gamma(1-\gamma)$. Thus, from Eq. (9), we find for small s ,

$$\tilde{B}(s) \simeq \frac{1}{s} + \frac{1}{br} s^{1-\gamma}, \quad (17)$$

which, upon inversion, yields the asymptotic behaviour for $1 < \gamma < 2$

$$B(t) \simeq 1 + \frac{(\gamma-1)\sin(\pi\gamma)}{Ar\pi} \frac{1}{t^{2-\gamma}}. \quad (18)$$

Comparing Eqs. (15) for $2 < \gamma$ and (18) for $1 < \gamma < 2$, we see that in both cases the amplitude $B(t)$ of the delta peak at $x = 0$ approaches its stationary value algebraically as $t^{-\theta}$ at late times, where the exponent $\theta = \gamma - 2$ for $\gamma > 2$ and $\theta = 2 - \gamma$ for $1 < \gamma < 2$. Interestingly, the exponent θ approaches 0 as $\gamma \rightarrow 2$ from either side, indicating the slowest relaxation exactly for $\gamma = 2$. Indeed, this marginal case $\gamma = 2$ can also be worked out explicitly. In this case, $W(\tau) \simeq A/\tau^2$ for large τ , and consequently, for small s , $\tilde{W}(s) \simeq 1 + A s \ln(s)$. Substituting this behaviour in Eq. (9), we get $\tilde{B}(s) \simeq 1/s + 1/[A s \ln s]$. Upon inverting, we obtain, for $\gamma = 2$

$$B(t) \simeq 1 - \frac{1}{A \ln t} \quad (19)$$

thus demonstrating an ultra-slow inverse logarithmic relaxation to the stationary state in this marginal case $\gamma = 2$. Such inverse logarithmic relaxation was found previously in a completely different context of an Ising-Glaber chain relaxing in the presence of kinetic disorder [25].

3. Correlated resetting and refractory period distribution

In the previous section the time t_1 to the first reset and the duration τ_2 of the subsequent refractory period were independent random variables (see equation (1)). It is of interest

to consider the more general situation where they are correlated. For example, it would be natural if after a long period of activity before the first reset the refractory period were longer than after a short period of activity.

Generally we can consider the joint probability density $H(t, \tau)$ for the the first reset event to occur at time t and the subsequent refractory period to be of duration τ . The marginal distributions for the time of the first reset event and the refractory period duration are

$$h(t) = \int d\tau H(t, \tau) \quad (20)$$

$$W(\tau) = \int dt H(t, \tau) . \quad (21)$$

We also require the probability $g(t)$ of no resetting up to time t which is given by

$$g(t) = \int_t^\infty h(t') dt' . \quad (22)$$

3.1. Examples of correlated resetting period/refractory period distributions

As an example let us consider the case

$$H(t, \tau) = r e^{-rt} \frac{1}{a_1 t} e^{-\tau/(a_1 t)} . \quad (23)$$

Thus the resetting period t and the refractory period τ are both exponentially distributed but the mean of τ for given t increases linearly with the time t to the reset. In this case we can compute explicitly the marginal distributions

$$h(t) = r e^{-rt} \quad (24)$$

$$W(\tau) = \frac{2r}{a_1} K_0(2^{3/2}(\tau r/a_1)^{1/2}) \quad (25)$$

where K_0 is the modified Bessel function of the second kind. Note that $W(\tau)$ decays for large τ as $W(\tau) \simeq \left(\frac{r^3}{2\pi^2 a_1^3 \tau}\right)^{1/4} e^{-2^{3/2}(\tau r/a_1)^{1/2}}$. Thus there is a slow stretched exponential decay of the refractory period distribution.

