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Reconciliation of subjective probabilities and frequencies in forensic science

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Abstract

There is a continuous flow of articles published in legal and scientific journals that recite outworn direct or subtle attacks on Bayesian reasoning and/or the use of the subjective or personalistic interpretation of probability. An example is the recent paper written by Kaplan et al. [1] who, by referring to Kafadar's review paper [2], opined, but did not justify, that there is a '[...] need to reduce subjectivity in the evaluation of forensic science' and argued that '[...] the view presented here supports the use of objective probabilities.' [1, at p. 108]. To understand why the objection on the use of subjective probability is not persuasive and why the widely claimed objective probabilities do not exist, one must first scrutinise the historically competing interpretations of probability and their associated definitions. The basis of the defence of the use of the subjectivist interpretation of probability is the understanding of the simple points, misunderstood by critics, that subjectivity is not a synonym for arbitrariness and that the implementation of subjectivism does not neglect the use of the acquired knowledge that is often available in terms of relative frequencies. We will illustrate these points by reference to practical applications in forensic science where probabilities are often represented by relative frequencies. In this regard, our discussion clarifies the connection and the distinction between probabilities and frequencies. Specifically, we emphasise that probability is an expression of our personal belief, an interpretation not to be equated with relative frequency as a mere summary of data. Our argument reveals the inappropriateness of attempts to interpret relative frequencies as probabilities, and naturally solves common problems that derive from such attempts. Further we emphasise that, despite the fact that they can be given an explicit role in probability assignments, neither are relative frequencies a necessary condition for such assignments nor, in forensic applications that consider events for which probabilities need to be specified, need they be meaningfully conceptualised in a frequentist perspective.

Keywords: Probability, Frequency, Degree of belief, Exchangeability.

1. Introduction

As mentioned by Lindley [3], any kind of uncertainty is assessed in the light of the knowledge possessed at the time of the assessment. This idea is not new. The Italian mathematician de Finetti (1930, reprinted in [4]) defined probability – the measure of uncertainty – as a degree of belief,

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insisting that probability is conditional on the status of information of the subject who assesses it. So, if a given person is interested in the probability of an event, say E, that person's probability, $\Pr(E)$, should be written as $\Pr(E \mid I_{s,t})$ where $I_{s,t}$ is the information available to subject s at time t.[1] The subjective nature of probability has lived in many scientific areas. In physics, for example, Schrödinger [6] wrote :

> Since the knowledge may be different with different persons or with the same person at different times, they may anticipate the same event with more or less confidence, and thus different numerical probabilities may be attached to the same event. (at p. 53)

He, then, added that :

> Thus whenever we speak loosely of the probability of an event, it is always to be understood: probability with regard to a certain given state of knowledge. (at p. 54)

Some people think that probability exists as an objective feature of phenomena that can happen following an intrinsic randomness, and they may think that quantum mechanic probabilities are instantiations of these 'objective' probabilities. Let us suppose that they are right. Then we should ask ourselves: what is the bearing of this intrinsic randomness associated with the observation of a particular DNA allele in the biological material from a person (e.g., blood)? or with the guilt of a defendant? A relevant consideration here is Laplace's well-known infamous assertion that if he would have known all the initial conditions, he would have been able to calculate the entire future of the universe. Suppose this assertion is true and that, in the future, a quantum supercomputer will allow us, if we knew all the microscopic objective (perhaps to be thought of as 'ultimate') probabilities, to calculate the exact probability of observing a DNA profile and even that of the defendant's liability, if you are sympathetic with reflections of this kind of science fiction. However, this conjecture still leaves unsolved the problem of what to do now[2]. Let us notice that not only we know today neither these ultimate probabilities nor the scientific laws (if any) for doing any associated calculations. We do not know for sure that these ultimate probabilities exist: we do not know that quantum mechanics is true, we can at best only believe that it is true – and we have strong reason to believe that something is not quite right, because we do know that quantum mechanics and general relativity are mutually incoherent. Thus, even if you agree with the idea of intrinsic randomness in nature, the existence of objective probabilities can only be a matter of belief, not of fact.

De Finetti [8] approached this aspect by affirming:

> [...] most confusion is often arising from mistaking subjective data (like Probabilities and Expectations, concerned always and exclusively with a single event and random quantity), with objective data, like observed successes or failures of single events or observed frequencies, correlations, etc.; many attempts to construct a meaningful notion of 'objective probability' (seemingly: a mythical idealization of an observed frequency)

[1] In [4] (English translation of [5]), de Finetti expressed the concept in the following terms: "Instead of asking ourselves 'what is probability' we shall examine the meaning implicit in the use we intend to make of it and, for this purpose, we shall ask the following three questions: 'probability of what?', and to this shall all answer 'of an event'; and then 'in what circumstances?', and here it is natural to answer 'taking into account all the circumstances known to be relevant at the time'; and lastly 'evaluated by whom?' to which the only possible answer is 'by the subject considering them'; if we like, we could each reply 'by me'. These are the three answers a subjectivist like me would give [...]".

[2] As insisted by Allen [7], the law does not have the privilege to postpone inference and decision: "The law has to decide at the moment in question; it cannot suspend belief while a series of tests is conducted to see if anything useful emerges." (at p. 139)

> have been done, but they could not and cannot lead to any outlet, as any strange attempt to construct a 'spherical cube'. (at p. 1)

In the same perspective, it is useful to remember de Finetti's widely known sentence 'Probability does not exist' (in things) ([9] in the Preface at p.x). Probability is not something that can be known or not known: probabilities are states of mind, not states of nature[3].

