Cylinders in rational surfaces

CYLINDERS IN RATIONAL SURFACES

IVAN CHELTSOV

Abstract. We answer a question of Ciro Ciliberto about cylinders in rational surfaces which are obtained by blowing up the plane at points in general position.

Let S be a smooth rational surface. A *cylinder* in S is an open subset $U \subset S$ such that

$$U \cong \mathbb{C}^1 \times Z$$

for an affine curve Z. Here, the curve Z is just $\mathbb{P}^1$ with finitely many missing points.

The surface S contains many cylinders, and it seems hopeless to describe all of them. Instead of doing this, one can consider a similar problem for *polarized* rational surfaces. This problem has a significant application in affine geometry (see [8, 9, 10]).

Fix an ample $\mathbb{Q}$-divisor A on the surface S.

Definition 1. An A-polar cylinder in S is a Zariski open subset U in S such that

(C) $U \cong \mathbb{C}^1 \times Z$ for some affine curve Z, i.e., U is a cylinder in S,
(P) there is an effective $\mathbb{Q}$-divisor D on S such that $D \sim_{\mathbb{Q}} A$ and $U = S \setminus \mathrm{Supp}(D)$.

One can always choose an ample divisor A such that S contains an A-polar cylinder. This follows from [8, Proposition 3.13]. One the other hand, we have

Theorem 2 ([10, 2])**.** *Suppose that S is a smooth del Pezzo surface and $A \in \mathbb{Q}_{>0}[-K_S]$. Then S contains an A-polar cylinder $\iff$ $K_S^2 \geqslant 4$.*

Theorem 3 ([11, 4])**.** *Suppose that S is a smooth del Pezzo surface and $A \notin \mathbb{Q}_{>0}[-K_S]$. If $K_S^2 \geqslant 3$, then S contains an A-polar cylinder.*

The paper [4] also contains a generalization of Theorem 2. To describe it, we put

$$\mu_A = \inf\Big\{\lambda \in \mathbb{Q}_{>0} \;\Big|\; \text{the } \mathbb{Q}\text{-divisor } K_S + \lambda A \text{ is pseudo-effective}\Big\}.$$

It is well-known that $\mu_A \in \mathbb{Q}$ (see, for example, [12, Theorem 2.1] or [13, Theorem 1]). The number μ_A is known as the Fujita invariant of the divisor A (see [7, Definition 3.1]). Let Δ_A be the smallest extremal face of the Mori cone $\overline{\mathbb{NE}}(S)$ that contains $K_S + \mu_A A$. Denote by r_A the dimension of the face Δ_A. Then $r_A + K_S^2 \leqslant 9$, since S is rational. The number r_A is known as the Fujita rank of the divisor A (see [4, Definition 2.1.1]).

Theorem 4 ([4, Theorem 2.2.3])**.** *Suppose that S is a smooth del Pezzo surface and*

$$r_A + K_S^2 \leqslant 3.$$

Then S does not contain A-polar cylinders.

All varieties are assumed to be algebraic, projective and defined over $\mathbb{C}$.

During the conference *Complex affine geometry, hyperbolicity and complex analysis*, which was held in Grenoble in October 2016, Ciro Ciliberto asked

Question 5. *Suppose that S is a blow up of $\mathbb{P}^2$ at points in general position and*
$$r_A + K_S^2 \leqslant 3.$$
Is it true that S does not contain A-polar cylinders?

Ciliberto also suggested to consider Question 5 modulo [5, Conjecture 2.3].

In this paper, we show that the answer to Question 5 is Yes. To be precise, we prove

Theorem 6. *Suppose that S satisfies the following generality condition:*

($\bigstar$) *the self-intersection of every smooth rational curve in S is at least -1.*

If $r_A + K_S^2 \leqslant 3$, then S does not contain A-polar cylinders.

By [6, Proposition 2.4], if S is obtained by blowing up $\mathbb{P}^2$ at points in general position, then the condition ($\bigstar$) in Theorem 6 is satisfied. Thus, the answer to Question 5 is Yes.

Remark 7. If S is a del Pezzo surface, then the condition ($\bigstar$) in Theorem 6 is satisfied. In fact, if $K_S^2 \geqslant 1$, then this condition is equivalent to the ampleness of the divisor $-K_S$. This shows that Theorem 6 is a generalization of Theorem 4.

By [9, Corollary 3.2], Theorem 13 implies

Corollary 8. *Suppose that S satisfies the condition ($\bigstar$), and A is a $\mathbb{Z}$-divisor. Put*
$$V = \mathrm{Spec}\left(\bigoplus_{n\geqslant 0} H^0\Big(S, \mathcal{O}_S\big(nA\big)\Big)\right).$$
If $r_A + K_S^2 \leqslant 3$, then V does not admit an effective action of the group $\mathbb{C}_+$.

The inequality $r_A + K_S^2 \leqslant 3$ in Theorem 6 is sharp:

Example 9. Suppose that S satisfies the condition ($\bigstar$) in Theorem 6, and $K_S^2 \leqslant 3$. Then there exists a blow up $f\colon S \to \mathbb{P}^2$ of $9 - K_S^2$ different points. Put $k = 4 - K_S^2 \geqslant 1$. Denote the $9 - K_S^2$ exceptional curves of the blow up f by E_1, E_2, E_3, E_4, E_5, $G_1, \ldots, G_k$. Let $\mathcal{C}$ be the unique conic in $\mathbb{P}^2$ that passes through $f(E_1)$, $f(E_2)$, $f(E_3)$, $f(E_4)$, $f(E_5)$. Let L be a general line in $\mathbb{P}^2$ tangent to $\mathcal{C}$, and let $\mathcal{P}$ be the pencil generated by $\mathcal{C}$ and $2L$. Let C_i be the conic in $\mathcal{P}$ that contains $f(G_i)$. Then
$$\mathbb{P}^2 \setminus \Big(\mathcal{C} \cup L \cup C_1 \cup \cdots \cup C_k\Big)$$
is a cylinder. Denote the proper transforms of $\mathcal{C}$ and L on S by $\widetilde{\mathcal{C}}$ and $\widetilde{L}$, respectively. Similarly, denote by $\widetilde{C}_i$ the proper transform of the conic C_i on the surface S. Then
$$S \setminus \Big(\widetilde{\mathcal{C}} \cup \widetilde{L} \cup E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup \widetilde{C}_1 \cup \cdots \cup \widetilde{C}_k \cup G_1 \cup \cdots \cup G_k\Big) \cong \mathbb{P}^2 \setminus \Big(\mathcal{C} \cup L \cup C_1 \cup \cdots \cup C_k\Big).$$

Let ϵ_1, ϵ_2 and x be rational numbers such that $\frac{1}{2} > \epsilon_1 > \frac{\epsilon_2}{2} > 0$ and $1 > x > 1 - \frac{1-2\epsilon_1}{2k}$. Put $A = -K_S + x(G_1 + \cdots + G_k)$. Then A is ample and $r_A = k$, since

$$A \sim_{\mathbb{Q}} \Big(1+\epsilon_1-\frac{\epsilon_2}{2}\Big)\widetilde{C}+\epsilon_2\widetilde{L}+\Big(\epsilon_1-\frac{\epsilon_2}{2}\Big)\sum_{i=1}^{5}E_i+\frac{1-2\epsilon_1}{2k}\sum_{i=1}^{k}\widetilde{C}_i+\Big(x+\frac{1-2\epsilon_1}{2k}-1\Big)\sum_{i=1}^{k}G_i.$$

Thus, the surface S contains an A-polar cylinder, and $r_A + K_S^2 = 4$.

