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Multilateral Bargaining in Networks: On the Prevalence of Inefficiencies

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We introduce a new noncooperative multilateral bargaining model for network-restricted environments in which players can bargain only with their neighbors. The main theorem characterizes a condition on network structures for efficient equilibria. If the underlying network is either *complete* or *circular*, an efficient stationary subgame perfect equilibrium exists for *all* discount factors – all the players always try to reach an agreement as soon as practicable, and hence no strategic delay occurs. In *any other* network, however, an efficient equilibrium is impossible if a discount factor is greater than a certain threshold, as some players strategically delay an agreement. We also provide an example of a *Braess-like paradox*, in which network improvements decrease social welfare.

Key words: noncooperative bargaining, coalition formation, cooperation restriction, strategic delay,

Braess's Paradox

Subject classifications: Games/group decisions: Bargaining, Noncooperative; Networks/graphs: Theory

Area of review: Games, Information, and Networks

1. Introduction

Bargaining occasionally requires an agreement among three or more players. Sometimes each player can directly cooperate with all the other players; while other times, some players can cooperate only with some of the other players for some informational, legal, political, or physical reasons. In many U.S. states, for instance, because of franchise laws, automobile manufacturers (such as General Motors and Ford) cannot sell their products directly to customers, but only through independent dealers. Accordingly, manufacturers are banned from using online marketplaces (such as autotrader.com) as their sales channels, and hence they can bargain or negotiate only with their franchised dealers, who can then sell their franchisees' cars through online marketplaces.

Such cooperation restrictions *per se* hinder an immediate agreement among all players. Additional delay, due to players' strategic concerns, may also occur in reaching a final agreement to generate the joint surplus – players may strategically bargain only with some of their neighbors first to postpone negotiating with the others. In the U.S. automobile industry example, each of the franchised dealers can bargain with the manufacturers as well as the online marketplaces, but the dealers may form a trade organization (such as National Automobile Dealers Association) to unify their negotiation channel, before bargaining against other industrial bodies. Although such a strategic delay in realization of the joint surplus yields an inefficient outcome, it could alter how the surplus might be allocated among the players.

To analyze the role of cooperation restriction in bargaining, this paper introduces a noncooperative bargaining model in which each player can bargain only with its directly connected players in a given network. In each period, a randomly selected proposer makes an offer specifying a coalition among the neighbors and monetary transfers to each member in the proposed coalition. If all the members in the coalition accept the offer, then the proposer thereafter controls the coalition including the members' network connections. Otherwise, the offer dissolves. The game repeats until the grand coalition forms, after which the player who controls the grand coalition wins the unit surplus. All the players have a common discount factor over discrete time.

The main result characterizes a condition on network structures for efficient equilibria. If the underlying network is either *complete* or *circular*, then for *any* discount factor there exists an efficient stationary subgame perfect equilibrium. In such an efficient equilibrium, all the players always strive to reach an agreement as soon as practicable. Hence, strategic delay never occurs. In *any other* network, however, strategic delay occurs and an efficient stationary subgame perfect equilibrium is impossible if a discount factor is greater than a certain threshold level – players can take a better position for bargaining in a network by deliberately excluding some of their neighbors from the bargaining partners.

We also provide an interesting example, in which adding a new communication link decreases social welfare. This observation is reminiscent of the *Braess's paradox* (Braess 1968), which refers

to a situation where constructing a new route reduces overall performance when players choose their route selfishly. Analogously in our model, the more links are available, the fewer links are actually used for bargaining, as each player strategically chooses only some of the links. As a result, network improvements may decrease social welfare.¹

In our bargaining model, inefficiency is caused by strategic delay, which has been a central issue in the literature. Many noncooperative bargaining models focus on incomplete information or uncertainty as a source of delay.² The coalitional bargaining literature, including Seidmann and Winter (1998), Okada (2000), Gomes (2005), and Lee (2016), has found that strategic delay occurs even under complete information settings except in cases of unanimity games – when subcoalitions generate a strictly positive surplus, players may form an inefficient coalition as an intermediate bargaining step. In this paper, however, we show that strategic delay is prevalent, even in unanimity games, if cooperation restrictions are imposed.³

The model proposed has two important features, which distinguish it from the existing non-cooperative bargaining models in networks.⁴ First, we allow *strategic coalition formation* so that players can choose their bargaining partners. In the literature of bargaining in networks, however, players are supposed to bargain within a random meeting.⁵ In those models, the players in the random meeting bargain over their joint surplus, but they cannot choose their bargaining partners. Second, we also allow players to *buy out* other players, enabling them to form a coalition gradually. In the literature of noncooperative bargaining, the idea of successive bargaining through buyout was firstly introduced by Gul (1989) for a sequence of bilateral bargainings and by Krishna and Serrano (1996) in which players can bargain with two or more players at the same time.⁶ To the best of our knowledge, this paper is the first to allow multilateral bargaining through coalition formation in networks.⁷

The paper is organized as follows. In Section 2, we introduce a noncooperative multilateral bargaining model for a network-restricted environment. Section 3 provides the main characterization result on efficient equilibria. Section 4 outlines the core part of the proof and further investigates

the players' strategic interactions, with some leading examples. Based on such examples, we discuss a *Braess-like paradox* in Section 5. Lastly, in Section 6, further discussions on the main result for robustness and extensions follow. The missing proofs and the technical verifications for the constructed equilibria are presented in the e-companion to this paper.

2. A Model

2.1. Networks

A *network* (or a graph) $g = (N, E)$ consists of a finite set $N = \{1, 2, \dots, n\}$ of *players* (or nodes) and a set E of *links* (or edges) of N . When $g = (N, E)$ is not the only network under consideration, the notations $N(g)$ and $E(g)$ are used for the player set and the link set rather than N and E to emphasize the underlying network g . Through this paper, we assume that g is simple and connected.⁸

Given $g = (N, E)$ and $S \subseteq N$, the *subgraph* restricted on S is $g_{|S} = (S, \{ij \in E \mid \{i, j\} \subseteq S\})$. The (closed) *neighborhood* of $i \in N$ is given by $N_i(g) \equiv \{j \in N \mid \exists ij \in E\} \cup \{i\}$. The *degree* of i in g , denoted by $\deg_i(g)$, is the cardinality of a set $\{j \in N \mid \exists ij \in E\}$. The (geodesic) *distance* between i and j in g , denoted by $d(i, j; g)$, is the number of links in a shortest path between them. The *diameter* of g , denoted by $\text{diam}(g)$, is the greatest distance between any two players in g .

A coalition $S \subseteq N$ is *dominating* in g if, for all $i \in N$, either $i \in S$ or there exists $j \in S$ such that $ij \in E$. A player $i \in N$ is a *dominating player* in g if $\{i\}$ is a dominating set. Let $D(g)$ be a set of dominating players in g . For any integer $k \geq 2$, a network is *k-regular* if $\deg_i(g) = k$ for all $i \in N(g)$. A network g is *complete* if it is $(|N(g)| - 1)$ -regular, or equivalently if $D(g) = N(g)$. A connected network g is *circular* if it is 2-regular.⁹

2.2. A Noncooperative Bargaining Game

A *noncooperative bargaining game*, or shortly a *game*, is a triple $\Gamma = (g, p, \delta)$, where g is a underlying network, $p \in \mathbb{R}_{++}^{|N|}$ is an initial *recognition probability* with $\sum_{i \in N} p_i = 1$, and $0 < \delta < 1$ is a common discount factor.¹⁰

A game $\Gamma = (g, p, \delta)$ proceeds as follows. In each period, one of the players is randomly selected as a proposer according to p . Then, the proposer i makes an offer; that is, i strategically chooses a

pair (S, y) consisting of a coalition $S \subseteq N_i(g)$ and monetary transfers $\{y_j\}_{j \in S \setminus \{i\}}$. Each respondent $j \in S \setminus \{i\}$ sequentially either accepts the offer or rejects it.¹¹ If any $j \in S \setminus \{i\}$ rejects the offer, then the offer dissolves and all the players repeat the same game in the next period. If each $j \in S \setminus \{i\}$ accepts the offer, then i *buys out* $S \setminus \{i\}$, that is, each respondent $j \in S \setminus \{i\}$ leaves the game with receiving y_j from the proposer i and the remaining players $(N \setminus S) \cup \{i\}$ play the subsequent game $\Gamma^{(i,S)}$ in the next period.¹²

After i buys out $S \setminus \{i\}$, or i forms a coalition S , the subsequent game $\Gamma^{(i,S)} = (g^{(i,S)}, p^{(i,S)}, \delta)$ is defined in the following way:

- i) The induced network $g^{(i,S)} = (N^{(i,S)}, E^{(i,S)})$, where $N^{(i,S)} = (N \setminus S) \cup \{i\}$ and

$$E^{(i,S)} = \{ij \mid (\exists i'j \in E) i' \in S \text{ and } j \in N \setminus S\} \cup \{jk \mid (\exists jk \in E) j, k \in N \setminus S\}. \quad (1)$$

That is, after i 's S -formation, $S \setminus \{i\}$ leaves the network, but i inherits all the network connections from S . Note that, in terms of graph theory, a coalition formation can be viewed as a *vertex contraction* and an induced network is called a *minor* of the original one.

