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On determinant representation and integrability of Nekrasov functions



A. Mironov^{a,b,c,*}, A. Morozov^{b,c}

^a Lebedev Physics Institute, Moscow 119991, Russia

^b ITEP, Moscow 117218, Russia

^c Institute for Information Transmission Problems, Moscow 127994, Russia

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ABSTRACT

Conformal blocks and their AGT relations to LMNS integrals and Nekrasov functions are best described by “conformal” (or Dotsenko–Fateev) matrix models, but in non-Gaussian Dijkgraaf–Vafa phases, where different eigenvalues are integrated along different contours. In such matrix models, the determinant representations and integrability are restored only after a peculiar Fourier transform in the numbers of integrations. From the point of view of conformal blocks, this is Fourier transform w.r.t. the intermediate dimensions, and this explains why such quantities are expressed through tau-functions in Miwa parametrization, with external dimensions playing the role of multiplicities. In particular, these determinant representations provide solutions to the Painlevé VI equation. We also explain how this pattern looks in the pure gauge limit, which is described by the Brezin–Gross–Witten matrix model.

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1. Introduction

AGT relations [1], identifying the conformal blocks and the Nekrasov functions, possess different interpretations. The most straightforward and useful one is through properly defined Dotsenko–Fateev-like (DF) integral representations of the conformal blocks [2–5], which can be interpreted as matrix models, with character decompositions [6,7] looking exactly like the Nekrasov sums over representations. Matrix models possess a lot of other nice properties, which can be then transmitted to either the Nekrasov functions or to the conformal blocks, especially at $c = 1$ (i.e. $\beta = 1$). Among these features are the closely related integrability and determinant representations, see [9] for comprehensive reviews and references. Particular pieces of this general pattern are constantly being rediscovered in particular studies of particular questions.

Let us note that the DF representation of the conformal block leads to quite a complicated matrix model: a β -ensemble in the non-Gaussian Dijkgraaf–Vafa (DV) phase [8]. This means that the different eigenvalues are integrated along different contours. In this paper, we restrict ourselves with the case of unit central charge, $c = 1$ in the conformal theory, which means that we deal not with a β -ensemble, but with an ordinary matrix model. In fact, a lot of general properties discussed below persist for the β -ensembles too.

In the Dijkgraaf–Vafa phases of matrix models, the determinant representations and integrability are restored only after a peculiar Fourier transform in the numbers of integrations [10]. From the point of view of conformal blocks, this is Fourier transform w.r.t. (the square roots of) the intermediate dimensions, and this explains why such quantities are expressed through tau-functions in Miwa parametrization, with external dimensions playing the role of multiplicities. Moreover, these determinant representations provide solutions to the Painlevé VI equation.

Strictly speaking, the matrix model representations exist only when two integrability conditions are imposed on the conformal momenta

$$N_1 = \alpha - \alpha_1 - \alpha_2, \quad N_2 = -\alpha - \alpha_3 - \alpha_4 \quad (1)$$

while the conformal block at generic values of the external dimensions is obtained by the analytic continuation. This analytic continuation is immediate for various expansions of the conformal block [11], but not that immediate for determinant representation, since it implies

* Corresponding author.

E-mail addresses: mironov@lpi.ru, mironov@itep.ru (A. Mironov), morozov@itep.ru (A. Morozov).

that determinant can be of a matrix of non-integer size. One possibility to handle this situation is to change a matrix determinant for an infinite-dimensional operator determinant.

This idea was realized on the other side of the AGT story, where there is a long program [12–23] of interpreting linear combinations of conventional conformal blocks in terms of Painlevé τ -functions. Two facts were revealed in these papers: that a Fourier transform of the conformal block in the intermediate conformal momentum admits a Fredholm determinant representation, and that it satisfies the Painlevé VI equation. These claims were actually made only for the case, when the conformal momenta satisfy

$$\alpha_1 \pm \alpha_2 + \alpha \notin \mathbb{Z}, \quad \alpha_1 \pm \alpha_2 - \alpha \notin \mathbb{Z}, \quad \alpha_3 \pm \alpha_4 + \alpha \notin \mathbb{Z}, \quad \alpha_3 \pm \alpha_4 - \alpha \notin \mathbb{Z} \tag{2}$$

which is a kind of complementary to (1) in the matrix model approach. In fact these complicated functional determinants are nothing more than generalizations of the *finite* ones, made from very simple hypergeometric functions, which arise at the “integer” locus (1).

It turns out that these properties persist [13,16,17,23] in the “pure-gauge” limit of AGT relations, which is somewhat peculiar in many respects. Most important, the relevant matrix model is the celebrated Brezin–Gross–Witten (BGW) model [24], which was studied in great detail in [25,26], where it was shown to possess determinant representation in terms of the Bessel functions, see also [27]. In fact, there is an even more interesting matrix model representation of the pure gauge limit, [28] which is, however, different from the framework described in this paper, and deserves a separate discussion.

In the pure gauge limit (PGL), the $\mathcal{N} = 2$ SUSY Yang–Mills theory is no longer conformal, due to dimensional transmutation one trades masses of the hypermultiplets for a new parameter Λ . From the point of view of conformal theory, this corresponds to pushing all external dimensions to infinity, while simultaneously approaching the singularity of the conformal block: this eliminates both external dimensions and puncture positions by a single Λ . Moreover, according to [29], in the PGL, the 4-point spherical and 1-point toric conformal blocks coincide:

$$B_*(\Delta|\Lambda) = \lim_{\substack{\Delta_1, \Delta_2, \Delta_3, \Delta_4 \rightarrow \infty, q \rightarrow 0 \\ q(\Delta_2 - \Delta_1)(\Delta_3 - \Delta_4) \equiv \Lambda^4}} B^{(0)}(\Delta_1, \Delta_2, \Delta_3, \Delta_4; \Delta, c|q) = \lim_{\substack{\Delta_{ext} \rightarrow \infty \\ e^{i\pi\tau} \Delta_{ext}^2 \equiv \Lambda^4}} B^{(1)}(\Delta_{ext}; \Delta, c|e^{i\pi\tau}) \tag{3}$$

As we explained above, the conformal block at $c = 1$ is described by the ordinary matrix model of Penner type in Dijkgraaf–Vafa phase. We discuss the model in section 2. In section 3, we explain that appropriate Fourier transform of the conformal block provides the determinant representation, and in section 4, we discuss its integrability properties. In section 5, we explain that the Fourier transform satisfies the Painlevé VI equation in the both cases of conditions (1) and (2).

