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ALPHA-INVARIANTS AND PURELY LOG TERMINAL BLOW-UPS

IVAN CHELTSOV, JIHUN PARK, AND CONSTANTIN SHRAMOV

Abstract. We prove that the sum of the α-invariants of two different Kollár components of a Kawamata log terminal singularity is less than 1.

Let V be a normal irreducible projective variety of dimension $n \geqslant 1$, and let Δ_V be an effective $\mathbb{Q}$-divisor on V. Write

$$\Delta_V = \sum_{i=1}^{r} a_i \Delta_i,$$

where each Δ_i is a prime divisor, and each a_i is a positive rational number. Suppose that the log pair (V, Δ_V) has at most Kawamata log terminal singularities. Then, in particular, each a_i does not exceed 1. Suppose also that the divisor $-(K_V + \Delta_V)$ is ample, so that (V, Δ_V) is a log Fano variety. Finally, suppose that V is faithfully acted on by a finite group G such that the divisor Δ_V is G-invariant. Let $\alpha_G(V, \Delta_V)$ be the real number

$$\sup\left\{\lambda \in \mathbb{Q} \,\middle|\, \begin{array}{l} \text{the pair } \big(V, \Delta_V + \lambda D_V\big) \text{ has Kawamata log terminal singularities} \\ \text{for every } G\text{-invariant and effective } \mathbb{Q}\text{-divisor } D_V \sim_{\mathbb{Q}} -\Big(K_V + \Delta_V\Big) \end{array}\right\}.$$

This number is known as the α-invariant of the log Fano variety (V, Δ_V), or its global log canonical threshold (see [12, Definition 3.1]). If G is trivial, we put $\alpha(V, \Delta_V) = \alpha_G(V, \Delta_V)$.

Example 1. The divisor $-(K_{\mathbb{P}^1} + \Delta_{\mathbb{P}^1})$ is ample if and only if $\sum_{i=1}^{r} a_i < 2$. One has

$$\alpha(\mathbb{P}^1, \Delta_{\mathbb{P}^1}) = \frac{1 - \max(a_1, \ldots, a_r)}{2 - \sum_{i=1}^{r} a_i}.$$

We put $\alpha_G(V) = \alpha_G(V, \Delta_V)$ if $\Delta_V = 0$.

Example 2. A finite group G acting faithfully on $\mathbb{P}^1$ is one of the following finite groups: the alternating group $\mathfrak{A}_5$, the symmetric group $\mathfrak{S}_4$, the alternating group $\mathfrak{A}_4$, a dihedral group D_{2m} of order $2m$, or a cyclic group $\boldsymbol{\mu}_m$ of order m (including the case $m = 1$, that is, the trivial group). The number $\frac{\alpha_G(\mathbb{P}^1)}{2}$ is equal to the length of the smallest G-orbit in $\mathbb{P}^1$, which gives

$$\alpha_G\big(\mathbb{P}^1\big) = \begin{cases} 6 \text{ if } G \cong \mathfrak{A}_5, \\ 3 \text{ if } G \cong \mathfrak{S}_4, \\ 2 \text{ if } G \cong \mathfrak{A}_4, \\ 1 \text{ if } G \cong \mathrm{D}_{2m}, \\ \dfrac{1}{2} \text{ if } G \cong \boldsymbol{\mu}_m. \end{cases}$$

We assume that all varieties are defined over the field $\mathbb{C}$.

If both $\Delta_V = 0$ and G is trivial, we put $\alpha(V) = \alpha_G(V, \Delta_V)$. This is the most classical case. Namely, if V is a smooth Fano variety, then by [11, Theorem A.3] the number $\alpha(V)$ coincides with the α-invariant of V defined by Tian in [45]. Its values were found or estimated in many cases. For example, in the toric case the explicit formula for $\alpha(V)$ is given by Cheltsov and Shramov in [11, Lemma 5.1]. It gives $\alpha(\mathbb{P}^n) = \frac{1}{n+1}$, which can also be verified by an easy explicit computation. The α-invariants of smooth del Pezzo surfaces were computed in [2].

Theorem 3. *Let V be a smooth del Pezzo surface. Then one has*

$$\alpha(V) = \begin{cases} 1 \text{ if } K_V^2 = 1 \text{ and } |-K_V| \text{ contains no cuspidal curves}, \\ \frac{5}{6} \text{ if } K_V^2 = 1 \text{ and } |-K_V| \text{ contains a cuspidal curve}, \\ \frac{5}{6} \text{ if } K_V^2 = 2 \text{ and } |-K_V| \text{ contains no tacnodal curves}, \\ \frac{3}{4} \text{ if } K_V^2 = 2 \text{ and } |-K_V| \text{ contains a tacnodal curve}, \\ \frac{3}{4} \text{ if } V \text{ is a cubic in } \mathbb{P}^3 \text{ with no Eckardt points}, \\ \frac{2}{3} \text{ if either } V \text{ is a cubic in } \mathbb{P}^3 \text{ with an Eckardt point, or } K_V^2 = 4, \\ \frac{1}{2} \text{ if } V \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } K_V^2 \in \{5, 6\}, \\ \frac{1}{3} \text{ in the remaining cases.} \end{cases}$$

The α-invariants of all del Pezzo surfaces with Du Val singularities were computed in [4, 43, 38, 37, 7].

Example 4. Let V be a singular cubic surface in $\mathbb{P}^3$ that has at most Du Val singularities. Then one has

$$\alpha(V) = \begin{cases} \frac{2}{3} \text{ if } V \text{ has unique singular point, and it is of type } \mathbb{A}_1, \\ \frac{1}{3} \text{ if } V \text{ contains singular point of type } \mathbb{A}_4, \\ \frac{1}{3} \text{ if } V \text{ has unique singular point, and it is of type } \mathbb{D}_4, \\ \frac{1}{3} \text{ if } V \text{ contains two singular points of type } \mathbb{A}_2, \\ \frac{1}{4} \text{ if } V \text{ contains singular point of type } \mathbb{A}_5, \\ \frac{1}{4} \text{ if } V \text{ has unique singular point, and it is of type } \mathbb{D}_5, \\ \frac{1}{6} \text{ if } V \text{ has unique singular point, and it is of type } \mathbb{E}_6, \\ \frac{1}{2} \text{ in all the remaining cases.} \end{cases}$$

The α-invariants of many non-Gorenstein singular del Pezzo surfaces that are quasi-smooth well-formed complete intersections in weighted projective spaces were computed in [9, 15, 24]. The α-invariants of many smooth and singular Fano threefolds were computed or estimated in [23, 11, 3, 5, 6, 25]. The α-invariants of smooth Fano hypersurfaces were estimated in [1, 8, 40, 10].

