Nonuniversality and finite dissipation in decaying magnetohydrodynamic turbulence

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A model equation for the Reynolds number dependence of the dimensionless dissipation rate in freely decaying homogeneous magnetohydrodynamic turbulence in the absence of a mean magnetic field is derived from the real-space energy balance equation, leading to $C_\varepsilon = C_{\varepsilon,\infty} + C/R_- + O(1/R_-^2))$, where R_- is a generalized Reynolds number. The constant $C_{\varepsilon,\infty}$ describes the total energy transfer flux. This flux depends on magnetic and cross helicities, because these affect the nonlinear transfer of energy, suggesting that the value of $C_{\varepsilon,\infty}$ is not universal. Direct numerical simulations were conducted on up to 2048^3 grid points, showing good agreement between data and the model. The model suggests that the magnitude of cosmological-scale magnetic fields is controlled by the values of the vector field correlations. The ideas introduced here can be used to derive similar model equations for other turbulent systems.

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Magnetohydrodynamic (MHD) turbulence is present in many areas of physics, ranging from industrial applications such as liquid metal technology to nuclear fusion and plasma physics, geo-, astro- and solar physics, and even cosmology. The numerous different MHD flow types that arise in different settings due to anisotropy, alignment, different values of the diffusivities, to name only a few, lead to the question of universality in MHD turbulence, which has been the subject of intensive research by many groups [1–12]. The behavior of the (dimensionless) dissipation rate is connected to this problem, in the sense that correlation (alignment) of the different vector fields could influence the energy transfer across the scales [2, 13, 14], and thus possibly the amount of energy that is eventually dissipated at the small scales.

For neutral fluids it has been known for a long time that the dimensionless dissipation rate in forced and freely decaying homogeneous isotropic turbulence tends to a constant with increasing Reynolds number. The first evidence for this was reported by Batchelor [15] in 1953, while the experimental results reviewed by Sreenivasan in 1984 [16], and subsequent experimental and numerical work by many groups, established the now well-known characteristic curve of the dimensionless dissipation rate against Reynolds number: see [17–20] and references therein. For statistically steady isotropic turbulence, the theoretical explanation of this curve was recently found to be connected to the energy balance equation for forced turbulent flows [19], where the asymptote describes the maximal inertial transfer flux in the limit of infinite Reynolds number.

For freely decaying MHD, recent results suggest that the temporal maximum of the total dissipation tends to a constant value with increasing Reynolds number. The first evidence for this behavior in MHD was put forward in 2009 by Mininni and Pouquet [21] using results from direct numerical simulations (DNSs) of isotropic MHD turbulence. The temporal maximum of the total dissipation rate $\varepsilon(t)$ became independent of Reynolds number at a Taylor-scale Reynolds number R_λ (measured at the peak of $\varepsilon(t)$) of about 200.

Dallas and Alexakis [22] measured the dimensionless dissipation rate C_ε from DNS data, where ε was nondimensionalized with respect to the initial values of the rms velocity $U(t)$ and the integral length scale $L(t)$ (here defined with respect to the total energy), for random initial fields with strong correlations between the velocity field and the current density. The authors compared data with Ref. [21], and again it was found that $C_\varepsilon \to const.$ with increasing Reynolds number. Interestingly the approach to the asymptote was slower than for the data of Ref. [21].

In this Letter we propose a model for the Reynolds number dependence of the dimensionless dissipation rate derived from the energy balance equation for MHD turbulence in terms of Elsässer fields [23], which predicts nonuniversal values of the dimensionless dissipation rate in the infinite Reynolds number limit. In order to compare the predictions of the model against data, we carried out a series of DNSs of decaying isotropic MHD turbulence. Firstly we explain the derivation of the model equation, then proceed to a description of our numerical simulations and subsequently compare the model to DNS results. We conclude with a discussion of the results and suggestions for further research.

The equations describing incompressible decaying MHD flows are

$$\partial_t \boldsymbol{u} = -\frac{1}{\rho}\nabla P - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{\rho}(\nabla\times\boldsymbol{b})\times\boldsymbol{b} + \nu\Delta\boldsymbol{u} \ , \quad (1)$$

$$\partial_t \boldsymbol{b} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} + \eta\Delta\boldsymbol{b} \ , \quad (2)$$

$$\nabla\cdot\boldsymbol{u} = 0 \ \text{ and } \ \nabla\cdot\boldsymbol{b} = 0 \ , \quad (3)$$

where $\boldsymbol{u}$ denotes the velocity field, $\boldsymbol{b}$ the magnetic induction expressed in Alfvén units, ν the kinematic viscosity, η the resistivity, P the pressure and $\rho = 1$ the density. For simplicity and in order to compare to results in the literature we consider the case of unit magnetic Prandtl number, that is $Pm = \nu/\eta = 1$.

For freely decaying MHD turbulence the decay rate of the total energy $\varepsilon_D = -\partial_t E_{tot}$ equals the total dissipation rate ε, and the time evolution of the total energy is governed by the energy balance equation of MHD turbulence in real space, which is derived from the MHD equations (1)-(3). This suggests that the energy balance equation can be used in order to derive the Reynolds number dependence of the total dissipation rate.