In the PGL, the Penner model is substituted by the BGW model, which is less investigated and requires a more detailed exposition. Hence, we remind the Shapovalov and character representations of Nekrasov functions in the PGL in ss. 6 and 7 respectively. In sections 8 and 9, we discuss the integrability of the PGL, and explain that it satisfies the equation Painlevé III. Section 10 contains concluding remarks.

2. Matrix-model description of conformal blocks [5]

As explained in detail in [5], by a suitable adjustment of Dotsenko–Fateev (DF) trick [2] (applying it to holomorphic quantities and making use of open rather than closed integration contours), conformal blocks can be converted into the matrix-model form. Emerging in this way are just the “conformal matrix models” of [3], which are also close to Penner models [30] and which are nowadays naturally called DF-models.

As the simplest example, the 4-point conformal block (=Nekrasov function) in the $c = 1$ CFT [31]

$$B(\Delta_i; \Delta; q) = q^{\Delta - \Delta_1 - \Delta_2} \cdot \left(1 + \frac{(\Delta_2 - \Delta_1 + \Delta)(\Delta_3 - \Delta_4 + \Delta)}{2\Delta} \cdot q + \mathcal{O}(q^2) \right) \tag{4}$$

can be realized via the matrix (eigenvalue) integral Z_{N_1, N_2} [4]

$$\boxed{Z_{N_1, N_2} = \mathfrak{Z} \cdot B(\Delta_i; \Delta; q)} \tag{5}$$

$$Z_{N_1, N_2} = q^{2\alpha_1\alpha_2} (1 - q)^{2\alpha_2\alpha_3} \cdot \frac{1}{N_1! N_2!} \int \prod_i dx_i \Delta^2(x) \prod_i x_i^{2\alpha_1} (1 - x_i)^{2\alpha_2} (q - x_i)^{2\alpha_3} \tag{6}$$

where the normalization factor

$$\mathfrak{Z} = \prod_{i=1}^{N_1} \frac{\Gamma(i)\Gamma(2\alpha_1 + i)\Gamma(2\alpha_2 + i)}{\Gamma(2\alpha - i + 1)} \times \prod_{i=1}^{N_2} \frac{\Gamma(i)\Gamma(2\alpha_3 + i)\Gamma(2\alpha_4 + i)}{\Gamma(-2\alpha - i + 1)} = C_{\alpha_1\alpha_2}^\alpha C_{\alpha_3\alpha_4}^{-\alpha} \tag{7}$$

with the structure constants

$$C_{\alpha_1\alpha_2}^\alpha = \frac{\mathfrak{G}(\alpha + \alpha_1 - \alpha_2 + 1)\mathfrak{G}(\alpha + \alpha_2 - \alpha_1 + 1)\mathfrak{G}(\alpha - \alpha_1 - \alpha_2 + 1)\mathfrak{G}(2\alpha)}{\mathfrak{G}(2\alpha_1 + 1)\mathfrak{G}(2\alpha_2 + 1)\mathfrak{G}(\alpha + \alpha_1 + \alpha_2 + 2)} \tag{8}$$

being nothing but the Selberg integrals [5,7] expressed through the Barnes G -functions $\mathfrak{G}(x)$ [32], and the matrix integral (6) depends on two integers, N_1 and N_2 that count the number of integrations over the contours $C_1 = [0, q]$ and $C_2 = [1, \infty)$ respectively:

$$N_1 = \alpha - \alpha_1 - \alpha_2, \quad N_2 = -\alpha - \alpha_3 - \alpha_4 \tag{9}$$

and

$$\Delta_i = \alpha_i^2, \quad \Delta = \alpha^2 \tag{10}$$

Thus,

$$N = N_1 + N_2 = -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 \equiv -\alpha_{1234} \tag{11}$$

parameterizes the fourth conformal dimension Δ_4 , while N_1 , the intermediate conformal dimension Δ . Generic values of α 's correspond to non-integer N_1 and N , but this analytical continuation is straightforward and unambiguous, because (6) belongs to the class of Selberg integrals, which are ratios of polynomials and thus are well-defined analytical functions of their variables [5,7,11] (see, however, [33,34] for description of more delicate situations).

However, at integer values of N_1 and N_2 , additional structures emerge, whose analytical continuation, though also straightforward is rather ugly. These are determinant formulas underlying integrability properties. We prefer to describe them in the “pure” case, at integer $N_{1,2}$, when the determinants are finite-dimensional and τ -functions look nice and simple. The analytical continuation converts them into functional determinants, for which there is still no nice terminology and commonly accepted condensed notation, thus, one needs to write overloaded and non-transparent explicit formulas, see [12–23] for examples. In the next sections, we present the clear version of this story: at integer values of N_1 and N_2 .

3. Determinant representation of Fourier-transformed matrix models [10]

One can consider instead of Z_{N_1, N_2} the standard N -fold matrix integral with all eigenvalues being integrated over the same contour that is a linear combination of the two contours C_1 and C_2 ,

$$Z_N = q^{2\alpha_1\alpha_2} (1-q)^{2\alpha_2\alpha_3} \cdot \frac{1}{N!} \int_C \prod_i dx_i \Delta^2(x) \prod_i x_i^{2\alpha_1} (1-x_i)^{2\alpha_2} (q-x_i)^{2\alpha_3} \tag{12}$$

with two generating parameters μ_1 and μ_2

$$\int_C = \mu_1 \int_{C_1} + \mu_2 \int_{C_2} \tag{13}$$

This integral is clearly a generation function of Z_{N_1, N_2} :

$$Z_N(\mu_1, \mu_2) = \sum_{N_1, N_2: N_1+N_2=N} \mu_1^{N_1} \mu_2^{N_2} \cdot Z_{N_1, N_2} \tag{14}$$

since the binomial coefficient is cancelled by the normalization factorials in (6) and (12).

For $Z_N(\mu_1, \mu_2)$ there is a determinant representation

$$Z_N(\mu_1, \mu_2) = q^{2\alpha_1\alpha_2} (1-q)^{2\alpha_2\alpha_3} \cdot \det_{1 \leq i, j \leq N} G(i+j-2) \tag{15}$$

where

$$\begin{aligned} G(k) &= \mu_1 \int_0^q x^{2\alpha_1+k} (1-x)^{2\alpha_2} (q-x)^{2\alpha_3} dx + \mu_2 \int_1^\infty x^{2\alpha_1+k} (1-x)^{2\alpha_2} (q-x)^{2\alpha_3} dx = \\ &= \mu_1 q^{2\alpha_{12}+k+1} \mathfrak{B}(2\alpha_1+k+1, 2\alpha_2+1) {}_2F_1(-2\alpha_3, 2\alpha_1+k+1; 2\alpha_{12}+k+2; q) + \\ &\quad + \mu_2 \mathfrak{B}(-2\alpha_{123}-k-1, 2\alpha_3+1) {}_2F_1(-2\alpha_{123}-k-1, -2\alpha_2; -2\alpha_{12}-k; q) \end{aligned} \tag{16}$$

where $\mathfrak{B}(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is the standard Beta-function [35], and the first and the second terms at the r.h.s. of (16) are obtained by taking integrals of $x^{2\alpha_1} (1-x)^{2\alpha_2} (q-x)^{2\alpha_3}$ over C_1 and C_2 respectively.

