Zeros of Ramanujan polynomials

ZEROS OF RAMANUJAN POLYNOMIALS

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Abstract. In this paper, we investigate the properties of Ramanujan polynomials, a family of reciprocal polynomials with real coefficients originating from Ramanujan's work. We begin by finding their number of real zeros, establishing a bound on their sizes, and determining their limiting values. Next, we prove that all nonreal zeros of Ramanujan polynomials lie on the unit circle, and are asymptotically uniformly distributed there. Finally, for each Ramunujan polynomial, we find all its zeros that are roots of unity.

1. Introduction

This paper investigates the properties of Ramanujan polynomials, which, for each $k \geq 0$, the authors of [2] define to be

$$R_{2k+1}(z) = \sum_{j=0}^{k+1} \frac{B_{2j} B_{2k+2-2j}}{(2j)!(2k+2-2j)!} z^{2j},$$

where B_j denotes the jth Bernoulli number. Of particular interest is the location of the zeros of these polynomials, whose knowledge will give rise to explicit formulas for the Riemann zeta function at odd arguments in terms of Eichler integrals.

Ramanujan polynomials are *reciprocal polynomials* with real coefficients, meaning that they satisfy the functional equation

$$R_{2k+1}(z) = z^{2k+2} R_{2k+1}\left(\frac{1}{z}\right),$$

where $2k+2 = \deg(R_{2k+1})$. This elegant property greatly simplifies the analysis of their zeros, the details of which will be unveiled in later sections.

To begin, this paper will derive certain basic properties of Ramanujan polynomials, including a bound on the sizes of their real zeros. Furthermore, we will show that the largest real zero of R_{2k+1} tends to 2 from above as k approaches infinity.

The subsequent section will give a proof that all nonreal zeros of Ramanujan polynomials lie on the unit circle. In particular, we prove that for each k, these zeros (which take the form $e^{i\theta}$) are interlaced between angles θ for which $\sin k\theta$ assumes the values ± 1. Hence, as k tends to infinity, the nonreal zeros of R_{2k+1} become uniformly distributed on the unit circle.

The final section of the paper will determine which zeros of R_{2k+1} are $2k$-th roots of unity. Specifically, the roots of unity that are zeros of R_{2k+1} are

- Both $\pm i$ if k is even;
- All four of $\pm\rho$, $\pm\bar{\rho}$ if k is a multiple of 3,

and no others. Here ρ is a primitive cube root of unity.

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2. Motivation

In Ramanujan's notebooks, we find the following remarkable formula involving the odd values of the Riemann-Zeta function (see [3]):

$$\begin{aligned}\alpha^{-k}\left\{\frac{1}{2}\zeta(2k+1)+\sum_{n=1}^{\infty}\frac{n^{-2k-1}}{e^{2\alpha n}-1}\right\} &= (-\beta)^{-k}\left\{\frac{1}{2}\zeta(2k+1)+\sum_{n=1}^{\infty}\frac{n^{-2k-1}}{e^{2\beta n}-1}\right\}\\ &\quad - 2^{2k}\sum_{j=0}^{k+1}(-1)^j\frac{B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!}\alpha^{k+1-j}\beta^j,\end{aligned} \tag{1}$$

where α, $\beta > 0$ with $\alpha\beta = \pi$, and k is any positive integer. We recognize immediately that the sum involving the Bernoulli numbers is

$$\alpha^{k+1}R_{2k+1}\left(i\sqrt{\frac{\beta}{\alpha}}\right).$$

A rigorous proof of this formula together with a generalization was obtained by Grosswald. He proved the following (see [1]):

Theorem 2.1 (Grosswald)**.** *Let*

$$\sigma_k(n) = \sum_{d|n} d^k$$

and set

$$F_k(z) = \sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^k}e^{2\pi i n z}$$

for $\Im(z) > 0$*. Then*

$$F_{2k+1}(z) - z^{2k}F_{2k+1}\left(-\frac{1}{z}\right) = \tfrac{1}{2}\zeta(2k+1)(z^{2k}-1) + \frac{(2\pi i)^{2k+1}}{2z}R_{2k+1}(z). \tag{2}$$

The function $F_k(z)$ is an example of an Eichler integral, and the above formula relates the values of two Eichler integrals to $\zeta(2k+1)$ through the Ramanujan polynomial. In particular, zeros of $R_{2k+1}(z)$ that lie in the upper half plane and that are not $2k$-th roots of unity give us a formula for $\zeta(2k+1)$ in terms of Eichler integrals. Indeed, the results of this paper tell us that, for each $k \geq 4$, there exists at least one algebraic number α with $|\alpha| = 1$, $\alpha^{2k} \neq 1$ lying in the upper half plane such that $R_{2k+1}(\alpha) = 0$ and hence

$$\tfrac{1}{2}\zeta(2k+1) = \frac{F_{2k+1}(\alpha) - \alpha^{2k}F_{2k+1}(-1/\alpha)}{\alpha^{2k}-1}.$$

In other words, there exists an explicit formula for the Riemann zeta function at odd arguments $9, 11, 13, \ldots$ in terms of the difference of two Eichler integrals.