Probability represents our assessment about the truth or otherwise of events that may be located in the past, the present or the future, about which our state of knowledge is generally incomplete and about which we are, therefore, uncertain. Probability is our expression of uncertainty about the truth of a proposition, that is about the occurrence of an event described by a proposition, either because that event will occur in the future or it occurred in the past, but the occurrence of which is unknown to us. The offer of a probability as an autonomous conception, independent from individuals, has no meaning. A telling expression of this view can be found in [14]:

> [...] Probability (with a capital P) as a metaphysical entity that exists in abstract is like thinking that it is possible (without being Alice in Wonderland) that the cat's smile can remain and continue to be visible even after the cat has disappeared. (at p. 199)

Several papers have been published in scientific and legal journals, such as *Science & Justice* and *Law, Probability & Risk*, offering a broad discussion about the role of probabilities in forensic science and their interpretation. The subjectivist point of view is supported by some quarters, while others perceive it with skepticism, especially whenever subjectivity is understood as a synonym for arbitrariness. However, suppose for the moment that we are able to overcome the divergences over opposing definitions of probability. Suppose also that, after incoherencies or limitations characterizing alternative definitions have been pointed out, scientists could find an agreement on a subjectivist interpretation of probability for forensic science applications. Then, there is still another potentially unresolved problem: the assignment of a subjective probability. So, even admitting the feasibility of this interpretation, how ought a degree of belief be quantified?[4] How is the strength of our belief in something to be quantified? A standard answer to this question is that subjective probabilities represent degrees of belief conditional on available information, though it needs to be acknowledged that the expression *available information* is rather vague. What does it mean? Available information may consist of personal experience, knowledge from past experience, witnesses and so on. Further, there are many examples of applications where available information takes the form of relative frequencies. Thus a supplementary question is 'how may knowledge in the form of relative frequencies be used to inform probabilities?' One tempting solution is the equivalence of the relative frequency with a probability, a shortcut widely observed and taught in forensic science.

Strictly speaking, frequency is a term that refers to data, whereas probability is a term which refers to personal belief. Statistical and forensic literature have already emphasized an antagonism

[3]This aphorism can also be found in de Morgan [10]: 'Probability is the feeling of the mind, not the inherent property of a set of circumstances' (at p. 7). Analogously, Maxwell [11] wrote: 'The true logic for this world is the calculus of Probabilities, which takes account of the magnitude of the probability which is, or ought to be, in a reasonable man's mind' (at p. 197). Jevons [12] explains that 'Probability belongs wholly to the mind. This is proved by the fact that different minds may regard the very same event at the same time with widely different degrees of probability. [...] Probability thus belongs to our mental condition, to the light in which we regard events, the occurrence or non-occurrence of which is certain in themselves' (at p. 198). More recently, Jaynes [13] reinforced this view by affirming that probabilities express ignorance, states of partial information and if one is ignorant of a phenomenon, that is a fact about one's state of mind, not a fact about the phenomenon.

[4]One possibility, not pursued here, is based on the interpretation of probability assertions as decisions, using proper scoring rules. See [15] for discussion in the context of forensic science.

between subjective probabilities and frequencies. As mentioned by La Caze [16, at p. 358], there are compelling reasons not to identify probabilities with hypothetical frequencies in infinite sequences. Probabilities and hypothetical frequencies are linked but they are not the same; they have a different meaning. There is a certain opposition between the subjectivist and the frequentist interpretation of probability, but this does not have to be interpreted as an opposition between the subjective probabilities and the relative frequencies. As mentioned by Cooke [17], such an opposition

> [...] is nonsense. Subjective probabilities can be, and often are, limiting relative frequency. In particular, this happens when a subject's belief state leads him to regard the past as relevant for the future in a special way. (at p. 108)

This point regarding the relationship between frequency and probability has also been raised in the discussion paper [18]. Forensic scientists often rely on the combination of data on the occurrence of target features, summarized in terms of relative frequencies, and personal knowledge of task-relevant circumstances for a particular case. Unfortunately, from analysis of the discussion and critiques of [18], the paper's message appears to have been understood in some quarters as a philosophical (and puzzling) ban on everything that it is objective. Such an understanding masks a confusion about the personal interpretation of probability, an interpretation that does not prevent the exploitation of information in terms of relative frequencies whenever available [19]. The distinction and the connection between probabilities and frequencies are the main themes of this paper.

To overcome misunderstandings and to clarify our position regarding these aspects, this paper is structured as follows: Section 2 presents a brief summary of the two major alternative definitions of probability other than that of subjectivity. The important role of frequencies is introduced in Section 3 to show that it makes sense to compute relative frequencies for observations and events of interest, and why and how relative frequencies should affect one's beliefs. As noted by de Finetti [5], the convergence of one's personal probabilities towards the value of observed relative frequencies, as the number of observations increases, is a logical consequence of Bayes' theorem if a condition called *exchangeability* is satisfied.

> The hypothesis, or, better, the condition which constitutes our starting point, is instead very clear and simple. From it follows all the conclusions of the ordinary theory of *a posteriori* probabilities, and particularly those that allow the probability to be evaluated on the basis of frequency. It will not be out of place to repeat that its task is this: to show that our mental disposition to expect the future frequency not to differ much from that of the past - unless the fact of having obtained that frequency appeared to us *a priori* as unlikely and exceptional - is *justified* as much as it is meaningful to ask for a justification, and is *explained* as much as it is meaningful to ask for an explanation, if we feel that we are in the following state of mind: of judging two sequences of trials which differ only in their order as equally probable. ([4] at p. 201)[5]

This concept is presented in Sections 4 and 5. Section 6 introduces the statistical syllogism that is required to clarify how a conclusion on a probability may be reached when there is information on a relative frequency and personal knowledge. Conclusions are presented in Section 7.

[5] See Sections 20 to 23 of [4] for an extended description.

2. Classical and frequentist definitions of probability and their limitations

According to the classical definition, probability is defined as the ratio between the number of favorable cases and the number of possible cases, provided that all cases are equally probable. To overcome the evident problem of circularity that affects this interpretation, the term probable is often replaced by the term *possible* (or *likely*), though it seems that the term *possible* (*likely*) is just a synonym of *probable*. In essence, the statement does not define probability, it only offers a way of evaluating it.

The frequentist definition of probability is the limit of the relative frequency of a target event that has occurred in a large number of trials if it is conceivable that the same experiment may be repeated under identical conditions a very large number of times. As well as the problem of circularity, it is clear that this definition limits the range of applications since to use frequency as a measure of probability it must be possible to repeat the experiment a large number of times under identical conditions. In a coin-toss scenario, this is equivalent to saying that the probability of a head (tail) is assessed by imagining that one is able to repeat the tosses a large number of times under identical conditions (e.g., with the same force), and to note the number of times a head has been observed.

