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Kaluza-Klein (KK) compactifications of higher-dimensional Yang-Mills theories contain a number of 4-dimensional scalars corresponding to the internal components of the gauge field. While at tree level the scalar zero modes are massless, it is well known that quantum corrections make them massive. We compute these radiative corrections at 1 loop in an effective field theory framework, using the background field method and proper Schwinger-time regularization. In order to clarify the proper treatment of the sum over KK modes in the effective field theory approach, we consider the same problem in two different UV completions of Yang-Mills: string theory and lattice field theory. In both cases, when the compactification radius R is much bigger than the scale of the UV completion ($R \gg \sqrt{\alpha'}$, a), we recover a mass renormalization that is independent of the UV scale and agrees with the one derived in the effective field theory approach. These results support the idea that the value of the mass corrections is, in this regime, universal for any UV completion that respects locality and gauge invariance. The string analysis suggests that this property holds also at higher loops. The lattice analysis suggests that the mass of the adjoint scalars appearing in $\mathcal{N} = 2, 4$ super Yang-Mills is highly suppressed, even if the lattice regularization breaks all supersymmetries explicitly. This is due to an interplay between the higher-dimensional gauge invariance and the degeneracy of bosonic and fermionic degrees of freedom.

I. INTRODUCTION

Gauge theories compactified on a circle or a torus appear in various different physical contexts. For instance, the reduction on a circle from four to three dimensions is relevant for studying the finite-temperature effects, while toroidal compactification from D to four dimensions provides the simplest possible toy model for extra-dimensional theories. The components of the gauge field along the compact dimensions appear as scalars in the effective field theory for the noncompact space. Imposing periodic boundary conditions, these scalars contain a massless zero mode. It has been known for a long time that these massless modes are lifted by radiative corrections [1]: in a Yang-Mills theory compactified on the circle S^1 it is possible to write a gauge invariant mass term and so we expect to find a nonzero 1-loop correction δm^2 that vanishes in the limit $R \to \infty$. The quantum mass corrections to the zero and higher Kaluza-Klein modes were thoroughly studied in the context of extra-dimensional field theories [2–4], with particular attention towards phenomenological applications; see for instance [5]. However this effective field theory approach has some shortfalls: since a higher-dimensional field theory is nonrenormalizable, a sensitivity to the UV physics can appear which depends on the regularization scheme; moreover it is not entirely clear how to treat rigorously the sum over the Kaluza-Klein modes. Different approaches for computing the vacuum polarization have been proposed and seem to give mostly consis-

tent results; see, e.g., Refs. [6–16]. The finiteness of the scalar mass can also be obtained in the effective potential approach, as discussed in Ref. [17]. The equivalence of the two approaches and a first step towards a 2-loop calculation were presented in Ref. [18].

In this paper we want to explore dimensional reduction as a tool for defining extended supersymmetric theories. In particular we try to find new hints for defining extended supersymmetry on the lattice without fine-tuning. In order to obtain some quantitative information, we concentrate on the 1-loop corrections to the mass of the adjoint scalar field that is obtained from the zero mode of the gauge field component along a compactified direction, and try to disentangle the high-energy (i.e., cutoff scale) contributions from the low-energy ones. Insight on this problem is obtained by considering the quantum corrections to the mass of the Kaluza-Klein zero modes in two different UV completions of Yang-Mills: string theory and lattice field theory. As we shall see below, both cases are concrete examples of finite theories, where explicit and unambiguous computations can be performed. Even though the computational techniques are different in the two cases, a clear physical picture emerges from the comparison of the two computations: the leading order correction to the scalar mass is universal and agrees with the result obtained from an effective field theory computation.

The string theory computation is most easily compared with the quantum field theory one if the latter is performed

using the background field method and a Schwinger-time regularization; see Ref. [19] and references therein. The setup for the background field calculation is discussed in Sec. II. However our goal in this work is to obtain a quantitative result for the string and lattice theory computations; we summarize the effective field theory computation mainly to set up a common notation and facilitate the comparison. Our results in Sec. II agree with the well-known results in Refs. [2–4] for the mass renormalization of the scalar mode, thereby providing a new consistency check. In string theory we follow the procedure outlined in Refs. [20–22] and relate the mass shift δm^2 to the correlator of two vertex operators. Note that the string computation requires a prescription to regulate the divergences that appear when the two vertices are very close on the world sheet. These divergences are automatically regulated when the soft insertions of the external states are re-summed and one derives the radiative mass corrections from the effective action, as done in Refs. [23,24]. Even if this approach is very efficient for untwisted string states such ours, it cannot be applied to the case of twisted states. Thus it is interesting to follow Refs. [20–22], as we do, and extract the mass renormalization from the 2-point function (see Ref. [25] for an application of this approach in the context of closed string theory). With the proper prescription for the short distance's divergences on the world sheet, we verify that the mass corrections to the components of the gauge field in the noncompactfied dimensions vanish, as dictated by the 4-dimensional gauge symmetry. Having regulated these divergences, the string theory techniques are readily extended to the case of compact space-time dimensions. In close analogy with the noncompact case, we find that the string calculation is easily mapped into the quantum field theory calculation and there is quantitative agreement between the two approaches when the string scale is much lower than the compactification scale ($\sqrt{\alpha'} \ll R$).

In order to study the theory defined on a discrete space-time lattice, we generalize the techniques developed in Ref. [26] in the context of finite-temperature field theories. Again, when the lattice scale a is much lower than the compactification scale, the mass generated by radiative corrections for the component of the gauge field in the compact dimension is found to be identical to the one obtained in the effective theory calculation and thus to the string theory one in the regime $\sqrt{\alpha'} \ll R$. Notice that the lattice and the string theory calculations deal with the sum over the Kaluza-Klein modes in a very different way: the string UV completion provides a setup where the so-called Kaluza-Klein regularization is implemented in a consistent way and the sums run over *all* the modes; on the contrary, lattice gauge theory provides a gauge invariant way of implementing a hard cutoff on the integrals and sums and only modes of energies up to $1/a$ are considered. The fact that these two different approaches yield the same

result in the limit $a, \sqrt{\alpha'} \ll R$ suggests that all UV completions that respect locality and gauge invariance yield a leading order contribution to the scalar mass that is completely captured by an effective field theory approach. The physical reason is that the high-energy modes in the UV completion see the extra dimensions as uncompact and so do not contribute to the mass renormalization because of the higher-dimensional gauge symmetry.

We find that a similar pattern holds also for the 1-loop contribution of fermions in lattice perturbation theory. The fermionic contribution can actually be written in a form that is very close to the bosonic one. As a consequence, we find that the leading terms in the bosonic and fermionic contributions to the mass renormalization of the adjoint scalar field cancel whenever the number of degrees of freedom are equal. Hence the mass renormalization of the scalar field is highly suppressed if a supersymmetric theory is dimensionally reduced. Our computation provides an explicit 1-loop realization of the mechanism suggested in Ref. [27], and supports the interesting possibility that Yang-Mills theories with extended supersymmetry can be defined on the lattice without any fine-tuning by dimensional reduction of a higher-dimensional $\mathcal{N} = 1$ theory, exactly as it happens in the continuum case [28].

The paper is organized as follows. Section II introduces the main ingredients in the calculation of the quantum corrections to the masses of the Kaluza-Klein zero modes, and derives the usual formula for the mass shifts in a new effective field theory framework, namely, in the background field method with a Schwinger-time regularization. The details of the string computation are described in Sec. III. Section IV deals with the details of the lattice computation, for the cases of bosonic and fermionic contributions in the loops. The possibility of an accidental extended supersymmetric is discussed at the end of Sec. IV. The main results of this work are summarized in the conclusion together with some open questions that could be addressed in future works.

II. MASS CORRECTIONS IN COMPACTIFIED FIELD THEORIES

This section concentrates on the study of the mass renormalization in the $SU(N)$ Yang-Mills gauge theory using the background field method in a space-time with compactified dimensions. We shall see below that even though our calculations are performed in a different setting, they reproduce the results that have already appeared in the literature. The correspondence between quantum field theory and string computations is apparent when amplitudes are expressed in terms of Schwinger parameters and an explicit mapping can be defined to relate the string moduli and the Schwinger parameters; see [19] and references therein. We shall therefore use the Schwinger parametrization in order to emphasize the connection with

the string theory approach. A similar approach has recently been developed in Refs. [29,30].

