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The Tamed Unadjusted Langevin Algorithm

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Abstract

In this article, we consider the problem of sampling from a probability measure π having a density on \mathbb{R}^d proportional to $x \mapsto e^{-U(x)}$. The Euler discretization of the Langevin stochastic differential equation (SDE) is known to be unstable, when the potential U is superlinear. Based on previous works on the taming of superlinear drift coefficients for SDEs, we introduce the Tamed Unadjusted Langevin Algorithm (TULA) and obtain non-asymptotic bounds in V -total variation norm and Wasserstein distance of order 2 between the iterates of TULA and π , as well as weak error bounds. Numerical experiments are presented which support our findings.

Keywords: Tamed Unadjusted Langevin Algorithm, Markov Chain Monte Carlo, Total variation distance, Wasserstein distance

2010 MSC: 65C05, 60F05, 62L10

1. Introduction

The Unadjusted Langevin Algorithm (ULA) first introduced in the physics literature by [1] and popularised in the computational statistics community by [2] and [3] is a technique to sample complex and high-dimensional probability distributions. This issue has far-reaching consequences in Bayesian statistics and

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machine learning [4], [5], aggregation of estimators [6] and molecular dynamics [7]. More precisely, let π be a probability distribution on \mathbb{R}^d which has density (also denoted by π) with respect to the Lebesgue measure given for all $x \in \mathbb{R}^d$ by,

$$\pi(x) = e^{-U(x)} / \int_{\mathbb{R}^d} e^{-U(y)} dy, \quad \text{with} \quad \int_{\mathbb{R}^d} e^{-U(y)} dy < +\infty.$$

Assuming that $U : \mathbb{R}^d \rightarrow \mathbb{R}$ is continuously differentiable, the overdamped Langevin stochastic differential equation (SDE) associated with π is given by

$$dY_t = -\nabla U(Y_t)dt + \sqrt{2}dB_t, \quad (1)$$

where $(B_t)_{t \geq 0}$ is a d -dimensional Brownian motion. The discrete time Markov chain associated with the ULA algorithm is obtained by the Euler-Maruyama discretization scheme of the Langevin SDE defined for $k \in \mathbb{N}$ by,

$$X_{k+1} = X_k - \gamma \nabla U(X_k) + \sqrt{2\gamma} Z_{k+1}, \quad X_0 = x_0, \quad (2)$$

where $x_0 \in \mathbb{R}^d$, $\gamma > 0$ and $(Z_k)_{k \in \mathbb{N}}$ are i.i.d. standard d -dimensional Gaussian variables. Under adequate assumptions on a globally Lipschitz ∇U , non-asymptotic bounds in total variation and Wasserstein distances between the distribution of $(X_k)_{k \in \mathbb{N}}$ and π can be found in [8], [9], [10]. However, the ULA algorithm is unstable if ∇U is superlinear *i.e.* $\liminf_{\|x\| \rightarrow +\infty} \|\nabla U(x)\| / \|x\| = +\infty$, see [11, Theorem 3.2], [12] and [13]. This is illustrated with a particular example in [12, Lemma 6.3] where, the SDE (1) is considered in one dimension with $U(x) = x^4/4$ along with the associated Euler discretization (2) and it is shown that for all $\gamma > 0$, if $\mathbb{E}[X_0^2] \geq 2/\gamma$, one obtains $\lim_{n \rightarrow +\infty} \mathbb{E}[X_n^2] = +\infty$. Moreover, the sample path $(X_n)_{n \in \mathbb{N}}$ diverges to infinity with positive probability.

Until recently, either implicit numerical schemes, e.g. see [12] and [14], or adaptive stepsize schemes, e.g. see [15], were used to address this problem. However, in the last few years, a new generation of explicit numerical schemes, which are computationally efficient, has been introduced by “taming” appropriately the superlinearly growing drift, see [16] and [17] for more details.

Nonetheless, with the exception of [12], these works focus on the discretization of SDEs with superlinear coefficients in finite time. We aim at extending

these techniques to sample from π , the invariant measure of (1). To deal with the superlinear nature of ∇U , we introduce a family of drift functions $(G_\gamma)_{\gamma>0}$ with $G_\gamma : \mathbb{R}^d \rightarrow \mathbb{R}^d$ indexed by the step size γ which are close approximations of ∇U in a sense made precise below. Consider then the following Markov chain $(X_k)_{k \in \mathbb{N}}$ defined for all $k \in \mathbb{N}$ by

$$X_{k+1} = X_k - \gamma G_\gamma(X_k) + \sqrt{2\gamma} Z_{k+1}, \quad X_0 = x_0. \quad (3)$$

We suggest two different explicit choices for the family $(G_\gamma)_{\gamma>0}$ based on previous studies on the tamed Euler scheme [16], [17], [18]. Define for all $\gamma > 0$, $H_\gamma, H_{\gamma,c} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ for all $x \in \mathbb{R}^d$ by

$$H_\gamma(x) = \frac{\nabla U(x)}{1 + \gamma \|\nabla U(x)\|} \quad \text{and} \quad H_{\gamma,c}(x) = \left(\frac{\partial_i U(x)}{1 + \gamma |\partial_i U(x)|} \right)_{i \in \{1, \dots, d\}}, \quad (4)$$

where $\partial_i U$ is the i^{th} -coordinate of ∇U . The Euler scheme (3) with $G_\gamma = H_\gamma$, respectively $G_\gamma = H_{\gamma,c}$, is referred to as the Tamed Unadjusted Langevin Algorithm (TULA), respectively the coordinate-wise Tamed Unadjusted Langevin Algorithm (TULAc).

Another line of work has focused on the Metropolis Adjusted Langevin Algorithm (MALA) that consists in adding a Metropolis Hastings step to the ULA algorithm. [19] provides a detailed analysis of MALA in the case where the drift coefficient is superlinear. Note also that a normalization of the gradient was suggested in [11, Section 1.4.3] calling it MALTA (Metropolis Adjusted Langevin Truncated Algorithm) and analyzed in [20] and [21].

The article is organized as follows. In Section 2, the Markov chain $(X_k)_{k \in \mathbb{N}}$ defined by (3) is shown to be V -geometrically ergodic w.r.t. an invariant measure π_γ . Non-asymptotic bounds between the distribution of $(X_k)_{k \in \mathbb{N}}$ and π in total variation and Wasserstein distances are provided, as well as weak error bounds. In Section 3, the methodology is illustrated through numerical examples. Finally, proofs of the main results appear in Section 4.

Notations

Let $\mathcal{B}(\mathbb{R}^d)$ denote the Borel σ -field of \mathbb{R}^d . Moreover, let $L^1(\mu)$ be the set of μ -integrable functions for μ a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Further, $\mu(f) = \int_{\mathbb{R}^d} f(x) d\mu(x)$ for an $f \in L^1(\mu)$. Given a Markov kernel R on \mathbb{R}^d , for all $x \in \mathbb{R}^d$ and f integrable under $R(x, \cdot)$, denote by $Rf(x) = \int_{\mathbb{R}^d} f(y) R(x, dy)$. Let $V : \mathbb{R}^d \rightarrow [1, \infty)$ be a measurable function. The V -total variation distance between μ and ν is defined as $\|\mu - \nu\|_V = \sup_{|f| \leq V} |\mu(f) - \nu(f)|$. If $V = 1$, then $\|\cdot\|_V$ is the total variation denoted by $\|\cdot\|_{TV}$. Let μ and ν be two probability measures on a state space Ω with a given σ -algebra. If $\mu \ll \nu$, we denote by $d\mu/d\nu$ the Radon-Nikodym derivative of μ w.r.t. ν . In that case, the Kullback-Leibler divergence of μ w.r.t. to ν is defined as

$$\text{KL}(\mu|\nu) = \int_{\Omega} \frac{d\mu}{d\nu} \log \left(\frac{d\mu}{d\nu} \right) d\nu.$$

We say that ζ is a transference plan of μ and ν if it is a probability measure on $(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d))$ such that for any Borel set A of \mathbb{R}^d , $\zeta(A \times \mathbb{R}^d) = \mu(A)$ and $\zeta(\mathbb{R}^d \times A) = \nu(A)$. We denote by $\Pi(\mu, \nu)$ the set of transference plans of μ and ν . Furthermore, we say that a couple of \mathbb{R}^d -random variables (X, Y) is a coupling of μ and ν if there exists $\zeta \in \Pi(\mu, \nu)$ such that (X, Y) are distributed according to ζ . For two probability measures μ and ν , we define the Wasserstein distance of order $p \geq 1$ as

$$W_p(\mu, \nu) = \left(\inf_{\zeta \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^p d\zeta(x, y) \right)^{1/p}.$$

By [22, Theorem 4.1], for all μ, ν probability measure on \mathbb{R}^d , there exists a transference plan $\zeta^* \in \Pi(\mu, \nu)$ such that for any coupling (X, Y) distributed according to ζ^* , $W_p(\mu, \nu) = \mathbb{E}[\|X - Y\|^p]^{1/p}$.

For $u, v \in \mathbb{R}^d$, define the scalar product $\langle u, v \rangle = \sum_{i=1}^d u_i v_i$ and the Euclidian norm $\|u\| = \langle u, u \rangle^{1/2}$. Denote by $\mathbb{S}^{d-1} = \{u \in \mathbb{R}^d : \|u\| = 1\}$. For $k \in \mathbb{N}$, $m, m' \in \mathbb{N}^*$ and Ω, Ω' two open sets of $\mathbb{R}^m, \mathbb{R}^{m'}$ respectively, denote by $C^k(\Omega, \Omega')$, the set of k -times continuously differentiable functions. For $f \in C^2(\mathbb{R}^d, \mathbb{R})$, denote by ∇f the gradient of f , $\partial_i f$ the i^{th} -coordinate of ∇f , Δf the

distance	order of the upper bound	assumptions
$\ \delta_x R_\gamma^n - \pi\ _{V^{1/2}}$	$n\gamma\lambda^{n\gamma}V(x) + \sqrt{\gamma}$	A1, A2, H1 and H2
$W_2^2(\delta_x R_\gamma^n, \pi)$	$n\gamma\lambda^{n\gamma}V(x) + \gamma$	A1, A2, H1, H2 and H3
$W_2^2(\delta_x R_\gamma^n, \pi)$	$n\gamma^{1+\beta}\lambda^{n\gamma}V(x) + \gamma^{1+\beta}$	A1, A2, H2, H3 and H4

Table 1: Summary of the upper bounds on the distances between the distribution of the n^{th} iteration of the Markov chain defined by (3) and π .

Laplacian of f and $\nabla^2 f$ the Hessian of f . Define then for $x \in \mathbb{R}^d$, $\|\nabla^2 f(x)\| = \sup_{u \in \mathbb{S}^{d-1}} \|\nabla^2 f(x)u\|$. For $k \in \mathbb{N}$ and $f \in C^k(\mathbb{R}^d, \mathbb{R})$, denote by $D^i f$ the i -th derivative of f for $i \in \{0, \dots, k\}$, *i.e.* $D^i f$ is a symmetric i -linear map defined for all $x \in \mathbb{R}^d$ and $j_1, \dots, j_i \in \{1, \dots, d\}$ by $D^i f(x)[e_{j_1}, \dots, e_{j_i}] = \partial_{j_1 \dots j_i} f(x)$ where e_1, \dots, e_d is the canonical basis of \mathbb{R}^d . For $x \in \mathbb{R}^d$ and $i \in \{1, \dots, k\}$, define $\|D^0 f(x)\| = |f(x)|$, $\|D^i f(x)\| = \sup_{u_1, \dots, u_i \in \mathbb{S}^{d-1}} D^i f(x)[u_1, \dots, u_i]$. Note that $\|D^1 f(x)\| = \|\nabla f(x)\|$ and $\|D^2 f(x)\| = \|\nabla^2 f(x)\|$. For $m, m' \in \mathbb{N}^*$, define

$$C_{\text{poly}}(\mathbb{R}^m, \mathbb{R}^{m'}) = \left\{ f \in C(\mathbb{R}^m, \mathbb{R}^{m'}) \mid \exists C_q, q \geq 0, \forall x \in \mathbb{R}^m, \right. \\ \left. \|f(x)\| \leq C_q(1 + \|x\|^q) \right\}.$$

For all $x \in \mathbb{R}^d$ and $M > 0$, we denote by $B(x, M)$ (respectively $\bar{B}(x, M)$), the open (respectively close) ball centered at x of radius M . In the sequel, we take the convention that for $n, p \in \mathbb{N}$, $n < p$ then $\sum_p^n = 0$ and $\prod_p^n = 1$.

2. Ergodicity and convergence analysis

In this Section, under appropriate assumptions on ∇U and G_γ , we show that the diffusion process $(Y_t)_{t \geq 0}$ defined by (1) and its discretization $(X_k)_{k \in \mathbb{N}}$ defined by (3) satisfy a Foster-Lyapunov drift condition and are V -geometrically ergodic, see Proposition 1 and Proposition 3. Second, for all $k \in \mathbb{N}^*$, non-asymptotic bounds in V -norm between the distribution of X_k and π are established. Our next results give non-asymptotic bounds in Wasserstein distance of order 2, under the additional assumption that U is strongly convex. A summary of our main contributions is given in Table 1, where $\lambda \in [0, 1)$. We conclude this

part by non-asymptotic bounds on the bias and the variance of the ergodic average $n^{-1} \sum_{k=0}^{n-1} f(X_k)$, $n \in \mathbb{N}^*$, used as an estimator of $\pi(f)$, for $f : \mathbb{R}^d \rightarrow \mathbb{R}$ sufficiently smooth.

Henceforth, it is assumed that U is continuously differentiable. Consider the following assumptions on U .

