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**THE DIVISIBILITY OF  $a^n - b^n$  BY POWERS OF  $n$** 

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**Abstract**

For given integers  $a, b$  and  $j \geq 1$  we determine the set  $R_{a,b}^{(j)}$  of integers  $n$  for which  $a^n - b^n$  is divisible by  $n^j$ . For  $j = 1, 2$ , this set is usually infinite; we determine explicitly the exceptional cases for which  $a, b$  the set  $R_{a,b}^{(j)}$  ( $j = 1, 2$ ) is finite. For  $j = 2$ , we use Zsigmondy's Theorem for this. For  $j \geq 3$  and  $\gcd(a, b) = 1$ ,  $R_{a,b}^{(j)}$  is probably always finite; this seems difficult to prove, however.

We also show that determination of the set of integers  $n$  for which  $a^n + b^n$  is divisible by  $n^j$  can be reduced to that of  $R_{a,b}^{(j)}$ .

**1. Introduction**

Let  $a, b$  and  $j$  be fixed integers, with  $j \geq 1$ . The aim of this paper is to find the set  $R_{a,b}^{(j)}$  of all positive integers  $n$  such that  $n^j$  divides  $a^n - b^n$ . For  $j = 1, 2, \dots$ , these sets are clearly nested, with common intersection  $\{1\}$ . Our first results (Theorems 1 and 2) describe this set in the case that  $\gcd(a, b) = 1$ . In Section 4 we describe (Theorem 15) the set in the general situation where  $\gcd(a, b)$  is unrestricted.

**Theorem 1.** *Suppose that  $\gcd(a, b) = 1$ . Then the elements of the set  $R_{a,b}^{(1)}$  consist of those integers  $n$  whose prime factorization can be written in the form*

$$n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r} \quad (p_1 < p_2 < \dots < p_r, \text{ all } k_i \geq 1), \quad (1)$$

where  $p_i \mid (a^{n_i} - b^{n_i})$  ( $i = 1, \dots, r$ ), with  $n_1 = 1$  and  $n_i = p_1^{k_1} p_2^{k_2} \dots p_{i-1}^{k_{i-1}}$  ( $i = 2, \dots, r$ ).

In this theorem, the  $k_i$  are arbitrary positive integers. This result is a more explicit version of that proved in Györy [5], where it was shown that if  $a - b > 1$  then for any positive integer  $r$  the number of elements of  $R_{a,b}^{(1)}$  having  $r$  prime factors is infinite. The result is also essentially contained in [11], which described

the indices  $n$  for which the generalized Fibonacci numbers  $u_n$  are divisible by  $n$ . However, we present a self-contained proof in this paper.

On the other hand, for  $j \geq 2$ , the exponents  $k_i$  are more restricted.

**Theorem 2.** *Suppose that  $\gcd(a, b) = 1$ , and  $j \geq 2$ . Then the elements of the set  $R_{a,b}^{(j)}$  consist of those integers  $n$  whose prime factorization can be written in the form (1), where*

$$p_1^{(j-1)k_1} \text{ divides } \begin{cases} a - b & \text{if } p_1 > 2; \\ \text{lcm}(a - b, a + b) & \text{if } p_1 = 2, \end{cases}$$

and  $p_i^{(j-1)k_i} \mid a^{n_i} - b^{n_i}$ , with  $n_i = p_1^{k_1} p_2^{k_2} \dots p_{i-1}^{k_{i-1}}$  ( $i = 2, \dots, r$ ).

Again, the result was essentially contained in [5], where it was proved that for  $a - b > 1$  and for any given  $r$ , there exists an  $n \in R_{a,b}^{(j)}$  with  $r$  distinct prime factors. Further, the number of these  $n$  is finite, and all of them can be determined. The paper [5] was stimulated by a problem from the 31st International Mathematical Olympiad, which asked for all those positive integers  $n > 1$  for which  $2^n + 1$  was divisible by  $n^2$ . (For the answer, see [5], or Theorem 16.)

Thus we see that construction of  $n \in R_{a,b}^{(j)}$  depends upon finding a prime  $p_i$  not used previously with  $a^{n_i} - b^{n_i}$  being divisible by  $p_i^{j-1}$ . This presents no problem for  $j = 2$ , so that  $R_{a,b}^{(2)}$ , as well as  $R_{a,b}^{(1)}$ , are usually infinite. See Section 5 for details, including the exceptional cases when they are finite. However, for  $j \geq 3$  the condition  $p_i^{j-1} \mid a^{n_i} - b^{n_i}$  is only rarely satisfied. This suggests strongly that in this case  $R_{a,b}^{(j)}$  is always finite for  $\gcd(a, b) = 1$ . This seems very difficult to prove, even assuming the ABC Conjecture. A result of Ribenboim and Walsh [10] implies that, under ABC, the powerful part of  $a^n - b^n$  cannot often be large. But this is not strong enough for what is needed here. On the other hand,  $R_{a,b}^{(j)}$  ( $j \geq 3$ ) can be made arbitrarily large by choosing  $a$  and  $b$  such that  $a - b$  is a powerful number. For instance, choosing  $a = 1 + (q_1 q_2 \dots q_s)^{j-1}$  and  $b = 1$ , where  $q_1, q_2, \dots, q_s$  are distinct primes, then  $R_{a,b}^{(j)}$  contains the  $2^s$  numbers  $q_1^{\varepsilon_1} q_2^{\varepsilon_2} \dots q_s^{\varepsilon_s}$  where the  $\varepsilon_i$  are 0 or 1. See Example 6 in Section 7.