Clearly, before considering any explicit computation, the gauge invariance of the theory should be used to constrain the form of the 2-gluon correlator. This is most easily done in a path integral approach and by using Becchi-Rouet-Stora invariance; see for instance [31]. In configuration space, this 2-point function must satisfy the Ward identity

$$\frac{\partial}{\partial x^M}\frac{\partial}{\partial y^N}\langle A^{aM}(x)A^{bN}(y)\rangle = -i\delta^{ab}\delta^D(x-y), \quad (2.1)$$

where, in a D-dimensional theory, $M = 0, \ldots, D-1$. If all dimensions are uncompact this leads to the usual conclusion that the gluon self-energy is transverse and no mass term can be generated. We will see this feature arising explicitly in our 1-loop computation. In a toroidal compactification the situation is different. Since we focus on the case of vanishing Wilson lines, all fields are periodic around the compact dimension and the associated momenta are discrete. Thus for the Kaluza-Klein zero modes, Eq. (2.1) reduces to a constraint involving only the gluon polarizations along the *uncompact* directions, since these modes have a nonzero momentum only along these directions. The mass correction of the other components (which are scalars from the lower dimensional point of view) is not constrained by any symmetry and can only be determined by performing an explicit computation. Let us notice that for the *higher* Kaluza-Klein modes these Ward identities yield again nontrivial constraints on the quantum mass corrections; see Sec. 3 of Ref. [15], where this point is discussed in detailed.

Quantitative information on the renormalization of zero-mode mass can only be obtained by explicit calculations. We consider first the case of a D-dimensional theory, without compact dimensions, in order to set up our framework, and check indeed the symmetry constraints are satisfied.

Starting from the Feynman rules detailed in Ref. [32], we compute the sum of 1-loop diagrams contributing to the gauge boson 2-point function at zero external momentum.[1] In a D-dimensional theory, without compact dimensions, there are four diagrams (as opposed to three in standard Yang-Mills theories, as a result of an extra Feynman rule of 2-ghost 2-gluon interaction), and their contributions are shown below:

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[1]Note that we have used different metrics in different contexts. The field theory computation employs a "mostly negative" metric, the string theory computation a "mostly positive" metric, while the lattice computation is performed in Euclidean space-time. The reader should keep these conventions in mind in comparing results in this paper.

$$A_1 = \quad \text{} \quad = 2Dg_D^2 N \int \frac{d^Dk}{(2\pi)^D}\frac{k_\mu k_\nu}{k^4},$$

$$A_2 = \quad \text{} \quad = -Dg_D^2 N g_{\mu\nu}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2},$$

$$A_3 = \quad \text{} \quad = -4g_D^2 N \int\frac{d^Dk}{(2\pi)^D}\frac{k_\mu k_\nu}{k^4},$$

$$A_4 = \quad \text{} \quad = 2g_D^2 N g_{\mu\nu}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2}.$$

$$(2.2)$$

The sum of these amplitudes amounts to

$$A = (D-2)g_D^2 N \int\frac{d^Dk}{(2\pi)^D}\left[\frac{2k_\mu k_\nu}{k^4} - \frac{g_{\mu\nu}}{k^2}\right]. \quad (2.3)$$

We suppress color indices, which appear only in a delta function.

This quantity is ultraviolet divergent and needs to be properly regularized in order to evaluate the mass corrections. The gauge coupling g has mass dimension $[g] = \frac{4-D}{2}$, and so any divergence contained in this amplitude which is to contribute to a mass shift of the gauge boson must have mass dimension $2 - (4 - D) = D - 2$. In four dimensions this is a quadratic divergence.

In dimensional regularization, the divergence appears as a factor $\Gamma(1 - \frac{D}{2})$, which has a first pole at $D = 2$. However, using the recursion relation for Gamma functions, this can be transformed into a factor $\Gamma(2 - \frac{D}{2})$ [because of the appearance of a $(D - 2)$ factor before the integral] which has its first pole at $D = 4$ as expected for a logarithmic divergence in four dimensions.

In this work a cutoff on the Schwinger time is used to regulate the divergences. This is again in order to compare in a straightforward way with perturbative string theory calculations, but also so that we can extract the divergences as powers of a mass scale Λ which, while we associate it with a momentum cutoff for the theory, does not break the gauge invariance. This procedure involves the exponentiation of the propagators in the momentum integrals using

$$\frac{1}{X^r} = \frac{1}{\Gamma(r)}\int_0^\infty dT\, T^{r-1}e^{-TX}, \quad (2.4)$$

where the variable T is termed a Schwinger-time parameter. As an example, this procedure yields for a tadpole diagram

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} = \int_0^\infty dT \int \frac{d^D k}{(2\pi)^D} e^{-Tk^2}$$

$$= \frac{1}{(4\pi)^{(D/2)}} \int_0^\infty dT T^{-(D/2)}, \qquad (2.5)$$

which is a divergent integral. The divergences arise from the $T \to 0$ region of the integral, where there is no exponential damping of the contribution from large momenta in the above expression, and so we can regulate by imposing a lower bound T_0 on the integration variable T. Doing this we see that the divergence appears in the result as a factor $T_0^{1-(D/2)}$ and so, as we expect, this divergence is of mass dimension $D - 2$. In order to associate the lower bound on the Schwinger time with a momentum cutoff Λ, we write $T_0 = \frac{1}{\Lambda^2}$. Thus the two integrals contained in Eq. (2.3) amount to
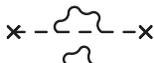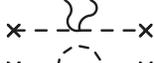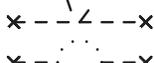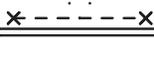
$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} = \frac{-i}{(4\pi)^{(D/2)}} \frac{\Lambda^{D-2}}{\frac{D}{2} - 1},$$

$$\int \frac{d^D k}{(2\pi)^D} \frac{k_\mu k_\nu}{k^4} = \frac{-i g_{\mu\nu}}{2(4\pi)^{(D/2)}} \frac{\Lambda^{D-2}}{\frac{D}{2} - 1}. \qquad (2.6)$$

Inserting this result into Eq. (2.3) we see that the two terms cancel and the expression vanishes as required by gauge invariance.

Let us now examine the effects of compactification on this cancellation. We restrict ourselves to the case where we compactify one of the D dimensions, leaving an effective theory in $d = D - 1$ dimensions. The resulting effective theory consists of a d component gauge boson, and a scalar field in the adjoint representation arising from the extra-dimensional component of the original D component gauge field. The momentum of the fields in the finite compactified dimension produces a tree-level mass for an infinite tower of fields called Kaluza-Klein (KK) modes. The gauge coupling is rescaled by $g_d^2 = \frac{g_D^2}{2\pi R}$.

The zero-mode gauge boson does not receives any 1-loop mass renormalization after compactification since the computation of the 2-point function is basically the one discussed above. The adjoint scalar however, does, as expected due to the breaking of the original gauge invariance. We will illustrate this here, and confirm agreement with the result obtained in Refs. [2–4]. Note that in Ref. [4] the relevant 2-point functions are computed not at zero external momentum p, but in the approximation $p^2 = r^2$, where $r = p_5 = \frac{n}{R}$ is the KK mass of the external particle. As a result of the Poisson resummation used to compute the sum over KK modes, inverse powers of the KK mode of the external particle are generated, which can yield extra contributions to the final result which would be missed in the $p = 0$ limit. This only affects the result for $r \neq 0$ external modes however, and therefore we can work at $p = 0$ for our purposes.

TABLE I. Diagrams yielding the quantum corrections to the adjoint scalar mass.

Diagram	$g_d^2 N I_1$	$g_d^2 N I_2$
	0	$2d$
	0	0
	d	0
	0	2
	1	0
	0	-4
	-2	0

By keeping a generic nonvanishing external momentum $p \neq 0$ for the zero modes, it would be possible to compute higher-derivative terms in the low-energy effective action which can be relevant in phenomenological applications [33]. However, in this paper, we focus on the mass correction terms which represent the most relevant contributions in the infrared, and which are most easily computed both in string theory and in the lattice field theory approach.

In computing the contributions to the scalar 2-point function at zero momentum, there are two integrals, summed over Kaluza-Klein modes, which arise. These are

$$I_1 = \sum_l \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - l^2}$$

$$= -\frac{i}{(4\pi)^{d/2}} \sum_l \int_0^\infty \frac{dT}{T^{(d/2)}} e^{-l^2 T},$$

$$I_2 = \sum_l \int \frac{d^d k}{(2\pi)^d} \frac{l^2}{(k^2 - l^2)^2}$$

$$= \frac{i}{(4\pi)^{(d/2)}} \sum_l l^2 \int_0^\infty \frac{dT}{T^{(d/2)-1}} e^{-l^2 T}, \qquad (2.7)$$

with $l = \frac{m}{R}$ where m is an integer denotes the mass of a KK mode, and the sum over l is a sum over the integers m.

There are seven diagrams contributing to the 2-point function for the adjoint scalar field; they are shown in Table I with their contributions in terms of I_1 and I_2.