3.2. Stationary distribution

Following the first-renewal approach presented in Section 2 it is easy to arrive at the Laplace transform for the probability distribution with resetting to the origin

$$\tilde{P}(x, s) = \frac{1}{1 - \tilde{H}(s, s)} \left[\int_0^\infty dt e^{-st} g(t) G_0(x, t) + \delta(x) \frac{1}{s} \left[\tilde{H}(s, 0) - \tilde{H}(s, s) \right] \right] , \quad (26)$$

where we define the double Laplace transform

$$\tilde{H}(s, m) = \int_0^\infty dt \int_0^\infty d\tau e^{-st - m\tau} H(t, \tau) . \quad (27)$$

As before the stationary state $P^{\text{st}}(x)$ is extracted as the coefficient of $1/s$ in the small s expansion of (26) and in the case where the time to first reset and refractory period have finite means $\langle R \rangle$ and $\langle \tau \rangle$, we obtain the stationary distribution

$$P^{\text{st}}(x) = \frac{1}{\langle R \rangle + \langle \tau \rangle} \left[\int_0^\infty dt g(t) G_0(x, t) + \delta(x) \langle \tau \rangle \right]. \quad (28)$$

Interestingly, the stationary distribution $P^{\text{st}}(x)$ depends on the joint distribution of the first resetting time and the refractory period $H(t, \tau)$ only through the marginal $h(t)$ of the first resetting time (through $g(t) = \int_t^\infty h(t') dt'$ in Eq. (22)). The correlation between t and τ present in the joint distribution $H(t, \tau)$ may determine the marginal $h(t) = \int_0^\infty H(t, \tau) d\tau$, and thereby the stationary state. However, the dependence of the stationary distribution on the correlation is *weak* in this sense. Note that taking $\langle \tau \rangle = 0$ we obtain the stationary distribution without the refractory period

$$P_{\tau=0}^{\text{st}}(x) = \frac{1}{\langle R \rangle} \int_0^\infty dt g(t) G_0(x, t), \quad (29)$$

see [19]. Again we see from (28) that a finite refractory period introduces a delta function at the resetting position with relative weight equal to the ratio of mean refractory period to mean resetting period.

3.3. Survival Probability

Let us now consider the process under resetting to X_r followed by a refractory period and with the addition of an absorbing boundary at the origin. The process is absorbed when it reaches the origin for the first time. In the context of a search process we can think of the time to absorption as the time to find a target located at the origin. More generally we can consider the time to reach the origin as the time to completion of some task. In the following we will refer to the survival probability and the mean first passage time (MFPT) of the process but with the general context of random time to completion of a task in mind.

We again take advantage of a renewal equation. We first define $Q_r(x_0, t)$ as the survival probability of the process in the *presence* of resetting and refractory period and $Q_0(x_0, t)$ as the survival probability in the *absence* of resetting, both starting from x_0 at $t = 0$. Note that we do not keep track of the final position of the particle and in fact, the final position has already been integrated over in the definition of the survival probability. It is well known that for the computation of the first-passage probabilities for stochastic processes, direct calculation of the survival probability treating the starting position x_0 as a variable (the so called backward approach) is much more advantageous [26, 27], compared to the ones where one keeps track of the final position and then integrates over it. Note that the initial position x_0 , treated as a variable here, may be set equal to the resetting position X_r at the end of the calculation.

We may write down a renewal equation using similar reasoning to that of (1)

$$Q_r(x_0, t) = g(t) Q_0(x_0, t)$$

$$\begin{aligned}
& + \int_0^\infty dt_1 Q_0(x_0, t_1) \int_0^\infty d\tau_2 H(t_1, \tau_2) \int_0^\infty dt_3 Q_r(X_r, t_3) \delta(t - (t_1 + \tau_2 + t_3)) \\
& + \int_0^t dt_1 \int_{t-t_1}^\infty d\tau_2 H(t_1, \tau_2) Q_0(x_0, t_1) .
\end{aligned} \tag{30}$$

The first term on the rhs is the contribution from survival trajectories up to time t in which there is no resetting, which occurs with probability $g(t)$. The second term on the rhs of (30) integrates the contributions from survival trajectories in which the first reset occurs at time t_1 and the refractory period τ_2 ends before time t , which has probability density $H(t_1, \tau_2)$; these trajectories have survival probability $Q_0(x_0, t_1)$ for the initial period until the first reset and survival probability $Q_r(X_r, t_3)$ from the end of the refractory period at $t_1 + \tau_2$ until final time t . The third term on the rhs of (30) integrates the contributions from trajectories which survive without resetting until time t_1 at which the first reset occurs, and for which the subsequent refractory period τ_2 is ongoing at time t .

