

Ratio Orderings and Comparative Statics

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Abstract

Monotone ratio orderings are refinements of first order stochastic dominance that allow monotone comparative statics results in games of incomplete information. We develop analogous refinements for second order stochastic dominance based on the monotonicity of the cumulative probability ratio and the unimodality of the likelihood and probability ratios. We go on to investigate comparative statics in first price auctions, both private and common value, of the effects of more precise information in the sense of the new orderings. We find that almost all types bid more aggressively under the new distribution than they did under the old, but the highest types may bid less. This leads to higher expected revenue in a simple common value auction, but to an ambiguous result in the private value case.

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1 Introduction

Various order relationships between distributions have long been of interest to economists. Those working in welfare economics rank income or wealth distributions in terms of inequality or dispersion, and tend to use (generalized) Lorenz order, while those working on decision making under risk and uncertainty tend to use stochastic dominance relationships. Stochastic dominance relationships have been also of use in games of incomplete information. However, even a strong ordering of two random variables - first order stochastic dominance - can be insufficient to ensure unambiguous comparisons in some games of incomplete information (see, for example, Maskin and Riley, 2000a, footnote 14). As a consequence, several strengthenings of first order stochastic dominance have been introduced, including the monotone likelihood ratio order used for a wide class of examples (Athey, 2002) and the monotone probability ratio order (also known as conditional stochastic dominance or the reverse hazard rate order) used in auctions (Lebrun, 1998; Maskin and Riley, 2000a).

Though being powerful analytical tools, both monotone orderings are very restrictive, ruling out many interesting cases. Being refinements on first order stochastic dominance, they offer no predictions for changes in the distributions that satisfy second order but not first order dominance. Informally speaking, this involves transformations leading to valuations (or signals, etc.) being “less dispersed” but not necessarily “higher” than before. There has been little work on the comparative statics arising from a change in distributions in terms of dispersion. For example, what happens in a game of incomplete information if the distribution of types is subject to a mean preserving spread?

With this question in mind, we develop new refinements for second order stochastic dominance based on the monotonicity of the cumulative probability ratio and the unimodality of the likelihood and probability ratios. These new orderings of distributions allow us to compare distributions of general rather than specific functional form, and include some existing orderings (such as conditional stochastic dominance and the monotone likelihood ratio) as special cases. We go on to investigate how they can be used for comparative statics in auctions. We show that these orderings are sufficient for some comparative statics predictions in that they imply a single crossing condition, if not monotonicity. This in turn leads to higher expected revenue in a common value context, but often to lower revenue with private values. We also show that the comparative statics results obtained under the existing monotone orderings can be considered as special cases of the comparative statics obtained under unimodality.

Suppose that the group of bidders becomes more homogenous (in a private value auction), or all bidders get more precise information about the true value of the object (in a common value auction). Intuitively, one would expect that such decrease in dispersion of types would lead to uniformly more aggressive bidding - that is, all bidders bid more aggressively under the new distribution than they did under the old. Yet, we show that, in first price auctions, a reduction in dispersion in the sense of the new

orderings prompts most types to bid more aggressively, but the highest types may bid less. This may turn into bad news for the sellers as lower bids by the higher types may translate into lower expected revenue. We show that this indeed should be of concern in private value auctions, as expected revenue may not necessary increase with greater homogeneity of buyers. Surprisingly though, the expected revenue in a simple common value auction is higher when all bidders have more precise information about the true value of the object.

The latter result echoes Milgrom (1989, p16), who writes, in the context of auctions with a common value element, that the Linkage Principle “implies that if the auctioneer/seller has private information about the item being sold..., then a policy of always revealing that information increases average receipts”. The Linkage Principle was first derived in the context of affiliated signals, that is, when the information that the seller might possess is positively correlated with the signals of the buyers.

The hypothesis that more precise information about the true value of the object should lead to uniformly more aggressive bidding and higher selling prices has been investigated in Kagel and Levin (1986) and in subsequent literature for specific functional forms of preferences and distributions of signals. More recently, Goeree and Offerman (1999) investigated the effects of more precise information on the competitive bidding in a framework that nests both private and common value cases. Yet, the major drawback of this literature is that providing agents with “more precise information” has been frequently analyzed by considering two uniform distributions with different support. While being analytically convenient, this assumption is relatively restrictive. We show that the unimodal ratio orderings could serve as an alternative technique allowing to analyze more general pairs of distributions.

It is worth reminding that measures of stochastic dominance are not confined to the economics of information. Since the famous work of Atkinson (1970) they have also been important in the literature on social welfare and the comparisons of income distributions (see Lambert (1989) for a survey). However, the ordering more commonly used in this literature is (generalized) Lorenz dominance, even though it is equivalent to second order stochastic dominance (Shorrocks, 1983; Kakwani, 1984; Thistle, 1989), and, thus, both measures can be interpreted in terms of inequality. In this paper, we add to the result of Ramos et al. (2000) by giving two further (and weaker) sufficient conditions for generalized Lorenz dominance.

More recently, income inequality and games of incomplete information have been considered together (Hopkins and Kornienko, 2003; Samuelson, forthcoming) in the context of strategic social interaction, where the question has been whether increasing equality leads to greater social competition. It is hoped that this paper will be of some interest to researchers in both fields as well as in their intersection.

We start with a brief survey of the existing refinements of first-order stochastic dominance relationships, develop new refinements of second-order stochastic dominance orderings, and establish relationships among the existing and new orderings (Section 2).

In Section 3, we use the new refinements to analyze the effect of changes in dispersion in private and (simple linear) common value auctions. Section 4 concludes.

2 Ratio Orderings of Distributions

In what follows, we consider two distinct non-negative variables Z_A and Z_B with finite means μ_A and μ_B respectively, having distribution functions F_A and F_B , respectively, with F_A and F_B both having support $[\underline{z}, \bar{z}]$ with $0 \leq \underline{z} < \bar{z}$. Assume that F_A and F_B are twice continuously differentiable and the densities f_A and f_B are strictly positive on the corresponding supports. We employ the following definition of unimodality.¹

Definition 1 *A function $f(z)$ is unimodal around \hat{z} if $f(z)$ is strictly increasing for $z < \hat{z}$ and $f(z)$ is strictly decreasing for $z > \hat{z}$.*

The following order of distributions was first introduced by Ramos, Ollero and Sordo (2000).