The second diagram in Table I vanishes only at zero momentum and in the Feynman type gauge.[2] The sum of all the diagrams produces

$$A = (d - 1)g_d^2 N[I_1 + 2I_2]. \qquad (2.8)$$

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[2]I.e. α the parameter of the background gauge fixing term is $\alpha = 1$; see [32]

We evaluate the integrals from (2.7) following a similar procedure to the noncompact case. After exponentiating propagators and performing the Gaussian momentum integral, we find quantities such as $\sum_l e^{-Tl^2}$ where the sum is over the KK mode of the loop particle. We evaluate such infinite sums through a Poisson resummation, transforming the sum over KK modes into a sum over the winding number of the path of the loop particle around the compact extra dimension

$$\sum_{l=(m/R)} e^{-Tl^2} = \frac{(2\pi R)}{\sqrt{4\pi T}} \sum_n e^{-((\pi^2 R^2 n^2)/T)}. \qquad (2.9)$$

The $n = 0$ term corresponds to the noncompact case, and so produces a Λ^{D-2} divergence. In all other terms we make the change of variables $t = \frac{\pi^2 R^2 n^2}{T}$ which then results in the integration over the Schwinger parameter producing Gamma functions with arguments away from the singularities. Omitting further details, we obtain

$$I_1 = -\frac{i}{2} \frac{(2\pi R)^{2-d}}{\pi^{((d+1)/2)}} \zeta(d-1) \Gamma\left(\frac{d-1}{2}\right)$$
$$- \frac{i}{(4\pi)^{(d/2)}} \frac{2\sqrt{\pi}R}{d-1} \Lambda^{d-1}, \qquad (2.10)$$

$$I_2 = \frac{i}{2} \frac{(2\pi R)^{2-d}}{\pi^{((d+1)/2)}} \zeta(d-1)\left[\frac{1}{2}\Gamma\left(\frac{d-1}{2}\right) - \Gamma\left(\frac{d+1}{2}\right)\right]$$
$$+ \frac{1}{2} \frac{i}{(4\pi)^{(d/2)}} \frac{2\sqrt{\pi}R}{d-1} \Lambda^{d-1}.$$

In Table II we show the final contribution of each diagram. We quote the coefficient of $\frac{i}{(4\pi)^{(d/2)}} \frac{2}{\sqrt{\pi}} (\pi R)^{2-d} \zeta(d-1)\Gamma(\frac{d-1}{2})$, and also $\frac{i}{(4\pi)^{(d/2)}} \frac{2\sqrt{\pi}R}{d-1} \Lambda^{d-1}$ that results for each diagram.

It is easily seen that the divergent contributions cancel each other, as expected by the higher-dimensional gauge invariance. The total contribution then becomes

TABLE II. Nonzero contributions from the diagrams in Fig. 1.

Diagram	$\sim R^{2-d}$	$\sim \Lambda^{d-1}$
	$d(2-d)$	d
	$-d$	$-d$
	$2-d$	1
	-1	-1
	$-2(2-d)$	-2
	2	2
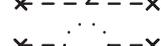
$$A = (d-1)g_d^2 N \sum_l \int \frac{d^d k}{(2\pi)^d}\left[\frac{2l^2}{(k^2-l^2)^2} + \frac{1}{k^2-l^2}\right]$$
$$= -(d-1)g_d^2 N \frac{i(2\pi R)^{2-d}}{\pi^{((d+1)/2)}} \zeta(d-1)\Gamma\left(\frac{d+1}{2}\right). \qquad (2.11)$$

This results in an additive mass renormalization of the adjoint scalar field of

$$\delta m^2 = \frac{g_d^2 N}{\pi^{((d+1)/2)}} \frac{(D-2)}{(2\pi R)^{d-2}} \zeta(d-1)\Gamma\left(\frac{d+1}{2}\right). \qquad (2.12)$$

For $D = 5$ and $d = 4$, this gives

$$\delta m^2 = \frac{9 g_4^2 N}{16\pi^4 R^2} \zeta(3). \qquad (2.13)$$

III. MASS CORRECTIONS IN THE OPEN BOSONIC STRING THEORY

The naive vacuum of the open bosonic string theory is unstable, as it is signalled by the presence of a tachyon excitation with mass square $M_t^2 = -1/\alpha'$. However, it is still useful to study formally the perturbation theory around this unstable point, since in this way it is possible to understand, in a simple setup, many properties of the string amplitudes of fully consistent theories. In practice, one can compute the loop amplitudes by using standard string techniques [34] and discard by hand the tachyon contributions before considering the loop integrals. This approach has been already used successfully in the past in the study of the low-energy limit of 1-loop string amplitudes; see for instance [19,35]. Moreover the analysis of the radiative corrections to the mass of the string states was initiated in the context of bosonic theory [20]. Most of the early studies of these radiative corrections were done in the context of closed string theory [21,22,25]. More recently [23,24], the same problem has been analyzed in an open string context by computing the effective action for two stacks of D-branes. In this section we will consider this (bosonic) D-brane setup, but we will follow the original approach of [20] and compute the 2-point function for open strings on the annulus. Even if we focus on the string states corresponding to the internal components of the gauge field, this approach can be used also when the vertex operators contain twist fields, a situation where the technique used in Refs. [23,24] cannot be applied.

Let us consider a stack of N space-filling D-branes in bosonic string theory and we take the space-time to be the product of the d-dimensional Minkowski space and s circles of radius R (in principle, bosonic string is critical only if $d + s \equiv D = 26$, however this constraint will play no role in most of our computations). We will focus on the massless open string states supported by these D-branes. The (on shell) 2-point amplitude with massless states requires one to take the external particles at zero momen-

tum, which is sufficient for computing the 1-loop mass corrections we are interested in. The vertex operator describing these states is simply[3]

$$V^a = ig_D T^a \partial X^I, \qquad (3.1)$$

where T^a is a $SU(N)$ generator,[4] g_D is the D-dimensional Yang-Mills coupling, and $I \equiv \mu = 0, \ldots, d-1$ for the vector boson, while $I \equiv i = 1, \ldots, s$ for the scalars arising from the Kaluza-Klein reduction of the higher-dimensional gauge field.

The radiative correction to the tree-level mass square (δm_I^2) is obtained from the planar 2-point amplitude A^I:

$$A^I = -g_D^2 N \operatorname{Tr}(T^a T^b) \int \langle \partial X^I(1) \partial X^I(y) \rangle d\mu, \qquad (3.2)$$

where in our conventions $\operatorname{Tr}(T^a T^b) = \delta^{ab}/2$ and of course the index I is not summed, but takes one of the values listed above. In our case the correlator $\langle \ldots \rangle$ is taken over the annulus topology, $d\mu$ is the 1-loop integration measure (3.7). Let us analyze these ingredients in some detail.

We will parametrize the annulus as the upper half complex plane (minus the point $z = 0$) modded out by the equivalence relation $z \to kz$, where k is a real number $k \in (0, 1)$. Each value of k corresponds to a different shape for the annulus and, in the amplitude (3.2), we need to integrate over all possibilities. The two borders of the annulus are the segments on the real axis $y \in [k, 1]$ and $y \in [-1, -k]$. We are free to choose the position of the first vertex operator and the second vertex operator has to stay on the same border, $y \in [k, 1]$ in our case. The correlator $\langle \partial X^I(1) \partial X^I(y) \rangle$ can be split in the contribution of the vibration modes of the string and the one of the center of mass and rigid motion (zero modes). By following the derivation in Chap. 8 of [34], one can compute these correlators. The nonzero-mode (nzm) part is expressed in terms of the Green function satisfying Dirichlet boundary conditions G_D

$$\langle \partial X^I(y_1) \partial X^J(y_2) \rangle_{\mathrm{nzm}} = -2\alpha' \eta^{IJ} \partial_{y_1} \partial_{y_2} G_D(y_1, y_2) \quad (3.3)$$

with

$$G_D(y_1, y_2) = \ln \left[(y_1 - y_2) \prod_{n=1}^{\infty} \frac{(1 - k^n y_1/y_2)(1 - k^n y_2/y_1)}{(1 - k^n)^2} \right] - \frac{1}{2} \ln y_1 - \frac{1}{2} \ln y_2, \qquad (3.4)$$

where the last two terms have been added so that G_D has

simpler periodicity properties, but obviously they do not contribute to (3.3). In the computation of the zero-mode part we use the expansion $\partial X(y) = -\mathrm{i}(2\alpha')\hat{p}/y + \ldots$, where the dots stand for the nonzero-mode we have already taken into consideration. Thus we get

$$\langle \partial X^I(y_1) \partial X^J(y_2) \rangle_{zm} = \frac{\mathrm{Vol}}{(2\pi R)^s} \sum_{n^i} \int \frac{d^d p}{(2\pi)^d}$$
$$\times \left[-(2\alpha')^2 \frac{p^I}{y_1} \frac{p^J}{y_2} + \langle \ldots \rangle_{\mathrm{nzm}} \right]$$
$$\times e^{\alpha'(\sum p_\mu^2 + \sum_i ((n_i^2)/(R^2))) \ln k}, \qquad (3.5)$$

where the volume is $\mathrm{Vol.} = (2\pi)^D \delta^d(0) R^s$. If we consider standard gauge bosons as external states, the index I lies in the noncompact space. After integrating over p, we can see that the zero-mode contribution combines with the nonzero-mode one and transforms the Dirichlet Green function into the Neumann one G_N:

$$G_N(y_1, y_2) = G_D(y_1, y_2) + \frac{(\ln y_1 - \ln y_2)^2}{2 \ln k}. \qquad (3.6)$$