Though Ramanujan polynomials have appeared in the work of Grosswald and others, they were never studied for their own sake. It turns out that they are of tremendous interest in their own right, and serve as motivation for further applications. Indeed, the authors of [2] study the function

$$G_{2k+1}(z) = \frac{2}{z^{2k}-1}\left(F_{2k+1}(z) - z^{2k}F_{2k+1}(-1/z)\right)$$

and show that the set

$$\{G_{2k+1}(z) \,|\, \Im(z) > 0,\ z \in \overline{\mathbb{Q}},\ z^{2k} \neq 1\}$$

contains at most one algebraic number.

3. Basic Properties of Ramanujan Polynomials

Let us begin with the following simple well-known observation.

Lemma 3.1. *Suppose that the polynomial*

$$P(z) = p_0 + p_1 z^2 + \cdots + p_d z^{2d}$$

has real coefficients p_j*, and satisfies* $p_{d-j} = p_j$, $j = 0, \ldots, d$. *Then* $P(z)$ *is a reciprocal polynomial and, for* z *on the unit circle,* $z^{-d}P(z)$ *is real.*

Proof. We have

$$z^{2d}P\left(\frac{1}{z}\right) = \sum_{j=0}^{d} p_j z^{2(d-j)} = \sum_{j=0}^{d} p_{d-j} z^{2j} = \sum_{j=0}^{d} p_j z^{2j} = P(z).$$

Hence for z on the unit circle,

$$z^{-d}P(z) = z^d P\left(\frac{1}{z}\right) = \bar{z}^{-d}P(\bar{z}).$$

□

Corollary 3.2. *Ramanujan polynomials are reciprocal.*

Proof. This follows from the fact that the coefficient of z^{2j} of $R_{2k+2}(z)$ is the same as that of $z^{2k+2-2j}$. □

From this corollary we see that replacing z by $-1/z$ in the identity (2) gives the same identity again.

Before moving on, let us take a moment to list the first few Ramanujan polynomials and their zeros (the values given in parentheses are approximations to exact solutions by radicals). Notice that, for $1 \le k \le 8$, $R_{2k+1}(z)$ has exactly 4 real zeros. Furthermore, the largest of the real zeros is always between 2 and 2.2 (and it seems to be approaching 2 as k increases). On the other hand, the nonreal zeros seem to lie exactly on the unit circle.

$R_1(z) = \frac{1}{2\cdot 3!}(z^2+1)$ (this is the trivial case) Zeros: $\pm\, i$

$R_3(z) = \frac{1}{6!}(-z^4+5z^2-1)$ Zeros: $\pm\sqrt{\frac{5\pm\sqrt{21}}{2}}$ $(\pm 2.1889, \pm 0.4569)$

$R_5(z) = \frac{1}{12\cdot 7!}(2z^6-7z^4-7z^2+2)$ Zeros: $\pm\, i, \pm\sqrt{\frac{9\pm\sqrt{65}}{4}}$ $(\pm 2.0653, \pm 0.4842)$

$R_7(z) = \frac{1}{10!}(-3z^8+10z^6+7z^4+10z^2-3)$ Zeros: $\pm\,\rho, \pm\bar{\rho}, \pm\sqrt{\frac{13\pm\sqrt{133}}{6}}$ $(\pm 2.0221, \pm 0.4945)$

$R_9(z) = \frac{1}{12!}(10z^{10}-33z^8-22z^6-22z^4-33z^2+10)$

Zeros: $\pm\, i, \pm\sqrt{\frac{43}{40}+\frac{3\sqrt{201}}{40}\pm\frac{1}{2}\sqrt{\frac{1029}{200}+\frac{129\sqrt{201}}{200}}}, \pm\sqrt{\frac{43}{40}-\frac{3\sqrt{201}}{40}\pm\frac{i}{2}\sqrt{\frac{-1029}{200}+\frac{129\sqrt{201}}{200}}}$
$(\pm 2.0071, \pm 0.4982, \pm 0.7112 \pm 0.7030i)$

And a few more cases (all zeros other than $\pm i$, $\pm\rho$, $\pm\bar{\rho}$ are approximations):

$R_{11}(z) = \frac{1}{2\cdot 15!}(-1382z^{12} + 4550z^{10} + 3003z^8 + 2860z^6 + 3003z^4 + 4550z^2 - 1382)$

Zeros: $\pm 2.0022, \pm 0.4995, \pm 0.3081 \pm 0.9513i, \pm 0.8146 \pm 0.5800i$

$R_{13}(z) = \frac{1}{12\cdot 15!}(210z^{14} - 691z^{12} - 455z^{10} - 429z^8 - 429z^6 - 455z^4 - 691z^2 + 210)$

Zeros: $\pm i, \pm\rho, \pm\bar{\rho}, \pm 2.0006, \pm 0.4998, \pm 0.8715 \pm 0.4904i$

$R_{15}(z) = \frac{1}{5\cdot 18!}(-10851z^{16} + 35700z^{14} + 23494z^{12} + 22100z^{10} + 21879z^8 + 22100z^6 + 23494z^4 + 35700z^2 - 10851)$

Zeros: $\pm 2.0002, \pm 0.5000, \pm 0.2219 \pm 0.9751i, \pm 0.9058 \pm 0.4238i, \pm 0.6247 \pm 0.7809i$

$R_{17}(z) = \frac{1}{21!}(438670z^{18} - 1443183z^{16} - 949620z^{14} - 892772z^{12} - 881790z^{10} - 881790z^8 - 892772z^6 - 949620z^4 - 1443183z^2 + 438670)$

Zeros: $\pm i, \pm 2.0001, \pm 0.5000, \pm 0.3822 \pm 0.9241i, \pm 0.9279 \pm 0.3729i, \pm 0.7091 \pm 0.7051i$.