- ii) The induced recognition probability $p^{(i,S)}$ defined on $N^{(i,S)}$:

$$p_j^{(i,S)} = \begin{cases} p_S & \text{if } j = i \\ p_j & \text{if } j \in N \setminus S. \end{cases} \quad (2)$$

That is, the proposer i takes the respondents' chances of being a proposer as well. In Section 6, we also consider an alternative protocol, in which players cannot inherit others' proposal power from buying out.

The game continues until only one last player remains, after which the last player acquires one unit of surplus.

2.3. Coalitional States

A (coalitional) *state* π is a partition of N , specifying a set of *active* (or *remaining*) players $N^\pi \subseteq N$. For each active player $i \in N^\pi$, i 's partition block $[i]_\pi$ represents the player i together with the other

players whom she has previously bought out. Denote π° by the initial state, that is, $N^{\pi^\circ} = N$ and $[i]_{\pi^\circ} = \{i\}$ for all $i \in N$.

A state π is *feasible* in g , if there exists a sequence of coalition formations $\{(i_\ell, S_\ell)\}_{\ell=1}^L$ such that $i_1 \in N$ and $S_{i_1} \subseteq N_{i_1}$; and $i_\ell \in N^{(i_1, S_1) \cdots (i_{\ell-1}, S_{\ell-1})}$ and $S_\ell \subseteq N_{i_\ell}^{(i_1, S_1) \cdots (i_{\ell-1}, S_{\ell-1})}$ for all $\ell = 2, \dots, L$; and $N^\pi = N^{(i_1, S_1) \cdots (i_L, S_L)}$. Let $\Pi(g)$ be a set of all feasible states in g . According to (1) and (2), for each $\pi \in \Pi(g)$, the induced network $g^\pi = (N^\pi, E^\pi)$ and the induced recognition probability p^π are uniquely determined as $E^\pi \equiv \bigcup_{i \in N^\pi} \{ij \mid \exists i' j' \in E \text{ (} i' \in [i]_\pi \text{ and } j' \in [j]_\pi \text{)}\}$ and $p_i^\pi = \sum_{j \in [i]_\pi} p_j$ for $i \in N^\pi$.

When there is no danger of confusion, we omit π° in notations, for instance, $g^{\pi^\circ} = g$, $g^{\pi^\circ(i, S)} = g^{(i, S)}$, and so on. The description of the underlying network g may also be omitted, when it is clear. For notational simplicity, for any $v \in \mathbb{R}^{|N|}$ and any $S \subseteq N$, we denote $v_S = \sum_{j \in S} v_j$.

2.4. Stationary Subgame Perfect Equilibria

We focus on stationary subgame perfect equilibria as usual in the literature of coalitional bargaining.¹³ A stationary strategy depends only on the current coalitional state and the within-period histories, but not the histories of past periods. The existence of a stationary subgame perfect equilibrium is known in the literature including Eraslan (2002) and Eraslan and McLennan (2013). See Lee (2014) for the formal description of stationary strategies, in which players can form intermediate coalitions.

We first consider a special class of stationary strategies, namely *cutoff strategies*. A cutoff strategy profile (\mathbf{x}, \mathbf{q}) consists of a *value profile* $\mathbf{x} = \{\{x_i^\pi\}_{i \in N^\pi}\}_{\pi \in \Pi}$ and a *coalition formation strategy profile* $\mathbf{q} = \{\{q_i^\pi\}_{i \in N^\pi}\}_{\pi \in \Pi}$, where $x_i^\pi \in \mathbb{R}$ and q_i^π is a probability measure in $\{S \mid i \in S \subseteq N_i^\pi\}$ for each $\pi \in \Pi(g)$. A cutoff strategy profile (\mathbf{x}, \mathbf{q}) specifies the behaviors of an active player $i \in N^\pi$ in the following way:

- i) player i proposes $(S, \{\delta x_j^\pi\}_{j \in S \setminus \{i\}})$ with probability $q_i^\pi(S)$; i.e., a proposer chooses bargaining partners according to her coalitional formation strategy and offers them their discounted stationary value;

ii) player i accepts any offer (S, y) with $i \in S$ if and only if $y_i \geq \delta x_i^\pi$; i.e., a respondent accepts any offer, which is no less than his discounted stationary value.

A cutoff strategy profile (\mathbf{x}, \mathbf{q}) induces a probability measure $\mu_{\mathbf{x}, \mathbf{q}}$ on the set of all possible histories. Given history h , let $\tilde{\pi}(h) = \{\pi^t(h)\}_{t=0}^T$ be a sequence of states, which is determined by h . Given (\mathbf{x}, \mathbf{q}) , define the set of *inducible states*:

$$\Pi_{\mathbf{x}, \mathbf{q}}(g) = \{\pi \in \Pi(g) \mid (\exists h \exists t) \mu_{\mathbf{x}, \mathbf{q}}(h) > 0 \text{ and } \pi = \pi^t(h)\}.$$

Given \mathbf{x} , for each $\pi \in \Pi(g)$, $i \in N^\pi$, and $S \subseteq N_i^\pi$, define a player i 's *excess surplus* of S -formation:

$$e_i^\pi(S, \mathbf{x}) = \begin{cases} \delta x_i^{\pi(i, S)} - \delta x_S^\pi & \text{if } S \subsetneq N^\pi \\ 1 - \delta x_{N^\pi}^\pi & \text{if } S = N^\pi. \end{cases}$$

A cutoff strategy profile (\mathbf{x}, \mathbf{q}) induces a *continuation payoff* in π for each active player $i \in N^\pi$:

$$u_i^\pi(\mathbf{x}, \mathbf{q}) = p_i^\pi \sum_{S \subseteq N^\pi} q_i^\pi(S) e_i^\pi(S, \mathbf{x}) + \sum_{j \in N^\pi} p_j^\pi \left(\sum_{S: i \in S \subseteq N^\pi} q_j^\pi(S) \delta x_i^\pi + \delta \left(\sum_{S: i \notin S \subseteq N^\pi} q_j^\pi(S) x_i^{\pi(j, S)} \right) \right). \quad (3)$$

The first term in the right-hand side of (3) means, when player i is recognized as a proposer she will expect a weighed sum of her excess surpluses $e_i^\pi(S, \mathbf{x})$. In addition, as the second term captures, no matter who makes an offer, if she is included in the proposed coalition she earns δx_i^π , she can otherwise still expect her stationary value $x_i^{\pi(j, S)}$ in the next period. Rearranging the above terms, (3) can be written as:

$$u_i^\pi(\mathbf{x}, \mathbf{q}) = p_i^\pi \sum_{S \subseteq N^\pi} q_i^\pi(S) e_i^\pi(S, \mathbf{x}) + \delta \left(\sum_{j \in N^\pi} p_j^\pi \sum_{S \subseteq N^\pi} q_j^\pi(S) \left(\mathbf{1}(i \in S) x_i^\pi + \mathbf{1}(i \notin S) x_i^{\pi(j, S)} \right) \right). \quad (4)$$

As Lemma 1 below shows, any stationary subgame perfect equilibrium can be uniquely represented by a cutoff strategy equilibrium in terms of the players' payoff. Thus, when we are interested in the players' equilibrium payoff or efficiency, without loss of generality, we may only consider cutoff strategy equilibria. Hence, in this paper, an equilibrium refers to a cutoff strategy equilibrium.

LEMMA 1. *For any stationary subgame perfect equilibrium, there exists a cutoff strategy equilibrium, which yields the same equilibrium payoff vector.*

REMARK 1. In a cutoff strategy equilibrium, proposers always make an *acceptable* offer and respondents always accept the offer proposed. The space of cutoff strategies is, however, rich enough to represent all the possible stationary strategies. For instance, player i 's strategy of choosing $S = \{i\}$ represents that the player i cannot make an acceptable offer and that any offer, which is profitable to i must be rejected by others – it can also be viewed that the player declines to be a proposer.