H1. *There exist $\ell, L \in \mathbb{R}_+$ such that for all $x, y \in \mathbb{R}^d$,*

$$\|\nabla U(x) - \nabla U(y)\| \leq L \left\{ 1 + \|x\|^\ell + \|y\|^\ell \right\} \|x - y\| .$$

H2. *i) $\liminf_{\|x\| \rightarrow +\infty} \|\nabla U(x)\| = +\infty$.*

$$ii) \liminf_{\|x\| \rightarrow +\infty} \left\langle \frac{x}{\|x\|}, \frac{\nabla U(x)}{\|\nabla U(x)\|} \right\rangle > 0.$$

Note that under **H2**, $\liminf_{\|x\| \rightarrow +\infty} U(x) = +\infty$, U has a minimum x^* and $\nabla U(x^*) = 0$. Without loss of generality, it is assumed that $x^* = 0$. It implies under **H1** that for all $x \in \mathbb{R}^d$,

$$\|\nabla U(x)\| \leq 2L \left\{ 1 + \|x\|^{\ell+1} \right\} . \quad (5)$$

Besides, under **H2**-ii), there exists $C \in \mathbb{R}$ such that for all $x \in \mathbb{R}^d$, $\langle -\nabla U(x), x \rangle \leq C$. By [23, Theorem 2.1], [24, Chapter IV, Theorems 2.3, 3.1] and [11, Theorem 2.1], (1) has a unique strong solution denoted $(Y_t)_{t \geq 0}$. By [25, Section 5.4.C, Theorem 4.20], one constructs the associated strongly Markovian semigroup $(P_t)_{t \geq 0}$ given for all $t \geq 0$, $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$ by $P_t(x, A) = \mathbb{E}[\mathbb{1}_A(Y_t) | Y_0 = x]$. Consider the infinitesimal generator \mathcal{A} associated with (1) defined for all $h \in C^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ by

$$\mathcal{A}h(x) = -\langle \nabla U(x), \nabla h(x) \rangle + \Delta h(x) , \quad (6)$$

and for any $a \in \mathbb{R}_+^*$, define the Lyapunov function $V_a : \mathbb{R}^d \rightarrow [1, +\infty)$ for all $x \in \mathbb{R}^d$ by

$$V_a(x) = \exp \left(a(1 + \|x\|^2)^{1/2} \right) . \quad (7)$$

Foster-Lyapunov conditions enable to control the moments of the diffusion process $(Y_t)_{t \geq 0}$, see e.g. [23, Section 6] or [11, Theorem 2.2].

Proposition 1. Assume **H1**, **H2** and let $a \in \mathbb{R}_+^*$. There exists $b_a \in \mathbb{R}_+$ (given explicitly in the proof) such that for all $x \in \mathbb{R}^d$

$$\mathcal{A}V_a(x) \leq -aV_a(x) + ab_a \quad (8)$$

and

$$\sup_{t \geq 0} P_t V_a(x) \leq V_a(x) + b_a .$$

Moreover, there exist $C_a \in \mathbb{R}_+$ and $\rho_a \in [0, 1)$ such that for all $t \in \mathbb{R}_+$ and probability measures μ_0, ν_0 on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ satisfying $\mu_0(V_a) + \nu_0(V_a) < +\infty$,

$$\|\mu_0 P_t - \nu_0 P_t\|_{V_a} \leq C_a \rho_a^t \|\mu_0 - \nu_0\|_{V_a} , \quad \|\mu_0 P_t - \pi\|_{V_a} \leq C_a \rho_a^t \mu_0(V_a) . \quad (9)$$

Proof. The proof is postponed to Section 4.1. \square

The Markov chain $(X_k)_{k \in \mathbb{N}}$ defined in (3) is a discrete-time approximation of the diffusion $(Y_t)_{t \geq 0}$. To control the total variation and Wasserstein distances of the marginal distributions of $(X_k)_{k \in \mathbb{N}}$ and $(Y_t)_{t \geq 0}$, it is necessary to assume that for $\gamma > 0$ small enough, G_γ and ∇U are close. This is formalized by **A1**. Under the additional assumption **A2**, we obtain the stability and ergodicity of $(X_k)_{k \in \mathbb{N}}$.

A1. For all $\gamma > 0$, G_γ is continuous. There exist $\alpha \geq 0$, $C_\alpha < +\infty$ such that for all $\gamma > 0$ and $x \in \mathbb{R}^d$,

$$\|G_\gamma(x) - \nabla U(x)\| \leq \gamma C_\alpha (1 + \|x\|^\alpha) .$$

Note that under **H1**, **A1** and by (5), we have for all $x \in \mathbb{R}^d$

$$\|G_\gamma(x)\| \leq 2L \left\{ 1 + \|x\|^{\ell+1} \right\} + \gamma C_\alpha (1 + \|x\|^\alpha) . \quad (10)$$

A2. For all $\gamma > 0$, $\liminf_{\|x\| \rightarrow +\infty} \left\langle \frac{x}{\|x\|}, G_\gamma(x) \right\rangle - \frac{\gamma}{2\|x\|} \|G_\gamma(x)\|^2 > 0$.

Lemma 2. Assume **H1** and **H2**. Let $\gamma > 0$ and G_γ be equal to H_γ or $H_{\gamma,c}$ defined in (4). Then **A1** and **A2** are satisfied.

Proof. The proof is postponed to Section 4.2. \square

The Markov kernel R_γ associated with (3) is given for all $\gamma > 0$, $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$ by

$$R_\gamma(x, A) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathbb{1}_A \left(x - \gamma G_\gamma(x) + \sqrt{2\gamma}z \right) e^{-\|z\|^2/2} dz . \quad (11)$$

We then obtain the counterpart of Proposition 1 for the Markov chain $(X_k)_{k \in \mathbb{N}}$.

Proposition 3. *Assume **H1**, **A1**, **A2** and let $\gamma \in \mathbb{R}_+^*$. There exist $M, \mathfrak{a}, b \in \mathbb{R}_+^*$ (given explicitly in the proof) satisfying for all $x \in \mathbb{R}^d$*

$$R_\gamma V_{\mathfrak{a}}(x) \leq e^{-\mathfrak{a}^2 \gamma} V_{\mathfrak{a}}(x) + \gamma b \mathbb{1}_{\overline{B}(0, M)}(x) . \quad (12)$$

In addition, R_γ has a unique invariant measure π_γ , R_γ is $V_{\mathfrak{a}}$ -geometrically ergodic w.r.t. π_γ .

Proof. The proof is postponed to Section 4.3. □

Note that a straightforward induction of (12) gives for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$,

$$R_\gamma^n V_{\mathfrak{a}}(x) \leq e^{-n\mathfrak{a}^2 \gamma} V_{\mathfrak{a}}(x) + \{(b\gamma)(1 - e^{-n\mathfrak{a}^2 \gamma})\} / (1 - e^{-\mathfrak{a}^2 \gamma}) .$$

Using $1 - e^{-\mathfrak{a}^2 \gamma} = \int_0^\gamma \mathfrak{a}^2 e^{-\mathfrak{a}^2 t} dt \geq \gamma \mathfrak{a}^2 e^{-\mathfrak{a}^2 \gamma}$, we get for all $n \in \mathbb{N}$

$$R_\gamma^n V_{\mathfrak{a}}(x) \leq e^{-\mathfrak{a}^2 n \gamma} V_{\mathfrak{a}}(x) + (b/\mathfrak{a}^2) e^{\mathfrak{a}^2 \gamma} . \quad (13)$$

In the following result, we compare the discrete and continuous time processes $(X_k)_{k \in \mathbb{N}}$ and $(Y_t)_{t \geq 0}$ using Girsanov's theorem and Pinsker's inequality, see [8] and [9, Theorem 10] for similar arguments.

Theorem 4. *Assume **H1**, **H2**, **A1** and **A2**. Let $\gamma_0 > 0$. There exist $C > 0$ and $\lambda \in (0, 1)$ such that for all $\gamma \in (0, \gamma_0]$, $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$,*

$$\|\delta_x R_\gamma^n - \pi\|_{V_{\mathfrak{a}}^{1/2}} \leq C (n\gamma \lambda^{n\gamma} V_{\mathfrak{a}}(x) + \sqrt{\gamma}) , \quad (14)$$

where \mathfrak{a} is defined in Proposition 3 and for all $\gamma \in (0, \gamma_0]$,

$$\|\pi_\gamma - \pi\|_{V_{\mathfrak{a}}^{1/2}} \leq C \sqrt{\gamma} . \quad (15)$$

Proof. The proof is postponed to Section 4.4. □

By adding strong convexity for the potential, one obtains the corresponding bounds for the Wasserstein distance of order 2.

H3. U is strongly convex, i.e. there exists $m > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$\langle \nabla U(x) - \nabla U(y), x - y \rangle \geq m \|x - y\|^2 .$$

By coupling $(Y_t)_{t \geq 0}$ and the linear interpolation of $(X_k)_{k \in \mathbb{N}}$ with the same Brownian motion, the following result is obtained.

Theorem 5. Assume **A 1**, **A 2**, **H 1**, **H 2** and **H 3**. Let $\gamma_0 > 0$. There exist $C > 0$ and $\lambda \in (0, 1)$ such that for all $x \in \mathbb{R}^d$, $\gamma \in (0, \gamma_0]$ and $n \in \mathbb{N}$,

$$W_2^2(\delta_x R_\gamma^n, \pi) \leq C (n\gamma\lambda^{n\gamma}V_\varkappa(x) + \gamma) , \quad (16)$$

where \varkappa is defined in Proposition 3 and for all $\gamma \in (0, \gamma_0]$,

$$W_2^2(\pi_\gamma, \pi) \leq C\gamma . \quad (17)$$

Proof. The proof is postponed to Section 4.5. \square

If $U \in C^2(\mathbb{R}^d, \mathbb{R})$ and under the following assumption on $\nabla^2 U$, the bound can be improved.

H 4. U is twice continuously differentiable and there exist $\nu, L_H \in \mathbb{R}_+$ and $\beta \in [0, 1]$ such that for all $x, y \in \mathbb{R}^d$,

$$\|\nabla^2 U(x) - \nabla^2 U(y)\| \leq L_H \{1 + \|x\|^\nu + \|y\|^\nu\} \|x - y\|^\beta .$$

It is shown in Section 4.5 that **H 4** implies **H 1**.

Theorem 6. Assume **A 1**, **A 2**, **H 2**, **H 3** and **H 4**. Let $\gamma_0 > 0$. There exist $C > 0$ and $\lambda \in (0, 1)$ such that for all $x \in \mathbb{R}^d$, $\gamma \in (0, \gamma_0]$ and $n \in \mathbb{N}$,

$$W_2^2(\delta_x R_\gamma^n, \pi) \leq C (n\gamma^{1+\beta}\lambda^{n\gamma}V_\varkappa(x) + \gamma^{1+\beta}) , \quad (18)$$

where \varkappa is defined in Proposition 3 and for all $\gamma \in (0, \gamma_0]$,

$$W_2^2(\pi_\gamma, \pi) \leq C\gamma^{1+\beta} . \quad (19)$$

Proof. The proof is postponed to Section 4.5. \square

The exponent of γ in (16) is improved from 1 to $1 + \beta$. In particular, if $\nabla^2 U$ is Lipschitz, $\nu = 0$, $\beta = 1$, and [10, Theorem 8] is recovered.

Let $(X_k)_{k \in \mathbb{N}}$ be the Markov chain defined in (3). To study the empirical average $(1/n) \sum_{k=0}^{n-1} \{f(X_k) - \pi(f)\}$ for $n \in \mathbb{N}^*$, we follow a method introduced in [26] and based on the Poisson equation. For f a π -integrable function, the Poisson equation associated with the generator \mathcal{A} defined in (6) is given for all $x \in \mathbb{R}^d$ by

$$\mathcal{A}\phi(x) = -(f(x) - \pi(f)) , \quad (20)$$

where ϕ , if it exists, is a solution of the Poisson equation. This equation has proved to be a useful tool to analyze additive functionals of diffusion processes, see e.g. [27] and references therein. The existence and regularity of a solution of the Poisson equation has been investigated in [28], [29], [30], [31]. In that purpose, the following additional assumption on U is necessary.

H5. $U \in C^4(\mathbb{R}^d, \mathbb{R})$ and $\|D^i U\| \in C_{\text{poly}}(\mathbb{R}^d, \mathbb{R}_+)$ for $i \in \{1, \dots, 4\}$.

Theorem 7. Assume **H2**, **H5**, **A1** and **A2**. Let $f \in C^3(\mathbb{R}^d, \mathbb{R})$ be such that $\|D^i f\| \in C_{\text{poly}}(\mathbb{R}^d, \mathbb{R}_+)$ for $i \in \{0, \dots, 3\}$. Let $\gamma_0 > 0$ and $(X_k)_{k \in \mathbb{N}}$ be the Markov chain defined by (3) and starting at $X_0 = 0$. There exists $C > 0$ such that for all $\gamma \in (0, \gamma_0]$ and $n \in \mathbb{N}^*$,

$$\left| \mathbb{E} \left[\frac{1}{n} \sum_{k=0}^{n-1} f(X_k) - \pi(f) \right] \right| \leq C \left(\gamma + \frac{1}{n\gamma} \right) \quad (21)$$

and

$$\mathbb{E} \left[\left(\frac{1}{n} \sum_{k=0}^{n-1} f(X_k) - \pi(f) \right)^2 \right] \leq C \left(\gamma^2 + \frac{1}{n\gamma} \right) . \quad (22)$$

Proof. The proof is postponed to Section 4.6. \square

Note that the standard rates of convergence are recovered, see [26, Theorems 5.1, 5.2].

3. Numerical examples

We illustrate our theoretical results using three numerical examples.

Multivariate Gaussian variable in high dimension. We first consider a multivariate Gaussian variable in dimension $d \in \{100, 1000\}$ of mean 0 and covariance matrix $\Sigma = \text{diag}(1, \dots, d)$. The potential $U : \mathbb{R}^d \rightarrow \mathbb{R}$ defined for all $x \in \mathbb{R}^d$ by $U(x) = (1/2)x^T \Sigma^{-1}x$ is d^{-1} -strongly convex and 1-gradient Lipschitz. The assumptions **H1**, **H2**, **H3**, **H4** with $\beta = 1$ and **H5** are thus satisfied. Note that in this case, ULA is stable and the analysis of [8], [9], [10] valid. Nevertheless, implementing TULA and TULAc on this example is still of interest. Indeed, some Bayesian posterior distributions have intricate expressions and identifying the superlinear part in the gradient ∇U may be a difficult task. Within this context, we check the robustness of TULA and TULAc with respect to (globally) Lipschitz ∇U .