The α-invariant plays an important role in Kähler geometry. If V is a smooth Fano variety, then V admits a G-invariant Kähler–Einstein metric provided that

$$\alpha_G\big(V\big) > \frac{\dim(V)}{\dim(V)+1}.$$

This was proved by Tian in [45]. In [19], this result was improved by Fujita. He proved that V admits a Kähler–Einstein metric if it is smooth and $\alpha(V) \geqslant \frac{\dim(V)}{\dim(V)+1}$. In particular, all smooth hypersurfaces in $\mathbb{P}^d$ of degree d are Kähler–Einstein, because their α-invariants are at least $\frac{d-1}{d}$ by [1, 8].

The K-stability of the log Fano variety (V, Δ_V) crucially depends on $\alpha(V, \Delta_V)$. For instance, if

$$\alpha\big(V, \Delta_V\big) < \frac{1}{\dim(V)+1},$$

then the log Fano variety (V, Δ_V) is K-unstable by [22, Theorem 3.5] and [21, Lemma 5.5]. This bound is sharp, since $\mathbb{P}^n$ is K-semistable and $\alpha(\mathbb{P}^n) = \frac{1}{n+1}$. Vice versa, if $\alpha(V, \Delta_V) \geqslant \frac{\dim(V)}{\dim(V)+1}$, then the log Fano variety (V, Δ_V) is K-semistable by [34, Theorem 1.4] and [20, Proposition 2.1].

The α-invariant also plays an important role in birational geometry. It was first observed by Park in [35], where he proved the following

Theorem 5 ([4, Theorem 5.7])**.** *Let X be a variety with at most terminal $\mathbb{Q}$-factorial singularities. Suppose that there is a flat morphism $\phi\colon X \to Z$ such that Z is a curve, and $-K_X$ is ϕ-ample. Let P be a point in Z, and let E_X be a scheme fiber of ϕ over P. Suppose that E_X is irreducible, reduced, normal, and has at most Kawamata log terminal singularities, so that E_X is a Fano variety by the adjunction formula. Suppose also that there is a commutative diagram*

$$\begin{array}{ccc} X & \overset{\rho}{\dashrightarrow} & Y \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & Z & \end{array}$$

such that Y is a variety with at most terminal $\mathbb{Q}$-factorial singularities, ψ is a flat morphism, the divisor $-K_Y$ is ψ-ample, and ρ is a birational map that induces an isomorphism

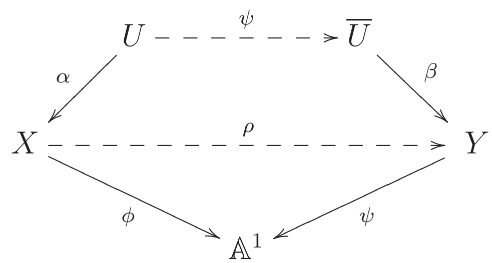
$$X \setminus \mathrm{Supp}\big(E_X\big) \cong Y \setminus \mathrm{Supp}\big(E_Y\big),$$

where E_Y is a scheme fiber of ψ over P. Suppose, in addition, that E_Y is irreducible. Then ρ is an isomorphism provided that $\alpha(E_X) \geqslant 1$. Moreover, if E_Y is reduced, normal and has at most Kawamata log terminal singularities, then ρ is an isomorphism provided that $\alpha(E_X) + \alpha(E_Y) > 1$.

Theorem 5 gives a necessary condition in terms of α-invariants for the existence of a non-biregular fiberwise birational transformation of a Mori fibre space over a curve. It follows from [29, Theorem 1.1] that this condition is not a sufficient condition. Nevertheless, the bound is sharp (one can find many examples in [35, 36]).

Example 6. Let S be a $\mathbb{P}^1$-bundle over a curve. Then we have an elementary transformation to another $\mathbb{P}^1$-bundle over the same curve. Note that the $\alpha(\mathbb{P}^1) = \frac{1}{2}$ by Example 2.

Example 7 ([18, Example 5.8])**.** Let S be a smooth cubic surface in $\mathbb{P}^3$ with an Eckardt point O. Denote by L_1, L_2, L_3 the lines in S passing through O. Put $X = S \times \mathbb{A}^1$, and let ϕ be the natural projection $X \to \mathbb{A}^1$. Let us identify S with a fiber of ϕ. Then there is commutative diagram



where α is the blow up of the point O, the map ψ is the anti-flip along the proper transforms of the curves L_1, L_2, L_3, and β is the contraction of the proper transform of the surface S. The scheme fiber of ψ over the point $\phi(S)$ is a cubic surface in $\mathbb{P}^3$ that has one singular point of type $\mathbb{D}_4$. Its α-invariant is $\frac{1}{3}$ by Example 4. On the other hand, we have $\alpha(S) = \frac{2}{3}$ by Theorem 3.

Example 8 ([35, Example 5.3])**.** Let X and Y be subvarieties in $\mathbb{A}^1\times\mathbb{P}^3$ given by eqautions

$$x^3 + y^2z + z^2w + t^{12}w^3 = 0 \quad \text{and} \quad x^3 + y^2z + z^2w + w^3 = 0,$$

respectively, where t is a coordinate on $\mathbb{A}^1$, and $(x : y : z : w)$ are homogeneous coordinates on $\mathbb{P}^3$. Then the projections $\phi\colon X \to \mathbb{A}^1$ and $\psi\colon Y \to \mathbb{A}^1$ are fibrations into cubic surfaces, and the map

$$(t, x, y, z, w) \mapsto (t, t^2x, t^3y, z, t^6w)$$

gives a non-biregular birational fiberwise map $\rho\colon X \dashrightarrow Y$ between them. The fiber of ϕ over the point $t = 0$ is a cubic surface that has one Du Val singular point of type $\mathbb{E}_6$, so that its α-invariant is $\frac{1}{6}$ by Example 4, and the scheme fiber of ψ over the point $t = 0$ is a smooth cubic surface with an Eckardt point, so that its α-invariant is $\frac{2}{3}$ by Theorem 3.