The inequality $r_A + K_S^2 \geqslant 4$ does not always imply the existence of A-polar cylinders:

Example 10. Let $f\colon S \to \mathbb{P}^2$ be a blow up of 9 points such that $|-K_S|$ is base point free. Then $|-K_S|$ is a pencil. Suppose that all curves in the pencil $|-K_S|$ are irreducible. This easily implies that the surface S satisfies the generality condition ($\bigstar$) in Theorem 6. Suppose, in addition, that all singular curves in the pencil $|-K_S|$ do not have cusps. Let E_1, E_2, E_3 and E_4 be any four f-exceptional curves. Fix $x \in \mathbb{Q}$ such that $0 < x < 1$. Put $A = -K_S + x(E_1 + E_2 + E_3 + E_4)$. Then A is ample. Moreover, we have $r_A = 4$. Furthermore, if $x > \frac{7}{8}$, then it follows from Example 9 that S contains an A-polar cylinder. But S does not contain A-polar cylinders for $x \leqslant \frac{1}{4}$ by Lemma 12 and Remark 13.

The condition ($\bigstar$) in Theorem 6 cannot be omitted:

Example 11 (cf. [3, Example 4.3])**.** Let L_1 and L_2 be two distinct lines in $\mathbb{P}^2$. Then

$$\mathbb{P}^2 \setminus \big(L_1 \cup L_2\big) \cong \mathbb{C}^1 \times \mathbb{C}^*.$$

Let P_1 be a point in $L_1 \setminus L_2$. Let P_2, P_3, P_4, P_5, P_6 and P_7 be general points in $L_2 \setminus L_1$. Let $f\colon \widehat{S} \to \mathbb{P}^2$ be the blow up of these seven points P_1, P_2, P_3, P_4, P_5, P_6 and P_7. Denote by F_1, F_2, F_3, F_4, F_5, F_6 and F_7 the f-exceptional curves such that $f(F_i) = P_i$. Let $g\colon \widetilde{S} \to \widehat{S}$ be the blow up of the point in F_1 contained in the proper transform of L_1. Denote by G the g-exceptional curve. Let $\widetilde{F}_1$ be the proper transform on $\widetilde{S}$ of the curve F_1. Let $h\colon \overline{S} \to \widetilde{S}$ be the blow up of the point $\widetilde{F}_1 \cap G$. Denote by H the h-exceptional curve. Let $e\colon \mathcal{S} \to \widetilde{S}$ be the blow up of a general point in H. Denote by $\mathcal{E}$ the e-exceptional curve. Denote the proper transforms of the curves H, G, F_1, F_2, F_3, F_4, F_5, F_6, F_7, L_1 and L_2 on the surface $\mathcal{S}$ by the symbols $\mathcal{H}$, $\mathcal{G}$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$, $\mathcal{F}_5$, $\mathcal{F}_6$, $\mathcal{F}_7$, $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Fix a positive rational number ϵ such that $\epsilon < \frac{1}{3}$. Then

$$-K_{\mathcal{S}} \sim_{\mathbb{Q}} (2-\epsilon)\mathcal{L}_1 + (1+\epsilon)\mathcal{L}_2 + (1-\epsilon)\mathcal{F}_1 + \epsilon\sum_{i=2}^{7}\mathcal{F}_i + (2-2\epsilon)\mathcal{G} + (2-3\epsilon)\mathcal{H} + (1-3\epsilon)\mathcal{E}.$$

We also have $\mathcal{S} \setminus (\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{G} \cup \mathcal{H} \cup \mathcal{E}) \cong \mathbb{P}^2 \setminus (L_1 \cup L_2)$. Let $\pi\colon \mathcal{S} \to S$ be the contraction of the curves $\mathcal{L}_1$, $\mathcal{G}$ and $\mathcal{H}$. Then S is a smooth surface. We have $K_S^2 = 2$, the divisor $-K_S$ is nef and big, and $\pi(\mathcal{E}) \cdot \pi(\mathcal{E}) = \pi(\mathcal{L}_2) \cdot \pi(\mathcal{L}_2) = -2$. In particular, the surface S does not satisfies the generality condition ($\bigstar$) in Theorem 6. Let L_{12} be the line in $\mathbb{P}^2$ that contains P_1 and P_2, and let $\mathcal{L}_{12}$ be its proper transform on $\mathcal{S}$.

Fix a positive rational number x such that $\epsilon > x > 3\epsilon - 1$. Then

$$\begin{aligned} -K_{\mathcal{S}} + x\mathcal{L}_{12} \sim_{\mathbb{Q}} & (2-\epsilon)\mathcal{L}_1 + (1+\epsilon)\mathcal{L}_2 + (1-\epsilon)\mathcal{F}_1 + (\epsilon - x)\mathcal{F}_2 + \\ & + \epsilon\Big(\mathcal{F}_3 + \mathcal{F}_4 + \mathcal{F}_5 + \mathcal{F}_6 + \mathcal{F}_7\Big) + (2+x-2\epsilon)\mathcal{G} + (2+x-3\epsilon)\mathcal{H} + (1+x-3\epsilon)\mathcal{E}. \end{aligned}$$

Put $A = -K_S + x\pi(\mathcal{L}_{12})$. Then the divisor A is ample and $r_A = 1$, so that $r_A + K_S^2 = 3$. On the other hand, the surface S contains an A-polar cylinder, since

$$A \sim_{\mathbb{Q}} (1+\epsilon)\pi\big(\mathcal{L}_2\big) + (1-\epsilon)\pi\big(\mathcal{F}_1\big) + (\epsilon - x)\pi\big(\mathcal{F}_2\big) + \epsilon\sum_{i=3}^{7}\pi\big(\mathcal{F}_i\big) + (1+x-3\epsilon)\pi\big(\mathcal{E}\big),$$

and $S \setminus (\pi(\mathcal{L}_2)\cup\pi(\mathcal{F}_1)\cup\pi(\mathcal{F}_2)\cup\pi(\mathcal{F}_3)\cup\pi(\mathcal{F}_4)\cup\pi(\mathcal{F}_5)\cup\pi(\mathcal{F}_6)\cup\pi(\mathcal{F}_7)\cup\pi(\mathcal{E})) \cong \mathbb{C}^1\times\mathbb{C}^*$.

