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# On explicit order 1.5 approximations with varying coefficients: the case of super-linear diffusion coefficients

Sotirios Sabanis and Ying Zhang  
*School of Mathematics,*  
*The University of Edinburgh, Edinburgh EH9 3FD, U.K.*

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## Abstract

A conjecture appears in [5], in the form of a remark, where it is stated that it is possible to construct, in a specified way, any high order explicit numerical schemes to approximate the solutions of SDEs with superlinear coefficients. We answer this conjecture affirmatively for the case of order 1.5 approximations and show that the suggested methodology works. Moreover, we explore the case of having Hölder continuous derivatives for the diffusion coefficients.

*AMS subject classifications:* Primary 60H35; secondary 65C30.

## 1 Introduction

A new type of explicit order 1.5 scheme is constructed in this article in order to approximate SDEs with super-linear growing drift and diffusion coefficients. The techniques used in [9] and [5] are extended and the optimal  $\mathcal{L}^2$  rate of convergence of the proposed order 1.5 scheme is obtained. The main idea is to follow the approach of [7] by using an appropriate Ito-Taylor series (known also as Wagner-Platen) expansion and the taming technique introduced in [9] and also used in [5]. In this way, it is demonstrated that any high order (explicit) scheme can be constructed with optimal rate of convergence.

Due to recent research (see [4], [2], [9] and references therein), new explicit Euler-type schemes have been developed to approximate SDEs with superlinearly growing coefficients following the observation in [3] that the classical (explicit) Euler scheme cannot be used for such approximations. This has been extended to Milstein-type schemes (see [5], [1] and references therein). Such schemes are explicit and therefore more computationally efficient compared to the implicit methods. Theorem 1 below gives the optimal rate of convergence in  $\mathcal{L}^2$  which is obtained under certain conditions (also given below).

Finally, there are the notations used in this article. The Euclidean norm of a vector  $b \in \mathbb{R}^d$  and the Hilbert-Schmidt norm of a matrix  $\sigma \in \mathbb{R}^{d \times m}$  are denoted by  $|b|$  and  $|\sigma|$  respectively.  $\sigma^*$  is the transpose matrix of  $\sigma$ . The  $i$ -th element of  $b$  and  $(i, j)$ -th element of  $\sigma$  are denoted respectively by  $b^i$  and  $\sigma^{(i,j)}$ , for every  $i = 1, \dots, d$  and  $j = 1, \dots, m$ . In addition,  $[a]$  denotes the integer part of a positive real number  $a$ . The inner product of two vectors  $x, y \in \mathbb{R}^d$  is denoted by  $xy$ . Furthermore, denote by  $D$  an operator such that for a function  $g(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $Dg(\cdot)$  gives a  $d \times d$  matrix whose  $(i, j)$ -th entry is given by  $\frac{\partial g^i(\cdot)}{\partial x^j}$ , for all  $i, j = 1, \dots, d$ .  $D^2$  is an operator such that for a function  $f(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $D^2 f(\cdot)$  gives a  $d \times d$  matrix whose  $(u, l)$ -th

entry is given by  $\frac{\partial^2 f(\cdot)}{\partial x^u \partial x^l}$ , for all  $u, l = 1, \dots, d$ . For every  $j = 1, \dots, m$ , denote by

$$L^0 = \frac{\partial}{\partial t} + \sum_{u=1}^d b^{(u)} \frac{\partial}{\partial x^u} + \frac{1}{2} \sum_{u,l=1}^d \sum_{j_1=1}^m \sigma^{(u,j_1)} \sigma^{(l,j_1)} \frac{\partial^2}{\partial x^u \partial x^l},$$

$$L^j = \sum_{u=1}^d \sigma^{(u,j)} \frac{\partial}{\partial x^u},$$

Also, for every  $j, j_1 = 1, \dots, m$ ,

$$L^j L^{j_1} = \sum_{u,l=1}^d \sigma^{(u,j)} \frac{\partial}{\partial x^u} \sigma^{(l,j_1)} \frac{\partial}{\partial x^l} + \sum_{u,l=1}^d \sigma^{(u,j)} \sigma^{(l,j_1)} \frac{\partial^2}{\partial x^u \partial x^l}.$$

## 2 Main results

Let  $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{F}, \mathbb{P})$  be a complete filtered probability space satisfying the usual conditions, which means that the filtration is right continuous and  $\mathcal{F}_0$  contains all  $P$ -null sets. Denote by  $(w_t)_{t \in [0, T]}$  an  $m$ -dimensional Wiener process. Moreover, assume that  $b(x)$  and  $\sigma(x)$  are  $\mathcal{B}(\mathbb{R}^d)$  measurable functions that take values in  $\mathbb{R}^d$  and  $\mathbb{R}^{d \times m}$  respectively.  $b(x)$  and  $\sigma(x)$  are continuously differentiable in  $x \in \mathbb{R}^d$ . Now, consider a  $d$ -dimensional SDE, for a fixed  $T > 0$ ,

$$x_t = x_0 + \int_0^t b(x_s) ds + \int_0^t \sigma(x_s) dw_s, \quad (2.1)$$

almost surely for any  $t \in [0, T]$  and  $x_0 \in \mathbb{R}^d$ . The following are the assumptions for the SDE, and note that  $p_0, p_1 \geq 2$ ,  $\rho \geq 2$  (or  $\rho = 0$ ).

**A-1**  $E|x_0|^{p_0} < \infty$ .

**A-2** There exists a constant  $K > 0$ , such that for any  $x \in \mathbb{R}^d$

$$2xb(x) + (p_0 - 1)|\sigma(x)|^2 \leq K(1 + |x|^2)$$

**A-3** There exists a constant  $K > 0$ , such that for any  $x, \bar{x} \in \mathbb{R}^d$

$$2(x - \bar{x})(b(x) - b(\bar{x})) + (p_1 - 1)|\sigma(x) - \sigma(\bar{x})|^2 \leq K|x - \bar{x}|^2$$

**A-4** There exists a constant  $K > 0$ , such that for  $i = 1, \dots, d$ ,

$$|D^2 b^i(x) - D^2 b^i(\bar{x})| \leq K(1 + |x| + |\bar{x}|)^{\rho-2} |x - \bar{x}|,$$

for all  $x, \bar{x} \in \mathbb{R}^d$ .

**A-5** There exists a constant  $K > 0$ , such that for  $i = 1, \dots, d$ ,  $j = 1, \dots, m$ ,

$$|D^2 \sigma^{(i,j)}(x) - D^2 \sigma^{(i,j)}(\bar{x})| \leq K(1 + |x| + |\bar{x}|)^{\frac{\rho-4}{2}} |x - \bar{x}|^\beta,$$

for  $\beta \in (0, 1]$  and for all  $x, \bar{x} \in \mathbb{R}^d$ .

**Remark 1.** By **A-4**, for  $i, u, l = 1, \dots, d$ , and  $x, \bar{x} \in \mathbb{R}^d$ , there exists a constant  $K > 0$ , such that

$$\left| \frac{\partial^2 b^i(x)}{\partial x^u \partial x^l} \right| \leq K(1 + |x|)^{\rho-1}.$$

In addition,

$$|Db(x) - Db(\bar{x})| \leq K(1 + |x| + |\bar{x}|)^{\rho-1} |x - \bar{x}|.$$

Furthermore, for  $i, u = 1, \dots, d$ , and  $x, \bar{x} \in \mathbb{R}^d$ , there is a constant  $K > 0$  such that

$$\left| \frac{\partial b^i(x)}{\partial x^u} \right| \leq K(1 + |x|)^\rho,$$

$$|b(x) - b(\bar{x})| \leq K(1 + |x| + |\bar{x}|)^\rho |x - \bar{x}|,$$

which implies

$$|b(x)| \leq K(1 + |x|)^{\rho+1}.$$

Similarly, by **A-5**, for  $i, u, l = 1, \dots, d$ ,  $j = 1, \dots, m$  and  $x, \bar{x} \in \mathbb{R}^d$ , there exists  $K > 0$ , such that

$$\left| \frac{\partial^2 \sigma^{(i,j)}(x)}{\partial x^u \partial x^l} \right| \leq K(1 + |x|)^{\frac{\rho-4}{2} + \beta}.$$

Moreover,

$$|D\sigma^{(j)}(x) - D\sigma^{(j)}(\bar{x})| \leq K(1 + |x| + |\bar{x}|)^{\frac{\rho-4}{2} + \beta} |x - \bar{x}|.$$

Furthermore, for  $i, u = 1, \dots, d$ ,  $j = 1, \dots, m$  and  $x, \bar{x} \in \mathbb{R}^d$ , there exists  $K > 0$ , such that

$$\left| \frac{\partial \sigma^{(i,j)}(x)}{\partial x^u} \right| \leq K(1 + |x|)^{\frac{\rho-2}{2} + \beta},$$

$$|\sigma(x) - \sigma(\bar{x})| \leq K(1 + |x| + |\bar{x}|)^{\frac{\rho-2}{2} + \beta} |x - \bar{x}|,$$

which implies

$$|\sigma(x)| \leq K(1 + |x|)^{\frac{\rho}{2} + \beta}.$$

Then, there exists a constant  $K > 0$ , such that

$$\begin{aligned} |L^0 b(x)| &\leq K(1 + |x|)^{2\rho+1}, \\ |L^j b(x)| &\leq K(1 + |x|)^{\frac{3}{2}\rho+1} \\ |L^0 \sigma(x)| &\leq K(1 + |x|)^{\frac{3}{2}\rho+1}, \\ |L^j \sigma(x)| &\leq K(1 + |x|)^{\rho+1}, \\ |L^j L^{j_1} \sigma(x)| &\leq K(1 + |x|)^{\frac{3}{2}\rho+1}. \end{aligned}$$

For every  $n \in \mathbb{N}$ , define

$$\begin{aligned} b^n(x) &= \frac{b(x)}{1 + n^{-\theta} |x|^{2\rho\theta}}, \\ \sigma^n(x) &= \frac{\sigma(x)}{1 + n^{-\theta} |x|^{2\rho\theta}}, \\ L^{n,0} b(x) &:= \frac{L^0 b(x)}{1 + n^{-\theta} |x|^{2\rho\theta}}, \end{aligned}$$

$$\begin{aligned}
L^{n,j}b(x) &:= \frac{L^j b(x)}{1 + n^{-\theta}|x|^{2\rho\theta}}, \\
L^{n,0}\sigma(x) &:= \frac{L^0 \sigma(x)}{1 + n^{-\theta}|x|^{2\rho\theta}}, \\
L^{n,j}\sigma(x) &:= \frac{L^j \sigma(x)}{1 + n^{-\theta}|x|^{2\rho\theta}}, \\
L^{n,j}L^{j_1}\sigma(x) &:= \frac{L^j L^{j_1} \sigma(x)}{1 + n^{-\theta}|x|^{2\rho\theta}},
\end{aligned}$$

where  $\theta \geq 1/2$  is a constant that corresponds the order of the scheme, and thus is taken to be  $3/2$ .

**Remark 2.** *Due to Remark 1, one observes that*

$$\begin{aligned}
|b^n(x)| &\leq \min(Cn^{\frac{1}{2}}(1 + |x|), |b(x)|), \\
|\sigma^n(x)|^2 &\leq \min(Cn^{\frac{1}{2}}(1 + |x|^2), |\sigma(x)|), \\
|L^{n,0}b(x)| &\leq \min(Cn(1 + |x|), |L^0 b(x)|), \\
|L^{n,j}b(x)| &\leq \min(Cn^{\frac{3}{4}}(1 + |x|), |L^j b(x)|), \\
|L^{n,0}\sigma(x)| &\leq \min(Cn^{\frac{3}{4}}(1 + |x|), |L^0 \sigma(x)|), \\
|L^{n,j}\sigma(x)| &\leq \min(Cn^{\frac{1}{2}}(1 + |x|), |L^j \sigma(x)|), \\
|L^{n,j}L^{j_1}\sigma(x)| &\leq \min(Cn^{\frac{3}{4}}(1 + |x|), |L^j L^{j_1} \sigma(x)|).
\end{aligned}$$

Define

$$\begin{aligned}
b_1^n(t, x) &= \int_{\kappa(n,t)}^t L^{n,0}b(x) ds, \\
b_2^n(t, x) &= \sum_j \int_{\kappa(n,t)}^t L^{n,j}b(x) dw_s^j, \\
\tilde{b}^n(t, x) &:= b^n(x) + b_1^n(t, x) + b_2^n(t, x), \\
\sigma_1^n(t, x) &= \sum_j \int_{\kappa(n,t)}^t L^{n,j}\sigma(x) dw_s^j, \\
\sigma_2^n(t, x) &= \int_{\kappa(n,t)}^t L^{n,0}\sigma(x) ds, \\
\sigma_3^n(t, x) &= \sum_j \sum_{j_1} \int_{\kappa(n,t)}^t \int_{\kappa(n,t)}^s L^{n,j}L^{j_1}\sigma(x) dw_r^j dw_s^{j_1}, \\
\tilde{\sigma}^n(t, x) &= \sigma^n(x) + \sigma_1^n(t, x) + \sigma_2^n(t, x) + \sigma_3^n(t, x).
\end{aligned}$$

Define  $\kappa(n, t) := \lfloor nt \rfloor / n$ , for any  $t \in [0, T]$ . The order 1.5 strong Taylor scheme is as follows:

$$x_t^n = x_0 + \int_0^t \tilde{b}^n(s, x_{\kappa(n,s)}^n) ds + \int_0^t \tilde{\sigma}^n(s, x_{\kappa(n,s)}^n) dw_s, \quad (2.2)$$

almost surely for any  $t \in [0, T]$ .

**Remark 3.** Throughout this article,  $C > 0$  may take different values at different places, but it is always independent of  $n \in \mathbb{N}$ .

**Theorem 1.** Assume **A-1** - **A-5** are satisfied with  $\rho \geq 2$ ,  $p_0 \geq 2(5\rho + 1)$  and  $p_1 > 2$ , then the explicit order 1.5 scheme (2.2) converges to the true solution of the SDE (2.1) in  $\mathcal{L}^2$  with a rate of convergence equal to  $1 + \beta/2$ , i.e., for any  $n \in \mathbb{N}$ ,

$$\sup_{0 \leq t \leq T} E|x_t - x_t^n|^2 \leq Cn^{-(2+\beta)}. \quad (2.3)$$

**Corollary 1.** In Theorem 1, assume that **A-1**, **A-2**, **A-4** and **A-5** hold with  $\rho = 0$ ,  $\beta = 1$  and  $p_0 \geq 4$ , then

$$\sup_{0 \leq t \leq T} E|x_t - x_t^n|^2 \leq Cn^{-3}.$$

which corresponds to the classical order 1.5 scheme.

### 3 Moment bounds

**Lemma 1.** Assume **A-1** - **A-3** hold. Then, there is a unique solution to the SDE (2.1), and the  $p_0$ -th moment of the solution is bounded uniformly in time, i.e. for any  $t \in [0, T]$ ,

$$\sup_{0 \leq t \leq T} E|x_t|^{p_0} \leq C.$$

*Proof.* It is a well-known result, and the proof can be found in [6]. □

**Remark 4.** By Remark 2, for each  $n \in \mathbb{N}$ , the norm of  $\tilde{b}^n$  and  $\tilde{\sigma}^n$  are growing at most linearly in  $x$ . Then, together with **A-1**, this guarantees that for each  $n \in \mathbb{N}$  and  $p \leq p_0$ ,

$$E \left[ \sup_{0 \leq t \leq T} |x_t^n|^p \right] < \infty.$$

**Lemma 2.** Let **A-4** - **A-5** be satisfied, the following inequalities hold

$$\begin{aligned} E|b_1^n(t, x_{\kappa(n,t)}^n)|^{p_0} &\leq C(1 + E|x_{\kappa(n,t)}^n|^{p_0}), \\ E|b_2^n(t, x_{\kappa(n,t)}^n)|^{p_0} &\leq Cn^{\frac{p_0}{4}}(1 + E|x_{\kappa(n,t)}^n|^{p_0}), \\ E|\sigma_1^n(t, x_{\kappa(n,t)}^n)|^{p_0} &\leq C(1 + E|x_{\kappa(n,t)}^n|^{p_0}), \\ E|\sigma_2^n(t, x_{\kappa(n,t)}^n)|^{p_0} &\leq C(1 + E|x_{\kappa(n,t)}^n|^{p_0}), \\ E|\sigma_3^n(t, x_{\kappa(n,t)}^n)|^{p_0} &\leq C(1 + E|x_{\kappa(n,t)}^n|^{p_0}), \end{aligned}$$

for any  $n \in \mathbb{N}$  and  $t \in [0, T]$ .