The α-invariant also plays an important role in singularity theory. Let $U \ni P$ be a germ of a Kawamata log terminal singularity. Then it follows from [47, Lemma 1] that there is a birational morphism $\phi\colon X \to U$ such that its exceptional locus consists of a single prime divisor E_X such that $\phi(E_X) = P$, the log pair (X, E_X) has purely log terminal singularities, and the divisor $-(K_X + E_X)$ is ϕ-ample. Then

$$-\big(K_X + E_X\big) \sim_{\mathbb{Q}} -\delta_X E_X$$

for some positive rational number δ_X. Recall from [39, Definition 2.1] that the birational morphism $\phi\colon X \to U$ is a purely log terminal blow-up of the singularity $U \ni P$.

By [26, Theorem 7.5], the divisor E_X is a normal variety that has rational singularities. Moreover, it can be naturally equipped with a structure of a log Fano variety. Let $R_1, \ldots, R_s$ be all the irreducible components of the locus $\mathrm{Sing}(X)$ of codimension 2 that are contained in E_X. Put

$$\mathrm{Diff}_{E_X}\big(0\big) = \sum_{i=1}^{s} \frac{m_i - 1}{m_i} R_i,$$

where m_i is the smallest positive integer such that the divisor m_iE_X is Cartier in a general point of R_i. Then $\mathrm{Diff}_{E_X}(0)$ is usually called the *different* of the pair (X, E_X). One has

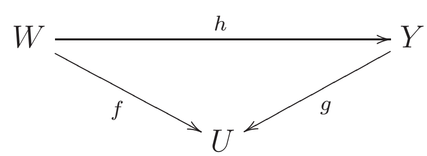
$$-\delta_X E_X\Big|_{E_X} \sim_{\mathbb{Q}} -\Big(K_X + E_X\Big)\Big|_{E_X} \sim_{\mathbb{Q}} -\left(K_{E_X} + \mathrm{Diff}_{E_X}(0)\right).$$

Furthermore, the singularities of the log pair $(E_X, \mathrm{Diff}_{E_X}(0))$ are Kawamata log terminal by Adjunction, see [44, 3.2] or [27, 17.6]. This means that $(E_X, \mathrm{Diff}_{E_X}(0))$ is a log Fano variety with Kawamata log terminal singularities, because $-E_X$ is ϕ-ample.

Definition 9 (cf. [31, Definition 1.1])**.** The log Fano variety $(E_X, \mathrm{Diff}_{E_X}(0))$ is a Kollár component of $U \ni P$.

Let us show how to compute $\alpha(E_X, \mathrm{Diff}_{E_X}(0))$ in three simple cases.

Example 10. Let $U \ni P$ be a germ of a Du Val singularity, and $f\colon W \to U$ be the minimal resolution of this singularity. Then the exceptional curves of f are smooth rational curves whose self-intersections are -2, and their dual graph is of type $\mathbb{A}_m$, $\mathbb{D}_m$, $\mathbb{E}_6$, $\mathbb{E}_7$, or $\mathbb{E}_8$. Let E_W be one of the exceptional curves that is chosen as follows. If $U \ni P$ is not a singularity of type $\mathbb{A}_m$, let E_W be the only f-exceptional curve that intersects three other f-exceptional curves, i.e., E_W is the "fork" of the dual graph. If $U \ni P$ is a singularity of type $\mathbb{A}_m$, choose E_W to be the k-th curve in the dual graph. In this case, we may assume that $k \leqslant \frac{m+1}{2}$. In all cases, there exists a commutative diagram



where h is the contraction of all f-exceptional curves except E_W, and g is the contraction of the proper transform of E_W on the surface Y. Denote the g-exceptional curve by E_Y. Then Y has at most Du Val singularities of type $\mathbb{A}$, the curve E_Y is smooth, and it contains all singular points of the surface Y, if any. One can check that the log pair (Y, E_Y) has purely log terminal singularities, see [28, Theorem 4.15(3)]. Also, the divisor $-(K_Y + E_Y)$ is g-ample. Thus, the curve E_Y is a Kollár component of the singularity $U \ni P$. Moreover, if $U \ni P$ is a singularity of type $\mathbb{A}_m$, then

$$\alpha\big(E_Y, \mathrm{Diff}_{E_Y}(0)\big) = \frac{k}{m+1} \leqslant \frac{1}{2}.$$

Indeed, if $U \ni P$ is a singularity of type $\mathbb{A}_1$, then h is an isomorphism and Y is smooth, so that $\mathrm{Diff}_{E_Y}(0) = 0$, which gives $\alpha(E_Y, \mathrm{Diff}_{E_Y}(0)) = \frac{1}{2}$. Similarly, if $U \ni P$ is a singularity of type $\mathbb{A}_m$, $m \geqslant 2$, and $k = 1$, then Y has a singular point P_1 that is a Du Val singular point of type $\mathbb{A}_{m-1}$. In this case, we have

$$\mathrm{Diff}_{E_Y}(0) = \frac{m-1}{m}P_1,$$

which gives $\alpha(E_Y, \mathrm{Diff}_{E_Y}(0)) = \frac{1}{m+1}$. Finally, if $U \ni P$ is a singularity of type $\mathbb{A}_m$, $m \geqslant 3$, and $2 \leqslant k \leqslant \frac{m+1}{2}$, then Y has two singular points P_1 and P_2, which are Du Val singular points of type $\mathbb{A}_{k-1}$ and $\mathbb{A}_{m-k}$. In this case, we have

$$\mathrm{Diff}_{E_Y}(0) = \frac{k-1}{k}P_1 + \frac{m-k}{m-k+1}P_2,$$

so that $\alpha(E_Y, \mathrm{Diff}_{E_Y}(0)) = \frac{k}{m+1}$. Likewise, if $U \ni P$ is a singularity of type $\mathbb{D}_m$ with $m \geqslant 4$, then $\alpha(E_Y, \mathrm{Diff}_{E_Y}(0)) = 1$. Indeed, in this case Y has three singular points P_1, P_2 and P_3 such that P_1 and P_2 are Du Val singular points of type $\mathbb{A}_1$, and P_3 is a Du Val singular point of type $\mathbb{A}_{m-3}$, so that

$$\mathrm{Diff}_{E_Y}(0) = \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{m-3}{m-2}P_3,$$