Recall that S is a smooth rational surface. Let $C_1, \ldots, C_n$ be irreducible curves on it. Fix non-negative rational numbers $\lambda_1, \ldots, \lambda_n$. Put $D = \lambda_1 C_1 + \cdots + \lambda_n C_n$.

In the rest of the paper, we prove Theorem 6. Before doing this, let us prove

Lemma 12. *In the assumptions of Example 10, suppose also that $D \sim_{\mathbb{Q}} A$ and $x \leqslant \frac{1}{4}$. Then the log pair (S, D) is log canonical.*

Proof. Suppose that (S, D) is not log canonical at some point $P \in S$. Then $\mathrm{mult}_P(D) > 1$. Let $\mathcal{C}$ be the curve in the pencil $|-K_S|$ that contains P. We may assume that $\mathcal{C} = C_1$. Put $\Delta = \lambda_2 C_2 + \cdots + \lambda_n C_n$. Then $\lambda_1 < 1$, because

$$C_1 + x\big(E_1 + E_2 + E_3 + E_4\big) \sim_{\mathbb{Q}} \lambda_1 C_1 + \Delta.$$

and the log pair $(S, C_1 + x(E_1 + E_2 + E_3 + E_4))$ is log canonical (by assumption).

We have $\lambda_1 > 0$. Indeed, if $\lambda_1 = 0$, then $1 \geqslant 4x = C_1 \cdot \Delta \geqslant \mathrm{mult}_P(D) > 1$.

If C_1 is smooth at P, then the inversion of adjunction (see [1, Theorem 7]) gives

$$1 \geqslant 4x = C_1 \cdot \Delta \geqslant \Big(C_1 \cdot \Delta\Big)_P > 1.$$

Thus, the curve C_1 has a simple node at P. Then $P \notin E_1 \cup E_2 \cup E_3 \cup E_4$.

We may assume that one of the curves E_1, E_2, E_3, E_4 is not contained in $\mathrm{Supp}(\Delta)$. Indeed, if this is not the case, then we can swap D with the divisor

$$(1+\mu)D - \mu\Big(C_1 + x\big(E_1 + E_2 + E_3 + E_4\big)\Big),$$

where μ is the largest positive rational number such that the latter divisor is effective. Without loss of generality, we may assume that $E_4 \not\subset \mathrm{Supp}(\Delta)$. Then

$$1 - x = E_4 \cdot \big(\lambda_1 C_1 + \Delta\big) = \lambda_1 + E_4 \cdot \Delta \geqslant \lambda_1.$$

Put $m = \mathrm{mult}_P(\Delta)$. Then $4x = C_1 \cdot \Delta \geqslant 2m$, so that $m \leqslant 2x$.

Let $f\colon \widetilde{S} \to S$ be the blow up of the point P. Denote by F the f-exceptional curve. Denote by $\widetilde{C}_1$ and $\widetilde{\Delta}$ the proper transforms on $\widetilde{S}$ of the divisors C_1 and Δ, respectively. Then $(\widetilde{S}, \lambda_1\widetilde{C}_1 + \widetilde{\Delta} + (2\lambda_1 + m - 1)F)$ is not log canonical at some point $Q \in F$, since

$$K_{\widetilde{S}} + \lambda_1\widetilde{C}_1 + \widetilde{\Delta} + \big(2\lambda_1 + m - 1\big)F \sim_{\mathbb{Q}} f^*\big(K_S + D\big).$$

Moreover, we have $2\lambda_1+m-1\leqslant 1$, since we already proved that $\lambda_1\leqslant 1-x$ and $m\leqslant 2x$.

If we have $Q\notin\widetilde{C}_1$, then $(\widetilde{S},\widetilde{\Delta}+F)$ is not log canonical at Q, so that

$$\frac{1}{2}\geqslant 2x\geqslant m=F\cdot\widetilde{\Delta}\geqslant\Big(F\cdot\widetilde{\Delta}\Big)_P>1$$

by the inversion of adjunction. This shows that $Q\in\widetilde{C}_1$.

The curve $\widetilde{C}_1$ is smooth and intersects F transversally at Q. We know that $m\leqslant 2x\leqslant 1$. Thus, we can apply [1, Theorem 13] to $(\widetilde{S},\lambda_1\widetilde{C}_1+\widetilde{\Delta}+(2\lambda_1+m-1)F)$. This gives

$$4x-2m=\widetilde{\Delta}\cdot\widetilde{C}_1\geqslant\Big(\widetilde{\Delta}\cdot\widetilde{C}_1\Big)_Q>2\Big(1-\big(2\lambda_1+m-1\big)\Big)$$

or

$$m=\widetilde{\Delta}\cdot F\geqslant\Big(\widetilde{\Delta}\cdot F\Big)_Q>2\big(1-\lambda_1\big).$$

In both cases we obtain a contradiction, since $m\leqslant 2x$ and $\lambda_1\leqslant 1-x$. □

Denote by U the complement in the surface S to the union of the curves $C_1,\ldots,C_n$. To prove Theorem 6, we need the following remark originated from [8, Lemma 4.11].

Remark 13 ([4]). Suppose that $U\cong\mathbb{C}^1\times Z$ for an affine curve Z. Then the exact sequence

$$\bigoplus_{i=1}^{n}\mathbb{Z}\big[C_i\big]\longrightarrow\mathrm{Pic}\big(S\big)\longrightarrow\mathrm{Pic}\big(U\big)$$

implies that $n\geqslant\mathrm{rank}\,\mathrm{Pic}(S)=10-K_S^2$, because we have $\mathrm{Pic}(U)=0$, since $\mathrm{Pic}(Z)=0$. Moreover, the embeddings $Z\hookrightarrow\mathbb{P}^1$ and $\mathbb{C}^1\hookrightarrow\mathbb{P}^1$ induce a commutative diagram

$$\begin{tikzcd}
\mathbb{P}^1\times\mathbb{P}^1 \arrow[dd, "\bar{p}_2"] & \mathbb{C}^1\times\mathbb{P}^1 \arrow[l, hook'] \arrow[dd, "p_2"] & \mathbb{C}^1\times Z\cong U \arrow[l, hook'] \arrow[r, hook] \arrow[d, "p_Z"] & S \arrow[dd, dashed, "\psi"] & \\
& & Z \arrow[dl, hook'] \arrow[dr, hook] & & \mathcal{S} \arrow[ul, "\pi"'] \arrow[dl, "\phi"] \\
\mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 \arrow[rr, equal] & & \mathbb{P}^1 &
\end{tikzcd}$$