*Proof.* Due to Remark 2, these inequalities follow immediately. □

**Corollary 2.** Assume **A-4** - **A-5** are satisfied, then for any  $n \in \mathbb{N}$  and  $t \in [0, T]$ ,

$$\begin{aligned} E|\tilde{b}^n(t, x_{\kappa(n,t)}^n)|^{p_0} &\leq Cn^{\frac{p_0}{2}}(1 + E|x_{\kappa(n,t)}^n|^{p_0}), \\ E|\tilde{\sigma}^n(t, x_{\kappa(n,t)}^n)|^{p_0} &\leq Cn^{\frac{p_0}{4}}(1 + E|x_{\kappa(n,t)}^n|^{p_0}). \end{aligned}$$

For  $p_0 \geq 4$ , one obtains the following result <sup>1</sup>.

**Lemma 3.** *If A-1 - A-5 hold, the order 1.5 scheme (2.2) satisfies*

$$\sup_{n \in \mathbb{N}} \sup_{0 \leq t \leq T} E|x_t^n|^{p_0} \leq C.$$

*Proof.* By Itô's formula,

$$\begin{aligned} |x_t^n|^{p_0} &= |x_0|^{p_0} + p_0 \int_0^t |x_s^n|^{p_0-2} x_s^n \tilde{b}^n(s, x_{\kappa(n,s)}^n) ds \\ &\quad + p_0 \int_0^t |x_s^n|^{p_0-2} x_s^n \tilde{\sigma}^n(s, x_{\kappa(n,s)}^n) dw_s \\ &\quad + \frac{p_0}{2} \int_0^t |x_s^n|^{p_0-2} |\tilde{\sigma}^n(s, x_{\kappa(n,s)}^n)|^2 ds \\ &\quad + \frac{p_0(p_0-2)}{2} \int_0^t |x_s^n|^{p_0-4} |\tilde{\sigma}^{n*}(s, x_{\kappa(n,s)}^n) x_s^n|^2 ds, \end{aligned}$$

for any  $t \in [0, T]$ . Then, since the expectation of the third term above is zero, one obtains

$$\begin{aligned} E|x_t^n|^{p_0} &\leq E|x_0|^{p_0} + p_0 E \int_0^t |x_s^n|^{p_0-2} (x_s^n - x_{\kappa(n,s)}^n) b^n(x_{\kappa(n,s)}^n) ds \\ &\quad + p_0 E \int_0^t |x_s^n|^{p_0-2} x_{\kappa(n,s)}^n b^n(x_{\kappa(n,s)}^n) ds \\ &\quad + p_0 E \int_0^t |x_s^n|^{p_0-2} x_s^n b_1^n(s, x_{\kappa(n,s)}^n) ds + p_0 E \int_0^t |x_s^n|^{p_0-2} x_s^n b_2^n(s, x_{\kappa(n,s)}^n) ds \\ &\quad + \frac{p_0(p_0-1)}{2} E \int_0^t |x_s^n|^{p_0-2} |\tilde{\sigma}^n(s, x_{\kappa(n,s)}^n)|^2 ds, \end{aligned}$$

which can be written as

$$\begin{aligned} E|x_t^n|^{p_0} &\leq E|x_0|^{p_0} + p_0 E \int_0^t |x_s^n|^{p_0-2} (x_s^n - x_{\kappa(n,s)}^n) b^n(x_{\kappa(n,s)}^n) ds \\ &\quad + \frac{p_0}{2} E \int_0^t |x_s^n|^{p_0-2} (2x_{\kappa(n,s)}^n b^n(x_{\kappa(n,s)}^n) + (p_0-1) |\sigma^n(x_{\kappa(n,s)}^n)|^2) ds \\ &\quad + p_0 E \int_0^t |x_s^n|^{p_0-2} x_s^n b_1^n(s, x_{\kappa(n,s)}^n) ds + p_0 E \int_0^t |x_s^n|^{p_0-2} x_s^n b_2^n(s, x_{\kappa(n,s)}^n) ds \\ &\quad + \frac{p_0(p_0-1)}{2} E \int_0^t |x_s^n|^{p_0-2} |\sigma_M^n(s, x_{\kappa(n,s)}^n)|^2 ds \\ &\quad + p_0(p_0-1) E \int_0^t |x_s^n|^{p_0-2} \sum_{i=1}^d \sum_{j=1}^m \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n) ds \\ &=: G_1 + G_2 + G_3 + G_4 + G_5 + G_6 + G_7, \end{aligned} \tag{3.1}$$

---

<sup>1</sup>When  $p_0 = 2$ , Lemma 3 can be proved using similar arguments. However, for  $2 < p_0 < 4$ , although the lemma is still valid, it typically requires additional effort by introducing different techniques. Since, this is not the main interest of this article, the proof is omitted.

where  $\sigma_M^n(t, x) = \sigma_1^n(t, x) + \sigma_2^n(t, x) + \sigma_3^n(t, x)$ . Then,  $G_1 = E|x_0|^{p_0}$ , to estimate  $G_2$ , one writes

$$\begin{aligned} G_2 &:= p_0 E \int_0^t |x_s^n|^{p_0-2} (x_s^n - x_{\kappa(n,s)}^n) b^n(x_{\kappa(n,s)}^n) ds \\ &= p_0 E \int_0^t |x_s^n|^{p_0-2} \int_{\kappa(n,s)}^s \tilde{b}^n(r, x_{\kappa(n,r)}^n) dr b^n(x_{\kappa(n,s)}^n) ds \\ &\quad + p_0 E \int_0^t |x_s^n|^{p_0-2} \int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x_{\kappa(n,r)}^n) dw_r b^n(x_{\kappa(n,s)}^n) ds, \end{aligned}$$

for any  $t \in [0, T]$ . By applying Young's inequality and Remark 2, the following estimate can be obtained

$$\begin{aligned} G_2 &\leq C + C \int_0^t \sup_{0 \leq r \leq s} E|x_r^n|^{p_0} ds + CE \int_0^t \left| n^{\frac{1}{2}} \int_{\kappa(n,s)}^s \tilde{b}^n(r, x_{\kappa(n,r)}^n) dr \right|^{p_0} ds \\ &\quad + p_0 E \int_0^t (|x_s^n|^{p_0-2} - |x_{\kappa(n,s)}^n|^{p_0-2}) \int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x_{\kappa(n,r)}^n) dw_r b^n(x_{\kappa(n,s)}^n) ds \\ &\quad + p_0 E \int_0^t |x_{\kappa(n,s)}^n|^{p_0-2} \int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x_{\kappa(n,r)}^n) dw_r b^n(x_{\kappa(n,s)}^n) ds, \end{aligned}$$

for any  $t \in [0, T]$ . The last term above is zero, and the application of Corollary 2 and Itô's formula gives

$$\begin{aligned} G_2 &\leq C + C \int_0^t \sup_{0 \leq r \leq s} E|x_r^n|^{p_0} ds \\ &\quad + CE \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{b}^n(r, x_{\kappa(n,r)}^n) dr \int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x_{\kappa(n,r)}^n) dw_r b^n(x_{\kappa(n,s)}^n) ds \\ &\quad + CE \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{\sigma}^n(r, x_{\kappa(n,r)}^n) dw_r \int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x_{\kappa(n,r)}^n) dw_r b^n(x_{\kappa(n,s)}^n) ds \\ &\quad + CE \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} |\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|^2 dr \int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x_{\kappa(n,r)}^n) dw_r |b^n(x_{\kappa(n,s)}^n)| ds, \end{aligned}$$

which implies due to Remark 2

$$\begin{aligned} G_2 &\leq C + C \int_0^t \sup_{0 \leq r \leq s} E|x_r^n|^{p_0} ds \\ &\quad + Cn^{\frac{1}{2}} E \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-3} (1 + |x_{\kappa(n,s)}^n|) |\tilde{b}^n(r, x_{\kappa(n,r)}^n)| dr \left| \int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x_{\kappa(n,r)}^n) dw_r \right| ds \\ &\quad + Cn^{\frac{1}{2}} E \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-3} (1 + |x_{\kappa(n,s)}^n|) |\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|^2 dr ds \\ &\quad + Cn^{\frac{1}{2}} E \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} (1 + |x_{\kappa(n,s)}^n|) |\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|^2 dr \left| \int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x_{\kappa(n,r)}^n) dw_r \right| ds, \end{aligned}$$



for any  $t \in [0, T]$ . Moreover, apply Young's inequality to obtain

$$\begin{aligned}
G_2 &\leq C + C \int_0^t \sup_{0 \leq r \leq s} E |x_r^n|^{p_0} ds \\
&+ CE \int_0^t n^{\frac{1}{4}} \int_{\kappa(n,s)}^s (1 + |x_r^n|^{p_0-2} + |x_{\kappa(n,s)}^n|^{p_0-2}) |\tilde{b}^n(r, x_{\kappa(n,r)}^n)| dr \\
&\quad \times n^{\frac{1}{4}} \left| \int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x_{\kappa(n,r)}^n) dw_r \right| ds \\
&+ CE \int_0^t \int_{\kappa(n,s)}^s n^{1-\frac{2}{p_0}} (1 + |x_r^n|^{p_0-2} + |x_{\kappa(n,s)}^n|^{p_0-2}) n^{-\frac{1}{2}+\frac{2}{p_0}} |\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|^2 dr ds \\
&+ CE \int_0^t n^{\frac{1}{4}} \int_{\kappa(n,s)}^s (1 + |x_r^n|^{p_0-3} + |x_{\kappa(n,s)}^n|^{p_0-3}) |\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|^2 dr \\
&\quad \times n^{\frac{1}{4}} \left| \int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x_{\kappa(n,r)}^n) dw_r \right| ds,
\end{aligned}$$

which can be further estimated as

$$\begin{aligned}
G_2 &\leq C + C \int_0^t \sup_{0 \leq r \leq s} E |x_r^n|^{p_0} ds \\
&+ CE \int_0^t \left( \int_{\kappa(n,s)}^s n^{\frac{3}{4}-\frac{1}{p_0}} (1 + |x_r^n|^{p_0-2} + |x_{\kappa(n,s)}^n|^{p_0-2}) n^{-\frac{1}{2}+\frac{1}{p_0}} |\tilde{b}^n(r, x_{\kappa(n,r)}^n)| dr \right)^{\frac{p_0}{p_0-1}} ds \\
&+ CnE \int_0^t \int_{\kappa(n,s)}^s (1 + |x_r^n|^{p_0} + |x_{\kappa(n,s)}^n|^{p_0}) dr ds \\
&+ Cn^{-\frac{p_0}{4}+1} E \int_0^t \int_{\kappa(n,s)}^s |\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|^{p_0} dr ds \\
&+ CE \int_0^t \left( \int_{\kappa(n,s)}^s n^{\frac{3}{4}-\frac{2}{p_0}} (1 + |x_r^n|^{p_0-3} + |x_{\kappa(n,s)}^n|^{p_0-3}) n^{-\frac{1}{2}+\frac{2}{p_0}} |\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|^2 dr \right)^{\frac{p_0}{p_0-1}} ds \\
&+ Cn^{\frac{p_0}{4}} \int_0^t E \left| \int_{\kappa(n,s)}^s \tilde{\sigma}^n(r, x_{\kappa(n,r)}^n) dw_r \right|^{p_0} ds
\end{aligned}$$

for any  $t \in [0, T]$ . The application of Young's inequality and Corollary 2 gives

$$\begin{aligned}
G_2 &\leq C + C \int_0^t \sup_{0 \leq r \leq s} E |x_r^n|^{p_0} ds \\
&+ CE \int_0^t \left( \int_{\kappa(n,s)}^s n^{\frac{3p_0-4}{4p_0} \times \frac{p_0-1}{p_0-2}} (1 + |x_r^n|^{p_0-1} + |x_{\kappa(n,s)}^n|^{p_0-1}) dr \right)^{\frac{p_0}{p_0-1}} ds \\
&+ CE \int_0^t \left( \int_{\kappa(n,s)}^s n^{\frac{(2-p_0) \times (p_0-1)}{2p_0}} |\tilde{b}^n(r, x_{\kappa(n,r)}^n)|^{p_0-1} dr \right)^{\frac{p_0}{p_0-1}} ds \\
&+ CE \int_0^t \left( \int_{\kappa(n,s)}^s n^{\frac{3p_0-8}{4p_0} \times \frac{p_0-1}{p_0-3}} (1 + |x_r^n|^{p_0-1} + |x_{\kappa(n,s)}^n|^{p_0-1}) dr \right)^{\frac{p_0}{p_0-1}} ds \\
&+ CE \int_0^t \left( \int_{\kappa(n,s)}^s n^{\frac{4-p_0}{2p_0} \times \frac{p_0-1}{2}} |\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|^{p_0-1} dr \right)^{\frac{p_0}{p_0-1}} ds \\
&+ Cn^{-\frac{p_0}{4}+1} \int_0^t \int_{\kappa(n,s)}^s E |\tilde{\sigma}^n(r, x_{\kappa(n,r)}^n)|^{p_0} dr ds,
\end{aligned}$$

which, due to Hölder's inequality and Corollary 2, can be written as

$$\begin{aligned}
G_2 &\leq C + C \int_0^t \sup_{0 \leq r \leq s} E |x_r^n|^{p_0} ds \\
&+ Cn^{\frac{3p_0-4}{4(p_0-2)} - \frac{1}{p_0-1}} \int_0^t E \int_{\kappa(n,s)}^s (1 + |x_r^n|^{p_0} + |x_{\kappa(n,s)}^n|^{p_0}) dr ds \\
&+ Cn^{-\frac{p_0}{2}+1 - \frac{1}{p_0-1}} \int_0^t E \int_{\kappa(n,s)}^s |\tilde{b}^n(r, x_{\kappa(n,r)}^n)|^{p_0} dr ds \\
&+ Cn^{\frac{3p_0-8}{4(p_0-3)} - \frac{1}{p_0-1}} \int_0^t E \int_{\kappa(n,s)}^s (1 + |x_r^n|^{p_0} + |x_{\kappa(n,s)}^n|^{p_0}) dr ds,
\end{aligned}$$

for any  $t \in [0, T]$ . Note that in the third and fifth term above,  $n^{\frac{3p_0-4}{4(p_0-2)}}$  and  $n^{\frac{3p_0-8}{4(p_0-3)}}$  are less or equal to  $n$  for all  $p_0 \geq 4$ . Thus, apply Corollary 2,

$$G_2 \leq C + C \int_0^t \sup_{0 \leq r \leq s} E |x_r^n|^{p_0} ds, \quad (3.2)$$

for any  $t \in [0, T]$ . For  $G_3$ , applying **A-2** gives

$$\begin{aligned}
G_3 &:= \frac{p_0}{2} E \int_0^t |x_s^n|^{p_0-2} (2x_{\kappa(n,s)}^n b(x_{\kappa(n,s)}^n) + (p_0-1)|\sigma^n(x_{\kappa(n,s)}^n)|^2) ds \\
&= \frac{p_0}{2} E \int_0^t |x_s^n|^{p_0-2} \frac{2x_{\kappa(n,s)}^n b(x_{\kappa(n,s)}^n) + (p_0-1)|\sigma(x_{\kappa(n,s)}^n)|^2}{1 + n^{-3/2}|x_{\kappa(n,s)}^n|^{3p_0}} ds \\
&\leq CE \int_0^t |x_s^n|^{p_0-2} (1 + |x_{\kappa(n,s)}^n|^2) ds,
\end{aligned}$$

which yields the desired result by using Young's inequality,

$$G_3 \leq C + C \int_0^t \sup_{0 \leq r \leq s} E |x_r^n|^{p_0} ds, \quad (3.3)$$

for any  $t \in [0, T]$ . Next, by Young's inequality,  $G_4$  can be estimated as

$$\begin{aligned} G_4 &:= p_0 E \int_0^t |x_s^n|^{p_0-2} x_s^n b_1^n(s, x_{\kappa(n,s)}^n) ds \\ &\leq CE \int_0^t |x_s^n|^{p_0} ds + CE \int_0^t |b_1^n(s, x_{\kappa(n,s)}^n)|^{p_0} ds, \end{aligned}$$

which implies due to Lemma 2,

$$G_4 \leq C + C \int_0^t \sup_{0 \leq r \leq s} E |x_r^n|^{p_0} ds, \quad (3.4)$$

for any  $t \in [0, T]$ . As for  $G_5$ , one writes

$$\begin{aligned} G_5 &:= p_0 E \int_0^t |x_s^n|^{p_0-2} x_s^n b_2^n(s, x_{\kappa(n,s)}^n) ds \\ &= p_0 E \int_0^t |x_s^n|^{p_0-2} (x_s^n - x_{\kappa(n,s)}^n) b_2^n(s, x_{\kappa(n,s)}^n) ds \\ &\quad + p_0 E \int_0^t (|x_s^n|^{p_0-2} - |x_{\kappa(n,s)}^n|^{p_0-2}) x_{\kappa(n,s)}^n b_2^n(s, x_{\kappa(n,s)}^n) ds \\ &\quad + p_0 E \int_0^t |x_{\kappa(n,s)}^n|^{p_0-2} x_{\kappa(n,s)}^n b_2^n(s, x_{\kappa(n,s)}^n) ds \\ &=: G_{51} + G_{52} + G_{53}. \end{aligned} \quad (3.5)$$