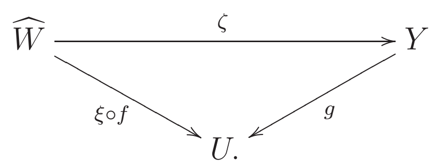
which easily gives $\alpha(E_Y, \mathrm{Diff}_{E_Y}(0)) = 1$. If $U \ni P$ is a singularity of type $\mathbb{E}_m$, then Y has three Du Val singular points P_1, P_2, and P_3 of types $\mathbb{A}_1$, $\mathbb{A}_2$, and $\mathbb{A}_{m-4}$, respectively. Thus, we have

$$\mathrm{Diff}_{E_Y}(0) = \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{m-4}{m-3}P_3.$$

This immediately implies

$$\alpha\big(E_Y, \mathrm{Diff}_{E_Y}(0)\big) = \begin{cases} 2 \text{ if } m = 6, \\ 3 \text{ if } m = 7, \\ 6 \text{ if } m = 8. \end{cases}$$

Example 11. Let $U \ni P$ be a germ of a Du Val singularity of type $\mathbb{A}_m$, and let $f\colon W \to U$ be the minimal resolution of this singularity. Let Q be a point on one of the two exceptional curves that correspond to "tails" of the dual graph such that Q is not contained in any other exceptional curve. Let $\xi\colon \widehat{W} \to W$ be the blow up at Q, and ζ be the contraction of the proper transforms of all the f-exceptional curves. Thus, there exists a commutative diagram



Denote the g-exceptional curve by E_Y. Then Y has a unique singular point O, the dual graph of the exceptional curves of its minimal resolution $\zeta\colon \widehat{W} \to Y$ is a chain, the self-intersection numbers of the exceptional curves of ζ are $-3, -2, \ldots, -2$, and the proper transform of E_Y intersects only the "tail" component of this chain. The curve E_Y is smooth, and it contains the singular point O. By [28, Theorem 4.15(3)] the log pair (Y, E_Y) has purely log terminal singularities. Also, the divisor $-(K_Y + E_Y)$ is g-ample. Thus, the curve E_Y is a Kollár component of the singularity $U \ni P$. Moreover, we have

$$\alpha\big(E_Y, \mathrm{Diff}_{E_Y}(0)\big) = \frac{1}{2m+2} < \frac{1}{2}.$$

Indeed, the surface Y has a cyclic quotient singularity at the point O, which is a quotient of $\mathbb{C}^2$ by the cyclic group $\boldsymbol{\mu}_{2m+1}$, so that

$$\mathrm{Diff}_{E_Y}(0) = \frac{2m}{2m+1}P,$$

which implies the required formula.

Example 12. Let $U \ni P$ be a germ of a Du Val singularity of type $\mathbb{A}_m$, $m \geqslant 2$, and let $f\colon W \to U$ be the minimal resolution of this singularity. Let Q be the intersection point of the k-th and $(k+1)$-th exceptional curves of f, where $1 \leqslant k \leqslant \frac{m}{2}$. Let $\xi\colon \widehat{W} \to W$ be the blow up at Q, and ζ be the contraction of the proper transforms of all the f-exceptional curves. As in Example 11, there is a commutative diagram

$$\begin{array}{ccc} \widehat{W} & \xrightarrow{\quad\zeta\quad} & Y \\ & {\scriptstyle \xi\circ f}\searrow \quad \swarrow{\scriptstyle g} & \\ & U. & \end{array}$$

Denote the g-exceptional curve by E_Y. Then Y has two singular points P_1 and P_2, the dual graphs of the exceptional curves of the minimal resolution of singularities $\zeta\colon \widehat{W} \to Y$ are chains such that the self-intersection numbers of the exceptional curves are $-3, -2, \ldots, -2$, and the proper transform of E_Y intersects only the "tail" components of these chains. The curve E_Y is smooth, and it contains both the points P_1 and P_2. By [28, Theorem 4.15(3)] the log pair (Y, E_Y) has purely log terminal singularities. Also, the divisor $-(K_Y + E_Y)$ is g-ample. Thus, the curve E_Y is a Kollár component of the singularity $U \ni P$. As in Example 11, one can check that each P_i is a cyclic quotient singularity of the surface Y, which is a quotient of $\mathbb{C}^2$ by the cyclic group $\boldsymbol{\mu}_{2n_i+1}$, where $n_1 = k$ and $n_2 = m - k$. This implies

$$\mathrm{Diff}_{E_Y}(0) = \frac{2k}{2k+1}P_1 + \frac{2(m-k)}{2(m-k)+1}P_2.$$

Therefore,

$$\alpha\big(E_Y, \mathrm{Diff}_{E_Y}(0)\big) = \frac{2k+1}{2m+2} \leqslant \frac{1}{2}.$$

In particular, we see that $\alpha(E_Y, \mathrm{Diff}_{E_Y}(0)) = \frac{1}{2}$ if and only if m is even, and Q is the "central point" of the configuration of the f-exceptional curves.

It is easy to see from [28, Theorem 4.15] that if $U \ni P$ is a Du Val singularity of type $\mathbb{D}$ or $\mathbb{E}$, and the exceptional curve E_W in Example 10 is not chosen to be the "fork" of the dual graph, then the corresponding curve E_Y is not a Kollár component. This is not a coincidence: we will see later that in these cases the singulrity $U \ni P$ has a unique Kollár component, which is described in Example 10. This is not true in general, i.e., a Kollár component of a singularity $U \ni P$ may not be unique, as one can see from Examples 10, 11, and 12. Nevertheless, Li and Xu established in [31, Theorem B] the following:

Theorem 13. *A K-semistable Kollár component of $U \ni P$ is unique if it exists.*

The K-semistable Kollár components of two-dimensional Du Val singularities are described in our Examples 10 and 12. They are precisely the Kollár components whose α-invariants are at least $\frac{1}{2}$ (cf. [32, Example 4.7]).

Note that Du Val singularities are two-dimensional rational quasi-homogeneous isolated hypersurface singularities. The K-semistable Kollár components of many three-dimensional rational quasi-homogeneous isolated hypersurface singularities have been described in [9, 15]. Similarly, the K-semistable Kollár components of many four-dimensional rational quasi-homogeneous isolated hypersurface singularities have been described in [23].