In the next section we give preliminary results needed for the proof of the theorems. We prove them in Section 3. In Section 4 we describe (Theorem 15)  $R_{a,b}^{(j)}$ , where  $\gcd(a, b)$  is unrestricted. In Section 5 we find all  $a, b$  for which  $R_{a,b}^{(2)}$  is finite (Theorem 16). In Section 6 we discuss the divisibility of  $a^n + b^n$  by powers of  $n$ . In Section 7 we give some examples, and make some final remarks in Section 8.

**2. Preliminary Results**

We first prove a version of Fermat’s Little Theorem that gives a little bit more information in the case  $x \equiv 1 \pmod{p}$ .

**Lemma 3.** *For  $x \in \mathbb{Z}$  and  $p$  an odd prime we have*

$$x^{p-1} + x^{p-2} + \dots + x + 1 \equiv \begin{cases} p \pmod{p^2} & \text{if } x \equiv 1 \pmod{p}; \\ 1 \pmod{p} & \text{otherwise.} \end{cases} \tag{2}$$

*Proof.* If  $x \equiv 1 \pmod{p}$ , say  $x = 1 + kp$ , then  $x^j \equiv 1 + jkp \pmod{p^2}$ , so that

$$x^{p-1} + x^{p-2} + \dots + x + 1 \equiv p + kp \sum_{j=0}^{p-1} j \equiv p \pmod{p^2}. \tag{3}$$

Otherwise

$$x(x - 1)(x^{p-2} + \dots + x + 1) = x^p - x \equiv 0 \pmod{p}, \tag{4}$$

so that for  $x \not\equiv 1 \pmod{p}$  we have  $x(x^{p-2} + \dots + x + 1) \equiv 0 \pmod{p}$ , and hence

$$x^{p-1} + x^{p-2} + \dots + x + 1 \equiv x(x^{p-2} + \dots + x + 1) + 1 \equiv 1 \pmod{p}. \tag{5}$$

□

The following is a result of Birkoff and Vandiver [2, Theorem III]. It is also special case of Lucas [9, p. 210], as corrected for  $p = 2$  by Carmichael [3, Theorem X].

**Lemma 4.** *Let  $\gcd(a, b) = 1$  and  $p$  be prime with  $p \mid (a - b)$ . Define  $t > 0$  by  $p^t \parallel (a - b)$  for  $p > 2$  and  $2^t \parallel \text{lcm}(a - b, a + b)$  if  $p = 2$ . Then for  $\ell > 0$*

$$p^{t+\ell} \parallel (a^{p^\ell} - b^{p^\ell}). \tag{6}$$

*On the other hand, if  $p \nmid a - b$  then for  $\ell \geq 0$*

$$p \nmid a(p^\ell - b^{p^\ell}). \tag{7}$$

*Proof.* Put  $x = a/b$ . First suppose that  $p$  is odd and  $p^t \parallel a - b$  for some  $t > 0$ . Then as  $\gcd(a, b) = 1$ ,  $b$  is not divisible by  $p$ , and we have  $x \equiv 1 \pmod{p^t}$ . Then from

$$a^p - b^p = (a - b)b^{p-1}(x^{p-1} + x^{p-2} + \dots + x + 1) \tag{8}$$

we have by Lemma 3 that  $p^{t+1} \parallel (a^p - b^p)$ . Applying this result  $\ell$  times, we obtain (6).

For  $p = 2$ , we have  $p^{t+1} \parallel a^2 - b^2$  and from  $a^2 \equiv b^2 \equiv 1 \pmod{8}$ , we obtain  $2^1 \parallel (a^2 + b^2)$ , and so  $p^{t+2} \parallel (a^4 - b^4)$ . An easy induction then gives the required result.

Now suppose that  $p \nmid (a - b)$ . Since  $\gcd(a, b) = 1$ , (7) clearly holds if  $p \mid a$  or  $p \mid b$ , as must happen for  $p = 2$ . So we can assume that  $p$  is odd and  $p \nmid b$ . Then  $x \not\equiv 1 \pmod{p}$  so that, by Lemma 3 and (8), we have  $p \nmid (a^p - b^p)$ . Applying this argument  $\ell$  times, we obtain (7).  $\square$

For  $n \in R_{a,b}^{(j)}$ , we now define the set  $\mathcal{P}_{a,b}^{(j)}(n)$  to be the set of all prime powers  $p^k$  for which  $np^k \in R_{a,b}^{(j)}$ . Our next result describes this set precisely. (Compare with [11, Theorem 1(a)]).