Since we are interested in the total dissipation $\varepsilon = \varepsilon_{mag} + \varepsilon_{kin}$, where $\varepsilon_{mag} = 2\eta \int_0^\infty dk\ k^2 E_{mag}(k)$, and $\varepsilon_{kin} = 2\nu \int_0^\infty dk\ k^2 E_{kin}(k)$ ($E_{mag}(k)$ and $E_{kin}(k)$ denoting magnetic and kinetic energy spectra), are the magnetic and kinetic dissipation rates, respectively, we could take two approaches, either formulating the energy balance in terms of the primary fields $\boldsymbol{u}$ and $\boldsymbol{b}$ or in terms of the Elsässer fields $\boldsymbol{z}^\pm = \boldsymbol{u} \pm \boldsymbol{b}$. Since

$$\partial_t \langle |\boldsymbol{z}^\pm|^2 \rangle = 2\partial_t E_{tot} \pm 2\partial_t H_c \ , \tag{4}$$

where $H_c = \langle \boldsymbol{u} \cdot \boldsymbol{b} \rangle$ is the cross helicity, we can describe the total dissipation either by the energy balance equations for $\langle |\boldsymbol{z}^\pm|^2 \rangle$ [23] or by the sum of the energy balance equations for $E_{mag}(t) = \int_0^\infty dk\ E_{mag}(k)$ and $E_{kin}(t) = \int_0^\infty dk\ E_{kin}(k)$ [24, 25].

This, however, is not the case if we are interested in the *dimensionless* dissipation rate. Unlike in hydrodynamics, there are several choices of scales with which to non-dimensionalize $\varepsilon(t)$ and thus with respect to which to define an MHD analogue to the Taylor surrogate expression [15, 18]. For example U and L could be used, or the rms $\boldsymbol{b}$ field B and L or U and L_{kin} etc., or scales defined with respect to $\boldsymbol{z}^\pm$. The physical interpretation is different for the different scaling quantities. Since the total dissipation must equal the total flux of energy passed through the scales by the kinetic and magnetic energy transfer terms, a scaling with U will be appropriate only for hydrodynamic transfer as this transfer term scales as U^3/L_{kin}. All other transfer terms include $\boldsymbol{b}$ and $\boldsymbol{u}$ and thus should be scaled accordingly. This also precludes the most straightforward generalization of the Taylor surrogate, which would be a scaling of ε with L and $\sqrt{U^2+B^2}$. A hydrodynamic transfer term would then be scaled partly with magnetic quantities, while the appropriate scaling should only involve kinetic quantities.

Instead we propose to define the dimensionless dissipation rate for MHD turbulence with respect to the Elsässer variables

$$C_\varepsilon = \frac{C_\varepsilon^+ + C_\varepsilon^-}{2} \equiv \frac{1}{2}\left(\frac{\varepsilon L_+}{{z^+}^2 z^-} + \frac{\varepsilon L_-}{{z^-}^2 z^+} \right) \ , \tag{5}$$

where $L_\pm = (3\pi \int_0^\infty dk\ k^{-1} \langle |\boldsymbol{z}^\pm|^2 \rangle)/(4\int_0^\infty dk\ \langle |\boldsymbol{z}^\pm|^2 \rangle)$ are the integral scales defined with respect to $\boldsymbol{z}^\pm$, and $z^\pm$ denote the rms values of $\boldsymbol{z}^\pm$ [26].

Using this definition we can now consistently non-dimensionalize the evolution equations of $\langle |\boldsymbol{z}^\pm|^2 \rangle$. For conciseness we outline the arguments for the $\langle |\boldsymbol{z}^+|^2 \rangle$ case, since the $\langle |\boldsymbol{z}^-|^2 \rangle$ case proceeds analogously [27].

Following [23] the energy balance for $\langle |\boldsymbol{z}^+|^2 \rangle$ reads for the case $Pm = 1$

$$-\frac{1}{2}\partial_t \langle |\boldsymbol{z}^+|^2 \rangle = -\frac{3}{4}\partial_t B_{LL}^{++} - \frac{\partial_r}{r^4}\left(\frac{3r^4}{2} C_{LL,L}^{+-+} \right) + \frac{3(\nu+\eta)}{2r^4}\partial_r \left(r^4 \partial_r B_{LL}^{++} \right) \ , \tag{6}$$

where $C_{LL,L}^{+-+}(r)$ and $B_{LL}^{++}(r)$ are the longitudinal third-order correlation function and the second-order longitudinal structure function of the Elsässer fields, respectively. The definitions of the these functions can be found in the Supplemental Material [27]. Using (4) one can express the LHS of (6) in terms of $\varepsilon(t)$ and $\partial_t H_c$.

If we now introduce the nondimensional variable $\sigma = r/L_+$ [3] and nondimensionalize equation (6) with respect to $z^\pm$ and L_+ as proposed in the definition of C_ε in eq. (5), we obtain

$$C_\varepsilon^+ = -\frac{\partial_\sigma}{\sigma^4}\left(\frac{3\sigma^4 C_{LL,L}^{+-+}}{2{z^+}^2 z^-} \right) - \frac{L_+}{{z^+}^2 z^-}\partial_t \frac{3B_{LL}^{++}}{4} + \frac{L_+}{{z^+}^2 z^-}\partial_t H_c + \frac{\nu+\eta}{L_+ z^-}\frac{3\partial_\sigma}{2\sigma^4}\left(\sigma^4 \partial_\sigma \frac{B_{LL}^{++}}{{z^+}^2} \right) \ . \tag{7}$$

In this way we arrive at a consistent scaling for each transfer term in (6) with the appropriate quantity, as the function $C_{LL,L}^{+-+}(r)$ scales with ${z^+}^2 z^-$.

Since the inverse of the coefficient in front of the dissipative term is similar to a Reynolds number, we introduce the generalized large-scale Reynolds number

$$R_- = \frac{2z^- L_+}{\nu+\eta} \ , \tag{8}$$

hence (7) suggests a dependence of C_ε^+ on $1/R_-$. However, the structure and correlation functions and the cross helicity flux also depend on Reynolds number.