The purpose of this paper is to prove the following analogue of Theorem 5.

Theorem 14. *Suppose that there is a commutative diagram*

$$\begin{array}{ccc} X & \overset{\rho}{\dashrightarrow} & Y \\ & {}_{\phi}\searrow \quad \swarrow_{\psi} & \\ & U & \end{array}$$

where ψ is a birational morphism such that its exceptional locus consists of a single prime divisor E_Y with $\psi(E_Y) = P$, the log pair (Y, E_Y) has purely log terminal singularities, and the divisor $-(K_Y + E_Y)$ is ψ-ample. Suppose also that

$$\alpha\big(E_X, \mathrm{Diff}_{E_X}(0)\big) + \alpha\big(E_Y, \mathrm{Diff}_{E_Y}(0)\big) \geqslant 1.$$

Then ρ is an isomorphism.

Before proving this result, let us consider its applications. Suppose that

$$\alpha\big(E_X, \mathrm{Diff}_{E_X}(0)\big) \geqslant \frac{\dim(U) - 1}{\dim(U)}. \tag{15}$$

By Theorem 14, this inequality implies that the α-invariant of another Kollár component of the singularity $U \ni P$, if any, must be less than $\frac{1}{\dim(U)}$, so that it should be K-unstable. Of course, this also follows from Theorem 13, because the inequality (15) implies that the log Fano variety $(E_X, \mathrm{Diff}_{E_X}(0))$ is K-semistable.

Theorem 14 also implies

Corollary 16. *If $\alpha(E_X, \mathrm{Diff}_{E_X}(0)) \geqslant 1$, then the Kollár component of $U \ni P$ is unique.*

This corollary is well known: it follows from [39, Theorem 4.3] and [30, Theorem 2.1]. Recall from [39, Definition 4.1] that the singularity $U \ni P$ is said to be *weakly exceptional* if it has a unique purely log terminal blow-up. This is equivalent to the condition that there is a Kollár component E_X of $U \ni P$ such that $\alpha(E_X, \mathrm{Diff}_{E_X}(0)) \geqslant 1$, see [39, Theorem 4.3], [30, Theorem 2.1], [12]. It follows from Example 10 that Du Val singularities of types $\mathbb{D}$ and $\mathbb{E}$ are weakly exceptional. On the other hand, Du Val singularities of type $\mathbb{A}$ are not weakly exceptional, since each of them admits several Kollár components (see Examples 10, 11, and 12), and thus has several purely log terminal blow ups.

Remark 17. Du Val singularities are special examples of two-dimensional quotient singularities. Note that quotient singularities are always Kawamata log terminal. For each of them, it is easy to describe one Kollár component. Let $\widehat{G}$ be a finite subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$. Suppose that $U \ni P$ is a quotient singularity $\mathbb{C}^{n+1}/\widehat{G}$. By the Chevalley–Shephard–Todd theorem, we may assume that the group $\widehat{G}$ does not contain quasi-reflections (cf. [13, Remark 1.16]). Let $\eta\colon \mathbb{C}^{n+1} \to U$ be the quotient map. Then there is a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\ \omega\ } & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathbb{C}^{n+1} & \xrightarrow[\ \eta\ ]{} & U \end{array}$$

where π is the blow up at the origin, the morphism ω is the quotient map that is induced by the action of $\widehat{G}$ lifted to the variety W, and ψ is a birational morphism. Denote by $\widetilde{E}$ the exceptional divisor of π, and denote by E_Y the exceptional divisor of ψ. Then $\widetilde{E} \cong \mathbb{P}^n$, and E_Y is naturally isomorphic to the quotient $\mathbb{P}^n/G$, where G is the image of the group

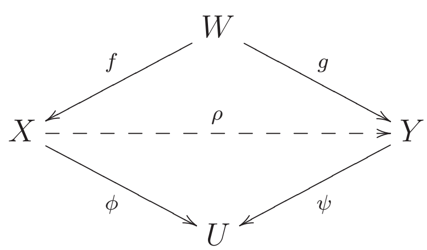
$\widehat{G}$ in $\mathrm{PGL}_{n+1}(\mathbb{C})$. Moreover, the log pair (Y, E_Y) has purely log terminal singularities, and the divisor $-(K_Y+E_Y)$ is ψ-ample. Thus, the log Fano variety $(E_Y, \mathrm{Diff}_{E_Y}(0))$ is a Kollár component of the singularity $U \ni P$. Also, it follows from [31, Example 7.1(1)] and [31, Theorem 1.2] that E_Y is K-semistable. Furthermore, one has

$$\alpha\big(E_Y, \mathrm{Diff}_{E_Y}(0)\big) = \alpha_G\big(\mathbb{P}^n\big),$$

see [12, Proof of Theorem 3.16]. Thus, if $\alpha_G(\mathbb{P}^n) \geqslant 1$, then this Kollár component is unique by Corollary 16. One can find many subgroups $G \subset \mathrm{PGL}_{n+1}(\mathbb{C})$ with $\alpha_G(\mathbb{P}^n) \geqslant 1$ in [33, 12, 13, 41, 14, 42, 16]. Note also that one always has $\alpha_G(\mathbb{P}^n) \leqslant 1184036$ by [46].

In the remaining part of the paper, we prove Theorem 14. Let us use its assumptions and notations. We have to show that ρ is an isomorphism. Suppose that this is not the case. Let us seek for a contradiction.

We may assume that U is affine. There exists a commutative diagram



such that W is a smooth variety, and f and g are birational morphisms. Denote by E_X^W and E_Y^W the proper transforms of the divisors E_X and E_Y on the variety W, respectively. Then E_X^W is g-exceptional, and E_Y^W is f-exceptional. We may assume that E_X^W, E_Y^W and the remaining exceptional divisors of f and g form a divisor with simple normal crossings.