**Proposition 5.** *Suppose that  $j \geq 1$ ,  $\gcd(a, b) = 1$ ,  $n \in R_{a,b}^{(j)}$  and*

$$a^n - b^n = 2^{e_2} \prod_{p>2} p^{e_p}, \quad n = \prod_p p^{k_p} \tag{9}$$

and define  $e_2$  by  $2^{e_2} \parallel \text{lcm}(a^n - b^n, a^n + b^n)$ . Then

$$\mathcal{P}^{(1)}(n) = \bigcup_{p \mid a^n - b^n} \{p^k, k \in \mathbb{N}\}, \tag{10}$$

and for  $j \geq 2$

$$\mathcal{P}_{a,b}^{(j)}(n) = \bigcup_{p:p^{j-1} \mid a^n - b^n} \left\{ p^k : 1 \leq k \leq \left\lfloor \frac{e_p - jk_p}{j-1} \right\rfloor \right\}. \tag{11}$$

Note that  $e_2$  is never 1. Consequently, if  $2m \in R_{a,b}^{(2)}$ , where  $m$  is odd, then  $4m \in R_{a,b}^{(2)}$ . Also,  $2 \in R_{a,b}^{(j)}$  for  $j \leq 3$  when  $a - b$  is even.

*Proof.* Taking  $n \in R_{a,b}^{(j)}$  we have, from (9) and the definition of  $e_2$ , that  $jk_p \leq e_p$  for all primes  $p$ . Hence, applying Lemma 4 with  $a, b$  replaced by  $a^n, b^n$  we have for  $p$  dividing  $a^n - b^n$  that for  $\ell > 0$

$$p^{e_p + \ell} \parallel (a^{np^\ell} - b^{np^\ell}). \tag{12}$$

So  $(np^\ell)^j \mid (a^{np^\ell} - b^{np^\ell})$  is equivalent to  $j(k_p + \ell) \leq e_p + \ell$ , or  $(j - 1)\ell \leq e_p - jk_p$ . Thus we obtain (10) for  $j \geq 2$ , with  $\ell$  unrestricted for  $j = 1$ , giving (10).

On the other hand, if  $p \nmid (a^n - b^n)$ , then by Lemma 4 again,  $p^\ell \nmid (a^{np^\ell} - b^{np^\ell})$ , so that certainly  $(np^\ell)^j \nmid (a^{np^\ell} - b^{np^\ell})$ .  $\square$

We now recall some facts about the order function  $\text{ord}$ . For  $m$  an integer greater than 1 and  $x$  an integer prime to  $m$ , we define  $\text{ord}_m(x)$ , the *order of  $x$  modulo  $m$* , to be the least positive integer  $h$  such that  $x^h \equiv 1 \pmod{m}$ . The next three lemmas, containing standard material on the  $\text{ord}$  function, are included for completeness.

**Lemma 6.** *For  $x \in \mathbb{N}$  and prime to  $m$ , we have  $m \mid (x^n - 1)$  if and only if  $\text{ord}_m(x) \mid n$ .*

*Proof.* Let  $\text{ord}_m(x) = h$ , and assume that  $m \mid (x^n - 1)$ . Then as  $m \mid (x^h - 1)$ , also  $m \mid (x^{\gcd(h,n)} - 1)$ . By the minimality of  $h$ ,  $\gcd(h, n) = h$ , i.e.,  $h \mid n$ . Conversely, if  $h \mid n$  then  $(x^h - 1) \mid (x^n - 1)$ , so that  $m \mid (x^n - 1)$ .  $\square$

**Corollary 7.** *Let  $j \geq 1$ . We have  $n^j \mid (x^n - 1)$  if and only if  $\gcd(x, n) = 1$  and  $\text{ord}_{n^j}(x) \mid n$ .*

**Lemma 8.** *For  $m = \prod_p p^{f_p}$  and  $x \in \mathbb{N}$  and prime to  $m$  we have*

$$\text{ord}_m(x) = \text{lcm}_p \text{ord}_{p^{f_p}}(x). \tag{13}$$

*Proof.* Put  $h_p = \text{ord}_{p^{f_p}}(x)$ ,  $h = \text{ord}_m(x)$  and  $h' = \text{lcm}_p h_p$ . Then by Lemma 6 we have  $p^{f_p} \mid (x^{h'} - 1)$  for all  $p$ , and hence  $m \mid (x^{h'} - 1)$ . Hence  $h \mid h'$ . On the other hand, as  $p^{f_p} \mid n$  and  $m \mid (x^h - 1)$ , we have  $p^{f_p} \mid (x^h - 1)$ , and so  $h_p \mid h$ , by Lemma 6. Hence  $h' = \text{lcm}_p h_p \mid h$ .  $\square$

Now put  $p_* = \text{ord}_p(x)$ , and define  $t > 0$  by  $p^t \parallel (x^{p_*} - 1)$ .

**Lemma 9.** *For  $\gcd(x, n) = 1$  and  $\ell > 0$  we have  $p_* \mid (p - 1)$  and  $\text{ord}_{p^\ell}(x) = p^{\max(\ell-t, 0)} p_*$ .*

*Proof.* Since  $p \mid (x^{p-1} - 1)$ , we have  $p_* \mid (p - 1)$ , by Lemma 6. Also, from  $p^\ell \mid (x^{\text{ord}_{p^\ell}(x)} - 1)$  we have  $p \mid (x^{\text{ord}_{p^\ell}(x)} - 1)$ , and so, by Lemma 6 again,  $p_* = \text{ord}_p(x) \mid \text{ord}_{p^\ell}(x)$ . Further, if  $\ell \leq t$  then from  $p^\ell \mid (x^{p_*} - 1)$  we have by Lemma 6 that  $\text{ord}_{p^\ell}(x) \mid p_*$ , so  $\text{ord}_{p^\ell}(x) = p_*$ . Further, by Lemma 4 for  $u \geq t$

$$p^u \parallel (x^{p^{u-t} p_*} - 1), \tag{14}$$

so that, taking  $u = \ell \geq t$  and using Lemma 6,  $\text{ord}_{p^\ell}(x) \mid p^{\ell-t} p_*$ . Also, if  $t \leq u < \ell$ , then, from (14),  $x^{p^{t-u} p_*} \not\equiv 1 \pmod{p^\ell}$ . Hence  $\text{ord}_{p^\ell}(x) = p^{\ell-t} p_*$  for  $\ell \geq t$ .  $\square$