We also consider in the supplementary document [32, ??] a badly conditioned multivariate Gaussian variable in dimension $d = 100$ of mean 0 and covariance matrix $\Sigma = \text{diag}(10^{-5}, 1, \dots, 1)$. In this example, ULA requires a step size of order 10^{-5} to be stable which implies a large number of iterations to obtain relevant results. On the other side, TULA and TULAc are applicable with a step size of order 10^{-2} and within a relatively small number of iterations, valid results for the axes 2 to 100 are obtained.

Double well. The potential is defined for all $x \in \mathbb{R}^d$ by $U(x) = (1/4)\|x\|^4 - (1/2)\|x\|^2$. We have $\nabla U(x) = (\|x\|^2 - 1)x$ and $\nabla^2 U(x) = (\|x\|^2 - 1)\text{Id} + 2xx^T$. We get $\|\nabla^2 U(x)\| = 3\|x\|^2 - 1$, $\langle x, \nabla U(x) \rangle = \|x\| \|\nabla U(x)\|$ for $\|x\| \geq 1$ and

$$\|\nabla^2 U(x) - \nabla^2 U(y)\| \leq 3(\|x\| + \|y\|)\|x - y\| ,$$

so that **H1**, **H2**, **H4** with $\beta = 1$ and **H5** are satisfied.

Ginzburg-Landau model. This model of phase transitions in physics [33, Section 6.2] is defined on a three-dimensional $d = p^3$ lattice for $p \in \mathbb{N}^*$ and the potential

is given for $x = (x_{ijk})_{i,j,k \in \{1, \dots, p\}} \in \mathbb{R}^d$ by

$$U(x) = \sum_{i,j,k=1}^p \left\{ \frac{1-\tau}{2} x_{ijk}^2 + \frac{\tau\alpha}{2} \left\| \tilde{\nabla} x_{ijk} \right\|^2 + \frac{\tau\lambda}{4} x_{ijk}^4 \right\},$$

where $\alpha, \lambda, \tau > 0$ and $\tilde{\nabla} x_{ijk} = (x_{i_+jk} - x_{ijk}, x_{ij_+k} - x_{ijk}, x_{ijk_+} - x_{ijk})$ with $i_{\pm} = i \pm 1 \pmod{p}$ and similarly for j_{\pm}, k_{\pm} . In the simulations, p is equal to 10.

We have

$$\begin{aligned} \nabla U(x) = & \left\{ \tau\alpha (6x_{ijk} - x_{i_+jk} - x_{ij_+k} - x_{ijk_+} - x_{i_-jk} - x_{ij_-k} - x_{ijk_-}) \right. \\ & \left. + (1-\tau)x_{ijk} + \tau\lambda x_{ijk}^3 \right\}_{i,j,k \in \{1, \dots, p\}}, \end{aligned}$$

and

$$\nabla^2 U(x) = \text{diag} \left((1-\tau + 6\tau\alpha + 3\tau\lambda x_{ijk}^2)_{i,j,k \in \{1, \dots, p\}} \right) + M,$$

where $M \in \mathbb{R}^{d \times d}$ is a constant matrix. **H1**, **H4** with $\beta = 1$ and **H5** are thus satisfied. Using that $x \mapsto \sum_{i,j,k=1}^p \left\| \tilde{\nabla} x_{ijk} \right\|^2$ is convex by composition of convex functions and its gradient evaluated in 0 is 0, we have for all $x \in \mathbb{R}^d$,

$$\langle x, \nabla U(x) \rangle \geq \sum_{i,j,k=1}^p \{(1-\tau)x_{ijk}^2 + \tau\lambda x_{ijk}^4\}.$$

By Cauchy-Schwarz inequality, $\left\{ \sum_{i,j,k=1}^p x_{ijk}^2 \right\}^2 \leq d \sum_{i,j,k=1}^p x_{ijk}^4$, and for all $x \in \mathbb{R}^d$, $\|x\|^2 \geq (2|1-\tau|d)/(\tau\lambda)$, we get $\langle x, \nabla U(x) \rangle \geq \{(\tau\lambda)/2\} \sum_{i,j,k=1}^p x_{ijk}^4$.

Besides, we have

$$\|\nabla U(x)\| \leq (|1-\tau| + 12\tau\alpha) \|x\| + \tau\lambda \left\| (x_{ijk}^3)_{i,j,k \in \{1, \dots, p\}} \right\|.$$

Let $a, b, c \in \{1, \dots, p\}$ be such that $|x_{abc}| = \max |x_{ijk}|$. We get

$$\|x\| \left\| (x_{ijk}^3)_{i,j,k \in \{1, \dots, p\}} \right\| \leq dx_{abc}^4 \leq d \sum_{i,j,k=1}^p x_{ijk}^4.$$

Finally, for $\|x\|^2 \geq \max\{1, (2|1-\tau|d)/(\tau\lambda)\}$, we obtain

$$\|x\| \|\nabla U(x)\| \leq \left\{ \frac{2d|1-\tau|}{\tau\lambda} + \frac{24\alpha d}{\lambda} + 2d \right\} \langle x, \nabla U(x) \rangle,$$

and **H2** is satisfied.

We benchmark TULA and TULAc against ULA given by (2), MALA and a Random Walk Metropolis-Hastings with a Gaussian proposal (RWM). TMALA (Tamed Metropolis Adjusted Langevin Algorithm) and TMALAc (coordinate-wise Tamed Metropolis Adjusted Langevin Algorithm), the Metropolized versions of TULA and TULAc, are also included in the numerical tests. Their theoretical analysis is similar to the one of MALTA [20, Proposition 2.1].

Since double well and Ginzburg-Landau models are coordinate-wise exchangeable, the results are provided only for their first coordinate. The Markov chains associated with these models are started at $X_0 = 0, (10, 0^{\otimes(d-1)}), (100, 0^{\otimes(d-1)}), (1000, 0^{\otimes(d-1)})$ and for the multivariate Gaussian at a random vector of norm 0, 10, 100, 1000. For the Gaussian and double well examples, for each initial condition, algorithm, step size $\gamma \in \{10^{-3}, 10^{-2}, 10^{-1}\}$, we run 100 independent Markov chains started at X_0 of 10^6 samples (respectively 10^5) in dimension $d = 100$ (respectively $d = 1000$). For the Ginzburg-Landau model, we run 100 independent Markov chains started at X_0 of 10^5 samples. For each run, we estimate the 1st and 2nd moment for the first and last coordinate, *i.e.* $\int_{\mathbb{R}^d} x_i \pi(x) dx$ for $i \in \{1, d\}$, by the empirical average and we compute the boxplots of the errors. For ULA, if the norm of X_k for $k \in \mathbb{N}$ exceeds 10^5 , the chain is stopped and for this step size γ the trajectory of ULA is not taken into account. For MALA, RWM, TMALA and TMALAc, if the acceptance ratio is below 0.05, we similarly do not take into account the corresponding trajectories.

For the three examples and for $i \in \{1, \dots, d\}$, $\int_{\mathbb{R}^d} x_i \pi(x) dx = 0$. By symmetry, for the double well, we have for $i \in \{1, \dots, d\}$ and $r \in \mathbb{R}_+$,

$$\mathbb{E} [X_i^2] = d^{-1} \int_{\mathbb{R}_+} r^2 \nu(r) dr / \int_{\mathbb{R}_+} \nu(r) dr, \quad \nu(r) = r^{d-1} \exp \{ (r^2/2) - (r^4/4) \} .$$

A Random Walk Metropolis run of 10^7 samples gives $\int_{\mathbb{R}^d} x_i^2 \pi(x) dx \approx 0.104 \pm 0.001$ for $d = 100$ and $\int_{\mathbb{R}^d} x_i^2 \pi(x) dx \approx 0.032 \pm 0.001$ for $d = 1000$.

Because of lack of space, we only display some boxplots in Figures 1 to 4. The Python code and all the figures are available at <https://github.com/nbrosse/TULA>. We remark that TULA, TULAc and to a lesser extent, TMALA and TMALAc, have a stable behaviour even with large step sizes and starting

far from the origin. This is particularly visible in Figures 2 and 4 where ULA diverges (*i.e.* $\liminf_{k \rightarrow +\infty} \mathbb{E}[\|X_k\|] = +\infty$) and MALA does not move even for small step sizes $\gamma = 10^{-3}$. Note however the existence of a bias for ULA, TULA and TULAc in Figure 3. Finally, comparison of the results shows that TULAc is preferable to TULA.

Note that other choices are possible for G_γ , depending on the model under study. For example, in the case of the double well, we could "tame" only the superlinear part of ∇U , *i.e.* consider for all $\gamma > 0$ and $x \in \mathbb{R}^d$,

$$G_\gamma(x) = \frac{\|x\|^2 x}{1 + \gamma \|x\|^2} - x. \quad (23)$$

A1 is satisfied and we have

$$\begin{aligned} \left\langle \frac{x}{\|x\|}, G_\gamma(x) \right\rangle - \frac{\gamma}{2\|x\|} \|G_\gamma(x)\|^2 &= \frac{\|x\|^3}{1 + \gamma \|x\|^2} \left\{ 1 + \gamma - \frac{\gamma}{2} \frac{\|x\|^2}{1 + \gamma \|x\|^2} \right\} \\ &\quad - \|x\| \{1 + (\gamma/2)\}, \\ \liminf_{\|x\| \rightarrow +\infty} \left\langle \frac{x}{\|x\|^2}, G_\gamma(x) \right\rangle - \frac{\gamma}{2\|x\|^2} \|G_\gamma(x)\|^2 &= \frac{\gamma^{-1} - \gamma}{2}. \end{aligned}$$

A2 is satisfied if and only if $\gamma \in (0, 1)$. It is striking to see that this theoretical threshold is clearly visible on the simulations. The algorithm (3) with G_γ defined by (23) obtains similar results as TULAc for $\gamma < 1$ but for $\gamma = 1$, the algorithm diverges.

Given the results of the numerical experiments, TULAc should be chosen over ULA to sample from general probability distributions. Indeed, TULAc has similar results as ULA when the step size is small and is more stable when using larger step sizes.

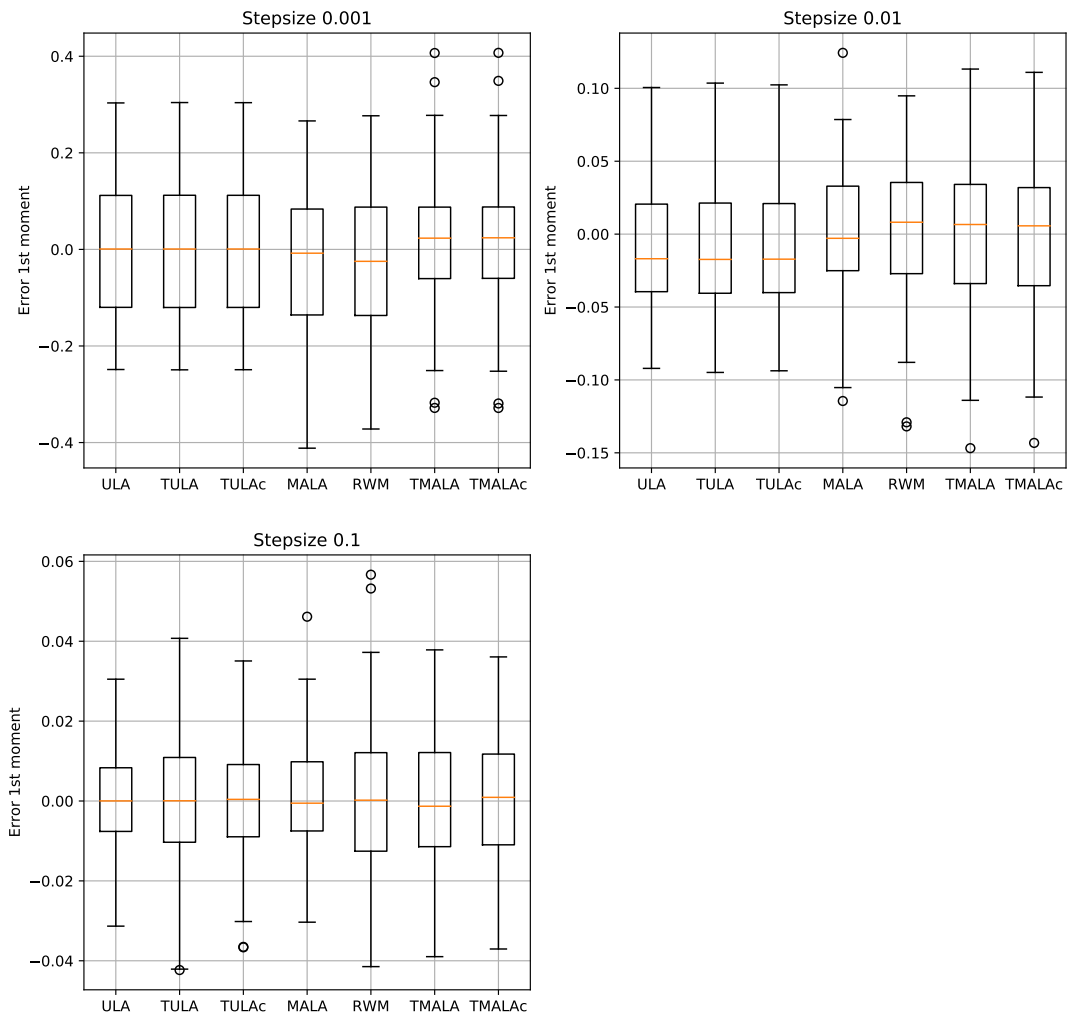


Figure 1: Boxplots of the error on the first moment for the multivariate Gaussian (first coordinate) in dimension 1000 starting at 0 for different step sizes.