In fact, the same trick with Fourier transform in the multiplicities N_i of contour integrations is applicable to description of DV phases of generic β -ensembles, i.e. to conformal blocks with $c \neq 1$.

4. Toda chain equation in Miwa variables

It is well known since [36,37] that the determinant (15) is a Toda chain τ -function, see [9] for detailed explanations. More exactly, if one considers a generic matrix model

$$Z_N = \frac{1}{N!} \int \prod_i df(x_i) \Delta^2(x) \prod_i e^{\sum_k t_k x_i^k} \tag{17}$$

with an arbitrary measure $f(x)$, then Z_N is a Toda chain τ -function with N being the discrete Toda time variable, it has the determinant representation

$$Z_N = \det_{1 \leq i, j \leq N} C_{i+j-2}, \quad C_k \equiv \int df(x) x^k \exp\left(-\frac{\mu x^2}{2} + \sum_k t_k x^k\right) \tag{18}$$

and Z_N satisfies the equations of the (forced) Toda chain hierarchy, the first of which is

$$Z_N \partial^2 Z_N - (\partial Z_N)^2 = Z_{N+1} Z_{N-1} \tag{19}$$

However, these equations are formulated in terms of the ordinary time variables, which are not present in (15). Instead of infinitely many times, G there depends just on three α -parameters, which are associated with the three points $\mu_1 = 0$, $\mu_2 = q$, $\mu_3 = 1$ accordingly. They can be treated either as the measure $f(x)$ in (17), or as the Miwa variables. Let us choose the second option and actually deal with the Toda chain τ -function in terms of Miwa variables (this statement is immediately extended with more Miwa variables to multipoint conformal blocks). Hence, it satisfies the integrable equations (bilinear identities) in Miwa variables which we briefly remind here (see [37,38] for details).

When converted from time to Miwa variables,

$$t_k = \frac{1}{k} \sum_a p_a \mu_a^{-k} \tag{20}$$

the Hirota bilinear equations become 3-term difference equations with respect to the multiplicities p_a [39]:

$$\begin{aligned} &(\mu_a - \mu_b) \tau[p_a, p_b, p_c + 1] \tau[p_a + 1, p_b + 1, p_c] + \\ &+ (\mu_b - \mu_c) \tau[p_a + 1, p_b, p_c] \tau[p_a, p_b + 1, p_c + 1] + \\ &+ (\mu_c - \mu_a) \tau[p_a, p_b + 1, p_c] \tau[p_a + 1, p_b, p_c + 1] = 0 \end{aligned} \tag{21}$$

and, at all unit multiplicities $p_a = 1$, they are solved by

$$\tau = Z_N = \frac{\det_{1 \leq i, j \leq N} \phi_i(\mu_j)}{\Delta(\mu)} \tag{22}$$

with arbitrary set of functions of a single variables $\{\phi_i(\mu)\}$ with asymptotics at large μ : $\phi_i(\mu) \sim \mu^{i-1}$. Transition from (22) to (21) involves taking a singular limit where several p_a variables μ_a coincide. As a byproduct of the study of this limit, one obtains another interesting equation [38]:

$$p_a Z_{N+1}[p_a + 1] Z_{N-1}[p_a - 1] = Z_N^2 \frac{\partial}{\partial \mu_a} \frac{\hat{Z}_N[p_a]}{Z_N[p_a]} \tag{23}$$

where $\hat{\tau}_N$ differs from τ_N in (22) by a substitution $\phi_N(\mu_j) \rightarrow \phi_{N+1}(\mu_j)$ in the last row of the matrix at the r.h.s.

Coming back to the conformal blocks, the function (16) would depend on time variables, if there was a factor $\exp(\sum_{k=1}^{\infty} t_k x^k)$ in the integration measure. Instead, the measure in (16) consists of three factors, and parameters $\alpha_{1,2,3}$ are exactly the multiplicities $p_{1,2,3}$ in (20) with $\mu_1 = 0, \mu_2 = q, \mu_3 = 1$. Note that in (21) the p -variables do not need to be integer, thus arbitrary complex-valued $\alpha_{1,2,3}$ are actually allowed.

Thus, we see that **the Fourier-like transform of the conformal block at $c = 1$ w.r.t. the square root α of the internal dimension is the Toda chain τ -function in Miwa parametrization, with square roots of external dimensions playing the role of the multiplicities**. This may be considered a kind of underlying first principle of the observations of [12–23].

5. Painlevé VI equation for Fourier transformed conformal blocks

In this section, we mention the most concrete realization of integrability of conformal blocks: in the case of the four external legs. Namely its Fourier transform

$$Z_N(\eta) = \sum_{k=-\infty}^{\infty} Z_{k, N-k} \cdot e^{ik\eta} \tag{24}$$

satisfies the equation Painlevé VI, this was discovered in [12] as an interpretation of the old result of [40] satisfies the Painlevé VI equation. Here $e^{i\eta} = \frac{\mu_1}{\mu_2}$. Summation over k is actually over $N_1 = \alpha - \alpha_1 - \alpha_2$, and, for integer N_1 and N , it is automatically restricted to the finite segment $0 \leq k \leq N$ due to the factorials (Gamma-functions) in the denominator of (6). Generically one can consider this is a sum over α , which parameterizes the internal dimension $\Delta = \alpha^2$:

$$Z(\alpha, \eta) = \sum_{k=-\infty}^{\infty} z(\alpha + k) \cdot e^{ik\eta} \tag{25}$$

and then it additionally depends on the non-integer part of α . Dependence on external dimensions $\alpha_1, \dots, \alpha_4$ (including not-obligatory-integer $N = -\sum_{i=1}^4 \alpha_i$) is suppressed in this formula.