4. On Real Zeros of Ramanujan Polynomials

Invoking the identity

$$\frac{B_{2j}}{(2j)!} = -\frac{2\zeta(2j)}{(2\pi i)^{2j}},$$

where ζ denotes the Riemann zeta function, we may define

$$M_{2k+1}(z) = \frac{(2k+2)!}{B_{2k+2}} R_{2k+1}(z) = z^{2k+2} + 1 - \sum_{j=1}^{k} \frac{2\zeta(2j)\zeta(2k+2-2j)}{\zeta(2k+2)} z^{2j},$$

which is just the monic companion of $R_{2k+1}(z)$.

We now verify the existence of a real zero of M_{2k+1}.

Theorem 4.1. *For $k \geq 1$ we have $M_{2k+1}(2) = -2k - 1$.*

Proof. Recall that the generating function for Bernoulli numbers is

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!},$$

and that all B_n with $n \geq 3$ and odd are 0. Hence

$$\left(\frac{2t}{e^{2t}-1}\right)\left(\frac{t}{e^t-1}\right) = \left(\sum_{j=0}^{\infty} \frac{B_j (2t)^j}{j!}\right)\left(\sum_{n=0}^{\infty} \frac{B_n t^n}{n!}\right),$$

and the coefficient of t^{2k+2} (for $k \geq 1$) in this product of sums is

$$\begin{aligned}\sum_{j=0}^{2k+2} \frac{B_j B_{2k+2-j}}{j!(2k+2-j)!} 2^j &= \sum_{j \text{ even}}^{2k+2} \frac{B_j B_{2k+2-j}}{j!(2k+2-j)!} 2^j \\ &= \sum_{j=0}^{k+1} \frac{B_{2j} B_{2k+2-2j}}{(2j)!(2k+2-2j)!} 2^{2j} \\ &= R_{2k+1}(2).\end{aligned}$$

Now, notice also that

$$\left(\frac{2t}{e^{2t}-1}\right)\left(\frac{t}{e^t-1}\right) = \frac{t^2}{(e^t-1)^2} - \frac{t}{2} \cdot \frac{2t}{e^{2t}-1},$$

and furthermore

$$\frac{d}{dt}\left(\frac{t^2}{e^t-1}\right) = \frac{2t}{e^t-1} - \frac{t^2}{e^t-1} - \frac{t^2}{(e^t-1)^2}.$$

Hence we can eliminate $\frac{t^2}{(e^t-1)^2}$ to obtain

$$\begin{aligned}\left(\frac{2t}{e^{2t}-1}\right)\left(\frac{t}{e^t-1}\right) &= \frac{2t}{e^t-1}-\frac{t^2}{e^t-1}-\frac{d}{dt}\left(\frac{t^2}{e^t-1}\right)-\frac{t}{2}\cdot\frac{2t}{e^{2t}-1}\\ &= \sum_{n=0}^{\infty}\frac{B_n}{n!}\left(2t^n-t^{n+1}-(n+1)t^n-\frac{t}{2}(2t)^n\right)\\ &= -\sum_{n=0}^{\infty}\frac{B_n}{n!}\left((n-1)t^n+(1+2^{n-1})t^{n+1}\right).\end{aligned}$$

So for $n=2k+2$ with $k>0$ the coefficient of t^{2k+2} is

$$\frac{-(2k+1)B_{2k+2}}{(2k+2)!}=R_{2k+1}(2),$$

since $B_{2k+1}=0$. Then

$$M_{2k+1}(2)=R_{2k+1}(2)\frac{(2k+2)!}{B_{2k+2}}=-(2k+1).$$

□

We mention in passing that the two formulae for $R_{2k+1}(2)$ in the above proof give, for $k>0$, the identity

$$\sum_{j=0}^{k+1}\frac{B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!}2^{2j}=-(2k+1)\frac{B_{2k+2}}{(2k+2)!}$$

between Bernoulli numbers.

Now, it is only a pleasure to state the following corollary:

Corollary 4.2. *For $k\geq 1$, M_{2k+1} has exactly four distinct real zeros.*

Proof. Now $M_{2k+1}(z)$ is positive for z real, positive and sufficiently large, so, by the intermediate value theorem, it has a real zero, z_0 say, greater than 2. As $M_{2k+1}(z)$ is reciprocal, $1/z_0\in(0,1/2)$ is also a zero. By Descartes' Rule of Signs, we see that $M_{2k+1}(z)$ can have at most 2 positive zeros, so z_0 and $1/z_0$ are the only positive zeros. Since M_{2k+1} is an even function, we may also conclude that $-z_0$ and $-1/z_0$ are the only negative zeros. □

We may in fact give an upper bound on the size of the largest real zero. This bound, coupled with the proof that all nonreal zeros of M_{2k+1} lie on the unit circle (which will be given in the next section), will tell us that the zeros of M_{2k+1} are uniformly bounded for all k.