Another benefit of using cutoff strategies is its greater tractability. Lemma 2 characterizes a cutoff strategy equilibrium with two tractable conditions, *optimality* and *consistency*.

LEMMA 2. *A cutoff strategy profile (\mathbf{x}, \mathbf{q}) is a stationary subgame perfect equilibrium if and only if, for all $\pi \in \Pi$ and $i \in N^\pi$, the following two conditions hold,*

$$\text{i) **Optimality:** } q_i^\pi(S) > 0 \implies (\forall S' \subseteq N_i^\pi) e_i^\pi(S, \mathbf{x}) \geq e_i^\pi(S', \mathbf{x}),$$

that is, proposers choose their bargaining partners to maximize their excess surplus;

$$\text{ii) **Consistency:** } x_i^\pi = u_i^\pi(\mathbf{x}, \mathbf{q}),$$

that is, players' stationary value coincides to their continuation payoff.

The proofs of the above lemmas are omitted as they can be found in Lee (2014).

3. The Main Result: Efficient Equilibria

In this section, we characterize a necessary and sufficient condition on network structures for efficient equilibria. First, we formally define efficiency under network restrictions. Given g , define a *maximum coalition formation strategy profile* $\bar{\mathbf{q}} = \{\{\bar{q}_i^\pi\}_{i \in N^\pi}\}_{\pi \in \Pi(g)}$ with

$$\bar{q}_i^\pi(S) = \begin{cases} 1 & \text{if } S = N_i^\pi \\ 0 & \text{otherwise,} \end{cases}$$

that is, for each state $\pi \in \Pi(g)$, each proposer $i \in N^\pi$ chooses a maximum coalition N_i^π to bargain with. Given $\Gamma = (g, p, \delta)$, let $\bar{u}(\Gamma)$ be a maximum welfare. Note that $\bar{u}(\Gamma)$ is obtained by any cutoff strategy profile involving a maximum coalition formation strategy profile. A strategy profile (\mathbf{x}, \mathbf{q}) is *efficient* if

$$\sum_{i \in N} u_i(\mathbf{x}, \mathbf{q}) = \bar{u}(\Gamma).$$

It is worth noting that an efficient strategy profile does not necessarily consist of maximum coalition formation strategies, as the maximum welfare depends only on the total number of periods for the final agreement. Example 1 below shows how non-maximum coalition formation strategies can achieve efficiency.

EXAMPLE 1. Consider a game Γ with a four-player circular network. Under the maximum coalition formation strategy profile, each proposer forms a three-player coalition in the first period, after which the two remaining players continue to bargain. Thus, forming a grand coalition requires two periods and the maximum welfare is $\bar{u}(\Gamma) = \delta$. Instead of forming three-player coalitions, however, the players can also obtain the maximum welfare by forming two-player coalitions in the first period, as the three remaining players can still form a grand coalition in the second period. Thus, such a strategy profile with non-maximum coalition formation also takes two periods to generate the surplus. \square

The following Lemma 3 characterizes the coalition formation strategies, which constitute an efficient equilibrium.

LEMMA 3. *Given $\Gamma = (g, p, \delta)$, an equilibrium (\mathbf{x}, \mathbf{q}) is efficient if and only if, for all $\pi \in \Pi_{\mathbf{x}, \mathbf{q}}(g)$ and all $i \in N^\pi$,*

$$q_i^\pi(S) > 0 \implies (\forall S' \subseteq N_i^\pi) \bar{u}(\Gamma^{\pi(i, S)}) \geq \bar{u}(\Gamma^{\pi(i, S')}).$$

Now we are ready to state the main theorem.

THEOREM 1. *An efficient stationary subgame perfect equilibrium exists for all discount factors if and only if the underlying network is either complete or circular.*

Under network restrictions, players tend to exclude some of the neighbors from their bargaining partners for the purposes of taking a better position for future bargaining in the induced networks. Such strategic behaviors, however, delay the realization of the joint surplus.

On the other hand, if the underlying network is either complete or circular, any coalition formation does not alter the network structure significantly – in terms of graph theory, any *minor* of

a complete (or circular) network is still complete (or circular). When players cannot change their position through coalition formations, the only way to increase their payoff is forming the largest possible coalition in each period in order to reach an agreement as soon as practicable, and hence no strategic delay occurs.

We prove the theorem through the four propositions. For the sufficient condition, we construct an efficient equilibrium in a complete network (Proposition 1) and in a circular network (Proposition 2). For the necessary condition, Proposition 3 shows the inefficiency result in a special class of networks, namely *pre-complete networks*. Proposition 4 then completes the necessary condition by showing that, for any game with an incomplete non-circular network, there must be a sequence of coalition formations, which induces a pre-complete non-circular network.

3.1. The Sufficient Condition: Complete or Circular Networks

In the literature, it is well-known that there is a unique efficient equilibrium in a unanimity game without network restrictions (Chatterjee et al. 1993, Okada 1996), even when gradual coalition formations are allowed (Seidmann and Winter 1998, Okada 2000). Furthermore, players' expected payoffs are the same as their recognition probabilities in *any* equilibrium and in *any* discount factor. Proposition 1 restates this well-known result in our setting.

PROPOSITION 1. *Consider a game (g, p, δ) with a complete network g . For any δ , the payoff vector in any equilibrium equals to the recognition probability p .*

The proof relies on the fact that any induced network of a complete network must be complete. Based on the existence and the uniqueness of an equilibrium in smaller (but still complete) networks, one can prove the result for larger networks using mathematical induction. The complete proof is in the e-companion to this paper.

The uniqueness of equilibrium payoff in a complete network will play an important role later in proving the result of inefficiency in other incomplete networks. It is worth noting that the uniqueness of equilibrium payoff holds even without imposing stationarity on strategies as Krishna and Serrano (1996) shows.¹⁴ In an incomplete network, however, the uniqueness may not hold any

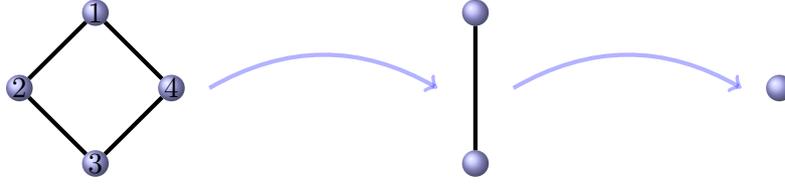


Figure 1 Bargaining in a Circular Network: It takes exactly two periods for a grand coalition in any equilibrium.
See Example 2 and Section 5.

longer even under the stationarity assumption. In a circular network, one can construct an efficient equilibrium for *any* recognition probability and *any* discount factor although it is not necessarily unique.

PROPOSITION 2. Consider a game (g, p, δ) with a circular network g . For any δ , there exists an equilibrium, in which the payoff vector equals to $\delta^{\text{diam}(g)-1}p$.

Recall that, for any game (g, p, δ) with a circular network g , the maximum welfare is $\bar{u}(g, p, \delta) = \delta^{\text{diam}(g)-1}$. That is, in a circular network, each proposer can form a three-player coalition to reduce the diameter by one in each period and hence the total number of periods for a unanimous agreement equals to the diameter of the underlying network. Thus, Proposition 2 shows that the maximum welfare is obtained by an equilibrium. Similarly to Proposition 1, the players' expected payoff is proportional to their recognition probability in the equilibrium constructed.¹⁵

Proposition 2 can also be proved by mathematical induction, as any induced network of a circular network must be circular (except for one- or two-player networks). The complete proof is presented in the e-companion and the following example illustrates the players' incentive on maximum coalition formation.