**Corollary 10.** *Let  $j \geq 1$ . For  $n = \prod_p p^{k_p}$  and  $x \in \mathbb{N}$  prime to  $n$  we have  $n^j \mid x^n - 1$  if and only if  $\gcd(x, n) = 1$  and*

$$\text{lcm}_p p^{k'_p} p_* \mid \prod_p p^{k_p}. \tag{15}$$

Here the  $k'_p = \max(jk_p - t_p, 0)$  are integers with  $t_p > 0$ .

Note that  $p_*$ ,  $k'_p$  and  $t_p$  in general depend on  $x$  and  $j$  as well as on  $p$ .

What we actually need in our situation is the following variant of Corollary 10.

**Corollary 11.** *Let  $j \geq 1$ . For  $n = \prod_p p^{k_p}$  and integers  $a, b$  with  $\gcd(a, b) = 1$  we have  $n^j \mid a^n - b^n$  if and only if  $\gcd(n, a) = \gcd(n, b) = 1$  and*

$$\text{lcm}_p p^{k'_p} p_* \mid \prod_p p^{k_p}. \tag{16}$$

Here the  $k'_p = \max(jk_p - t_p, 0)$  are integers with  $t_p > 0$ .

In this corollary, the  $x$  used to define  $p_*$  and  $t = t_p$  (see after Lemma 8) is chosen to satisfy  $bx \equiv a \pmod{n^j}$ . The result is then easily deduced from Corollary 10.

By contrast with Proposition 5, our next proposition allows us to *divide* an element  $n \in R_{a,b}^{(j)}$  by a prime, and remain within  $R_{a,b}^{(j)}$ .

**Proposition 12.** *Let  $n \in R_{a,b}^{(j)}$  with  $n > 1$ , and suppose that  $p_{\max}$  is the largest prime factor of  $n$ . Then  $n/p_{\max} \in R_{a,b}^{(j)}$ .*

*Proof.* Suppose  $n \in R_{a,b}^{(j)}$ , so that (15) holds, with  $x = a/b$ , and put  $q = p_{\max}$ . Then, since for every  $p$  all prime factors of  $p_*$  are less than  $p$ , the only possible term on the left-hand side that divides  $q^{k_q}$  on the right-hand side is the term  $q^{k'_q}$ . Now reducing  $k_q$  by 1 will reduce  $k'_q$  by at least 1, unless it is already 0, when it does not change. In either case (15) will still hold with  $n$  replaced by  $n/q$ , and so  $n/q \in R_{a,b}^{(j)}$ . □

Various versions and special cases of Proposition 12 for  $j = 1$  have been known for some time, in the more general setting of Lucas sequences, due to Somer [12, Theorem 5(iv)], Jarden [7, Theorem E], Hoggatt and Bergum [6], Walsh [14], André-Jeannin [1] and others. See also Smyth [11, Theorem 3].

In order to work out for which  $a, b$  the set  $R_{a,b}^{(j)}$  is finite, we need the following classical result. Recall that  $a^n - b^n$  is said to have a *primitive prime divisor*  $p$  if the prime  $p$  divides  $a^n - b^n$  but does not divide  $a^k - b^k$  for any  $k$  with  $1 \leq k < n$ .

**Theorem 13** (Zsigmondy [15]). *Suppose that  $a$  and  $b$  are nonzero coprime integers with  $a > b$  and  $a + b > 0$ . Then, except when*

- $n = 2$  and  $a + b$  is a power of 2

*or*

- $n = 3, a = 2, b = -1$

*or*

- $n = 6, a = 2, b = 1,$

$a^n - b^n$  has a primitive prime divisor.

(Note that in this statement we have allowed  $b$  to be negative, as did Zsigmondy. His theorem is nowadays often quoted with the restriction  $a > b > 0$  and so has the second exceptional case omitted.)

**3. Proof of Theorems 1 and 2**

Let  $n \in R_{a,b}^{(j)}$  have a factorisation (1), where  $p_1 < p_2 < \dots < p_r$  and all  $k_i > 0$ . First take  $j \geq 1$ . Then, by Proposition 12,  $n/p_r^{k_r} = n_r \in R_{a,b}^{(j)}$ , and hence

$$(n/p_r^{k_r})/p_{r-1}^{k_{r-1}} = n_{r-1}, \quad \dots, \quad p_1^{k_1} = n_2, \quad 1 = n_1$$

are all in  $R_{a,b}^{(j)}$ . Now separate the two cases  $j = 1$  and  $j \geq 2$  for Theorems 1 and 2 respectively. Now for  $j = 1$  Proposition 5 gives us that  $p_i \mid a^{n_i} - b^{n_i}$  ( $i = 1, \dots, r$ ), while for  $j \geq 2$  we have, again from Proposition 5, that

$$p_1^{(j-1)k_1} \text{ divides } \begin{cases} a - b & \text{if } p_1 > 2; \\ \text{lcm}(a - b, a + b) & \text{if } p_1 = 2, \end{cases}$$

and  $p_i^{(j-1)k_i} \mid a^{n_i} - b^{n_i}$  ( $i = 2, \dots, r$ ). Here we have used the fact that  $\text{gcd}(p_i, n_i) = 1$ , so that if  $p_i^{k_i} \mid (a^{n_i} - b^{n_i})/n_i^2$  then  $p_i^{k_i} \mid a^{n_i} - b^{n_i}$  (i.e., we are applying Proposition 5 with all the exponents  $k_p$  equal to 0.)