Theorem 4.3. *The largest real zero of M_{2k+1} does not exceed 2.2 (for any $k\geq 0$), and approaches 2 as $k\to\infty$.*

Note that, for $0\leq k\leq 4$, we already have explicit expressions for the real zeros of M_{2k+1}, and they are certainly bounded above by 2.2. Hence for the rest of this section, we focus on the case $k\geq 5$.

Now, to prove the theorem above, we require three lemmas.

Lemma 4.4. *For $n\geq 2$, we have the inequalities*

$$1+2^{-n}<\zeta(n)<1+\frac{n+1}{n-1}2^{-n}.$$

Proof. The lower bound is immediate from the definition of $\zeta(n)$. As for the upper bound,

$$\begin{aligned}\zeta(n) &< 1+\frac{1}{2^n}+\int_2^{\infty}x^{-n}dx\\ &= 1+\frac{n+1}{n-1}2^{-n}.\end{aligned}$$

□

The second lemma presents two known results concerning series involving the zeta-function.

Lemma 4.5 (see [4, equations (45) p. 163 and (193) p.178])**.** *We have*

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{4^j} = \tfrac{1}{2}$$

and

$$\sum_{j=1}^{\infty} (\zeta(2j) - 1) = \tfrac{3}{4}.$$

Proof. Now

$$\begin{aligned}
\sum_{j=1}^{\infty} \frac{\zeta(2j)}{4^j} &= \sum_{j=1}^{\infty} 4^{-j} \sum_{k=1}^{\infty} \frac{1}{k^{2j}} \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{(2k)^{2j}} \\
&= \sum_{k=1}^{\infty} \frac{1}{(2k)^2 - 1} \\
&= \tfrac{1}{2} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) \\
&= \tfrac{1}{2},
\end{aligned}$$

where the second equality follows from changing the order of summation, and last equality follows from telescoping series.

The second result is proved in a similar manner. □

Next, we need the following estimate.

Lemma 4.6. *For $k \geq 1$ and $j = 1, \ldots, k$, we have*

$$\frac{\zeta(2k+2-2j)}{\zeta(2k+2)} - 1 < 3 \cdot 4^{j-(k+1)}.$$

Proof.

$$\begin{aligned}
\frac{\zeta(2k+2-2j)}{\zeta(2k+2)} - 1 &< \zeta(2k+2-2j) - 1 \\
&< \frac{2k+3-2j}{2k+1-2j} 4^{j-(k+1)} \\
&\leq 3 \cdot 4^{j-(k+1)},
\end{aligned}$$

using Lemma 4.4, and the fact that $j \leq k$. □

Equipped with these lemmas, we are ready to prove Theorem 4.3.

Proof of Theorem 4.3. We have

$$\frac{M_{2k+1}(\sqrt{4+t})}{(4+t)^{k+1}} = 1 - 2 \sum_{j=1}^{k} \frac{\zeta(2j)}{(4+t)^j} \frac{\zeta(2k+2-2j)}{\zeta(2k+2)} + \frac{1}{(4+t)^{k+1}},$$

which, on replacing 1 by $2 \sum_{j=1}^{\infty} \frac{\zeta(2j)}{4^j}$ using Lemma 4.5, gives

$$2 \sum_{j=k+1}^{\infty} \frac{\zeta(2j)}{4^j} + \frac{1}{(4+t)^{k+1}} + 2 \sum_{j=1}^{k} \zeta(2j) \left(\frac{1}{4^j} - \frac{1}{(4+t)^j} \frac{\zeta(2k+2-2j)}{\zeta(2k+2)} \right).$$

We now claim that $M_{2k+1}(\sqrt{4+t})$ is positive for some small $t > 0$ that goes to 0 as $k \to \infty$. For this to hold, we see from the above expression that a sufficient condition is

$$4^{-j} > (4+t)^{-j} \frac{\zeta(2k+2-2j)}{\zeta(2k+2)} \text{ for } j = 1, \ldots, k.$$

Using the upper bound in Lemma 4.6, it is therefore sufficient that

$$\left(1 + \frac{t}{4}\right)^j > 1 + 3 \cdot 4^{j-(k+1)},$$

or equivalently

$$\frac{t}{4} > (1 + 3 \cdot 4^{j-(k+1)})^{1/j} - 1.$$

Since for $a \geq 0$ and $0 < \delta \leq 1$ we have $(1+a)^\delta \leq 1 + a\delta$, we replace this condition by

$$t > \frac{3 \cdot 4^{j-k}}{j}.$$

This lower bound attains its maximum at $j = k$, and hence we obtain our final sufficient condition for $M_{2k+1}(\sqrt{4+t})$ to be positive, namely that

$$t > \frac{3}{k}.$$

Hence for $k \geq 5$ the zero z_0 of M_{2k+1} lies in the open interval $\left(2, \sqrt{4 + \frac{3}{k}}\right)$. It follows that $z_0 < 2.15$ for $k \geq 5$ and $z_0 \to 2$ as $k \to \infty$. □

5. Zeros of Ramanujan Polynomials on the Unit Circle

We now ascertain the location of the nonreal zeros of M_{2k+1}.