EXAMPLE 2 (A FOUR-PLAYER CIRCULAR NETWORK). Let g be a four-player circular network as in Figure 1. We show that $(\mathbf{x}, \bar{\mathbf{q}})$ constitutes an equilibrium, where $x = \delta p$ and $x^\pi = p^\pi$ for any π with $2 \leq |N^\pi| \leq 3$. For any $i \in N$ and any $S \subseteq N_i$ such that $|S| \geq 2$, as $g^{(i,S)}$ is complete, Proposition 1 implies $x_i^{(i,S)} = x_S$ and hence $e_i(S, \mathbf{x}) = p_S \delta - \delta x_S = \delta(1 - \delta)p_S$, which strictly increases in p_S . Thus, it follows that $e_i(N_i, \mathbf{x}) > e_i(S, \mathbf{x})$, which confirms the optimality condition. Since player i is

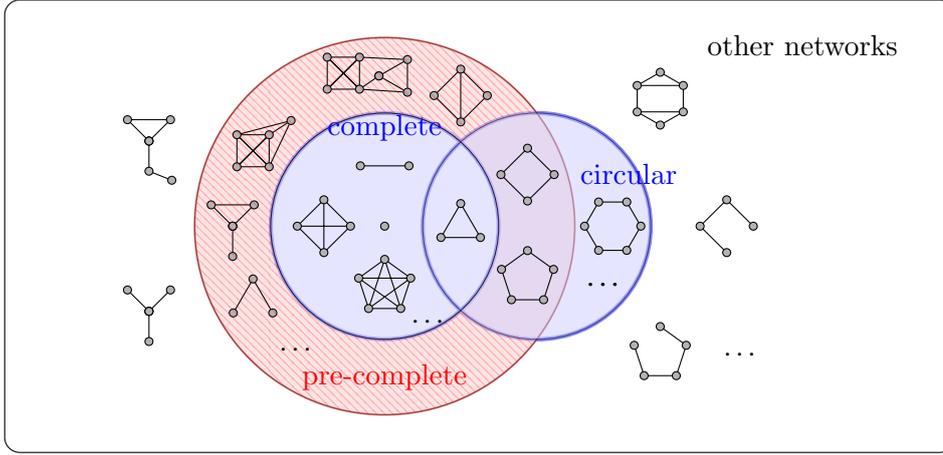


Figure 2 Classification of Networks: In either a complete or a circular network (the shaded area in blue), Proposition 1 and Proposition 2 show that an efficient equilibrium exists for any discount factor. In any other network, however, an efficient equilibrium is impossible for discount factors high enough. Proposition 3 firstly shows this inefficiency result for pre-complete non-circular networks (the slashed area in red). In any incomplete and non-circular, Proposition 4 then finds that there exists a sequence of coalition formations, which induces a pre-complete non-circular network so that delay occurs with a positive probability.

also selected as a bargaining partner as long as her adjacent players become a proposer, we have

$\sum_{\ell \in N} p_{\ell} \sum_{S \ni i} q_{\ell}(S) = p_{N_i}$ and $\sum_{\ell \in N} p_{\ell} \sum_{S \not\ni i} q_{\ell}(S) = 1 - p_{N_i}$. Therefore, i 's expected payoff is:

$$u_i(\mathbf{x}, \bar{\mathbf{q}}) = p_i \cdot \delta(1 - \delta)p_{N_i} + \delta [p_{N_i} \cdot \delta p_i + (1 - p_{N_i}) \cdot p_i] = \delta p_i,$$

which satisfies the consistency condition. For any non-initial state, the proposed strategies constitute an equilibrium due to Proposition 1. \square

3.2. The Necessary Condition: Prevalence of Inefficiency

The challenging part of Theorem 1 is its necessary condition – the impossibility of efficiency equilibria in *any* incomplete non-circular network. First, we prove it in a special class of networks, namely *pre-complete networks*, in which all the players can induce a complete network.

DEFINITION 1. A network g is *pre-complete* if g is not complete, but for all $i \in N(g)$ there exists $S \subseteq N_i(g)$ such that $g^{(i,S)}$ is complete.

Figure 2 illustrates the classification of networks particularly in relation to pre-complete networks. In the class of pre-complete networks, there are only two circular networks, 4-player and 5-player circle. In such networks, as Proposition 2, there exists an efficient equilibrium for any discount factor. In any other pre-complete network, however, Proposition 3 below shows that an efficient equilibrium is impossible for discount factors higher than a certain threshold.

PROPOSITION 3. *Let g be a pre-complete non-circular network. For any p , there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, any efficient strategy profile (\mathbf{x}, \mathbf{q}) cannot be an equilibrium in $\Gamma = (g, p, \delta)$.*

As Proposition 3 is the core of the main result and its proof provides novel insights on strategic interactions in multilateral bargaining, in Section 4 we present the outline of its proof and construct an inefficient equilibrium in some leading examples. Here, a simple example below illustrates why an efficient equilibrium is impossible in a three-player chain.

EXAMPLE 3 (IMPOSSIBILITY OF EFFICIENT EQUILIBRIA IN A CHAIN). Consider a three-player chain with $N = \{1, 2, 3\}$ and $E = \{12, 13\}$. Suppose there exists an efficient equilibrium (\mathbf{x}, \mathbf{q}) . Then player 1 is always included in a proposed coalition, that is, $q_1(N) = q_2(\{1, 2\}) = q_3(\{1, 3\}) = 1$. Thus, player 1's expected payoff is $u_1(\mathbf{x}, \mathbf{q}) = p_1(1 - \delta x_N) + \delta x_1$. Since $x_1 = u_1(\mathbf{x}, \mathbf{q})$ and $x_N = p_1 + (1 - p_1)\delta$, it follows $(1 - \delta)x_1 = p_1(1 - \delta(p_1 + (1 - p_1)\delta))$, or equivalently,

$$x_1 = p_1(1 + (1 - p_1)\delta). \quad (5)$$

On the other hand, player 2's expected payoff is

$$u_2(\mathbf{x}, \mathbf{q}) \geq p_2 \max_{S \subseteq N_2} e_2(S, \mathbf{x}) + \delta((p_1 + p_2)x_2 + p_3 p_2) \geq \delta(1 - p_3)x_2 + p_3 p_2 \delta.$$

By the consistency, we have $x_2 \geq \frac{\delta p_2 p_3}{1 - \delta(1 - p_3)}$ and similarly $x_3 \geq \frac{\delta p_2 p_3}{1 - \delta(1 - p_2)}$. Together with (5), it requires that

$$x_N \geq p_1(1 + (1 - p_1)\delta) + \frac{\delta p_2 p_3}{1 - \delta(1 - p_3)} + \frac{\delta p_2 p_3}{1 - \delta(1 - p_2)}.$$

To see a contradiction, as δ converges to 1, observe that the right-hand side converges to $1 + p_1(1 - p_1)$, which is strictly greater than 1 as long as $p_1 > 0$. However, x_N never exceeds 1. Thus, for a sufficiently high δ , the efficient strategy profile (\mathbf{x}, \mathbf{q}) cannot be an equilibrium. Its actual equilibrium will be constructed in Example 5. \square

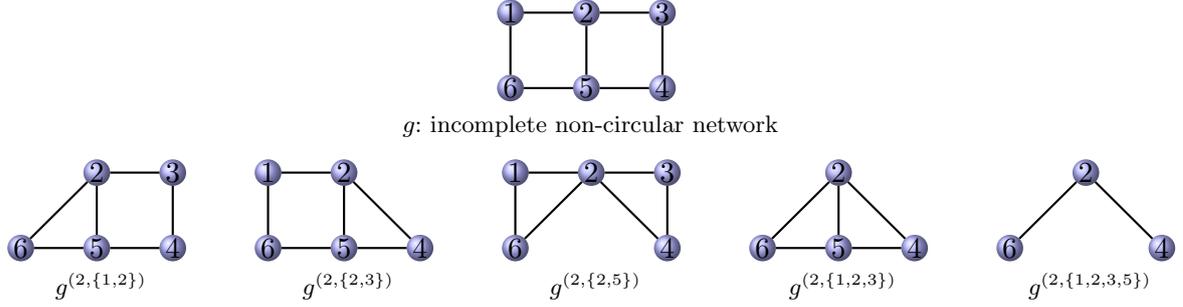


Figure 3 In an incomplete non-circular network g , if player 2 is selected as a proposer, then delay occurs with a positive probability. If player 2 forms $\{2\}$, then delay obviously occurs as $g^{(2, \{2\})} = g$. If she forms any $S \subseteq N_2(g)$ with $|S| \geq 2$, then $g^{(2, S)}$ is pre-complete non-circular, and hence delay occurs in the next period due to Proposition 3.

Now consider *any* incomplete non-circular network. If any coalition formation yields either a complete network or a circular network, then Proposition 1 and Proposition 2 imply that it must result in an efficient equilibrium. However, we can always find a player who cannot induce either a complete network or a circular network, as Proposition 4 below shows. Such a process can be repeated until a pre-complete non-circular network emerges; otherwise, the surplus can never be realized. Once a pre-complete non-circular network is induced, delay occurs due to Proposition 3. That is, in any game with an incomplete non-circular network, there must be a sequence of coalition formations, which involves a delay.

PROPOSITION 4. *Let g be an incomplete non-circular network. If g is not pre-complete, there exists $i \in N(g)$ such that, for all $S \subseteq N_i(g)$, $g^{(i, S)}$ is incomplete and non-circular.*

The next example illustrates how pre-complete non-circular networks must be induced when a particular player becomes a proposer.