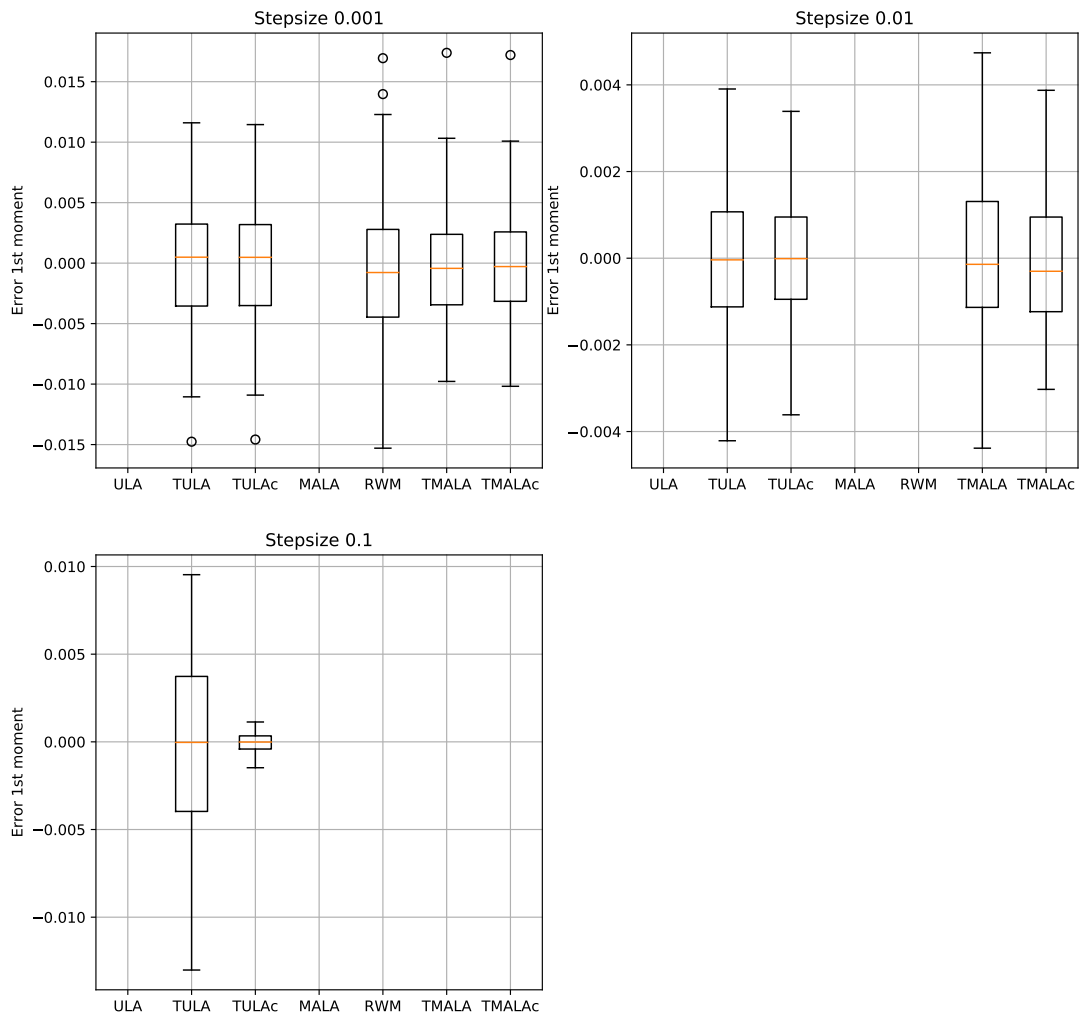


Figure 2: Boxplots of the error on the first moment for the double well in dimension 100 starting at $(100, 0^{\otimes 99})$ for different step sizes.

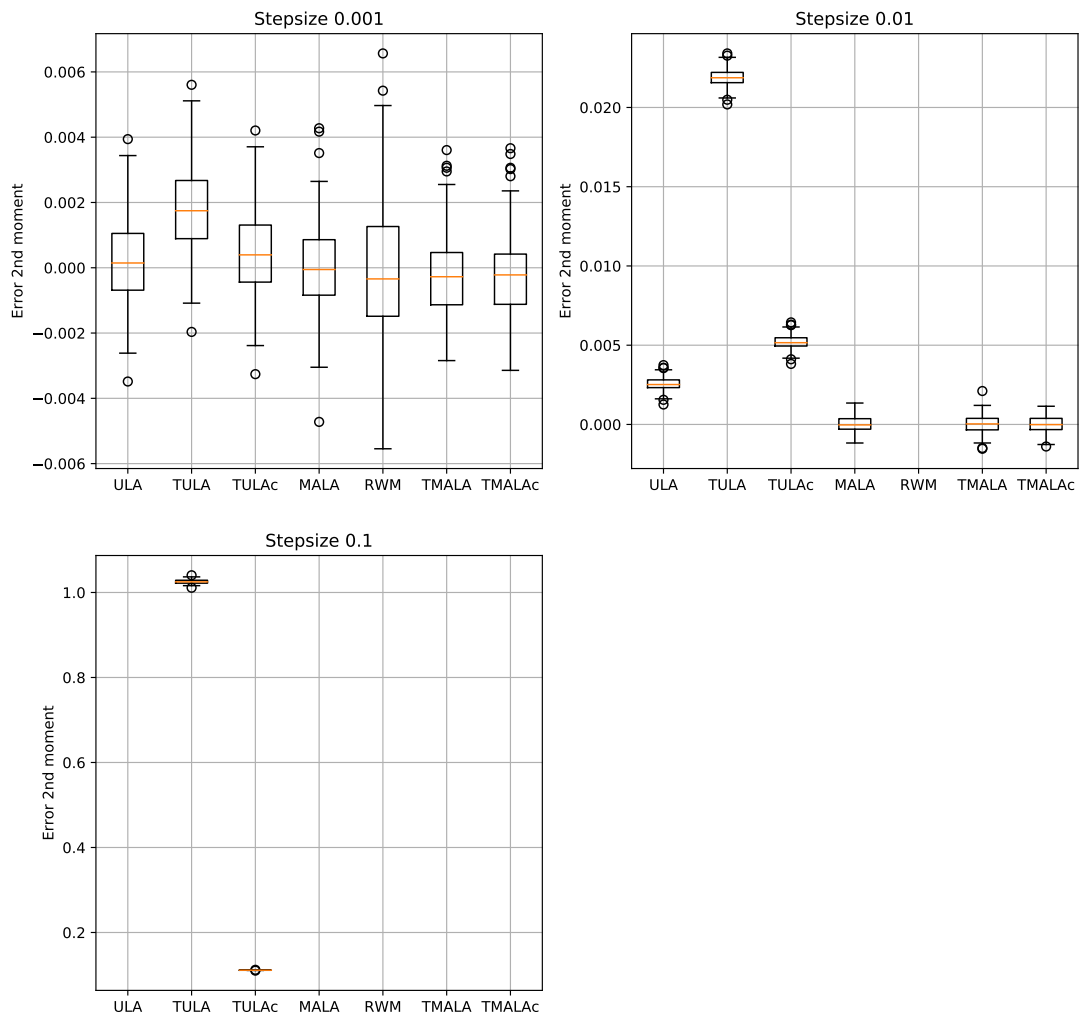


Figure 3: Boxplots of the error on the second moment for the double well in dimension 100 starting at 0 for different step sizes.

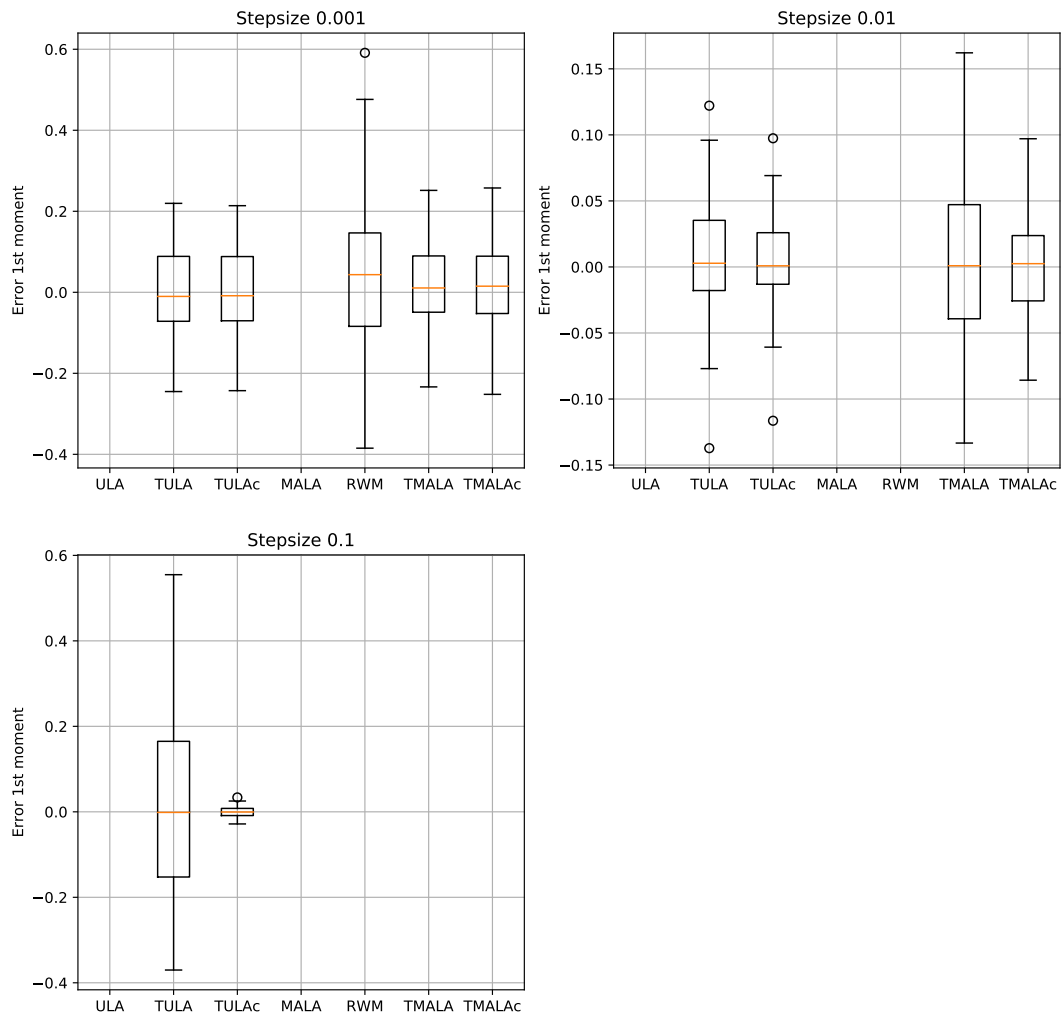


Figure 4: Boxplots of the error on the first moment for the Ginzburg-Landau model in dimension 1000 starting at $(100, 0^{\otimes 999})$ for different step sizes.

4. Proofs

4.1. Proof of Proposition 1

We have for all $x \in \mathbb{R}^d$,

$$\frac{\mathcal{A}V_a(x)}{aV_a(x)} = - \left\langle \nabla U(x), \frac{x}{(1 + \|x\|^2)^{1/2}} \right\rangle + \frac{a \|x\|^2}{1 + \|x\|^2} + \frac{d}{(1 + \|x\|^2)^{1/2}} - \frac{\|x\|^2}{(1 + \|x\|^2)^{3/2}}. \quad (24)$$

By **H2**-ii) and using $s \mapsto s/(1 + s^2)^{1/2}$ is non-decreasing for $s \geq 0$, there exist $M_1, \kappa \in \mathbb{R}_+^*$ such that for all $x \in \mathbb{R}^d$, $\|x\| \geq M_1$, $\left\langle \nabla U(x), x(1 + \|x\|^2)^{-1/2} \right\rangle \geq \kappa \|\nabla U(x)\|$. By **H2**-i), there exists $M_2 \geq M_1$ such that for all $x \in \mathbb{R}^d$, $\|x\| \geq M_2$, $\|\nabla U(x)\| \geq \kappa^{-1}\{1 + a + d(1 + M_1^2)^{-1/2}\}$. We then have for all $x \in \mathbb{R}^d$, $\|x\| \geq M_2$, $\mathcal{A}V_a(x) \leq -aV_a(x)$. Define

$$b_a = \exp(a(1 + M_2^2)^{1/2})\{2L(1 + M_2^{\ell+1}) + a + d\}.$$

Combining (5) and (24) gives (8). By [23, Theorem 1.1], we get $P_t V_a(x) \leq e^{-at}V_a(x) + b_a(1 - e^{-at})$. The second statement is a consequence of [11, Theorem 2.2] and [23, Theorem 6.1].

4.2. Proof of Lemma 2

Let $\gamma > 0$. We have for all $x \in \mathbb{R}^d$, $\|H_\gamma(x) - \nabla U(x)\| \leq \gamma \|\nabla U(x)\|^2$ and

$$\|H_{\gamma,c}(x) - \nabla U(x)\| \leq \gamma \left\{ \sum_{i=1}^d (\partial_i U(x))^4 \right\}^{1/2} \leq \gamma \|\nabla U(x)\|^2.$$

By (5), **A1** is satisfied. Define for all $x \in \mathbb{R}^d$, $x \neq 0$,

$$A_\gamma(x) = \left\langle \frac{x}{\|x\|}, H_\gamma(x) \right\rangle - \frac{\gamma}{2\|x\|} \|H_\gamma(x)\|^2.$$

By **H2**-ii), there exist $M_1, \kappa > 0$ such that for all $x \in \mathbb{R}^d$, $\|x\| \geq M_1$, $\langle x, \nabla U(x) \rangle \geq \kappa \|x\| \|\nabla U(x)\|$. We get then for all $x \in \mathbb{R}^d$, $\|x\| \geq M_1$,

$$\begin{aligned} A_\gamma(x) &= \frac{1}{2\|x\| \{1 + \gamma \|\nabla U(x)\|\}} \left\{ 2 \langle x, \nabla U(x) \rangle - \|\nabla U(x)\| \frac{\gamma \|\nabla U(x)\|}{1 + \gamma \|\nabla U(x)\|} \right\} \\ &\geq \frac{\|\nabla U(x)\|}{1 + \gamma \|\nabla U(x)\|} \frac{2\kappa \|x\| - 1}{2\|x\|}. \end{aligned}$$

By **H2-i**), there exist $M_2, C > 0$ such that for all $x \in \mathbb{R}^d$, $\|x\| \geq M_2$, $\|\nabla U(x)\| \geq C$. Using that $s \mapsto s(1+\gamma s)^{-1}$ is non-decreasing for $s \geq 0$, we get for all $x \in \mathbb{R}^d$, $\|x\| \geq \max(\kappa^{-1}, M_1, M_2)$, $A_\gamma(x) \geq (\kappa C)/\{2(1+\gamma C)\}$.