This frequentist account of probability is inconceivable operationally for applications in forensic science. Two well-known challenges to the frequentist view is given by Lindley and by de Finetti:

> There is nothing wrong with the frequency interpretation, or chance. It has not been used in this treatment because it is often useless. What is the chance that the defendant is guilty? Are we to imagine a sequence of trials in which the judgements, 'guilty' or 'not guilty', are made and the frequency of the former found? It will not work because it confuses the judgement of guilt, but, more importantly, because it is impossible to conceive of a suitable sequence. Do we repeat the same trial with a different jury; or with the same jury but different lawyers; or do we take all Scottish trials; or only Scottish trials for the same offence? The whole idea of chance is preposterous in this context. ([3] at p. 48)

and

> Finally, even granting the legitimacy of evaluating the frequency-limit by the observed frequency, one would only get back to an intermediate conclusion, which would not constitute a goal having any practical value. Indeed, even those who define probability as the limiting value of the frequency, apply these notions in life and practical examples with a sense of thereby justifying the likelihood of certain forecasts concerning single events, or of combinations of a finite number of single events (that is to say still of single events). On this account, the theory of probability, even for those who do not admit it, will always have as its object the probability of single events; what is only concealed in the steps of the arguments criticized, in which one substitutes for direct arguments about subjective probabilities so defined, formal calculations of fictitious entities (frequency-limits), is rejoined in the premises as much as in practical conclusions to considerations which can only be incomplete as long as one tries to ignore subjective value. ([20] at pp. 188–189).

These examples clarify that the long-run relative frequency definition of probability is inapplicable in many situations arising in real life.

There are implicit assumptions that must apply in each of the classical and frequentist definitions. These assumptions are that according to our state of knowledge, all cases are equally likely and it is theoretically conceivable to perform an experiment a large number of times under identical conditions. Use of these assumptions to assign a numerical value to a probability implies a judgement that these assumptions are satisfied. Definitions that seek to avoid subjectivism are based on acceptance of conventions that are inherently subjective. Scozzafava [21] emphasised this aspect in the following terms:

> Emphasis is usually given to the wider meaning of subjective probability to make up for the lack of objectivity: but even the so-called objective approach involves genuinely subjective aspects (sometimes in a disguised form), and so subjective probability must not be seen as the opposite of the 'objective' one. (at p. 685).

Historically, a critical remark has also been provided by Poincaré [22]:

> The definition, it will be said, is very simple. The probability of an event is the ratio of the number of cases favourable to the event to the total number of possible cases. A simple example will show how incomplete this definition is: ... We are therefore bound to complete the definition by saying '... to the total number of possible cases, provided the cases are equally probable.' So we are compelled to define the probable by the probable. How can we know that two possible cases are equally probable? Will it be by convention? If we insert at the beginning of every problem an explicit convention, well and good! We then have nothing to do but to apply the rules of arithmetic and algebra, and we complete our calculation, when our result cannot be called in question. But if we wish to make the slightest application of this result, we must prove that our convention is legitimate, and we shall find ourselves in the presence of the very difficulty we thought we had avoided. (at p. 185)

The frequentist view presumes the possibility of the performance of a long sequence of trials[6] under identical conditions, with each trial being physically independent of all other trials. These assumptions are typically unachievable in many different applied contexts such as history, law, economy, medicine and, especially, forensic science. In these contexts the entities or events of interest are usually *not* the result of repetitive or replicable processes. On the contrary, they are unique. Such complications do not arise with the personalistic interpretation of probability because it does not consider probability as a feature of the external world. Instead probability is understood as a notion that describes the relationship between a person, you (the reader, us and anybody else), issuing a statement of uncertainty, and the real world to which that statement relates, and in which a person acts. For the subjectivist perspective, it is therefore perfectly reasonable to assign probability to non-repeatable events as in given judicial contexts.

A direct implication of the consideration of probability as a personal degree of belief is that all the knowledge, experience and information on which the individual relies become important.

[6]Note that 'trial' here means an experiment, that is any procedure that can be infinitely repeated and has a well-defined set of possible outcomes, not a judicial trial.

More explicitly formulated, this means that an individual assessment of the degree of belief in the truth of a given statement or in the occurrence of an event (i) depends on information, (ii) may change as the information changes, and (iii) may vary amongst individuals because different individuals may have different information or assessment criteria. A relevant question is how we can quantify our beliefs. Literature on this topic is abundant; for example [23, 24] deal with a deep discussion about different methodologies to offer an operational perspective for subjective probabilities[7]. A widely known possibility is to measure probabilities maintained by an individual in terms of bets that the individual is willing to accept (e.g., the probability of a proposition can be elicited by comparing two lotteries of the same price). The betting scheme is particularly useful here as it allows the introduction and analysis of the very closely related question of the appropriateness of a probability (or set of probabilities related to a sequence of events) held by a particular individual. De Finetti [25] showed that coherence, a simple economic behavioral criterion, implies that a given individual should avoid a combination of probability assignments that is guaranteed to lead to loss. All that is needed to ensure such an avoidance is for uncertainty to be represented and manipulated using the theory of probability. In this context, the possibility of representing subjective degrees of belief in terms of betting odds is often forwarded as part of a line of argument to require that subjective degrees of belief should satisfy the laws of probability. This line of argument takes two parts. The first is that betting odds should be coherent, in the sense that they should not be open to a sure-loss contract. The second part is that a set of betting odds is coherent if and only if it satisfies the laws of probability. The Dutch Book argument encompasses both parts: the proof that betting odds are not open to a sure-loss contract if and only if they are probabilities is called the 'Dutch book Theorem'. Thus, if an individual translates his state of knowledge in such a manner that the assigned probabilities, as a whole, do not respect the laws of probability (standard probability axioms), then his assignments are not coherent.