Observe that $E_X^W \neq E_Y^W$. Indeed, if $E_X^W = E_Y^W$, then ρ is small, which is impossible, because $-E_X$ is ϕ-ample, and $-E_Y$ is ψ-ample (see [17, Proposition 2.7]). Let $F_1, \ldots, F_m$ be the prime divisors on W that are contracted by both f and g. Then

$$K_W + E_X^W + aE_Y^W + \sum_{i=1}^{m} a_i F_i \sim_{\mathbb{Q}} f^*\Big(K_X + E_X\Big)$$

for some rational numbers $a, a_1, \ldots, a_m$. Since the log pair (X, E_X) has purely log terminal singularities, all numbers $a, a_1, \ldots, a_m$ are strictly less than 1. Also, we have

$$E_X^W \sim_{\mathbb{Q}} f^*\big(E_X\big) - bE_Y^W - \sum_{i=1}^{m} b_i F_i,$$

where $b, b_1, \ldots, b_m$ are non-negative rational numbers. Then $b > 0$, since $f(E_Y^W) \subset E_X$.

Fix an integer $n \gg 0$. Put $\mathcal{M}_X = |-nE_X|$. Then $\mathcal{M}_X$ does not have base points. Denote its proper transforms on Y and W by $\mathcal{M}_X^Y$ and $\mathcal{M}_X^W$, respectively. Then

$$\mathcal{M}_X^W \sim_{\mathbb{Q}} -f^*\big(nE_X\big) \sim_{\mathbb{Q}} -nE_X^W - nbE_Y^W - \sum_{i=1}^{m} nb_i F_i,$$

which implies that $\mathcal{M}_X^Y \sim_{\mathbb{Q}} -nbE_Y$. On the other hand, we have $-(K_Y+E_Y) \sim_{\mathbb{Q}} -\delta_Y E_Y$ for some positive rational number δ_Y. Put $\epsilon_X = \frac{\delta_Y}{nb}$. Then $\epsilon_X \mathcal{M}_X^Y \sim_{\mathbb{Q}} -(K_Y+E_Y)$, so that

$$K_W + E_Y^W + \epsilon_X \mathcal{M}_X^W + \alpha E_X^W + \sum_{i=1}^{m} \alpha_i F_i \sim_{\mathbb{Q}} g^*\Big(K_Y + E_Y + \epsilon_X \mathcal{M}_X^Y\Big) \sim_{\mathbb{Q}} 0$$

for some rational numbers $\alpha, \alpha_1, \ldots, \alpha_m$. Similarly, let $\mathcal{M}_Y$ be the base point free linear system $|-nE_Y|$. Denote by $\mathcal{M}_Y^X$ and $\mathcal{M}_Y^W$ its proper transforms on X and W, respectively. Then there is a positive rational number ϵ_Y such that $\epsilon_Y \mathcal{M}_Y^X \sim_{\mathbb{Q}} -(K_X + E_X)$, so that

$$K_W + E_X^W + \epsilon_Y \mathcal{M}_Y^W + \beta E_Y^W + \sum_{i=1}^{m} \beta_i F_i \sim_{\mathbb{Q}} f^*\Big(K_X + E_X + \epsilon_Y \mathcal{M}_Y^X\Big) \sim_{\mathbb{Q}} 0$$

for some rational numbers $\beta, \beta_1, \ldots, \beta_m$.

Lemma 18. *One has $\alpha > 1$ and $\beta > 1$. In particular, the singularities of the log pairs $(Y, E_Y + \epsilon_X \mathcal{M}_X^Y)$ and $(X, E_X + \epsilon_Y \mathcal{M}_Y^X)$ are not log canonical.*

Proof. It is enough to show that $\alpha > 1$. We have

$$E_Y^W + \epsilon_X \mathcal{M}_X^W + \alpha E_X^W + \sum_{i=1}^{m} \alpha_i F_i \sim_{\mathbb{Q}} 0 \sim_{\mathbb{Q}} E_X^W + a E_Y^W + \sum_{i=1}^{m} a_i F_i - f^*\Big(K_X + E_X\Big).$$

This gives

$$\epsilon_X \mathcal{M}_X^W \sim_{\mathbb{Q}} (1-\alpha) E_X^W + (a-1) E_Y^W + \sum_{i=1}^{m} (a_i - \alpha_i) F_i - f^*\Big(K_X + E_X\Big). \tag{19}$$

It implies that

$$\epsilon_X \mathcal{M}_X \sim_{\mathbb{Q}} -(K_X + E_X) - (\alpha - 1) E_X.$$

Recall that $-(K_X + E_X) \sim_{\mathbb{Q}} -\delta_X E_X$. We then obtain

$$\epsilon_X \mathcal{M}_X \sim_{\mathbb{Q}} -\Big(K_X + E_X\Big) - (\alpha - 1) E_X \sim_{\mathbb{Q}} -t_X\Big(K_X + E_X\Big),$$

where $t_X = 1 + \frac{1}{\delta_X} > 1$. On the other hand, from (19) we obtain

$$(1-\alpha) E_X^W + \sum_{i=1}^{m} (a_i - \alpha_i) F_i \sim_{\mathbb{Q}} (1-a) E_Y^W + (1 - t_X) f^*\Big(K_X + E_X\Big).$$

Since $a < 1$, Negativity Lemma (see [28, Lemma 3.39]) implies $\alpha > 1$. ☐

As in the proof of Lemma 18, put $t_Y = 1 + \frac{1}{\delta_Y} > 1$. Then

$$\epsilon_Y \mathcal{M}_Y \sim_{\mathbb{Q}} -t_Y\big(K_Y + E_Y\big).$$

Now take any non-negative rational numbers λ and μ such that $\lambda + \mu \geqslant 1$. One has

$$K_X + E_X + \lambda \epsilon_Y \mathcal{M}_Y^X + \mu \epsilon_X \mathcal{M}_X \sim_{\mathbb{Q}} -\big(\lambda + \mu t_X - 1\big)\Big(K_X + E_X\Big),$$

so that $K_X + E_X + \lambda \epsilon_Y \mathcal{M}_Y^X + \mu \epsilon_X \mathcal{M}_X$ is ϕ-ample. Similarly, we see that

$$K_Y + E_Y + \lambda \epsilon_Y \mathcal{M}_Y + \mu \epsilon_X \mathcal{M}_X^Y \sim_{\mathbb{Q}} -\big(\lambda t_Y + \mu - 1\big)\Big(K_Y + E_Y\Big),$$

so that $K_Y + E_Y + \lambda \epsilon_Y \mathcal{M}_Y + \mu \epsilon_X \mathcal{M}_X^Y$ is ψ-ample.