Define for all $x \in \mathbb{R}^d$, $x \neq 0$,

$$B_\gamma(x) = \left\langle \frac{x}{\|x\|}, H_{\gamma,c}(x) \right\rangle - \frac{\gamma}{2\|x\|} \|H_{\gamma,c}(x)\|^2 .$$

We have for all $x \in \mathbb{R}^d$, $\gamma \|H_{\gamma,c}(x)\| \leq \sqrt{d}$ and for all $x \in \mathbb{R}^d$, $\|x\| \geq M_1$,

$$\left\langle x, \left(\frac{\partial_i U(x)}{1 + \gamma |\partial_i U(x)|} \right)_{i \in \{1, \dots, d\}} \right\rangle \geq \frac{\kappa \|x\| \|\nabla U(x)\|}{1 + \gamma \max_{i \in \{1, \dots, d\}} |\partial_i U(x)|}$$

and

$$\left\| \left(\frac{\partial_i U(x)}{1 + \gamma |\partial_i U(x)|} \right)_{i \in \{1, \dots, d\}} \right\| \leq \frac{\|\nabla U(x)\|}{1 + \gamma \max_{i \in \{1, \dots, d\}} |\partial_i U(x)|} .$$

Combining these inequalities, we get for all $x \in \mathbb{R}^d$, $\|x\| \geq \max(\kappa^{-1}\sqrt{d}, M_1)$,

$$B_\gamma(x) \geq \frac{\|\nabla U(x)\|}{1 + \gamma \max_{i \in \{1, \dots, d\}} |\partial_i U(x)|} \frac{1}{2\|x\|} \left\{ 2\kappa \|x\| - \sqrt{d} \right\} \geq \frac{\|\nabla U(x)\|}{1 + \gamma \|\nabla U(x)\|} \frac{\kappa}{2} ,$$

and for all $x \in \mathbb{R}^d$, $\|x\| \geq \max(\kappa^{-1}\sqrt{d}, M_1, M_2)$, we get $B_\gamma(x) \geq (\kappa C)/\{2(1+\gamma C)\}$.

4.3. Proof of Proposition 3

Let $\gamma, a \in \mathbb{R}_+^*$. Note that the function $x \mapsto (1 + \|x\|^2)^{1/2}$ is Lipschitz continuous with Lipschitz constant equal to 1. By the log-Sobolev inequality [34, Proposition 5.5.1], and the Cauchy-Schwarz inequality, we have for all $x \in \mathbb{R}^d$ and $a > 0$

$$\begin{aligned} R_\gamma V_a(x) &\leq e^{a^2\gamma} \exp \left\{ a \int_{\mathbb{R}^d} (1 + \|y\|^2)^{1/2} R_\gamma(x, dy) \right\} \\ &\leq e^{a^2\gamma} \exp \left\{ a \left(1 + \|x - \gamma G_\gamma(x)\|^2 + 2\gamma d \right)^{1/2} \right\} . \end{aligned} \quad (25)$$

We now bound the term inside the exponential in the right hand side. For all $x \in \mathbb{R}^d$,

$$\|x - \gamma G_\gamma(x)\|^2 = \|x\|^2 - 2\gamma \left(\langle G_\gamma(x), x \rangle - (\gamma/2) \|G_\gamma(x)\|^2 \right) . \quad (26)$$

By **A2**, there exist $M_1, \kappa \in \mathbb{R}_+^*$ such that for all $x \in \mathbb{R}^d$, $\|x\| \geq M_1$, $\langle x, G_\gamma(x) \rangle - (\gamma/2) \|G_\gamma(x)\|^2 \geq \kappa \|x\|$. Denote by $M = \max(M_1, 2d\kappa^{-1})$. For all $x \in \mathbb{R}^d$, $\|x\| \geq M$, we have

$$\|x - \gamma G_\gamma(x)\|^2 + 2\gamma d \leq \|x\|^2 - \gamma \kappa \|x\| .$$

Using for all $t \in [0, 1]$, $(1-t)^{1/2} \leq 1-t/2$ and $s \mapsto s/(1+s^2)^{1/2}$ is non-decreasing for $s \geq 0$, we have for all $x \in \mathbb{R}^d$, $\|x\| \geq M$,

$$\begin{aligned} \left(1 + \|x - \gamma G_\gamma(x)\|^2 + 2\gamma d\right)^{1/2} &\leq \left(1 + \|x\|^2\right)^{1/2} \left(1 - \frac{\gamma \kappa \|x\|}{1 + \|x\|^2}\right)^{1/2} \\ &\leq \left(1 + \|x\|^2\right)^{1/2} - \frac{\gamma \kappa M}{2(1 + M^2)^{1/2}} . \end{aligned}$$

Plugging this result in (25) shows that for all $x \in \mathbb{R}^d$, $\|x\| \geq M$,

$$R_\gamma V_{\mathfrak{a}}(x) \leq e^{-\mathfrak{a}^2 \gamma} V_{\mathfrak{a}}(x) \quad \text{for} \quad \mathfrak{a} = \frac{\kappa M}{4(1 + M^2)^{1/2}} . \quad (27)$$

By (10), we have

$$\max_{\|x\| \leq M} \|G_\gamma(x)\| \leq 2L \left\{1 + \|M\|^{\ell+1}\right\} + \gamma C_\alpha (1 + \|M\|^\alpha) .$$

Combining it with (25), (26), $s \mapsto s/(1+s^2)^{1/2}$ is non-decreasing for $s \geq 0$ and $(1+t_1+t_2)^{1/2} \leq (1+t_1)^{1/2} + t_2/2$ for $t_1 = \|x\|^2$, $t_2 = \gamma^2 \|G_\gamma(x)\|^2 + 2\gamma \|x\| \|G_\gamma(x)\| + 2\gamma d$, we have for all $x \in \mathbb{R}^d$, $\|x\| \leq M$,

$$R_\gamma V_{\mathfrak{a}}(x) \leq e^{\gamma c} V_{\mathfrak{a}}(x) , \quad (28)$$

where

$$\begin{aligned} c = \mathfrak{a}^2 + \mathfrak{a} \left[M \left\{ 2L \left\{ 1 + \|M\|^{\ell+1} \right\} + \gamma C_\alpha (1 + \|M\|^\alpha) \right\} \right. \\ \left. + \frac{\gamma}{2} \left\{ 2L \left\{ 1 + \|M\|^{\ell+1} \right\} + \gamma C_\alpha (1 + \|M\|^\alpha) \right\}^2 + d \right] . \end{aligned}$$

Then, using that for all $t \geq 0$, $1 - e^{-t} \leq t$, we get for all $x \in \mathbb{R}^d$, $\|x\| \leq M$,

$$R_\gamma V_{\mathfrak{a}}(x) - e^{-\mathfrak{a}^2 \gamma} V_{\mathfrak{a}}(x) \leq e^{\gamma c} (1 - e^{-\gamma(\mathfrak{a}^2 + c)}) V_{\mathfrak{a}}(x) \leq \gamma e^{\gamma c} (\mathfrak{a}^2 + c) V_{\mathfrak{a}}(x) , \quad (29)$$

which combined with (27) gives (12) with $b = e^{\gamma c} (\mathfrak{a}^2 + c) e^{\kappa M/4}$. Finally, using Jensen's inequality and $(s+t)^\varsigma \leq s^\varsigma + t^\varsigma$ for $\varsigma \in (0, 1)$, $s, t \geq 0$ in (12), by [11, Section 3.1], for all $\gamma > 0$, R_γ has a unique invariant probability measure π_γ and R_γ is $V_{\mathfrak{a}}^\varsigma$ -geometrically ergodic w.r.t. π_γ .

4.4. Proof of Theorem 4

The proof is adapted from [6, Proposition 2] and [9, Theorem 10]. We first state a lemma.

Lemma 8. *Assume **H1**, **H2**, **A1** and **A2**. Let $\gamma_0 > 0$, $p \in \mathbb{N}^*$ and ν_0 be a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. There exists $C > 0$ such that for all $\gamma \in (0, \gamma_0]$*

$$\text{KL}(\nu_0 R_\gamma^p | \nu_0 P_{p\gamma}) \leq C \gamma^2 \int_{\mathbb{R}^d} \sum_{i=0}^{p-1} \left\{ \int_{\mathbb{R}^d} V_{\infty}(z) R_\gamma^i(y, dz) \right\} \nu_0(dy) .$$

Proof. Let $y \in \mathbb{R}^d$ and $\gamma > 0$. Denote by $(Y_t, \bar{Y}_t)_{t \geq 0}$ the unique strong solution of

$$\begin{cases} dY_t = -\nabla U(Y_t) dt + \sqrt{2} dB_t & , \quad Y_0 = y , \\ d\bar{Y}_t = -G_\gamma(\bar{Y}_{\lfloor t/\gamma \rfloor \gamma}) dt + \sqrt{2} dB_t & , \quad \bar{Y}_0 = y , \end{cases} \quad (30)$$

and by $(\mathcal{F}_t)_{t \geq 0}$ the filtration associated with $(B_t)_{t \geq 0}$. Denote by μ_p^y and $\bar{\mu}_p^y$ the marginal distributions on $\mathcal{C}([0, p\gamma], \mathbb{R}^d)$ of $(Y_t, \bar{Y}_t)_{t \geq 0}$. By (5), (10) and Propositions 1 and 3, we have

$$\begin{aligned} \mathbb{P} \left(\int_0^{p\gamma} \|\nabla U(Y_t)\|^2 + \|G_\gamma(Y_{\lfloor t/\gamma \rfloor \gamma})\|^2 dt < +\infty \right) &= 1 , \\ \mathbb{P} \left(\int_0^{p\gamma} \|\nabla U(\bar{Y}_t)\|^2 + \|G_\gamma(\bar{Y}_{\lfloor t/\gamma \rfloor \gamma})\|^2 dt < +\infty \right) &= 1 . \end{aligned}$$

By [35, Theorem 7.19], μ_p^y and $\bar{\mu}_p^y$ are equivalent and \mathbb{P} -almost surely,

$$\begin{aligned} \frac{d\mu_p^y}{d\bar{\mu}_p^y}((\bar{Y}_t)_{t \in [0, p\gamma]}) &= \exp \left(\frac{1}{2} \int_0^{p\gamma} \langle -\nabla U(\bar{Y}_s) + G_\gamma(\bar{Y}_{\lfloor s/\gamma \rfloor \gamma}), d\bar{Y}_s \rangle \right. \\ &\quad \left. - \frac{1}{4} \int_0^{p\gamma} \left\{ \|\nabla U(\bar{Y}_s)\|^2 - \|G_\gamma(\bar{Y}_{\lfloor s/\gamma \rfloor \gamma})\|^2 \right\} ds \right) . \end{aligned}$$

We get then

$$\begin{aligned} \text{KL}(\bar{\mu}_p^y | \mu_p^y) &= \mathbb{E} \left[-\log \left\{ \frac{d\mu_p^y}{d\bar{\mu}_p^y}((\bar{Y}_t)_{t \in [0, p\gamma]}) \right\} \right] \\ &= (1/4) \int_0^{p\gamma} \mathbb{E} \left[\|\nabla U(\bar{Y}_s) - G_\gamma(\bar{Y}_{\lfloor s/\gamma \rfloor \gamma})\|^2 \right] ds \\ &= (1/4) \sum_{i=0}^{p-1} \int_{i\gamma}^{(i+1)\gamma} \mathbb{E} \left[\|\nabla U(\bar{Y}_s) - G_\gamma(\bar{Y}_{i\gamma})\|^2 \right] ds . \end{aligned}$$

For $i \in \{0, \dots, p-1\}$ and $s \in [i\gamma, (i+1)\gamma)$, we have $\|\nabla U(\bar{Y}_s) - G_\gamma(\bar{Y}_{i\gamma})\|^2 \leq 2(A_1 + A_2)$ where

$$A_1 = \|\nabla U(\bar{Y}_s) - \nabla U(\bar{Y}_{i\gamma})\|^2, \quad A_2 = \|\nabla U(\bar{Y}_{i\gamma}) - G_\gamma(\bar{Y}_{i\gamma})\|^2.$$

By **A1**, $A_2 \leq \gamma^2 C_\alpha^2 (1 + \|\bar{Y}_{i\gamma}\|^\alpha)^2$ and by **H1**,

$$A_1 \leq L^2 \left(1 + \|\bar{Y}_s\|^\ell + \|\bar{Y}_{i\gamma}\|^\ell\right)^2 \|\bar{Y}_s - \bar{Y}_{i\gamma}\|^2. \quad (31)$$

On the other hand for $s \in [i\gamma, (i+1)\gamma)$,

$$\begin{aligned} \|\bar{Y}_s - \bar{Y}_{i\gamma}\|^2 &= (s - i\gamma)^2 \|G_\gamma(\bar{Y}_{i\gamma})\|^2 + 2\|B_s - B_{i\gamma}\|^2 \\ &\quad - 2^{3/2}(s - i\gamma) \langle B_s - B_{i\gamma}, G_\gamma(\bar{Y}_{i\gamma}) \rangle, \end{aligned} \quad (32)$$

$$\|\bar{Y}_s\| \leq \|\bar{Y}_{i\gamma}\| + \gamma \|G_\gamma(\bar{Y}_{i\gamma})\| + \sqrt{2} \|B_s - B_{i\gamma}\|. \quad (33)$$