Let us analyze the gauge boson mass corrections first and show that we get a vanishing mass correction as required by gauge invariance. The 1-loop measure is

$$\frac{dk}{k^2} dy [\mu(k)] = \frac{dk}{k^2} dy \left[\prod_{n=1}^{\infty} (1 - k^n)^{2-D} \right] \left(-\frac{\pi}{\alpha' \ln k} \right)^{d/2}, \qquad (3.7)$$

where the last factor follows from the Gaussian integration in (3.5) and the product over n is the contribution of the string vibration modes. Then, from (3.2) we read

$$\delta m_\mu^2 = -\alpha' \frac{g_d^2 N}{(2\pi)^d} \int_0^1 dk [\mu(k)]$$
$$\times \int_k^1 dy \left[\theta \left(0 \left| \frac{-\mathrm{i}\alpha' \ln k}{\pi R^2} \right. \right) \right]^s \partial_{y_1} \partial_y G_N(y_1, y) \Big|_{y_1 = 1}, \qquad (3.8)$$

where g_d is the d-dimensional Yang-Mills coupling $g_d^2 = g_D^2/(2\pi R)^s$ and

$$\theta(\nu|\tau) = \sum_n \exp[\pi i n^2 \tau + 2\pi i n\nu]. \qquad (3.9)$$

Apparently δm_μ^2 is trivially zero, since the integrand is a total derivative. However, as discussed in Ref. [25], one has to keep in mind two points: first the integrand is quadratically divergent as $y \to 1$ or $y \to k$ so it has to be regularized, then after regularization (3.8) is zero only if the integral over y is single valued on the boundary of the annulus (i.e., periodic when $y \to ky$). By using the explicit expression for the Green functions (3.4) and (3.6) one can check the following properties:

---

[3]We use the same conventions of Ref. [36]. Equation (3.1) basically states that open string endpoints are minimally coupled to the gauge field. This also fixes the overall normalization of the vertex operator. Alternatively the normalization can be determined by using unitarity and by matching the low-energy behavior of the tree-level 3-point function against the Yang-Mills 3-gluon vertex.

[4]At the full string level the gauge group is $U(N)$, however all amplitudes with external $U(1)$ massless states vanish.

$$\partial_{y_1} G_N(y_1, ky_2) = \partial_{y_1} G_N(y_1, y_2) = \partial_{y_1} G_N(y_2, y_1),$$
$$G_N(y_2^{-1}, y_1^{-1}) = G_N(y_1, y_2). \tag{3.10}$$

Then we regularize the integral (3.8) simply by cutting away the dangerous region around $y = 1 \sim k$ and, by using (3.10), we get

$$\int_k^1 dy \partial_y \partial G_N(1, y) \rightarrow \int_{k/(1-\epsilon)}^{1-\epsilon} dy \partial_y \partial G_N(1, y)$$
$$= 2\partial G_N(1, 1 - \epsilon) \sim \frac{2}{\epsilon} + \mathcal{O}(\epsilon). \tag{3.11}$$

The divergent contribution is due to the exchange of an off shell zero-momentum tachyon. It can be renormalized away by redefining the 2-dimensional cosmological constant, that is by adding to the world-sheet sigma model a coupling $C \int \sqrt{h}$, where h is the metric on the world sheet and C is an appropriate constant. As usual [37], we will discard this divergent contribution without leaving any additional finite part. After this regularization Eq. (3.11) vanishes and no radiative mass correction for the gauge boson is generated at 1-loop.

The situation is very different if we consider the scalars arising from the Kaluza-Klein compactification $I = i = 1, \ldots, s$. Let us first focus on the case $R \rightarrow 0$, where the analysis simplifies (by applying a T duality this limit is equivalent to a lower dimensional D-brane in the uncompact space). In this limit the sum in Eq. (3.5) vanishes and thus we have:

$$\delta m_i^2(R \rightarrow 0) = -\alpha' \frac{g_d^2 N}{(2\pi)^d} \int_0^1 dk [\mu(k)]$$
$$\times \int_k^1 dy \partial_{y_1} \partial_y G_D(y_1, y) \Big|_{y_1=1}. \tag{3.12}$$

By using Eq. (3.6) we can see that the integral over y now yields also a finite term

$$\int_{k/(1-\epsilon)}^{1-\epsilon} dy \partial_y \partial G_D(1, y)$$
$$= \int_{k/(1-\epsilon)}^{1-\epsilon} dy \partial_y \Big[\partial G_N(1, 1 - \epsilon) + \frac{\ln y}{\ln k} \Big] \sim \frac{2}{\epsilon} - 1 + \mathcal{O}(\epsilon). \tag{3.13}$$

Thus, by implementing the same subtraction used in the gluon case, we are left with a nonzero contribution to the scalar mass

$$\delta m_i^2(R \rightarrow 0) = \alpha' \frac{g_d^2 N}{(2\pi)^d} \int_0^1 \frac{dk}{k^2} \Big[\prod_{n=1}^{\infty} (1 - k^n)^{2-D} \Big]$$
$$\times \Big(\frac{-\pi}{\alpha' \ln k} \Big)^{d/2}. \tag{3.14}$$

This result is still divergent as $k \rightarrow 0$ (and $k \rightarrow 1$), but these are physical poles that correspond to the propagation of the

open (and closed) string tachyon. In a tachyon-free string theory these poles will be automatically absent, in the present case we will subtract them by hand.

Let us now consider the case of a compactification with finite radius.

$$\delta m_i^2 = -\alpha' \frac{g_d^2 N}{(2\pi)^d} \int_0^1 \frac{dk}{k^2} \prod_{n=1}^{\infty} (1 - k^n)^{2-D} \Big(-\frac{\pi}{\alpha' \ln k} \Big)^{d/2}$$
$$\times \Big[\theta\Big(0 \Big| \frac{-i\alpha' \ln k}{\pi R^2} \Big) \Big]^{s-1}$$
$$\times \int_k^1 dy \Big[\partial \partial_y G_D(1, y) \theta\Big(0 \Big| -\frac{i\alpha'}{\pi R^2} \ln k \Big)$$
$$- \frac{2\alpha'}{4\pi^2 R^2} \frac{1}{y} \theta''\Big(0 \Big| -\frac{i\alpha'}{\pi R^2} \ln k \Big) \Big], \tag{3.15}$$

where $\theta'(0|\tau) = \partial_\nu \theta(\nu|\tau)|_{\nu=0}$. By using the regularization (3.13), the integral over y can be performed explicitly. Then one can see that the first term is the stringy generalization of the field theory term proportional to $\mathcal{I}_1$, while the second one generalizes the contribution $2\mathcal{I}_2$ in Eq. (2.8). Both integrands are now dressed with the Dedekind function (3.19) η function which takes into account the contribution of the stringy modes. In order to compute the mass shift it is convenient to invert the modular parameter in (3.15) so that the two terms combine in a single contribution. Under this transformation, the θ function transforms as follows:

$$\theta(\nu|\tau) = \frac{1}{\sqrt{-i\tau}} e^{-\pi i \nu^2/\tau} \theta\Big(\frac{\nu}{\tau} \Big| -\frac{1}{\tau}\Big), \tag{3.16}$$

which implies

$$\theta''(0|\tau) = \frac{1}{\sqrt{-i\tau}} \frac{1}{\tau^2} \theta''\Big(0 \Big| -\frac{1}{\tau}\Big) - \frac{1}{\sqrt{-i\tau}} \frac{2\pi i}{\tau} \theta\Big(0 \Big| -\frac{1}{\tau}\Big). \tag{3.17}$$