**4. Finding  $R_{a,b}^{(j)}$  When  $\text{gcd}(a, b) > 1$ .**

For  $a > 1$ , define the set  $\mathcal{F}_a$  to be the set of all  $n \in \mathcal{N}$  whose prime factors all divide  $a$ . To find  $R_{a,b}^{(j)}$  in general, we first consider the case  $b = 0$ .



**Proposition 14.** *We have  $R_{a,0}^{(1)} = R_{a,0}^{(2)} = \mathcal{F}_a$ , while for  $j \geq 3$  the set  $R_{a,0}^{(j)} = \mathcal{F}_a \setminus S_a^{(j)}$ , where  $S_a^{(j)}$  is a finite set.*

*Proof.* From the condition  $n^j \mid a^n$ , all prime factors of  $n$  divide  $a$ , so  $R_{a,0}^{(j)} \subset \mathcal{F}_a$ , say  $R_{a,0}^{(j)} = \mathcal{F}_a \setminus S_a^{(j)}$ . We need to prove that  $S_a^{(j)}$  is finite. Suppose that  $a = p_1^{a_1} \dots p_r^{a_r}$ , with  $p_1$  the smallest prime factor of  $a$ . Then  $n = p_1^{k_1} \dots p_r^{k_r}$  for some  $k_i \geq 0$ . From  $n^j \mid a^n$  we have

$$k_i \leq \frac{a_i}{j} p_1^{k_1} \dots p_r^{k_r} \quad (i = 1, \dots, r). \tag{17}$$

For these  $r$  conditions to be satisfied it is sufficient that

$$\sum_{i=1}^r k_i \leq \frac{\min_{i=1}^r a_i}{j} p_1^{\sum_{i=1}^r k_i}. \tag{18}$$

Now (18) holds if  $j = 1$  or  $2$ , as in this case, from the simple inequality  $k \leq 2^{k-1}$  valid for all  $k \in \mathbb{N}$ , we have

$$\sum_{i=1}^r k_i \leq \frac{1}{2} 2^{\sum_{i=1}^r k_i} \leq \frac{\min_{i=1}^r a_i}{j} p_1^{\sum_{i=1}^r k_i}. \tag{19}$$

Hence  $S_a^{(j)}$  is empty if  $j = 1$  or  $2$ .

Now take  $j \geq 3$ , and let  $K = K_a^{(j)}$  be the smallest integer such that  $K p_1^{-K} \leq (\min_{i=1}^r a_i)/j$ . Then (18) holds for  $\sum_{i=1}^r k_i \geq K$ , and  $S_a^{(j)}$  is contained in the finite set  $S'' = \{n \in \mathbb{N}, n = p_1^{k_1} \dots p_r^{k_r} : \sum_{i=1}^r k_i < K\}$ . (To compute  $S_a^{(j)}$  precisely, one need just check for which  $r$ -tuples  $(k_1, \dots, k_r)$  with  $\sum_{i=1}^r k_i < K$  any of the  $r$  inequalities of (17) is violated.)  $\square$

One (at first sight) curious consequence of the equality  $R_{a,0}^{(1)} = R_{a,0}^{(2)}$  above is that  $n \mid a^n$  implies  $n^2 \mid a^n$ .

Now let  $g = \gcd(a, b)$  and  $a = a_1 g, b = b_1 g$ . Write  $n = G n_1$ , where all prime factors of  $G$  divide  $g$  and  $\gcd(n_1, g) = 1$ . Then we have the following general result.

**Theorem 15.** *The set  $R_{a,b}^{(j)}$  is given by*

$$R_{a,b}^{(j)} = \{n = G n_1 : G \in \mathcal{F}_g, n_1 \in R_{a_1^G, b_1^G}^{(j)} \text{ and } \gcd(g, n_1) = 1\} \setminus R, \tag{20}$$

where  $R$  is a finite set. Specifically, all  $n = Gn_1 \in R$  have  $1 \leq n_1 < j/2$  and

$$G = q_1^{\ell_1} \dots q_m^{\ell_m}, \tag{21}$$

where

$$\sum_{i=1}^m \ell_i < K_{g^{n_1}}^{(j)}. \tag{22}$$

Here the  $q_i$  are the primes dividing  $g$ , and  $K_{g^{n_1}}^{(j)}$  is the constant in the proof of Proposition 14 above.