The Painlevé VI equation is, in fact, a homogeneous equation in Z of degree 4, but in these terms it is rather long. Its condensed form used in [12] is in terms of $\zeta(q) = q(q-1) \frac{\partial \log Z}{\partial q}$ and looks like

$$\left(q(q-1)\zeta'' \right)^2 = -2 \det \begin{pmatrix} 2\alpha_1^2 & q\zeta' - \zeta & \zeta' + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \alpha_4^2 \\ q\zeta' - \zeta & 2\alpha_2^2 & (q-1)\zeta' - \zeta \\ \zeta' + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \alpha_4^2 & (q-1)\zeta' - \zeta & 2\alpha_3^2 \end{pmatrix} \quad (26)$$

where prime denotes a q -derivative (for conformal block q is the position of external leg, when the other two are located at 0, 1 and ∞ , or, in general a double ratio of the four positions). Substitutions $\alpha_4^2 = (\alpha_1 + \alpha_2 + \alpha_3)^2 - \rho$ and $\zeta = \alpha_1\alpha_2(q-1) + \alpha_2\alpha_3q + \xi$ convert (26) into

$$\begin{aligned} & \xi'(q\xi' - \xi) \left((q-1)\xi' - \xi \right) + \left(\frac{q(q-1)\xi''}{2} \right)^2 + 2\alpha_2\rho \left((\alpha_1 - \alpha_3)(q\xi' - \xi) - (\alpha_1 + \alpha_2)\xi' \right) - \alpha_2^2\rho^2 + \\ & + \underbrace{(\rho - (\alpha_1 + \alpha_3)^2)}_{\alpha_2^2 + 2\alpha_2(\alpha_1 + \alpha_3) - \alpha_4^2} (q\xi' - \xi)^2 + \underbrace{(2\alpha_1^2 + 2\alpha_1\alpha_2 + 2\alpha_1\alpha_3 - 2\alpha_2\alpha_3 - \rho)}_{\alpha_1^2 - \alpha_2^2 - 4\alpha_2\alpha_3 - \alpha_3^2 + \alpha_4^2} \xi'(q\xi' - \xi) - (\alpha_1 + \alpha_2)^2(\xi')^2 = 0 \end{aligned} \quad (27)$$

One can easily check that (15) at $N=0$, i.e.

$$Z_{N=0} = q^{2\alpha_1\alpha_2} (q-1)^{2\alpha_2\alpha_3} \quad \text{with} \quad N = - \sum_{i=1}^4 \alpha_i = 0 \quad (28)$$

solves (26): both sides of (26) vanish in this case, while in (27) both $\rho=0$ and $\xi=0$. This trivial solution provides only a “perturbative” prefactor in front of the conformal block at non-zero N . However, one can make a computer check that the first terms of q -expansion (15) at non-zero N , i.e. the Fourier transform (24) of the conformal block at arbitrary N , also satisfy (26) (see also [41]). Moreover, one can also check that (26) is fulfilled iff the coefficients in front of the poly-linear combinations of hypergeometric functions are, indeed, unit, as implied by (14). It is appealing to interpret (15) as a kind of a non-linear transform relating the Painlevé VI and the much simpler hypergeometric equation.

The check that the Fourier transform (25) satisfies (26), which was suggested in [12], is less sophisticated: one just looks for a solution of (26) in the form (first proposed in [40])

$$Z = \sum_k e^{ik\eta} \cdot q^{(\alpha+k)^2 - \alpha_1^2 - \alpha_2^2} \cdot \sum_{i=0} F_i(\alpha+k) \cdot q^i \quad (29)$$

and realizes, term by term in q , that the ratios $F_i(\alpha)/F_0(\alpha)$ are nothing but the coefficients of expansion of the conformal block (4), while $F_0(\alpha)$ is a product of the Barnes G -functions \mathfrak{G} . We comment on this check in a little bit simpler example of the Painlevé III equation in s. 8 below. Two different proofs that (25) satisfies (26) were provided in [15] and [17]. However, they are valid only for the case (2), while the application/extension of these proofs to the mostly interesting case of integer N_1 and N requires some care.

The Painlevé equation looks like a sophisticated non-linear equation of a rather strange form. However, it just a particular example of a set of Toda τ -functions satisfying the usual bilinear Hirota relation [42]

$$\tau_n \partial^2 \tau_n - (\partial \tau_n)^2 = \tau_{n+1} \tau_{n-1}, \quad \partial = q(1-q) \frac{\partial}{\partial q} \quad (30)$$

and often possess determinant representations (see, for example, [43]). In fact, the Painlevé equation can be considered as a counterpart of the string equation, which picks up a distinguished subset of τ -functions, and reflect the *super*-integrability of matrix models. We give a little more details about the interplay between the bilinear and Painlevé equations in discussion of a simpler Painlevé III example in sec. 9 below, which is associated with the pure gauge limit (PGL) of conformal blocks. To describe the PGL at the level of the Painlevé equations, one makes a slightly different substitution $\zeta = \alpha_1\alpha_2(q-1) + \alpha_2\alpha_3q + (\alpha_1 + \alpha_2)^2 - \xi$ which converts (26) into

$$\begin{aligned} & \frac{1}{4} \left(q(q-1)\xi'' \right)^2 - \xi'(q\xi' - \xi) \left((q-1)\xi' - \xi \right) - 2\alpha_2(\alpha_1 - \alpha_3)\rho(q\xi' - \xi) - \alpha_2^2\rho^2 + \\ & + \underbrace{(\rho - (\alpha_1 + \alpha_3)^2)}_{\alpha_2^2 + 2\alpha_2(\alpha_1 + \alpha_3) - \alpha_4^2} (q\xi' - \xi)^2 + \underbrace{(2\alpha_1^2 + 2\alpha_1\alpha_2 - 2\alpha_1\alpha_3 + 2\alpha_2\alpha_3 + \rho)}_{\alpha_1^2 - \alpha_2^2 - 4\alpha_2\alpha_3 - \alpha_3^2 + \alpha_4^2} \xi'(q\xi' - \xi) + \\ & \quad - 2\alpha_2\rho(\alpha_1 + \alpha_2)^2(\alpha_1 - \alpha_3) - \underbrace{(\alpha_1^2 - \alpha_2^2)(\alpha_3^2 - \alpha_4^2)}_{\alpha_2^2 + 2\alpha_2(\alpha_1 + \alpha_3) - \alpha_4^2} \xi' = 0 \end{aligned} \quad (31)$$

Underlined are the terms of the order q^{-2} or $\alpha^4 q^{-1}$, which survive in the pure gauge limit (PGL), when $\alpha_{1,2,3,4} \rightarrow \infty$ and $q = \frac{t}{(\alpha_1^2 - \alpha_2^2)(\alpha_3^2 - \alpha_4^2)} \rightarrow 0$ with finite t .

6. PGL from Virasoro representation theory

We now switch to the theory in the pure gauge limit (PGL). This is quite straightforward at the level of conformal blocks.

According to [44], the 4-point block can be expressed through inverse **Shapovalov matrix** with $\Delta = \frac{n^2}{4}$:

$$B_* = \sum_{n=0}^{\infty} \Lambda^{4n} \cdot Q_{\Delta}^{-1}([1^n], [1^n]) \quad (32)$$

(which allows one to treat it as a norm of the peculiar Gaiotto state [45]).