EXAMPLE 4. Consider a network g with $N = \{1, 2, 3, 4, 5, 6\}$ and $E = \{12, 23, 34, 45, 56, 16, 25\}$ as in Figure 3. If player 1 is selected as a proposer and she forms a coalition $\{1, 2, 6\}$, then delay will not occur as $g^{(1, \{1, 2, 6\})}$ is circular. However, if player 2 becomes a proposer – it must happen with a positive probability as $p_2 > 0$, then delay occurs. In particular, for any $S \subseteq N_2(g)$ with $|S| \geq 2$, $g^{(2, S)}$ is pre-complete (See the second row in Figure 3). Then, delay occurs in the next period due

to Proposition 3. If he forms a singleton $\{2\}$, that is, he declines to be a proposer, then delay obviously occurs. \square

4. Sources of Strategic Delay

In this section, we outline the proof of Proposition 3 and investigate the role of dominating players in strategic delay, by explicitly constructing an inefficient equilibrium in some examples. We divide pre-complete non-circular networks into three subclasses based on the number of dominating players: networks with a single dominating player, networks with multiple dominating players, and networks with no dominating player. Each subclass provides a different insight on strategic delay in the following subsections.

4.1. Networks with a Single Dominating Player

When there is a single dominating player, the unique dominating player has a significant advantage over the other players. Hence the dominating player demands a substantial payoff. To be specific, Lemma 4 provides a lower bound of the unique dominating player's equilibrium payoff.

LEMMA 4. *Let g be a pre-complete network with $D(g) = \{i^*\}$. If (\mathbf{x}, \mathbf{q}) is an equilibrium of $\Gamma = (g, p, \delta)$, then*

$$x_{i^*} \geq p_{i^*} + p_{i^*}(1 - p_{i^*})\delta. \quad (6)$$

Example 5 is an extreme case, in which only the dominating player takes a positive surplus. It is remarkable that the non-dominating players in the three-player chain takes nothing for any recognition probability, even for discount factors strictly less than one.

EXAMPLE 5 (INEFFICIENT EQUILIBRIUM IN A CHAIN). Let $g = (\{1, 2, 3\}, \{12, 13\})$ as in Example 3. We construct an inefficient equilibrium. Let $\bar{\delta} = \max \left\{ \frac{p_2}{(p_1 + p_2)(1 - p_1)}, \frac{p_3}{(p_1 + p_3)(1 - p_1)} \right\}$ so that $\bar{\delta} < 1$. Consider a strategy profile (\mathbf{x}, \mathbf{q}) such that

- $x_1 = \frac{p_1}{1 - (1 - p_1)\bar{\delta}}$; $x_2 = x_3 = 0$; and
- $q_1(N) = q_2(\{2\}) = q_3(\{3\}) = 1$,

and in any two-player subgame the active players follow the strategy according to Proposition 1. Since both player 2 and player 3 decline to be a proposer in the initial state, the strategy profile

is inefficient. To see that (\mathbf{x}, \mathbf{q}) constructs an equilibrium for $\delta > \bar{\delta}$, due to Lemma 2, it suffices to verify the following two conditions.

- i) *Optimality*: Calculate each player's excess surpluses. It is easy to see that $e_1(N, \mathbf{x}) > 0$ and $e_i(\{i\}, \mathbf{x}) = 0$ for all $i \in N$. For all $i \in \{1, 2\}$, due to Proposition 1, $x_i^{(i, \{1, 2\})} = p_1 + p_2$, and hence

$$\begin{aligned} e_i(\{1, 2\}, \mathbf{x}) &= \delta(p_1 + p_2) - \delta(x_1 + x_2) = \delta(p_1 + p_2) - \delta\left(\frac{p_1}{1-(1-p_1)\delta} + 0\right) \\ &= \frac{\delta}{1-(1-p_1)\delta} (p_2 - (p_1 + p_2)(1 - p_1)\delta). \end{aligned}$$

Then, $\delta > \bar{\delta}$ implies $e_i(\{1, 2\}, \mathbf{x}) < 0$. Similarly, we have $e_i(\{1, 3\}, \mathbf{x}) < 0$ for all $i \in \{1, 3\}$. Therefore, $\arg \max_{S \subseteq N_1} e_1(S, \mathbf{x}) = \{N\}$, $\arg \max_{S \subseteq N_i} e_i(S, \mathbf{x}) = \{i\}$ for $i = 2, 3$, and hence the optimality is confirmed.

- ii) *Consistency*: Compute each player's expected payoff:

- $u_1(\mathbf{x}, \mathbf{q}) = p_1 e(N, \mathbf{x}) + \delta x_1 = p_1(1 - \delta x_1) + \delta x_1 = \frac{p_1}{1-(1-p_1)\delta}$
- $u_2(\mathbf{x}, \mathbf{q}) = p_2 e(\{2\}, \mathbf{x}) + \delta x_2 = p_2 \cdot 0 + \delta \cdot 0 = 0$
- $u_3(\mathbf{x}, \mathbf{q}) = p_3 e(\{3\}, \mathbf{x}) + \delta x_3 = p_3 \cdot 0 + \delta \cdot 0 = 0$.

Therefore, $u_i(\mathbf{x}, \mathbf{q}) = x_i$ for all $i \in N$ and the consistency is confirmed. \square

It is important to observe that the non-dominating players in Example 5 have a limited power in bargaining. For a non-dominating player to form a coalition with the dominating player, he has to guarantee the dominating player her discounted stationary value. However, even if he declines to form a coalition with her, since they face the same network in the next period, she is still expecting her discounted stationary value. Thus, the dominating player's equilibrium payoff does not depend on non-dominating players' strategies. Similarly in any other pre-complete network with a single dominating player, even if the dominating player is not selected as a bargaining partner by the non-dominating players, she will still be the unique dominating player in the subsequent game.

On the other hand, any efficient equilibrium requires non-dominating players to induce a complete network. In a complete network, however, the dominating-player advantage disappears as all the

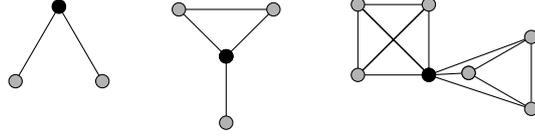


Figure 4 Pre-complete Networks with a Single Dominating Player: A dark node represents dominating player. The unique dominating player has an advantage in bargaining. Hence, non-dominating players are reluctant to induce a complete network, because the dominating player is too expensive to buy out.

players become a dominating player. That is, a non-dominating player should pay a premium when he buys out the unique dominating player, but the dominating-player advantage will disappear soon after. Therefore, non-dominating players are reluctant to induce a complete network, as the dominating player is *too expensive to buy out*, yet the dominating-player advantage will be shared with the other non-dominating players. See Figure 4 for some examples of pre-complete networks with a single dominating player.

Using Lemma 4, now we prove Proposition 3 in the case with a single dominating player.

PROOF OF PROPOSITION 3 (CASE 1: $|D(g)| = 1$)

Let $\mathcal{C}_i(g) = \{S \subseteq N_i(g) \mid g^{(i,S)} \text{ is complete}\}$. Since g is pre-complete, there exists j_1 and j_2 such that $d(j_1, j_2; g) = 2$. Let $J_1(g) = N_{j_1}(g) \setminus D(g)$, $J_2(g) = N_{j_2}(g) \setminus D(g)$, and $J(g) = J_1(g) \cup J_2(g)$. Suppose that an equilibrium (\mathbf{x}, \mathbf{q}) is efficient. Take any $j \in J_1$. Since (\mathbf{x}, \mathbf{q}) is efficient, we have $(\forall j' \in J_2) \sum_{S \in \mathcal{C}_{j'}} q_{j'}(S) = 1$ and $(\forall i \in D \cup J_1) \sum_{j \in S \subseteq N} q_i(S) = 1$. Thus, player j 's payoff is

$$\begin{aligned} u_j(\mathbf{x}, \mathbf{q}) &= p_j \max_{S \subseteq N_j} e_j(S, \mathbf{x}) + \delta(p_D + p_{J_1})x_j + \delta \sum_{j' \in J_2} p_{j'} \sum_{S \subseteq N} q_{j'}(S) x_j^{(j', S)} \\ &\geq \delta(p_D + p_{J_1})x_j + \delta p_{J_2} p_j, \end{aligned}$$

which implies that $x_j \geq \frac{p_j p_{J_2} \delta}{1 - (1 - p_{J_2}) \delta}$. Summing j over J_1 , we have $x_{J_1} \geq \frac{p_{J_1} p_{J_2} \delta}{1 - (1 - p_{J_2}) \delta}$. Similarly for J_2 , we have $x_{J_2} \geq \frac{p_{J_1} p_{J_2} \delta}{1 - (1 - p_{J_1}) \delta}$. Letting $D(g) = \{i^*\}$, on the other hand, Lemma 4 implies that $x_{i^*} \geq p_{i^*} + p_{i^*}(1 - p_{i^*})\delta$. Therefore, it follows

$$x_N = x_{J_1} + x_{J_2} + x_{i^*} \geq p_{J_1} p_{J_2} \delta \left(\frac{1}{1 - (1 - p_{J_1}) \delta} + \frac{1}{1 - (1 - p_{J_2}) \delta} \right) + p_{i^*} + p_{i^*}(1 - p_{i^*})\delta. \quad (7)$$