Theorem 5.1. *For $k \geq 0$, all nonreal zeros of Ramanujan polynomials lie on the unit circle.*

From Section 3, we know that this result is true for $k \leq 4$. We are therefore again entitled to assume that $k \geq 5$.

The idea of the proof is to approximate $M_{2k+1}(z)$ by the polynomial

$$A(z) = B(z)(z^4 - 4z^2 + 1), \tag{3}$$

where

$$B(z) = \frac{z^{2k} - 1}{z^2 - 1}. \tag{4}$$

Not only does A have integer coefficients, but we know its zeros exactly. We then define $\Delta(z)$ by

$$M_{2k+1}(z) = A(z) - \Delta(z). \tag{5}$$

To proceed, we need two lemmas, which enable us to describe quantitatively the approximation of $M_{2k+1}(z)$ by $A(z)$.

Lemma 5.2. *For $k \geq 5$ and $k - 1 \geq j \geq 2$, we have*

$$\frac{(2j+1)(2j'+1)}{(2j-1)(2j'-1)} < 2.5,$$

where $j' = k + 1 - j$.

Sketch of Proof. Treat this as a calculus problem involving a function of j. Then the left hand side is maximized at $j = 2$, and setting $k \geq 5$ gives the desired result. □

Lemma 5.3. *For $k \geq 5$, the polynomial $\Delta(z)$ satisfies $|\Delta(z)| < 1.3$ for z on the unit circle. Furthermore, writing*

$$\Delta(z) = (\varepsilon_1 - 1)(z^2 + z^{2k}) + \sum_{j=2}^{k-1} \varepsilon_j z^{2j},$$

we have, for $1 \leq j \leq k$,

$$\varepsilon_j = 2\left(\frac{\zeta(2j)\zeta(2k+2-2j)}{\zeta(2k+2)} - 1\right). \tag{6}$$

Proof. The formula for ε_j follows from the easily-verified fact that

$$A(z) = z^{2k+2} - 3z^{2k} - 2z^{2k-2} - \cdots - 2z^4 - 3z^2 + 1.$$

To bound $\Delta(z)$, we first note that, since $\varepsilon_1 - 1$ and all the ε_j for $2 \leq j \leq k-1$ are positive,

$$|\Delta(z)| < 2(\varepsilon_1 - 1) + \sum_{j=2}^{k-1} \varepsilon_j$$

for z on the unit circle. We therefore need to bound this sum from above.

Recalling that $\varepsilon_j = \varepsilon_{k+1-j}$, invoking Lemma 4.4 gives

$$\begin{aligned} \varepsilon_1 = \varepsilon_k &= \frac{\pi^2}{6}\frac{2\zeta(2k)}{\zeta(2k+2)} - 2 \\ &< \frac{\pi^2}{3}\left(1 + \frac{2k+1}{2k-1}4^{-k}\right) - 2 \\ &< 1.3 \end{aligned}$$

for $k \geq 5$, since the right hand side of the inequality in Lemma 4.4 is strictly decreasing.

Now, once again invoking Lemma 4.4, we have

$$\begin{aligned} \varepsilon_j &< 2\left(\left(1 + \frac{2j+1}{2j-1}4^{-j}\right)\left(1 + \frac{2j'+1}{2j'-1}4^{-j'}\right) - 1\right) \\ &= 2\left(\frac{2j+1}{2j-1}4^{-j} + \frac{2j'+1}{2j'-1}4^{-j'} + \frac{(2j+1)(2j'+1)}{(2j-1)(2j'-1)}4^{-(k+1)}\right). \end{aligned}$$

Summing both sides over j and using Lemma 5.2 gives us

$$\begin{aligned} \sum_{j=2}^{k-1} \varepsilon_j &< 4\sum_{j=2}^{\infty} \frac{2j+1}{2j-1}4^{-j} + \frac{5}{4}(k-2)4^{-k} \\ &= 4\sum_{j=2}^{\infty}\left(1 + \frac{2}{2j-1}\right)4^{-j} + \frac{5}{4}(k-2)4^{-k} \\ &= 2\log 3 - \frac{5}{3} + \frac{5}{4}(k-2)4^{-k} \\ &< 0.7 \end{aligned}$$

for $k \geq 5$, using the fact that

$$\log\left(\frac{1+x}{1-x}\right) = 2\sum_{j=1}^{\infty} \frac{x^{2j-1}}{2j-1},$$

and so

$$\log 3 = 1 + 4\sum_{j=2}^{\infty} \frac{1}{2j-1}4^{-j},$$

on setting $x = 1/2$. Hence

$$\begin{aligned} 2(\varepsilon_1 - 1) + \sum_{j=2}^{k-1} \varepsilon_j &< 2 \times 0.3 + 0.7 \\ &= 1.3, \end{aligned}$$

□

We are now ready prove Theorem 5.1:

Proof of Theorem 5.1. We first note that the polynomial $A(z)$, which approximates $M_{2k+1}(z)$, can, using (3), be written as

$$A(z) = z^{k+1} \left(\frac{z^k - z^{-k}}{z - z^{-1}} \right) ((z - z^{-1})^2 - 2),$$

Restricting z to the unit circle (in other words, putting $z = e^{i\theta}$) gives us

$$z^{-(k+1)} A(z) = -(2 + 4\sin^2\theta) \frac{\sin k\theta}{\sin\theta} = f(\theta),$$

say. In particular, f is a real-valued function of θ. Next, consider

$$g(\theta) = e^{-(k+1)i\theta} \Delta(e^{i\theta}).$$

Note that, by Lemma 3.1, $\Delta(z)$ is reciprocal, and $g(\theta)$ is real-valued; also, by Lemma 5.3, it is bounded above by 1.3.