For the sake of convenience, we list the first entries of Shapovalov matrix in the Appendix, where boxed are the matrix elements contributing to the conformal block in the PGL. This gives the answer

$$Z_*^{(1)} = 1 + \frac{2\Lambda^4}{n^2} + \frac{(2n^2 + 1)\Lambda^8}{n^2(n^2 - 1)^2} + \frac{2(2n^4 - 5n^2 + 12)\Lambda^{12}}{3n^2(n^2 - 1)^2(n^2 - 4)^2} + \frac{(4n^8 - 52n^6 + 243n^4 - 177n^2 + 324)\Lambda^{16}}{6n^4(n^2 - 1)^2(n^2 - 4)^2(n^2 - 9)^2} + O(\Lambda^{20}) \quad (33)$$

For further applications, (33) can be re-expanded as

$$Z_*^{(1)} = 1 + \frac{2\Lambda^4}{n^2} + \left(\frac{1}{n^2} + \frac{3}{4(n+1)^2} + \frac{3}{4(n-1)^2} + \frac{5}{4(n+1)} - \frac{5}{4(n-1)} \right) \Lambda^8 + \left(\frac{1}{2n^2} + \frac{1}{6(n+1)^2} + \frac{1}{6(n-1)^2} + \frac{1}{36(n+2)^2} + \frac{1}{36(n-2)^2} + \frac{17}{54(n+1)} - \frac{17}{54(n-1)} + \frac{7}{108(n+2)} - \frac{7}{108(n-2)} \right) \Lambda^{12} + O(\Lambda^{16}) \quad (34)$$

or

$$Z_*^{(1)} = 1 + \frac{2\Lambda^4}{n^2} + \left(\frac{1}{n^2} - \frac{1}{n^2 - 1} + \frac{3}{(n^2 - 1)^2} \right) \Lambda^8 + \left(\frac{1}{2n^2} - \frac{8}{27(n^2 - 1)} + \frac{2}{3(n^2 - 1)^2} - \frac{11}{54(n^2 - 4)} + \frac{4}{9(n^2 - 4)^2} \right) \Lambda^{12} + \left(\frac{235}{32 \cdot 81n^2} + \frac{1}{24n^4} - \frac{403}{256 \cdot 27(n^2 - 1)} + \frac{19}{64 \cdot 3(n^2 - 1)^2} - \frac{25}{32 \cdot 27(n^2 - 4)} + \frac{1}{18(n^2 - 4)^2} - \frac{71}{256 \cdot 81(n^2 - 9)} + \frac{5}{64 \cdot 9(n^2 - 9)^2} \right) \Lambda^{16} + O(\Lambda^{20}) \quad (35)$$

In this form it can be used to interpret the Fourier transform of conformal block n as s series

$$\text{FT}(\text{conf.block}) = \left(1 + \sum_{k=0} \frac{\Lambda^{4k+4} \beta_k(\Lambda^4)}{\mathbf{d} - k^2} + \sum_{k=0} \frac{\Lambda^{4k+4} \gamma_k(\Lambda^4)}{(\mathbf{d} - k^2)^2} \right) \Theta \quad (36)$$

with $\mathbf{d} = d/d \log(\Lambda^4)$ and Θ depends on the choice of the $U(1)$ -prefactor in front of (33), but we will use a slightly different method in s. 8 below.

7. PGL via unitary models

As further explained in [29], this quantity can be alternatively expressed as a **BGW matrix model**

$$Z_{BGW}(n|\Psi) = \frac{1}{\text{Vol}_{\beta}(n)} \int [dU]_{\beta} e^{\beta(\text{Tr } U^{\dagger} + \text{Tr } \Psi U)} \quad (37)$$

where β refers to a β -deformation of unitary integrals and volumes, which we do not need below, because will actually deal only with the case of $\beta = 1$. The measure $[dU]$ is normalized to unity: $\int [dU] = 1$.

At $\beta = 1$,

$$Z_*^{(1)} = \int [dU] \int [dV] Z_{BGW}(m_+|U) Z_{BGW}(m_-|V) \det(1 - \Lambda^4 U^{\dagger} \otimes V^{\dagger})^2 = \sum_{R,Q} \frac{d_R^2}{D_R(m_+)} \cdot \frac{d_Q^2}{D_Q(m_-)} \cdot \sum_{X,Y} (-\Lambda^4)^{|X|+|Y|} \left(\int [dU] \chi_R[U] \chi_X[U^{\dagger}] \chi_Y[U^{\dagger}] \right) \left(\int [dV] \chi_Q[V] \chi_{X^{tr}}[V^{\dagger}] \chi_{Y^{tr}}[V^{\dagger}] \right) \quad (38)$$

where χ_R are the characters of the linear group (the Schur polynomials). Here we used that

$$\det(1 - \Lambda^4 \cdot U \otimes V) = \sum_R (-1)^{|R|} \chi_R(U) \chi_R^{tr}(V) \quad (39)$$

where X^{tr} denotes the conjugated Young diagram, and the character expansion of (37) valid at $\beta = 1$:

$$Z_{BGW}(n|U) = \sum_R \frac{d_R^2}{D_R(n)} \chi_R(U) \tag{40}$$

Since

$$\chi_X \chi_Y = \sum_Z C_{Z,X,Y} \cdot \chi_Z \tag{41}$$

and

$$\int_{n \times n} [dU] \chi_R(U) \chi_Z(U^\dagger) = \delta_{R,Z} \cdot \theta(n - l_R) \quad \theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{42}$$

the matrix integral (38) is actually equal to

$$Z_*^{(1)} = \sum_{R,Q} (-\Lambda^4)^{|R|} \delta_{|R|,|Q|} \cdot K_{RQ} \cdot \frac{d_R^2 d_Q^2}{D_R(m_+) D_Q(m_-)} \tag{43}$$

with

$$K_{RQ} = \sum_{X,Y} C_{R,X,Y} C_{Q,X^{tr},Y^{tr}} \tag{44}$$

Note that, since at $\beta = 1$ the (analytically continued) “size” $m_- = -m_+ = -n$ is negative, the expansion (43) actually involves a transposed matrix K defined as a reflection w.r.t. the vertical axis:

$$\boxed{Z_*^{(1)} \stackrel{\beta=1}{=} \sum_{R,Q} \Lambda^{4|R|} \delta_{|R|,|Q|} \cdot K_{RQ} \cdot \frac{d_R^2 d_Q^2}{D_R(n) D_{Q^{tr}}(n)} = \sum_{R,Q} \Lambda^{4|R|} \delta_{|R|,|Q|} \cdot K_{RQ}^{tr} \cdot \frac{d_R^2 d_Q^2}{D_R(n) D_Q(n)}} \tag{45}$$

The first examples of the matrices K, K^{tr} can be found in the Appendix. Inserting them into (45) gives

$$\boxed{Z_*^{(1)} = 1 + \frac{2\Lambda^4}{n^2} + \frac{(2n^2 + 1)\Lambda^8}{n^2(n^2 - 1)^2} + \frac{2(2n^4 - 5n^2 + 12)\Lambda^{12}}{3n^2(n^2 - 1)^2(n^2 - 4)^2} + \frac{(4n^8 - 52n^6 + 243n^4 - 177n^2 + 324)\Lambda^{16}}{6n^4(n^2 - 1)^2(n^2 - 4)^2(n^2 - 9)^2} + O(\Lambda^{20}) = (33)}$$

At $\beta \neq 1$ ($c \neq 1$), the formulas are a little more involved, and the corresponding characters are the Jack rather than the Schur polynomials.