Define $\mathbf{P}_{\gamma,1} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ for all $t \in \mathbb{R}_+$ by

$$\begin{aligned} \mathbf{P}_{\gamma,1}(t) &= (2\pi)^{-d/2} L^2 \int_{\mathbb{R}^d} \left[2\|z\|^2 + \gamma \{2L(1 + t^{\ell+1}) + \gamma C_\alpha(1 + t^\alpha)\}^2\right] \\ &\quad \times \left[1 + t^\ell + \left\{t + \gamma(2L(1 + t^{\ell+1}) + \gamma C_\alpha(1 + t^\alpha)) + \sqrt{2\gamma}\|z\|\right\}^\ell\right]^2 e^{-\|z\|^2/2} dz. \end{aligned} \quad (34)$$

By (10), (31), (32) and (33), we have for $i \in \{0, \dots, p-1\}$

$$\int_{i\gamma}^{(i+1)\gamma} \mathbb{E}^{\mathcal{F}_{i\gamma}} [A_1] ds \leq (\gamma^2/2) \mathbf{P}_{\gamma,1}(\|\bar{Y}_{i\gamma}\|)$$

and we get

$$\int_{i\gamma}^{(i+1)\gamma} \mathbb{E}^{\mathcal{F}_{i\gamma}} \left[\|\nabla U(\bar{Y}_s) - G_\gamma(\bar{Y}_{i\gamma})\|^2\right] ds \leq \gamma^2 \left\{ \mathbf{P}_{\gamma,1}(\|\bar{Y}_{i\gamma}\|) + 2\gamma \mathbf{P}_2(\|\bar{Y}_{i\gamma}\|) \right\},$$

where $\mathbf{P}_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined for all $t \in \mathbb{R}_+$ by

$$\mathbf{P}_2(t) = C_\alpha^2 (1 + t^\alpha)^2. \quad (35)$$

By [36, Theorem 4.1, Chapter 2], we obtain

$$\text{KL}(\delta_y R_\gamma^p | \delta_y P_{p\gamma}) \leq \text{KL}(\bar{\mu}_p^y | \mu_p^y) \leq (\gamma^2/4) \sum_{i=0}^{p-1} \mathbb{E} \left[\mathbf{P}_{\gamma,1}(\|\bar{Y}_{i\gamma}\|) + 2\gamma \mathbf{P}_2(\|\bar{Y}_{i\gamma}\|) \right].$$

By (34) and (35), there exists $C > 0$ such that for all $\gamma \in (0, \gamma_0]$ and $x \in \mathbb{R}^d$, $P_{\gamma,1}(\|x\|) + 2\gamma P_2(\|x\|) \leq 4CV_{\mathfrak{a}}(x)$. Combining it with the chain rule for the Kullback-Leibler divergence concludes the proof. \square

Proof of Theorem 4. Let $\gamma \in (0, \gamma_0]$. By Proposition 1, we have for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$,

$$\|\delta_x R_\gamma^n - \pi\|_{V_{\mathfrak{a}}^{1/2}} \leq C_{\mathfrak{a}/2} \rho_{\mathfrak{a}/2}^{n\gamma} V_{\mathfrak{a}}^{1/2}(x) + \|\delta_x R_\gamma^n - \delta_x P_{n\gamma}\|_{V_{\mathfrak{a}}^{1/2}}.$$

Denote by $k_\gamma = \lceil \gamma^{-1} \rceil$ and by q_γ, r_γ the quotient and the remainder of the Euclidian division of n by k_γ . We have $\|\delta_x R_\gamma^n - \delta_x P_{n\gamma}\|_{V_{\mathfrak{a}}^{1/2}} \leq A + B$ where

$$\begin{aligned} A &= \|\delta_x R_\gamma^{q_\gamma k_\gamma} P_{r_\gamma \gamma} - \delta_x R_\gamma^n\|_{V_{\mathfrak{a}}^{1/2}} \\ B &= \sum_{i=1}^{q_\gamma} \left\| \delta_x R_\gamma^{(i-1)k_\gamma} P_{(n-(i-1)k_\gamma)\gamma} - \delta_x R_\gamma^{ik_\gamma} P_{(n-ik_\gamma)\gamma} \right\|_{V_{\mathfrak{a}}^{1/2}} \\ &\leq \sum_{i=1}^{q_\gamma} C_{\mathfrak{a}/2} \rho_{\mathfrak{a}/2}^{(n-ik_\gamma)\gamma} \left\| \delta_x R_\gamma^{(i-1)k_\gamma} P_{k_\gamma \gamma} - \delta_x R_\gamma^{ik_\gamma} \right\|_{V_{\mathfrak{a}}^{1/2}}. \end{aligned} \quad (36)$$

For $i \in \{1, \dots, q_\gamma\}$ we have by [9, Lemma 24],

$$\begin{aligned} \left\| \delta_x R_\gamma^{(i-1)k_\gamma} P_{k_\gamma \gamma} - \delta_x R_\gamma^{ik_\gamma} \right\|_{V_{\mathfrak{a}}^{1/2}}^2 &\leq 2 \left\{ \delta_x R_\gamma^{(i-1)k_\gamma} P_{k_\gamma \gamma}(V_{\mathfrak{a}}) + \delta_x R_\gamma^{ik_\gamma}(V_{\mathfrak{a}}) \right\} \\ &\quad \times \text{KL}(\delta_x R_\gamma^{ik_\gamma} | \delta_x R_\gamma^{(i-1)k_\gamma} P_{k_\gamma \gamma}). \end{aligned} \quad (37)$$

By Proposition 3, Lemma 8 and $k_\gamma \leq 1 + \gamma^{-1}$, we have for all $i \in \{1, \dots, q_\gamma\}$

$$\begin{aligned} \text{KL}(\delta_x R_\gamma^{ik_\gamma} | \delta_x R_\gamma^{(i-1)k_\gamma} P_{k_\gamma \gamma}) &\leq C\gamma^2 \sum_{j=0}^{k_\gamma-1} \int_{\mathbb{R}^d} V_{\mathfrak{a}}(z) \delta_x R_\gamma^{(i-1)k_\gamma+j}(dz) \\ &\leq C\gamma^2 (1 + \gamma^{-1}) \left\{ e^{-\mathfrak{a}^2 \gamma k_\gamma (i-1)} V_{\mathfrak{a}}(x) + \frac{b}{\mathfrak{a}^2} e^{\mathfrak{a}^2 \gamma} \right\}, \end{aligned} \quad (38)$$

where C is the constant defined in Lemma 8. By Proposition 1, we have for $x \in \mathbb{R}^d$, $P_{k_\gamma \gamma} V_{\mathfrak{a}}(x) \leq V_{\mathfrak{a}}(x) + b_{\mathfrak{a}}$ and by Proposition 3, we get for all $i \in \{1, \dots, q_\gamma\}$

$$\delta_x R_\gamma^{(i-1)k_\gamma} P_{k_\gamma \gamma}(V_{\mathfrak{a}}) + \delta_x R_\gamma^{ik_\gamma}(V_{\mathfrak{a}}) \leq 2 \left\{ e^{-\mathfrak{a}^2 \gamma k_\gamma (i-1)} V_{\mathfrak{a}}(x) + \frac{b}{\mathfrak{a}^2} e^{\mathfrak{a}^2 \gamma} + b_{\mathfrak{a}} \right\}. \quad (39)$$

By (36), (37), (38) and (39), we obtain

$$B \leq 2C_{\mathfrak{a}/2} C^{1/2} \gamma (1 + \gamma^{-1})^{1/2} \\ \times \sum_{i=1}^{q_\gamma} \rho_{\mathfrak{a}/2}^{(q_\gamma - i)\gamma k_\gamma} \left\{ e^{-(i-1)\gamma k_\gamma \mathfrak{a}^2} V_{\mathfrak{a}}(x) + \left(b_{\mathfrak{a}} + \frac{b}{\mathfrak{a}^2} e^{\mathfrak{a}^2 \gamma} \right) \right\}$$

and we get

$$B \left\{ 2C_{\mathfrak{a}/2} C^{1/2} \gamma (1 + \gamma^{-1})^{1/2} \right\}^{-1} \leq \left(b_{\mathfrak{a}} + \frac{b}{\mathfrak{a}^2} e^{\mathfrak{a}^2 \gamma} \right) \frac{1}{1 - \rho_{\mathfrak{a}/2}^{k_\gamma \gamma}} \\ + V_{\mathfrak{a}}(x) q_\gamma \max(\rho_{\mathfrak{a}/2}, e^{-\mathfrak{a}^2})^{(q_\gamma - 1)\gamma k_\gamma}.$$

Bounding A along the same lines and using $k_\gamma \gamma \geq 1$, we get (14). By Proposition 3 and taking the limit $n \rightarrow +\infty$, we obtain (15). \square

4.5. Proofs of Theorems 5 and 6

We first state preliminary technical lemmas on the diffusion $(Y_t)_{t \geq 0}$. The proofs are postponed to the supplementary document [32]. Define for all $p \in \mathbb{N}^*$ and $k \in \{0, \dots, p\}$,

$$a_{k,p} = m^{k-p} \prod_{i=k+1}^p \{i(d + 2(i-1))(i-k)^{-1}\}. \quad (40)$$

Lemma 9. *Assume **H3**. Let $p \in \mathbb{N}^*$, $x \in \mathbb{R}^d$ and $(Y_t)_{t \geq 0}$ be the solution of (1) started at x . For all $t \geq 0$,*

$$\mathbb{E} \left[\|Y_t\|^{2p} \right] \leq a_{0,p} (1 - e^{-2pmt}) + \sum_{k=1}^p a_{k,p} e^{-2kmt} \|x\|^{2k},$$

where for $k \in \{0, \dots, p\}$, $a_{k,p}$ is given in (40).

Proof. The proof is postponed to the supplementary document [32]. \square

Lemma 10. *Assume **H3** and let $p \in \mathbb{N}^*$. We have $\int_{\mathbb{R}^d} \|y\|^{2p} \pi(dy) \leq a_{0,p}$.*

Proof. The proof is postponed to the supplementary document [32]. \square

Let $\gamma > 0$ and under **H1** set

$$N = \lceil (\ell + 1)/2 \rceil . \quad (41)$$

Consider $P_{\gamma,3} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined for all $s \in \mathbb{R}_+$ by

$$P_{\gamma,3}(s) = 2d + 8L^2(1 + s^{\ell+1}) \left\{ \frac{\gamma}{2} \left(2 + \sum_{k=1}^N a_{k,N} s^{2k} \right) + Nma_{0,N} \frac{\gamma^2}{3} \right\} . \quad (42)$$

Lemma 11. *Assume **H1** and **H3**. Let $x \in \mathbb{R}^d$, $\gamma > 0$ and $(Y_t)_{t \geq 0}$ be the solution of (1) started at x . For all $t \in [0, \gamma]$, we have $\mathbb{E} \left[\|Y_t - x\|^2 \right] \leq tP_{\gamma,3}(\|x\|)$, where $P_{\gamma,3}$ is defined in (42).*

Proof. The proof is postponed to the supplementary document [32]. \square

For $p \in \mathbb{N}$ and $\gamma > 0$, define $Q_{\gamma,p} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ for all $s \in \mathbb{R}_+$ by,

$$\begin{aligned} Q_{\gamma,p}(s) = & \left\{ \prod_{i=1}^p 2i(d + 3i - 2) \right\} \left[2d \frac{\gamma^p}{(p+1)!} + 8L^2(1 + s^{\ell+1}) \right. \\ & \times \left. \left\{ \left(2 + \sum_{k=1}^N a_{k,N} s^{2k} \right) \frac{\gamma^{p+1}}{(p+2)!} + 2Nma_{0,N} \frac{\gamma^{p+2}}{(p+3)!} \right\} \right] \\ & + 2 \sum_{k=1}^p \left\{ \prod_{i=k+1}^p 2i(d + 3i - 2) \right\} \left\{ d + 4 + \frac{L^2(1 + s^{\ell+1})^2}{m(k+1)} \right\} \\ & \times \left\{ \left(\sum_{i=1}^k a_{i,k} s^{2i} \right) \frac{\gamma^{p-k}}{(p+1-k)!} + 2kma_{0,k} \frac{\gamma^{p+1-k}}{(p+2-k)!} \right\} \quad (43) \end{aligned}$$

where N is defined in (41).

Lemma 12. *Assume **H1** and **H3**. Let $p \in \mathbb{N}$, $\gamma > 0$, $x \in \mathbb{R}^d$ and $(Y_t)_{t \geq 0}$ be the solution of (1) started at x . For all $t \in [0, \gamma]$, we have $\mathbb{E} \left[\|Y_t\|^{2p} \|Y_t - x\|^2 \right] \leq tQ_{\gamma,p}(\|x\|)$, where $Q_{\gamma,p}$ is defined in (43).*

Proof. The proof is postponed to the supplementary document [32]. \square

Lemma 13. *Assume **H4**.*

a) *For all $x \in \mathbb{R}^d$, $\|\nabla^2 U(x)\| \leq C_H \{1 + \|x\|^{\nu+\beta}\}$ where $C_H = \max(2L_H, \|\nabla^2 U(0)\|)$.*

b) For all $x, y \in \mathbb{R}^d$,

$$\|\nabla U(x) - \nabla U(y) - \nabla^2 U(y)(x - y)\| \leq \frac{2L_H}{1 + \beta} \{1 + \|x\|^\nu + \|y\|^\nu\} \|x - y\|^{1+\beta} .$$

Proof. a) By **H4**, we get for all $x \in \mathbb{R}^d$

$$\begin{aligned} \|\nabla^2 U(x)\| &\leq \|\nabla^2 U(x) - \nabla^2 U(0)\| + \|\nabla^2 U(0)\| \\ &\leq L_H \{1 + \|x\|^\nu\} \|x\|^\beta + \|\nabla^2 U(0)\| . \end{aligned}$$

The proof then follows from the upper bound for all $x \in \mathbb{R}^d$, $\|x\|^\beta \leq 1 + \|x\|^{\nu+\beta}$.

b) Let $x, y \in \mathbb{R}^d$. By **H4**,

$$\begin{aligned} \|\nabla U(x) - \nabla U(y) - \nabla^2 U(y)(x - y)\| &\leq \int_0^1 \|\nabla^2 U(tx + (1-t)y) - \nabla^2 U(y)\| dt \|x - y\| \\ &\leq L_H \int_0^1 \{1 + \|y\|^\nu + \|tx + (1-t)y\|^\nu\} \|t(x - y)\|^\beta dt \|x - y\| , \end{aligned}$$

and the proof follows from $\|tx + (1-t)y\|^\nu \leq \|x\|^\nu + \|y\|^\nu$.