Definition 2 *Two distributions F_A, F_B satisfy the Unimodal Likelihood Ratio (ULR) order and we write $F_A \succ_{ULR} F_B$ if the likelihood ratio $L(z) = f_A(z)/f_B(z)$ is unimodal and $E[Z_A] \geq E[Z_B]$.*²

It is well-known (see, for example, Dharmadhikari and Joag-Dev (1988)) that all logconcave functions are unimodal.³ Thus, if $\log L(z)$ is concave and $\mu_A \leq \mu_B$, then $F_A \succ_{ULR} F_B$. From our definition of unimodality, there is a unique value of z which we denote \hat{z}_L which maximizes the likelihood ratio $L(z)$, with $\hat{z}_L \leq \bar{z}$. If the mode of the ratio is located at the upper bound, that is, $\hat{z}_L = \bar{z}$, we arrive at a monotone order as a special case.

Definition 3 *The two distributions F_A, F_B satisfy the Monotone Likelihood Ratio (MLR) order and we write $F_A \succ_{MLR} F_B$, if the ratio of their densities $L(z)$ is strictly increasing.*

¹This is a slight strengthening of standard definitions of unimodality - for example, by Dharmadhikari and Joag-Dev (1988, Chapter 1) and by An (1998). In the first source, a function $f(z)$ is unimodal if $\int_{\underline{z}}^z f(t)dt$ is convex on (\underline{z}, \hat{z}) and concave on (\hat{z}, \bar{z}) . In the second, the function $f(z)$ has to satisfy the following: for all $\delta > 0$, the set $D_\delta = \{z \in \Omega : f(z) \geq \delta\}$ is a convex set in Ω .

²Note that in this definition, as in a forthcoming definition of the unimodality of the probability ratio, we impose the condition on the means so that Z_B does not first order dominate Z_A , and to rule out the possibility of the mode at the lower bound.

³For review of logconcave and logconvex functions see Ann (1998).

Milgrom (1981) introduced the MLR order to the economics of information. More recently, Athey (2002) employs the MLR order to obtain monotone comparative statics in games of incomplete information. It is well-known (see, for example, Wolfstetter (1999, Chapter 4)) that the MLR order implies first order stochastic dominance.

We now turn to the ratios of distribution functions.

Definition 4 *Two distributions F_A, F_B satisfy the Unimodal Probability Ratio order and we write $F_A \succ_{UPR} F_B$ if the ratio of their distribution functions $P(z) = F_A(z)/F_B(z)$ is unimodal and $E[Z_A] \geq E[Z_B]$.*

As in case of the likelihood ratio, if $P(z)$ is logconcave and $\mu_A \leq \mu_B$, then $F_A \succ_{UPR} F_B$. Again, denote the unique value of z which maximizes the probability ratio $P(z)$ as \hat{z}_P , with $\hat{z}_P \leq \bar{z}$. If the mode of the ratio is located at the upper bound, that is, $\hat{z}_P = \bar{z}$, we arrive at a monotone order as a special case.

Definition 5 *The two distributions F_A, F_B satisfy the Monotone Probability Ratio (MPR) order and we write $F_A \succ_{MPR} F_B$, if the probability ratio $P(z)$ is strictly increasing on $(\underline{z}, \bar{z}]$.*

That is, if for all $x < y$ in $(\underline{z}, \bar{z}]$

$$\frac{F_A(x)}{F_B(x)} < \frac{F_A(y)}{F_B(y)} \quad (1)$$

Note that the relationship (1) implies that $P'(z) > 0$ for all z in $(\underline{z}, \bar{z}]$, or that

$$\sigma_A(z) = \frac{f_A(z)}{F_A(z)} > \frac{f_B(z)}{F_B(z)} = \sigma_B(z), \quad (2)$$

The ratio $\sigma(z)$ is known as the “reverse hazard rate” in the statistics literature. It is clear that the inequality (1) holds (for differentiable distribution functions) if and only if (2) holds. The MPR order has therefore also been called the reverse hazard rate order (see, for example, Shaked and Shanthikumar (1994)). The monotone ratio of distribution functions was introduced to economics by Eeckhoudt and Gollier (1995) in the context of decision making under uncertainty, and, independently, by Lebrun (1998) and Maskin and Riley (2000a) in the first price auction literature. Maskin and Riley (2000a) call their version of the ordering “conditional stochastic dominance”.⁴ This is because the MPR order implies that for all $x < y$ in $(\underline{z}, \bar{z}]$, rearranging (1),

$$\Pr(Z_A < x \mid Z_A < y) = \frac{F_A(x)}{F_A(y)} < \frac{F_B(x)}{F_B(y)} = \Pr(Z_B < x \mid Z_B < y) \quad (3)$$

⁴Maskin and Riley’s (2000a) definition is more general, allowing for the possibility of different supports and atoms at the lower bound.

The MPR order also implies (strict) first order stochastic dominance. To see this, note that $F_A(\bar{z}) = F_B(\bar{z}) = 1$ and the ratio $P(z)$ is increasing so that $F_A(z) < F_B(z)$ for all $z \in (\underline{z}, \bar{z})$. This in turn implies that $\mu_A > \mu_B$.

The next proposition shows that the Unimodal Likelihood Ratio order implies the Unimodal Probability Ratio order.