We now take the Laplace transform with Laplace variable s of (30). The Laplace transform of the third term yields

$$\begin{aligned}
& \int_0^\infty dt e^{-st} \int_0^t dt_1 \int_{t-t_1}^\infty d\tau_2 H(t_1, \tau_2) Q_0(x_0, t_1) \\
& = \int_0^\infty dt_1 \int_0^\infty d\tau_2 \int_{t_1}^{t_1+\tau_2} dt e^{-st} H(t_1, \tau_2) Q_0(x_0, t_1) \\
& = \int_0^\infty dt_1 \frac{1}{s} [e^{-st_1} - e^{-s(t_1+\tau_2)}] H(t_1, \tau_2) Q_0(x_0, t_1) \\
& = \frac{1}{s} \left[\int_0^\infty dt_1 e^{-st_1} h(t_1) Q_0(x_0, t_1) - \int_0^\infty dt_1 \int_0^\infty d\tau_2 e^{-s(t_1+\tau_2)} H(t_1, \tau_2) Q_0(x_0, t_1) \right] .
\end{aligned}$$

Putting this result together with computations similar to those used to obtain (4), we find

$$\begin{aligned}
\tilde{Q}_r(x_0, s) & = \int_0^\infty dt e^{-st} g(t) Q_0(x_0, t) \\
& + \tilde{Q}_r(X_r, s) \int_0^\infty dt_1 \int_0^\infty d\tau_2 e^{-s(t_1+\tau_2)} H(t_1, \tau_2) Q_0(x_0, t_1) \\
& + \frac{1}{s} \left[\int_0^\infty dt_1 e^{-st_1} h(t_1) Q_0(x_0, t_1) - \int_0^\infty dt_1 \int_0^\infty d\tau_2 e^{-s(t_1+\tau_2)} H(t_1, \tau_2) Q_0(x_0, t_1) \right] .
\end{aligned} \tag{31}$$

Setting $x_0 = X_r$ and rearranging yields

$$\begin{aligned}
\tilde{Q}_r(X_r, s) & = \left[1 - \int_0^\infty dt \int_0^\infty d\tau e^{-s(t+\tau)} H(t, \tau) Q_0(X_r, t) \right]^{-1} \\
& \times \left\{ \int_0^\infty dt e^{-st} g(t) Q_0(X_r, t) \right. \\
& \left. + \frac{1}{s} \left[\int_0^\infty dt e^{-st} h(t) Q_0(x_0, t) - \int_0^\infty dt \int_0^\infty d\tau e^{-s(t+\tau)} H(t, \tau) Q_0(X_r, t) \right] \right\} .
\end{aligned} \tag{32}$$

This is the main result of this section. It gives the Laplace transform of the survival probability under resetting with refractory period in terms of the Laplace transforms

of the refractory period distribution and survival probability without resetting. It generalises a previous result for the case of uncorrelated refractory period and resetting period [9, 18, 10, 11] (although note that the initial condition in those works is different to ours leading to slightly different expressions). To check the case of uncorrelated resetting period and refractory period we can set $H(t, \tau) = h(t)W(\tau)$ and obtain

$$\begin{aligned} \tilde{Q}_r(X_r, s) &= \left[1 - \tilde{W}(s) \int_0^\infty dt e^{-st} h(t) Q_0(X_r, t) \right]^{-1} \\ &\times \left\{ \int_0^\infty dt e^{-st} g(t) Q_0(X_r, t) \right. \\ &\quad \left. + \frac{1 - \tilde{W}(s)}{s} \int_0^\infty dt e^{-st} h(t) Q_0(x_0, t) \right\}. \end{aligned} \quad (33)$$

We can further check the vanishing refractory period case where $\tilde{W}(s) = 1$ in which case we recover the result [19]

$$\tilde{Q}_r(X_r, s) = \frac{\int_0^\infty dt e^{-st} g(t) Q_0(X_r, t)}{1 - \int_0^\infty dt e^{-st} h(t) Q_0(x_0, t)}. \quad (34)$$

3.4. Mean first passage time

The mean first passage time to the origin (or equivalently the mean time to absorption at the origin), $T(X_r)$, is conveniently given by

$$T(X_r) = \tilde{Q}_r(X_r, s \rightarrow 0). \quad (35)$$

In the $s \rightarrow 0$ limit it can be checked that (33) reduces to

$$T(X_r) = \frac{\int_0^\infty dt g(t) Q_0(X_r, t) + \int_0^\infty dt \langle \tau | t \rangle h(t) Q_0(X_r, t)}{1 - \int_0^\infty dt h(t) Q_0(X_r, t)}. \quad (36)$$

where $\langle \tau | t \rangle = \int_0^\infty d\tau \tau H(t, \tau) / h(t)$ is the average of the refractory period conditioned on the resetting period t .