In many forensic fields (e.g., DNA) the bets and lottery schemes to assign probabilities are – from a practical point of view – difficult to apply, especially when the events of interest are considered rare. A typical example for this is the assignment of a probability for the event of observing a particular DNA profile in a person of interest. It is thus tempting to ask if this signifies the end of an approach to probability based on personal beliefs. We do not think so even if probability elicitation is not a simple task for practitioners [26]. Some of the current controversies are linked to this question. Setting aside these controversies, in many cases the so-called 'state of knowledge' consists of a relative frequency (e.g., the relative frequency of individuals in a sample from a population, presenting a target characteristic of interest). Discarding such information on the grounds of a philosophical objection to a frequentist interpretation of probability would be unwise, and is not what is meant by embracing a subjectivist position. It thus becomes fundamental and unavoidable to take a closer look at how relative frequencies can be used to help individuals in their belief assignment. It is also important to distinguish between a frequentist interpretation of probability and a relative frequency that can be used to help quantify one's degrees of belief.

[7]De Finetti framed the operational perspective as follows: 'However, it must be stated explicitly *how* these subjective probabilities are *defined*, i.e. in order to give an *operative* (and not an empty verbalistic) definition, it is necessary to indicate a procedure, albeit idealized but not distorted, an (effective or conceptual) experiment for its *measurement*.' [14, at p. 212]

3. Frequency and belief

Many scientists equate subjective probability with the arbitrary expression of belief of an individual about events or characteristics of interest, quantified somehow, according to their state of knowledge[8]. Such an equation amounts to the consideration of the assignment of a probability as an unfounded (i.e., arbitrary) guess. Following the same argument, acceptance of such a probability could be considered a statement of faith, with all the associated personal bias. The characterisation of the personalistic view of probability as arbitrary suggests that the use of relative frequencies to inform a probabilistic view would be an implicit rejection of the personalistic view. On the contrary, consideration of relative frequencies within a subjective view of probability is to use available information to help quantify personal degrees of beliefs in probabilistic terms. A key misunderstanding is concerned with the apparent relationship between frequencies and beliefs, a misunderstanding which regards frequencies and beliefs as equivalent, since frequency data can be used to inform probabilities. Lindley has noted [3] :

> There are, as we have seen with the defendant's guilt, occasions where probability exists but [frequency] does not. There are other situations where both exist but are different [...] All that you can do is use the chance [...] as data, or evidence, to help assess your belief. (at p. 49)

He also remarks:

> There are occasions where probability and [frequency] are numerically the same. A case was encountered earlier. A forensic scientist's belief that a stain will be of a particular blood type may be the frequency of that type in the population. (The population acting as a sequence of people.) There is nothing wrong with this. What is happening is that the (frequency) data of blood type is being used as a basis for the beliefs. In our notation, if I is the event of having that blood type, $\Pr(I \mid K) = p$, where K is the data on the population and p is the frequency from that data. (at p. 49)

In the Bayesian framework, the expression of degrees of belief as proportions (i.e., relative frequencies) is not only fully acknowledged as reasonable but takes the form of a mathematical theorem, the *de Finetti representation theorem* [28]. The theorem says that the convergence of one's personal probabilities of an event E towards the values of observed proportions (e.g. the probability to observe a given DNA genotype of interest converges towards a relative frequency), as the number of observations increases[9], is a logical consequence of Bayes' theorem if a condition called *exchangeability* is satisfied by our degrees of belief prior to observations.

De Finetti's theorem enables one to say that a probability assignment located around the value of an empirical proportion (relative frequency) is *objective* in the sense that several persons, whose *a priori* probabilities were different, would converge towards the same posterior probabilities, were they to know the same data and share the same likelihoods. This usually happens in the conduct of statistical inference, where likelihoods are provided by the choice of appropriate probability distributions, that is statistical models, so that they are the same for any observer who agrees on the choice of the statistical model.

[8] See also [27].

[9] See D'Agostini [29] for a discussion on this aspect. He noticed that 'It is a matter of fact that (relative) frequency and probability are somehow connected within probability theory, without the need for identifying the two concepts.' (at p. 13).

4. Bayes' paradigm and subjective probability

The fundamental element of the Bayesian paradigm states that all uncertainties characterizing a problem can be described by probabilities or probability distributions. A probability is interpreted as a conditional measure of uncertainty associated with the occurrence of a particular event, a proposition of interest (e.g., the values that a hypothetical unknown quantity of interest may assume), or a future or a missing observation. Probability is conditioned on the available information, the observed data, whenever available, and in some cases the underlying statistical model describing the randomness of the observed data. Probabilities provide a measure of personal degrees of belief in the occurrence of an event under these conditions, or in the range of values a quantity of interest may assume, and so on.

Imagine the quantity of interest is the proportion, say θ, of individuals sharing a given trait of interest. A Bayesian statistical model may be specified by means of two ingredients: a parametric statistical model, $f(x \mid \theta), \theta \in \Theta$, modelling the randomness that is associated with observations x, and a prior probability distribution $\pi(\theta)$ on the parameter, θ. Hence, under the Bayesian paradigm, the uncertainty about a parameter θ is modeled through a probability distribution π on θ, called the prior distribution, that summarizes the knowledge that is available on the value that the parameter θ may assume, before data are obtained. While the prior distribution is a measure of personal degree of belief about θ prior to observing the data, the function $f(x \mid \theta)$ represents the probability of observing data taking certain values given the hypothetical information that θ takes a certain value. Sometimes the Bayesian approach is criticised because it involves a prior distribution which influences the posterior distribution. Others consider the Bayesian approach to be of value precisely *because* it offers the possibility to take into account prior information. Although there are problems of inference where the available information is minimal, or where the results of scientific experiments must be reported as general results, minimally dependent on personal prior beliefs, this is not a hindrance in principle for the Bayesian approach. In fact, efforts have been devoted towards the proposal and construction of prior distributions on the choice of which most observers might agree, so-called non-informative, vague or objective priors. It is important to note, however, that the latter terminology is misleading in the sense that there are no 'non-informative' or 'objective' priors [30]: each prior, by definition, just reflects a well defined opinion which is as distinctive as any other way of expressing prior belief. It should also be pointed out that, even when theoretically admitting objectivity to a prior distribution thus obtained, the model choice itself still represents another source of subjectivity[10]

Bayesian statistical inference about θ is described as the modification of the uncertainty about its unknown value in the light of evidence, and Bayes' theorem specifies how this should be done. Bayes' theorem allows initial information about the parameter θ, represented by the prior distribution $\pi(\theta)$, to be updated by incorporating information contained in the observations, say x. Inference is then based on the posterior distribution, $\pi(\theta \mid x)$, the distribution of θ conditional on x:

[10] 'Bayesian probabilities can be viewed as 'subjective' or 'personal' but it is not necessary. Or, to put it another way, if you want to label my posterior distribution as 'personal' because it is based on my personal choice of prior distribution, you should also label inferences from the proportionnel hagards model as 'personal' because it is based on the user's choice of the parametrization of Cox (1972); you should also label any linear regression (classical or otherwise) as 'personal' as based on the assumptions of additivity, linearity, variance function and error distribution; and so on for all the very simplest models in existence.' [31] (at pp. 114-115)

$$\pi(\theta \mid x) = \frac{f(x \mid \theta)\pi(\theta)}{\int f(x \mid \theta)\pi(\theta)d\theta} = \frac{f(x \mid \theta)\pi(\theta)}{f(x)}, \tag{1}$$

where $f(x)$ is the marginal distribution of x. Statistical inference about the parameter θ is based, thus, on the modification of the uncertainty about its value in the light of evidence.

Consider the following hypothetical case example. A laboratory receives a consignment of discrete items whose attributes may be relevant within the context of a criminal investigation. The laboratory is requested to conduct analyses in order to gather information that should allow one to draw an inference about, for instance, the proportion of items in the consignment that are of a certain kind (e.g., counterfeit products). The term 'positive' is used here to refer to an item's property that is of interest (e.g., counterfeit); otherwise the result of the analysis is termed 'negative'. This allows the introduction of a random variable X that takes the value 1 (i.e. success) if the analyzed unit is positive and 0 (i.e. failure) otherwise. This is a generic type of case which applies well to many situations, such as surveys or, more generally, sampling procedures conducted to infer the proportion of individuals or items in a population who share a given property or possess certain characteristics (e.g., that of being counterfeit). Suppose now that $n = 10$ units are analyzed, so that there are $2^n = 1024$ possible outcomes. The forensic scientist should be able to assign a probability to each of the 1024 possible outcomes. At this point, if it was reasonable to assume that only the observed values $x_1, x_2, \ldots, x_n$ matter and not the order in which they appear, the forensic scientist would have a sensibly simplified task. In fact, the total number of probability assignments would reduce from 1024 to 11, since it is assumed that all sequences are assigned the same probability if they have the same number of 1's, (i.e., successes). This is possible if it is thought that all the items are indistinguishable in the sense that it does not matter which particular item produced a success (i.e., a positive response) or a failure (i.e., a negative response). Stated otherwise, this means that one's probability assignment is invariant under changes in the order of successes and failures. If the outcomes were permuted in any way, assigned probabilities would be unchanged. For a coin-tossing experiment, Lindley has expressed this as follows [24]:

> One way of expressing this is to say that any one toss, with its resulting outcome, may be exchanged for any other with the same outcome, in the sense that the exchange will not alter your belief, expressing the idea that the tosses were done under conditions that you feel very identical. (at p. 148)

Formally, this is captured by the notion of *exchangeability*. The set of observations $x_1, ..., x_n$ is said to be exchangeable – for you, under a knowledge base – if their joint distribution is invariant under permutation. A formal definition is as follows:

> The random quantities $x_1, ..., x_n$, are said to be judged exchangeable under a probability measure $\Pr$ if the implied joint degree of belief distribution satisfies $\Pr(x_1, ..., x_n) = \Pr(x_{\pi(1)}, ..., x_{\pi(n)})$ for all permutations π defined on the set $\{1, ..., n\}$. [32, at p. 169]

Exchangeability is discussed here because it allows one, as will be illustrated in the next section, to understand how the implementation of relative frequencies to inform subjective beliefs can be justified.

The notion of exchangeability can be generalized to one of partial exchangeability. An example of the simplest extension of exchangeability, that of marginal partial exchangeability, is given by [32] and concerns laboratory measurements :

> Suppose that $x_1, x_2, ...$ are real-valued measurements of a physical or chemical property of a given substance, all made on the same sample with the same measurement procedure. Under such conditions, many individuals might judge the complete sequence of measurements to be exchangeable. Suppose, however, that sequences of such measurements are combined from k different laboratories, the substance being identical but the measurement procedures varying from laboratory to laboratory. In this case, judgments of exchangeability for each laboratory sequence separately might be appropriate, whereas such a judgment for the combined sequence might not be. (at p. 170)

Such a situation is known as one of 'marginal partial exchangeability' and, with slight adjustments, can also be considered in forensic science applications. Consider, for the sake of illustration, that $x_1, x_2, ...$ are real-valued measurements or observations of a genetical response in human subjects when a particular analysis is applied. If there are different ethnic groups, most individuals would be very reluctant to make a judgment of exchangeability for the entire sequence of results in the general population. However, within each sub-population, a judgment of exchangeability might be regarded as reasonable. Similarly, such a judgment may be made, depending on the type of analysis performed, within each combination of gender and defined age-group[11].

The assumption of exchangeability of events is based upon the state of information of the subject: that is, the subject has no information on which to consider the order of the observations to be relevant, though this consideration is subject to revision when available evidence changes (i.e., the person's knowledge base changes). Exchangeability is a qualified judgment of symmetry based on information, and for this reason it does not suffer of the circularity of the classical definition of probability mentioned above in Section 2.

5. De Finetti's representation theorem

An important consequence of exchangeability is that it provides an existence theorem for a probability distribution $\pi(\theta)$ on a parameter space Θ. Dawid [33] illustrates the relevance of the concept in the following terms :

> When considering a sequence of coin-tosses, for example, de Finetti does not assume – as would typically be done automatically and uncritically – that these must have the probabilistic structure of Bernoulli trials[12]. Instead, he attempts to understand when and why this Bernoulli model might be reasonable. In accordance with his positivist position, he starts by focusing attention directly on Your personal joint probability distribution for the potentially infinite sequence of *outcomes* $(X_1, X_2, ...)$ of the tosses – this distribution being numerically fully determined (and so, in particular, having no

[11] Analogously, see also [24]: 'The records of the doctor observing the presence or absence of a symptom with a disease, you might think exchangeable, though if you knew the sexes of the patients and thought the disease was sex-related, you might not. This example also serves to illustrate an important point, that since the definition of exchangeable depends on your probabilities, it depends on your knowledge base, and a series exchangeable under one base, without knowledge of sex, may fail to be under another, with knowledge of sex.' (at p. 150)

[12] A Bernoulli trial is an experiment with two, and only two, possible outcomes; e.g., the coin toss will result in either a head or a tail.