Proposition 1

$$F_A \succ_{ULR} F_B \Rightarrow F_A \succ_{UPR} F_B \quad (4)$$

Proof: By assumption $\hat{z}_L = \operatorname{argmax} L(z)$ is unique. We need to show that there exist a unique point $\hat{z}_P = \operatorname{argmax} P(z)$. Notice that

$$\frac{dP(z)}{dz} = \frac{f_B(z)}{F_B(z)} (L(z) - P(z)) \quad (5)$$

Thus, $L(z)$ and $P(z)$ cross at points of internal extremum of $P(z)$, with $L(z)$ crossing $P(z)$ from above at the internal maximum of $P(z)$ and from below at the internal minimum of $P(z)$. Since $L(z)$ is unimodal, $L(z)$ and $P(z)$ cross at most twice, which we suppose. Denote these points z_- and z_+ , with $z_- < \hat{z}_L < z_+$. This would imply that the sequence of signs of the difference $L(z) - P(z)$ is $(-, +, -)$, so that $P(z)$ is (decreasing, increasing, decreasing), with a maximum at the boundary \underline{z} , internal minimum at z_- and internal maximum at z_+ . However, this can not be true because $P(z)$ is increasing on $(\underline{z}, \hat{z}_L)$. To see that, apply the second mean value theorem for integrals,⁵ that implies that for any $\omega \in (\underline{z}, \bar{z})$, there exists an $\epsilon \in (\underline{z}, \omega)$ such that

$$F_A(\omega) = \int_{\underline{z}}^{\omega} f_A(t)dt = \int_{\underline{z}}^{\omega} \frac{f_A(t)}{f_B(t)} f_B(t)dt = \frac{f_A(\epsilon)}{f_B(\epsilon)} \int_{\underline{z}}^{\omega} f_B(t)dt = \frac{f_A(\epsilon)}{f_B(\epsilon)} F_B(\omega)$$

Rearranging,

$$P(\omega) = \frac{F_A(\omega)}{F_B(\omega)} = \frac{f_A(\epsilon)}{f_B(\epsilon)} = L(\epsilon)$$

Since $L(z)$ is strictly increasing on $(\underline{z}, \hat{z}_L)$, $\epsilon < \omega$ implies that $P(\omega) < L(\omega)$ for all $\omega \in (\underline{z}, \hat{z}_L]$. Thus, by (5), $P(z)$ is increasing on $(\underline{z}, \hat{z}_L]$. That implies that there is at most one internal maximum of $P(z)$. ■

Eeckhoudt and Gollier (1995) showed that the Monotone Likelihood Ratio order implies the Monotone Probability Ratio order. Since the monotone order is a special case of the unimodal order, we present their result as a corollary.⁶

⁵See for example, Theorem 7.2 in Sahoo and Riedel (1998) which say that if f and g are continuous on $[a, b]$ and g is strictly positive on (a, b) , then there exists a number ϵ in (a, b) , depending on a and b such that $\int_a^b f(t)g(t)dt = g(\epsilon(a, b)) \int_a^b f(t)dt$.

⁶This result had been proved earlier in the statistics literature in terms of the reverse hazard rate order. The result is one of many in Shaked and Shanthikumar (1994).

Corollary 1

$$F_A \succ_{MLR} F_B \Rightarrow F_A \succ_{MPR} F_B \tag{6}$$

The proof is straightforward, since $P(\omega) < L(\omega)$ for all $\omega \in (\underline{z}, \bar{z}]$ by the integral mean value theorem. Note that the converse does not hold, and it is easy to construct pairs of distributions where probability ratio is monotone but the likelihood ratio is not.

Example 1 *If f_B is uniform on $[0, 1]$ and $f_A = (4 + 16z - 9z^2)/9$, the ratio f_A/f_B is not monotone. However, the ratio $F_A/F_B = (4 + 8z - z^2)/9$ is strictly increasing on $(0, 1]$.*

As Milgrom (1981) points out, many well known families of distributions, for example, the normal and the exponential satisfy the MLR order. A larger set of families of distributions satisfy UPR order, including mean preserving spreads. One can easily verify that, for example, if F_A and F_B are both normal or both lognormal, with $\mu_A \geq \mu_B$ and with $\sigma_A < \sigma_B$ then $F_A \succ_{UPR} F_B$.

We now turn to the relationship between unimodal orderings and second order stochastic dominance, or, equivalently, generalized Lorenz order.⁷ We first show that the unimodal probability ratio order implies a single crossing property.

Lemma 1 *If $F_A(z) \succ_{UPR} F_B(z)$ and the maximum of $P(z)$ is interior then $F_A(z)$ and $F_B(z)$ are single crossing: there is a unique $\tilde{z} \in (\underline{z}, \bar{z})$ such that $F_A(\tilde{z}) = F_B(\tilde{z})$.*

Proof: As $P(\bar{z}) = 1$ and as $P(z)$ is unimodal, it must be that $P(\underline{z}) \leq 1$ or F_A would be everywhere greater than F_B which would imply $E[Z_A] < E[Z_B]$. It reaches a unique maximum at some point \hat{z}_P , and $P(\hat{z}_P) \geq 1$, with equality only if $\hat{z}_P = \bar{z}$. Therefore, if the maximum is interior, that is, $\hat{z}_P < \bar{z}$, then necessarily there is a unique point $\tilde{z} < \hat{z}_P$ such that $P(\tilde{z}) = 1$. ■

The next proposition says that if $F_A(z)$ and $F_B(z)$ satisfy the unimodal probability ratio order then $F_A(z)$ second order stochastically (generalized Lorenz) dominates $F_B(z)$.

Proposition 2

$$F_A(z) \succ_{UPR} F_B(z) \Rightarrow F_A(z) \succ_{2SD} F_B(z) \tag{7}$$

Proof: Note that $F_A(z) \succ_{2SD} F_B$, (that is, $F_A(z)$ second order stochastically dominates $F_B(z)$) if and only if $\int_{\underline{z}}^z F_B(t) - F_A(t) dt \geq 0$ for all z . By Lemma 1, $F_A(\tilde{z}) = F_B(\tilde{z})$,

⁷For the equivalence of second order stochastic dominance and generalized Lorenz dominance, see Shorrocks (1983), Kakwani (1984) and Thistle (1989).