To estimate  $G_{51}$ ,

$$\begin{aligned} G_{51} &:= p_0 E \int_0^t |x_s^n|^{p_0-2} (x_s^n - x_{\kappa(n,s)}^n) b_2^n(s, x_{\kappa(n,s)}^n) ds \\ &= p_0 E \int_0^t |x_s^n|^{p_0-2} \int_{\kappa(n,s)}^s \tilde{b}(r, x_{\kappa(n,r)}^n) dr b_2^n(s, x_{\kappa(n,s)}^n) ds \\ &\quad + p_0 E \int_0^t |x_s^n|^{p_0-2} \int_{\kappa(n,s)}^s \tilde{\sigma}(r, x_{\kappa(n,r)}^n) dw_r b_2^n(s, x_{\kappa(n,s)}^n) ds, \end{aligned}$$

which implies due to Young's inequality,

$$\begin{aligned} G_{51} &\leq CE \int_0^t |x_s^n|^{p_0} ds + CE \int_0^t \left| n^{\frac{1}{4}} \int_{\kappa(n,s)}^s \tilde{b}(r, x_{\kappa(n,r)}^n) dr n^{-\frac{1}{4}} b_2^n(s, x_{\kappa(n,s)}^n) \right|^{\frac{p_0}{2}} ds \\ &\quad + CE \int_0^t \left| n^{\frac{1}{4}} \int_{\kappa(n,s)}^s \tilde{\sigma}(r, x_{\kappa(n,r)}^n) dw_r n^{-\frac{1}{4}} b_2^n(s, x_{\kappa(n,s)}^n) \right|^{\frac{p_0}{2}} ds, \end{aligned}$$

for any  $t \in [0, T]$ . Then, by Young's inequality

$$\begin{aligned} G_{51} &\leq CE \int_0^t |x_s^n|^{p_0} ds + C n^{\frac{p_0}{4}} E \int_0^t \left| \int_{\kappa(n,s)}^s \tilde{b}(r, x_{\kappa(n,r)}^n) dr \right|^{p_0} ds \\ &\quad + C n^{\frac{p_0}{4}} E \int_0^t \left| \int_{\kappa(n,s)}^s \tilde{\sigma}(r, x_{\kappa(n,r)}^n) dw_r \right|^{p_0} ds + C n^{-\frac{p_0}{4}} E \int_0^t |b_2^n(s, x_{\kappa(n,s)}^n)|^{p_0} ds, \end{aligned}$$

which on the application of Hölder's inequality and Lemma 2 yields

$$\begin{aligned} G_{51} &\leq C + C \int_0^t \sup_{0 \leq r \leq s} E |x_r^n|^{p_0} ds \\ &\quad + C n^{\frac{p_0}{4} - p_0 + 1} \int_0^t \int_{\kappa(n,s)}^s E |\tilde{b}(r, x_{\kappa(n,r)}^n)|^{p_0} dr ds \\ &\quad + C n^{\frac{p_0}{4} - \frac{p_0}{2} + 1} \int_0^t \int_{\kappa(n,s)}^s E |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)|^{p_0} dr ds, \end{aligned}$$

for any  $t \in [0, T]$ . By using Corollary 2, one obtains

$$G_{51} \leq C + C \int_0^t \sup_{0 \leq r \leq s} E |x_r^n|^{p_0} ds, \quad (3.6)$$

for any  $t \in [0, T]$ . As for  $G_{52}$ , Itô's formula gives

$$\begin{aligned} G_{52} &:= p_0 E \int_0^t (|x_s^n|^{p_0-2} - |x_{\kappa(n,s)}^n|^{p_0-2}) x_{\kappa(n,s)}^n b_2^n(s, x_{\kappa(n,s)}^n) ds \\ &\leq C E \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{b}(r, x_{\kappa(n,r)}^n) dr x_{\kappa(n,s)}^n b_2^n(s, x_{\kappa(n,s)}^n) ds \\ &\quad + C E \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{\sigma}(r, x_{\kappa(n,r)}^n) dw_r x_{\kappa(n,s)}^n \sum_j \int_{\kappa(n,s)}^s L^{n,j} b(x_{\kappa(n,r)}^n) dw_r^j ds \\ &\quad + C E \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)|^2 dr |x_{\kappa(n,s)}^n| |b_2^n(s, x_{\kappa(n,s)}^n)| ds, \end{aligned}$$

which, by Young's inequality, can be expressed as

$$\begin{aligned} G_{52} &\leq C \int_0^t E \int_{\kappa(n,s)}^s n^{\frac{3}{4} - \frac{1}{p_0}} (1 + |x_r^n|^{p_0-2} + |x_{\kappa(n,s)}^n|^{p_0-2}) n^{-\frac{1}{2} + \frac{1}{p_0}} |\tilde{b}(r, x_{\kappa(n,r)}^n)| dr \\ &\quad \times n^{-\frac{1}{4}} |b_2^n(s, x_{\kappa(n,s)}^n)| ds \\ &\quad + C \sum_{j=1}^m \int_0^t E \int_{\kappa(n,s)}^s (1 + |x_r^n|^{p_0-2} + |x_{\kappa(n,s)}^n|^{p_0-2}) |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)| |L^{n,j} b(x_{\kappa(n,r)}^n)| dr ds \\ &\quad + C \int_0^t E \int_{\kappa(n,s)}^s n^{\frac{3}{4} - \frac{2}{p_0}} (1 + |x_r^n|^{p_0-3} + |x_{\kappa(n,s)}^n|^{p_0-3}) n^{-\frac{1}{2} + \frac{2}{p_0}} |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)|^2 dr \\ &\quad \times n^{-\frac{1}{4}} |b_2^n(s, x_{\kappa(n,s)}^n)| ds, \end{aligned}$$

for any  $t \in [0, T]$ . One uses Young's inequality and Remark 2 to obtain

$$\begin{aligned} G_{52} &\leq C \int_0^t E \left( \int_{\kappa(n,s)}^s n^{\frac{3}{4} - \frac{1}{p_0}} (1 + |x_r^n|^{p_0-2} + |x_{\kappa(n,s)}^n|^{p_0-2}) n^{-\frac{1}{2} + \frac{1}{p_0}} |\tilde{b}(r, x_{\kappa(n,r)}^n)| dr \right)^{\frac{p_0}{p_0-1}} ds \\ &\quad + C \int_0^t E \int_{\kappa(n,s)}^s n^{1 - \frac{1}{p_0}} (1 + |x_r^n|^{p_0-1} + |x_{\kappa(n,s)}^n|^{p_0-1}) n^{-\frac{1}{4} + \frac{1}{p_0}} |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)| dr ds \\ &\quad + C \int_0^t E \left( \int_{\kappa(n,s)}^s n^{\frac{3}{4} - \frac{2}{p_0}} (1 + |x_r^n|^{p_0-3} + |x_{\kappa(n,s)}^n|^{p_0-3}) n^{-\frac{1}{2} + \frac{2}{p_0}} |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)|^2 dr \right)^{\frac{p_0}{p_0-1}} ds \\ &\quad + C n^{-\frac{p_0}{4}} \int_0^t E |b_2^n(s, x_{\kappa(n,s)}^n)|^{p_0} ds, \end{aligned}$$

which implies due to Young's inequality and Lemma 2

$$\begin{aligned}
G_{52} &\leq C \int_0^t E \left( \int_{\kappa(n,s)}^s n^{\frac{3p_0-4}{4p_0} \times \frac{p_0-1}{p_0-2}} (1 + |x_r^n|^{p_0-1} + |x_{\kappa(n,s)}^n|^{p_0-1}) dr \right)^{\frac{p_0}{p_0-1}} ds \\
&\quad + C \int_0^t E \left( \int_{\kappa(n,s)}^s n^{\frac{(2-p_0) \times (p_0-1)}{2p_0}} |\tilde{b}(r, x_{\kappa(n,r)}^n)|^{p_0-1} dr \right)^{\frac{p_0}{p_0-1}} ds \\
&\quad + C \int_0^t E \int_{\kappa(n,s)}^s n(1 + |x_r^n|^{p_0} + |x_{\kappa(n,s)}^n|^{p_0}) dr ds \\
&\quad + C \int_0^t E \int_{\kappa(n,s)}^s n^{\frac{4-p_0}{4p_0} \times p_0} |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)|^{p_0} dr ds \\
&\quad + C \int_0^t E \left( \int_{\kappa(n,s)}^s n^{\frac{3p_0-8}{4p_0} \times \frac{p_0-1}{p_0-3}} (1 + |x_r^n|^{p_0-1} + |x_{\kappa(n,s)}^n|^{p_0-1}) dr \right)^{\frac{p_0}{p_0-1}} ds \\
&\quad + C \int_0^t E \left( \int_{\kappa(n,s)}^s n^{\frac{4-p_0}{2p_0} \times \frac{p_0-1}{2}} |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)|^{p_0-1} dr \right)^{\frac{p_0}{p_0-1}} ds \\
&\quad + C + C \int_0^t \sup_{0 \leq r \leq s} E |x_r^n|^{p_0} ds,
\end{aligned}$$

for any  $t \in [0, T]$ . The application of Hölder's inequality and Corollary 2 gives

$$\begin{aligned}
G_{52} &\leq C n^{\frac{3p_0-4}{4(p_0-2)} - \frac{1}{p_0-1}} \int_0^t E \int_{\kappa(n,s)}^s (1 + |x_r^n|^{p_0} + |x_{\kappa(n,s)}^n|^{p_0}) dr ds \\
&\quad + C n^{-\frac{p_0}{2} + 1 - \frac{1}{p_0-1}} \int_0^t E \int_{\kappa(n,s)}^s |\tilde{b}(r, x_{\kappa(n,r)}^n)|^{p_0} dr ds \\
&\quad + C n^{\frac{3p_0-8}{4(p_0-3)} - \frac{1}{p_0-1}} \int_0^t E \int_{\kappa(n,s)}^s (1 + |x_r^n|^{p_0} + |x_{\kappa(n,s)}^n|^{p_0}) dr ds \\
&\quad + C n^{-\frac{p_0}{4} + 1 - \frac{1}{p_0-1}} \int_0^t E \int_{\kappa(n,s)}^s |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)|^{p_0} dr ds \\
&\quad + C + C \int_0^t \sup_{0 \leq r \leq s} E |x_r^n|^{p_0} ds,
\end{aligned}$$

for any  $t \in [0, T]$ . One observes that  $n^{\frac{3p_0-4}{4(p_0-2)} - \frac{1}{p_0-1}}$  and  $n^{\frac{3p_0-8}{4(p_0-3)} - \frac{1}{p_0-1}}$  are less or equal to  $n$  for all  $p_0 \geq 4$ , then by applying Corollary 2

$$G_{52} \leq C + C \int_0^t \sup_{0 \leq r \leq s} E |x_r^n|^{p_0} ds, \quad (3.7)$$

for any  $t \in [0, T]$ . In addition, note that

$$G_{53} := p_0 E \int_0^t |x_{\kappa(n,s)}^n|^{p_0-2} x_{\kappa(n,s)}^n b_2^n(s, x_{\kappa(n,s)}^n) ds = 0, \quad (3.8)$$

for any  $t \in [0, T]$ . Then, substitute (3.6), (3.7) and (3.8) into (3.5),

$$G_5 \leq C + C \int_0^t \sup_{0 \leq r \leq s} E |x_r^n|^{p_0} ds, \quad (3.9)$$

for any  $t \in [0, T]$ . In order to estimate  $G_6$ , apply Young's inequality to obtain

$$\begin{aligned}
G_6 &:= \frac{p_0(p_0 - 1)}{2} E \int_0^t |x_s^n|^{p_0-2} |\sigma_M^n(s, x_{\kappa(n,s)}^n)|^2 ds \\
&\leq CE \int_0^t |x_s^n|^{p_0} ds + CE \int_0^t |\sigma_M^n(s, x_{\kappa(n,s)}^n)|^{p_0} ds \\
&\leq CE \int_0^t |x_s^n|^{p_0} ds + CE \int_0^t |\sigma_1^n(s, x_{\kappa(n,s)}^n)|^{p_0} ds \\
&\quad + CE \int_0^t |\sigma_2^n(s, x_{\kappa(n,s)}^n)|^{p_0} ds + CE \int_0^t |\sigma_3^n(s, x_{\kappa(n,s)}^n)|^{p_0} ds,
\end{aligned}$$

which implies due to Lemma 2

$$G_6 \leq C + C \int_0^t \sup_{0 \leq r \leq s} E |x_r^n|^{p_0} ds, \quad (3.10)$$

for any  $t \in [0, T]$ . Finally, for  $G_7$ , one writes

$$\begin{aligned}
G_7 &:= p_0(p_0 - 1) E \int_0^t |x_s^n|^{p_0-2} \sum_{i=1}^d \sum_{j=1}^m \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n) ds \\
&= p_0(p_0 - 1) E \int_0^t (|x_s^n|^{p_0-2} - |x_{\kappa(n,s)}^n|^{p_0-2}) \sum_{i=1}^d \sum_{j=1}^m \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n) ds \\
&\quad + p_0(p_0 - 1) E \int_0^t |x_{\kappa(n,s)}^n|^{p_0-2} \sum_{i=1}^d \sum_{j=1}^m \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n) ds \\
&=: G_{71} + G_{72}. \tag{3.11}
\end{aligned}$$