For conciseness we introduce dimensionless versions of all terms present on the RHS of (7), such that

$$C_{LL,L}^{+-+}(r,t) = {z^+}^2 z^- g^{+-+}(\sigma,t), \tag{9}$$

$$B_{LL}^{++}(r,t) = {z^+}^2 h_2^{++}(\sigma,t), \tag{10}$$

$$\partial_t B_{LL}^{++}(r,t) = \frac{(z^+)^2 z^-}{L_+} F^+(\sigma,t), \tag{11}$$

$$\partial_t H_c(t) = \frac{(z^+)^2 z^-}{L_+} G^+(t), \tag{12}$$

which leads to a dimensionless version of the $\langle |\boldsymbol{z}^+|^2 \rangle$ energy balance equation for freely decaying MHD turbulence

$$C_\varepsilon^+ = \frac{\varepsilon L_+}{{z^+}^2 z^-} = -\frac{\partial_\sigma}{\sigma^4}\left(\frac{3\sigma^4}{2} g^{+-+} \right) - \frac{3}{4}F^+ + G^+ + \frac{3}{R_-}\frac{\partial_\sigma}{\sigma^4}\left(\sigma^4 \partial_\sigma h_2^{++} \right) \ . \tag{13}$$

After non-dimensionalization the highest derivative in the differential equation is multiplied with the small parameter $1/R_-$, suggesting that this can be viewed as a singular perturbation problem [28]; and thus we consider asymptotic expansions of the dimensionless functions in inverse powers of R_- [19, 29].

The formal asymptotic series of a generic function f (used for conciseness in place of the functions on the RHS of (7)) up to second order in $1/R_-$ reads

$$f = f_0 + \frac{1}{R_-} f_1 + \frac{1}{R_-^2} f_2 + O(R_-^{-3}) \ . \tag{14}$$

After substitution of the expansions into (13) and following the same steps for the evolution equation for $\langle |\boldsymbol{z}^-|^2 \rangle$, we arrive at model equations for C_ε^+ and C_ε^-

$$C_\varepsilon^\pm = C_{\varepsilon,\infty}^\pm + \frac{C^\pm}{R_\mp} + \frac{D^\pm}{R_\mp^2} + O(R_\mp^{-3}) \ , \tag{15}$$

up to third order in $1/R_\mp$, where we defined the coefficients $C_{\varepsilon,\infty}^\pm$, $C^\pm$ and $D^\pm$

$$C_{\varepsilon,\infty}^\pm = -\frac{\partial_\sigma}{\sigma^4} \left(\frac{3\sigma^4}{2} g_0^{\pm\mp\pm} \right) - \frac{3}{4} F_0^\pm \pm G_0^\pm \ , \tag{16}$$

$$C^\pm = \frac{3\partial_\sigma}{\sigma^4} \left[\sigma^4 \left(\partial_\sigma h_{2,0}^{\pm\pm} - \frac{g_1^{\pm\mp\pm}}{2} \right) \right] - \frac{3}{4} F_1^\pm \pm G_1^\pm \ , \tag{17}$$

$$D^\pm = \frac{3\partial_\sigma}{\sigma^4} \left[\sigma^4 \left(\partial_\sigma h_{2,1}^{\pm\pm} - \frac{g_2^{\pm\mp\pm}}{2} \right) \right] - \frac{3}{4} F_2^\pm \pm G_2^\pm \ , \tag{18}$$

in order to write (13) in a more concise way. Using $R_+ = (L_-/L_+)(z^+/z^-)R_-$ to define

$$C = \frac{1}{2} \left(C^+ + \frac{L_-}{L_+} \frac{z^+}{z^-} C^- \right) \ , \tag{19}$$

(D is defined analoguously), finally one obtains for the dimensionless dissipation rate C_ε

$$C_\varepsilon = C_{\varepsilon,\infty} + \frac{C}{R_-} + \frac{D}{R_-^2} + O(R_-^{-3}) \ . \tag{20}$$

Since the time dependence of the various quantities in this problem has been suppressed for conciseness, we stress that (20) is time dependent, including the Reynolds number R_-. A normalization using initial values of $z^\pm$ and $L^\pm$ would have resulted in a dependence of $C_\varepsilon(t)$ on initial values of R_-, which only describe the initial conditions and not the evolved flow for which C_ε is measured.

At the peak of $\varepsilon(t)$ the additional terms $F_0^\pm$ should in fact vanish for constant flux of cross helicity (that is, $\partial_t^2 H_c = 0$), since in the infinite Reynolds number limit the second-order structure function will have its inertial range form at all scales. By self-similarity the spatial and temporal dependences of e.g. B_{LL}^{++} should be separable in the inertial range, that is $B_{LL}^{++}(r,t) \sim (\varepsilon^+(t)r)^\alpha$ for some value α, and $\partial_t B_{LL}^{++} \sim \alpha \varepsilon^+(t)^{\alpha-1} \partial_t \varepsilon^+ r^\alpha$. At the peak of dissipation $\partial_t \varepsilon^+|_{t_{peak}} = \partial_t \varepsilon|_{t_{peak}} - \partial_t^2 H_c = \partial_t \varepsilon|_{t_{peak}} = 0$, and we obtain $F_0^+(t_{peak}) = 0$. As the terms $G_0^\pm$ which describe the flux of cross helicity in the infinite Reynolds number limit, cancel the corresponding contribution from the transfer terms [27], the asymptotes $C_{\varepsilon,\infty}^\pm$ describe the flux of total energy provided the model (15) is applied at t_{peak}.