By using (3.16) and (3.17) in (3.15), we can combine the terms proportional to θ and reconstruct again the Neumman Green function (3.6). As we have seen above, in this case the (properly regularized) integral over y vanishes. Thus

$$\delta m_i^2 = \frac{-g_d^2 N}{(2\alpha')^{(d/2)-2}} \frac{1}{(2\pi)^{d+1}} \frac{1}{2R^2} \int_0^1 \frac{dk}{k^2} \prod_{n=1}^{\infty} (1 - k^n)^{2-D}$$
$$\times \Big(\frac{-2\pi}{\ln k} \Big)^{(d/2)-1} [\theta(0|iT_R)]^{s-1} \frac{\theta''(0|\frac{i}{T_R})}{T_R^{(5/2)}}, \tag{3.18}$$

where $T_R = -(\alpha' \ln k)/(\pi R^2)$. If we work with a critical theory ($D = 26$) and consider the case of a single compact dimension ($s = 1$), we recover Eq. (81) of [24]. In order to match the results, one needs to perform a modular transformation and use

$$k^{1/24} \prod_{n=1}^{\infty} (1 - k^n) = \eta\left(\frac{\ln k}{2\pi i}\right) = \left(\frac{-2\pi}{\ln k}\right)^{1/2} \eta\left(-\frac{2\pi i}{\ln k}\right).$$

$$(3.19)$$

Then in this case we can write (3.18) in the closed string channel $t_c = -1/\ln k$ and we obtain[5]

$$\frac{1}{2} \frac{g_{26}^2}{(2\alpha')^{13}} \frac{(2\pi R)^2}{(2\pi)^{25}} \int_0^{\infty} dt_c \, \eta^{-24}(2\pi i t_c)$$

$$\times \sum_{w=-\infty}^{\infty} w^2 e^{-w^2 t_c \pi^2 R^2/\alpha'}. \quad (3.20)$$

Let us go back to Eq. (3.18) and study the compactification on a circle ($s = 1$) for a generic dimension d. If we discard by hand the tachyon poles, the leading contribution comes from the region $T_R > 1$ (i.e., $|\ln k| > \pi R^2/\alpha'$) where the θ'' is not suppressed. In the regime where the string scale is much higher than the compactification scale $R^2 \gg \alpha'$, this implies also $|\ln k| \gg 1$. In this limit the string amplitudes reduce to the field theory result; see [19] and references therein. Thus we expect that, when $R^2 \gg \alpha'$, the string result automatically reduces to the field theory one (2.12). Let us check that this is indeed the case. Since k is small we can expand the product over n in (3.18) and keep only the second term that cancels the tachyonic pole. Then we have

$$\delta m_i^2 \sim \frac{g_d^2 N}{(2\alpha')^{(d/2)-2}} \frac{1}{(2\pi)^{d-2}} \frac{D-2}{2R^2} \left(\frac{2\alpha'}{R^2}\right)^{(d/2)-2}$$

$$\times \int_0^{\infty} dT_R T_R^{-(3/2)-(d/2)} \sum_{w=-\infty}^{\infty} w^2 e^{-\pi w^2/T_R}. \quad (3.21)$$

By means of a change of variable the integral reduces to the Euler formula of the Gamma function and the sum to the definition of the Riemann zeta function

$$\delta m_i^2 \sim \frac{g_d^2 N}{\pi^{((d+1)/2)}} \frac{D-2}{(2\pi R)^{d-2}} \Gamma\left(\frac{d+1}{2}\right) \zeta(d-1). \quad (3.22)$$

Let us close this section by noting that the mechanism we have just discussed actually holds at any order in perturbation theory. The explicit expressions for the Green functions with Dirichlet or Neumann boundary condition become more involved on a world sheet with an arbitrary number of holes or handles. However everything can be written in terms of classical functions defined on the appropriate Riemann surface, such as the Abelian differentials and the prime form (see for instance [38]). The main ingredient used in the string computation is the periodicity of the integrand when the relative position of the two vertex operators changes. It is possible to generalize step by step the procedure described in this section and check

that even with the higher loop Green functions the integral over the relative position of the punctures yields the same results as in (3.11) and (3.13). Thus the vector states are protected against a mass renormalization because in the relevant string 2-point function the Neumann Green function appears. On the contrary, the internal polarizations of the gauge field are not protected and the higher loop contributions to the mass shift are given by a generalization of (3.18) which involves Riemann's θ functions instead of the Jacobi ones. Still we expect that the same mechanism described above is at work: when $R^2 \gg \alpha'$ the elements of the period matrix, which generalize the 1-loop parameter $\ln k$, must be large otherwise the result is suppressed. In this limit, we expect[6] that the string answer reduces to the field theory one and all factors of α' cancel. At first sight this seems to be in agreement with results obtained at 2-loop in quantum field theory [18,39]. A more careful investigation is needed in order to clarify this issue.

IV. MASS CORRECTIONS ON THE LATTICE

In this section we consider $(d + 1)$-dimensional gauge theories regularized on an asymmetric lattice. In particular, we consider one dimension to be much smaller than the remaining ones so as to recover in the continuum limit a theory compactified on a circle. We use N_s to indicate the number of points in the compact dimension, then its radius R is $2\pi R = N_s a$, where a is the lattice spacing. By using standard lattice perturbation theory, we compute (again) the 1-loop radiative corrections to the mass of the gluon and the scalar states. A similar computation for standard 4-dimensional theories was performed in Ref. [40] in order to check that there is no mass renormalization for the vector bosons, as required by gauge invariance. We want to see how this result changes when the finite size effects of the S^1 compactification are taken into account. In this case, gauge invariance does not protect the mass of the gauge boson polarized along the S^1. The compactification from four to three dimensions has been analyzed in detail in Ref. [26]. As already discussed in the Introduction, this case is relevant for studying the thermal behavior of the Yang-Mills theory: the nonzero mass corrections to the time component of the vector boson are interpreted as a screening effect for the (electric) components of the force.

A. Bosonic contribution

We start by focusing on pure Yang-Mills theory. Technically we mix the approaches of [26,40]. We compute the five 1-loop Feynman diagrams contributing to the 2-point function, see Fig. 1, when one of the D dimensions of the lattice is compact, and focus on the component of the 2-point function in this direction in order to examine the

---

[5]Contrary to what is claimed in [24], Eq. (3.20) does not vanish in the limit $R \to 0$. In this limit the sum over w becomes an integral and one recovers (3.14).

[6]When one wants to focus on the contributions from the massless states in the loops, as we have done in (3.21), it is rather difficult to explicitly check this point even at two loops.

(a) gluon sunset:

(b) gluon tadpole:

(c) measure:
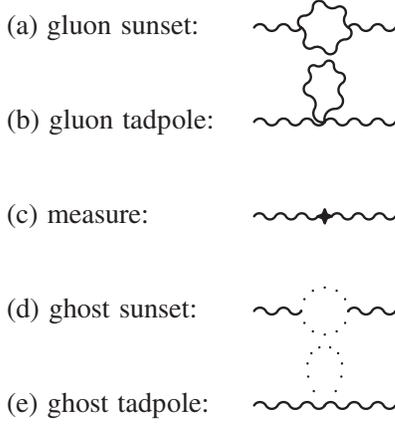
(d) ghost sunset:

(e) ghost tadpole:

FIG. 1.   Contributions to the gluon 2-point function in the pure gauge theory.

mass correction to the generated adjoint scalar. Since we are interested only in extracting the mass correction we can put the external momenta to zero, which simplies drastically the computation with the 4-particle vertices. Then we combine these contributions together by using a discrete version of the partial integration introduced in Ref. [40]. Let us see how this works in details.

By using the Feynman rules listed in Refs. [26,40], it is straightforward to construct the contribution of the diagram in Figure 1(a):

$$
A_a = \frac{g_{d+1}^2 N}{a^{d-1}} \frac{1}{N_s} \sum_{n=0}^{N_s-1} \int_{-\pi}^{\pi} \frac{d^d q}{(2\pi)^d}
$$
$$
\times \left\{ \frac{\cos^2 \frac{\pi n}{N_s}}{D_n} + \left[\frac{d-2}{2} + \frac{1}{4} \right] \frac{\sin^2 \frac{2\pi n}{N_s}}{D_n^2} \right\}, \quad (4.1)
$$

where g_{d+1} is the Yang-Mills coupling of the higher-dimensional theory and $1/D_n$ is the bosonic propagator

$$
D_n = 4\sin^2 \frac{\pi n}{N_s} + 4\sum_{i=1}^{d} \sin^2 \frac{q_i}{2}. \quad (4.2)
$$

The factor of $1/4$ in the square parenthesis of Eq. (4.1) cancels against the ghost loop depicted in Fig. 1(d):

$$
A_d = \frac{g_{d+1}^2 N}{a^2} \frac{1}{N_s} \sum_{n=0}^{N_s-1} \int_{-\pi}^{\pi} \frac{d^d q}{(2\pi)^d} \left(-\frac{1}{4} \frac{\sin^2 \frac{2\pi n}{N_s}}{D_n^2} \right). \quad (4.3)
$$