To estimate  $G_{71}$ , Itô's formula is used to obtain the following

$$\begin{aligned}
G_{71} &:= CE \int_0^t (|x_s^n|^{p_0-2} - |x_{\kappa(n,s)}^n|^{p_0-2}) \sum_{i=1}^d \sum_{j=1}^m \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n) ds \\
&\leq CE \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{b}(r, x_{\kappa(n,r)}^n) dr \sum_{i=1}^d \sum_{j=1}^m \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n) ds \\
&\quad + CE \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{\sigma}(r, x_{\kappa(n,r)}^n) dw_r \sum_{i=1}^d \sum_{j=1}^m \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n) ds \\
&\quad + CE \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)|^2 dr \left| \sum_{i=1}^d \sum_{j=1}^m \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n) \right| ds,
\end{aligned}$$

which by using Remark 2 yields

$$\begin{aligned}
G_{71} &\leq Cn^{\frac{1}{4}}E \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-3} (1 + |x_{\kappa(n,s)}^n|) |\tilde{b}(r, x_{\kappa(n,r)}^n)| dr |\sigma_M^n(s, x_{\kappa(n,s)}^n)| ds \\
&\quad + CE \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{\sigma}(r, x_{\kappa(n,r)}^n) dw_r \\
&\quad \quad \times \sum_{i=1}^d \sum_{j=1}^m \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sum_j \int_{\kappa(n,s)}^s L^{n,j} \sigma(x_{\kappa(n,r)}^n) dw_r^j ds \\
&\quad + CE \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{\sigma}(r, x_{\kappa(n,r)}^n) dw_r \\
&\quad \quad \times \sum_{i=1}^d \sum_{j=1}^m \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \int_{\kappa(n,s)}^s L^{n,0} \sigma(x_{\kappa(n,r)}^n) dr ds \\
&\quad + CE \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} x_r^n \tilde{\sigma}(r, x_{\kappa(n,r)}^n) dw_r \\
&\quad \quad \times \sum_{i=1}^d \sum_{j=1}^m \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sum_j \sum_{j_1} \int_{\kappa(n,s)}^s \int_{\kappa(n,r)}^r L^{n,j} L^{j_1} \sigma(x_{\kappa(n,\gamma)}^n) dw_{\gamma}^{j_1} dw_r^j ds \\
&\quad + Cn^{\frac{1}{4}}E \int_0^t \int_{\kappa(n,s)}^s |x_r^n|^{p_0-4} (1 + |x_{\kappa(n,s)}^n|) |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)|^2 dr |\sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n)| ds,
\end{aligned}$$

for any  $t \in [0, T]$ . One then observes that, since  $L^{n,0} \sigma(x_{\kappa(n,r)}^n)$  takes the same value for all  $r \in [\kappa(n,s), s]$ , it can be taken out of the integral in the third term above, and thus the third term is zero. Moreover, by Young's inequality and Remark 2,

$$\begin{aligned}
G_{71} &\leq CE \int_0^t \int_{\kappa(n,s)}^s n^{\frac{1}{4}} (1 + |x_r^n|^{p_0-2} + |x_{\kappa(n,s)}^n|^{p_0-2}) |\tilde{b}(r, x_{\kappa(n,r)}^n)| dr |\sigma_M^n(s, x_{\kappa(n,s)}^n)| ds \\
&\quad + CE \int_0^t \int_{\kappa(n,s)}^s n^{\frac{3}{4}} |x_r^n|^{p_0-3} (1 + |x_{\kappa(n,s)}^n|)^2 |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)| dr ds \\
&\quad + CE \int_0^t \int_{\kappa(n,s)}^s n^{\frac{3}{4} - \frac{2}{p_0}} |x_r^n|^{p_0-3} (1 + |x_{\kappa(n,s)}^n|) n^{-\frac{1}{4} + \frac{1}{p_0}} |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)| \\
&\quad \quad \times n^{-\frac{1}{4} + \frac{1}{p_0}} \left| \sum_{j_1} \int_{\kappa(n,r)}^r L^{n,j} L^{j_1} \sigma(x_{\kappa(n,\gamma)}^n) dw_{\gamma}^{j_1} \right| dr ds \\
&\quad + CE \int_0^t \int_{\kappa(n,s)}^s n^{\frac{1}{4}} (1 + |x_r^n|^{p_0-3} + |x_{\kappa(n,s)}^n|^{p_0-3}) |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)|^2 dr |\sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n)| ds,
\end{aligned}$$

which on the application of Young's inequality gives

$$\begin{aligned}
G_{71} &\leq C \int_0^t E \left( \int_{\kappa(n,s)}^s n^{\frac{3}{4}-\frac{1}{p_0}} (1 + |x_r^n|^{p_0-2} + |x_{\kappa(n,s)}^n|^{p_0-2}) n^{-\frac{1}{2}+\frac{1}{p_0}} |\tilde{b}(r, x_{\kappa(n,r)}^n)| dr \right)^{\frac{p_0}{p_0-1}} ds \\
&\quad + CE \int_0^t \int_{\kappa(n,s)}^s n^{1-\frac{1}{p_0}} (1 + |x_r^n|^{p_0-1} + |x_{\kappa(n,s)}^n|^{p_0-1}) n^{-\frac{1}{4}+\frac{1}{p_0}} |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)| dr ds \\
&\quad + CE \int_0^t \int_{\kappa(n,s)}^s \left( n^{\frac{3}{4}-\frac{2}{p_0}} (1 + |x_r^n|^{p_0-2} + |x_{\kappa(n,s)}^n|^{p_0-2}) n^{-\frac{1}{4}+\frac{1}{p_0}} |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)| \right)^{\frac{p_0}{p_0-1}} dr ds \\
&\quad + C \int_0^t E \left( \int_{\kappa(n,s)}^s n^{\frac{3}{4}-\frac{2}{p_0}} (1 + |x_r^n|^{p_0-3} + |x_{\kappa(n,s)}^n|^{p_0-3}) n^{-\frac{1}{2}+\frac{2}{p_0}} |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)|^2 dr \right)^{\frac{p_0}{p_0-1}} ds \\
&\quad + C n^{-\frac{p_0}{4}+1} E \int_0^t \int_{\kappa(n,s)}^s \left| \sum_{j_1} \int_{\kappa(n,r)}^r L^{n,j} L^{j_1} \sigma(x_{\kappa(n,\gamma)}^n) dw_{\gamma}^{j_1} \right|^{p_0} dr ds \\
&\quad + C \int_0^t E |\sigma_M^n(s, x_{\kappa(n,s)}^n)|^{p_0} ds,
\end{aligned}$$

for any  $t \in [0, T]$ . By Young's inequality, Hölder's inequality and Lemma 2,

$$\begin{aligned}
G_{71} &\leq C n^{\frac{3p_0-4}{4(p_0-2)}-\frac{1}{p_0-1}} \int_0^t E \int_{\kappa(n,s)}^s (1 + |x_r^n|^{p_0} + |x_{\kappa(n,s)}^n|^{p_0}) dr ds \\
&\quad + C n^{-\frac{p_0}{2}+1-\frac{1}{p_0-1}} \int_0^t \int_{\kappa(n,s)}^s E |\tilde{b}(r, x_{\kappa(n,r)}^n)|^{p_0} dr ds \\
&\quad + C n E \int_0^t \int_{\kappa(n,s)}^s (1 + |x_r^n|^{p_0} + |x_{\kappa(n,s)}^n|^{p_0}) dr ds \\
&\quad + C n^{-\frac{p_0}{4}+1} \int_0^t \int_{\kappa(n,s)}^s E |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)|^{p_0} dr ds \\
&\quad + C n^{\frac{3p_0-8}{4(p_0-2)}} E \int_0^t \int_{\kappa(n,s)}^s (1 + |x_r^n|^{p_0} + |x_{\kappa(n,s)}^n|^{p_0}) dr ds \\
&\quad + C n^{-\frac{p_0}{4}+1} \int_0^t \int_{\kappa(n,s)}^s E |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)|^{p_0} dr ds \\
&\quad + C n^{\frac{3p_0-8}{4(p_0-3)}-\frac{1}{p_0-1}} \int_0^t E \int_{\kappa(n,s)}^s (1 + |x_r^n|^{p_0} + |x_{\kappa(n,s)}^n|^{p_0}) dr ds \\
&\quad + C n^{-\frac{p_0}{4}+1-\frac{1}{p_0-1}} \int_0^t \int_{\kappa(n,s)}^s E |\tilde{\sigma}(r, x_{\kappa(n,r)}^n)|^{p_0} dr ds \\
&\quad + C n^{-\frac{3p_0}{4}+2} \int_0^t \int_{\kappa(n,s)}^s \int_{\kappa(n,r)}^r E |L^{n,j} L^{j_1} \sigma(x_{\kappa(n,\gamma)}^n)|^{p_0} d\gamma dr ds \\
&\quad + C + C \int_0^t \sup_{0 \leq r \leq s} E |x_r^n|^{p_0} ds,
\end{aligned}$$

for any  $t \in [0, T]$ . Due to Corollary 2 and Remark 2, it can be shown that

$$G_{71} \leq C + C \int_0^t \sup_{0 \leq r \leq s} E |x_r^n|^{p_0} ds, \quad (3.12)$$



for any  $t \in [0, T]$ . In order to estimate  $G_{72}$ , one writes

$$\begin{aligned}
G_{72} &:= p_0(p_0 - 1)E \int_0^t |x_{\kappa(n,s)}^n|^{p_0-2} \sum_{i=1}^d \sum_{j=1}^m \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n) ds \\
&= p_0(p_0 - 1)E \int_0^t |x_{\kappa(n,s)}^n|^{p_0-2} \sum_{i=1}^d \sum_{j=1}^m \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sum_j \int_{\kappa(n,s)}^s L^{n,j} \sigma(x_{\kappa(n,r)}^n) dw_r^j ds \\
&\quad + p_0(p_0 - 1)E \int_0^t |x_{\kappa(n,s)}^n|^{p_0-2} \sum_{i=1}^d \sum_{j=1}^m \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \int_{\kappa(n,s)}^s L^{n,0} \sigma(x_{\kappa(n,r)}^n) dr ds \\
&\quad + p_0(p_0 - 1)E \int_0^t |x_{\kappa(n,s)}^n|^{p_0-2} \sum_{i=1}^d \sum_{j=1}^m \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \\
&\quad \quad \times \sum_j \sum_{j_1} \int_{\kappa(n,s)}^s \int_{\kappa(n,r)}^r L^{n,j} L^{j_1} \sigma(x_{\kappa(n,\gamma)}^n) dw_{\gamma}^{j_1} dw_r^j ds,
\end{aligned}$$

which implies due to Remark 2 and the fact that the first and third terms are zero

$$G_{72} \leq CE \int_0^t \int_{\kappa(n,s)}^s n(1 + |x_{\kappa(n,s)}^n|^{p_0}) dr ds,$$

for any  $t \in [0, T]$ . Then, one obtains

$$G_{72} \leq C + C \int_0^t \sup_{0 \leq r \leq s} E|x_r^n|^{p_0} ds, \quad (3.13)$$

for any  $t \in [0, T]$ . Furthermore, substituting (3.12) and (3.13) into (3.14) yields

$$G_7 \leq C + C \int_0^t \sup_{0 \leq r \leq s} E|x_r^n|^{p_0} ds, \quad (3.14)$$

for any  $t \in [0, T]$ . Therefore, for any  $n \in \mathbb{N}$ ,

$$\sup_{0 \leq s \leq t} E|x_s^n|^{p_0} \leq C + C \int_0^t \sup_{0 \leq r \leq s} E|x_r^n|^{p_0} ds < \infty,$$

and applying Gronwall's lemma completes the proof.  $\square$

## 4 Proof of main result

**Lemma 4.** *Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a twice continuously differentiable function that satisfies*

$$|D^2 f(x) - D^2 f(\bar{x})| \leq (1 + |x| + |\bar{x}|)^\alpha |x - \bar{x}|^\beta$$

for all  $x, \bar{x} \in \mathbb{R}$  and for any fixed  $\alpha \in \mathbb{R}$ . Then, there is a constant  $K > 0$  such that

$$\left| \frac{\partial f(x)}{\partial y^i} - \frac{\partial f(\bar{x})}{\partial y^i} - \sum_{j=1}^d \frac{\partial^2 f(\bar{x})}{\partial y^i \partial y^j} (x^j - \bar{x}^j) \right| \leq K(1 + |x| + |\bar{x}|)^\alpha |x - \bar{x}|^{1+\beta},$$

for any for all  $x, \bar{x} \in \mathbb{R}$ , and  $i = 1, \dots, d$ .

*Proof.* One uses mean value theorem to obtain that, for all  $i = 1, \dots, d$ ,

$$\frac{\partial f(x)}{\partial y^i} - \frac{\partial f(\bar{x})}{\partial y^i} = \sum_{j=1}^d \frac{\partial^2 f((qx + (1-q)\bar{x}))}{\partial y^i \partial y^j} (x^j - \bar{x}^j),$$

for some  $q \in (0, 1)$ . Then for a fixed  $q \in (0, 1)$ ,

$$\begin{aligned} & \left| \frac{\partial f(x)}{\partial y^i} - \frac{\partial f(\bar{x})}{\partial y^i} - \sum_{j=1}^d \frac{\partial^2 f(\bar{x})}{\partial y^i \partial y^j} (x^j - \bar{x}^j) \right| \\ &= \left| \sum_{j=1}^d \frac{\partial^2 f((qx + (1-q)\bar{x}))}{\partial y^i \partial y^j} (x^j - \bar{x}^j) - \sum_{j=1}^d \frac{\partial^2 f(\bar{x})}{\partial y^i \partial y^j} (x^j - \bar{x}^j) \right| \\ &\leq \sum_{j=1}^d \left| \frac{\partial^2 f((qx + (1-q)\bar{x}))}{\partial y^i \partial y^j} - \frac{\partial^2 f(\bar{x})}{\partial y^i \partial y^j} \right| |x^j - \bar{x}^j| \\ &\leq K(1 + |x| + |\bar{x}|)^\alpha |x - \bar{x}|^{1+\beta}. \end{aligned}$$

□

**Lemma 5.** *Assume A-1 to A-5 hold, then, for every  $n \in \mathbb{N}$ ,*

$$\begin{aligned} \sup_{0 \leq t \leq T} E |b_1^n(t, x_{\kappa(n,t)}^n)|^p &\leq Cn^{-p}, \\ \sup_{0 \leq t \leq T} E |b_2^n(t, x_{\kappa(n,t)}^n)|^p &\leq Cn^{-\frac{p}{2}}, \\ \sup_{0 \leq t \leq T} E |\sigma_1^n(t, x_{\kappa(n,t)}^n)|^p &\leq Cn^{-\frac{p}{2}}, \\ \sup_{0 \leq t \leq T} E |\sigma_2^n(t, x_{\kappa(n,t)}^n)|^p &\leq Cn^{-p}, \\ \sup_{0 \leq t \leq T} E |\sigma_3^n(t, x_{\kappa(n,t)}^n)|^p &\leq Cn^{-p}, \end{aligned}$$

for any  $p \leq \frac{p_0}{2\rho+1}$ .

*Proof.* By applying Hölder's inequality and Remark 1, one obtains

$$\begin{aligned} E |b_2^n(t, x_{\kappa(n,t)}^n)|^p &\leq E \left| \int_{\kappa(n,t)}^t L^{n,0} b(x_{\kappa(n,s)}^n) ds \right|^p \\ &\leq Cn^{-p+1} \int_{\kappa(n,t)}^t E |L^{n,0} b(x_{\kappa(n,s)}^n)|^p ds \\ &\leq Cn^{-p+1} \int_{\kappa(n,t)}^t E (1 + |x_{\kappa(n,s)}^n|)^{(2\rho+1)p} ds \\ &\leq Cn^{-p}, \end{aligned}$$

due to Lemma 3. Other results can be proved by using similar arguments. □

**Corollary 3.** *Assume A-1 to A-5 hold, then, for every  $n \in \mathbb{N}$ ,*

$$\begin{aligned} \sup_{0 \leq t \leq T} E |\tilde{b}^n(t, x_{\kappa(n,t)}^n)|^p &\leq C, \\ \sup_{0 \leq t \leq T} E |\tilde{\sigma}^n(t, x_{\kappa(n,t)}^n)|^p &\leq C, \end{aligned}$$

for any  $p \leq \frac{p_0}{2\rho+1}$ .

**Lemma 6.** Assume **A-1** to **A-5** hold, then, for every  $n \in \mathbb{N}$ ,

$$\sup_{0 \leq t \leq T} E|x_t^n - x_{\kappa(n,t)}^n|^p \leq Cn^{-\frac{p}{2}}$$

for any  $p \leq \frac{p_0}{2\rho+1}$ .

*Proof.* One writes

$$\begin{aligned} E|x_t^n - x_{\kappa(n,t)}^n|^p &\leq CE \left| \int_{\kappa(n,t)}^t \tilde{b}^n(s, x_{\kappa(n,s)}^n) ds \right|^p + CE \left| \int_{\kappa(n,t)}^t \tilde{\sigma}^n(s, x_{\kappa(n,s)}^n) dw_s \right|^p \\ &\leq n^{-p+1} CE \int_{\kappa(n,t)}^t |\tilde{b}^n(s, x_{\kappa(n,s)}^n)|^p ds \\ &\quad + Cn^{-\frac{p}{2}+1} E \int_{\kappa(n,t)}^t |\tilde{\sigma}^n(s, x_{\kappa(n,s)}^n)|^p ds, \end{aligned}$$

which on the application of corollary 3 completes the proof.  $\square$

**Lemma 7.** Assume **A-1** to **A-5** hold, then, for every  $n \in \mathbb{N}$ ,

$$\sup_{0 \leq t \leq T} E|b(x_{\kappa(n,t)}^n) - b^n(x_{\kappa(n,t)}^n)|^p \leq Cn^{-\frac{3}{2}p},$$

$$\sup_{0 \leq t \leq T} E|\sigma(x_{\kappa(n,t)}^n) - \sigma^n(x_{\kappa(n,t)}^n)|^p \leq Cn^{-\frac{3}{2}p},$$

for any  $p \leq \frac{p_0}{4\rho+1}$ .