□

For all $n \in \mathbb{N}$, we now bound the Wasserstein distance W_2 between π and the distribution of the n^{th} iterate of X_n defined by (3). The strategy consists given two initial conditions (x, y) , in coupling X_n and $Y_{\gamma n}$ solution of (1) at time γn , using the same Brownian motion. Similarly to (30), for $\gamma > 0$, consider the unique strong solution $(Y_t, \bar{Y}_t)_{t \geq 0}$ of

$$\begin{cases} dY_t = -\nabla U(Y_t)dt + \sqrt{2}dB_t & , \quad Y_0 = y , \\ d\bar{Y}_t = -G_\gamma(\bar{Y}_{\lfloor t/\gamma \rfloor \gamma}) dt + \sqrt{2}dB_t & , \quad \bar{Y}_0 = x , \end{cases} \quad (44)$$

where $(B_t)_{t \geq 0}$ is a d -dimensional Brownian motion. Note that for $n \in \mathbb{N}$, $\bar{Y}_{n\gamma} = X_n$ and let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration associated with $(B_t)_{t \geq 0}$.

Lemma 14. *Assume **A1**, **A2**, **H1** and **H3**. Let $\gamma_0 > 0$. Define $(Y_t)_{t \geq 0}$, $(\bar{Y}_t)_{t \geq 0}$ by (44). Then there exists $C > 0$ such that for all $n \in \mathbb{N}$ and $\gamma \in (0, \gamma_0]$, almost surely,*

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\|Y_{(n+1)\gamma} - \bar{Y}_{(n+1)\gamma}\|^2 \right] \leq e^{-m\gamma} \|Y_{n\gamma} - \bar{Y}_{n\gamma}\|^2 + C\gamma^2 V_{\mathfrak{a}}(\bar{Y}_{n\gamma}) .$$

Proof. Using the Markov property, we only need to show the result for $n = 0$. Define for $t \in [0, \gamma)$, $\Theta_t = Y_t - \bar{Y}_t$. By Itô's formula, we have for all $t \in [0, \gamma)$,

$$\|\Theta_t\|^2 = \|y - x\|^2 - 2 \int_0^t \langle \Theta_s, \nabla U(Y_s) - G_\gamma(x) \rangle ds .$$

By (5) and Lemma 9, the family of random variables $(\langle \Theta_s, \nabla U(Y_s) - G_\gamma(x) \rangle)_{s \in [0, \gamma)}$ is uniformly integrable. Pathwise continuity implies then for $s \in [0, \gamma)$ the continuity of $s \mapsto \mathbb{E}[\langle \Theta_s, \nabla U(Y_s) - G_\gamma(x) \rangle]$. Taking the expectation and deriving, we have for $t \in [0, \gamma)$,

$$\begin{aligned} \frac{d}{dt} \mathbb{E} [\|\Theta_t\|^2] &= -2\mathbb{E} [\langle \Theta_t, \nabla U(Y_t) - G_\gamma(x) \rangle] \\ &= -2\mathbb{E} [\langle \Theta_t, \nabla U(Y_t) - \nabla U(\bar{Y}_t) \rangle] - 2A_1 - 2A_2 \\ &\leq -2m\mathbb{E} [\|\Theta_t\|^2] - 2A_1 - 2A_2 , \end{aligned} \quad (45)$$

where

$$A_1 = \mathbb{E} [\langle \Theta_t, \nabla U(\bar{Y}_t) - \nabla U(x) \rangle] , \quad A_2 = \mathbb{E} [\langle \Theta_t, \nabla U(x) - G_\gamma(x) \rangle] . \quad (46)$$

Using that $|\langle a, b \rangle| \leq (m/4)\|a\|^2 + m^{-1}\|b\|^2$ for all $a, b \in \mathbb{R}^d$,

$$|A_1| \leq (m/4)\mathbb{E} [\|\Theta_t\|^2] + m^{-1}\mathbb{E} [\|\nabla U(\bar{Y}_t) - \nabla U(x)\|^2] .$$

Similarly to the proof of Lemma 8, we have $\mathbb{E} [\|\nabla U(\bar{Y}_t) - \nabla U(x)\|^2] \leq t\mathbf{P}_{\gamma,1}(\|x\|)$ where $\mathbf{P}_{\gamma,1}$ is defined in (34). For A_2 , we have

$$|A_2| \leq (m/4)\mathbb{E} [\|\Theta_t\|^2] + m^{-1}\|\nabla U(x) - \nabla G_\gamma(x)\|^2 \quad (47)$$

and $\|\nabla U(x) - G_\gamma(x)\|^2 \leq \gamma^2\mathbf{P}_2(\|x\|)$ where \mathbf{P}_2 is defined in (35). We get for $t \in [0, \gamma)$,

$$\frac{d}{dt} \mathbb{E} [\|\Theta_t\|^2] \leq -m\mathbb{E} [\|\Theta_t\|^2] + 2m^{-1} \{t\mathbf{P}_{\gamma,1}(\|x\|) + \gamma^2\mathbf{P}_2(\|x\|)\} .$$

Using Grönwall's lemma and $1 - e^{-s} \leq s$ for all $s \geq 0$, we obtain

$$\mathbb{E} [\|Y_\gamma - \bar{Y}_\gamma\|^2] \leq e^{-m\gamma} \|y - x\|^2 + m^{-1}\gamma^2 \{ \mathbf{P}_{\gamma,1}(\|x\|) + 2\gamma\mathbf{P}_2(\|x\|) \} .$$

Finally, by (34) and (35), there exists $C > 0$ such that for all $x \in \mathbb{R}^d$, $\mathbf{P}_{\gamma,1}(\|x\|) + 2\gamma\mathbf{P}_2(\|x\|) \leq CmV_{\mathfrak{a}}(x)$. \square

Lemma 15. *Assume **A1**, **A2**, **H3** and **H4**. Let $\gamma_0 > 0$. Define $(Y_t)_{t \geq 0}$, $(\bar{Y}_t)_{t \geq 0}$ by (44). Then there exists $C > 0$ such that for all $n \in \mathbb{N}$ and $\gamma \in (0, \gamma_0]$, almost surely,*

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\left\| Y_{(n+1)\gamma} - \bar{Y}_{(n+1)\gamma} \right\|^2 \right] &\leq e^{-m\gamma} \left\| Y_{n\gamma} - \bar{Y}_{n\gamma} \right\|^2 \\ &\quad + C\gamma^{2+\beta} V_{\mathfrak{a}}(\bar{Y}_{n\gamma}) + C\gamma^3 V_{\mathfrak{a}}(\bar{Y}_{n\gamma}) . \end{aligned}$$

Remark 16. *The calculations in the proof show that the dependence w.r.t. $\bar{Y}_{n\gamma}$ and $Y_{n\gamma}$ is in fact polynomial but their exact expressions are very involved. For the sake of simplicity, we bound these polynomials by $V_{\mathfrak{a}}$. The same remark applies equally to Lemma 14.*

Proof. Note first that by Lemma 13-a), **H4** implies **H1** with $L = C_H$ and $\ell = \nu + \beta$. By the Markov property, we only need to show the result for $n = 0$. The proof is a refinement of Lemma 14 and we use the same notations. We have to improve the bound on A_1 defined in (46). We decompose $A_1 = A_{11} + A_{12}$ where

$$\begin{aligned} A_{11} &= \mathbb{E} \left[\langle \Theta_t, \nabla U(\bar{Y}_t) - \nabla U(x) - \nabla^2 U(x)(\bar{Y}_t - x) \rangle \right] , \\ A_{12} &= \mathbb{E} \left[\langle \Theta_t, \nabla^2 U(x)(\bar{Y}_t - x) \rangle \right] . \end{aligned}$$

Using $|\langle a, b \rangle| \leq (m/6) \|a\|^2 + \{3/(2m)\} \|b\|^2$ for all $a, b \in \mathbb{R}^d$,

$$|A_{11}| \leq \frac{m}{6} \mathbb{E} \left[\|\Theta_t\|^2 \right] + \frac{3}{2m} \mathbb{E} \left[\left\| \nabla U(\bar{Y}_t) - \nabla U(x) - \nabla^2 U(x)(\bar{Y}_t - x) \right\|^2 \right] . \quad (48)$$

By Lemma 13-b),

$$\begin{aligned} &\left\| \nabla U(\bar{Y}_t) - \nabla U(x) - \nabla^2 U(x)(\bar{Y}_t - x) \right\|^2 \\ &\quad \leq \frac{4L_H^2}{(1+\beta)^2} \left(1 + \|x\|^\nu + \|\bar{Y}_t\|^\nu \right)^2 \|\bar{Y}_t - x\|^{2(1+\beta)} . \end{aligned}$$

Following the proof of Lemma 8, using (32) and (33), we have

$$\mathbb{E} \left[\left\| \nabla U(\bar{Y}_t) - \nabla U(x) - \nabla^2 U(x)(\bar{Y}_t - x) \right\|^2 \right] \leq t^{1+\beta} \mathbf{P}_{\gamma,4}(\|x\|) . \quad (49)$$

where $\mathbf{P}_{\gamma,4} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined for all $s \in \mathbb{R}_+$ by,

$$\begin{aligned} \mathbf{P}_{\gamma,4}(s) &= \frac{4L_H^2}{(1+\beta)^2} \int_{\mathbb{R}^d} \left[\sqrt{2} \|z\| + \sqrt{\gamma} \{2L(1+s^{\ell+1}) + \gamma C_\alpha(1+s^\alpha)\} \right]^{2(1+\beta)} \\ &\times \left[1 + s^\nu + \left\{ s + \gamma (2L(1+s^{\ell+1}) + \gamma C_\alpha(1+s^\alpha)) + \sqrt{2\gamma} \|z\| \right\}^\nu \right]^2 \frac{e^{-\|z\|^2/2}}{(2\pi)^{d/2}} dz . \end{aligned} \quad (50)$$

We decompose A_{12} in $A_{12} = A_{121} + A_{122}$ where

$$A_{121} = \mathbb{E} [\langle \Theta_t, -t \nabla^2 U(x) G_\gamma(x) \rangle] , \quad A_{122} = \sqrt{2} \mathbb{E} [\langle \Theta_t, \nabla^2 U(x) B_t \rangle] .$$

Define $\mathbf{P}_{\gamma,5} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ for $s \in \mathbb{R}_+$ by,

$$\mathbf{P}_{\gamma,5}(s) = C_H^2 (1 + s^{\nu+\beta})^2 \{2L(1+s^{\ell+1}) + \gamma C_\alpha(1+s^\alpha)\}^2 . \quad (51)$$

By Lemma 13-a) and (10),

$$|A_{121}| \leq (m/6) \mathbb{E} [\|\Theta_t\|^2] + \{3/(2m)\} t^2 \mathbf{P}_{\gamma,5}(\|x\|) . \quad (52)$$

By Cauchy-Schwarz inequality and Lemma 13-a),

$$\begin{aligned} |A_{122}| &= \sqrt{2} \left| \mathbb{E} \left[\left\langle \int_0^t \{\nabla U(Y_s) - \nabla U(y)\} ds, \nabla^2 U(x) B_t \right\rangle \right] \right| \\ &\leq \sqrt{2t} C_H (1 + \|x\|^{\nu+\beta}) \mathbb{E} \left[\left\| \int_0^t \{\nabla U(Y_s) - \nabla U(y)\} ds \right\|^2 \right]^{1/2} . \end{aligned} \quad (53)$$

By **H1**, Cauchy-Schwarz inequality and using $(1 + \|y\|^\ell + \|Y_s\|^\ell)^2 \leq 3(2 + \|y\|^{2\ell} + \|Y_s\|^{2\ell})$ for $s \in [0, \gamma)$, we have

$$\begin{aligned} \mathbb{E} \left[\left\| \int_0^t \{\nabla U(Y_s) - \nabla U(y)\} ds \right\|^2 \right] &\leq 3tL^2(2 + \|y\|^{2\ell}) \int_0^t \mathbb{E} [\|Y_s - y\|^2] ds \\ &\quad + 3tL^2 \int_0^t \mathbb{E} [\|Y_s\|^{2\ell} \|Y_s - y\|^2] ds . \end{aligned}$$

By Lemmas 11 and 12, we get

$$\mathbb{E} \left[\left\| \int_0^t \{\nabla U(Y_s) - \nabla U(y)\} ds \right\|^2 \right] \leq \frac{3t^3 L^2}{2} \left\{ (2 + \|y\|^{2\ell}) \mathbf{P}_{\gamma,3}(\|y\|) + \mathbf{Q}_{\gamma, \lceil \ell \rceil}(\|y\|) \right\} ,$$

where $\mathbf{P}_{\gamma,3}, \mathbf{Q}_{\gamma, \lceil \ell \rceil} \in \mathbf{C}_{\text{poly}}(\mathbb{R}_+, \mathbb{R}_+)$ are defined in (42) and (43). Plugging this result in (53), we obtain

$$|A_{122}| \leq t^2 \sqrt{3d} C_{HL} \left(1 + \|x\|^{\nu+\beta}\right) \left\{ \left(2 + \|y\|^{2\ell}\right) \mathbf{P}_{\gamma,3}(\|y\|) + \mathbf{Q}_{\gamma, \lceil \ell \rceil}(\|y\|) \right\}^{1/2}. \quad (54)$$