Let us now consider the tadpole contributions [Figs. 1(b) and 1(e)]:

$$
A_b = \frac{g_{d+1}^2 N}{a^{d-1}} \frac{1}{N_s} \left[\frac{1}{12} - \sum_{n=0}^{N_s-1} \int_{-\pi}^{\pi} \frac{d^d q}{(2\pi)^d} \right.
$$
$$
\left. \times \frac{d\cos^2 \frac{2\pi n}{N_s} - \cos^2 \frac{\pi n}{N_s} + \frac{4}{3}\sin^2 \frac{\pi n}{N_s}}{D_n} \right] \quad (4.4)
$$

for the gluon loop, and

$$
A_e = -\frac{2}{3} \frac{g_{d+1}^2 N}{a^{d-1}} \frac{1}{N_s} \sum_{n=0}^{N_s-1} \int_{-\pi}^{\pi} \frac{d^d q}{(2\pi)^d} \frac{\sin^2 \frac{\pi n}{N_s}}{D_n} \quad (4.5)
$$

for the ghost loop. The factor of $1/12$ in (4.4) cancels against the diagram in Fig. 1(c), which arises in the lattice regularization from the integration measure. Thus, by combining all these contributions, we obtain a simple expression for the gluon for the complete amplitude:

$$
A = \frac{g_{d+1}^2 N}{a^{d-1}} \frac{d-1}{N_s} \sum_{n=0}^{N_s-1} \int_{-\pi}^{\pi} \frac{d^d q}{(2\pi)^d} \left[\frac{2\sin^2 \frac{2\pi n}{N_s}}{D_n^2} - \frac{\cos^2 \frac{2\pi n}{N_s}}{D_n} \right].
$$
$$
(4.6)
$$

We can combine the two terms in this equation by using a discrete version of the integration by parts discussed in Ref. [40]. First we need the (forward) derivative of D_n

$$
D_{n+1} - D_n \equiv \nabla D_n
$$
$$
= 2 \sin\frac{2\pi n}{N_s} \sin\frac{2\pi}{N_s} + 4\cos\frac{2\pi n}{N_s} \sin^2 \frac{\pi}{N_s}. \quad (4.7)
$$

By using this result, we can rewrite the first term in (4.6) as follows:

$$
\sum_{n=0}^{N_s-1} \frac{2\sin^2 \frac{2\pi n}{N_s}}{D_n^2} = \sum_{n=0}^{N_s-1} \frac{\sin \frac{2\pi n}{N_s} \nabla D_n}{\sin \frac{2\pi}{N_s} D_n^2}
$$
$$
= - \sum_{n=0}^{N_s-1} \frac{\sin \frac{2\pi n}{N_s}}{\sin \frac{2\pi}{N_s}} \left[\nabla \frac{1}{D_n} + \frac{\nabla D_n}{D_n} \nabla\left(\frac{1}{D_n} \right) \right],
$$
$$
(4.8)
$$

where the term added in the second step vanishes due to the periodicity of D_n. At the first order in the continuum limit ($N_s \to \infty$) the second term of this equation vanishes and we obtain the relation used in Ref. [40]. It is easy to see that the discrete analogue of an integration by parts involves the backward derivative $\nabla^* g_n \equiv g_n - g_{n-1}$

$$
\sum_n [\nabla f_n] g_n = - \sum_n f_n [\nabla^* g_n]. \quad (4.9)
$$

By using this relation for the first term of Eq. (4.6) we see that it cancels the second term, thus the total amplitudes become

$$
A = \frac{g_{d+1}^2 N}{a^{d-1}} (d-1) \int_{-\pi}^{\pi} \frac{d^d q}{(2\pi)^d}
$$
$$
\times \left[\frac{1}{N_s} \sum_{n=0}^{N_s-1} \frac{\sin \frac{2\pi n}{N_s}}{\sin \frac{2\pi}{N_s}} \frac{\nabla D_n}{D_n} \nabla\left(\frac{1}{D_n} \right) \right]. \quad (4.10)
$$

It is possible to perform explicitly the sum over the discrete modes of the momentum in the compact dimension. The idea is to rewrite the sum as a contour integral; this can be

done by promoting the combination $e^{2\pi i/N_s}$ to a complex variable z. Then D_n is substituted by the function $D(z) = 4\sum \sin^2 \frac{q_i}{2} - (z + z^{-1} - 2)$ and D_{n+1} by $D(e^{((2i\pi)/(N_s))}z)$. Then we multiply the complex function obtained from (4.10) by the function $1/(z^{N_s} - 1)$ which has poles at $z = e^{2\pi i n/N_s}$ for any integer n. Then the square parenthesis in (4.10) is equal to

$$[\ldots] = \frac{1}{2\pi i} \oint_C \frac{dz}{z} \frac{z - z^{-1}}{e^{((2i\pi)/(N_s))} - e^{-((2i\pi)/(N_s))}} \left(\frac{2}{D(z)}\right.$$
$$\left. - \frac{1}{D(e^{((2i\pi)/(N_s))}z)} - \frac{D(e^{((2i\pi)/(N_s))}z)}{D^2(z)}\right) \frac{1}{z^{N_s} - 1},$$
$$(4.11)$$

where the contour C is the union of an anticlockwise circle of radius slightly bigger than 1, and a clockwise circle with radius slightly smaller than 1. By supposing that the function in the parenthesis of (4.11) does not contain additional poles on the circle of unit radius (this is certainly the case for generic values of q_i), one can apply the Cauchy theorem and recover the sum in its original form. Since the integrand is well behaved at infinity, we can also deform the contours and sum all the residues whose modulus is different from 1. In this case the relevant poles are $z = 0$, at $z = e^{\pm\tilde{\phi}}$ for the terms $1/D(z)$, and at $z = e^{\pm\tilde{\phi}}e^{-((2\pi i)/(N_s))}$ for the term containing $D(e^{((2i\pi)/(N_s))}z)$, where

$$\tilde{\phi} = \text{arccosh}\left(1 + 2\sum_{i=1}^{d} \sin^2 \frac{q_i}{2}\right). \quad (4.12)$$

The residues of the poles of the first term [proportional to $1/D(z)$] in (4.11) sum up to zero, while the remaining contributions combine to yield a very simple expression

$$A = \frac{g_{d+1}^2 N}{a^{d-1}} (D - 2)\left[N_s \int_{-\pi}^{\pi} \frac{d^d q}{(2\pi)^d} \frac{e^{N_s\tilde{\phi}}}{(e^{N_s\tilde{\phi}} - 1)^2}\right]. \quad (4.13)$$

Clearly, in the large N_s limit, this integral is dominated by the infrared region of low momenta; in fact when $q \to 0$ then also $\tilde{\phi} \to 0$, while for physical momenta of the order $1/a$ (i.e., finite q) the integrand is exponentially suppressed. So, in this limit, we can approximate the square parenthesis in (4.13) as follows:

$$[\ldots] = N_s \int \frac{d^d q}{(2\pi)^d} \frac{e^{N_s q}}{(e^{N_s q} - 1)^2}$$
$$= -\int \frac{dq}{(2\pi)^d} \Omega_{d-1} q^{d-1} \frac{d}{dq}\left(\frac{1}{e^{N_s q} - 1}\right), \quad (4.14)$$

where $\Omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of the d-dimensional sphere of unit radius. Then we can integrate by parts and use

$$\int_0^\infty dx \frac{x^{a-1}}{e^x - 1} = \Gamma(a)\zeta(a) \quad (4.15)$$

to obtain a compact formula for the 2-point function

$$\delta m^2 \sim \frac{D - 2}{(2\pi)^d} \frac{g_{d+1}^2 N}{(2\pi R)^{d-1}} \frac{2\pi^{(d/2)}}{\Gamma(\frac{d}{2})} \Gamma(d)\zeta(d - 1). \quad (4.16)$$

By using Legendre's duplication formula

$$\Gamma(d) = \frac{2^{d-(1/2)}}{\sqrt{2\pi}} \Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d + 1}{2}\right), \quad (4.17)$$

we can bring the lattice result to the same form found in the string theory derivation of the previous section (3.22)

$$\delta m^2 \sim \frac{g_d^2 N}{\pi^{((d+1)/2)}} \frac{(D - 2)}{(2\pi R)^{d-2}} \zeta(d - 1)\Gamma\left(\frac{d + 1}{2}\right). \quad (4.18)$$

B. Wilson fermions in the loop

We expect that the same pattern seen in the previous section arises for the loop contribution of any massless particle coupled in a way that respects the higher-dimensional gauge invariance. In this section we show that this is indeed the case when considering Wilson fermions minimally coupled to the higher-dimensional gauge field. For the sake of simplicity we will choose the Wilson parameter to be 1 ($r = 1$). Of course, in the lattice Lagrangian for the Wilson fermions, the chiral symmetry is broken and thus a mass term for these fermions is generated through quantum corrections. In order to have a vanishing effective mass one would need to add fine-tuned counterterms that cancel these corrections. Since here we will focus only on the 1-loop contribution to the scalar mass, the counterterms for the fermion mass do not play any role and we will neglect this point. The fermion contribution to the 1-loop function with two external scalars is given by the diagrams in Fig. 2. Even if the lattice Feynman rules for Wilson fermions are rather different from those of the gluons, we see that the computation can be done by following closely the same steps described in the previous sections. Again we focus on the case of the zero-momentum external particle, since we want to extract the mass corrections from the 2-point function.