Lemma 20. *At least one of the log pairs $(X, E_X + \lambda \epsilon_Y \mathcal{M}_Y^X)$ and $(Y, E_Y + \mu \epsilon_X \mathcal{M}_X^Y)$ is not log canonical.*

Proof. Suppose that both $(X, E_X + \lambda\epsilon_Y\mathcal{M}_Y^X)$ and $(Y, E_Y + \mu\epsilon_X\mathcal{M}_X^Y)$ are log canonical. Then the log pairs $(X, E_X + \lambda\epsilon_Y\mathcal{M}_Y^X + \mu\epsilon_X\mathcal{M}_X)$ and $(Y, E_Y + \lambda\epsilon_Y\mathcal{M}_Y + \mu\epsilon_X\mathcal{M}_X^Y)$ are also log canonical. On the other hand, we have

$$K_W + E_X^W + \lambda\epsilon_Y\mathcal{M}_Y^W + \mu\epsilon_X\mathcal{M}_X^W + cE_Y^W + \sum_{i=1}^m c_iF_i \sim_{\mathbb{Q}} f^*\Big(K_X + E_X + \lambda\epsilon_Y\mathcal{M}_Y^X + \mu\epsilon_X\mathcal{M}_X\Big)$$

for some rational numbers $c, c_1, \ldots, c_m$ that do not exceed 1. Similarly, we have

$$K_W + E_Y^W + \lambda\epsilon_Y\mathcal{M}_Y^W + \mu\epsilon_X\mathcal{M}_X^W + dE_X^W + \sum_{i=1}^m d_iF_i \sim_{\mathbb{Q}} g^*\Big(K_Y + E_Y + \lambda\epsilon_Y\mathcal{M}_Y + \mu\epsilon_X\mathcal{M}_X^Y\Big),$$

where $d, d_1, \ldots, d_m$ are rational numbers that do not exceed 1. Denote by D_W the boundary $\lambda\epsilon_Y\mathcal{M}_Y^W + \mu\epsilon_X\mathcal{M}_X^W + E_X^W + E_Y^W + \sum_{i=1}^m F_i$. Then

$$K_W + D_W \sim_{\mathbb{Q}} f^*\Big(K_X + E_X + \lambda\epsilon_Y\mathcal{M}_Y^X + \mu\epsilon_X\mathcal{M}_X\Big) + (1-c)E_Y^W + \sum_{i=1}^m (1-c_i)F_i \sim_{\mathbb{Q}}$$

$$\sim_{\mathbb{Q}} g^*\Big(K_Y + E_Y + \lambda\epsilon_Y\mathcal{M}_Y + \mu\epsilon_X\mathcal{M}_X^Y\Big) + (1-d)E_X^W + \sum_{i=1}^m (1-d_i)F_i.$$

Moreover, the log pair (W, D_W) is log canonical, since W is smooth, the linear systems $\mathcal{M}_Y^W$ and $\mathcal{M}_X^W$ are free from base points, and the divisors E_X^W, E_Y^W, $F_1, \ldots, F_m$ form a simple normal crossing divisor. Since $K_X + E_X + \lambda\epsilon_Y\mathcal{M}_Y^X + \mu\epsilon_X\mathcal{M}_X$ is ϕ-ample, it follows from [28, Corollary 3.53] that the log pair $(X, E_X + \lambda\epsilon_Y\mathcal{M}_Y^X + \mu\epsilon_X\mathcal{M}_X)$ is the canonical model of the log pair (W, D_W). Similarly, the log pair $(Y, E_Y + \lambda\epsilon_Y\mathcal{M}_Y + \mu\epsilon_X\mathcal{M}_X^Y)$ is also the canonical model of the log pair (W, D_W), because $K_Y + E_Y + \lambda\epsilon_Y\mathcal{M}_Y + \mu\epsilon_X\mathcal{M}_X^Y$ is ψ-ample. Since the canonical model is unique by [28, Theorem 3.52], we see that ρ is an isomorphism. Since ρ is not an isomorphism by assumption, we obtain a contradiction. This completes the proof of the lemma. □

Let $\lambda = \alpha(E_X, \mathrm{Diff}_{E_X}(0))$ and $\mu = \alpha(E_Y, \mathrm{Diff}_{E_Y}(0))$. We may assume that the log pair $(X, E_X + \lambda\epsilon_Y\mathcal{M}_Y^X)$ is not log canonical. Then $(E_X, \mathrm{Diff}_{E_X}(0) + \lambda\epsilon_Y\mathcal{M}_Y^X|_{E_X})$ is not log canonical by Inversion of adjunction, see [27, 17.6]. On the other hand, we have

$$\epsilon_Y\mathcal{M}_Y^X\Big|_{E_X} \sim_{\mathbb{Q}} -\Big(K_X + E_X\Big)\Big|_{E_X} \sim_{\mathbb{Q}} -\Big(K_{E_X} + \mathrm{Diff}_{E_X}(0)\Big).$$

This is impossible by the definition of the α-invariant $\alpha(E_X, \mathrm{Diff}_{E_X}(0))$.

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References

[1] I. Cheltsov, *Log canonical thresholds on hypersurfaces*, Sb. Math. **192** (2001), 1241–1257.

[2] I. Cheltsov, *Log canonical thresholds of del Pezzo surfaces*, Geom. Funct. Anal. **18** (2008), 1118–1144.

[3] I. Cheltsov, *Fano varieties with many selfmaps*, Adv. Math. **217** (2008), 97–124.

[4] I. Cheltsov, *On singular cubic surfaces*, Asian J. Math. **19** (2009), 191–214.

[5] I. Cheltsov, *Log canonical thresholds of three-dimensional Fano hypersurfaces*, Izv. Math. **73** (2009), 727–795.

[6] I. Cheltsov, *Extremal metrics on two Fano varieties*, Sb. Math. **200** (2009), 95–132.

[7] I. Cheltsov, D. Kosta, *Computing α-invariants of singular del Pezzo surfaces*, J. Geom. Anal. **24** (2014), 798–842.

[8] I. Cheltsov, J. Park, *Global log-canonical thresholds and generalized Eckardt points*, Sb. Math. **193** (2002), 779–789.

[9] I. Cheltsov, J. Park, C. Shramov, *Exceptional del Pezzo hypersurfaces*, J. Geom. Anal. **20** (2010), 787–816.