Due to selective decay, that is the faster decay of the total energy compared to H_c and H_{mag} [14], one could perhaps expect $\partial_t H_c$ to be small compared to ε in the infinite Reynolds number limit in most situations. In this case we obtain $G_0^\pm \simeq 0$ and

$$C_{\varepsilon,\infty}^\pm(t_{peak}) = -\frac{\partial_\sigma}{\sigma^4} \left(\frac{3\sigma^4}{2} g_0^{\pm\mp\pm} \right) \ , \tag{21}$$

which recovers the inertial-range scaling results of Ref. [23] and reduces to Kolmogorov's 4/5th law for $\boldsymbol{b} = 0$.

Since $C_{\varepsilon,\infty}$ is a measure of the flux of total energy across different scales in the inertial range, differences for the value of this asymptote should be expected for systems with different initial values for the ideal invariants H_c and magnetic helicity $H_{mag} = \langle \boldsymbol{a} \cdot \boldsymbol{b} \rangle$, where $\boldsymbol{a}$ is the vector potential $\boldsymbol{b} = \nabla \times \boldsymbol{a}$. In case of $H_{mag} \neq 0$, the value of $C_{\varepsilon,\infty}$ should be *less* than for $H_{mag} = 0$ due to a more pronounced reverse energy transfer in the helical case [13][30], the result of which is *less* forward transfer and thus a smaller value of the flux of total energy. For $H_c \neq 0$ we expect $C_{\varepsilon,\infty}$ to be smaller than for $H_c = 0$, since alignment of $\boldsymbol{u}$ and $\boldsymbol{b}$ weakens the coupling of the two fields in the induction equation, which leads to less transfer of magnetic energy across different scales and presumably also less transfer of kinetic to magnetic energy. In short, one should expect nonuniversal values of $C_{\varepsilon,\infty}$.

Before we compare the model equation with DNS data and address this question of nonuniversality numerically, we briefly outline our numerical method. Equations (1)-(3) are solved numerically in a periodic box of length $L_{box} = 2\pi$ using a fully de-aliased pseudospectral MHD code [31, 32]. All simulations satisfy $k_{max}\eta_{mag,kin} \geqslant 1$, where $\eta_{mag,kin}$ are the magnetic and kinetic Kolmogorov scales, respectively. We do not impose a background magnetic field, and both the initial magnetic and velocity fields are random Gaussian with zero mean, with initial magnetic and kinetic energy spectra of the form $E_{mag,kin}(k) \sim k^4 \exp(-k^2/(2k_0)^2)$, where $k_0 \geqslant 5$ and further simulation details are specified in Table 1 of [27]. The initial relative magnetic helicity is $\rho_{mag}(k) = kH_{mag}(k)/2E_{mag}(k) = 1$ for all runs of series H and zero for the runs labelled NH. The initial relative cross helicity was $\rho_c(0) = H_c(0)/(|\boldsymbol{u}(0)||\boldsymbol{b}(0)|) = 0$ for runs of the H and NH series and $\rho_c(0) = 0.6$ for series CH06H and CH06NH, while initial magnetic and kinetic energies were in equipartition. All spectral quantities have been shell- and ensemble-averaged, with ensemble sizes restricted by computational resources to up to 10 runs per ensemble. The total dissipation rate ε was measured at its maximum.

Figure 1 shows fits of the model equation to DNS data for datasets that differ in the initial value of H_{mag} and H_c. As can be seen, the model fits the data very well. For the series H runs and for $R_- > 70$ it is sufficient to consider terms of first order in R_-, while for the series NH the first-order approximation is valid for $R_- > 100$. The cross-helical CH06H runs gave consistently lower values of C_ε compared to the series H runs, while little difference was observed between series CH06NH and NH. The asymptotes were $C_{\varepsilon,\infty} = 0.241 \pm 0.008$ for the H series, $C_{\varepsilon,\infty} = 0.265 \pm 0.013$ for the NH series, $C_{\varepsilon,\infty} = 0.193 \pm 0.006$ for the CH06H series and $C_{\varepsilon,\infty} = 0.268 \pm 0.005$ for the CH06NH series.

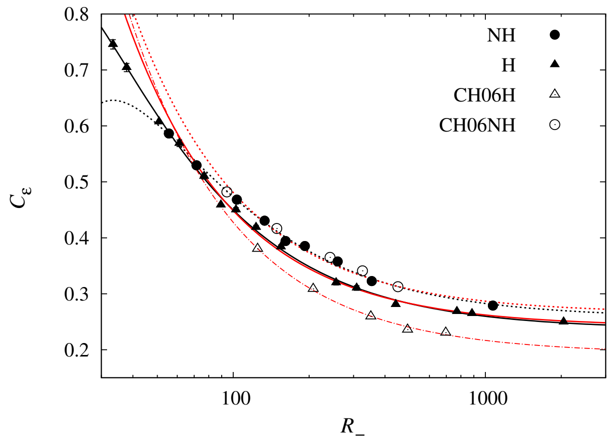

FIG. 1. (Color online) The solid and dotted and dash-dotted lines show (20) fitted to helical, non-helical and cross-helical DNS data, respectively. The red (grey) lines refer to fits using the first-order model equation, the black lines use the model equation up to second order in $1/R_-$. As can be seen, the respective asymptotes differ for the data sets.

As predicted by the qualitative theoretical arguments outlined before, the measurements show that the asymptote calculated from the nonhelical runs is larger than for the helical case, as can be seen in Fig. 1. The asymptotes of the series H and NH do not lie within one standard error of one another. Simulations carried out with $H_c \neq 0$ suggest little difference in C_ε for magnetic fields with initially zero magnetic helicity. For initially helical magnetic fields C_ε is further quenched if $H_c \neq 0$. In view of nonuniversality, an even larger variance of $C_{\varepsilon,\infty}$ can be expected once other parameters such as external forcing, plasma β, Pm, etc., are taken into account. Here we have restricted ourselves to nonuniversality caused by different values of vector field correlations.