One can easily obtain explicit formulas for the matrices K_{RQ} and K_{RQ}^{tr} using the expansion of the characters (the Schur polynomials) in accordance with the Frobenius formula

$$\chi_R(U) = \sum_{\Delta} \frac{\psi_R(\Delta)}{z_{\Delta}} \prod_i \text{Tr } U^{\delta_i} = \sum_{\Delta} d_R \varphi_R(\Delta) \prod_i \text{Tr } U^{\delta_i} \tag{46}$$

where Δ is the Young diagram with $l(\Delta)$ lines with lengths $\delta_1 \geq \delta_2 \geq \dots \delta_{l(\Delta)} \geq 0$ so that $|\Delta| = |R|$, $\psi_R(\Delta)$ is the character of the symmetric group $S_{|R|}$, and z_{Δ} is the standard symmetric factor of the Young diagram Δ (order of automorphism) [46]. One will also need the orthogonality relations

$$\sum_R \psi_R(\Delta_1) \psi_R(\Delta_2) = z_{\Delta_1} \delta_{\Delta_1, \Delta_2} \tag{47}$$

$$\sum_R \psi_R(\Delta_1) \psi_{R^{tr}}(\Delta_2) = (-1)^{l(\Delta_1) + |\Delta_1|} z_{\Delta_1} \delta_{\Delta_1, \Delta_2} \tag{48}$$

in order to obtain

$$K_{RQ} = (-1)^{|R|} \sum_{\Delta_1, \Delta_2: |\Delta_1| + |\Delta_2| = |R|} (-1)^{l(\Delta_1 + \Delta_2)} \frac{\psi_R(\Delta_1 + \Delta_2) \psi_Q(\Delta_1 + \Delta_2)}{z_{\Delta_1} z_{\Delta_2}} \tag{49}$$

$$K_{RQ}^{tr} = \sum_{\Delta_1, \Delta_2: |\Delta_1| + |\Delta_2| = |R|} \frac{\psi_R(\Delta_1 + \Delta_2) \psi_Q(\Delta_1 + \Delta_2)}{z_{\Delta_1} z_{\Delta_2}}$$

Here the sum of two Young diagrams $\Delta_1 + \Delta_2$ is defined to be an ordered set of union of lines of the two diagrams and summation includes $\Delta_1 = \emptyset$ and $\Delta_2 = \emptyset$.

A more interesting question is what is a non-technical reason for Shapovalov and Littlewood–Richardson formalisms to give the same answers, it remains beyond the scope of the present paper.

8. Fourier transform and Painlevé III

Since integrability behind the conformal blocks (15) gets explicit only after the Fourier transform in the internal α -parameter, one can expect the same to happen in the pure gauge limit. This expectation is indeed true as observed recently in [16,17,23]. In what follows, we describe our understanding of this story.

That is, the Fourier transform of (33), the PGL of conformal block satisfies the equation Painlevé III, which can be written in many different forms [20]. In the PGL, (27) turns into

$$\frac{1}{4}(t\ddot{\zeta})^2 + \dot{\zeta}^2(t\dot{\zeta} - \zeta) = -\dot{\zeta} \tag{50}$$

where dot denotes the derivative w.r.t. $t = \Lambda^4$ and $\zeta = t \frac{d}{dt} \log Z$. In fact, this is a quartic homogeneous equation in Z :

$$\begin{aligned} t^4(Z^2\ddot{Z}^2 + 4Z\ddot{Z}^3 - 6Z\dot{Z}\ddot{Z}\ddot{Z} - 3\dot{Z}^2\ddot{Z}^2 + 4\dot{Z}^3\ddot{Z}) + 4t^3(Z^2\ddot{Z}\ddot{Z} - Z\dot{Z}\ddot{Z}^2 - Z\dot{Z}^2\ddot{Z} + \dot{Z}^3\ddot{Z}) + 4t^2(Z^2\dot{Z}^2 - Z\dot{Z}^2\dot{Z}) = \\ = 4t(Z^2\dot{Z}^2 - Z^3\ddot{Z}) - 4Z^3\dot{Z} \end{aligned} \tag{51}$$

(note that the two sides are also homogeneous in t , but of different degrees, -2 at the l.h.s. and -1 at the r.h.s.).

Following [13,16,17,23,40], we look for its solution in the form of a Fourier transform (25) of some series $\sum_i F_i(a) \cdot t^i$:

$$Z = \sum_{k \in \mathbb{Z}} t^{(a+k)^2} \cdot \left(\sum_{i=0} F_i(a+k) \cdot t^i \right) \tag{52}$$

Because of the presence of a , which does not need to be integer, in the exponential, this is actually a double series in integer powers of two independent parameters t and t^{2a} . Thus vanishing should be all the coefficients of this *double* expansion, i.e. coefficients in front of any $t^{4a^2 \pm 2k_1 a + k_2}$ with $k_1, k_2 \in \mathbb{Z}_{\geq 0}$. This imposes an enormously big set of constraints on the functions $F_i(a)$, but it has a solution. To illustrate how this works, consider, for example, the coefficient in front of t^{4a^2+2a} to see that

$$F_1(a) = \frac{1}{2a^2} F_0(a) \tag{53}$$

It simultaneously cancels the coefficient of t^{4a^2+4a+1} . Similarly, looking at t^{4a^2} , one obtains that

$$F_0(a+1)F_0(a-1) = \left[\frac{1}{4a^2(4a^2-1)} \right]^2 F_0(a)^2 \tag{54}$$

The same condition cancels the coefficient in front of t^{4a^2+1} . The next degrees already give a condition for $F_2(a)$: vanishing the coefficient in front of t^{4a^2+2a+1} gives

$$F_2(a) = \frac{(8a^2+1)}{4a^2(4a^2-1)^2} F_0(a) \tag{55}$$

which simultaneously guarantees cancelling of t^{4a^2+2} and t^{4a^2+4a+2} . Further, one can determine from the coefficient of t^{4a^2+2a+2} that

$$F_3(a) = \frac{8a^4 - 5a^2 + 3}{24a^2(a-1)^2(2a+1)^2(2a-1)^2(a+1)^2} F_0(a) \tag{56}$$