$F_A(z) < F_B(z)$ on $(\underline{z}, \tilde{z})$ and $F_A(z) > F_B(z)$ on (\tilde{z}, \bar{z}) with $\underline{z} < \tilde{z} \leq \bar{z}$. This implies $\int_{\underline{z}}^z F_B(t) - F_A(t)dt$ is positive and increasing on $[\underline{z}, \tilde{z}]$ and decreases on $[\tilde{z}, \bar{z}]$ (possibly empty). So if $\int_{\underline{z}}^z F_B(t) - F_A(t)dt$ is negative anywhere it is negative at \bar{z} . However, $\int_{\underline{z}}^{\bar{z}} F_B(t) - F_A(t)dt = \mu_A - \mu_B$ which is non negative by assumption. ■

Propositions 1 and 2 together imply the result by Ramos et al. (2000) that Unimodal Likelihood Ratio (ULR) order implies generalised Lorenz order (or equivalently, second order stochastic dominance), and that logconcavity of $L(z)$ together with the restriction on the means implies the generalized Lorenz order. Note that here, in the UPR and MCR orders, we have two new and weaker conditions that are nonetheless sufficient for second order stochastic dominance and the generalised Lorenz order.

We now introduce a further novel refinement of second order stochastic dominance (2SD).

Definition 6 *The two distributions F_A, F_B satisfy the Monotone Cumulative Probability Ratio (MCR) order and we write $F_A \succ_{MCR} F_B$ if the cumulative probability ratio*

$$C(z) = \frac{\int_{\underline{z}}^z F_A(t)dt}{\int_{\underline{z}}^z F_B(t)dt}$$

is strictly increasing on (\underline{z}, \bar{z}) .

That is, $F_A \succ_{MCR} F_B$ if and only if for all z in (\underline{z}, \bar{z})

$$\frac{\int_{\underline{z}}^z F_A(t)dt}{F_A(z)} < \frac{\int_{\underline{z}}^z F_B(t)dt}{F_B(z)}. \quad (8)$$

This ordering has a particular interpretation in terms of ranking the conditional expectations of the two variables. Define

$$h(z) = E[Z|Z < z] = \frac{\int_{\underline{z}}^z t dF(t)}{F(z)} = z - \frac{\int_{\underline{z}}^z F(t)dt}{F(z)}, \quad (9)$$

then, we have immediately:

Lemma 2 *$F_A \succ_{MCR} F_B$ if and only if $h_{F_A}(z) = E[Z_A|Z_A < z] > E[Z_B|Z_B < z] = h_{F_B}(z)$ for all $z \in (\underline{z}, \bar{z})$.*

Proof: This follows directly from inspection of (8) and (9). ■

The Monotone Cumulative Probability Ratio order implies second order stochastic dominance as the next Lemma shows.

Lemma 3 *If $F_A \succ_{MCR} F_B$, then $F_A \succ_{SSD} F_B$.*

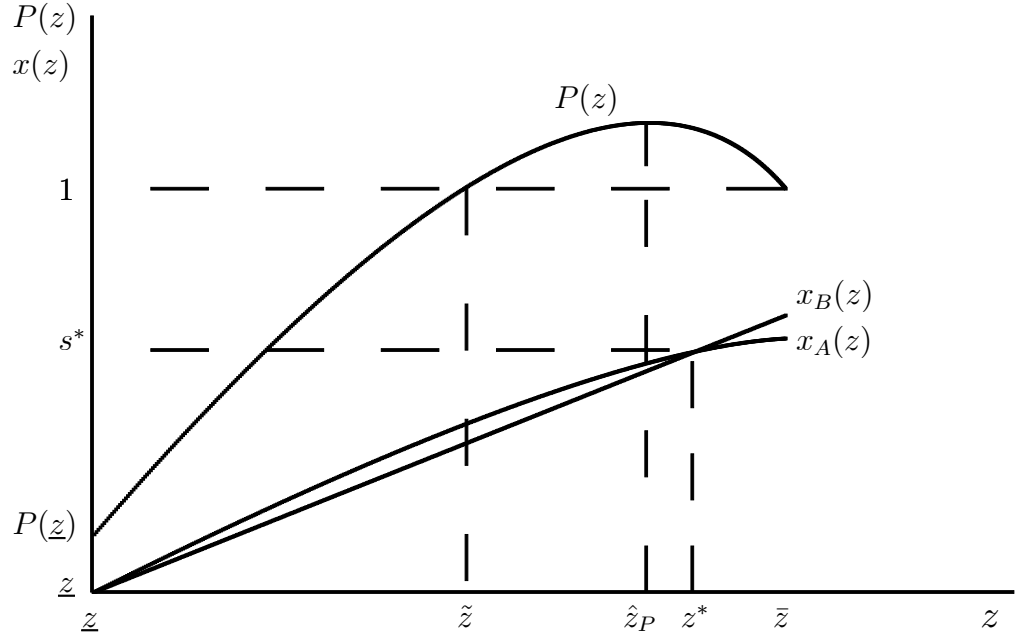


Figure 1: Solutions to the differential equation (10) under unimodality of the probability ratio.

Proof: By definition, F_A second order stochastically dominates F_B , if and only if $C(z) \leq 1$ for all $z \in [\underline{z}, \bar{z}]$. Since $C(z)$ is increasing, we need only establish that $C(\bar{z}) \leq 1$. Notice that $C(\bar{z}) = (\bar{z} - \mu_A)/(\bar{z} - \mu_B)$. Lemma 2, together with the continuity of the function h , implies that $E[Z_A] = h_{F_A}(\bar{z}) \geq h_{F_B}(\bar{z}) = E[Z_B]$. Thus, we have established that $\mu_A \geq \mu_B$, so that $C(\bar{z}) \leq 1$. ■

We now will establish that the UPR order implies the MCR order. We first consider a particular form of differential equation which depends on the reverse hazard ratio $f(z)/F(z)$. Next Theorem shows that the unimodal ratio order implies single crossing of corresponding solutions to a particular differential equation. That is, if $F_A(z) \succ_{UPR} F_B(z)$, so that $F_A(z)$ and $F_B(z)$ cross at most once at some point \tilde{z} on the interior of their support, then the corresponding solutions to the differential equation will cross no more than once and only to the right of \tilde{z} . Moreover, if the solutions cross, they cross to the right of the maximum of $P(z) = F_A(z)/F_B(z)$. This possibility is illustrated in Figure 1.

Theorem 1 Consider the following differential equation:

$$\frac{dx(z)}{dz} = \psi(x(z), z) \frac{f(z)}{F(z)}, \quad x(\underline{z}) = \underline{z} \quad (10)$$

where $\psi_1 < 0$ and $\psi_2 > 0$.