> "unknown parameters"). Exchangeability holds when this joint distribution is symmetric, in the sense that Your uncertainty would not be changed even if the tosses were first to be relabelled in some fixed but arbitrary way (so that, e.g., X_1 now refers to toss 5, X_2 to toss 21, X_3 to toss 1, etc.). In many applied contexts You would be willing to regard this as an extremely weak and reasonable condition to impose on Your personal joint distribution, at least to an acceptable approximation. De Finetti's famous representation theorem now implies that, assuming *only* exchangeability, we can deduce that Your joint distribution is exactly the same *as if* You believed in a model of Bernoulli trials, governed by some unknown parameter p, and had personal uncertainty about p (expressed by some probability distribution on $[0, 1]$). In particular, You would give probability 1 to the existence of a limiting relative frequency of H in the sequence of tosses, and could take this limit as the definition of the "parameter" p. (at p. 45)[13]

This above provides a concise statement of de Finetti's representation theorem, a mathematically simple but powerful result providing a connection between the subjectivist and the objectivist perspective.

Consider a binary sequence $(X_1, X_2, \dots)$, that is values are taking only values 1 (i.e. success) and 0 (i.e. failure) representing, for example, the 'presence' or 'absence' of a given feature such as a given DNA genotype, a Y-chromosome sequence, or a target substance in an item from a consignment. Note that experiments which lead to such events are called Bernoulli trials and the sequence of X_is a Bernoulli sequence. The theorem can be stated informally as follows. If such sequence of observations is exchangeable, then any finite subset can be considered as a random sample from a Bernoulli distribution, denoted $\mathrm{Ber}(\theta)$, and there exists a prior distribution for θ, where θ is the limiting relative frequency of the number of successes.

A note on terminology is necessary to clarify the language and to avoid misleading conclusions. In particular, there may arise a confusion between the meaning of exchangeability and that of independence. The two concepts are not equivalent, and in fact the notion of exchangeability is stronger than the notion of independence. Under a frequentist perspective, consider a sequence of trials $(X_1, X_2, \dots)$ and treat it as a sequence of independent and identically distributed Bernoulli trials, having some unknown success probability θ. Consider, for sake of illustration, a coin-toss scenario. If we consider the results, heads and tails, and consider such results as independent and identically distributed, this means that $\Pr(X_n = x_n \mid X_1 = x_1, X_2 = x_2, \dots, X_{n-1} = x_{n-1}) = \Pr(X_n = x_n)$, so that the results on the first $n-1$ tosses, will not alter my uncertainty on the successive n^{th} toss. Independence implies that one believes that it is impossible to learn from experience. In contrast, in the subjectivist point of view, probability is an expression of the degree of belief (uncertainty) of the experimenter rather than an attribute of the experiment itself, may be evaluated by using available information, and will be updated as soon as new information is available (future trials). In this respect, [8] wrote:

> 'independence' does not exist, because the result of any 'trial' (to use once the current terminology) is informative, so that it modifies the probability of the future 'trials' as evaluated by everybody in such a situation. (at p. 5)

A simple learning scheme to revise personal probabilities after each trial is represented by the Polya urn model. Consider an urn containing b black balls and w white balls. According to this

[13] The personal pronoun 'You' is capitalised to indicate that it is the personal knowledge and probabilities of 'You', the reader, that are to be used.

method, after each draw, the extracted ball is replaced and a ball of the same colour is also added, increasing the number of balls in the urn by 1. In this way, the observation of a ball of a given colour, increases the degree of belief that a ball of the same colour is observed at the next draw. Let $X_i = 1$ if the i^{th} draw yields a white ball, and $X_i = 0$ otherwise. We can show that the original sequence is exchangeable, but events are not independent. Note for example the following:

$$\Pr(1,0,1,0) = \frac{w}{w+b} \times \frac{b}{w+b+1} \times \frac{w+1}{w+b+2} \times \frac{b+1}{w+b+3}$$

$$\Pr(0,0,1,1) = \frac{b}{w+b} \times \frac{b+1}{w+b+1} \times \frac{w}{w+b+2} \times \frac{w+1}{w+b+3}$$

The events are not independent since the observation of a ball of a given colour will alter the probability a ball of the same colour is observed at the next trial, but events are exchangeable as their joint probability is invariant under permutation. Past results modify the current assignment of probability. Assumption of a process of successive extraction with replacement of a ball from an urn of unknown composition implies that, as noticed by [34] 'if it is hypothesised that the successive extractions represents an "independent and identically distributed" process, we cannot, by logic, use the results of the previous extractions to infer the contents of the urn. We can do it if we hypothesise that the process of the successive extractions is an exchangeable process, accepting not independence but only the condition that however you change the order of the random variables of the process, the probabilities that characterise the process do not change.' (at p. 51)[14] The reader is referred to the Appendix for an example related to forensic science, where independence is not needed for a solution to be obtained.

More formally, de Finetti's representation theorem states that if it is reasonable, for you, to assume that, for any indefinitely extendible sequence of binary random quantities $(x_1, x_2, ...)$, the order of the labelling of trials is irrelevant, then there exists a probability distribution F on the interval $[0, 1]$ such that, for any finite sequence of observations $(x_1, ..., x_n)$,

$$\Pr(x_1, x_2, ..., x_n) = \int_0^1 \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} d\mathrm{F}(\theta),$$

where F is the function of the limiting relative frequency

$$\mathrm{F}(\theta) = \Pr(Y \le \theta), \qquad \text{with } Y = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i.$$

The long-run relative frequency is treated itself as a random variable. This result allows it to be shown that a subjective assessment regarding a sequence of exchangeable observations is equivalent to placing a prior probability distribution on the long-run relative frequency. The probabilities