*Proof.*

$$|b(x_{\kappa(n,t)}^n) - b^n(x_{\kappa(n,t)}^n)| = n^{-\frac{3}{2}} \frac{|x_{\kappa(n,t)}^n|^{3\rho} |b(x_{\kappa(n,t)}^n)|}{1 + n^{-\frac{3}{2}} |x_{\kappa(n,t)}^n|^{3\rho}} \leq n^{-\frac{3}{2}} (1 + |x_{\kappa(n,t)}^n|)^{4\rho+1},$$

which, by using Lemma 3 and the same argument for  $\sigma$  completes the proof.  $\square$

**Lemma 8.** Assume **A-1** to **A-5** hold, then, for every  $n \in \mathbb{N}$  and  $t \in [0, T]$ ,

$$\sup_{0 \leq t \leq T} E|\sigma(x_t^n) - \sigma(x_{\kappa(n,t)}^n) - \sigma_M^n(t, x_{\kappa(n,t)}^n)|^2 \leq Cn^{-(2+\beta)},$$

for any  $p_0 \geq 2(5\rho + 1)$ .

*Proof.* For every  $k = 1, \dots, d$ ,  $v = 1, \dots, m$ , applying Itô's formula to  $\sigma^{(k,v)}(x_t^n) - \sigma^{(k,v)}(x_{\kappa(n,t)}^n)$  gives

$$\begin{aligned} &\sigma^{(k,v)}(x_t^n) - \sigma^{(k,v)}(x_{\kappa(n,t)}^n) \\ &= \sum_{i=1}^d \int_{\kappa(n,t)}^t \frac{\partial \sigma^{(k,v)}(x_s^n)}{\partial x^i} \tilde{b}^{n,i}(s, x_{\kappa(n,s)}^n) ds \\ &\quad + \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial \sigma^{(k,v)}(x_s^n)}{\partial x^i} \tilde{\sigma}^{n,(i,j)}(s, x_{\kappa(n,s)}^n) dw_s^j \\ &\quad + \frac{1}{2} \sum_{i,l=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial^2 \sigma^{(k,v)}(x_s^n)}{\partial x^i \partial x^l} \tilde{\sigma}^{n,(i,j)}(s, x_{\kappa(n,s)}^n) \tilde{\sigma}^{n,(l,j)}(s, x_{\kappa(n,s)}^n) ds, \end{aligned}$$

which can be further expanded as

$$\begin{aligned}
& \sigma^{(k,v)}(x_t^n) - \sigma^{(k,v)}(x_{\kappa(n,t)}^n) \\
&= \sum_{i=1}^d \int_{\kappa(n,t)}^t \left( \frac{\partial \sigma^{(k,v)}(x_s^n)}{\partial x^i} - \frac{\partial \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i} \right) b^{n,i}(x_{\kappa(n,s)}^n) ds \\
&+ \sum_{i=1}^d \int_{\kappa(n,t)}^t \frac{\partial \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i} b^{n,i}(x_{\kappa(n,s)}^n) ds \\
&+ \sum_{i=1}^d \int_{\kappa(n,t)}^t \frac{\partial \sigma^{(k,v)}(x_s^n)}{\partial x^i} (b_1^{n,i}(s, x_{\kappa(n,s)}^n) + b_2^{n,i}(s, x_{\kappa(n,s)}^n)) ds \\
&+ \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \left( \frac{\partial \sigma^{(k,v)}(x_s^n)}{\partial x^i} - \frac{\partial \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i} \right. \\
&\quad \left. - \sum_{l=1}^d \frac{\partial^2 \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i \partial x^l} (x_s^{n,i} - x_{\kappa(n,s)}^{n,i}) \right) \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) dw_s^j \\
&+ \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \sum_{l=1}^d \frac{\partial^2 \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i \partial x^l} \left( \int_{\kappa(n,s)}^s \tilde{b}^{n,i}(r, x_{\kappa(n,r)}^n) dr \right. \\
&\quad \left. + \sum_{j_1=1}^m \int_{\kappa(n,s)}^s \sigma_M^{n,(i,j_1)}(r, x_{\kappa(n,r)}^n) dw_r^{j_1} \right) \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) dw_s^j \\
&+ \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \sum_{l=1}^d \frac{\partial^2 \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i \partial x^l} \sum_{j_1=1}^m \int_{\kappa(n,s)}^s \sigma^{n,(i,j_1)}(x_{\kappa(n,r)}^n) dw_r^{j_1} \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) dw_s^j \\
&+ \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \left( \frac{\partial \sigma^{(k,v)}(x_s^n)}{\partial x^i} - \frac{\partial \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i} \right) \sigma_1^{n,(i,j)}(s, x_{\kappa(n,s)}^n) dw_s^j \\
&+ \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i} (\sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) + \sigma_1^{n,(i,j)}(s, x_{\kappa(n,s)}^n)) dw_s^j \\
&+ \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial \sigma^{(k,v)}(x_s^n)}{\partial x^i} (\sigma_2^{n,(i,j)}(s, x_{\kappa(n,s)}^n) + \sigma_3^{n,(i,j)}(s, x_{\kappa(n,s)}^n)) dw_s^j \\
&+ \frac{1}{2} \sum_{i,l=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \left( \frac{\partial^2 \sigma^{(k,v)}(x_s^n)}{\partial x^i \partial x^l} - \frac{\partial^2 \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i \partial x^l} \right) \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma^{n,(l,j)}(x_{\kappa(n,s)}^n) ds \\
&+ \frac{1}{2} \sum_{i,l=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial^2 \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i \partial x^l} \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma^{n,(l,j)}(x_{\kappa(n,s)}^n) ds \\
&+ \frac{1}{2} \sum_{i,l=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial^2 \sigma^{(k,v)}(x_s^n)}{\partial x^i \partial x^l} (\sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma_M^{n,(l,j)}(s, x_{\kappa(n,s)}^n) \\
&\quad + \sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n) \tilde{\sigma}^{n,(l,j)}(s, x_{\kappa(n,s)}^n)) ds \\
&:= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 + J_9 + J_{10} + J_{11} + J_{12}.
\end{aligned}$$

It can be observed that

$$\begin{aligned}
& E|J_2 + J_6 + J_8 + J_{11} - \sigma_M^{n,(i,j)}(t, x_{\kappa(n,t)}^n)|^2 \\
& \leq E|J_2 + J_{11} - \sigma_2^{n,(i,j)}(t, x_{\kappa(n,t)}^n)|^2 + E|J_6 + J_8 - \sigma_1^{n,(i,j)}(t, x_{\kappa(n,t)}^n) - \sigma_3^{n,(i,j)}(t, x_{\kappa(n,t)}^n)|^2 \\
& \leq \sum_{i,l=1}^d \sum_{j=1}^m E \left| -\frac{n^{-3/2}|x_{\kappa(n,s)}^n|^{3\rho}}{(1+n^{-3/2}|x_{\kappa(n,s)}^n|^{3\rho})^2} \int_{\kappa(n,t)}^t \frac{\partial^2 \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i \partial x^l} \sigma^{(i,j)}(x_{\kappa(n,s)}^n) \sigma^{(l,j)}(x_{\kappa(n,s)}^n) ds \right|^2 \\
& + E \left| -\frac{n^{-3/2}|x_{\kappa(n,s)}^n|^{3\rho}}{(1+n^{-3/2}|x_{\kappa(n,s)}^n|^{3\rho})^2} \sum_{i,l=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial^2 \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i \partial x^l} \right. \\
& \quad \left. \times \int_{\kappa(n,s)}^s \sigma^{n,(i,j_1)}(x_{\kappa(n,r)}^n) dw_r^{j_1} \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) dw_s^j \right|^2,
\end{aligned}$$

which implies due to Remark 1 and Lemma 3 that

$$\begin{aligned}
E|J_2 + J_6 + J_8 + J_{11} - \sigma_M^{n,(i,j)}(t, x_{\kappa(n,t)}^n)|^2 & \leq Cn^{-3} E|n^{-1}|x_{\kappa(n,s)}^n|^{3\rho}(1+|x_{\kappa(n,s)}^n|^{3/2\rho+1})|^2 \\
& + Cn^{-5} E||x_{\kappa(n,s)}^n|^{3\rho}(1+|x_{\kappa(n,s)}^n|^{3/2\rho+1})|^2 \\
& \leq Cn^{-5},
\end{aligned}$$

for  $p_0 \geq 9\rho + 2$ . Then, one writes

$$\begin{aligned}
E|\sigma(x_t^n) - \sigma(x_{\kappa(n,t)}^n) - \sigma_M^n(t, x_{\kappa(n,t)}^n)|^2 & \leq E|J_1 + J_3 + J_4 + J_5 + J_7 + J_9 + J_{10} + J_{12}|^2 \\
& + E|J_2 + J_6 + J_8 + J_{11} - \sigma_M^{n,(i,j)}(t, x_{\kappa(n,t)}^n)|^2,
\end{aligned}$$

for any  $t \in [0, T]$ . For  $E|J_1|^2$ , by using Cauchy–Schwarz inequality, it can be estimated as

$$\begin{aligned}
E|J_1|^2 & := E \left| \sum_{i=1}^d \int_{\kappa(n,t)}^t \left( \frac{\partial \sigma^{(k,v)}(x_s^n)}{\partial x^i} - \frac{\partial \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i} \right) b^{n,i}(x_{\kappa(n,s)}^n) ds \right|^2 \\
& \leq n^{-1} \sum_{i=1}^d \int_{\kappa(n,t)}^t E \left| \left( \frac{\partial \sigma^{(k,v)}(x_s^n)}{\partial x^i} - \frac{\partial \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i} \right) b^n(x_{\kappa(n,s)}^n) \right|^2 ds,
\end{aligned}$$

which on the application of Young’s inequality, Remark 1 and Hölder’s inequality yields

$$\begin{aligned}
E|J_1|^2 & \leq Cn^{-1} \int_{\kappa(n,t)}^t E(1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{\rho-4+2\beta} (1 + |x_{\kappa(n,s)}^n|)^{2\rho+2} |x_s^n - x_{\kappa(n,s)}^n|^2 ds \\
& \leq Cn^{-1} \int_{\kappa(n,t)}^t E(1 + |x_s^n|^{3\rho} + |x_{\kappa(n,s)}^n|^{3\rho}) |x_s^n - x_{\kappa(n,s)}^n|^2 ds \\
& \leq Cn^{-1} \int_{\kappa(n,t)}^t \left( E(1 + |x_s^n|^{p_0} + |x_{\kappa(n,s)}^n|^{p_0}) \right)^{\frac{3\rho}{p_0}} \left( E|x_s^n - x_{\kappa(n,s)}^n|^{\frac{2p_0}{p_0-3\rho}} \right)^{\frac{p_0-3\rho}{p_0}} ds,
\end{aligned}$$

for any  $t \in [0, T]$ . One uses Lemma 3 and Lemma 6 to obtain

$$E|J_1|^2 \leq Cn^{-3},$$

for every  $n \in \mathbb{N}$ . To estimate  $E|J_3|^2$ , one applies Cauchy–Schwarz inequality and Remark 1 to obtain

$$\begin{aligned}
E|J_3|^2 & := E \left| \sum_{i=1}^d \int_{\kappa(n,t)}^t \frac{\partial \sigma^{(k,v)}(x_s^n)}{\partial x^i} (b_1^{n,i}(s, x_{\kappa(n,s)}^n) + b_2^{n,i}(s, x_{\kappa(n,s)}^n)) ds \right|^2 \\
& \leq Cn^{-1} \int_{\kappa(n,t)}^t E(1 + |x_s^n|)^{\rho-2+2\beta} (|b_1^n(s, x_{\kappa(n,s)}^n)|^2 + |b_2^n(s, x_{\kappa(n,s)}^n)|^2) ds,
\end{aligned}$$

which implies due to Hölder's inequality

$$E|J_3|^2 \leq Cn^{-1} \int_{\kappa(n,t)}^t (E(1 + |x_s^n|^{p_0}))^{\frac{\rho}{p_0}} (E(|b_1^n(s, x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-\rho}} + |b_2^n(s, x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-\rho}}))^{\frac{p_0-\rho}{p_0}} ds,$$

for any  $t \in [0, T]$ . By Lemma 3 and Lemma 5, it becomes

$$E|J_3|^2 \leq Cn^{-3},$$

for every  $n \in \mathbb{N}$ . As for  $E|J_4|^2$ , by using Young's inequality, Cauchy–Schwarz inequality, Remark 1 and Lemma 4,

$$\begin{aligned} E|J_4|^2 &:= E \left| \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \left( \frac{\partial \sigma^{(k,v)}(x_s^n)}{\partial x^i} - \frac{\partial \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i} \right. \right. \\ &\quad \left. \left. - \sum_{l=1}^d \frac{\partial^2 \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i \partial x^l} (x_s^{n,i} - x_{\kappa(n,s)}^{n,i}) \right) \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) dw_s^j \right|^2 \\ &\leq C \int_{\kappa(n,t)}^t E((1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{\rho-4} (1 + |x_{\kappa(n,s)}^n|)^{\rho+2\beta} |x_s^n - x_{\kappa(n,s)}^n|^{2+2\beta}) ds \\ &\leq C \int_{\kappa(n,t)}^t E((1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{2\rho-2} |x_s^n - x_{\kappa(n,s)}^n|^{2+2\beta}) ds, \end{aligned}$$

which implies due to Hölder's inequality

$$E|J_4|^2 \leq C \int_{\kappa(n,t)}^t \left( E(1 + |x_s^n|^{p_0} + |x_{\kappa(n,s)}^n|^{p_0}) \right)^{\frac{2\rho-2}{p_0}} \left( E|x_s^n - x_{\kappa(n,s)}^n|^{\frac{(2+2\beta)p_0}{p_0-2\rho+2}} \right)^{\frac{p_0-2\rho+2}{p_0}} ds,$$

for any  $t \in [0, T]$ . Then, on the application of Lemma 6 and Lemma 3 yields

$$E|J_4|^2 \leq Cn^{-(2+\beta)},$$

for every  $n \in \mathbb{N}$ . In order to estimate  $E|J_5|^2$ , one uses Young's inequality and Cauchy–Schwarz inequality to obtain

$$\begin{aligned} E|J_5|^2 &:= E \left| \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \sum_{l=1}^d \frac{\partial^2 \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i \partial x^l} \left( \int_{\kappa(n,s)}^s \tilde{b}^{n,i}(r, x_{\kappa(n,r)}^n) dr \right. \right. \\ &\quad \left. \left. + \sum_{j_1=1}^m \int_{\kappa(n,s)}^s \sigma_M^{n,(i,j_1)}(r, x_{\kappa(n,r)}^n) dw_r^{j_1} \right) \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) dw_s^j \right|^2 \\ &\leq C \int_{\kappa(n,t)}^t E \left| \int_{\kappa(n,s)}^s |\tilde{b}^n(r, x_{\kappa(n,r)}^n)| dr + \sum_{j_1=1}^m \int_{\kappa(n,s)}^s \sigma_M^{n,(i,j_1)}(r, x_{\kappa(n,r)}^n) dw_r^{j_1} \right|^2 \\ &\quad \times (1 + |x_{\kappa(n,s)}^n|)^{2\rho+4\beta-4} ds \\ &\leq C \int_{\kappa(n,t)}^t E \left| \int_{\kappa(n,s)}^s |\tilde{b}^n(r, x_{\kappa(n,r)}^n)| dr + \sum_{j_1=1}^m \int_{\kappa(n,s)}^s \sigma_M^{n,(i,j_1)}(r, x_{\kappa(n,r)}^n) dw_r^{j_1} \right|^2 \\ &\quad \times (1 + |x_{\kappa(n,s)}^n|)^{2\rho} ds, \end{aligned}$$

which yields on the application of Hölder's inequality

$$E|J_5|^2 \leq C \int_{\kappa(n,t)}^t \left( n^{-\frac{2p_0}{p_0-2\rho}+1} \int_{\kappa(n,s)}^s E|\tilde{b}^n(r, x_{\kappa(n,r)}^n)|^{\frac{2p_0}{p_0-2\rho}} ds \right. \\ \left. + n^{-\frac{p_0}{p_0-2\rho}+1} \int_{\kappa(n,s)}^s E|\sigma_M^n(r, x_{\kappa(n,r)}^n)|^{\frac{2p_0}{p_0-2\rho}} ds \right)^{\frac{p_0-2\rho}{p_0}} \left( E(1 + |x_{\kappa(n,s)}^n|^{p_0}) \right)^{\frac{2\rho}{p_0}} ds,$$

for any  $t \in [0, T]$ . One uses Corollary 3 and Lemma 5 to obtain

$$E|J_5|^2 \leq Cn^{-3},$$

for every  $n \in \mathbb{N}$ . As for  $E|J_7|^2$ , it can be estimated by using Cauchy–Schwarz inequality

$$E|J_7|^2 := E \left| \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \left( \frac{\partial \sigma^{(k,v)}(x_s^n)}{\partial x^i} - \frac{\partial \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i} \right) \sigma_1^{n,(i,j)}(s, x_{\kappa(n,s)}^n) dw_s^j \right|^2 \\ \leq \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t E \left| \frac{\partial \sigma^{(k,v)}(x_s^n)}{\partial x^i} - \frac{\partial \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i} \right|^2 |\sigma_1^n(s, x_{\kappa(n,s)}^n)|^2 ds,$$

which, for  $\rho > 2$ , yields by using Remark 1 and Hölder's inequality

$$E|J_7|^2 \leq C \int_{\kappa(n,t)}^t E(1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{\rho-4+2\beta} |x_s^n - x_{\kappa(n,s)}^n|^2 |\sigma_1^n(s, x_{\kappa(n,s)}^n)|^2 ds \\ \leq C \int_{\kappa(n,t)}^t \left( E(1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{p_0} \right)^{\frac{\rho-2}{p_0}} \\ \times \left( E|x_s^n - x_{\kappa(n,s)}^n|^{\frac{2p_0}{p_0-\rho+2}} |\sigma_1^n(s, x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-\rho+2}} \right)^{\frac{p_0-\rho+2}{p_0}} ds,$$

for any  $t \in [0, T]$ . Then, one needs to apply Cauchy–Schwarz inequality and Lemma 3 to obtain

$$E|J_7|^2 \leq C \int_{\kappa(n,t)}^t \left( E|x_s^n - x_{\kappa(n,s)}^n|^{\frac{4p_0}{p_0-\rho+2}} E|\sigma_1^n(s, x_{\kappa(n,s)}^n)|^{\frac{4p_0}{p_0-\rho+2}} \right)^{\frac{p_0-\rho+2}{2p_0}} ds,$$