Suppose $x_A(z)$ and $x_B(z)$ are solutions to this differential equation for distributions

$F_A(z)$ and $F_B(z)$, respectively. If $F_A(z) \succ_{UPR} F_B(z)$, then either $x_A(z) > x_B(z)$ almost everywhere, or there exists a point $z^* > \operatorname{argmax} P(z)$ such that $x_A(z^*) = x_B(z^*)$, $x_A(z) > x_B(z)$ for all $z \in (\underline{z}, z^*)$ and $x_A(z) < x_B(z)$ for all $z \in (z^*, \bar{z})$.

Proof: Denote $\operatorname{argmax} P(z)$ as \hat{z}_P so that $P'(z) > 0$ for z in $[\underline{z}, \hat{z}_P)$ and $P'(z) < 0$ for z in $(\hat{z}_P, \bar{z}]$. Examining the differential equation (10), one can see that if x_A and x_B cross at all on the interior of $[\underline{z}, \bar{z}]$, then at such crossing points $\psi(x_A, z) = \psi(x_B, z)$. Now, if $P'(z) > 0$ then $f_A/F_A > f_B/F_B$. Hence, if $x_A(z) = x_B(z)$ on $(\underline{z}, \hat{z}_P)$, then $x'_A(z) > x'_B(z)$ and if $x_A(z) = x_B(z)$ on (\hat{z}_P, \bar{z}) then $x'_A(z) < x'_B(z)$. Hence, if $x_A(z)$ and $x_B(z)$ cross at all on (\underline{z}, \bar{z}) , it must be that $x_A(z)$ crosses $x_B(z)$ at most twice - from below to the left of \hat{z}_P and from above to the right of \hat{z} . Denote these points of intersection z_+ and z_- with $z_+ < \hat{z}_P < z_-$. Since x_A crosses x_B from below at z_+ and from above at z_- , so that the sequence of sign of the difference $x_A - x_B$ is $-, +, -$. Under the boundary conditions to (10), $x_A(\underline{z})$ equals $x_B(\underline{z})$. Then there must exist a point $\check{z} \in (\underline{z}, z_+)$ where the difference $x_B - x_A$ is maximized. At this point, the slopes of x_A and x_B must be equal, i.e. $x'_A(\check{z}) = x'_B(\check{z})$. Since $\partial\psi(x, z)/\partial x < 0$ we have $\psi(x_A(\check{z}), \check{z}) > \psi(x_B(\check{z}), \check{z})$. But this implies that $f_A(\check{z})/F_A(\check{z}) < f_B(\check{z})/F_B(\check{z})$, which contradicts $P(z)$ being increasing on $[\underline{z}, \hat{z}_P)$. Thus, x_A crosses x_B at most once, from above, and to the right of \hat{z}_P . ■

We can now use Theorem 1 to link the MCR and the UPR order.

Lemma 4 *If $F_A(z) \succ_{UPR} F_B(z)$, then $h_{F_A}(z) > h_{F_B}(z)$ for all $z \in (\underline{z}, \bar{z})$. Thus, if $F_A(z) \succ_{UPR} F_B(z)$ then $F_A(z) \succ_{MCR} F_B(z)$.*

Proof: One can calculate that $h'(z) = (z - h(z))f(z)/F(z)$ or in other words, $h(z)$ is a solution to a special case of the differential equation (10). Thus, by Theorem 1, on (\underline{z}, \bar{z}) , $h_{F_A}(z)$ is initially greater than $h_{F_B}(z)$ and then crosses $h_{F_B}(z)$ at most once. But such a crossing is not possible as $h_{F_A}(\bar{z}) = \mu_A \geq \mu_B = h_{F_B}(\bar{z})$. Thus, given Lemmas 2 and 3, the UPR order implies the MCR order. ■

The summary of the relationships between distributions $F_A(z)$ and $F_B(z)$ with $\mu_A \geq \mu_B$ is presented below (note that the abbreviations ‘‘GL’’, ‘‘1SD’’ and ‘‘2SD’’ stand for generalized Lorenz order, first order stochastic dominance and second order stochastic dominance respectively).

$$\begin{array}{ccccc}
F_A(z) \succ_{MLR} F_B(z) & \Rightarrow & F_A(z) \succ_{MPR} F_B(z) & \Rightarrow & F_A(z) \succ_{1SD} F_B(z) \\
\Downarrow & & \Downarrow & & \Downarrow \\
F_A(z) \succ_{ULR} F_B(z) & \Rightarrow & F_A(z) \succ_{UPR} F_B(z) & \Rightarrow & F_A(z) \succ_{2SD} F_B(z) \\
\Uparrow & & \Uparrow & & \Downarrow \\
L(z) \text{ is logconcave} & & P(z) \text{ is logconcave} & & F_A(z) \succ_{GL} F_B(z)
\end{array}$$

3 Application to Comparative Statics for Auctions

In this section we will show how ratio orderings of distributions allow comparative static predictions in some games of incomplete information, in particular, in symmetric first price auctions.⁸ In what follows, we consider an auction with n bidders. Each player has a type z_i drawn from a common distribution $F(z)$, which is twice differentiable with strictly positive density on its support $[\underline{z}, \bar{z}]$. All agents have a continuous action space which we take to be some subset of the real line and strategies will therefore be of the form $x_i(z_i)$, a mapping from type to action. The payoff to bidder i from winning the auction is $u_i = U(\phi_i(z_i, z_{-i}) - x_i)$, where $U(\cdot)$ is an increasing von Neumann-Morgenstern utility function, and $\phi_i(\cdot)$ gives the value of the object conditional on winning the auction (losing bidders are assumed to get zero utility). As Maskin and Riley (2000b) show, in such a model there will exist a monotone pure strategy equilibrium if preferences are monotone, i.e. $\partial\phi_i/\partial z_i > 0$, bidders are risk averse or risk neutral and types are independently distributed.