Note that for these calculations it was sufficient to keep only Fourier modes with $k = 0, \pm 1, \pm 2$ in (52). Note also that (54) implies that

$$F_0(a) = \frac{1}{\mathfrak{G}(1+2a)\mathfrak{G}(1-2a)} \tag{57}$$

As soon as the polynomial in front of t^{4a^2+2ka} vanishes at each k separately, one obtains that the equation (51) is satisfied by a more general function

$$Z = \sum_{k \in \mathbb{Z}} t^{(a+k)^2} \sum_{i=0} F_i(a+k) \cdot t^i \cdot e^{ik\eta} \tag{58}$$

It remains to note that the coefficients of the expansion (53), (55), (56) coincide with those of $Z_*^{(1)}$ in (33), i.e. finally the solution to the equation (51) can be written as

$$Z = \sum_{k \in \mathbb{Z}} \mathfrak{Z}(a+k) e^{ik\eta}, \quad \mathfrak{Z} = \frac{t^{a^2} Z_*^{(1)}(n=2a)}{\mathfrak{G}(1+2a)\mathfrak{G}(1-2a)} \tag{59}$$

or the Fourier transform of **the PGL conformal block satisfies the Painlevé III equation**, as claimed in [13,16,17,23].

9. Relation to Toda integrability

Our last task in the present paper is to explain what has Painlevé III to do with the ordinary KP/Toda integrability, typical for the eigenvalue matrix models.

For this we note that the homogeneity of the equation (51) makes it much similar to the Hirota equation. However, the Hirota equation is bilinear, while (51) is quartic. Hence, one may expect that there is a bilinear Bäcklund transformation to another function Z_1 , in an analogy with the mKdV case, when the standard infinite set of Hirota equations for the Toda τ -function

$$\tau_n \partial^2 \tau_n - (\partial \tau_n)^2 = \tau_{n+1} \tau_{n-1} \quad (60)$$

reduces to a pair of equations

$$\begin{aligned} \tau_0 \partial^2 \tau_0 - (\partial \tau_0)^2 &= \tau_1^2 \\ \tau_1 \partial^2 \tau_1 - (\partial \tau_1)^2 &= \tau_0^2 \end{aligned} \quad (61)$$

provided by the reduction

$$\tau_{n+2} = \tau_n \quad (62)$$

Indeed, it turns out that the equation (51) can be rewritten in a much similar form of two equations [20]

$$\begin{aligned} Z \partial^2 Z - (\partial Z)^2 &= t Z_1^2 \\ Z_1 \partial^2 Z_1 - (\partial Z_1)^2 &= Z^2 \end{aligned} \quad (63)$$

with a Bäcklund transformed function Z_1 . The derivative here is taken w.r.t. $\log t$: $\partial = \frac{\partial}{\partial \log t}$. Because of this one can easily multiply Z or Z_1 by powers of t , and change the pair of coefficients at the r.h.s. from $(t, 1)$ to, say $(t^{1/2}, t^{1/2})$. Moreover, one can obtain such Z from a quasi-periodic solution¹ to the Toda chain (60) with the periodicity condition $\tau_{n+2} = \sqrt{t} \cdot \tau_n$, then $Z = \tau_1$ and $Z_1 = t^{1/4} \cdot \tau_0$. To prove the equivalence of (61) to (50) one needs to express Z_1 through Z from the first equation:

$$Z_1^2 = \frac{1}{t} Z^2 \cdot \partial^2 \log Z = Z^2 \cdot \frac{\partial \zeta}{\partial t} \quad (64)$$

and then substitute into the second equation. Emerging equation differs from (51), in particular, it contains several terms with the fourth derivative of Z . They can be easily eliminated, because the derivative of (50) factorizes:

$$\left(t^2 \zeta'''' + t \zeta'' + 6t \zeta'^2 - 4\zeta \zeta' + 2 \right) \cdot \zeta = 0 \quad (65)$$

This allows one to express ζ'''' and thus the fourth derivative of Z through lower derivatives and check that (63) is indeed equivalent to (51). For Z of the form (58), the Bäcklund transform (64) implies:

$$\begin{aligned} Z_1 = t^{a^2} \left[\frac{C_0 F_0(a)}{a} \left(1 + \frac{2(4a^2 + 1)}{(4a^2 - 1)^2} t + \dots \right) + a C_1 F_0(a+1) t^{2a} \left((2a+1)^2 + \frac{2(4a^2 - 8a + 1)}{(2a-1)^2} t + \dots \right) + \right. \\ \left. + a C_1 F_0(a-1) t^{-2a} \left((2a-1)^2 + \frac{2(4a^2 + 8a + 1)}{(2a+1)^2} t + \dots \right) + \right. \\ \left. + a^3 (2a+1)^2 C_2 t^{4a} \frac{F_0(a+1)^2}{F_0(a)} \left((2a+1)^2 + \frac{128(4a^2 - 16a + 1)}{(2a-1)^2} t + \dots \right) + \right. \\ \left. + a^3 (2a-1)^2 C_2 t^{-4a} \frac{F_0(a-1)^2}{F_0(a)} \left((2a-1)^2 + \frac{128(4a^2 + 16a + 1)}{(2a+1)^2} t + \dots \right) + \dots \right] \end{aligned} \quad (66)$$

Here the constants C_i are expressed through the root x of the equation $49x^4 - 308x^3 + 580x^2 - 128x + 2 = 0$:

$$\begin{aligned} C_0 &= \sqrt{x} \frac{29400x^3 - 185927x^2 + 355476x - 90512}{17084} \\ C_1 &= \sqrt{x} \frac{5635x^3 - 37380x^2 + 77102x - 28168}{4271} \\ C_2 &= 4\sqrt{x} \\ &\dots \end{aligned} \quad (67)$$

¹ It can be also obtained as an automodel solution of the sine-Gordon or 2-periodic two-dimensional Toda equation [20].