In summary, a definition for the dimensionless dissipation rate C_ε for MHD turbulence has been proposed, where ε was non-dimensionalized with respect to the Elsässer fields instead of the rms velocity. For this definition of C_ε and the case of unit Prandtl number we derived a model for the dependence of C_ε on a generalized Reynolds number R_-. The model predicts that $C_\varepsilon \to const$ with increasing R_-, in analogy to hydrodynamics, and the asymptote is a measure of the total energy transfer flux. The model was compared to DNS data for datasets which differ in their initial values of magnetic and cross helicities. At moderate to high R_-, we found good agreement to data with the model only using terms up to first order in $1/R_-$. However, at low R_- terms of second order in R_- cannot be neglected, in fact these terms improve the fit specifically at low R_-. This is expected from adding another term in the expansion and thus provides further justification of the validity of eq. (20).

As predicted, the values of the respective asymptotes from the datasets differ, suggesting a dependence of C_ε on different values of the helicities, and thus a connection to the question of universality in MHD turbulence. This presents an interesting point for further research concerning the influence of other vector field correlations on the dissipation rate. Other questions concern the generalization of this approach to more general MHD systems such as flows with magnetic Prandtl numbers $Pm \neq 1$, compressive fluctuations, and to the presence of a background magnetic field, as well as to turbulent systems where the flow carries other quantities such as temperature or pollutants; and also the application to decaying hydrodynamic turbulence [20]. In the most general case in plasmas there will be a mean magnetic field, which leads to spectral anisotropy and the breakdown of the conservation of magnetic helicity [33] and thus might introduce several difficulties to be overcome when generalizing this method, as the spectral flux will then depend on the direction of the mean field [3, 34] and a more generalized description and role for the magnetic helicity would be needed.

Our model shows that different degrees of correlation in a turbulent plasma control the amount of energy that can effectively be transferred into the smallest scales. It could have several possible applications, e. g. for heating rates in the solar wind, especially as high values of the cross helicity inhibit such transfer to some extent. For situations where one is interested in sustaining a magnetic field over long times, thus trying to minimize dissipative effects, one could estimate from (16)-(17) what type of correlations produce not only a low asymptotic value of the dissipation rate but also a fast approach to this asymptote. This would have relevance to cosmological and astrophysical [35] magnetic fields as well as terrestial plasmas, such as in a tokamak reactor. Our results suggest that in cosmology, where a topical problem is the origin of large-scale magnetic fields, it is not only a nonzero value of magnetic helicity, but perhaps also the parameter range of other correlations such as the cross and kinetic helicities, that facilitate the presence of long-time magnetic fields. Moreover, this raises questions about the possible generation mechanisms for cosmological magnetic fields leading to different correlations between the vector fields such that they can sustain long evolution times.

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In this Supplemental Material we include definitions of the structure and correlation functions used in the Letter. For simulations with zero cross helicity we give numerical verification that the two model equations for C_ε^+ and C_ε^- in fact lead to identical results, as expected. Moreover, we provide a table of simulation details with information on the resolution of our simulations, Reynolds numbers and measured values of the dimensionless dissipation rate.

DEFINITIONS OF THE STRUCTURE AND CORRELATION FUNCTIONS

In order to keep this material self-consistent, we include here the definitions of the Elsässer structure and correlation functions used in the Letter. The third-order correlation and structure functions $C_{LL,L}^{\pm\pm\mp}(r)$, $C_{LL,L}^{\pm\mp\pm}(r)$ and $B_{LL,L}^{\pm\mp\pm}(r)$ and the second-order structure functions $B_{LL}^{\pm\pm}(r)$ are defined as follows:

$$B_{LL,L}^{\pm\mp\pm}(r) = \langle (\delta_L z^\pm(r))^2 \delta_L z^\mp(r)\rangle, \tag{1}$$

$$C_{LL,L}^{\pm\pm\mp}(r) = \langle z_L^\pm(\boldsymbol{x}) z_L^\pm(\boldsymbol{x}) z_L^\mp(\boldsymbol{x}+\boldsymbol{r})\rangle, \tag{2}$$

$$C_{LL,L}^{\pm\mp\pm}(r) = \langle z_L^\pm(\boldsymbol{x}) z_L^\mp(\boldsymbol{x}) z_L^\pm(\boldsymbol{x}+\boldsymbol{r})\rangle, \tag{3}$$

$$B_{LL}^{\pm\pm}(r) = \langle (\delta_L z^\pm(r))^2\rangle, \tag{4}$$

where $v_L = \boldsymbol{v}\cdot\boldsymbol{r}/r$ denotes the longitudinal component of a vector field $\boldsymbol{v}$, that is its component parallel to the displacement vector $\boldsymbol{r}$, and

$$\delta_L v(r) = [\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{x})]\cdot\frac{\boldsymbol{r}}{r}\ , \tag{5}$$

its longitudinal increment.

The transfer term in the energy balance equations for $\boldsymbol{z}^\pm$ are written in the Letter in terms of the function $C_{LL,L}^{\pm\pm\mp}(r)$ for reasons of conciseness. It is common in the literature to use the relation

$$C_{LL,L}^{\pm\mp\pm} = \frac{1}{4}\left(B_{LL,L}^{\pm\mp\pm} - 2C_{LL,L}^{\pm\pm\mp}\right)\ , \tag{6}$$

in order to express the transfer terms through the functions $B_{LL,L}^{\pm\mp\pm}(r)$ and $C_{LL,L}^{\pm\pm\mp}(r)$ instead [1].