Some work has been done on comparative statics of the bidding behavior by Lebrun (1998) and Maskin and Riley (2000a), and, for a wider class of examples, by Athey (2002). For a very general specifications for the primitives, that is, preferences and the distributions of types, these researchers derive sufficient conditions for the existence of monotone comparative statics - conditional stochastic dominance and monotonicity of the likelihood ratio, respectively. In other words, a “higher” distribution of valuations in a sense of either of the orderings should lead to a uniformly more aggressive bidding. A result of this sort is given as a corollary below.

Another type of comparative static result has attracted much less attention, though it potentially has a number of applications. What happens if the distribution of types becomes less dispersed? For private value auction, this would mean that the group of bidders becomes more homogenous, while in a private value auction this would represent all participants receiving more precise information about the value of the object being auctioned. The obvious hypothesis is that bidding will be more competitive. This is certainly the case for equilibrium bidding functions calculated for particular functional forms in Kagel and Levin (1986). However, we show that in general this is not true, even under quite strong regularity conditions. That is, there are plausible circumstances in which more precise information will induce some agents to bid less.

In what follows, we consider comparative statics of the bidding behavior and seller’s expected revenue in symmetric auctions for changes in distributions of types that satisfy ratio orderings. We consider two particular cases. We start with the independent private value auctions and turn later to a (linear) common value model.

⁸Our analysis also can be applied to those other games of incomplete information that can be solved by the methods developed in the auction literature. These include games of status considered in Hopkins and Kornienko (2003), oligopoly games with private information about costs and signalling games with a continuous message space.

3.1 Private Values

The first is the independent private value model where the value a bidder places on the object for sale is simply her type, or $\phi_i(z_i, z_{-i}) = z_i$. In this model, the symmetric equilibrium is given by the solution to the following differential equation:

$$\frac{dx(z)}{dz} = \frac{(n-1)U(\cdot)}{U'(\cdot)} \frac{f(z)}{F(z)} = \psi(x(z), z) \frac{f(z)}{F(z)}. \quad (11)$$

The standard boundary condition in the independent private value case is that $x(\underline{z}) = \underline{z}$.⁹ Note that if $U(\cdot)$ is strictly increasing and (weakly) concave then $\psi_1 < 0$ and $\psi_2 > 0$.

We show next that the unimodal probability ratio order can be used to obtain comparative statics for general utility functions. In particular, it implies that more accurate information will always lead to more aggressive bidding for those with relatively low signals, and may lead to less aggressive bidding for only those with relatively high signals. To be more precise, if $F_A(z) \succ_{UPR} F_B(z)$, so that $F_A(z)$ and $F_B(z)$ cross at most once at some point \tilde{z} on the interior of their support, then the corresponding bidding functions will cross no more than once and only to the right of \tilde{z} . Moreover, if the functions cross, they cross to the right of the maximum of $P(z) = F_A(z)/F_B(z)$. In other words, equilibrium bidding functions $x_A(z)$ and $x_B(z)$, respectively, behave exactly like solutions to the differential equation (10) in Figure 1.

Proposition 3 *Suppose $x_A(z)$ and $x_B(z)$ are the equilibrium bidding functions for distributions $F_A(z)$ and $F_B(z)$, respectively. If $F_A(z) \succ_{UPR} F_B(z)$, then either $x_A(z) > x_B(z)$ almost everywhere, or there exists a point $z^* > \operatorname{argmax} P(z)$ such that $x_A(z^*) = x_B(z^*)$, $x_A(z) > x_B(z)$ for all $z \in (\underline{z}, z^*)$ and $x_A(z) < x_B(z)$ for all $z \in (z^*, \bar{z})$.*

Proof: The proof follows directly from the Proposition 1 by observing that differential equation (11) is a special case of differential equation (10). ■

Lebrun (1998) and Maskin and Riley (2001a) showed that if the two distributions satisfy the monotone probability ratio property, one can obtain monotone comparative statics in asymmetric first-price auctions. The corollary below is a similar result for symmetric first-price auctions. If the maximum of the ratio is at the upper bound, i.e. $\hat{z}_P = \bar{z}$, then the ratio is monotone and the proposition implies that the solutions will not cross.

Corollary 2 *Suppose $x_A(z)$ and $x_B(z)$ are the equilibrium bidding functions for distributions $F_A(z)$ and $F_B(z)$, respectively. If $F_A(z) \succ_{MPR} F_B(z)$, then $x_A(z) > x_B(z)$ almost everywhere.*

⁹See the appendix of Lebrun (1998).

Athey (2002) showed that if the two distributions satisfy the monotone likelihood ratio property, one can obtain monotone comparative statics. We arrive to this result in a corollary below. As the monotone likelihood ratio implies the monotone probability ratio order, which is the special case of the unimodal probability ratio order, it is not surprising that the monotone comparative statics one obtains with the MLR order are a special case of those obtained with UPR.

Corollary 3 *Suppose $x_A(z)$ and $x_B(z)$ are the equilibrium bidding functions for distributions $F_A(z)$ and $F_B(z)$, respectively. If $F_A(z) \succ_{MLR} F_B(z)$, then $x_A(z) > x_B(z)$ almost everywhere.*

The intuition behind the failure of monotonicity disclosed in Proposition 3 can be expressed in the following tradeoff. As the distribution of types becomes more compressed, the marginal return to raising one's bid rises, inducing more aggressive bidding. However, for those with types above \tilde{z} , the point of intersection of $F_A(z)$ and $F_B(z)$, the probability of winning has risen as $F_A(z) > F_B(z)$ for $z > \tilde{z}$. This has the opposite effect. Thus monotonicity may fail for some z above \tilde{z} . One can also remark that this second effect is increasing with the competitiveness of the auction, that is, the more risk averse are bidders or the larger the number of participants. This can be illustrated with an example.