*Proof.* Supposing that  $n \in R_{a,b}^{(j)}$  we have

$$n^j \mid a^n - b^n \tag{23}$$

and so  $n^j \mid g^n(a_1^n - b_1^n)$ . Writing  $n = Gn_1$ , as above, we have

$$n_1^j \mid (a_1^G)^{n_1} - (b_1^G)^{n_1} \tag{24}$$

and

$$G^j \mid g^{Gn_1} ((a_1^G)^{n_1} - (b_1^G)^{n_1}). \tag{25}$$

Thus (23) holds with  $n, a, b$  replaced by  $n_1, a_1^G, b_1^G$ . So we have reduced the problem of (23) to a case where  $\gcd(a, b) = 1$ , which we can solve for  $n_1$  prime to  $g$ , along with the extra condition (25). Now, from the fact that  $R_{g,0}^{(2)} = \mathcal{F}_g$  from Proposition 14, we have  $G^2 \mid g^G$  and hence  $G^j \mid g^{Gn_1}$  for all  $G \in \mathcal{F}_g$ , provided that  $n_1 \geq j/2$ . Hence (25) can fail to hold for all  $G \in \mathcal{F}_g$  only for  $1 \leq n_1 < j/2$ .

Now fix  $n_1$  with  $1 \leq n_1 < j/2$ . Then note that by Proposition 14,  $G^j \mid g^{Gn_1}$  and hence (23) holds for all  $G \in \mathcal{F}_{g^{n_1}} \setminus S$ , where  $S$  is a finite set of  $G$ 's contained in the set of all  $G$ 's given by (21) and (22).  $\square$

Note that (taking  $n_1 = 1$  and using (25)) we always have  $R_{g,0}^{(j)} \subset R_{a,b}^{(j)}$ . See example in Section 7.

### 5. When Are $R_{a,b}^{(1)}$ and $R_{a,b}^{(2)}$ Finite?

First consider  $R_{a,b}^{(1)}$ . From Theorem 1 it is immediate that  $R_{a,b}^{(1)}$  contains all powers of any primes dividing  $a - b$ . Thus  $R_{a,b}^{(1)}$  is infinite unless  $a - b = \pm 1$ , in which case  $R_{a,b}^{(1)} = \{1\}$ . This was pointed out earlier by André-Jeannin [1, Corollary 4].

Next, take  $j = 2$ . Let us denote by  $\mathcal{P}_{a,b}^{(2)}$  the set of primes that divide some  $n \in R_{a,b}^{(2)}$  and, as before, put  $g = \gcd(a, b)$ .

**Theorem 16.** *The set  $R_{a,b}^{(2)} = \{1\}$  if and only if  $a$  and  $b$  are consecutive integers, and  $R_{a,b}^{(2)} = \{1, 3\}$  if and only if  $ab = -2$ . Otherwise,  $R_{a,b}^{(2)}$  is infinite.*

*If  $R_{a/g,b/g}^{(2)} = \{1\}$  (respectively,  $= \{1, 3\}$ ) then  $\mathcal{P}_{a,b}^{(2)}$  is the set of all prime divisors of  $g$  (respectively,  $3g$ ). Otherwise  $\mathcal{P}_{a,b}^{(2)}$  is infinite.*

For coprime positive integers  $a, b$  with  $a - b > 1$ , the infiniteness of  $R_{a,b}^{(2)}$  already follows from the above-mentioned results of [5].

The application of Zsigmondy’s Theorem that we require is the following.

**Proposition 17.** *If  $R_{a,b}^{(2)}$  contains some integer  $n \geq 4$  then both  $R_{a,b}^{(2)}$  and  $\mathcal{P}_{a,b}^{(2)}$  are infinite sets.*

*Proof.* First note that if  $a = 2, b = 1$  (or more generally  $a - b = \pm 1$ ) then by Theorem 2,  $R^{(2)} = \{1\}$ . Hence, taking  $n \in R_{a,b}^{(2)}$  with  $n \geq 4$  we have, by Zsigmondy’s Theorem, that  $a^n - b^n$  has a primitive prime divisor,  $p$  say. Now if  $p \mid n$  then, by applying Proposition 12 as many times as necessary we find  $p \mid n'$ , where  $n' \in R_{a,b}^{(2)}$  and now  $p$  is the maximal prime divisor of  $n'$ . Hence, by Proposition 12 again,  $n'' = n'/p \in R_{a,b}^{(2)}$  and so, from  $n' = pn''$  and Proposition 5 we have that  $p \mid a^{n''} - b^{n''}$ , contradicting the primitivity of  $p$ .

Now using Proposition 5 again,  $np \in R_{a,b}^{(2)}$ . Repeating the argument with  $n$  replaced by  $np$  and continuing in this way we obtain an infinite sequence

$$n, \quad np, \quad npp_1, \quad npp_1p_2, \quad \dots, \quad npp_1p_2 \dots p_\ell, \quad \dots$$

of elements of  $R_{a,b}^{(2)}$ , where  $p < p_1 < p_2 < \dots < p_\ell < \dots$  are primes. □

*Proof of Theorem 16.* Assume  $\gcd(a, b) = 1$ , and, without loss of generality, that  $a > 0$  and  $a > b$ . (We can ensure this by interchanging  $a$  and  $b$  and/or changing both their signs.) If  $a - b$  is even, then  $a$  and  $b$  are odd, and  $a^2 - b^2 \equiv 1 \pmod{2^{t+1}}$ , where  $t \geq 2$ . Hence  $4 \in R_{a,b}^{(2)}$ , by Proposition 5, and so both  $R_{a,b}^{(2)}$  and  $\mathcal{P}_{a,b}^{(2)}$  are infinite sets, by Proposition 17.