10. Conclusion

In this paper, we reminded a piece of the old theory from [9] and [4,5] and once again emphasized the importance of matrix model techniques for modern studies of the AGT relations and other subjects in representation theory. That is, we explained that because conformal blocks are described by non-trivial (Dijkgraaf–Vafa [8]) phases of conformal matrix models, their integrability can be seen only after a Fourier transform in (square roots of) the intermediate dimensions. We also stressed that the Painlevé equations, which these Fourier transforms were discovered to satisfy in [12–23], can in fact be naturally embedded (which is, in no way, a surprise) into the KP/Toda context usual for matrix models. Moreover, we argued that not only a rather formal Fourier transform of [12], but also the very explicit one, necessarily emerging from the matrix model representation of the conformal block and expressed as a poly-linear combination of hypergeometric functions, satisfies the same Painlevé VI equation. This reveals a connection between the Painlevé and hypergeometric equations which deserves separate investigation.

Further, we considered the pure gauge limit addressed also in [13,16,17,23]. What we especially like about the recent [23] is that it addresses the subject actually related to the Brezin–Gross–Witten (BGW) model, which does not attract attention it deserves, especially, among other matrix models. Hopefully, [23] together with our comments in the present paper would help to change this attitude. From the point of view of Painlevé theory, this case is even simpler, because emerging is the Painlevé III equation rather than the usual Painlevé VI one. For all these reasons, we thoroughly considered the PGL example in the present paper. For further developments in the theory of BGW matrix models, see [26]. For an alternative matrix model description, see [28].

Other obvious next steps in the study include:

- Generalization from 4-point to arbitrary conformal blocks, which is absolutely immediate in terms of determinant representations and integrability, however, counterparts of the Painlevé equations still need to be found (cf. [18]);
- Generalization to $c \neq 1$: as mentioned at the end of s. 3, determinant representation is lost, but the idea of Fourier transform survives and it remains to work out a language adequate for application to β -ensembles;
- Further generalization to the balanced networks and other DIM-related models of [47], which is related to the $5d$ generalizations of the AGT correspondence [48] and q -Painlevé [19,49,50];
- Generalization to elliptic/toric conformal blocks of [51,52].

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Appendix A

In this Appendix, we list the Shapovalov matrices and the K -, K^{tr} -matrices at the first 4 levels. These formulas are necessary for reproducing the first terms of expansion of the PGL conformal block.

The Shapovalov matrices and their inverse

$$\begin{aligned}
 \text{level 1:} \quad & Q = 2\Delta \quad Q^{-1} = \frac{1}{2\Delta} = \boxed{\frac{2}{n^2}} \\
 \text{level 2:} \quad & Q = \frac{1}{2} \left(\begin{array}{c|cc} & [2] & [1, 1] \\ \hline & & \\ [2] & 2n^2 + 1 & 3n^2 \\ \hline [1, 1] & 3n^2 & n^2(n^2 + 2) \end{array} \right) \quad Q^{-1} = \frac{1}{n^2(n^2 - 1)} \left(\begin{array}{c|cc} & [2] & [1, 1] \\ \hline & & \\ [2] & n^2(n^2 + 2) & -3n^2 \\ \hline [1, 1] & -3n^2 & \boxed{2n^2 + 1} \end{array} \right) \\
 \text{level 3:} \quad & Q = \frac{1}{4} \left(\begin{array}{c|ccc} & [3] & [2, 1] & [1, 1, 1] \\ \hline & & & \\ [3] & 2(3n^2 + 4) & 8(2n^2 + 1) & 24n^2 \\ [2, 1] & 8(2n^2 + 1) & 2n^4 + 35n^2 + 8 & 9n^2(n^2 + 4) \\ \hline [1, 1, 1] & 24n^2 & 9n^2(n^2 + 4) & 3n^2(n^2 + 2)(n^2 + 4) \end{array} \right) \\
 & Q^{-1} = \frac{1}{3n^2(n^2 - 1)^2(n^2 - 4)^2} \left(\begin{array}{c|ccc} & [3] & [2, 1] & [1, 1, 1] \\ \hline & & & \\ [3] & 2(n^2 - 1)^2(n^2 + 4)(n^2 + 8) & -16(n^2 - 1)^2(n^2 + 4) & 32(n^2 - 1)^2 \\ [2, 1] & -16(n^2 - 1)^2(n^2 + 4) & 6n^6 + 44n^4 - 96n^2 + 64 & -18n^4 + 32n^2 - 32 \\ \hline [1, 1, 1] & 32(n^2 - 1)^2 & -18n^4 + 32n^2 - 32 & \boxed{4n^4 - 10n^2 + 24} \end{array} \right)
 \end{aligned}$$

...

Boxed are the matrix elements, contributing to conformal block in the PGL.

Matrices K and K^{tr}

level 1 : $K_{[1],[1]} = 2$

$$\text{level 2 : } K = \left(\begin{array}{c|cc} & [2] & [1, 1] \\ \hline [2] & 1 & 3 \\ \hline [1, 1] & 3 & 1 \end{array} \right) \quad K^{tr} = \left(\begin{array}{c|cc} & [2] & [1, 1] \\ \hline [2] & 3 & 1 \\ \hline [1, 1] & 1 & 3 \end{array} \right)$$

$$\text{level 3 : } K = \left(\begin{array}{c|ccc} & [3] & [2, 1] & [1, 1, 1] \\ \hline [3] & 0 & 2 & 4 \\ [2, 1] & 2 & 6 & 2 \\ \hline [1, 1, 1] & 4 & 2 & 0 \end{array} \right) \quad K^{tr} = \left(\begin{array}{c|ccc} & [3] & [2, 1] & [1, 1, 1] \\ \hline [3] & 4 & 2 & 0 \\ [2, 1] & 2 & 6 & 2 \\ \hline [1, 1, 1] & 0 & 2 & 4 \end{array} \right)$$

$$\text{level 4 : } K = \left(\begin{array}{c|ccccc} & [4] & [3, 1] & [2, 2] & [2, 1, 1] & [1, 1, 1, 1] \\ \hline [4] & 0 & 0 & 1 & 3 & 5 \\ [3, 1] & 0 & 4 & 3 & 9 & 3 \\ [2, 2] & 1 & 3 & 6 & 3 & 1 \\ [2, 1, 1] & 3 & 9 & 3 & 4 & 0 \\ \hline [1, 1, 1, 1] & 5 & 3 & 1 & 0 & 0 \end{array} \right)$$

$$K^{tr} = \left(\begin{array}{c|ccccc} & [4] & [3, 1] & [2, 2] & [2, 1, 1] & [1, 1, 1, 1] \\ \hline [4] & 5 & 3 & 1 & 0 & 0 \\ [3, 1] & 3 & 9 & 3 & 4 & 0 \\ [2, 2] & 1 & 3 & 6 & 3 & 1 \\ [2, 1, 1] & 0 & 4 & 3 & 9 & 3 \\ \hline [1, 1, 1, 1] & 0 & 0 & 1 & 3 & 5 \end{array} \right)$$

...

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