Example 2 *Consider a symmetric n -bidder first price private value auction. Let $F_A(z)$ be $3z^2 - 2z^3$ and $F_B(z) = z$ both on $[0, 1]$ and both having expected values of $1/2$. Then, as F_B represents a uniform distribution and F_A has a unimodal density, it implies that $F_A \succ_{ULR} F_B$, which implies that $F_A \succ_{UPR} F_B$. If the bidders are risk-neutral, the equilibrium bidding functions are $x_A(z) = z(3z - 4)/(4z - 6)$ and $x_B(z) = z/2$ for $n = 2$. These do not cross on $(0, 1)$, but they meet at the boundaries. However, for three risk neutral bidders, or equivalently if $n = 2$ but $U(\cdot) = \sqrt{(\cdot)}$,¹⁰ the solutions are $x_A(z) = 2z(126 - 175z + 60z^2)/(35(3 - 2z)^2)$ and $x_B(z) = 2z/3$. These cross much as in Figure 1. Note that \hat{z} , the maximum of F_A/F_B , is equal to 0.75 and the solutions cross at approximately 0.93.*

For the simplest case of risk-neutral bidders (that is for linear $U(\cdot)$), monotone comparative statics can be obtained under weaker assumptions on the distribution functions. In particular, the MCR order on the power transformations of the distribution functions is both necessary and sufficient for monotone comparative statics. To see that, let $G(z) = F(z)^{n-1}$. Then, note the solution to the differential equation (11) under risk neutrality is exactly

$$x(z) = h_G(z) = z - \frac{\int_{\underline{z}}^z G(t) dt}{G(z)}.$$

¹⁰As Simmons (1996) showed, the equilibrium strategies of risk-neutral n bidders are equivalent to the ones for 2 bidders with constant relative risk aversion with a parameter $1/(n - 1)$.

This solution is well known (see, for example, Krishna (2002, pp17-19)). However, what is new is that Lemma 2 implies the following.

Corollary 4 *Consider a symmetric first price auction with n risk neutral bidders. If $x_A(z)$ and $x_B(z)$ are the equilibrium bidding functions under F_A and F_B , respectively, then $x_A(z) > x_B(z)$ for all $z \in (\underline{z}, \bar{z})$ if and only if $G_A \succ_{MCR} G_B$.*

The next question of interest is a comparative analysis of the seller's expected revenue in equilibrium. Unfortunately, as in Example 2 above, we know that in quite simple settings the bidding functions can cross.

Example 3 *Consider a symmetric first price private value auction with n risk neutral bidders. Take our two distributions from Example 2, where $F_A \succ_{UPR} F_B$. Under the uniform distribution F_B , revenue can be evaluated as $(n-1)/(n+1)$. Under the unimodal distribution F_A , revenue can be calculated as $13/35 > 1/3$ for $n = 2$ and $2867/5005 < 3/5$ for $n = 4$. That is, for $n = 2$ the more concentrated distribution of valuations gives a higher expected revenue, but for a larger number of bidders, the uniform distribution gives a higher expected revenue.*

This is symptomatic of the following tradeoff. Proposition 3 established that as the distribution of values or signals becomes less dispersed, individuals bid mostly more aggressively. This will tend to raise revenue. However, the revenue generated by an auction is determined by the bid of the buyer with the highest signal. And as the distribution becomes less dispersed, the tails of the distribution become thinner and the expected value of the highest signal falls.¹¹ Equally, an agent selling an object by auction would like to see signals being more dispersed: it may make individual buyers bid less aggressively but it increases the likelihood of a buyer with a particularly high signal. Thus, in private value auctions, this competition-likelihood tradeoff results in an ambiguous effect of a less dispersed distribution of types even for the risk-neutral case. Yet, as we will see next, common value auctions are different.

3.2 Common Values

We now examine the case where the object to be sold has a single common value V and the buyers are risk neutral and have independent signals drawn from a common distribution $F(z)$. In the symmetric equilibrium of such an auction the winning bid comes from the buyer with the highest signal. The insight of Milgrom and Weber (1982) is that therefore to estimate the value of the object conditional on winning

¹¹Indeed, this effect is familiar from the literature on optimal job search. There, if the distribution of wages is subject to a mean preserving spread it increases the expected return to search, as it increases the likelihood of a particularly good offer.

the auction one has to calculate its expected value conditional on the signals of rival bidders being lower. Following Milgrom and Weber, we define the function which gives the expected value of the object conditional on the signal of the i th bidder and Y_1 , the highest signal of the other $n - 1$ bidders, as $v(z, y) = E[V|Z = z, Y_1 = y]$. That is, the expected value of the object conditional on winning the auction is equal to its value conditional on the highest signal of one's opponents being equal to one's own and all other signals being lower.

Here we assume independent signals, a simplification that has become current in recent applications. The most common additional simplifying assumption is that the value V of the object to bidder i is a linear function of his own and the other bidders' signals, i.e.¹²

$$V = \alpha z_i + \beta \sum_{k \neq i} z_k \quad \text{for some } \alpha, \beta > 0. \quad (12)$$

Then the expected value of the object conditional on winning will be $v(z, z) = (\alpha + \beta)z + \beta(n - 2)E[Z|Z < z]$, or, by (9),

$$v(z, z) = (\alpha + \beta)z + \beta(n - 2)h(z) \quad (13)$$

The symmetric equilibrium is given by the solution to the following differential equation:

$$\frac{dx(z)}{dz} = (n - 1)(v(z, z) - x) \frac{f(z)}{F(z)}. \quad (14)$$

with boundary condition $x(\underline{z}) = v(\underline{z}, \underline{z})$. This differential equation is derived from Milgrom and Weber (1982, Section 6).

Suppose now that prior to bidding bidders' signals are distributed as F_B and that the seller has some information represented by the random variable U . Suppose that seller releases u , the realization of U , to the public, and denote the new distribution of signals after the seller's announcement as $F_A = F(z|u)$. We are particularly interested in the case of the release of the information that decreases the dispersion of the distribution of signals in a sense of unimodal probability ratio order. For this simple model, we can achieve an interesting preliminary result. Lemma 2 established that the function $h(z)$ defined in (9) above, which gives the expected value of a signal conditional on it being lower than z , is higher under a less dispersed distribution. This in turn means that $v(z, z)$ will be higher under a less dispersed distribution.¹³

Lemma 5 *Consider the common value which is a linear function of bidders' signals as in (12). Let v_A and v_B be the expected values generated by distributions F_A and F_B respectively. Then, for $n \geq 3$, $v_A(z, z) > v_B(z, z)$ for all $z \in (\underline{z}, \bar{z})$ if and only if $F_A \succ_{MCR} F_B$.*

¹²This form of common value auction is discussed in more detail in Klemperer (1999, Appendix D).