[14]Note also, as done by [20] for the example of an urn of unknown composition, that 'what is unknown is the composition of the urn, not the probability: the latter is always known and depends on the subjective opinion about the composition, which opinion is modified as new drawings are made, and observed frequencies are taken into account' (at p. 214). On the same line of reasoning, see [35] who wrote 'we must not speak of independent events with constant but unknown probability, but of drawing out of an urn with constant but unknown percentage, which is independent - subordinate to the hypothesis - of a given percentage (for example, the real one if we could say which it is)' (at p. 14).

a forensic scientist is interested in can then be assigned with the use of a relative frequency. Formally, from a frequentist perspective, a forensic scientist will assign

$$\Pr(X = x \mid \theta) = \theta, \text{where} \quad \theta = \frac{\sum_{i=1}^{n} x_i}{n}.$$

Alternatively, from a Bayesian perspective, the forensic scientist will assign $\Pr(X = x)$ as

$$\Pr(X = x) = \int_0^1 \theta d\mathrm{F}(\theta),$$

where F is a beta prior distribution with parameters $\alpha = \sum_{i=1}^{n} x_i$ and $\beta = n - \sum_{i=1}^{n} x_i$. It can be shown (see the Appendix in [18]) that

$$\Pr(X = x) = \frac{\sum_{i=1}^{n} x_i}{n}.$$

In other words, if you assign the same probability to all the sequences of length n with the same number of 1s, that is, only the number of 1s in n trials matters, and not the order in which they are observed, then any finite sequence of observations can be considered as a unique 'weighted' mixture of draws with replacement from a possibly infinite set of urns, containing balls marked 1 and 0 in different proportions θ (these fictional 'proportions' are mathematical limits but, as such, the theorem proves that they exist). The 'weights' are the probabilities F expressing your beliefs about the 'true' urn from which the drawing is made.

De Finetti's approach provides a subjective explanation for the existence of objective chance. In this respect, Zabell [36] noticed:

> The existence of limiting frequencies (Z) emerges as a mathematical consequence of the qualitative symmetry assumption of exchangeability, rather than as a dubious (in part because untestable) physical assumption about the existence of infinite limiting frequencies. (at p. 328)

6. Exchangeability in practice

Comprehension of the nature of probability as an individual's belief, reflecting a person's knowledge base, is both an insightful and valuable perspective for forensic scientists. This approach to probability can cope with a combination of personal knowledge of circumstances for a particular case and, where available, relative frequencies observed in relevant databases. Clearly, if relative frequencies are available, personal degrees of belief should take them into account [19]. However, consideration of relative frequencies and databases implies strong and highly idealised assumptions, for example of replicable experiments under identical conditions and of a meaningful selection of databases from relevant populations. While such thinking may be feasible where the focus is on the occurrence (e.g. rarity) of selected features of interest, there is a large number of practical situations both in forensic science and beyond, where events for which probabilities need to be assigned are conditioned on unique and non-replicable sets of circumstances, requiring other methods based on, for example, scoring rules [15].

The use of available data does *not* mean that a probability assignment based upon relative frequencies is objective in the sense that there exists some kind of intrinsic link between frequencies and probabilities which is independent of all non-frequency related information. Any probability judgment in a particular case, even when the judgment is frequency-based, has a component based on personal knowledge. A singular probability judgment is subsumed under a statistical law by an argument which is sometimes called the *statistical syllogism*, namely that there is a major premiss, a minor premiss and a conclusion. Consider situations involving individuals where, for example, the practitioners are interested in a given DNA genotype, or situations involving objects such as footwear mark of a given size. The framework of the argument is as follows, where (1)–(3) are premisses and (4) is the conclusion:

1. The relative frequency of property Q in a sample from a population R is γ.
2. a is an individual or object in the population R.
3. Nothing else is known about a which is relevant with respect to possession of property Q by a.
4. The probability that a has property Q is γ.

A practical example is the following:

1. The relative frequency of DNA profile Q in a sample from the population R is γ.
2. The unknown criminal is an individual in the population R.
3. Nothing else is known about the criminal which is relevant with respect to possession of profile Q.
4. The probability that the criminal has profile Q is γ.

It is at the third premiss that personal knowledge, separate from the relative frequency, may be used in consideration of the probability. In these examples, the premiss is that there is no relevant personal knowledge. The conclusion given then follows. The conclusion does not follow necessarily from the premisses even though it is the conclusion that would be accepted by most people. If someone believes that the probability is different from γ, this would sound unreasonable. However, it would not be a logical contradiction because people are free to make their own choice of probability. The subjectivist Bayesian version of the statistical syllogism is as follows:

1. The relative frequency of property Q in a sample from the population R is γ.
2. a is an individual or object in the population R.
3. a has, *for you*, the same probability of possessing property Q as any other individual in the population R (exchangeability condition, a qualified judgement of symmetry).
4. The probability, for You, that a has property Q is γ.

In the Bayesian version of the statistical syllogism, when the population is finite, the conclusion does follow from the premisses because, if you believe (3), then (4) is the conclusion of a mathematical deduction; (4) is the only coherent conclusion with premiss (3) and the laws of probability calculus. An example is presented in the Appendix.

Therefore, if probabilities are given a subjectivist interpretation, relative frequencies are known and the probabilities of possession of property of Q for all members of R are taken to be equal, then relative frequencies determine the individual probabilities. On the other hand, if one would have evidence that individual a_i belongs to a subpopulation S of R, with different characteristics which are relevant with respect to Q, then the probability that a_i is Q would not be equal to the probability that an individual in the general population R is Q. One should change one's premisses to obtain a new, valid, argument of the same form (as presented in footnote 9), by substituting S for R and a new value, say γ', for γ, corresponding to the relative frequency of Q's in S, if it is known.

Note that a relative frequency (something observed from an available database from a relevant population) of a particular characteristic is not to be equated with the proportion of the relevant population with that characteristic. The proportion is not known, nor can it be known in many cases, and the relative frequency is only an estimate of it.

7. Conclusion

In the Bayesian paradigm, the tendency to base degrees of belief upon relative frequencies, as previously discussed, is not only fully acknowledged as reasonable, but takes the form of a mathematical theorem, de Finetti's *representation theorem*. The theorem says that the convergence of one's personal probabilities towards the values of observed frequencies, as the number of observations increases, is a logical consequence of Bayes' theorem if a condition called *exchangeability* is satisfied by our degrees of belief, prior to observations.