If  $a - b = 1$  then  $R^{(2)} = \{1\}$ , as we have just seen, above.

If  $a - b$  is odd and at least 5, then  $a - b$  must either be divisible by 9 or by a prime  $p \geq 5$ . Hence 9 or  $p$  belong to  $R_{a,b}^{(2)}$ , by Proposition 5, and again both  $R_{a,b}^{(2)}$  and  $\mathcal{P}_{a,b}^{(2)}$  are infinite sets, by Proposition 17.

If  $a - b = 3$  then  $3 \in R_{a,b}^{(2)}$ , and  $a^3 - b^3 = 9(b^2 + 3b + 3)$ . If  $b = -1$  (and  $a = 2, ab = -2$ ) or  $-2$  (and  $a = 1, ab = -2$ ) then  $a^3 - b^3 = 9$  and

so, by Theorem 2, so  $R^{(2)} = \{1, 3\}$ . Otherwise, using  $\gcd(a, b) = 1$  we see that  $a^3 - b^3 \geq 5$ , and so the argument for  $a - b \geq 5$  but with  $a, b$  replaced by  $a^3, b^3$  applies.  $\square$

**6. The Powers of  $n$  Dividing  $a^n + b^n$**

Define  $R_{a,b}^{(j)+}$  to be the set  $\{n \in \mathbb{N} : n^j \text{ divides } a^n + b^n\}$ . Take  $j \geq 1$ , and assume that  $\gcd(a, b) = 1$ . (The general case  $\gcd(a, b) \geq 1$  can be handled as in Section 4.) We then have the following result.

**Theorem 18.** *Suppose that  $j \geq 1$ ,  $\gcd(a, b) = 1$ ,  $a > 0$  and  $a \geq |b|$ . Then*

- (a)  $R_{a,b}^{(1)+}$  consists of the odd elements of  $R_{a,-b}^{(1)}$ , along with the numbers of the form  $2n_1$ , where  $n_1$  is an odd element of  $R_{a^2,-b^2}^{(1)}$ ;
- (b) If  $j \geq 2$  the set  $R_{a,b}^{(j)+}$  consists of the odd elements of  $R_{a,-b}^{(j)}$  only .

Furthermore, for  $j = 1$  and  $2$ , the set  $R_{a,b}^{(j)+}$  is infinite, except in the following cases:

- If  $a + b$  is 1 or a power of 2,  $(j, a, b) \neq (1, 1, 1)$ , when it is  $\{1\}$ ;
- $R_{1,1}^{(1)+} = \{1, 2\}$ ;
- $R_{2,1}^{(2)+} = \{1, 3\}$ .

*Proof.* If  $n$  is even and  $j \geq 2$ , or if  $4 \mid n$  and  $j = 1$ , then  $n^j \mid a^n + b^n$  implies that  $4 \mid a^n + b^n$ , contradicting the fact that, as  $a$  and  $b$  are not both even,  $a^n + b^n \equiv 1$  or  $2 \pmod{8}$ . So either

- $n$  is odd, in which case  $n^j \mid a^n + b^n$  is equivalent to finding the odd elements of the set  $R_{a,-b}^{(j)}$ ;
- or
- $j = 1$  and  $n = 2n_1$ , where  $n_1$  is odd, and belongs to  $R_{a^2,-b^2}^{(1)}$ .

Now suppose that  $j = 1$  or  $2$ . If  $a + b$  is  $\pm 1$  or  $\pm 2^i$  for some  $i > 0$ , then, by Theorem 2, all  $n \in R_{a,-b}^{(j)}$  with  $n > 1$  are even, so for  $j = 2$  there are no  $n > 1$  with  $n^j \mid a^n + b^n$  in this case. Otherwise,  $a + b$  will have an odd prime factor, and so at least one odd element greater than 1. By Theorem 16 and its proof, we see that  $R_{a,-b}^{(2)}$  will have infinitely many odd elements unless  $a(-b) = -2$ , i.e.,  $a = 2, b = 1$  (using  $a > 0$  and  $a \geq |b|$ ).

For  $j = 1$  there will be infinitely many  $n$  with  $n \mid a^n + b^n$ , except when both  $a + b$  and  $a^2 + b^2$  are 1 or a power of 2. It is an easy exercise to check that, this can happen only for  $a = b = 1$  or  $a = 1, b = 0$ .  $\square$

If  $g = \gcd(a, b) > 1$ , then, since  $R_{a,b}^{(j)+}$  contains the set  $R_{g,0}^{(j)}$ , it will be infinite, by Proposition 14. For  $j \geq 3$  and  $\gcd(a, b) = 1$ , the finiteness of the set  $R_{a,b}^{(j)+}$  would follow from the finiteness of  $R_{a,b}^{(j)}$ , using Theorem 16(b).

**7. Examples**

The set  $R_{a,b}^{(j)}$  has a natural labelled, directed-graph structure, as follows: take the vertices to be the elements of  $R_{a,b}^{(j)}$ , and join a vertex  $n$  to a vertex  $np$  as  $n \rightarrow_p np$ , where  $p \in \mathcal{P}_{a,b}^{(j)}(n)$ . We reduce this to a spanning tree of this graph by taking only those edges  $n \rightarrow_p np$  for which  $p$  is the largest prime factor of  $np$ . For our first example we draw this tree (Figure 1).