By using the Feynman rules listed in Ref. [26], we obtain for the first diagram in Fig. 2(a)

(a) fermion sunset:

(b) fermion tadpole:

FIG. 2. Fermionic contributions to the gluon 2-point function.

$$A_a = -\frac{g_{d+1}^2 T(F)}{a^{d-1}} \frac{c_D}{N_s} \sum_{n=0}^{N_s-1} \int_{-\pi}^{\pi} \frac{d^d q}{(2\pi)^d}$$

$$\times \left[\frac{1}{2}\left(\sin\frac{4\pi n}{N_s} + D_n \sin\frac{2\pi n}{N_s}\right)^2 \frac{1}{D_n^f} - 1\right]\frac{1}{D_n^f}, \quad (4.19)$$

where $T(F)$ is the index of the fermion representation, c_D counts the physical polarizations of the fermion, and D_n^f is the Wilson propagator

$$D_n^f = \sin^2 \frac{2\pi n}{N_s} + \sum_{i=1}^d \sin^2 q_i + \frac{1}{4}D_n^2. \quad (4.20)$$

The contribution of the tadpole diagram [see Fig. 2(b)] is

$$A_2 = -\frac{g_{d+1}^2 T(F)}{a^{d-1}} \frac{c_D}{N_s} \sum_{n=0}^{N_s-1} \int_{-\pi}^{\pi} \frac{d^d q}{(2\pi)^d}$$

$$\times \frac{\sin^2 \frac{2\pi n}{N_s} - \frac{1}{2}\cos\frac{2\pi n}{N_s}D_n}{D_n^f}. \quad (4.21)$$

By combining the two diagrams we obtain an expression that has a structure similar to Eq. (4.6)

$$A = -\frac{g_{d+1}^2 T(F)}{a^{d-1}} \frac{c_D}{N_s} \sum_{n=0}^{N_s-1} \int_{-\pi}^{\pi} \frac{d^d q}{(2\pi)^d}\left[\frac{1}{2}\right.$$

$$\times \left(\frac{\sin\frac{4\pi n}{N_s} + D_n \sin\frac{2\pi n}{N_s}}{D_n^f}\right)^2 - \frac{\cos^2\frac{2\pi n}{N_s} + \frac{1}{2}\cos\frac{2\pi n}{N_s}D_n}{D_n^f}\right]. \quad (4.22)$$

Then we will follow the same approach used in computing the bosonic loop: we start by focusing on the first term and rewrite it in terms of the variation of the fermionic propagator $D_{n+1}^f - D_n^f \equiv \nabla D_n^f$

$$\sum_{n=0}^{N_s-1}\frac{1}{2}\left(\frac{\sin\frac{4\pi n}{N_s} + D_n \sin\frac{2\pi n}{N_s}}{D_n^f}\right)^2$$

$$= \frac{1}{2\sin\frac{2\pi n}{N_s}}\sum_{n=0}^{N_s-1}\frac{(\sin\frac{4\pi n}{N_s} + D_n \sin\frac{2\pi n}{N_s})\nabla D_n^f}{(D_n^f)^2}, \quad (4.23)$$

where we have discarded all the terms that sum up to zero due to the periodicities of the trigonometric functions. The we can rewrite (4.23) as follows:

$$\sum_{n=0}^{N_s-1}\frac{\sin\frac{4\pi n}{N_s} + D_n \sin\frac{2\pi n}{N_s}}{2\sin\frac{2\pi n}{N_s}}\left(-\nabla\frac{1}{D_n^f} - \frac{\nabla D_n^f}{D_n^f}\nabla\frac{1}{D_n^f}\right). \quad (4.24)$$

We can now use (4.9) and "integrate" by parts the first term in the parenthesis. In this way we see that it precisely cancels the second term in (4.22). Thus the full 2-point amplitude is

$$A = -\frac{g_{d+1}^2 T(F)}{a^{d-1}} \frac{c_D}{N_s} \sum_{n=0}^{N_s-1} \int_{-\pi}^{\pi} \frac{d^d q}{(2\pi)^d} \frac{\sin\frac{4\pi n}{N_s} + D_n \sin\frac{2\pi n}{N_s}}{2\sin\frac{2\pi n}{N_s}}$$

$$\times \left(\frac{2}{D_n^f} - \frac{1}{D_{n+1}^f} - \frac{D_{n+1}^f}{(D_n^f)^2}\right). \quad (4.25)$$

As in the previous section we can rewrite this sum as a contour integral. Before doing this, it is convenient to rewrite the propagator for the Wilson fermion in the following form:

$$D_n^f = \left(1 + 2\sum_{\mu=1}^d \sin^2\frac{q_\mu}{2}\right)$$

$$\times \left[4\sin^2\frac{\pi n}{N_s} + \frac{\sum_{i=1}^d \sin^2 q_i + 4(\sum_{i=1}^d \sin^2 q_i)^2}{1 + 2\sum_{i=1}^d \sin^2 q_i}\right]. \quad (4.26)$$

In this way the fermionic result (4.25) will take a form that is very similar to the one encountered in the bosonic case. In particular the first parenthesis combines with the other *sin*'s in (4.25) and the contour integral we find has the same analytical structure as (4.11)

$$A = -\frac{g_{d+1}^2 T(F)c_D}{a^{d-1}} \int_{-\pi}^{\pi} \frac{d^d q}{(2\pi)^d} \oint_C \frac{dz}{2\pi i z}$$

$$\times \frac{z - z^{-1}}{e^{((2i\pi)/(N_s))} - e^{-((2i\pi)/(N_s))}}\left(\frac{2}{\hat{D}(z)} - \frac{1}{\hat{D}(e^{((2i\pi)/(N_s))}z)}\right.$$

$$\left. - \frac{\hat{D}(e^{((2i\pi)/(N_s))}z)}{\hat{D}^2(z)}\right)\frac{1}{z^{N_s} - 1}, \quad (4.27)$$

where $1/\hat{D}(z)$ has poles at $z = e^{\pm\phi^f}$ with

$$\tilde{\phi}^f = \text{arccosh}\left(1 + \frac{1}{2}\frac{\sum_{i=1}^d \sin^2 q_i + 4(\sum_{i=1}^d \sin^2 q_i)^2}{1 + 2\sum_{i=1}^d \sin^2 q_i}\right). \quad (4.28)$$

Thus we have now rewritten the 1-loop fermionic contribution in the same form as encountered in the bosonic computation and the only difference is in the explicit relation between the position of the poles and the momenta q. Thus the 1-loop fermion contribution to the scalar mass is

$$\delta m^2 = -\frac{g_{d+1}^2 T(F)c_D}{a^{d-1}}\left[N_s \int_{-\pi}^{\pi} \frac{d^d q}{(2\pi)^d} \frac{e^{N_s \tilde{\phi}^f}}{(e^{N_s \tilde{\phi}^f} - 1)^2}\right]. \quad (4.29)$$

As we have already seen, in the large N_s limit, only the low-energy momenta contribute significantly to this integral $q \sim 1/N_s$. Then we can expand (4.12) for small q's and we see that, for low-energy momenta, we have again $\phi^f \sim |q|$. Thus, in this limit, the fermion contribution to the scalar mass reduces to the result derived in the effective field theory [4]

$$\delta m^2 \sim -\frac{c_D T(F)}{(2\pi)^d} \frac{g_{d+1}^2}{(2\pi R)^{d-1}} \frac{2\pi^{(d/2)}}{\Gamma(\frac{d}{2})} \Gamma(d)\zeta(d-1).$$

(4.30)

C. Accidental extended supersymmetry on the lattice?

Even if our lattice analysis has been restricted to perturbation theory, the results of this section suggest the exciting possibility of realizing super Yang-Mills (SYM) theories with extended supersymmetry on the lattice in an accidental way. As is well known, lattice regularization breaks most of the symmetries of the Poincaré group and all supersymmetries that are present in the continuum version of the same theory. However Poincaré symmetries arise automatically in the continuum limit, because all relevant or marginal operators that could violate them are prohibited by some of the symmetries that are present in the theory at finite lattice spacing. A similar observation applies also to the 4-dimensional $\mathcal{N} = 1$ SYM theory [27,41,42]: if one adds to the standard Yang-Mills theory a *chiral* fermion in the adjoint representation, then no dangerous operator, such as a mass term for the fermions, can be dynamically generated and, at low energies, one automatically recovers a supersymmetric theory. Of course from a lattice prospective the challenging aspect of this program is to simulate dynamical chiral fermions.