As $\delta \rightarrow 1$, the right-hand side of (7) converges to $p_{J_1} + p_{J_2} + p_{i^*} + p_{i^*}(1 - p_{i^*}) = 1 + p_{i^*}(1 - p_{i^*})$. Since $0 < p_{i^*} < 1$, there exists $\bar{\delta} < 1$ such that (7) contradicts to the fact of $x_N \leq 1$ for $\delta > \bar{\delta}$. Q.E.D.

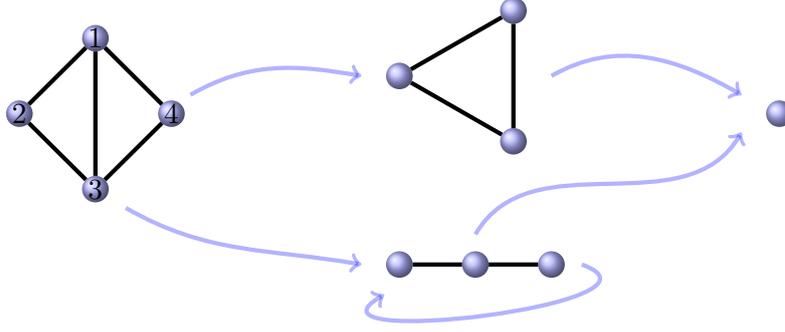


Figure 5 Bargaining in a Chordal Network: The expected periods for a grand coalition is strictly greater than two. See Example 6 and Section 5.

4.2. Networks with Multiple Dominating Players

An efficient equilibrium requires each dominating player to form a grand coalition immediately. However, if the advantage for being a unique dominating player is substantial, the dominating players may form a smaller coalition to be a unique collective dominating player by excluding the other non-dominating players.

In the next example, we construct an equilibrium, in which the dominating players form a coalition only with each other.

EXAMPLE 6 (A CHORDAL NETWORK). Let $g = (\{1, 2, 3, 4\}, \{12, 23, 34, 41, 13\})$ and $p = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Let $\bar{\delta} \approx 0.91$ be a solution to $\delta(8 - 8\delta + \delta^2) = (4 - \delta)(1 - \delta)(4 + 2\delta - \delta^2)$. For $\delta > \bar{\delta}$, one can construct an equilibrium (\mathbf{x}, \mathbf{q}) such that

- $x_1 = x_3 = \frac{(6-\delta)\delta}{4(4-\delta)(2-\delta)}$; $x_2 = x_4 = \frac{(6-6\delta+\delta^2)\delta}{4(4-\delta)(2-\delta)}$;
- $q_1(\{1, 3\}) = q_3(\{1, 3\}) = 1$; $q_2(\{1, 2\}) = q_2(\{2, 3\}) = q_4(\{1, 4\}) = q_4(\{3, 4\}) = \frac{1}{2}$.

In any subgame in which the number of active players is less than or equal to three, they follow the equilibrium strategies according to Proposition 1 and Example 5. See the e-companion to verify that the proposed strategy profile constitutes an equilibrium. In the initial state, the two dominating players form a coalition with each other and exclude the other non-dominating players, after which delay occurs by the non-dominating players in the induced chain network, as in Figure 5. Note that the equilibrium welfare is $x_N = \frac{\delta(3-\delta)}{2(2-\delta)}$. The equilibrium payoff vector converges to $(\frac{5}{12}, \frac{1}{12}, \frac{5}{12}, \frac{1}{12})$ as δ goes to one. \square

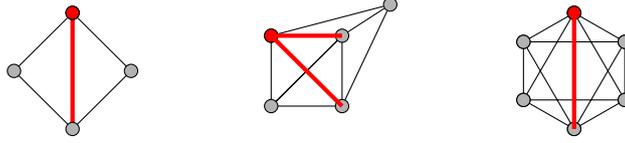


Figure 6 Examples of Pre-complete Networks with Multiple Dominating Players: As the thick red lines illustrate, dominating players form a coalition by themselves to be a unique dominating player and split the other players, rather than immediately forming a grand coalition.

See Figure 6 for some pre-complete networks with multiple dominating players. Again due to Lemma 4, Proposition 3 is proved for the case with multiple dominating players.

PROOF OF PROPOSITION 3 (CASE 2: $|D(g)| \geq 2$)

Suppose (\mathbf{x}, \mathbf{q}) is an efficient equilibrium and define $J_1(g)$, $J_2(g)$, and $J(g)$ as in Case 1. Take any $i \in D$. Player i 's optimality implies $e_i(N, \mathbf{x}) \geq e_i(D, \mathbf{x})$, or equivalently,

$$1 - \delta x_N \geq \delta x_i^{(i,D)} - \delta x_D. \quad (8)$$

Since $g^{(i,D)}$ has a single dominating player, Lemma 4 implies $x_i^{(i,D)} \geq p_D + p_D(1 - p_D)\delta$ and it follows from (8) that

$$1 - p_D\delta(1 + \delta - p_D\delta) \geq \delta x_J.$$

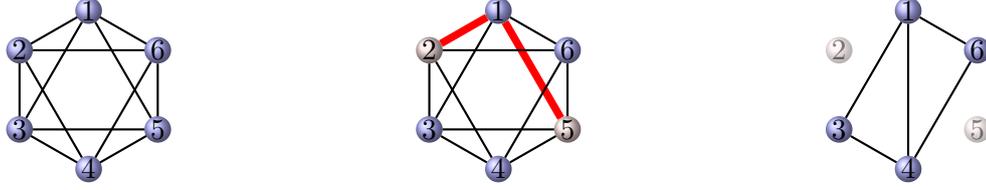
By (7) in Case 1, we have

$$1 - p_D\delta(1 + \delta - p_D\delta) \geq p_{J_1}p_{J_2}\delta^2 \left(\frac{1}{1 - (1 - p_{J_1})\delta} + \frac{1}{1 - (1 - p_{J_2})\delta} \right). \quad (9)$$

As δ goes to one, the right hand side of (9) converges to p_J ; while the left hand side converges to p_J^2 . Since $p_J < 1$, there exists $\bar{\delta} < 1$ such that the inequality (9) yields a contradiction for $\delta > \bar{\delta}$. Q.E.D.

4.3. Networks with No Dominating Player

Now we consider a network without a dominating player. In such a case, some players can be a dominating player in the induced network by buying out only a part of their neighbors. As in Example 7 below, it is particularly interesting that strategic exclusion and delay occurs even in regular networks, where all the players are initially identical, as they strategically exclude some of



(a) 6-player 4-regular network (b) Players form a 3-player coalition (c) A chordal network is induced

Figure 7 Strategic Delay in a Regular Network: Each player forms a three-player coalition to induce a chordal network, instead of forming a four- or five-player coalition, which induces a complete network. See Example 7.

their neighbors to create a dominating-player advantage by destroying the regularity in the initial network.