Finally, by confining our attention to $\theta \in (0, \pi)$, we see that $f(\theta) > 2$ whenever $\sin k\theta = -1$ and $f(\theta) < -2$ whenever $\sin k\theta = 1$. This is because

$$\begin{aligned} \frac{2 + 4\sin^2\theta}{\sin\theta} &> 2 + 4\sin^2\theta \\ &> 2 \end{aligned}$$

on the interval $(0, \pi)$.

Therefore the function

$$e^{(k+1)i\theta} M_{2k+1}(e^{i\theta}) = f(\theta) + g(\theta)$$

has a zero between every consecutive pairs of values of $\theta \in (0, \pi)$ with $\sin k\theta = \pm 1$. There are precisely k values for which $\sin k\theta = \pm 1$, namely

$$\frac{(2l+1)\pi}{2k}, \, l = 1, \ldots, k,$$

giving $k - 1$ zeros of M_{2k+1} on the upper half of the unit circle. Taking complex conjugates yields another $k - 1$ zeros on the lower half of the unit circle. Hence M_{2k+1} has exactly $2k - 2$ zeros on the unit circle, which is what we wanted to show. □

Observe that the above theorem not only gives the modulus of the nonreal zeros, but also gives some restriction on their distribution. In particular, these zeros are interlaced between angles θ for which $\sin k\theta$ assumes the values ± 1, which means that, asymptotically, they are uniformly distributed on the unit circle.

We also observe another useful result concerning zeros of Ramanujan polynomials:

Corollary 5.4. *Ramanujan polynomials have no repeated zeros.*

Proof. The real zeros of M_{2k+1} were already shown to be distinct. The nonreal zeros on the unit circle are strictly interlaced between angles θ for which $\sin k\theta$ assumes the values ± 1, and hence never coincide with each other. □

6. Zeros of Ramanujan Polynomials that are Roots of Unity

In this section we find all zeros of the Ramanujan polynomial R_{2k+1} that are roots of unity. We do this in three stages. First, we show that any such zero must be a $2k$-th root of unity. Next, we show that such a zero must in fact be a primitive 3rd, 4th, or 6th root of unity. Finally, we calculate for which values of k these three cases occur.

Let

$$\rho = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

a primitive cube root of unity, and $\phi_\ell(z)$ be the ℓ-th cyclotomic polynomial, whose zeros are the primitive ℓ-th roots of unity.

Our main result is the following.

Theorem 6.1. *The roots of unity that are zeros of M_{2k+1} are*
- *Both $\pm i$ if k is even;*
- *All four of $\pm\rho$, $\pm\bar{\rho}$ if k is a multiple of 3,*

and no others.

In terms of polynomial factors, this says that the only cyclotomic factors of R_{2k+1} are z^2+1 when k is even, and

$$(z^2+z+1)(z^2-z+1)$$

when k is a multiple of 3.

Proposition 6.2. *Any zero of R_{2k+1} that is a root of unity must be a $2k$-th root of unity.*

Proof. As in the previous section, we work with M_{2k+1} rather than R_{2k+1}, and write

$$M_{2k+1}(z) = A(z) - \Delta(z),$$

where A and B are given by (3) and (4). Note that $A(\pm 1) = -2k$, so that any $2k$-th root of unity that is a zero of A is also a zero of B. Suppose that $M_{2k+1}(z) = 0$ at a primitive ℓ-th root of unity – call it ω – so that ϕ_ℓ is a factor of M_{2k+1} (ϕ_l must be a factor since M_{2k+1} has rational coefficients). Then

$$A(\omega) = \Delta(\omega)$$

and since, by Lemma 5.3, $|\Delta(z)| < 1.3$ for z on the unit circle, we have $|A(\omega)| < 1.3$. If ω is not a $2k$-th root of unity, then the resultant $\mathrm{Res}(\phi_\ell, B)$ must be a nonzero integer, and so at least 1 in modulus. But (see [5, Section 5.9]) this resultant is equal to

$$\prod_{z:\,\phi_l(z)=0} B(z).$$

Hence, choosing ω to be such a z where $|B(z)|$ is largest, we must have $|B(\omega)| \geq 1$. Since

$$|z^4 - 4z^2 + 1| \geq 2$$

for z on the unit circle, we see from (3) that $|A(\omega)| \geq 2$, a contradiction. □

Next, we need the following lemma.