where p_Z, p_2 and $\bar{p}_2$ are projections to the second factors, ψ is the map induced by p_Z, the map π is a birational morphism resolving the indeterminacy of ψ, and ϕ is a morphism. Let $\mathcal{E}_1,\ldots,\mathcal{E}_m$ be the π-exceptional curves (if π is an isomorphism, we simply put $m=0$). Let C be the section of the projection $\bar{p}_2$, which is the complement of $\mathbb{C}^1\times\mathbb{P}^1$ in $\mathbb{P}^1\times\mathbb{P}^1$. Denote by $\mathcal{C}_1,\ldots,\mathcal{C}_n$ the proper transforms on $\mathcal{S}$ of the curves $C_1,\ldots,C_n$, respectively. Similarly, denote by the symbol $\mathcal{C}$ the proper transform of the curve C on the surface $\mathcal{S}$. By construction, a general fiber of the morphism ϕ is $\mathbb{P}^1$, and the curve $\mathcal{C}$ is its section. Then the curve $\mathcal{C}$ is either one of the curves $\mathcal{C}_1,\ldots,\mathcal{C}_n$ or one of the curves $\mathcal{E}_1,\ldots,\mathcal{E}_m$. All the other curves among $\mathcal{C}_1,\ldots,\mathcal{C}_n$ and $\mathcal{E}_1,\ldots,\mathcal{E}_m$ are mapped by ϕ to points in $\mathbb{P}^1$. Without loss of generality, we may assume that either $\mathcal{C}=\mathcal{C}_1$ or $\mathcal{C}=\mathcal{E}_m$. We have

$$K_{\mathcal{S}}+\sum_{i=1}^{n}\lambda_i\mathcal{C}_i+\sum_{i=1}^{m}\mu_i\mathcal{E}_i=\pi^*\Big(K_S+\sum_{i=1}^{n}\lambda_iC_i\Big)$$

for some rational numbers $\mu_1,\ldots,\mu_m$. Let $\mathcal{F}$ be a general fiber of ϕ. If $\mathcal{C}=\mathcal{C}_1$, then

$$-2+\lambda_1=\Big(K_{\mathcal{S}}+\sum_{i=1}^{n}\lambda_i\mathcal{C}_i+\sum_{i=1}^{m}\mu_i\mathcal{E}_i\Big)\cdot\mathcal{F}=\pi^*\Big(K_S+\sum_{i=1}^{n}\lambda_iC_i\Big)\cdot\mathcal{F}=\Big(K_S+\sum_{i=1}^{n}\lambda_iC_i\Big)\cdot\pi\big(\mathcal{F}\big).$$

Similarly, if $\mathcal{C}=E_m$, then

$$-2+\mu_m=\Big(K_{\mathcal{S}}+\sum_{i=1}^{n}\lambda_i\mathcal{C}_i+\sum_{i=1}^{m}\mu_i\mathcal{E}_i\Big)\cdot\mathcal{F}=\pi^*\Big(K_S+\sum_{i=1}^{n}\lambda_iC_i\Big)\cdot\mathcal{F}=\Big(K_S+\sum_{i=1}^{n}\lambda_iC_i\Big)\cdot\pi\big(\mathcal{F}\big).$$

In particular, if the divisor $K_S+\sum_{i=1}^{n}\lambda_iC_i$ is pseudo-effective, then

- if $\mathcal{C}=\mathcal{C}_1$, then $\lambda_1\geqslant 2$, so that (S,D) is not log canonical along $C_1=\pi(\mathcal{C})$,
- if $\mathcal{C}=E_m$, then (S,D) is not log canonical at the point $\pi(\mathcal{C})$.

To prove Theorem 6, we also need

Theorem 14 ([2, Theorem 1.12])**.** *Suppose that S is a del Pezzo surface, $K_S^2\leqslant 3$, and*

$$D\sim_{\mathbb{Q}}-K_S.$$

Let P be a point in S. Suppose that the log pair (S,D) is not log canonical at the point P. Then $|-K_S|$ contains a unique divisor T such that (S,T) is not log canonical at P. Moreover, the support of D contains all the irreducible components of $\mathrm{Supp}(T)$*.*

Let $G_1,\ldots,G_r$ be disjoint smooth rational curves on S such that $G_1^2=\cdots=G_r^2=-1$. Let $\epsilon_1,\ldots,\epsilon_r$ be arbitrary non-negative rational numbers. Suppose that

$$-K_S+\sum_{i=1}^{r}\epsilon_iG_i\sim_{\mathbb{Q}}D=\sum_{i=1}^{n}\lambda_iC_i.$$

Lemma 15. *If $C_i\neq G_j$ for all i and j, then (S,D) is log canonical along $G_1\cup\cdots\cup G_r$.*

Proof. Suppose that the log pair (S,D) is not log canonical at some point $P\in G_1\cup\cdots\cup G_r$. Then $\mathrm{mult}_P(D)>1$. If $P\in G_1$ and $G_1\not\subset\mathrm{Supp}(D)$, then

$$1\geqslant 1-\epsilon_1=\Big(-K_S+\sum_{i=1}^{r}\epsilon_iG_i\Big)\cdot G_1=D\cdot G_1\geqslant\mathrm{mult}_P\big(D\big)>1.$$

Similarly, we see that $P\notin G_2\cup\cdots\cup G_r$ provided that $C_i\neq G_j$ for all i and j. □

Let $g\colon S\to\overline{S}$ be the contraction of the curves $G_1,\ldots,G_r$. Observe that $K_{\overline{S}}^2=r+K_S^2$. Put $\overline{C}_1=g(C_1),\ldots,\overline{C}_n=g(C_n)$, and put $\overline{D}=\lambda_1\overline{C}_1+\cdots+\lambda_n\overline{C}_n$. Then $\overline{D}\sim_{\mathbb{Q}}-K_{\overline{S}}$.

Lemma 16. *Suppose that $r+K_S^2=1$, and S satisfies the condition (★) in Theorem 6. Suppose also that $\lambda_i>0$ for every i. Then U is not a cylinder.*

Proof. First we observe that the surface $\overline{S}$ also satisfies the condition (★) in Theorem 6. Then $-K_{\overline{S}}$ is ample by Remark 7. Thus, if $K_S^2=1$, then U is not a cylinder by Theorem 2.

Let us prove the assertion by induction on $K_S^2\leqslant 1$. We may assume that $K_S^2\leqslant 0$.

Suppose that $C_1 = G_1$. Then there exists a commutative diagram

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \widehat{S} \arrow[rr, "h"] & & \overline{S} \end{tikzcd}$$

where $f\colon S \to \widehat{S}$ is a contraction of the curve $C_1 = G_1$, and h is a birational morphism. Denote by $\widehat{G}_2, \ldots, \widehat{G}_r$ the proper transforms on $\widehat{S}$ of the curves $G_2, \ldots, G_r$, respectively. Denote by $\widehat{C}_2, \ldots, \widehat{C}_n$ the proper transforms on $\widehat{S}$ of the curves $C_2, \ldots, C_n$, respectively. Then $K^2_{\widehat{S}} = K^2_S + 1$ and

$$-K_{\widehat{S}} + \sum_{i=2}^{r} \epsilon_i \widehat{G}_i \sim_{\mathbb{Q}} \sum_{i=2}^{n} \lambda_i \widehat{C}_i.$$

By induction $\widehat{S} \setminus (\widehat{C}_2 \cup \cdots \cup \widehat{C}_n) \cong U$ is not a cylinder. Thus, we may assume that $C_1 \neq G_1$. Similarly, we may assume that $C_i \neq G_j$ for all possible i and j.