EXAMPLE 7 (A 6-PLAYER 4-REGULAR NETWORK). Consider a 6-player 4-regular network g as in Figure 7 (a) and let $p_i = \frac{1}{6}$ for all $i = 1, 2, \dots, 6$. Each proposer can form a 4- or 5-player coalition for an efficient outcome. However, players form a 3-player coalition in equilibrium rather than pursuing an efficient outcome. To see this, suppose there exists an efficient equilibrium (\mathbf{x}, \mathbf{q}) . Since $\bar{u}(g, p, \delta) = \delta$, it is easy to see $x_i = \frac{\delta}{6}$ for all $i = 1, 2, \dots, 6$. If a player forms a 3-player coalition as in Figure 7 (b), then a chordal network is induced. For the induced game $\Gamma^{(1, \{1, 2, 5\})}$, for sufficiently high δ , one can construct an equilibrium, in which player 1 and player 4 form a cut coalition with each other and player 3 and player 6 form a coalition with one of the connected players as similar in Example 6. In this induced game, the equilibrium payoffs are:

- $x_1^{(1, \{1, 2, 5\})} = \frac{-\delta^2 + 21\delta + 18}{6(3-\delta)(6-\delta)}$,
- $x_3^{(1, \{1, 2, 5\})} = x_6^{(1, \{1, 2, 5\})} = \frac{\delta(\delta^2 - 11\delta + 12)}{6(3-\delta)(6-\delta)}$, and
- $x_4^{(1, \{1, 2, 5\})} = \frac{-\delta^2 + 13\delta - 6}{2(3-\delta)(6-\delta)}$,

and converge to $\frac{19}{30}$, $\frac{1}{30}$, and $\frac{3}{10}$, as δ goes to one. Go back to the initial game to compare the excess surpluses. For any $S \subseteq N_1$ with $|S| \geq 4$, player 1's S -formation induces a complete network and hence

$$e_1(S, \mathbf{x}) = \delta x_1^{(1, S)} - \delta x_S = \delta p_S - \delta x_S = \delta(1 - \delta) \frac{|S|}{6},$$

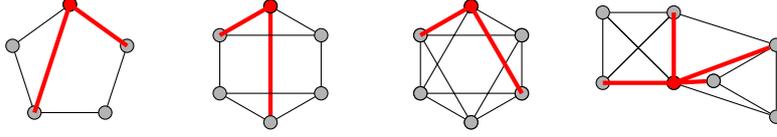


Figure 8 Examples of Pre-complete Networks with No Dominating Player: As the thick red lines illustrate, some players strategically induce an incomplete network to become a dominating player in the following period.

which converges to zero, as δ goes to one. On the other hand,

$$e_1(\{1, 2, 5\}, \mathbf{x}) = \delta x_1^{(1, \{1, 2, 5\})} - \delta(x_1 + x_2 + x_5) = \delta \left(\frac{-\delta^2 + 21\delta + 18}{6(3 - \delta)(6 - \delta)} - \frac{\delta}{3} \right),$$

which converges to $\frac{3}{10}$, as δ goes to one. Thus, the optimality condition is violated for sufficiently high δ , and hence an efficient equilibrium is impossible. \square

See Figure 8 for other examples of pre-complete networks with no dominating player. Proving Proposition 3 for the case with no dominating player is not straightforward. First, for any non-circular pre-complete network with no dominating player, Lemma 5 shows that there is a pair of players who can be a collective dominating player in the subsequent game.

LEMMA 5. *Let g be a pre-complete non-circular network with $D(g) = \emptyset$. There exist $i, j \in N(g)$ such that $i \in D(g^{(i, \{i, j\})}) \subsetneq N(g^{(i, \{i, j\})})$.*

Whenever there is a dominating player in an pre-complete network, Lemma 6 then shows *any* dominating player has a strict advantage compared to her recognition probability.

LEMMA 6. *Let g be a pre-complete network with $\emptyset \subsetneq D(g) \subsetneq N(g)$ and (\mathbf{x}, \mathbf{q}) be an equilibrium of (g, p, δ) . For any $i \in D(g)$, there exists $\Delta_i > 0$ such that $x_i - p_i \geq \Delta_i$ as δ converges to 1.*

Under a hypothetically efficient equilibrium in a network with no dominating player, however, Lemma 7 shows that each player's payoff should be strictly less than her recognition probability.

LEMMA 7. *Let g be a pre-complete network with $D(g) = \emptyset$. If (\mathbf{x}, \mathbf{q}) is an efficient equilibrium of $\Gamma = (g, p, \delta)$, then for all $i \in N$, $x_i = \delta p_i$.*

Combining those lemmas, therefore, when an efficient equilibrium is assumed, there exist some players who are strictly better off by strategically delaying a unanimous agreement – however, this contradicts to the assumption of efficient equilibrium. We complete the proof of Proposition 3 for the last case.

PROOF OF PROPOSITION 3 (CASE 3: $D(g) = \emptyset$)

As in Case 1, let $\mathcal{C}_i(g) = \{S \subseteq N_i(g) \mid g^{(i,S)} \text{ is complete}\}$. Suppose (\mathbf{x}, \mathbf{q}) is an efficient equilibrium.

Due to Lemma 7, for all $i \in N$ and all $S \in \mathcal{C}_i$,

$$e_i(S, \mathbf{x}) = \delta \left(x_i^{(i,S)} - x_S \right) = \delta (p_S - \delta p_S) = \delta(1 - \delta)p_S,$$

which converges to 0 as $\delta \rightarrow 1$. By Lemma 5, there exists $i, j \in N(g)$ such that $i \in D(g^{(i,\{i,j\})})$ and $\{i, j\} \notin \mathcal{C}_i$. Due to Lemma 6, there exists Δ_i such that $x_i^{(i,\{i,j\})} - p_i^{(i,\{i,j\})} \geq \Delta_i$. By Lemma 7, then we have

$$\begin{aligned} e_i(\{i, j\}, \mathbf{x}) &= \delta \left(x_i^{(i,\{i,j\})} - (x_i + x_j) \right) \\ &\geq \delta \left((p_i^{(i,\{i,j\})} + \Delta_i) - \delta(p_i + p_j) \right) \\ &= \delta\Delta_i + \delta(1 - \delta)(p_i + p_j). \end{aligned}$$

As $\delta \rightarrow 1$, note that $e_i(\{i, j\}, \mathbf{x}) \geq \Delta_i > 0$. For a sufficiently high δ , therefore, it follows that $e_i(\{i, j\}, \mathbf{x}) > e_i(S, \mathbf{x})$ for all $S \in \mathcal{C}_i$, which contradicts to optimality of player i . Q.E.D.

5. Braess's Paradox

Comparing Example 2 with Example 6, we observe a negative welfare effect of adding a new communication link. In the four-player circle with $p_i = \frac{1}{4}$ for all $i \in N$, the maximum welfare level δ is achieved in an equilibrium as in Example 2. Suppose a link between player 1 and player 3 is added in the circular network so that it becomes a chordal network. Since player 1 and player 3 can form a grand coalition immediately, the maximum welfare is now $\frac{1}{2}(1 + \delta)$, which is strictly greater than δ , which is the maximum welfare level in the circle. As Example 6 shows, however, the equilibrium welfare is $\frac{\delta(3-\delta)}{2(2-\delta)}$, which is strictly less than δ . In fact, this result holds for any recognition probability p , as long as $p_2 + p_4 > 0$.

The negative welfare effect of adding a new link can also be investigated by computing the expected periods for a unanimous agreement, as Figure 1 and Figure 5 depict. In the circular network, for any δ and any p , it takes exactly two periods for a grand coalition in the equilibrium. Note that all the players fully use their communication links whenever they are recognized as a proposer. In the chordal network, as the odd players can form a grand coalition immediately, the expected periods for a unanimous agreement can be $1 \times (p_1 + p_3) + 2 \times (p_2 + p_4) = 1 + p_2 + p_4$, which is strictly less than two.

An equilibrium, however, results in the expected periods for a unanimous agreement being strictly greater than two. If the even players are recognized as a proposer in the first period, then they choose one of the odd players as a bargaining partner to induce a three-player circle, after which a grand coalition immediately forms. However, if the odd players are initially recognized as a proposer, they induce a three-player chain. Once in the chain, the leaf players will then decline to make an offer and hence an additional delay occurs with positive probability. As such, the expected periods for a unanimous agreement in equilibrium is $2 + p_2 + p_4$ as δ converges to one.

REMARK 2. In the Braess's paradox with the original traffic network context, all the players are worse off with network improvement; while in this bargaining game, some players may be better off despite overall performance deteriorating.

REMARK 3. In this random-proposer bargaining model, the equilibrium may not be unique even in the class of stationary subgame perfect equilibria.¹⁶ However, the equilibrium constructed in Example 2, Example 5, and Example 6 is unique in the class of *symmetric* cutoff-strategy equilibria, in which identical players, in terms of their positions in a network and their recognition probabilities, play an identical cutoff strategy.

6. Discussions and Extensions

6.1. Robustness

While non-cooperative bargaining models explicitly take players' strategic interactions into account, one of the limitations compared to cooperative models is that the outcome may depend on the detail of the bargaining protocol. Accordingly, in such random proposer models, the equilibrium

payoffs generically depend on the recognition probability, which captures players' exogenously given relative bargaining power. The main result in this paper is, however, detail-free, that is, for *any* recognition probability p , Theorem 1 holds, as long as each player has a positive chance of being recognized.