Lemma 6.3. *Every cyclotomic polynomial ϕ_ℓ except for ϕ_3, ϕ_4 and ϕ_6 has a zero $z = e^{i\theta}$ with $\theta \in [0, \frac{14}{45}\pi] \cup [\pi - \frac{14}{45}\pi, \pi]$.*

Proof. If $\ell = 1$, $\ell = 2$ or $\ell \geq 8$, then the zero $e^{2\pi i/\ell}$ of ϕ_ℓ has its argument θ in the required range. Also ϕ_5 has the zero $e^{4\pi i/5}$ and ϕ_7 has the zero $e^{6\pi i/7}$, with both these zeros also having their arguments in the required range. □

Proof of Theorem 6.1. The theorem holds for $k < 8$ by the computations of Section 2; we can therefore assume that $k \geq 8$. Suppose that ω is a root of unity lying in the upper half plane such that $M_{2k+1}(\omega) = 0$. By Proposition 6.2, ω must be a $2k$-th root of unity. Then, since $A(\omega) = 0$, we have $\Delta(\omega) = 0$. If ω is an ℓ-th root of unity then, since Δ has rational coefficients,

ϕ_ℓ divides Δ. Hence, using Lemma 6.3, unless $\ell = 3$, 4 or 6 we can assume, by appropriate choice of zero of ϕ_ℓ, that $\omega = e^{i\theta}$ with $\theta \in [0, \frac{14}{45}\pi] \cup [\pi - \frac{14}{45}\pi, \pi]$.

Now from (5) we have

$$\omega^{-1}\Delta(\omega) = (\varepsilon_1 - 1)(\omega + \omega^{2k-1}) + \sum_{j=2}^{\lfloor \frac{k+1}{2} \rfloor}{}^{*} \varepsilon_j(\omega^{2j-1} + \omega^{2(k+1-j)-1}),$$

where the ε_j are given by (6), and where the $*$ indicates that the final term is halved for k odd. So, on putting $\omega = e^{i\theta}$ we obtain

$$\left(\frac{\pi^2}{3} \cdot \frac{\zeta(2k)}{\zeta(2k+2)} - 3\right) 2\cos\theta + \sum_{j=2}^{\lfloor \frac{k+1}{2} \rfloor}{}^{*} \left(\frac{2\zeta(2j)\zeta(2k+2-2j)}{\zeta(2k+2)} - 2\right) 2\cos((2j-1)\theta) = 0.$$

We now introduce an integer parameter r, to be chosen later, lying in the range $1 \le r < \lfloor \frac{k+1}{2} \rfloor$. Then we have

$$\begin{aligned}\left(\frac{\pi^2}{3} - 3\right)\cos\theta \quad &+ \quad \frac{\pi^2}{3} \cdot \left(\frac{\zeta(2k)}{\zeta(2k+2)} - 1\right)\cos\theta + 2\sum_{j=2}^{r}(\zeta(2j) - 1)\cos((2j-1)\theta) \\ &+ \quad \sum_{j=2}^{r} 2\zeta(2j)\left(\frac{\zeta(2k+2-2j)}{\zeta(2k+2)} - 1\right)\cos((2j-1)\theta) \\ &+ \quad 2\sum_{j=r+1}^{\lfloor \frac{k+1}{2} \rfloor}{}^{*} \left(\frac{\zeta(2j)\zeta(2k+2-2j)}{\zeta(2k+2)} - 1\right)\cos((2j-1)\theta) \qquad = \qquad 0.\end{aligned}$$

Hence, defining

$$h_r(\theta) = \left(\frac{\pi^2}{3} - 3\right)\cos\theta + 2\sum_{j=2}^{r}(\zeta(2j) - 1)\cos((2j-1)\theta),$$

we have

$$\begin{aligned}|h_r(\theta)| \qquad &< \qquad \frac{\pi^2}{3} \cdot \left(\frac{\zeta(2k)}{\zeta(2k+2)} - 1\right) + 2\sum_{j=2}^{r}\zeta(2j)\left(\frac{\zeta(2k+2-2j)}{\zeta(2k+2)} - 1\right) \\ &\qquad\qquad +2\sum_{j=r+1}^{\lfloor \frac{k+1}{2} \rfloor}\left(\frac{\zeta(2j)\zeta(2k+2-2j)}{\zeta(2k+2)} - 1\right) \\ &< \qquad \frac{\pi^2}{3} \cdot (\zeta(2k) - 1) + 2\sum_{j=2}^{r}\zeta(2j)(\zeta(2k+2-2r) - 1) \\ &\qquad\qquad +2\sum_{j=r+1}^{\lfloor \frac{k+1}{2} \rfloor}(\zeta(2j)\zeta(k+1) - 1) \\ &< \qquad \frac{\pi^2}{3} \cdot \frac{2k+1}{2k-1}4^{-k} + 2\sum_{j=2}^{r}\zeta(2j)\frac{2k+3-2r}{2k+1-2r}4^{-(k+1-r)} \\ &\qquad\qquad +2\sum_{j=r+1}^{\lfloor \frac{k+1}{2} \rfloor}\left(\zeta(2j)\left(1 + \frac{k+2}{k}2^{-(k+1)}\right) - 1\right),\end{aligned}$$

using Lemma 4.4.