Thus, by using Lemma 6 and Lemma 5

$$E|J_7|^2 \leq Cn^{-3},$$

for every  $n \in \mathbb{N}$ . Note that, for the case that  $\rho = 2$ , one obtains the same result immediately by using Cauchy–Schwarz inequality. As for  $E|J_9|^2$ , applying Remark 1 gives

$$E|J_9|^2 := E \left| \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial \sigma^{(k,v)}(x_s^n)}{\partial x^i} (\sigma_2^{n,(i,j)}(s, x_{\kappa(n,s)}^n) + \sigma_3^{n,(i,j)}(s, x_{\kappa(n,s)}^n)) dw_s^j \right|^2 \\ \leq C \int_{\kappa(n,t)}^t E(1 + |x_s^n|)^{\rho-2+2\beta} (|\sigma_2^n(s, x_{\kappa(n,s)}^n)|^2 + |\sigma_3^n(s, x_{\kappa(n,s)}^n)|^2) ds \\ \leq C \int_{\kappa(n,t)}^t E(1 + |x_s^n|)^{\rho} (|\sigma_2^n(s, x_{\kappa(n,s)}^n)|^2 + |\sigma_3^n(s, x_{\kappa(n,s)}^n)|^2) ds,$$

which on the application of Hölder's inequality gives

$$E|J_9|^2 \leq C \int_{\kappa(n,t)}^t (E(1 + |x_s^n|^{p_0}))^{\frac{\rho}{p_0}} (E(|\sigma_2^n(s, x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-\rho}} + |\sigma_3^n(s, x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-\rho}}))^{\frac{p_0-\rho}{p_0}} ds,$$

for any  $t \in [0, T]$ . By Lemma 5, one obtains

$$E|J_9|^2 \leq Cn^{-3},$$

for every  $n \in \mathbb{N}$ . To estimate  $E|J_{10}|^2$ , by Young's inequality and Remark 1

$$\begin{aligned} E|J_{10}|^2 &:= E \left| \frac{1}{2} \sum_{i,l=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \left( \frac{\partial^2 \sigma^{(k,v)}(x_s^n)}{\partial x^i \partial x^l} - \frac{\partial^2 \sigma^{(k,v)}(x_{\kappa(n,s)}^n)}{\partial x^i \partial x^l} \right) \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma^{n,(l,j)}(x_{\kappa(n,s)}^n) ds \right|^2 \\ &\leq Cn^{-1} \int_{\kappa(n,t)}^t E(1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{3\rho+4\beta-4} |x_s^n - x_{\kappa(n,s)}^n|^{2\beta} ds \\ &\leq Cn^{-1} \int_{\kappa(n,t)}^t E(1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{3\rho} |x_s^n - x_{\kappa(n,s)}^n|^{2\beta} ds, \end{aligned}$$

which implies due to Hölder's inequality

$$E|J_{10}|^2 \leq Cn^{-1} \int_{\kappa(n,t)}^t \left( E(1 + |x_{\kappa(n,s)}^n|^{p_0}) \right)^{\frac{3\rho}{p_0}} \left( E|x_s^n - x_{\kappa(n,s)}^n|^{\frac{2\beta p_0}{p_0-3\rho}} \right)^{\frac{p_0-3\rho}{p_0}} ds,$$

for any  $t \in [0, T]$ . Lemma 6 is used to obtain

$$E|J_{10}|^2 \leq Cn^{-(2+\beta)},$$

for every  $n \in \mathbb{N}$ . Finally for  $E|J_{12}|^2$ , on the application of Young's inequality, Cauchy-Schwarz inequality and Remark 1 gives

$$\begin{aligned} E|J_{12}|^2 &:= E \left| \frac{1}{2} \sum_{i,l=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial^2 \sigma^{(k,v)}(x_s^n)}{\partial x^i \partial x^l} (\sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma_M^{n,(l,j)}(s, x_{\kappa(n,s)}^n) \right. \\ &\quad \left. + \sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n) \tilde{\sigma}^{n,(l,j)}(s, x_{\kappa(n,s)}^n) \right) ds \Big|^2 \\ &\leq Cn^{-1} \int_{\kappa(n,t)}^t E(1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{2\rho+4\beta-4} |\sigma_M^n(s, x_{\kappa(n,s)}^n)|^2 ds \\ &\quad + Cn^{-1} \int_{\kappa(n,t)}^t E(1 + |x_s^n|)^{\rho-4+2\beta} |\tilde{\sigma}^n(x_{\kappa(n,s)}^n)|^2 |\sigma_M^n(s, x_{\kappa(n,s)}^n)|^2 ds, \end{aligned}$$

which implies due to Hölder's inequality

$$\begin{aligned} E|J_{12}|^2 &\leq Cn^{-1} \int_{\kappa(n,t)}^t \left( E(1 + |x_s^n|^{p_0} + |x_{\kappa(n,s)}^n|^{p_0}) \right)^{\frac{2\rho}{p_0}} \left( E|\sigma_M^n(s, x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-2\rho}} \right)^{\frac{p_0-2\rho}{p_0}} ds \\ &\quad + Cn^{-1} \int_{\kappa(n,t)}^t (E(1 + |x_s^n|^{p_0}))^{\frac{\rho-2}{p_0}} \left( E|\tilde{\sigma}^n(x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-\rho+2}} |\sigma_M^n(x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-\rho+2}} \right)^{\frac{p_0-\rho+2}{p_0}} ds, \end{aligned}$$

for any  $t \in [0, T]$ . By applying Lemma 3 and Lemma 5 to the first term and Cauchy-Schwarz inequality to the second term gives

$$E|J_{12}|^2 \leq Cn^{-3} + Cn^{-1} \int_{\kappa(n,t)}^t \left( E|\tilde{\sigma}^n(x_{\kappa(n,s)}^n)|^{\frac{4p_0}{p_0-\rho+2}} E|\sigma_M^n(x_{\kappa(n,s)}^n)|^{\frac{4p_0}{p_0-\rho+2}} \right)^{\frac{p_0-\rho+2}{2p_0}} ds,$$



which by using Lemma 5 yields the desired result, i.e.

$$E|J_{12}|^2 \leq Cn^{-3},$$

for every  $n \in \mathbb{N}$ . Therefore,

$$\sup_{0 \leq t \leq T} E|\sigma(x_t^n) - \sigma(x_{\kappa(n,t)}^n) - \sigma_M^n(t, x_{\kappa(n,t)}^n)|^2 \leq Cn^{-(2+\beta)} + Cn^{-3} + Cn^{-5} \leq Cn^{-(2+\beta)},$$

for every  $n \in \mathbb{N}$ ,  $\beta \in (0, 1]$  and  $p_0 \geq 10\rho + 2$ . □

**Lemma 9.** *Assume A-1 to A-5 hold, then, for every  $n \in \mathbb{N}$  and  $t \in [0, T]$ ,*

$$\sup_{0 \leq t \leq T} E|b(x_t^n) - b(x_{\kappa(n,t)}^n) - b_1^n(t, x_{\kappa(n,t)}^n) - b_2^n(t, x_{\kappa(n,t)}^n)|^2 \leq Cn^{-2},$$

for any  $p_0 \geq 2(5\rho + 1)$ .

*Proof.* For every  $k = 1, \dots, d$ , applying Itô's formula to  $b^k(x_t^n) - b^k(x_{\kappa(n,t)}^n)$  gives

$$\begin{aligned} & b^k(x_t^n) - b^k(x_{\kappa(n,t)}^n) \\ &= \sum_{i=1}^d \int_{\kappa(n,t)}^t \frac{\partial b^k(x_s^n)}{\partial x^i} \tilde{b}^{n,i}(s, x_{\kappa(n,s)}^n) ds + \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial b^k(x_s^n)}{\partial x^i} \tilde{\sigma}^{n,(i,j)}(s, x_{\kappa(n,s)}^n) dw_s^j \\ &+ \frac{1}{2} \sum_{i,l=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial^2 b^k(x_s^n)}{\partial x^i \partial x^l} \tilde{\sigma}^{n,(i,j)}(s, x_{\kappa(n,s)}^n) \tilde{\sigma}^{n,(l,j)}(s, x_{\kappa(n,s)}^n) ds, \end{aligned}$$

which can be further written as

$$\begin{aligned}
& b^k(x_t^n) - b^k(x_{\kappa(n,t)}^n) \\
&= \sum_{i=1}^d \int_{\kappa(n,t)}^t \left( \frac{\partial b^k(x_s^n)}{\partial x^i} - \frac{\partial b^k(x_{\kappa(n,s)}^n)}{\partial x^i} \right) b^{n,i}(x_{\kappa(n,s)}^n) ds \\
&+ \sum_{i=1}^d \int_{\kappa(n,t)}^t \frac{\partial b^k(x_{\kappa(n,s)}^n)}{\partial x^i} b^{n,i}(x_{\kappa(n,s)}^n) ds \\
&+ \sum_{i=1}^d \int_{\kappa(n,t)}^t \frac{\partial b^k(x_s^n)}{\partial x^i} (b_1^{n,i}(s, x_{\kappa(n,s)}^n) + b_2^{n,i}(s, x_{\kappa(n,s)}^n)) ds \\
&+ \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \left( \frac{\partial b^k(x_s^n)}{\partial x^i} - \frac{\partial b^k(x_{\kappa(n,s)}^n)}{\partial x^i} \right) \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) dw_s^j \\
&+ \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial b^k(x_{\kappa(n,s)}^n)}{\partial x^i} \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) dw_s^j \\
&+ \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial b^k(x_s^n)}{\partial x^i} \sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n) dw_s^j \\
&+ \frac{1}{2} \sum_{i,l=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \left( \frac{\partial^2 b^k(x_s^n)}{\partial x^i \partial x^l} - \frac{\partial^2 b^k(x_{\kappa(n,s)}^n)}{\partial x^i \partial x^l} \right) \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma^{n,(l,j)}(x_{\kappa(n,s)}^n) ds \\
&+ \frac{1}{2} \sum_{i,l=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial^2 b^k(x_{\kappa(n,s)}^n)}{\partial x^i \partial x^l} \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma^{n,(l,j)}(x_{\kappa(n,s)}^n) ds \\
&+ \frac{1}{2} \sum_{i,l=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial^2 b^k(x_s^n)}{\partial x^i \partial x^l} (\sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma_M^{n,(l,j)}(s, x_{\kappa(n,s)}^n) \\
&\quad + \sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n) \tilde{\sigma}^{n,(l,j)}(s, x_{\kappa(n,s)}^n)) ds \\
&:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9.
\end{aligned} \tag{4.1}$$

Note that

$$\begin{aligned}
& E|I_2 + I_8 - b_1^{n,(i,j)}(t, x_{\kappa(n,t)}^n)|^2 \\
&\leq \frac{1}{2} \sum_{i,l=1}^d \sum_{j=1}^m E \left| -\frac{n^{-3/2} |x_{\kappa(n,s)}^n|^{3\rho}}{(1 + n^{-3/2} |x_{\kappa(n,s)}^n|^{3\rho})^2} \int_{\kappa(n,t)}^t \frac{\partial^2 b^k(x_{\kappa(n,s)}^n)}{\partial x^i \partial x^l} \sigma^{(i,j)}(x_{\kappa(n,s)}^n) \sigma^{(l,j)}(x_{\kappa(n,s)}^n) ds \right|^2,
\end{aligned}$$

which on the application of Remark 1 and Lemma 3 yields

$$E|I_2 + I_8 - b_1^{n,(i,j)}(t, x_{\kappa(n,t)}^n)|^2 \leq C n^{-5} E |x_{\kappa(n,s)}^n|^{3\rho} (1 + |x_{\kappa(n,s)}^n|^{2\rho+2\beta-1})^2 \leq C n^{-5}, \tag{4.2}$$

for  $p_0 \geq 10\rho + 2$ . Moreover, notice that

$$I_5 = b_2^{n,(i,j)}(t, x_{\kappa(n,t)}^n). \tag{4.3}$$

Then,

$$\begin{aligned}
E|b(x_t^n) - b(x_{\kappa(n,t)}^n) - b_1^n(t, x_{\kappa(n,t)}^n) - b_2^n(t, x_{\kappa(n,t)}^n)|^2 &\leq E|I_1 + I_3 + I_4 + I_6 + I_7 + I_9|^2 \\
&\quad + E|I_2 + I_8 - b_1^{n,(i,j)}(t, x_{\kappa(n,t)}^n)|^2,
\end{aligned}$$

for any  $t \in [0, T]$ . To estimate  $E|I_1|^2$ , one uses Cauchy–Schwarz inequality and Remark 1 to obtain

$$\begin{aligned} E|I_1|^2 &:= E \left| \sum_{i=1}^d \int_{\kappa(n,t)}^t \left( \frac{\partial b^k(x_s^n)}{\partial x^i} - \frac{\partial b^k(x_{\kappa(n,s)}^n)}{\partial x^i} \right) b^{n,i}(x_{\kappa(n,s)}^n) ds \right|^2 \\ &\leq Cn^{-1} \int_{\kappa(n,t)}^t E(1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{2\rho-2} (1 + |x_{\kappa(n,s)}^n|)^{2\rho+2} |x_s^n - x_{\kappa(n,s)}^n|^2 ds, \end{aligned}$$

which further implies due to Young’s inequality and Hölder’s inequality

$$E|I_1|^2 \leq Cn^{-1} \int_{\kappa(n,t)}^t \left( E(1 + |x_s^n|^{p_0} + |x_{\kappa(n,s)}^n|^{p_0}) \right)^{\frac{4\rho}{p_0}} \left( E|x_s^n - x_{\kappa(n,s)}^n|^{\frac{2p_0}{p_0-4\rho}} \right)^{\frac{p_0-4\rho}{p_0}} ds,$$

for any  $t \in [0, T]$ . By Lemma 6, one obtains

$$E|I_1|^2 \leq Cn^{-3},$$

for any  $n \in \mathbb{N}$ . As for  $E|I_3|^2$ , applying Cauchy–Schwarz inequality and Remark 1 gives

$$\begin{aligned} E|I_3|^2 &:= E \left| \sum_{i=1}^d \int_{\kappa(n,t)}^t \frac{\partial b^k(x_s^n)}{\partial x^i} (b_1^{n,i}(s, x_{\kappa(n,s)}^n) + b_2^{n,i}(s, x_{\kappa(n,s)}^n)) ds \right|^2 \\ &\leq Cn^{-1} \int_{\kappa(n,t)}^t E(1 + |x_s^n|^{2\rho}) (|b_1^n(s, x_{\kappa(n,s)}^n)|^2 + |b_2^n(s, x_{\kappa(n,s)}^n)|^2) ds, \end{aligned}$$