VALIDATION FOR DIFFERENT SCALINGS FOR ZERO CROSS HELICITY

The definition of C_ε as in eq. (5) of the Letter was proposed in order to arrive at a consistent scaling for the transfer terms in the energy balance equations for $\boldsymbol{z}^\pm$, since the functions $C_{LL,L}^{\pm\mp\pm}(r)$ which describe the transfer terms scale as $(z^\pm)^2 z^\mp$. This led to separate model equations for C_ε^+ and C_ε^-. In the Letter, we outlined the derivation of the model equation for C_ε^+ from rescaling the evolution equation for $\langle|\boldsymbol{z}^+|^2\rangle$ leading to

$$C_\varepsilon^+ = C_{\varepsilon,\infty}^+ + \frac{C^+}{R_{z^-}} + \frac{D^+}{R_{z^-}^2} + O(R_{z^-}^{-3})\ . \tag{7}$$

In the same way a model equation for

$$C_\varepsilon^- = \frac{\varepsilon L_{z^-}}{{z^-}^2 z^+}\ , \tag{8}$$

is derived through nondimensionalizing the energy balance equation for $\boldsymbol{z}^-$. The flux terms in this equation are given in terms of the function $C_{LL,L}^{-+-}(r)$ (or equivalently $C_{LL,L}^{--+}(r)$ and $B_{LL,L}^{-+-}(r)$), and the dissipative term in terms of $B_{LL}^{--}(r)$.

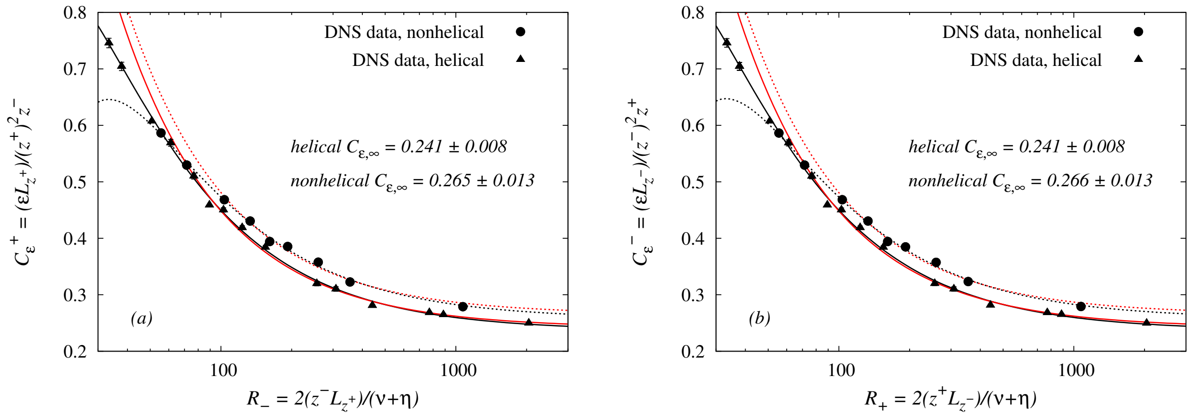

FIG. 1. The solid and dotted lines show the model equations fitted to helical and non-helical DNS data, respectively. Figure (a) shows C_ε^+ and Fig. (b) C_ε^-. In both figures, the red lines refer to fits using the first-order model equations, the black lines use the model equations up to second order in $1/R_{z^\pm}$. By comparing Figs. (a) and (b) it can clearly be seen that both scalings give the same results.

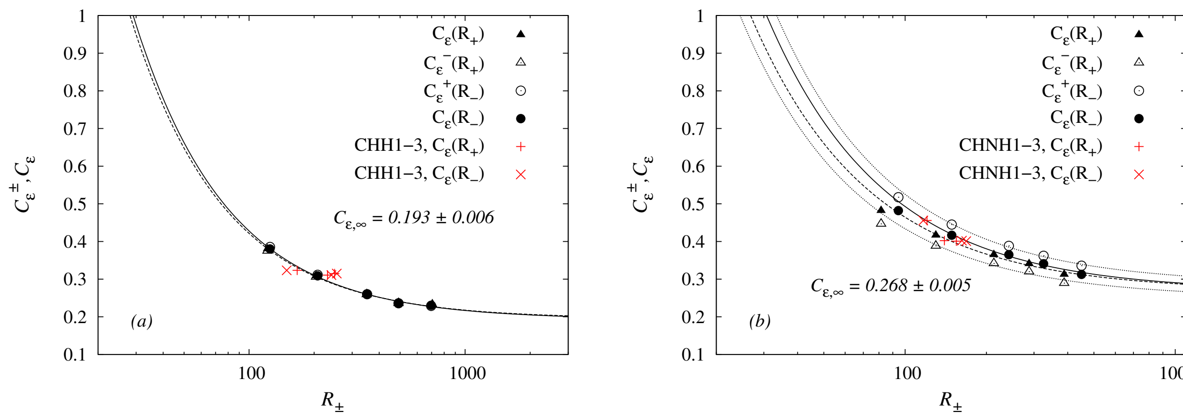

FIG. 2. The solid lines show the model equations for C_ε, C_ε^+ and C_ε^- fitted to data for (a) series CH06H and (b) series CH06NH with initial relative cross helicity $\rho_c(0) = 0.6$. The red crosses refer to additional runs with $0.2 \leqslant \rho_c(0) \leqslant 0.8$.

For zero cross helicity one should expect $C_\varepsilon^+ = C_\varepsilon^-$, since

$$\langle |\boldsymbol{z}^+|^2 \rangle = 2E_{tot} + 2H_c = 2E_{tot} \quad \text{and} \quad \langle |\boldsymbol{z}^-|^2 \rangle = 2E_{tot} - 2H_c = 2E_{tot} \ . \tag{9}$$

Therefore all quantities defined with respect to the rms fields z^+ and z^- should be the same, and thus for zero cross helicity it is possible to use, C_ε^+ or C_ε^- representatively for C_ε.