We have $\lambda_i \leqslant 1$ for each i. Moreover, if $\lambda_i = 1$, then $i = 1$ and $n = 1$, so that $D = C_1$. This follows from the equality $-K_{\overline{S}} \cdot \overline{D} = 1$, because the divisor $-K_{\overline{S}}$ is ample.

Suppose that U is a cylinder. Then $n \geqslant 10 - K^2_S$ by Remark 13. Then $\lambda_i < 1$ for every i.

Let us use notations of Remark 13. Then $\pi(\mathcal{C})$ is a point. Denote this point by P. By Remark 13 and Lemma 15, (S, D) is not log canonical at P and $P \notin G_1 \cup \cdots \cup G_r$.

Put $\overline{P} = g(P)$. Then $(\overline{S}, \overline{D})$ is not log canonical at $\overline{P}$.

By Theorem 14, there is $\overline{T} \in |-K_{\overline{S}}|$ such that $(\overline{S}, \overline{T})$ is not log canonical at the point $\overline{P}$. Moreover, Theorem 14 also implies that the curve $\overline{T}$ is one of the curves $\overline{C}_1, \ldots, \overline{C}_n$. Therefore, we may assume that $\overline{T} = \overline{C}_1$. Note that the curve $\overline{T}$ is singular at $\overline{P}$.

Denote by $m_1, \ldots, m_r$ the multiplicities of the curve $\overline{T}$ at $g(G_1), \ldots, g(G_r)$, respectively. Since the curve $\overline{T}$ is singular at the point $\overline{P}$ by construction, it is smooth away of this point. This implies that either $m_i = 0$ (and $g(G_i) \notin \overline{T}$) or $m_i = 1$ (and $g(G_i) \in \overline{T}$) for all i. Denote by T the proper transform of the curve $\overline{T}$ on the surface S. Then

$$T \sim g^*\big(\overline{T}\big) - \sum_{i=1}^{r} m_i G_i \sim -K_S + \sum_{i=1}^{r} (1 - m_i) G_i.$$

Let us replace D by a divisor $(1+\mu)D - \mu T$ for an appropriate rational number $\mu > 0$. The goal is to obtain an effective divisor that do not contain $T = C_1$ in its support. Put

$$D' = \frac{1}{1-\lambda_1} D - \frac{\lambda_1}{1-\lambda_1} C_1 = \sum_{i=2}^{n} \frac{\lambda_i}{1-\lambda_1} C_i.$$

Then D' is an effective divisor that does not contain $T = C_1$ in its support. But

$$D' \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^{r} \frac{\epsilon_i + (m_i - 1)\lambda_1}{1-\lambda_1} G_i.$$

Thus, if $\frac{\epsilon_i+(m_i-1)\lambda_1}{1-\lambda_1}\geqslant 0$ for every i, then (S, D') is not log canonical at P by Remark 13. In this case, the singularities of the log pair

$$\left(\overline{S}, \sum_{i=2}^{n}\frac{\lambda_i}{1-\lambda_1}\overline{C}_i\right)$$

are not log canonical at $\overline{P}$, since $P\not\in G_1\cup\cdots\cup G_r$. The latter is impossible by Theorem 14. Therefore, at least one of the numbers

$$\frac{\epsilon_1+(m_1-1)\lambda_1}{1-\lambda_1}, \frac{\epsilon_2+(m_2-1)\lambda_1}{1-\lambda_1}, \ldots, \frac{\epsilon_r+(m_r-1)\lambda_1}{1-\lambda_1}$$

must be negative. Thus, we may assume that there exists $k\leqslant r$ such that

$$\frac{\epsilon_i+(m_i-1)\lambda_1}{1-\lambda_1}<0$$

for every $i\leqslant k$, and $\frac{\epsilon_i+(m_i-1)\lambda_1}{1-\lambda_1}\geqslant 0$ for every $i>k$ (if any). Then $m_1=\cdots=m_k=0$. We may also assume that $\epsilon_1\leqslant\cdots\leqslant\epsilon_k$. Put

$$D''=\frac{1}{1-\epsilon_1}D-\frac{\epsilon_1}{1-\epsilon_1}C_1=\frac{\lambda_1-\epsilon_1}{1-\epsilon_1}C_1+\sum_{i=2}^{n}\frac{\lambda_i}{1-\epsilon_1}C_i.$$

Then D'' is an effective divisor such that

$$D''\sim_{\mathbb{Q}}-K_S+\sum_{i=2}^{r}\frac{\epsilon_i-\epsilon_1(1-m_i)}{1-\epsilon_1}G_i=\sum_{i=2}^{k}\frac{\epsilon_i-\epsilon_1}{1-\epsilon_1}G_i+\sum_{i=k+1}^{r}\frac{\epsilon_i-\epsilon_1(1-m_i)}{1-\epsilon_1}G_i.$$

Let $e\colon\widetilde{S}\to\overline{S}$ be the blow up of the point $g(G_1)$, and let $\widetilde{G}_1$ be its exceptional curve. Denote by $\widetilde{C}_1,\ldots,\widetilde{C}_n$ the proper transforms on $\widetilde{S}$ of the curves $C_1,\ldots,C_n$, respectively. Denote by $\widetilde{D}''$ the proper transform of the divisor D'' on the surface $\widetilde{S}$. Then

$$\widetilde{D}''=\frac{\lambda_1-\epsilon_1}{1-\epsilon_1}\widetilde{C}_1+\sum_{i=2}^{n}\frac{\lambda_i}{1-\epsilon_1}\widetilde{C}_i\sim_{\mathbb{Q}}-K_{\widetilde{S}}.$$

Since we have $g(G_1)\not\in\overline{T}$, the point $g(G_1)$ is not the base point of the pencil $|-K_{\overline{S}}|$. Thus, the pencil $|-K_{\overline{S}}|$ contains a unique irreducible curve $\overline{R}$ that passes through $g(G_1)$. Let $\widetilde{R}$ and R be the proper transforms of this curve on the surfaces $\widetilde{S}$ and S, respectively. If $\overline{R}$ is singular at $g(G_1)$, then R is a smooth rational curve such that

$$R^2\leqslant\widetilde{R}^2=-3.$$

This is impossible by our assumption. Then the curve R is smooth at $g(G_1)$ and $\widetilde{R}\sim -K_{\widetilde{S}}$. Since $\widetilde{R}^2=0$ and $\widetilde{C}_1\cdot\widetilde{R}=1$, the divisor $\widetilde{R}$ is nef. Then

$$0=K_{\widetilde{S}}^2=\widetilde{D}''\cdot\widetilde{R}=\frac{\lambda_1-\epsilon_1}{1-\epsilon_1}\widetilde{C}_1\cdot\widetilde{R}+\sum_{i=2}^{n}\frac{\lambda_i}{1-\epsilon_1}\widetilde{C}_i\cdot\widetilde{R}\geqslant\frac{\lambda_1-\epsilon_1}{1-\epsilon_1}\widetilde{C}_1\cdot\widetilde{R}=\frac{\lambda_1-\epsilon_1}{1-\epsilon_1},$$

so that $\epsilon_1\geqslant\lambda_1$. This is a contradiction, since $\epsilon_1<\lambda_1$. $\square$

Lemma 17. *Suppose that $r+K_S^2=2$, and S satisfies the condition ($\bigstar$) in Theorem 6. Suppose also that $\lambda_i>0$ for every i. Then U is not a cylinder.*

Proof. By Remark 7, the surface $\overline{S}$ is a del Pezzo surface. We also have $K_{\overline{S}}^2=K_S^2+r=2$. Therefore, if $K_S^2=2$, then $r=0$ and $S\cong\overline{S}$, so that U is not a cylinder by Theorem 2. Now arguing as in the proof of Lemma 16, we may assume that $C_i\neq G_j$ for all i and j.