In the zero refractory period limit the second term in the numerator vanishes and one recovers the result of [19]

$$T_{\tau=0}(X_r) \rightarrow \frac{\int_0^\infty dt g(t) Q_0(X_r, t)}{1 - \int_0^\infty dt h(t) Q_0(X_r, t)}. \quad (37)$$

Thus (36) shows that the mean time to absorption is that for the case of zero refractory period plus a second term coming from the occurrence of a refractory period. To understand this term better we note first that the quantity $\int_0^\infty dt h(t) Q_0(X_r, t)$ is the probability of survival of the process up to the first reset. Thus, the denominator of (36) $1 - \int_0^\infty dt h(t) Q_0(X_r, t)$ is the complementary probability of absorption of the process before the first reset occurs. Now consider the term in the numerator of (36) $\int_0^\infty dt \langle \tau | t \rangle h(t) Q_0(X_r, t)$. The integrand is the mean of τ conditioned on the first reset time t multiplied by the joint probability of first reset at t and survival of the process

up to time t . Thus the integral is simply the mean of τ conditioned on survival of the process up to τ multiplied by the probability of survival of the process.

We may write the second term in (36) as $\langle \tau | \text{survival} \rangle \frac{P}{Q}$ where P is the probability of the process surviving up to the first reset and $Q = 1 - P$. Then, considering each reset period as a Bernoulli trial which is successful if the process survives, the ratio of success to failure is the mean number of successful trials before a failure. Thus

$$T(X_r) = T_{\tau=0}(X_r) + \frac{\int_0^\infty d\tau \langle \tau | t \rangle h(t) Q_0(X_r, t)}{1 - \int_0^\infty dt h(t) Q_0(X_r, t)} \quad (38)$$

where the second term on the rhs, which results from the occurrence of a refractory period is simply the product of the mean duration of a refractory period after a reset period during which the process survives with the mean number of successful reset periods before absorption.

3.5. Poissonian reset

Here let us return to the case of Poissonian reset with rate r but now with a correlated refractory period

$$H(t, \tau) = r e^{-rt} W(\tau | t) \quad (39)$$

We assume that one can express the conditional mean appearing in (36,38) as a power series in t

$$\langle \tau | t \rangle = \sum_{n=0} a_n t^n. \quad (40)$$

Then we can express the quantities in (36) explicitly in terms of the Laplace transform of the survival probability without resetting and its derivatives

$$T(X_r) = \frac{\tilde{Q}_0(X_r, r) + r \sum_{n=0} (-1)^n a_n \tilde{Q}_0^{(n)}(X_r, r)}{1 - r \tilde{Q}_0(X_r, r)}. \quad (41)$$

For example, the case of independent refractory period has $a_0 = \langle \tau \rangle$ and $a_n = 0$ for $n > 0$. The example of Section 3.1 has $\langle \tau \rangle = a_1 t$ with the other coefficients $a_n = 0$ for $n \neq 1$.

4. Joint active time and first passage time distribution

Finally we consider the active time of a particle which is the time elapsed *excluding* refractory periods. The knowledge of the distribution of active time and time spent in the quiescent state may be useful when one has to optimise activity. We can write down a recursion relation for the joint probability density $P_r(t_a, t)$ of the particle being absorbed at time t and having spent active time t_a under resetting dynamics (here we just consider Poissonian resetting with rate r and independent refractory period). The

recursion relies on dividing up trajectories according to how many resets have occurred:

$$\begin{aligned}
P_r(t_a, t) &= e^{-rt} F_0(t) \delta(t - t_a) \\
&+ r \int dt_1 dt_2 dt_3 e^{-rt_1} Q_0(t_1) W(t_2) e^{-rt_3} F_0(t_3) \\
&\quad \times \delta(t - (t_1 + t_2 + t_3)) \delta(t_a - (t_1 + t_3)) \\
&+ r^2 \int dt_1 dt_2 dt_3 dt_4 dt_5 e^{-rt_1} Q_0(t_1) W(t_2) e^{-rt_3} W(t_4) e^{-rt_5} F_0(t_5) \\
&\quad \times \delta(t - (t_1 + t_2 + t_3 + t_4 + t_5)) \delta(t_a - (t_1 + t_3 + t_5)) \\
&+ \dots
\end{aligned} \tag{42}$$