Combining (48), (49), (52) and (54), we get

$$\begin{aligned} |A_1| &\leq (m/3) \mathbb{E} \left[\|\Theta_t\|^2 \right] + \{3/(2m)\} \left\{ t^{1+\beta} \mathbf{P}_{\gamma,4}(\|x\|) + t^2 \mathbf{P}_{\gamma,5}(\|x\|) \right\} \\ &\quad + t^2 \sqrt{3d} C_{HL} \left(1 + \|x\|^{\nu+\beta}\right) \left\{ \left(2 + \|y\|^{2\ell}\right) \mathbf{P}_{\gamma,3}(\|y\|) + \mathbf{Q}_{\gamma, \lceil \ell \rceil}(\|y\|) \right\}^{1/2}, \end{aligned}$$

and by (47), $|A_2| \leq (m/6) \mathbb{E} \left[\|\Theta_t\|^2 \right] + \{3/(2m)\} \gamma^2 \mathbf{P}_2(\|x\|)$, where $\mathbf{P}_2 \in \mathbf{C}_{\text{poly}}(\mathbb{R}_+, \mathbb{R}_+)$ is defined in (35). Combining these inequalities in (45), we get

$$\begin{aligned} \frac{d}{dt} \mathbb{E} \left[\|\Theta_t\|^2 \right] &\leq -m \mathbb{E} \left[\|\Theta_t\|^2 \right] + 3m^{-1} \left\{ \gamma^2 \mathbf{P}_2(\|x\|) + t^{1+\beta} \mathbf{P}_{\gamma,4}(\|x\|) + t^2 \mathbf{P}_{\gamma,5}(\|x\|) \right\} \\ &\quad + 2t^2 \sqrt{3d} C_{HL} \left(1 + \|x\|^{\nu+\beta}\right) \left\{ \left(2 + \|y\|^{2\ell}\right) \mathbf{P}_{\gamma,3}(\|y\|) + \mathbf{Q}_{\gamma, \lceil \ell \rceil}(\|y\|) \right\}^{1/2}. \end{aligned}$$

Using Grönwall's lemma and $1 - e^{-s} \leq s$ for all $s \geq 0$, we obtain

$$\begin{aligned} \mathbb{E} \left[\|Y_\gamma - \bar{Y}_\gamma\|^2 \right] &\leq e^{-m\gamma} \|y - x\|^2 \\ &\quad + 3m^{-1} \left\{ \gamma^3 \mathbf{P}_2(\|x\|) + \frac{\gamma^{2+\beta}}{2+\beta} \mathbf{P}_{\gamma,4}(\|x\|) + \frac{\gamma^3}{3} \mathbf{P}_{\gamma,5}(\|x\|) \right\} \\ &\quad + 2\gamma^3 \sqrt{d/3} C_{HL} \left(1 + \|x\|^{\nu+\beta}\right) \left\{ \left(2 + \|y\|^{2\ell}\right) \mathbf{P}_{\gamma,3}(\|y\|) + \mathbf{Q}_{\gamma, \lceil \ell \rceil}(\|y\|) \right\}^{1/2}. \end{aligned}$$

Finally, by (35), (42), (50), (51) and (43), there exists $C > 0$ such that for all $x \in \mathbb{R}^d$ and $\gamma \in (0, \gamma_0]$,

$$\begin{aligned} 3m^{-1} \left\{ \gamma^3 \mathbf{P}_2(\|x\|) + \frac{\gamma^{2+\beta}}{2+\beta} \mathbf{P}_{\gamma,4}(\|x\|) + \frac{\gamma^3}{3} \mathbf{P}_{\gamma,5}(\|x\|) \right\} &\leq C \gamma^{2+\beta} V_{\mathfrak{a}}(x), \\ 2\sqrt{d/3} C_{HL} \left(1 + \|x\|^{\nu+\beta}\right) &\leq C^{1/2} V_{\mathfrak{a}}(x)^{1/2}, \\ \left(2 + \|x\|^{2\ell}\right) \mathbf{P}_{\gamma,3}(\|x\|) + \mathbf{Q}_{\gamma, \lceil \ell \rceil}(\|x\|) &\leq C V_{\mathfrak{a}}(x). \end{aligned}$$

□

Proof of Theorem 5. Let $\gamma \in (0, \gamma_0]$. Define $(Y_t)_{t \geq 0}$, $(\bar{Y}_t)_{t \geq 0}$ by (44) and $X_n =$

$\bar{Y}_{n\gamma}$ for $n \in \mathbb{N}$. By Lemma 14 and Proposition 3, we have for all $n \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E} \left[\|Y_{n\gamma} - X_n\|^2 \right] &\leq e^{-nm\gamma} \|y - x\|^2 + C\gamma^2 \sum_{k=0}^{n-1} e^{-m\gamma(n-1-k)} \mathbb{E} [V_{\mathfrak{a}}(X_k)] \\ &\leq e^{-nm\gamma} \|y - x\|^2 + \frac{C\gamma^2}{1 - e^{-m\gamma}} \frac{b}{\mathfrak{a}^2} e^{\mathfrak{a}^2\gamma} + C\gamma^2 V_{\mathfrak{a}}(x) \sum_{k=0}^{n-1} e^{-m\gamma(n-1-k)} e^{-\mathfrak{a}^2\gamma k} . \end{aligned} \quad (55)$$

Note that

$$\sum_{k=0}^{n-1} e^{-m\gamma(n-1-k)} e^{-\mathfrak{a}^2\gamma k} \leq \frac{n}{1 - \max(e^{-m}, e^{-\mathfrak{a}^2})\gamma}$$

and $1 - s^\gamma \geq -\gamma \log(s) e^{\gamma \log(s)}$ for $s \in (0, 1)$. In eq. (55), integrating y with respect to π , for all $n \in \mathbb{N}$, $(Y_{n\gamma}, X_n)$ is a coupling between π and $\delta_x R_\gamma^n$. By Lemma 10, we get (16). By Proposition 3 and [22, Corollary 6.11], we have for all $x \in \mathbb{R}^d$, $\lim_{n \rightarrow +\infty} W_2(\delta_x R_\gamma^n, \pi) = W_2(\pi_\gamma, \pi)$ and we obtain (17). \square

Proof of Theorem 6. Let $\gamma \in (0, \gamma_0]$. Define $(Y_t)_{t \geq 0}$, $(\bar{Y}_t)_{t \geq 0}$ by (44) and $X_n = \bar{Y}_{n\gamma}$ for $n \in \mathbb{N}$. By Lemma 15, we have for all $n \in \mathbb{N}$,

$$\mathbb{E} \left[\|Y_{n\gamma} - X_n\|^2 \right] \leq e^{-nm\gamma} \|y - x\|^2 + A_n + B_n ,$$

where

$$\begin{aligned} A_n &= C\gamma^{2+\beta} \sum_{k=0}^{n-1} e^{-m\gamma(n-1-k)} \mathbb{E} [V_{\mathfrak{a}}(X_k)] , \\ B_n &= C\gamma^3 \sum_{k=0}^{n-1} e^{-m\gamma(n-1-k)} \mathbb{E} [V_{\mathfrak{a}}(Y_{k\gamma})] . \end{aligned}$$

Analysis similar to the proof of Theorem 5 using Proposition 1 instead of Proposition 3 for B_n shows then the result. \square

4.6. Proof of Theorem 7

We first state a lemma on the existence and regularity of a solution of the Poisson equation (20) which is adapted from [29, Theorem 1].

Lemma 17. *Assume **H2** and **H5**. Let $f \in C^3(\mathbb{R}^d, \mathbb{R})$ be such that $\|D^i f\| \in C_{\text{poly}}(\mathbb{R}^d, \mathbb{R}_+)$ for $i \in \{0, \dots, 3\}$. Then, there exists a solution of the Poisson equation (20) $\phi \in C^4(\mathbb{R}^d, \mathbb{R})$, such that $\|D^i \phi\| \in C_{\text{poly}}(\mathbb{R}^d, \mathbb{R}_+)$ for $i \in \{0, \dots, 4\}$.*

Proof. The proof is postponed to the supplementary document [32]. \square

Proof of Theorem 7. The proof is adapted from [26, Section 5.1] Let $\gamma \in (0, \gamma_0]$. In this Section, C is a positive constant which can change from line to line but does not depend on γ . For $k \in \mathbb{N}$, denote by

$$\delta_{k+1} = X_{k+1} - X_k = -\gamma G_\gamma(X_k) + \sqrt{2\gamma} Z_{k+1} .$$

By **H2**, **H5** and Lemma 17, there exists a solution to the Poisson equation (20) $\phi \in C^4(\mathbb{R}^d, \mathbb{R})$, such that for all $x \in \mathbb{R}^d$ and $i \in \{0, \dots, 4\}$,

$$\mathcal{A}\phi(x) = -(f(x) - \pi(f)) \quad \text{and} \quad \|\mathbb{D}^i \phi\| \in C_{\text{poly}}(\mathbb{R}^d, \mathbb{R}_+) . \quad (56)$$

By Taylor's formula, we have for $k \in \mathbb{N}$,

$$\begin{aligned} \phi(X_{k+1}) &= \phi(X_k) + \mathbb{D}\phi(X_k)[\delta_{k+1}] + (1/2)\mathbb{D}^2\phi(X_k)[\delta_{k+1}, \delta_{k+1}] \\ &\quad + (1/6)\mathbb{D}^3\phi(X_k)[\delta_{k+1}, \delta_{k+1}, \delta_{k+1}] + r_k , \\ r_k &= (1/6) \int_0^1 (1-s)^3 \mathbb{D}^4\phi(X_k + s\delta_{k+1})[\delta_{k+1}, \delta_{k+1}, \delta_{k+1}, \delta_{k+1}] ds . \end{aligned}$$

Using the expression of δ_{k+1} and (6), we get

$$\begin{aligned} \phi(X_{k+1}) &= \phi(X_k) + \gamma \mathcal{A}\phi(X_k) + \sqrt{2\gamma} \mathbb{D}\phi(X_k)[Z_{k+1}] \\ &\quad + \gamma \{ \mathbb{D}^2\phi(X_k)[Z_{k+1}, Z_{k+1}] - \Delta\phi(X_k) \} + \gamma \mathbb{D}\phi(X_k)[\nabla U(X_k) - G_\gamma(X_k)] \\ &\quad + (\gamma^2/2) \mathbb{D}^2\phi(X_k)[G_\gamma(X_k), G_\gamma(X_k)] - \sqrt{2\gamma} \gamma^{3/2} \mathbb{D}^2\phi(X_k)[G_\gamma(X_k), Z_{k+1}] \\ &\quad + (1/6) \mathbb{D}^3\phi(X_k)[\delta_{k+1}, \delta_{k+1}, \delta_{k+1}] + r_k . \end{aligned}$$

Summing from $k = 0$ to $n - 1$ for $n \in \mathbb{N}^*$, dividing by $n\gamma$, we get

$$\frac{1}{n} \sum_{k=0}^{n-1} (f(X_k) - \pi(f)) = \frac{\phi(X_0) - \phi(X_n)}{n\gamma} + \frac{1}{n\gamma} \left(\sum_{i=0}^3 M_{i,n} + \sum_{i=0}^3 S_{i,n} \right) ,$$

where

$$M_{0,n} = ((\sqrt{2}\gamma^{3/2})/6) \sum_{k=0}^{n-1} \{2 D^3 \phi(X_k)[Z_{k+1}, Z_{k+1}, Z_{k+1}] \\ + 3\gamma D^3 \phi(X_k)[G_\gamma(X_k), G_\gamma(X_k), Z_{k+1}]\},$$

$$M_{1,n} = \gamma \sum_{k=0}^{n-1} (D^2 \phi(X_k)[Z_{k+1}, Z_{k+1}] - \Delta \phi(X_k)),$$

$$M_{2,n} = \sqrt{2}\gamma \sum_{k=0}^{n-1} D \phi(X_k)[Z_{k+1}],$$

$$M_{3,n} = -\sqrt{2}\gamma^{3/2} \sum_{k=0}^{n-1} D^2 \phi(X_k)[G_\gamma(X_k), Z_{k+1}],$$

and

$$S_{0,n} = -(\gamma^2/6) \sum_{k=0}^{n-1} \{6 D^3 \phi(X_k)[G_\gamma(X_k), Z_{k+1}, Z_{k+1}] \\ + \gamma D^3 \phi(X_k)[G_\gamma(X_k), G_\gamma(X_k), G_\gamma(X_k)]\},$$

$$S_{1,n} = \gamma \sum_{k=0}^{n-1} D \phi(X_k)[\nabla U(X_k) - G_\gamma(X_k)],$$

$$S_{2,n} = (\gamma^2/2) \sum_{k=0}^{n-1} D^2 \phi(X_k)[G_\gamma(X_k), G_\gamma(X_k)],$$

$$S_{3,n} = \sum_{k=0}^{n-1} r_k.$$

By **A1**, we calculate for $n \in \mathbb{N}^*$, $|S_{1,n}| \leq \gamma^2 C_\alpha \sum_{k=0}^{n-1} \|D \phi(X_k)\| (1 + \|X_k\|^\alpha)$.

By **H5**, (10) and (56), there exist $p, q \geq 1$ and $C_q > 0$ such that the summands of $(M_{i,n})_{n \in \mathbb{N}}$ and $(S_{i,n})_{n \in \mathbb{N}}$ for $i \in \{0, \dots, 3\}$ are dominated by $C_q (1 + \|X_k\|^q) (1 + \|Z_{k+1}\|^p)$ for $k \in \{0, \dots, n-1\}$. Therefore, by Proposition 1, for $i \in \{0, \dots, 3\}$, $(M_{i,n})_{n \in \mathbb{N}}$ are martingales and for $n \in \mathbb{N}^*$, $\mathbb{E}[S_{i,n}^2] \leq Cn^2\gamma^4$,

$$\mathbb{E}[M_{0,n}^2] \leq Cn\gamma^3, \quad \mathbb{E}[M_{1,n}^2] \leq Cn\gamma^2, \quad \mathbb{E}[M_{2,n}^2] \leq Cn\gamma, \quad \mathbb{E}[M_{3,n}^2] \leq Cn\gamma^3,$$

which yield the result. \square

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