For the last sum of this upper bound, we have, using Lemma 4.5, that, for $r \le (k+1)/2$,

$$2\sum_{j=r+1}^{\lfloor\frac{k+1}{2}\rfloor}\left(\zeta(2j)\left(1+\frac{k+2}{k}2^{-(k+1)}\right)-1\right)$$
$$<\frac{3}{2}-2\sum_{j=1}^{r}(\zeta(2j)-1)+\frac{k+2}{k2^k}\zeta(2r+2)\left(\frac{k+1}{2}-r\right),$$

giving finally that

$$\begin{aligned}|h_r(\theta)| &< \frac{\pi^2}{3}\cdot\frac{2k+1}{2k-1}4^{-k}+2\sum_{j=2}^{r}\zeta(2j)\frac{2k+3-2r}{2k+1-2r}4^{-(k+1-r)}\\ &\quad+\frac{3}{2}-2\sum_{j=1}^{r}(\zeta(2j)-1)+\frac{k+2}{k2^k}\zeta(2r+2)\left(\frac{k+1}{2}-r\right).\end{aligned}$$

The big term in this upper bound is

$$\frac{3}{2}-2\sum_{j=1}^{r}(\zeta(2j)-1);$$

all other terms go to 0 as $k\to\infty$.

We now choose $r=4$. Then a Maple plot of $h_4(\theta)$ shows that it is a decreasing function of θ for $0\le\theta\le\pi/3$, and takes the value 0.01398 at $\theta=\frac{14}{45}\pi$. However, the upper bound for $|h_r(\theta)|$, calculated above, is a decreasing function of k, and, for $r=4$ and $k=8$, equals 0.01255. So at this value we must have $\theta>\frac{14}{45}\pi$. As $|h_4(\theta)|$ is an even function of θ, we see that θ lies in the interval $(\frac{14}{45}\pi,\pi-\frac{14}{45}\pi)$. So ℓ must be 3, 4 or 6.

The above discussion tells us that for $k\ge 8$, if $\omega=e^{i\theta}$ is a root of unity with $M_{2k+1}(\omega)=0$, then ω is a primitive 3rd, 4th, or 6th root of unity.

We now proceed to prove that $M_{2k+1}(i)=0$ if and only if k is even, while $M_{2k+1}(\rho)=0$ if and only if $3\mid k$. To begin, we see from (3) that for k odd, $A(i)=6$, so that $|\Delta(i)|<1.3$ implies that $M_{2k+1}(i)=A(i)-\Delta(i)\ne 0$. Similarly, if k is not a multiple of 3, then $|A(\rho)|=5$, which again shows that $M_{2k+1}(\rho)\ne 0$.

On the other hand, for k even we have $\frac{1}{2}\deg(M_{2k+1})=k+1$ is odd, so by the evenness and the functional equation for M_{2k+1}we have

$$M_{2k+1}(i)=(-1)^{k+1}M_{2k+1}(i),$$

giving $M_{2k+1}(i)=0$.

Next, fix $k\equiv 0 \pmod 3$ and recall that ρ satisfies

$$-\rho^2=1+\rho=-\frac{1}{\rho}.$$

Now, since ρ lies in the upper half plane, we may evaluate Grosswald's formula at ρ, yielding

$$F_{2k+1}(\rho)-\rho^{2k}F_{2k+1}\left(-\frac{1}{\rho}\right) = \frac{1}{2}\zeta(2k+1)(\rho^{2k}-1)+\frac{(2\pi i)^{2k+1}}{2\rho}R_{2k+1}(\rho),$$

or equivalently

$$F_{2k+1}(\rho)-F_{2k+1}(1+\rho)=\frac{(2\pi i)^{2k+1}}{2\rho}R_{2k+1}(\rho).$$

But since

$$\begin{aligned} F_{2k+1}(1+\rho) &= \sum_{n=1}^{\infty} \frac{\sigma_{2k+1}(n)}{n^{2k+1}} e^{2\pi i n(1+\rho)} \\ &= \sum_{n=1}^{\infty} \frac{\sigma_{2k+1}(n)}{n^{2k+1}} e^{2\pi i n\rho} \\ &= F_{2k+1}(\rho), \end{aligned}$$

the left-hand side of the above equality is 0, giving $R_{2k+1}(\rho) = 0$.

Now, from $M_{2k+1}(-i) = \overline{M_{2k+1}(i)}$ we see that $M_{2k+1}(i) = 0$ if and only if $M_{2k+1}(i) = 0$. Similarly, we have $M_{2k+1}(\bar{\rho}) = \overline{M_{2k+1}(\rho)}$ and $M_{2k+1}(-\rho) = M_{2k+1}(\rho)$, since M_{2k+1} is a polynomial in z^2. So all four of $\pm\rho$, $\pm\bar{\rho}$ are zeros of M_{2k+1} if any one of them is. Hence both of $\pm i$ are zeros of M_{2k+1} if and only if k is even, and all four of $\pm\rho$, $\pm\bar{\rho}$ are zeros of M_{2k+1} if and only if k is a multiple of 3. These are the only zeros of M_{2k+1} that are roots of unity. □

As a final observation, for k even, the fact that $R_{2k+1}(i) = 0$ allows us to deduce that,

$$\sum_{j=0}^{k+1} (-1)^j \frac{B_{2j} B_{2k+2-2j}}{(2j)!(2k+2-2j)!} = 0,$$

which agrees with claim (2) in [2]. This also follows (1), on putting $\alpha = \beta = \sqrt{\pi}$.

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