By [2, Lemma 3.1] applied to $(\overline{S},\overline{D})$, we have $\lambda_i\leqslant 1$ for each i.

Suppose that U is a cylinder. Then $n\geqslant 10-K_S^2\geqslant 9$ by Remark 13.

Let us use notations of Remark 13. Then $\pi(\mathcal{C})$ is a point. Denote this point by P. Then (S,D) is not log canonical at P and $P\notin G_1\cup\cdots\cup G_r$ by Remark 13 and Lemma 2. Put $\overline{P}=g(P)$. Then $(\overline{S},\overline{D})$ is not log canonical at $\overline{P}$.

By Theorem 14, there is a curve $\overline{T}\in|-K_{\overline{S}}|$ such that $(\overline{S},\overline{T})$ is not log canonical at $\overline{P}$, and all irreducible components of the curve $\overline{T}$ are among the irreducible curves $\overline{C}_1,\ldots,\overline{C}_n$. In particular, the curve $\overline{T}$ is singular at $\overline{P}$. It is uniquely determined by this property.

Note that $\overline{T}$ has at most two irreducible component. Thus, we may assume that

- either $\overline{T}=\overline{C}_1$,
- or $\overline{T}=\overline{C}_1+\overline{C}_2$ and $\lambda_1\leqslant\lambda_2$.

If $\overline{T}=\overline{C}_1$, then $\overline{T}$ has a cusp at $\overline{P}$. If $\overline{T}=\overline{C}_1+\overline{C}_2$, then $\overline{T}$ has a tacknode at $\overline{P}$. In both cases, the point $\overline{P}$ is the unique singular point of the curve $\overline{T}$.

If $\overline{T}=\overline{C}_1$, then $\lambda_1<1$, because

$$2=-K_{\overline{S}}\cdot\overline{D}=\sum_{i=1}^{n}\lambda_i\Big(-K_{\overline{S}}\cdot\overline{C}_i\Big)=2\lambda_1+\sum_{i=2}^{n}\lambda_i\Big(-K_{\overline{S}}\cdot\overline{C}_i\Big)\geqslant 2\lambda_1+\sum_{i=2}^{n}\lambda_i>2\lambda_1.$$

Similarly, if $\overline{T}=\overline{C}_1+\overline{C}_2$, then $\lambda_1<1$, because

$$2=-K_{\overline{S}}\cdot\overline{D}=\lambda_1+\lambda_2+\sum_{i=3}^{n}\lambda_i\Big(-K_{\overline{S}}\cdot\overline{C}_i\Big)>\lambda_1+\lambda_2\geqslant 2\lambda_1.$$

Denote by $m_1,\ldots,m_r$ the multiplicities of the curve $\overline{T}$ at $g(G_1),\ldots,g(G_r)$, respectively. Then either $m_i=0$ or $m_i=1$. Let T be the proper transform of the curve $\overline{T}$ on S. Then

$$T\sim g^*\big(\overline{T}\big)-\sum_{i=1}^{r}m_iG_i\sim -K_S+\sum_{i=1}^{r}(1-m_i)G_i.$$

Put $D'=\frac{1}{1-\lambda_1}D-\frac{\lambda_1}{1-\lambda_1}T$ and $\overline{D}'=\frac{1}{1-\lambda_1}\overline{D}-\frac{\lambda_1}{1-\lambda_1}\overline{T}$. If $\overline{T}=\overline{C}_1$, then

$$D'\sim_{\mathbb{Q}}\sum_{i=2}^{n}\frac{\lambda_i}{1-\lambda_1}C_i.$$

Similarly, if $\overline{T}=\overline{C}_1+\overline{C}_2$, then

$$D'\sim_{\mathbb{Q}}\frac{\lambda_2-\lambda_1}{1-\lambda_1}C_2+\sum_{i=3}^{n}\frac{\lambda_i}{1-\lambda_1}C_i.$$

In both cases D' is an effective divisor that does not contain the curve C_1 in its support. Similarly, the divisor $\overline{D}'$ is effective and does not contain $\overline{C}_1$ in its support. But

$$D' \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^{r} \frac{\epsilon_i + (m_i - 1)\lambda_1}{1-\lambda_1} G_i.$$

Thus, if $\frac{\epsilon_i+(m_i-1)\lambda_1}{1-\lambda_1} \geqslant 0$ for every i, then (S, D') is not log canonical at P by Remark 13. Then $\overline{D}' \sim_{\mathbb{Q}} -K_{\overline{S}}$ and $(\overline{S}, \overline{D}')$ is not log canonical at $\overline{P}$, which contradicts Theorem 14. Hence, at least one of the numbers $\frac{\epsilon_1+(m_1-1)\lambda_1}{1-\lambda_1}, \ldots, \frac{\epsilon_r+(m_r-1)\lambda_1}{1-\lambda_1}$ is negative.

We may assume that $\frac{\epsilon_i+(m_i-1)\lambda_1}{1-\lambda_1} < 0 \iff i \leqslant k$ for some $k \leqslant r$, and $\epsilon_1 \leqslant \cdots \leqslant \epsilon_k$. Then $m_i = 0$ and $\epsilon_i < \lambda_1$ for every $i \leqslant k$.