The first term on the rhs is the contribution from trajectories in which there is no resetting which occurs with probability e^{-rt} . For these trajectories $t_a = t$ and given there is no resetting the survival probability is that of the process without resetting which we denote $Q_0(t)$. The second term is the contribution from trajectories with one reset which occurs at time t_1 , a refractory period of duration t_2 and a period with no resetting of duration t_3 at the end of which the particle is absorbed which occurs with first passage probability of the process without resetting which we denote $F_0(t)$. Similarly, the third term is the contribution from trajectories with two resets which occur at time t_1 followed by refractory period of duration t_2 and at $t_1 + t_2 + t_3$ followed by a refractory period of duration t_4 , and with no resetting of duration t_5 at the end of which the particle is absorbed. Clearly there is an infinite sequence of contributions corresponding to higher numbers of resets.

Now taking the Laplace transform with Laplace variable s conjugate to t and Laplace variable λ conjugate to t_a

$$\begin{aligned}
\tilde{P}_r(\lambda, s) &= \tilde{F}_0(\lambda + r + s) + r \tilde{Q}_0(\lambda + r + s) \tilde{W}(s) \tilde{F}_0(\lambda + r + s) \\
&\quad + r^2 \tilde{Q}_0(\lambda + r + s)^2 \tilde{W}(s)^2 \tilde{F}_0(\lambda + r + s) + \dots
\end{aligned} \tag{43}$$

Summing the geometric series yields

$$\tilde{P}_r(\lambda, s) = \frac{\tilde{F}_0(\lambda + r + s)}{1 - r \tilde{Q}_0(\lambda + r + s) \tilde{W}(s)}. \tag{44}$$

First let us consider the marginal distributions for t (obtained by setting $\lambda = 0$) and for t_a (obtained by setting $s = 0$)

$$\tilde{P}_r(0, s) = \frac{\tilde{F}_0(r + s)}{1 - r \tilde{Q}_0(r + s) \tilde{W}(s)} \tag{45}$$

$$\tilde{P}_r(\lambda, 0) = \frac{\tilde{F}_0(\lambda + r)}{1 - r \tilde{Q}_0(\lambda + r)}. \tag{46}$$

The first equation (45) recovers (33) when we identify

$$\tilde{P}_r(0, s) = \tilde{F}_r(s) = 1 - s \tilde{Q}_r(s) \tag{47}$$

The second equation (46) shows that distribution of active time is precisely the first passage time distribution in the absence of resetting, as expected.

5. Conclusion

In this note we have considered the effects of a cost for resetting in the form of a refractory period generalising some previous results [9, 10, 11]. We have shown how calculations of the probability distribution and survival probability in the presence of an absorbing boundary may be carried out by using a renewal equation based on the first reset. The calculations yield some interesting novel results: (i) As a result of a non-zero refractory period the stationary state develops a delta function at the resetting position with weight proportional to the mean refractory period and how this peak emerges in the long-time behaviour. For a power-law distribution of refractory periods one can obtain slow (15,18) power-law relaxation or ultra-slow (19) inverse-logarithmic relaxation to the stationary behaviour. (ii) The survival probability and MFPT is strongly affected by correlated resetting and refractory periods. (iii) A joint active time and first passage time distribution may be computed.

It should be straightforward to extend our renewal approach to processes in higher dimensions in the presence of a random refractory period. The joint active time and mean first passage time distribution should allow one to calculate other quantities of interest, for example the mean first passage time conditioned on the active time. It would also be of interest to see how the presence of a finite refractory period affects other observables associated to reset dynamics [20, 11, 28]. Finally, the refractory period may have interesting consequences for bacterial population dynamics in the presence of ‘catastrophic’ events that are similar to reset events [29].

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