[10] I. Cheltsov, J. Park, J. Won, *Log canonical thresholds of certain Fano* 3*-folds*, Math. Z. **276** (2014), 51–79.

[11] I. Cheltsov, C. Shramov, *Log canonical thresholds of smooth Fano threefolds*, Russian Math. Surveys **63** (2008), 73–180.

[12] I. Cheltsov, C. Shramov, *On exceptional quotient singularities*, Geom. Topol. **15** (2011), 1843–1882.

[13] I. Cheltsov, C. Shramov, *Six-dimensional exceptional quotient singularities*, Math. Res. Lett. **18** (2011), 1121–1139.

[14] I. Cheltsov, C. Shramov, *Sporadic simple groups and quotient singularities*, Izv. Math. **77** (2013), 846–854.

[15] I. Cheltsov, C. Shramov, *Del Pezzo zoo*, Exp. Math. **22** (2013), 313–326.

[16] I. Cheltsov, C. Shramov, *Weakly-exceptional singularities in higher dimensions*, J. Reine Angew. Math. **689** (2014), 201–241.

[17] A. Corti, *Factoring birational maps of threefolds after Sarkisov*, J. Algebraic Geom. **4** (1995), 223–254.

[18] A. Corti, *Del Pezzo surfaces over Dedekind schemes*, Ann. of Math. **144** (1996), 641–683.

[19] K. Fujita, *K-stability of Fano manifolds with not small alpha invariants*, to appear in J. Inst. Math. Jussieu.

[20] K. Fujita, *Uniform K-stability and plt blowups of log Fano pairs*, arXiv:1701.00203 (2017).

[21] K. Fujita, *Openness results for uniform K-stability*, to appear in arXiv.

[22] K. Fujita, Y. Odaka, *On the K-stability of Fano varieties and anticanonical divisors*, to appear in Tohoku Math. J.

[23] J. Johnson, J. Kollár, *Fano hypersurfaces in weighted projective* 4*-spaces*, Exp. Math. **10** (2001), 151–158.

[24] I. Kim, J. Park, *Log canonical thresholds of complete intersection log del Pezzo surfaces*, Proc. Edinb. Math. Soc. **58** (2015), 445–483.

[25] I. Kim, T. Okada, J. Won, *Alpha invariants of birationally rigid Fano threefolds*, to appear in Int. Math. Res. Not.

[26] J. Kollár, *Singularities of pairs*, Proc. Sympos. Pure Math. **62** (1997), 221–287.

[27] J. Kollár et al., *Flips and abundance for algebraic threefolds*, Societe Mathematique de France, Astérisque **211**, 1992.

[28] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, Cambridge University Press (1998).

[29] I. Krylov, *Rationally connected non-Fano type varieties*, arXiv:1406.3752 (2014).

[30] S. Kudryavtsev, *Pure log terminal blow-Ups*, Math. Notes **69**:6, 814–819 (2001).

[31] C. Li, C. Xu, *Stability of valuations and Kollár components*, arXiv:1604.05398 (2016).

[32] Y. Liu, *The volume of singular Kähler–Einstein Fano varieties*, arXiv:1605.01034 (2016).

[33] D. Markushevich, Yu. Prokhorov, *Exceptional quotient singularities*, Amer. J. Math. **121** (1999), 1179–1189.

[34] Y. Odaka, Y. Sano, *Alpha invariants and K-stability of $\mathbb{Q}$-Fano varieties*, Adv. Math. **217** (2008), 97–124.

[35] J. Park, *Birational maps of del Pezzo fibrations*, J. Reine Angew. Math. **538** (2001), 213–221.

[36] J. Park, *A note on del Pezzo fibrations of degree* 1, Comm. Algebra **31** (2003), 5755–5768.
[37] J. Park, J. Won, *Log canonical thresholds on Gorenstein canonical del Pezzo surfaces*, Proc. Edinb. Math. Soc. **54** (2011), 187–219.
[38] J. Park, J. Won, *Log canonical thresholds on del Pezzo surfaces of degree* $\geqslant 2$, Nagoya Math. J. **200** (2010), 1–26.
[39] Yu. Prokhorov, *Blow-ups of canonical singularities*, Algebra (Moscow, 1998), de Gruyter, Berlin (2000), 301–317.
[40] A. Pukhlikov, *Birational geometry of Fano direct products*, Izv. Math. **69** (2005), 1225–1255.
[41] D. Sakovics, *Weakly-exceptional quotient singularities*, Cent. Eur. J. Math. **10** (2012), 885–902.
[42] D. Sakovics, *Five-dimensional weakly exceptional quotient singularities*, Proc. Edinb. Math. Soc. **57** (2014), 269–279.
[43] Y. Shi, *On the α-invariants of cubic surfaces with Eckardt points*, Adv. Math. **225** (2010), 1285–1307.
[44] V. Shokurov, *Three-fold log flips*, Russian Acad. Sci. Izv. Math. **40** (1993), 95–202.
[45] G. Tian, *On Kähler–Einstein metrics on certain Kähler manifolds with* $c_1(M) > 0$, Invent. Math. **89** (1987), 225–246.
[46] P. Tiep, *The α-invariant and Thompson's conjecture*, Forum Math. Pi **4** (2016), e5, 28 pp.
[47] C. Xu, *Finiteness of algebraic fundamental groups*, Compos. Math. **150** (2014), 409–414.

Ivan Cheltsov
School of Mathematics, The University of Edinburgh
Edinburgh EH9 3JZ, UK

National Research University Higher School of Economics, Laboratory of Algebraic Geometry,
6 Usacheva street, Moscow, 117312, Russia
I.Cheltsov@ed.ac.uk

Jihun Park
Center for Geometry and Physics, Institute for Basic Science
77 Cheongam-ro, Nam-gu, Pohang, Gyeongbuk, 37673, Korea

Department of Mathematics, POSTECH
77 Cheongam-ro, Nam-gu, Pohang, Gyeongbuk, 37673, Korea
wlog@postech.ac.kr

Constantin Shramov
Steklov Mathematical Institute of Russian Academy of Sciences
8 Gubkina street, Moscow, 119991, Russia

National Research University Higher School of Economics, Laboratory of Algebraic Geometry,
6 Usacheva street, Moscow, 117312, Russia
costya.shramov@gmail.com