¹³First, if $n = 2$ then $v(z, z) = (\alpha + \beta)z$ and is thus not affected by changes in the distribution of types. Second, if V were a non-linear function of the bidders' signals, then one would need the MPR order to ensure a monotone shift in $v(z, z)$. This is because the MPR order implies the inequality (3), that is, the conditional distribution functions are ordered in terms of first order stochastic dominance.

Proof: Given the definition of $v(z, z)$ given in (13), Lemma 2 implies that, if $n \geq 3$, $v_A(z, z) > v_B(z, z)$ almost everywhere if and only if $F_A \succ_{MCR} F_B$. ■

What implications does this have for equilibrium bidding functions? Using this Lemma, we can obtain another comparative static result. Just as in the private value case, agents with low signals, specifically those with signals less than \hat{z}_P , the point where the probability ratio achieves a maximum (see Figure 1), will bid more. Those with signals above that point, may or may not bid more. The following proposition shows that UPR order implies that the solutions do not cross below the maximum of the probability ratio \hat{z}_P .

Proposition 4 *Consider a first price common value auction where a single common value V is a linear function of independent signals given by (12), and suppose $x_A(z)$ and $x_B(z)$ are the equilibrium bidding functions for distributions $F(z)$ and $F(z|u)$ respectively. If $F(z|x) \succ_{UPR} F(z)$, then $x_A(z) > x_B(z)$ for all $z \in (\underline{z}, \hat{z}_P]$.*

Proof: From the UPR order, $f(z|u)/F(z|u)$ will be greater than $f(z)/F(z)$ on $(\underline{z}, \hat{z}_P)$. Furthermore, from Lemmas 4 and 5, we know that $v_A > v_B$ on (\underline{z}, \bar{z}) . Second, one can calculate that $\lim_{z \rightarrow \underline{z}} h(z) = \underline{z}$, so that given our boundary condition $x_A(\underline{z}) = x_B(\underline{z}) = (\alpha + \beta)\underline{z} + \beta(n - 2)\underline{z}$. Given the differential equation (14), if $x_B > x_A$ for all $(\underline{z}, \check{z})$ for some $\check{z} < \hat{z}_P$, then $x'_A > x'_B$ on for all $(\underline{z}, \check{z})$ which is a contradiction. Thus, it must be the case that $x_A > x_B$ for some interval (\underline{z}, z^+) . Suppose that the first interior crossing point z^+ is in $(\underline{z}, \hat{z}_P)$, then again from (14) we have $x'_A(z^+) > x'_B(z^+)$ and so x_A must cross from below, which is a contradiction. ■

So, in a first price auction, the effect of more precise information again leads most, but possibly not all, types to bid more. However, in a second price auction, the result is stronger.¹⁴

Proposition 5 *Consider a second price (Vickrey) common value auction where a single common value V is a linear function of independent signals given by (12). Then, revelation of information such that $F(z|x) \succ_{UPR} F(z)$ leads to uniformly higher bids and higher revenue.*

Proof: Note that the optimal bid in a second price auction (Milgrom and Weber (1982)) for an agent with signal z is exactly $v(z, z)$. Let $F_2(z)$ be the distribution function of the second highest order statistic of the distribution F . Then expected revenue with no information revelation in the second price auction is $\int_{\underline{z}}^{\bar{z}} v(z, z) dF_2(z)$, that is, the expected bid of the bidder with the second highest signal. Let $v(z, z, u) = (\alpha + \beta)z + \beta(n - 2)h(z, F(z|u))$. If $F(z|x) \succ_{UPR} F(z)$, by Lemma 5, we have $v_A =$

¹⁴Our conjecture is that revenue is higher in first price auctions as well as second price, even though bidding is not uniformly higher. We have not been able to prove this, however.

$v(z, z, u) > v(z, z) = v_B$ for all $z \in (\underline{z}, \bar{z})$. Revenue with information revelation is equal to $\int_{\underline{z}}^{\bar{z}} v(z, z, x) dF_2(z)$ and the result follows. ■

Why is the the effect of information on expected revenues in common value auctions quite different than in the private value case? In any auction, if information becomes less dispersed, it is less likely that the highest signal will be particularly high. This would seem to depress revenue. However, in the common value case, the optimal bid with such a signal will also depend on the estimate of the expected value of the other bidders' signals conditional on their being lower. Since the compression of signals also raises the expected value of low signals, expected revenue consequently rises.

Note that this result is somewhat different from the original results on the effect of information on revenue due to Milgrom and Weber (1982). Their result was that if the seller's information was affiliated with the signals of the buyers its release would raise expected revenue. That is, in effect it is an *ex ante* result: before the seller knows what her information is, she would be better off committing to release it once it is in her hands. The result here is *ex post*: once the seller knows that he has information that would cause buyers' estimates to become less dispersed if it were released, he would be better off by releasing it. This may seem restrictive. However, take as an example the standard practice in art and antique auctions of allowing potential buyers to view the items for sale before the auction. There, the buyer can be reasonably certain that the ability to examine the different lots, and the common knowledge that this facility is available to all buyers, will reduce uncertainty on the part of buyers. Then, the result obtained above, suggests that the seller has an active incentive to offer this facility.

4 Conclusions

In this paper, we review the existing methods to order distributions and introduce some new ones. We investigate the implications of these results for comparative statics in games of incomplete information, in particular, for symmetric first price auctions, with both private and common values. We find that more precise information about the value of the good for sale does not necessarily lead to uniformly more aggressive bidding. However, in the case of some simple common value auctions, expected revenue is higher.

We hope also that the stochastic orderings introduced in this paper will find wider use. For example, while we do not investigate asymmetric auctions in this paper, the orderings we introduce should be useful in determining the effects of one bidder having more precise information than other bidders. Finally, as Athey (2002) has shown, there is the potential to use the same methodology to obtain comparative statics in many other games of incomplete information besides auctions.

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