In the case of extended supersymmetry an additional complication arises: even pure SYM contains scalars (the complex scalar of the vector multiplet). Once supersymmetry is broken by the lattice regularization, one expects that a relevant mass term for these scalars is dynamically generated and thus apparently there is no hope to get a supersymmetric theory at low energies without fine-tuning [27]. This is indeed the case if the scalars are described by site variables in a 4-dimensional lattice. Various approaches have been suggested to overcome this problem, such as deconstruction, or the idea of realizing some of the supersymmetric generators at finite lattice spacing; see, e.g., Refs. [43–47], a recent review with extensive bibliographic references can be found in Ref. [47].

The results of this section suggests a different possibility: one can use the Kaluza-Klein reduction on the lattice to engineer 4-dimensional SYM theories with extended supersymmetries from a higher-dimensional $\mathcal{N} = 1$ theory. After all, also in the continuum field theories, this is the easiest way to construct SYM theories with extended supersymmetry [28]. In this approach the scalar fields are the internal components of the higher-dimensional gauge field and so are described by link variables in the compact directions of a higher-dimensional asymmetric lattice. Then, at distances shorter than the compactification scale R, the scalar fields and the gauge field are on the same footing and both are constrained by the higher-dimensional gauge invariance. Thus no dangerous contribution to the mass of the scalar fields can come from the high-energy

modes (i.e., modes with energies bigger than $1/R$). On the contrary the quantum corrections to the scalar mass are purely due to finite size effects and only modes with energies lower than $1/R$ can contribute. In the limit $a \ll R$, these modes are completely blind to the effects of the lattice regulator and thus to the supersymmetry breaking effects of the regularized theory. This is clearly visible in Eqs. (4.13) and (4.29): when $1 \ll N_s$ the two expressions reduce to those obtained in the continuum effective field theory and thus they cancel when expected. The overall normalization in these results basically counts the number of bosonic and fermionic degrees of freedom that can contribute to the mass corrections. For instance, if we choose $D = 6$ and two compact dimensions, there is a fermion/boson degeneracy and we obtain $\mathcal{N} = 2$ SYM.[7]

Thus there is hope to describe extended SYM theories on the lattice without having to fine-tune the scalar masses by using a higher-dimensional lattice with a different number of sites in the compact and uncompact dimension. For scales that are bigger than the compactification radius, but smaller that the size of the "uncompact" dimensions ($N_s \ll x \ll L$) the lattice theory should reduce to a standard 4-dimensional gauge theory. Of course an obvious drawback of this approach is that simulations might be very expensive when the number of dimensions of the lattice is big (for instance, we would need 6 compact dimensions to simulate $\mathcal{N} = 4$ SYM). Moreover there are several points that require further study in order to see whether this proposal can be realized in a practical way. A first obvious question is whether the pattern we described is general or is just a peculiarity of the 1-loop perturbation theory. There are actually some indications that this mechanism is indeed general. The distinction between high-energy modes, constrained by the higher-dimensional gauge invariance, and the low-energy ones, constrained by the tree-level supersymmetry, does not seem to rely on the 1-loop approximation. So one would expect that the higher loop radiative corrections to the scalar mass follow the same pattern and the leading contribution in the large R limit is independent of the UV cutoff. Indications in this sense come from the string analysis, where the 1-loop case is not special. A more fundamental question to be addressed is to check whether this approach can be used to study the strongly coupled regime of an $\mathcal{N} = 2$ supersymmetric theory on the lattice. Doubts in this respect were raised in Ref. [27], where it was noticed that, starting from a weakly-coupled 6-dimensional theory, the dynamically generated scale in 4 dimensions Λ_4 is exponentially suppressed in the large N_s limit. In this case, the a-dependent corrections to

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[7]The explicit expressions (4.16) and (4.30) are valid in the case of a single compact dimension $s = 1$. For two compact dimensions one of the integrals in (4.13) and (4.29) becomes a sum, however the mechanism described here still applies: the leading order contribution is independent of the lattice spacing and cancel between fermion and boson loops when expected.

Eqs. (4.18) and (4.30) are likely to spoil the accidental supersymmetry at the scale Λ_4. However a *strongly coupled* 6-dimensional starting point is needed in order to take a continuum limit of the lattice description which keeps the radius R and the 4-dimensional coupling g_4 finite. Thus the problem mentioned above does not appear in the scaling limit that is relevant to study of a fixed 4-dimensional supersymmetric physics. Another open question concerns the other types of fine-tuning that might be necessary in order to recover a theory with extended supersymmetry. For instance, one would expect to fine-tune the quartic coupling among scalars and the Yukawa couplings so that they are all related to the gauge coupling constant. The interplay between higher-dimensional gauge invariance and tree-level supersymmetry described here should be helpful also to avoid the fine-tuning of the couplings.

On a more practical side, one should worry about the subleading corrections to Eqs. (4.18) and (4.16). These corrections will certainly spoil the low-energy supersymmetry and we can suppress them only in the large N_s limit which is of course computationally very expensive. If these corrections are small for a moderate number of lattice points in the compact dimensions, then this approach can be really transformed into a practical tool for analyzing $\mathcal{N} = 2$ or even $\mathcal{N} = 4$ SYM on the lattice. Some indications in this direction come from [26], where it is pointed out that already for $N_s = 8$ the lattice artifact effects are only of the order of 2%.

V. CONCLUSIONS

In this paper we studied the quantum corrections to the mass of the internal components of the gauge field in Kaluza-Klein compactifications. Within an effective field approach this problem was analyzed in detail in Refs. [1–4,6,12,13]. The main feature of this result is that it depends only on the compactification scale R and is independent of the UV cutoff Λ necessary to define the higher-dimensional gauge theory (of course we assume $\Lambda \gg 1/R$). Even if this is the case, it is natural to wonder whether an effective field theory approach is reliable, since one is summing over the whole tower of Kaluza-Klein states which at a certain point will have masses bigger that then UV cutoff itself. In order to answer this question we studied the same problem in two different UV finite theories: string theory and lattice gauge theories. In the first case the UV cutoff is set by the string length $\sqrt{\alpha'}$, while in the second case the same role is played by the lattice spacing a; both of these theories represent local and gauge invariant UV completions of higher-dimensional Yang-Mills theories.

The interesting result is that, in the regime $R \gg \sqrt{\alpha'}, a$, both the lattice and the string computations reproduce exactly the same result found in field theory, thus justifying *a posteriori* the approach used in Refs. [2–4]. This analysis clarifies also the mechanism that protects the effective field theory results from the contributions of the modes with an energy of the order of the UV cutoff: since $R \gg \sqrt{\alpha'}, a$, these very energetic modes see all dimensions on the same footing and the constraints of the higher-dimensional gauge invariance should be taken into account. Thus, if we want to compute radiative corrections to terms that would violate the higher-dimensional gauge invariance, we do not really need to know the UV details of the string or lattice theories. It is sufficient to know that these UV completions respect locality and gauge invariance and this ensures that the leading order contribution to these terms is completely captured by an effective field theory approach. By carrying out the computation in the full UV finite theory, we see explicitly that the suppression of the UV modes is of the order of $e^{-R/\sqrt{\alpha'}}$ (or $e^{-R/a}$), while the contribution of the low-energy modes reproduces the expected effective field theory result.

The string analysis can be relevant for phenomenological applications in the context of models with large extra dimensions. In particular it would be interesting to generalize our computation to the amplitudes involving external states with a nonzero Kaluza-Klein charge. This case has been discussed in detail from the field theory point of view [15,16,27]. The string analysis can either support the picture emerging from the field theory computations or maybe indicate subtleties due to the high-energy modes. Of course it would be interesting to carry out the same quantitative analysis in the case of tachyon-free string theory. This might be directly useful in the string phenomenological scenarios where the standard model is engineered on D-branes, which usually contain nonchiral exotic matter fields.

In the context of lattice gauge theories the problem of the radiative corrections to the Kaluza-Klein scalars is interesting because of its connection with the possibility of obtaining an accidentally supersymmetric theory at low energies. This is why we have considered explicitly also the contribution of (Wilson) fermions. Even if technically the computation is more involved than its bosonic counterpart, we do not find any particular surprise and the pattern described in Sec. IV arises. There is certainly the need of more study to see whether this proposal can be turned into a concrete approach to supersymmetry on the lattice. In general, we hope that setups suggested by D-brane constructions and/or compactifications can provide useful suggestions on how to realize supersymmetric theories on the lattice also beyond the case of super Yang-Mills theories. Of course it would be very interesting to try and also include chiral multiplets and construct a lattice realization of more complicated supersymmetric theories.

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