then one writes by using Hölder’s inequality that

$$E|I_3|^2 \leq Cn^{-1} \int_{\kappa(n,t)}^t (E(1 + |x_s^n|^{p_0}))^{\frac{2\rho}{p_0}} (E|b_1^n(s, x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-2\rho}} + E|b_2^n(s, x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-2\rho}})^{\frac{p_0-2\rho}{p_0}} ds,$$

for any  $t \in [0, T]$ . On the application of Lemma 5 yields

$$E|I_3|^2 \leq Cn^{-3},$$

for any  $n \in \mathbb{N}$ . Next, Cauchy–Schwarz inequality and Remark 1 are used to estimate  $E|I_4|^2$ ,

$$\begin{aligned} E|I_4|^2 &:= E \left| \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \left( \frac{\partial b^k(x_s^n)}{\partial x^i} - \frac{\partial b^k(x_{\kappa(n,s)}^n)}{\partial x^i} \right) \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) dw_s^j \right|^2 \\ &\leq C \int_{\kappa(n,t)}^t E(1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{2\rho-2} (1 + |x_{\kappa(n,s)}^n|)^{\rho+2\beta} |x_s^n - x_{\kappa(n,s)}^n|^2 ds \\ &\leq C \int_{\kappa(n,t)}^t E(1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{3\rho} |x_s^n - x_{\kappa(n,s)}^n|^2 ds, \end{aligned}$$

which implies due to Young’s inequality and Hölder’s inequality

$$E|I_4|^2 \leq C \int_{\kappa(n,t)}^t \left( E(1 + |x_s^n|^{p_0} + |x_{\kappa(n,s)}^n|^{p_0}) \right)^{\frac{3\rho}{p_0}} \left( E|x_s^n - x_{\kappa(n,s)}^n|^{\frac{2p_0}{p_0-3\rho}} \right)^{\frac{p_0-3\rho}{p_0}} ds,$$

for any  $t \in [0, T]$ . One applies Lemma 6 to obtain

$$E|I_4|^2 \leq Cn^{-2},$$

for any  $n \in \mathbb{N}$ . As for  $E|I_6|^2$ , it can be written as

$$\begin{aligned} E|I_6|^2 &:= E \left| \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial b^k(x_s^n)}{\partial x^i} \sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n) dw_s^j \right|^2 \\ &\leq C \int_{\kappa(n,t)}^t E(1 + |x_s^n|^{2\rho}) |\sigma_M^n(s, x_{\kappa(n,s)}^n)|^2 ds, \end{aligned}$$

which on the application of Hölder's inequality yields

$$E|I_6|^2 \leq C \int_{\kappa(n,t)}^t (E(1 + |x_s^n|^{p_0}))^{\frac{2\rho}{p_0}} \left( E |\sigma_M^n(s, x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-2\rho}} \right)^{\frac{p_0-2\rho}{p_0}} ds,$$

for any  $t \in [0, T]$ . By using Lemma 3 and Lemma 5, one obtains

$$E|I_6|^2 \leq Cn^{-2},$$

for any  $n \in \mathbb{N}$ . In order to estimate  $E|I_7|^2$ , one uses Cauchy–Schwarz inequality and Remark 1,

$$\begin{aligned} E|I_7|^2 &:= E \left| \frac{1}{2} \sum_{i,l=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \left( \frac{\partial^2 b^k(x_s^n)}{\partial x^i \partial x^l} - \frac{\partial^2 b^k(x_{\kappa(n,s)}^n)}{\partial x^i \partial x^l} \right) \sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma^{n,(l,j)}(x_{\kappa(n,s)}^n) ds \right|^2 \\ &\leq Cn^{-1} \int_{\kappa(n,t)}^t E(1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{2\rho-4} (1 + |x_{\kappa(n,s)}^n|)^{2\rho+4\beta} |x_s^n - x_{\kappa(n,s)}^n|^2 ds \\ &\leq Cn^{-1} \int_{\kappa(n,t)}^t E(1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{4\rho} |x_s^n - x_{\kappa(n,s)}^n|^2 ds, \end{aligned}$$

which on the use of Young's inequality and Hölder's inequality gives

$$E|I_7|^2 \leq Cn^{-1} \int_{\kappa(n,t)}^t \left( E(1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{p_0} \right)^{\frac{4\rho}{p_0}} \left( E |x_s^n - x_{\kappa(n,s)}^n|^{\frac{2p_0}{p_0-4\rho}} \right)^{\frac{p_0-4\rho}{p_0}} ds,$$

for any  $t \in [0, T]$ . Then one applies Lemma 3 and Lemma 5 to obtain

$$E|I_7|^2 \leq Cn^{-3},$$

for any  $n \in \mathbb{N}$ . Finally for  $E|I_9|^2$ , one writes

$$\begin{aligned} E|I_9|^2 &:= E \left| \frac{1}{2} \sum_{i,l=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial^2 b^k(x_s^n)}{\partial x^i \partial x^l} (\sigma^{n,(i,j)}(x_{\kappa(n,s)}^n) \sigma_M^{n,(l,j)}(s, x_{\kappa(n,s)}^n) \right. \\ &\quad \left. + \sigma_M^{n,(i,j)}(s, x_{\kappa(n,s)}^n) \tilde{\sigma}^{n,(l,j)}(s, x_{\kappa(n,s)}^n) ds \right|^2 \\ &\leq Cn^{-1} \int_{\kappa(n,t)}^t E(1 + |x_s^n|)^{2\rho-2} (1 + |x_{\kappa(n,s)}^n|)^{\rho+2\beta} |\sigma_M^n(s, x_{\kappa(n,s)}^n)|^2 ds \\ &\quad + Cn^{-1} \int_{\kappa(n,t)}^t E(1 + |x_s^n|)^{2\rho-2} |\sigma_M^n(s, x_{\kappa(n,s)}^n)|^2 |\tilde{\sigma}^n(s, x_{\kappa(n,s)}^n)|^2 ds \end{aligned}$$

which implies due to Young's inequality and Hölder's inequality

$$\begin{aligned} E|I_9|^2 &\leq Cn^{-1} \int_{\kappa(n,t)}^t \left( E(1 + |x_s^n| + |x_{\kappa(n,s)}^n|)^{p_0} \right)^{\frac{3\rho}{p_0}} \left( E|\sigma_M^n(s, x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-3\rho}} \right)^{\frac{p_0-3\rho}{p_0}} ds \\ &\quad + Cn^{-1} \int_{\kappa(n,t)}^t \left( E(1 + |x_s^n|)^{p_0} \right)^{\frac{2\rho-2}{p_0}} \left( E|\tilde{\sigma}^n(s, x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-2\rho+2}} |\sigma_M^n(s, x_{\kappa(n,s)}^n)|^{\frac{2p_0}{p_0-2\rho+2}} \right)^{\frac{p_0-2\rho+2}{p_0}} ds \end{aligned}$$

for any  $t \in [0, T]$ . One needs to apply Lemma 3, 5 to the first term, and apply Cauchy–Schwarz inequality to the second term to obtain

$$E|I_9|^2 \leq Cn^{-3} + Cn^{-1} \int_{\kappa(n,t)}^t \left( E|\tilde{\sigma}^n(s, x_{\kappa(n,s)}^n)|^{\frac{4p_0}{p_0-2\rho+2}} E|\sigma_M^n(s, x_{\kappa(n,s)}^n)|^{\frac{4p_0}{p_0-2\rho+2}} \right)^{\frac{p_0-2\rho+2}{2p_0}} ds$$

for any  $t \in [0, T]$ . Then, by using Lemma 5, the following can be obtained

$$E|I_9|^2 \leq Cn^{-3},$$

for any  $n \in \mathbb{N}$ . Therefore,

$$\sup_{0 \leq t \leq T} E|b(x_t^n) - b(x_{\kappa(n,t)}^n) - b_1^n(t, x_{\kappa(n,t)}^n) - b_2^n(t, x_{\kappa(n,t)}^n)|^2 \leq Cn^{-2} + Cn^{-5} \leq Cn^{-2},$$

for any  $p_0 \geq 10\rho + 2$ , and the proof is complete.  $\square$

Denote by  $e_t^n := x_t - x_t^n$  for any  $t \in [0, T]$ , and define the stopping times as follows

$$\tau_R := \inf\{t \geq 0 : |x_t| \geq R\}, \quad \tau'_{n,R} := \inf\{t \geq 0 : |x_t^n| \geq R\}, \quad \nu_{n,R} := \tau_R \wedge \tau'_{n,R}.$$

**Lemma 10.** *Assume A-1 to A-5 hold, then, for every  $n \in \mathbb{N}$  and  $t \in [0, T]$ ,*

$$\begin{aligned} E \int_0^{t \wedge \nu_{n,R}} e_s^n (b(x_s^n) - b(x_{\kappa(n,s)}^n) - b_1^n(s, x_{\kappa(n,s)}^n) - b_2^n(s, x_{\kappa(n,s)}^n)) ds \\ \leq C \int_0^t \sup_{0 \leq r \leq s} E|e_{r \wedge \nu_{n,R}}^n|^2 ds + Cn^{-\frac{5+\beta}{2}}, \end{aligned}$$

for  $p_0 \geq 2(5\rho + 1)$ .

*Proof.* First, applying Itô's formula to  $b(x_t^n) - b(x_{\kappa(n,t)}^n)$  gives (4.1). Then, by (4.2) and (4.3),

one obtains

$$\begin{aligned}
& E \int_0^{t \wedge \nu_{n,R}} e_s^n (b(x_s^n) - b(x_{\kappa(n,s)}^n) - b_1^n(s, x_{\kappa(n,s)}^n) - b_2^n(s, x_{\kappa(n,s)}^n)) ds \\
& \leq E \int_0^{t \wedge \nu_{n,R}} e_s^n \left( \sum_{i=1}^d \int_{\kappa(n,s)}^s \left( \frac{\partial b^k(x_r^n)}{\partial x^i} - \frac{\partial b^k(x_{\kappa(n,r)}^n)}{\partial x^i} \right) b^{n,i}(x_{\kappa(n,r)}^n) dr \right. \\
& \quad + \sum_{i=1}^d \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} (b_1^{n,i}(r, x_{\kappa(n,r)}^n) + b_2^{n,i}(r, x_{\kappa(n,r)}^n)) dr \\
& \quad + \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,s)}^s \left( \frac{\partial b^k(x_r^n)}{\partial x^i} - \frac{\partial b^k(x_{\kappa(n,r)}^n)}{\partial x^i} \right) \sigma^{n,(i,j)}(x_{\kappa(n,r)}^n) dw_r^j \\
& \quad + \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma_M^{n,(i,j)}(r, x_{\kappa(n,r)}^n) dw_r^j \\
& \quad + \frac{1}{2} \sum_{i,l=1}^d \sum_{j=1}^m \int_{\kappa(n,s)}^s \left( \frac{\partial^2 b^k(x_r^n)}{\partial x^i \partial x^l} - \frac{\partial^2 b^k(x_{\kappa(n,r)}^n)}{\partial x^i \partial x^l} \right) \\
& \quad \quad \quad \times \sigma^{n,(i,j)}(x_{\kappa(n,r)}^n) \sigma^{n,(l,j)}(x_{\kappa(n,r)}^n) dr \\
& \quad + \frac{1}{2} \sum_{i,l=1}^d \sum_{j=1}^m \int_{\kappa(n,s)}^s \frac{\partial^2 b^k(x_r^n)}{\partial x^i \partial x^l} (\sigma^{n,(i,j)}(x_{\kappa(n,r)}^n) \sigma_M^{n,(l,j)}(r, x_{\kappa(n,r)}^n) \\
& \quad \quad \quad + \sigma_M^{n,(i,j)}(r, x_{\kappa(n,r)}^n) \tilde{\sigma}^{n,(l,j)}(r, x_{\kappa(n,r)}^n)) dr \Big) ds \\
& \quad + C \int_0^t \sup_{0 \leq r \leq s} E |e_{r \wedge \nu_{n,R}}^n|^2 ds + Cn^{-5} \\
& := T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8,
\end{aligned}$$

where  $T_7 = C \int_0^t \sup_{0 \leq r \leq s} E |e_{r \wedge \nu_{n,R}}^n|^2 ds$  and  $T_8 = Cn^{-5}$ . To estimate  $T_1$ , one applies Young's inequality and Remark 1 to obtain

$$\begin{aligned}
T_1 & := E \int_0^{t \wedge \nu_{n,R}} e_s^n \sum_{i=1}^d \int_{\kappa(n,s)}^s \left( \frac{\partial b^k(x_r^n)}{\partial x^i} - \frac{\partial b^k(x_{\kappa(n,r)}^n)}{\partial x^i} \right) b^{n,i}(x_{\kappa(n,r)}^n) dr ds \\
& \leq C \int_0^t \sup_{0 \leq r \leq s} E |e_{r \wedge \nu_{n,R}}^n|^2 ds \\
& \quad + Cn^{-1} \int_0^t \int_{\kappa(n,s)}^s E (1 + |x_r^n|^{4\rho} + |x_{\kappa(n,r)}^n|^{4\rho}) |x_r^n - x_{\kappa(n,r)}^n|^2 dr ds,
\end{aligned}$$

which on the use of Hölder's inequality yields

$$\begin{aligned}
T_1 & \leq C \int_0^t \sup_{0 \leq r \leq s} E |e_{r \wedge \nu_{n,R}}^n|^2 ds \\
& \quad + Cn^{-1} \int_0^t \int_{\kappa(n,s)}^s \left( E(1 + |x_r^n|^{p_0} + |x_{\kappa(n,r)}^n|^{p_0}) \right)^{\frac{4\rho}{p_0}} \left( E |x_r^n - x_{\kappa(n,r)}^n|^{\frac{2p_0}{p_0-4\rho}} \right)^{\frac{p_0-4\rho}{p_0}} dr ds.
\end{aligned}$$

Thus, by Lemma 6 and Lemma 3

$$T_1 \leq C \int_0^t \sup_{0 \leq r \leq s} E |e_{r \wedge \nu_{n,R}}^n|^2 ds + Cn^{-3},$$

for any  $n \in \mathbb{N}$ . For  $T_2$ ,  $T_5$  and  $T_6$ , the same results can be obtained by the direct application of Cauchy–Schwarz inequality combining with previous Lemmas and Remarks. The rest of the proof will mainly focus on  $T_3$  and  $T_4$ . In order to estimate  $T_3$  and  $T_4$ , the following notation is introduced

$$\mathbb{T}_r := \left( \frac{\partial b^k(x_r^n)}{\partial x^i} - \frac{\partial b^k(x_{\kappa(n,r)}^n)}{\partial x^i} \right) \sigma^{n,(i,j)}(x_{\kappa(n,r)}^n) + \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma_M^{n,(i,j)}(r, x_{\kappa(n,r)}^n).$$

Notice that, it has been shown in the proof of Lemma 9, that  $I_4$  and  $I_6$  can be estimated as follows

$$E \left| \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,s)}^s \mathbb{T}_r dw_r^j \right|^2 \leq Cn^{-2}. \quad (4.4)$$

Then, on the application of Remark 1 and Hölder’s inequality, one obtains

$$\begin{aligned} E|\mathbb{T}_r|^p &= E \left| \left( \frac{\partial b^k(x_r^n)}{\partial x^i} - \frac{\partial b^k(x_{\kappa(n,r)}^n)}{\partial x^i} \right) \sigma^{n,(i,j)}(x_{\kappa(n,r)}^n) + \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma_M^{n,(i,j)}(r, x_{\kappa(n,r)}^n) \right|^p \\ &\leq C \left( E(1 + |x_r^n| + |x_{\kappa(n,r)}^n|)^{p_0} \right)^{\frac{3pp}{2p_0}} \left( E|x_r^n - x_{\kappa(n,r)}^n|^{\frac{2pp_0}{2p_0-3pp}} \right)^{\frac{2p_0-3pp}{2p_0}} \\ &\quad + C (E(1 + |x_r^n|)^{p_0})^{\frac{pp}{p_0}} (E|\sigma_M^{n,(i,j)}(r, x_{\kappa(n,r)}^n)|^{\frac{pp_0}{p_0-\rho p}})^{\frac{p_0-\rho p}{p_0}}, \end{aligned}$$

which by Lemma 6 and Lemma 5

$$\sup_{r \leq T} E|\mathbb{T}_r|^p \leq Cn^{-\frac{p}{2}}. \quad (4.5)$$

Furthermore, one writes

$$\begin{aligned} T_3 + T_4 &:= E \int_0^{t \wedge \nu_{n,R}} e_s^n \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,s)}^s \mathbb{T}_r dw_r^j ds \\ &= E \int_0^{t \wedge \nu_{n,R}} (e_s^n - e_{\kappa(n,s)}^n) \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,s)}^s \mathbb{T}_r dw_r^j ds \\ &\quad + E \int_0^{t \wedge \nu_{n,R}} e_{\kappa(n,s)}^n \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,s)}^s \mathbb{T}_r dw_r^j ds. \end{aligned}$$