Put $D'' = \frac{1}{1-\epsilon_1}D - \frac{\epsilon_1}{1-\epsilon_1}T$. Then D'' is an effective divisor. Indeed, if $\overline{T} = \overline{C}_1$, then

$$D'' = \frac{\lambda_1 - \epsilon_1}{1-\epsilon_1} C_1 + \sum_{i=2}^{n} \frac{\lambda_i}{1-\epsilon_1} C_i.$$

Similarly, if $\overline{T} = \overline{C}_1 + \overline{C}_2$, then

$$D'' = \frac{\lambda_1 - \epsilon_1}{1-\epsilon_1} C_1 + \frac{\lambda_2 - \epsilon_1}{1-\epsilon_1} C_2 + \sum_{i=3}^{n} \frac{\lambda_i}{1-\epsilon_1} C_i.$$

Note that $\mathrm{Supp}(D'') = \mathrm{Supp}(D)$. On the other hand, we have

$$D'' \sim_{\mathbb{Q}} -K_S + \sum_{i=2}^{r} \frac{\epsilon_i - \epsilon_1(1-m_i)}{1-\epsilon_1} G_i.$$

Applying Lemma 16 to D'', we see that U is not a cylinder. This is a contradiction. □

Lemma 18. *Suppose that $r + K_S^2 = 3$, and S satisfies the condition ($\bigstar$) in Theorem 6. Suppose also that $\lambda_i > 0$ for every i. Then U is not a cylinder.*

Proof. By Remark 7, the surface $\overline{S}$ is a cubic surface in $\mathbb{P}^3$, because $K_{\overline{S}}^2 = r + K_S^2 = 3$. Moreover, if $K_S^2 = 3$, then $r = 0$ and $S \cong \overline{S}$, so that U is not a cylinder by Theorem 2. Arguing as in the proof of Lemma 16, we may assume that $C_i \neq G_j$ for all possible i and j.

Applying [2, Lemma 4.1] to $(\overline{S}, \overline{D})$, we see that $\lambda_i \leqslant 1$ for each i.

Suppose that U is a cylinder. Then $n \geqslant 10 - K_S^2 \geqslant 8$ by Remark 13.

Let us use notations of Remark 13. Then $\pi(\mathcal{C})$ is a point. Denote this point by P. By Remark 13 and Lemma 2, (S, D) is not log canonical at P and $P \notin G_1 \cup \cdots \cup G_r$.

By Theorem 14, there is a curve $\overline{T} \in |-K_{\overline{S}}|$ such that $(\overline{S}, \overline{T})$ is not log canonical at $\overline{P}$, and all irreducible components of the curve $\overline{T}$ are among the irreducible curves $\overline{C}_1, \ldots, \overline{C}_n$. Note that $\overline{T}$ is cut out on $\overline{S} \subset \mathbb{P}^3$ by a hyperplane that is tangent to $\overline{S}$ at the point $\overline{P}$.

The curve $\overline{T}$ has at most three irreducible component. Thus, we may assume that

- either $\overline{T} = \overline{C}_1$,
- or $\overline{T} = \overline{C}_1 + \overline{C}_2$ and $\lambda_1 \leqslant \lambda_2$.
- or $\overline{T} = \overline{C}_1 + \overline{C}_2 + \overline{C}_3$ and $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$.

If $\overline{T}=\overline{C}_1$, then $\overline{T}$ has a cusp at $\overline{P}$. If $\overline{T}=\overline{C}_1+\overline{C}_2$, then $\overline{T}$ has a tacknode at $\overline{P}$. Finally, if $\overline{T}=\overline{C}_1+\overline{C}_2+\overline{C}_3$, then the curves $\overline{C}_1$, $\overline{C}_2$ and $\overline{C}_3$ are lines passing through $\overline{P}$. In all possible cases, the point $\overline{P}$ is the unique singular point of the curve $\overline{T}$.

If $\overline{T}=\overline{C}_1$, then $\lambda_1<1$, because

$$3=-K_{\overline{S}}\cdot\overline{D}=\sum_{i=1}^{n}\lambda_i\Big(-K_{\overline{S}}\cdot\overline{C}_i\Big)=3\lambda_1+\sum_{i=2}^{n}\lambda_i\Big(-K_{\overline{S}}\cdot\overline{C}_i\Big)\geqslant 3\lambda_1+\sum_{i=2}^{n}\lambda_i>3\lambda_1.$$

Similarly, if $\overline{T}=\overline{C}_1+\overline{C}_2$, then $\lambda_1<1$, because

$$3=\lambda_1\mathrm{deg}\big(\overline{C}_1\big)+\lambda_2\mathrm{deg}\big(\overline{C}_2\big)+\sum_{i=3}^{n}\lambda_i\Big(-K_{\overline{S}}\cdot\overline{C}_i\Big)>\lambda_1\Big(\mathrm{deg}\big(\overline{C}_1\big)+\mathrm{deg}\big(\overline{C}_2\big)\Big)=3\lambda_1.$$

Finally, if $\overline{T}=\overline{C}_1+\overline{C}_2+\overline{C}_3$, then we also have $\lambda_1<1$, because

$$3=-K_{\overline{S}}\cdot\overline{D}=\lambda_1+\lambda_2+\lambda_3+\sum_{i=4}^{n}\lambda_i\Big(-K_{\overline{S}}\cdot\overline{C}_i\Big)>\lambda_1+\lambda_2+\lambda_3\geqslant 3\lambda_1.$$

Denote by $m_1,\ldots,m_r$ the multiplicities of the curve $\overline{T}$ at $g(G_1),\ldots,g(G_r)$, respectively. Then either $m_i=0$ or $m_i=1$. Let T be the proper transform of the curve $\overline{T}$ on S. Then

$$T\sim g^*\big(\overline{T}\big)-\sum_{i=1}^{r}m_iG_i\sim -K_S+\sum_{i=1}^{r}(1-m_i)G_i.$$

Put $D'=\frac{1}{1-\lambda_1}D-\frac{\lambda_1}{1-\lambda_1}T$ and $\overline{D}'=\frac{1}{1-\lambda_1}\overline{D}-\frac{\lambda_1}{1-\lambda_1}\overline{T}$. If $\overline{T}=\overline{C}_1$, then

$$D'\sim_{\mathbb{Q}}\sum_{i=2}^{n}\frac{\lambda_i}{1-\lambda_1}C_i.$$

Similarly, if $\overline{T}=\overline{C}_1+\overline{C}_2$, then

$$D'\sim_{\mathbb{Q}}\frac{\lambda_2-\lambda_1}{1-\lambda_1}C_2+\sum_{i=3}^{n}\frac{\lambda_i}{1-\lambda_1}C_i.$$

Finally, if $\overline{T}=\overline{C}_1+\overline{C}_2+\overline{C}_3$, then

$$D'\sim_{\mathbb{Q}}\frac{\lambda_2-\lambda_1}{1-\lambda_1}C_2+\frac{\lambda_3-\lambda_1}{1-\lambda_1}C_3+\sum_{i=4}^{n}\frac{\lambda_i}{1-\lambda_1}C_i.$$

In all cases D' is an effective divisor that does not contain the curve C_1 in its support. Similarly, the divisor $\overline{D}'$ is effective and does not contain $\overline{C}_1$ in its support. But

$$D'\sim_{\mathbb{Q}}-K_S+\sum_{i=1}^{r}\frac{\epsilon_i+(m_i-1)\lambda_1}{1-\lambda_1}G_i.$$

Thus, if $\frac{\epsilon_i+(m_i-1)\lambda_1}{1-\lambda_1}\geqslant 0$ for every i, then (S,D') is not log canonical at P by Remark 13. Then $\overline{D}'\sim_{\mathbb{Q}}-K_{\overline{S}}$ and $(\overline{S},\overline{D}')$ is not log canonical at $\overline{P}$, which contradicts Theorem 14. Hence, at least one of the numbers $\frac{\epsilon_1+(m_1-1)\lambda_1}{1-\lambda_1},\ldots,\frac{\epsilon_r+(m_r-1)\lambda_1}{1-\lambda_1}$ is negative.