1. Consider the set

$$R_{3,1}^{(2)} = \{1, 2, 4, 20, 220, 1220, 2420, 5060, 13420, 14740, 23620, 55660, 145420, 147620, 162140, 237820, 259820, 290620, 308660, 339020, 447740, 847220, 899140, 1210220, \dots \}$$

(sequence A127103 in Neil Sloane’s Integer Sequences website). Now

$$3^{20} - 1 = 2^4 \cdot 5^2 \cdot 11^2 \cdot 61 \cdot 1181,$$

showing that  $\mathcal{P}_{3,1}^{(2)}(20) = \{11, 11^2, 61, 1181\}$ . Also

$$3^{220} - 1 = 2^4 \cdot 5^2 \cdot 11^3 \cdot 23 \cdot 61 \cdot 67 \cdot 661 \cdot 1181 \cdot 1321 \cdot 3851 \cdot 5501 \cdot 177101 \cdot 570461 \cdot 659671 \cdot 24472341743191 \cdot 560088668384411 \cdot 927319729649066047885192700193701,$$

so that the elements of  $\mathcal{P}_{3,1}^{(2)}(220)$  less than  $10^6/220$ , needed for Figure 1, are

$$11, 23, 61, 67, 661, 1181, 1321, 3851.$$



than 9 is of one of the forms  $21m$ ,  $63m$ ,  $147m$ , or  $441m$ , where  $m$  is prime to 21;

$$R_{11,2}^{(2)} = \{1, 3, 9, 21, 63, 147, 441, 609, 1827, 4137, 4263, 7959, \\ 8001, 12411, 12789, 23877, 28959, 35931, 55713, 56007, \\ 86877, 107793, 119973, 167139, 212541, 216237, 230811, \\ 232029, 251517, 359919, 389403, \dots\}.$$

6.  $R_{27001,1}^{(4)} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ . This is because  $27001 - 1 = 2^3 \cdot 3^3 \cdot 5^3$ , and none of  $27001^n - 1$  has a factor  $p^3$  for any prime  $p > 5$  for any  $n = 1, 2, 3, 5, 6, 10, 15, 30$ .
7.  $R_{19,1}^{(3)} = \{1, 2, 3, 6, 42, 1806\}$ ? Is this the entire set? Yes, unless  $19^{1806} - 1$  is divisible by  $p^2$  for some prime  $p$  prime to 1806, in which case  $1806p$  would also be in the set. But determining whether or not this is the case seems to be a hard computational problem.
8.  $R_{56,2}^{(4)}$ , an example with  $\gcd(a, b) > 1$ . It seems highly probable that

$$R_{56,2}^{(4)} = (\mathcal{F}_2 \setminus \{2, 4, 8\}) \cup (3\mathcal{F}_2) \\ = 1, 3, 6, 12, 16, 24, 32, 48, 64, 96, 128, 192, 256, 384, 512, 768, 1024, \dots$$

However, in order to prove this, Theorem 15 tells us that we need to know that  $28^{2^\ell} \not\equiv 1 \pmod{p^3}$  for every prime  $p > 3$  and every  $\ell > 0$ . This seems very difficult! Note that  $R_{2,0}^{(4)} = \mathcal{F}_2 \setminus \{2, 4, 8\}$  and  $R_{28,1}^{(4)} = \{1, 3\}$ .

**8. Final Remarks**

1. By finding  $R_{a,b}^{(j)}$ , we are essentially solving the exponential Diophantine equation  $x^j y = a^x - b^x$ , since any solutions with  $x \leq 0$  are readily found.
2. It is known that

$$R_{a,b}^{(1)} = \left\{ n \in \mathbb{N} : n \text{ divides } \frac{a^n - b^n}{a - b} \right\}.$$

See [11, Proposition 12] (and also André-Jeannin [1, Theorem 2] for some special cases.) This result shows that  $R_{a,b}^{(1)} = \{n \in \mathbb{N} : n \text{ divides } u_n\}$ , where the  $u_n$  are the generalized Fibonacci numbers of the first kind defined by the recurrence  $u_0 = 1, u_1 = 1$ , and  $u_{n+2} = (a + b)u_{n+1} - abu_n$  ( $n \geq 0$ ). This provides a link between Theorem 1 of the present paper and the results of [11].

The set  $R_{a,b}^{(1)+}$  is a special case of a set  $\{n \in \mathbb{N} : n \text{ divides } v_n\}$ , also studied in [11]. Here  $(v_n)$  is the sequence of generalized Fibonacci numbers of the second kind. For earlier work on this topic see Somer [13].

3. *Earlier and related work.* The study of factors of  $a^n - b^n$  dates back at least to Euler, who proved that all primitive prime factors of  $a^n - b^n$  were  $\equiv 1 \pmod{n}$ . See [2, Theorem 1]. Chapter 16 of Dickson [4] is devoted to the literature on factors of  $a^n \pm b^n$ .

More specifically, Kennedy and Cooper [8] studied the set  $R_{10,1}^{(1)}$ . André-Jeannin [1, Corollary 4] claimed (erroneously – see Theorem 18) that the congruence  $a^n + b^n \equiv 0 \pmod{n}$  always has infinitely many solutions  $n$  for  $\gcd(a, b) = 1$ .

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