Since the second term above is zero, it can be further expressed as

$$\begin{aligned} T_3 + T_4 &= E \int_0^{t \wedge \nu_{n,R}} \int_{\kappa(n,s)}^s \bar{b}^n(r, x_{\kappa(n,r)}^n) dr \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,s)}^s \mathbb{T}_r dw_r^j ds \\ &\quad + E \int_0^{t \wedge \nu_{n,R}} \int_{\kappa(n,s)}^s \bar{\sigma}^n(r, x_{\kappa(n,r)}^n) dw_r \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,s)}^s \mathbb{T}_r dw_r^j ds, \end{aligned}$$

where  $\bar{b}^n(t, x_{\kappa(n,t)}^n) = b(x_t) - \tilde{b}^n(t, x_{\kappa(n,t)}^n)$  and  $\bar{\sigma}^n(t, x_{\kappa(n,t)}^n) = \sigma(x_t) - \tilde{\sigma}^n(t, x_{\kappa(n,t)}^n)$ . One

observes that  $T_3 + T_4$  can be expanded in the following way

$$\begin{aligned}
T_3 + T_4 &= E \int_0^{t \wedge \nu_{n,R}} \int_{\kappa(n,s)}^s (b(x_r) - b(x_r^n)) dr \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,s)}^s \mathbb{T}_r dw_r^j ds \\
&\quad + E \int_0^{t \wedge \nu_{n,R}} \int_{\kappa(n,s)}^s (b(x_r^n) - b(x_{\kappa(n,r)}^n) - b_1^n(r, x_{\kappa(n,r)}^n) - b_2^n(r, x_{\kappa(n,r)}^n)) dr \\
&\quad \quad \quad \times \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,s)}^s \mathbb{T}_r dw_r^j ds \\
&\quad + E \int_0^{t \wedge \nu_{n,R}} \int_{\kappa(n,s)}^s (b(x_{\kappa(n,r)}^n) - b^n(x_{\kappa(n,r)}^n)) dr \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,s)}^s \mathbb{T}_r dw_r^j ds \\
&\quad + \sum_{i=1}^d \sum_{j=1}^m E \int_0^{t \wedge \nu_{n,R}} \int_{\kappa(n,s)}^s (\sigma(x_r) - \sigma(x_r^n)) \mathbb{T}_r dr ds \\
&\quad + \sum_{i=1}^d \sum_{j=1}^m E \int_0^{t \wedge \nu_{n,R}} \int_{\kappa(n,s)}^s (\sigma(x_r^n) - \sigma(x_{\kappa(n,r)}^n) - \sigma_M^n(r, x_{\kappa(n,r)}^n)) \mathbb{T}_r dr ds \\
&\quad + \sum_{i=1}^d \sum_{j=1}^m E \int_0^{t \wedge \nu_{n,R}} \int_{\kappa(n,s)}^s (\sigma(x_{\kappa(n,r)}^n) - \sigma^n(x_{\kappa(n,r)}^n)) \mathbb{T}_r dr ds
\end{aligned}$$

which implies due to Remark 1, Young's inequality and Cauchy–Schwarz inequality

$$\begin{aligned}
T_3 + T_4 &\leq E \int_0^{t \wedge \nu_{n,R}} \left( \int_{\kappa(n,s)}^s (1 + |x_r| + |x_r^n|)^{2\rho} dr \int_{\kappa(n,s)}^s |e_r^n|^2 dr \right)^{\frac{1}{2}} \left| \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,s)}^s \mathbb{T}_r dw_r^j \right| ds \\
&\quad + C \int_0^t \left( E \left| \int_{\kappa(n,s)}^s (b(x_r^n) - b(x_{\kappa(n,r)}^n) - b_1^n(r, x_{\kappa(n,r)}^n) - b_2^n(r, x_{\kappa(n,r)}^n)) dr \right|^2 \right. \\
&\quad \quad \quad \left. \times E \left| \int_{\kappa(n,s)}^s \mathbb{T}_r dw_r^j \right|^2 \right)^{1/2} ds \\
&\quad + C \int_0^t n \times n^{-1} E \int_{\kappa(n,s)}^s |b(x_{\kappa(n,r)}^n) - b^n(x_{\kappa(n,r)}^n)|^2 dr ds \\
&\quad + C n^{-1} \int_0^t E \left| \int_{\kappa(n,s)}^s \mathbb{T} dw_r^j \right|^2 ds + C E \int_0^{t \wedge \nu_{n,R}} \int_{\kappa(n,s)}^s (1 + |x_r| + |x_r^n|)^{\frac{\rho}{2}} |e_r^n| |\mathbb{T}_r| dr ds \\
&\quad + C \int_0^t \int_{\kappa(n,s)}^s \sqrt{E |\sigma(x_r^n) - \sigma(x_{\kappa(n,r)}^n) - \sigma_M^n(r, x_{\kappa(n,r)}^n)|^2 E |\mathbb{T}_r|^2} dr ds \\
&\quad + C \int_0^t \int_{\kappa(n,s)}^s E \sqrt{|\sigma(x_{\kappa(n,r)}^n) - \sigma^n(x_{\kappa(n,r)}^n)|^2 E |\mathbb{T}_r|^2} dr ds,
\end{aligned}$$

for any  $t \in [0, T]$ . Then, by (4.4), (4.5), Lemma 9, Lemma 7, Lemma 8, Hölder's inequality



and Cauchy–Schwarz inequality, one obtains

$$\begin{aligned}
T_3 + T_4 &\leq Cn^{-1}E \int_0^t \int_{\kappa(n,s)}^s (1 + |x_r| + |x_r^n|)^{2\rho} dr \left| \sum_{i=1}^d \sum_{j=1}^m \int_{\kappa(n,s)}^s \mathbb{T}_r dw_r^j \right|^2 ds \\
&\quad + Cn^{-1} \int_0^t \int_{\kappa(n,s)}^s (E(1 + |x_r| + |x_r^n|)^{p_0})^{\frac{\rho}{p_0}} (E|\mathbb{T}_r|^{\frac{2p_0}{p_0-\rho}})^{\frac{p_0-\rho}{p_0}} dr ds \\
&\quad + CE \int_0^t \sup_{0 \leq r \leq s} E|e_{r \wedge \nu_{n,R}}^n|^2 ds + Cn^{-\frac{5+\beta}{2}} + Cn^{-3},
\end{aligned}$$

for  $\beta \in (0, 1]$ . Applying Hölder’s inequality gives

$$\begin{aligned}
T_3 + T_4 &= Cn^{-1} \int_0^t \left( n^{-\frac{p_0}{2\rho}+1} E \int_{\kappa(n,s)}^s (1 + |x_r| + |x_r^n|)^{p_0} dr \right)^{\frac{2\rho}{p_0}} \\
&\quad \times \left( n^{-\frac{p_0}{p_0-2\rho}+1} E \int_{\kappa(n,s)}^s |\mathbb{T}_r|^{\frac{2p_0}{p_0-2\rho}} dr \right)^{\frac{p_0-2\rho}{p_0}} ds \\
&\quad + CE \int_0^t \sup_{0 \leq r \leq s} E|e_{r \wedge \nu_{n,R}}^n|^2 ds + Cn^{-\frac{5+\beta}{2}},
\end{aligned}$$

for any  $t \in [0, T]$ . Thus, by using Lemma 3 and (4.5), one obtains

$$T_3 + T_4 \leq C \int_0^t \sup_{0 \leq r \leq s} E|e_{r \wedge \nu_{n,R}}^n|^2 ds + Cn^{-\frac{5+\beta}{2}},$$

for  $p_0 \geq 2(5\rho + 1)$ ,  $\beta \in (0, 1]$  and for any  $n \in \mathbb{N}$ , and the proof is complete.  $\square$

**Proof of Theorem 1.** Applying Itô’s formula to  $|e_{t \wedge \nu_{n,R}}^n|^2$  gives

$$\begin{aligned}
|e_{t \wedge \nu_{n,R}}^n|^2 &= 2 \int_0^{t \wedge \nu_{n,R}} e_s^n \bar{b}^n(s, x_{\kappa(n,s)}^n) ds \\
&\quad + 2 \int_0^{t \wedge \nu_{n,R}} e_s^n \bar{\sigma}^n(s, x_{\kappa(n,s)}^n) dw_s \\
&\quad + \int_0^{t \wedge \nu_{n,R}} |\bar{\sigma}^n(s, x_{\kappa(n,s)}^n)|^2 ds,
\end{aligned}$$

where  $\bar{b}^n(t, x_{\kappa(n,t)}^n) = b(x_t) - \tilde{b}^n(t, x_{\kappa(n,t)}^n)$  and  $\bar{\sigma}^n(t, x_{\kappa(n,t)}^n) = \sigma(x_t) - \tilde{\sigma}^n(t, x_{\kappa(n,t)}^n)$ . It can be expressed as

$$\begin{aligned}
E|e_{t \wedge \nu_{n,R}}^n|^2 &\leq 2E \int_0^{t \wedge \nu_{n,R}} e_s^n (b(x_s) - b(x_s^n)) ds \\
&\quad + 2E \int_0^{t \wedge \nu_{n,R}} e_s^n (b(x_s^n) - b(x_{\kappa(n,s)}^n) - b_1^n(s, x_{\kappa(n,s)}^n) - b_2^n(s, x_{\kappa(n,s)}^n)) ds \\
&\quad + 2E \int_0^{t \wedge \nu_{n,R}} e_s^n (b(x_{\kappa(n,s)}^n) - b^n(x_{\kappa(n,s)}^n)) ds \\
&\quad + E \int_0^{t \wedge \nu_{n,R}} |\sigma(x_s) - \sigma(x_s^n)|^2 ds \\
&\quad + E \int_0^{t \wedge \nu_{n,R}} |\sigma(x_s^n) - \sigma(x_{\kappa(n,s)}^n) - \sigma_M^n(s, x_{\kappa(n,s)}^n)|^2 ds \\
&\quad + E \int_0^{t \wedge \nu_{n,R}} |\sigma(x_{\kappa(n,s)}^n) - \sigma^n(x_{\kappa(n,s)}^n)|^2 ds.
\end{aligned}$$

Then, by using Cauchy–Schwarz inequality, one obtains

$$\begin{aligned}
E|e_{t \wedge \nu_{n,R}}^n|^2 &\leq E \int_0^{t \wedge \nu_{n,R}} (2e_s^n (b(x_s) - b(x_s^n)) + |\sigma(x_s) - \sigma(x_s^n)|^2) ds \\
&\quad + 2E \int_0^{t \wedge \nu_{n,R}} e_s^n (b(x_s^n) - b(x_{\kappa(n,s)}^n) - b_1^n(s, x_{\kappa(n,s)}^n) - b_2^n(s, x_{\kappa(n,s)}^n)) ds \\
&\quad + E \int_0^{t \wedge \nu_{n,R}} |b(x_{\kappa(n,s)}^n) - b^n(x_{\kappa(n,s)}^n)|^2 ds \\
&\quad + E \int_0^{t \wedge \nu_{n,R}} |\sigma(x_s^n) - \sigma(x_{\kappa(n,s)}^n) - \sigma_M^n(s, x_{\kappa(n,s)}^n)|^2 ds \\
&\quad + E \int_0^{t \wedge \nu_{n,R}} |\sigma(x_{\kappa(n,s)}^n) - \sigma^n(x_{\kappa(n,s)}^n)|^2 ds \\
&\quad + C \int_0^t \sup_{0 \leq r \leq s} E|e_{r \wedge \nu_{n,R}}^n|^2 ds,
\end{aligned}$$

since  $p_1 > 2$ , one can use **A-2** for the first term, and then on the application of Lemma 10, Lemma 7 and Lemma 8 yields

$$\sup_{0 \leq s \leq t} E|e_{s \wedge \nu_{n,R}}^n|^2 \leq C \int_0^t \sup_{0 \leq r \leq s} E|e_{r \wedge \nu_{n,R}}^n|^2 ds + Cn^{-(2+\beta)} < \infty,$$

for any  $n \in \mathbb{N}$ . Finally, the proof is complete by applying Gronwall’s lemma and Fatou’s lemma.

## 5 Simulation results

In this section, simulation results are provided to support the theoretical results in the previous sections. The following one dimensional SDE is considered

$$dx_t = x_t(1 - |x_t|^3)dt + \xi|x_t|^{\frac{5}{2}}dw_t, \quad \forall t \in [0, T], \quad (5.1)$$

where  $x_0 = 3$ ,  $\xi = 0.02$ , and  $T = 1$ . The discrete scheme used to obtain the numerical results is expressed as:

$$\begin{aligned}
X_{n+1} &= X_n + b^n \Delta + \sigma^n \Delta W + L^{n,1} b \Delta Z + \frac{1}{2} L^{n,0} b \Delta^2 \\
&\quad + \frac{1}{2} L^{n,1} \sigma ((\Delta W)^2 - \Delta) + L^{n,0} \sigma (\Delta W \Delta - \Delta Z) \\
&\quad + \frac{1}{2} L^{n,1} L^1 \sigma \left( \frac{1}{3} (\Delta W)^2 - \Delta \right) \Delta W,
\end{aligned}$$

where  $b^n$  and  $\sigma^n$  are the tamed drift and diffusion coefficients of the SDE (5.1), and

$$\Delta Z = \int_{t_n}^{t_{n+1}} \int_{t_n}^s dW_r ds.$$

Note that  $\Delta Z$  is normally distributed with mean zero, variance  $\frac{1}{3} \Delta^3$ , and covariance

$$E(\Delta Z \Delta W) = \frac{1}{2} \Delta^2.$$

In Figure 1, 1000 paths are simulated to produce the results. It illustrates that, for the

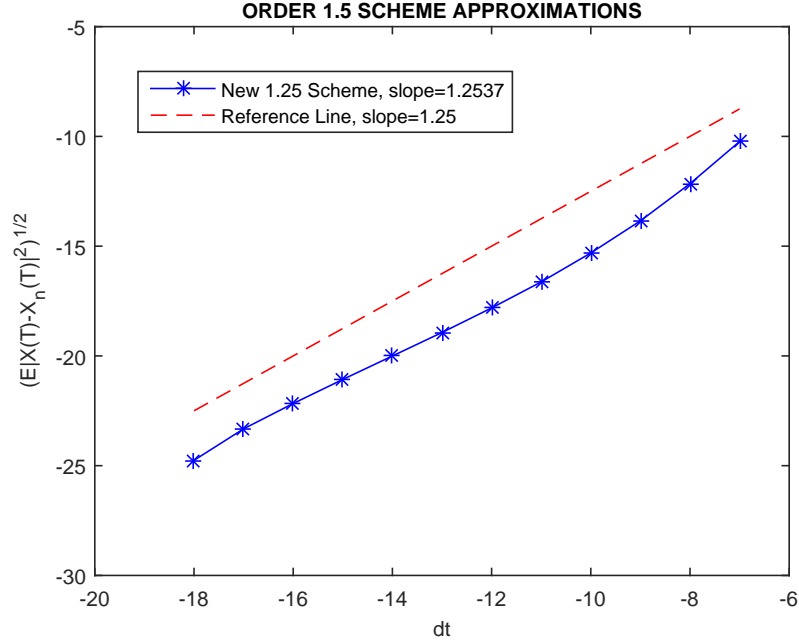


Figure 1: Rate of convergence for the case  $\beta = 0.5$

case  $\beta = 0.5$ , the new explicit order 1.5 scheme has a rate of convergence estimate close to the theoretical result 1.25, and slope obtained using least square method is 1.2537. However, notice that for  $d$  ( $d \geq 2$ ) dimensional case, in order to do the numerical implementation, the diffusion matrix  $\sigma$  is assumed to satisfy the commutativity conditions

$$L^j \sigma^{i,j_1} = L^{j_1} \sigma^{i,j} \quad (5.2)$$

and

$$L^j L^{j_1} \sigma^{i,j_2} = L^{j_1} L^j \sigma^{i,j_2}, \quad (5.3)$$

for all  $j, j_1, j_2 = 1, \dots, m$  and  $i = 1, \dots, d$ .

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