We may assume that $\frac{\epsilon_i+(m_i-1)\lambda_1}{1-\lambda_1}<0 \iff i\leqslant k$ for some $k\leqslant r$, and $\epsilon_1\leqslant\cdots\leqslant\epsilon_k$. Then $m_i=0$ and $\epsilon_i<\lambda_1$ for every $i=1,\ldots,k$.

Put $D''=\frac{1}{1-\epsilon_1}D-\frac{\epsilon_1}{1-\epsilon_1}T$. Then D'' is an effective divisor. Indeed, if $\overline{T}=\overline{C}_1$, then

$$D''=\frac{\lambda_1-\epsilon_1}{1-\epsilon_1}C_1+\sum_{i=2}^{n}\frac{\lambda_i}{1-\epsilon_1}C_i.$$

Similarly, if $\overline{T}=\overline{C}_1+\overline{C}_2$, then

$$D''=\frac{\lambda_1-\epsilon_1}{1-\epsilon_1}C_1+\frac{\lambda_2-\epsilon_1}{1-\epsilon_1}C_2+\sum_{i=3}^{n}\frac{\lambda_i}{1-\epsilon_1}C_i.$$

Finally, if $\overline{T}=\overline{C}_1+\overline{C}_2+\overline{C}_3$, then

$$D''=\frac{\lambda_1-\epsilon_1}{1-\epsilon_1}C_1+\frac{\lambda_2-\epsilon_1}{1-\epsilon_1}C_2+\frac{\lambda_3-\epsilon_1}{1-\epsilon_1}C_3+\sum_{i=4}^{n}\frac{\lambda_i}{1-\epsilon_1}C_i.$$

In all cases $\mathrm{Supp}(D'')=\mathrm{Supp}(D)$. On the other hand, we have

$$D''\sim_{\mathbb{Q}}-K_S+\sum_{i=2}^{r}\frac{\epsilon_i-\epsilon_1(1-m_i)}{1-\epsilon_1}G_i.$$

Applying Lemma 17 to D'', we see that U is not a cylinder. This is a contradiction. □

Let us show that Theorem 6 follows from Lemmas 16, 17 and 18.

Proof of Theorem 6. Let A be an ample $\mathbb{Q}$-divisor on S, let μ_A be its Fujita invariant, and let r_A be its the Fujita rank. Suppose that $r_A+K_S^2\leqslant 3$. Then

$$\mu_A A\sim_{\mathbb{Q}}-K_S+\sum_{i=1}^{r_A}a_iE_i,$$

where $E_1,\ldots,E_{r_A}$ are smooth rational curves on S, and $a_1,\ldots,a_{r_A}$ are positive rational numbers. Moreover, there is a birational morphism $f\colon S\to\widehat{S}$ such that

- the curves $E_1,\ldots,E_{r_A}$ are all f-exceptional curves,
- the surface $\widehat{S}$ is smooth.

This is well-known. See, for example, [12, Theorem 2.1] or [13, Theorem 1].

Suppose, in addition, that the surface S satisfies the condition (★) in Theorem 6. Then the curves $E_1,\ldots,E_{r_A}$ are disjoint and $E_1^2=E_2^2=\cdots=E_{r_A}^2=-1$.

To prove Theorem 6, we have to show that S does not contain A-polar cylinders. If $r_A+K_S^2\geqslant 1$, then S does not contain A-polar cylinders by Lemmas 16, 17 and 18. Therefore, we may assume that $r_A+K_S^2\leqslant 0$. Put $k=1-K_S^2-r_A$.

There is a birational morphism $h\colon\widehat{S}\to\widetilde{S}$ such that $\widetilde{S}$ is smooth and $K_{\widetilde{S}}^2=1$. Let $H_1,\ldots,H_k$ be the proper transforms of the h-exceptional curves on the surface S.

Then the curves $H_1, \ldots, H_k$ are smooth rational curves such that $H_1^2 = \cdots = H_k^2 = -1$. Moreover, the curves $E_1, \ldots, E_{r_A}, H_1, \ldots, H_k$ are disjoint. On the other hand, we have

$$\mu_A A \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^{r_A} a_i E_i = -K_S + \sum_{i=1}^{r_A} a_i E_i + 0 \times H_1 + \cdots + 0 \times H_k.$$

Therefore, the surface S does not contain A-polar cylinders by Lemma 16. $\square$

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References

[1] I. Cheltsov, *Del Pezzo surfaces and local inequalities*, Springer Proceedings in Mathematics and Statistics **79** (2014), 83–101.

[2] I. Cheltsov, J. Park, J. Won, *Affine cones over smooth cubic surfaces*, Journal of the European Mathematical Society **18** (2016), 1537–1564.

[3] I. Cheltsov, J. Park, J. Won, *Cylinders in singular del Pezzo surfaces*, Compositio Mathematica **152** (2016), 1198–1224.

[4] I. Cheltsov, J. Park, J. Won, *Cylinders in del Pezzo surfaces*, to appear in International Mathematics Research Notices.

[5] C. Ciliberto, B. Harbourne, R. Miranda, J. Roé, *Variations on Nagata's conjecture*, Clay Mathematics Proceedings **18** (2013), 185–203.

[6] T. de Fernex, *Negative curves on very general blow-ups of* $\mathbb{P}^2$, Projective Varieties with Unexpected Properties, 199–207, de Gruyter, Berlin, 2005.

[7] B. Lehmann, S. Tanimoto, Yu. Tschinkel, *Balanced line bundles on Fano varieties*, to appear in Journal für die reine und angewandte Mathematik.

[8] T. Kishimoto, Yu. Prokhorov, M. Zaidenberg, *Group actions on affine cones*, CRM Procedings Lecture Notes **54** (2011), 123–163.

[9] T. Kishimoto, Yu. Prokhorov, M. Zaidenberg, $\mathbb{G}_a$*-actions on affine cones*, Transformation Groups **18** (2013), 1137–1153.

[10] T. Kishimoto, Yu. Prokhorov, M. Zaidenberg, *Unipotent group actions on del Pezzo cones*, Algebraic Geometry **1** (2014), 46–56.

[11] A. Perepechko, *Flexibility of affine cones over del Pezzo surfaces of degree* 4 *and* 5, Functional Analysis and its Applications **47** (2013), 284289.

[12] M. Reid, *Surfaces of small degree*, Mathematische Annalen **275** (1986), 71–80.

[13] F. Sakai, *On polarized normal surfaces*, Manuscripta Mathematica **59** (1987), 109–127.

Ivan Cheltsov

School of Mathematics, University of Edinburgh, Edinburgh EH9 3FD, Scotland

I.Cheltsov@ed.ac.uk