Despite the mathematical elegance of this argument, it is important to avoid shortcomings in practical proceedings. First and foremost, the possibility to give relative frequencies an explicit role in probability assignment *does not imply* that probabilities can *only* be given when relative frequencies are available. Typically, relative frequency information is not available in cases of non-replicable, singular events, so that other methods of elicitation (based, e.g. on scoring rules) should be implemented to manage these situations. Second, and related to the previous observation, is the understanding that relative frequency, strictly speaking, is a summary of data. It conditions probability assignments, but does not define them [19]. In essence, thus, the mere feasibility of using relative frequencies in the assignment of probabilities should not be taken as a suggestion that the debate about the role of probability in forensic science is a debate about the role of relative frequencies – at best the latter can help where they are available, but they are by no means a necessary condition.

Besides de Finetti's theorem, other so-called convergence theorems have been proved lately demonstrating a generalization of de Finetti's exchangeability [37, 38, 39, 40]. These results justify the assertion that a probability assignment close to the value of an empirical frequency is objective in the sense that several persons, whose *a priori* probabilities were different, would converge towards the same *a posteriori* probabilities, were they to know the same data and share the same likelihoods. This usually happens in the conduct of statistical inference, where likelihoods are provided by the choice of appropriate probability distributions (i.e., statistical models), so that they are the same for any observer who agrees on the choice of the model. This confirms what Mondadori [41] has elegantly expressed as follows:

> It is not possible to break every link between probability and frequency. After all, each of us feels a degree of confidence in the occurrence of a future event as the number of 'analogous' events that have occurred in the past grows; more generally, it is simply a fact that we tend to evaluate the probability of a future event very close to the frequency of 'analogous' events passed by now. (at p. xx, translation by the authors)[15]

[15] Original quote: "Ma non è possibile spezzare ogni legame tra probabilità e frequenza. Dopo tutto, ciascuno di noi *sente* crescere il proprio

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grado di fiducia nel verificarsi di un evento futuro al crescere del numero di eventi ad esso 'analoghi' che si sono verificati in passato; più in generale, è semplicemente un fatto che tendiamo a valutare la probabilità di un evento futuro molto vicina alla frequenza degli eventi passati ad esso 'analoghi'." (at p. xx)

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Appendix

Two examples are given of exchangeability in practice. In the first example, there is a population that consists of three individuals, A, B and C. You[16] only know that two have a property Q and one does not. It is not known which two possess Q. In the second example, there is a population of small size N of which R possess Q and $(N - R)$ do not but R is not known. A sample of size n is taken.

Example 1:

The population consists of three individuals, A, B and C. Two have a property Q and one does not. For You, the three individuals have the same probability of possessing Q. Notice that You do not make any assumption about the numerical value of this probability: You are only saying that $\Pr(-)$ is the same, whatever the value of $\Pr(-)$.

(i) $\Pr(A) = \Pr(B) = \Pr(C)$.

There are three possible combinations of observations (outcomes), given the knowledge You have about the population R (the known frequency):

(ii) $C_1 = (A \text{ is } Q, B \text{ is } Q, \text{ and } C \text{ is not } Q)$; $C_2 = (A \text{ is not } Q, B \text{ is } Q, \text{ and } C \text{ is } Q)$; $C_3 = (A \text{ is } Q, B \text{ is not } Q, \text{ and } C \text{ is } Q)$.

Therefore, by propositional logic and the addition law of probability for mutually exclusive events:

(iii) $\Pr(A) = \Pr(C_1 \text{ or } C_3) = \Pr(C_1) + \Pr(C_3)$; $\Pr(B) = \Pr(C_1 \text{ or } C_2) = \Pr(C_1) + \Pr(C_2)$; $\Pr(C) = \Pr(C_2 \text{ or } C_3) = \Pr(C_2) + \Pr(C_3)$.

From (i) and (iii) it follows immediately:

(iv) $\Pr(C_1) + \Pr(C_3) = \Pr(C_1) + \Pr(C_2) = \Pr(C_2) + \Pr(C_3)$,

and thus (the exchangeability condition):

(v) $\Pr(C_1) = \Pr(C_2) = \Pr(C_3)$.

One and only one of the combinations is true, so that, by the probability axioms, it must hold:

(vi) $\Pr(C_1) + \Pr(C_2) + \Pr(C_3) = 1$.

Therefore, the only numerical evaluation of the individual probability that is coherent with the probability laws and Your opinion (Premise 3) is:

[16]The personal pronoun 'You' is capitalised to indicate that it is the personal knowledge and probabilities of 'You', the reader, that are to be used.

(vii) $\Pr(C_1) = \Pr(C_2) = \Pr(C_3) = 1/3$,

and hence, from (iii), conclusion (4) follows:

(viii) $\Pr(A) = \Pr(B) = \Pr(C) = 2/3$.

Any other numerical estimate would be logically incoherent. The same argument can be extended to any larger but finite population. No assumption of independence of the characteristics of A, B and C has been made.

Example 2

There is a population of small size N of which R possess Q and $(N - R)$ do not but unlike Example 1 R is not known. A sample of size n is taken. As an example of what Q might be, consider tablets in a consignment of drugs; the tablets may be either illicit (Q) or licit. The descriptor 'small' for the population size is used to indicate that removal of a member from the population, as in selection without replacement, effects the probability of possession of Q when the next member is selected for removal. If the first member selected from the population possesses Q, the probability the next member selected also possesses Q is $(R-1)/(N-1)$. The population size N is sufficiently small that $(R-1)/(N-1)$ cannot be approximated meaningfully by R/N. Successive draws from the consignment are not independent.

Let X be the number of members of the sample of size n that possess Q. The probability distribution for X is the hypergeometric distribution and

$$\Pr(X = x) = \frac{\binom{R}{x}\binom{N-R}{n-x}}{\binom{N}{n}}.$$

This distribution does not depend on the order in which the n members are drawn from the population, only on the number x which possess Q and the number $(n-x)$ which do not. The property that the distribution is independent of the order is that of *exchangeability*. As R is not known, it is not possible to determine $\Pr(X = x)$. However, it is possible given values for n, N and x to make inferences about R. A comparison of the frequentist and Bayesian approaches to this small consignment sampling problem is given in [42].