Figure 1 shows separate fits of the model equations for C_ε^+ and C_ε^- (obtained by the two nondimensionalizations) to DNS data for zero cross helicity. The measured asymptotes are identical and the curves are practically indistinguishable from each other and from those shown in Fig. (1) of the Letter, which was obtained by fitting eq. (20) of the Letter to the data. This provides numerical verification for the representative use of C_ε^+ (or C_ε^-) to describe C_ε.

However, for nonzero cross helicity the difference in the rms values z^+ and z^- precludes the use of C_ε^+ (or C_ε^-) to represent the dimensionless dissipation rate C_ε, and both C_ε^+ and C_ε^- have to be taken into account separately, which is reflected in the definition of the dimensionless dissipation rate in eq. (5) of the Letter. Figure 2 shows data for nonzero cross helicity H_c, that is runs of series CH06H and CH06NH, as well as the additional series CHH and CHNH. Interestingly, very little difference between C_ε^+ and C_ε^- is observed for series CH06H. For series CH06NH, note that the asymptote $C_{\varepsilon,\infty}$ is the same, no matter if extrapolated using R_- or R_+.

SIMULATION SPECIFICATIONS

Run id	N^3	$k_{max}\eta_{mag}$	R_-	R_L	R_λ	η	k_0	#	C_ε	σ	$\rho_c(0)$
H1	128^3	1.30	33.37	25.28	14.87	0.009	5	10	0.756	0.008	0
H2	256^3	2.42	37.77	27.81	15.85	0.008	5	10	0.704	0.007	0
H3	512^3	1.38	50.81	35.08	18.55	0.002	15	10	0.608	0.001	0
H4	256^3	1.80	61.14	40.63	20.34	0.005	5	10	0.569	0.006	0
H5	256^3	1.59	76.72	48.73	23.11	0.004	5	10	0.510	0.005	0
H6	1024^3	1.38	89.32	55.51	25.76	0.00075	23	10	0.4589	0.0003	0
H7	256^3	1.29	102.53	60.65	26.91	0.003	5	10	0.450	0.004	0
H8	512^3	2.33	123.17	69.40	29.67	0.0025	5	10	0.419	0.003	0
H9	512^3	2.01	154.67	83.06	33.84	0.002	5	10	0.384	0.003	0
H10	512^3	1.45	255.89	123.97	45.21	0.0012	5	10	0.320	0.004	0
H11	528^3	1.31	308.69	143.71	50.18	0.001	5	10	0.310	0.004	0
H12	1024^3	2.03	441.25	194.38	61.39	0.0007	5	5	0.281	0.002	0
H13	1032^3	1.38	771.34	309.08	82.97	0.0004	5	5	0.268	0.001	0
H14	1024^3	1.24	885.05	358.72	88.76	0.00035	5	5	0.265	0.002	0
H15	2048^3	1.35	2042.52	724.71	136.25	0.00015	5	1	0.250	-	0
CH06H1	512^3	2.17	124.89	108.81	49.88	0.002	5	1	0.380	-	0.6
CH06H2	512^3	1.57	207.61	171.87	68.57	0.0012	5	5	0.309	0.002	0.6
CH06H3	1024^3	2.21	351.52	277.21	95.31	0.0007	5	1	0.260	-	0.6
CH06H4	1024^3	1.76	491.50	380.70	116.85	0.0005	5	1	0.236	-	0.6
CH06H5	1024^3	1.37	696.19	523.08	132.48	0.00035	5	1	0.231	-	0.6
CHH1	512^3	1.46	254.64	129.83	47.55	0.0012	5	5	0.315	0.002	0.2
CHH2	512^3	1.50	240.35	147.68	55.25	0.0012	5	5	0.311	0.003	0.4
CHH3	512^3	1.74	149.28	195.27	90.60	0.0012	5	5	0.323	0.03	0.8
NH1	256^3	1.51	55.57	53.89	25.57	0.004	5	10	0.587	0.005	0
NH2	256^3	1.26	71.51	68.60	30.11	0.003	5	10	0.530	0.004	0
NH3	512^3	1.86	103.41	96.69	37.68	0.002	5	10	0.468	0.004	0
NH4	512^3	1.51	133.14	122.51	43.94	0.0015	5	10	0.431	0.004	0
NH5	512^3	1.29	161.35	151.76	50.73	0.0012	5	10	0.394	0.004	0
NH6	1024^3	2.28	192.40	168.28	54.44	0.001	5	5	0.358	0.002	0
NH7	1024^3	1.76	259.58	232.10	65.42	0.0007	5	5	0.358	0.002	0
NH8	1024^3	1.40	354.30	301.71	76.73	0.0005	5	5	0.323	0.002	0
NH9	2048^3	1.15	1071.44	823.58	134.73	0.00015	5	1	0.279	-	0
CH06NH1	512^3	2.02	94.39	113.02	49.29	0.002	5	1	0.482	-	0.6
CH06NH2	512^3	1.41	148.86	174.61	65.48	0.0012	5	5	0.417	0.003	0.6
CH06NH3	1024^3	1.93	242.06	272.85	87.50	0.0007	5	1	0.365	-	0.6
CH06NH4	1024^3	1.52	325.62	365.54	104.45	0.0005	5	1	0.341	-	0.6
CH06NH5	1024^3	1.16	450.01	515.23	127.09	0.00035	5	1	0.313	-	0.6
CHNH1	512^3	1.31	168.31	152.84	51.40	0.0012	5	5	0.401	0.003	0.2
CHNH2	512^3	1.34	162.27	157.48	55.52	0.0012	5	5	0.402	0.006	0.4
CHNH3	512^3	1.58	116.71	204.78	87.52	0.0012	5	5	0.456	0.002	0.8

TABLE I. Specifications of simulations. H refers to an initially helical magnetic field, NH to an initially non-helical magnetic field. R_L denotes the integral-scale Reynolds number, R_λ the Taylor-scale Reynolds number, R_- the generalized Reynolds number as in given in eq. (8) of the Letter, η the magnetic resistivity, k_0 the peak wavenumber of the initial energy spectra, k_{max} the largest resolved wavenumber, η_{mag} the Kolmogrov microscale associated with the magnetic field at the peak ot total dissipation, # the ensemble size, C_ε the dimensionless total dissipation rate defined in eq. (5) of the Letter, σ the standard error on C_ε and $\rho_c(0)$ the initial relative cross helicity. All Reynolds numbers are measured at the peak of total dissipation.

ENERGY AND CROSS-HELICITY FLUXES

In the Letter it is pointed out that the asymptotes $C^{\pm}_{\varepsilon}$ describe the total energy flux, as the contribution of the cross-helicity flux to the Elsässer flux is cancelled by the respective terms $G^{\pm}_0$. Here we provide some further details. The asymptotes $C^{\pm}_{\varepsilon,\infty}$ were given in the Letter as

$$C^{\pm}_{\varepsilon,\infty} = -\frac{\partial_\sigma}{\sigma^4}\left(\frac{3\sigma^4}{2} g_0^{\pm\mp\pm}\right) - \frac{3}{4}F^{\pm}_0 \pm G^{\pm}_0 \ , \tag{10}$$

which reduced at the peak of dissipation to

$$C^{\pm}_{\varepsilon,\infty} = -\frac{\partial_\sigma}{\sigma^4}\left(\frac{3\sigma^4}{2} g_0^{\pm\mp\pm}\right) \pm G^{\pm}_0 \ . \tag{11}$$

The dimensional version of this equation is

$$\varepsilon = -\frac{\partial_r}{r^4}\left(\frac{3r^4}{2} C^{\pm\mp\pm}_{LL,L}(r)\right) \pm \partial_t H_c \ , \tag{12}$$

where we assume that the function $C^{\pm\mp\pm}_{LL,L}$ has its inertial range form corresponding to $g_0^{\pm\mp\pm}$. The function $C^{\pm\mp\pm}_{LL,L}$ can be written in terms of $B^{\pm\mp\pm}_{LL,L}$ and $C^{\pm\pm\mp}_{LL,L}$

$$C^{\pm\mp\pm}_{LL,L} = \frac{1}{4}\left(B^{\pm\mp\pm}_{LL,L} - 2C^{\pm\pm\mp}_{LL,L}\right) = \frac{1}{4}\left(\langle(\delta_L z^{\pm}(r))^2\delta_L z^{\mp}(r)\rangle - 2\langle z^{\pm}_L(\boldsymbol{x}) z^{\pm}_L(\boldsymbol{x}) z^{\mp}_L(\boldsymbol{x}+\boldsymbol{r})\rangle\right) \ , \tag{13}$$

which can be written in terms of the primary fields $\boldsymbol{u}$ and $\boldsymbol{b}$ as

$$\begin{aligned} C^{\pm\mp\pm}_{LL,L} &= \frac{1}{4}\left(\langle(\delta_L z^{\pm}(r))^2\delta_L z^{\mp}(r)\rangle - 2\langle z^{\pm}_L(\boldsymbol{x}) z^{\pm}_L(\boldsymbol{x}) z^{\mp}_L(\boldsymbol{x}+\boldsymbol{r})\rangle\right) \\ &= \frac{1}{4}\frac{2}{3}\langle(\delta_L u(r))^3 - 6b_L(\boldsymbol{x})^2 u_L(\boldsymbol{x}+\boldsymbol{r})\rangle \mp \frac{1}{4}\frac{2}{3}\langle(\delta_L(b(r))^3 - 6u_L(\boldsymbol{x})^2 b_L(\boldsymbol{x}+\boldsymbol{r})\rangle \ , \end{aligned} \tag{14}$$

see e.g. [1], and the two terms on the last line of (14) are related to the flux terms in the evolution equations of the total energy and the cross helicity [1].

Now going back to (12), we can write in the inertial range

$$\begin{aligned} \varepsilon &= -\frac{\partial_r}{r^4}\left(\frac{3r^4}{2} C^{\pm\mp\pm}_{LL,L}(r)\right) \pm \partial_t H_c \\ &= -\frac{\partial_r}{r^4}\left(\frac{r^4}{4}\left[\langle(\delta_L u(r))^3 - 6b_L(\boldsymbol{x})^2 u_L(\boldsymbol{x}+\boldsymbol{r})\rangle \mp \langle(\delta_L(b(r))^3 - 6u_L(\boldsymbol{x})^2 b_L(\boldsymbol{x}+\boldsymbol{r})\rangle\right]\right) \pm \partial_t H_c \\ &= \varepsilon_T \pm \varepsilon_C \pm \partial_t H_c = \varepsilon_T \ , \end{aligned} \tag{15}$$

where ε_T is the flux of total energy and ε_C the cross-helicity flux, which must equal $-\partial_t H_c$ in the inertial range for freely decaying MHD turbulence, thus we see that the contribution from the third-order correlator $C^{\pm\mp\pm}_{LL,L}$ resulting in ε_C is cancelled by $\partial_t H_c$, or, after nondimensionalization, the cross helicity flux $\varepsilon_C L_{\pm}/[(z^{\pm})^2 z^{-}]$ is cancelled by $G^{\pm}_0$.

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[1] H. Politano and A. Pouquet, Phys. Rev. E, **57**, R21 (1998).