The general inefficiency result is also robust in terms of the updating rule of recognition probabilities. We have assumed that recognition probabilities are transferable – when players form a coalition (or a partnership), its representative player takes its members' chances of being a proposer as well. Alternatively, players' bargaining power can be viewed as their innate right, which cannot be traded. In such environments where players' recognition probabilities are not transferable, instead of (2), one may define the induced recognition probability $p^{(i,S)}$:

$$p_j^{(i,S)} = \begin{cases} \frac{p_j}{1-p_S+p_i} & \text{if } j \notin S \setminus \{i\} \\ 0 & \text{if } j \in S \setminus \{i\}. \end{cases} \quad (10)$$

REMARK 4. If all the active players are always recognized with equal probability in each state, it is a special case of non-transferable recognition probabilities.

With non-transferable recognition probabilities, it turns out that an efficient equilibrium is impossible for higher discount factors *even in circular networks*. This result highlights another source of inefficiency in multilateral bargaining due to positive externalities on recognition probabilities. That is, a player forms an intermediate coalition, even the other players who are not in the coalition get higher chances of being a proposer later, and hence players reluctant to form larger coalitions. With non-transferable recognition probabilities, Theorem 1 is then re-stated as follows.

THEOREM 2. *Suppose players cannot transfer their recognition probabilities. An efficient stationary subgame perfect equilibrium exists for all discount factors if and only if the underlying network is complete.*

The proof can be found in Lee (2014). The following example constructs an inefficient equilibrium in a four-player circular network.

EXAMPLE 8 (A CIRCULAR NETWORK WITH NON-TRANSFERABLE RECOGNITION PROBABILITIES).

Let $g = (\{1, 2, 3, 4\}, \{12, 23, 34, 14\})$ and $p = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. If $\delta > \frac{2}{3}$, then one can construct an equilibrium (\mathbf{x}, \mathbf{q}) such that

- $x_1 = x_2 = \frac{\delta}{3(2-\delta)}$; $x_3 = x_4 = \frac{\delta}{6(2-\delta)}$;
- $q_1(\{1\}) = q_2(\{2\}) = 1$; $q_3(\{3, 4\}) = q_4(\{3, 4\}) = 1$.

Here, player 1 and player 2 decline to make an offer and wait for a three-player complete network induced by either player 3 or player 4. The equilibrium payoff vector converges to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$. See the e-companion for details.¹⁷ □

6.2. How Inefficient?

It is worth noting that inefficiency occurs for high discount factors but it is still *asymptotically efficient*. As the discount factor increases over a certain threshold, delay occurs more and more frequently but it becomes less and less costly. As such, the inefficiency eventually disappears as the discount factor converges to one. On the other hand, if the discount factor is low enough, then any equilibrium must be efficient no matter what the underlying network is, as the impatient players will undoubtedly try to reach an agreement as soon as possible. In summary, the efficiency loss occurs if the discount factor is strictly greater than a certain threshold but strictly less than one. If the network is either complete or circular, then there is no such threshold and an efficient equilibrium exists for *all* discount factors.

Although inefficiency disappears as the discount factor converges to one, it could be substantial for a range of discount factors. To measure the efficiency loss, define the *Price of Anarchy* (PoA) for an environment (g, p) :

$$PoA(g, p) = \max_{\delta} \frac{\bar{u}(g, p, \delta)}{\min_{(\mathbf{x}, \mathbf{q}) \in \mathcal{E}} \sum_{i \in N} u_i(\mathbf{x}, \mathbf{q})},$$

where \mathcal{E} is the set of cutoff strategy equilibria. If g is a three-player chain and $p = (1/3, 1/3, 1/3)$ for example, then one can verify $PoA(g, p) = 1.25$.¹⁸ One can also compute that the upper bound of *PoA* of a three-player chain is two, which can be obtained when the central player's recognition

probability is close to one. This is contrast to the fact that the upper bound of PoA of either any complete network or any circular network is one, as an efficient outcome can be obtained in equilibrium.

6.3. Power Indices in Networks

In addition to investigating efficiency, analyzing the equilibrium payoff vector in the noncooperative model and comparing it with cooperative solution concepts would be important for future research. It is particularly noteworthy that the limiting equilibrium payoff vector in the model proposes a plausible *power index in networks*, which captures players' bargaining power based on their position in networks. For instance, in a 4-player chordal network the equilibrium outcome assigns $\frac{5}{12}$ to the dominating players and $\frac{1}{12}$ to the non-dominating players; while the Myerson-Shapley value (Myerson 1977) assigns the same value $\frac{1}{4}$ to all the players.¹⁹ In this regard, it would be the subject of future research to develop an algorithm for finding an equilibrium payoff vector and to study its normative properties.

7. Conclusion

We have introduced a new noncooperative coalitional bargaining model for network-restricted environments. Players in a different position in a network have different bargaining powers, and hence they strategically choose their bargaining partners to take a better position in future bargaining. Such strategic behaviors cause delay in reaching an agreement to generate a joint surplus. Only two types of networks support efficient equilibria for all discount factors: complete networks and circular networks. In such networks, the players are all symmetric in terms of their position not only in the initial network but also in any subsequent network induced by coalition formations. Thus, they have a common interest to reach an agreement as soon as practicable, leading to efficiency.

Endnotes

1. Gofman (2011) provides a similar result that adding a new trading relationship decreases efficiency in a different environment. In his model, inefficiency is caused by misallocation due to intermediaries because prices are bilaterally determined. In our model, players can bargain with more than two players at the same time and inefficiency is caused by players' strategic delay.

2. There are huge amounts of literature in this area from earlier studies in bilateral bargaining, such as Admati and Perry (1987), to a recent work in trading networks, including Bedayo et al. (2013).

3. Another strand of literature, such as Wright and Wong (2014), Nguyen et al. (2016), and Nguyen (2017), studies the role of transaction cost in delay where physical goods are traded among buyers, sellers, and intermediaries in given network. In their models, the network does not change after each transaction. Compared to this, we adopt a coalition-formation approach, in which merging assets are required to generate surplus, and a sequence of coalition formations alters the network structure. Therefore, in our model, players' competition to take a better position in networks is the main source of delay.

4. Cooperation restrictions have been an important issue in economic theory and operations research, and hence it has been studied extensively in the literature of cooperative games since Aumann and Dreze (1974) and Myerson (1977). Myerson (1977) adopts a network to describe a structure of cooperation restrictions and proposes a cooperative solution using the Shapley value. Other one-point solution concepts for cooperative games in networks are introduced by Owen (1986), Borm et al. (1992), Hamiache (1999), Borkotokey et al. (2015) among others. Core allocations for cooperative games in graph structure are also studied by Herings et al. (2000). Saad et al. (2009) classify these environments as coalitional graph games and introduce some applications in computer science. In addition to theoretical studies, a growing body of experimental research, including Bolton et al. (2003), investigates the role of communication restrictions in multilateral bargaining.

5. Various types of random meetings have been considered in literature – a bilateral meeting (Rubinstein and Wolinsky 1985, Calvó-Armengol 2001, Gale and Sabourian 2006, Gale and Kariv 2007, Gofman 2011, Manea 2011, Abreu and Manea 2012a,b), a multilateral meeting (Nguyen 2015, Polanski and Lazarova 2015), or a trading route (Bedayo et al. 2013, Siedlerek 2015). As Hart and Mas-Colell (1996) pointed out, however, a random-meeting model does not entirely capture players'

strategic behaviors and strategic decisions on coalition formation should also be considered. Only some recent papers such as Abreu and Manea (2012b) (Section 4) and Elliott and Nava (2015) allow players to choose their bargaining partner but at most one, as it is limited to bilateral bargaining. In our model, players can make an offer to multiple bargaining partners at once.

6. In Gul (1989), multilateral bargaining can be done only through a sequence of random bilateral meetings, and hence players can bargain with only one partner at a time and they cannot choose their bargaining partner. Some other papers Calvó-Armengol (2001), Gale and Kariv (2007), and Manea (2015), in which players repeatedly trade or bargain in a sequence of bilateral meetings, but we allow sequential multilateral bargaining by buying out other players' resources and rights. In addition to Krishna and Serrano (1996), the models of gradual multilateral bargaining include Seidmann and Winter (1998), Gomes (2005), and Lee (2016).

7. In the existing network bargaining models such as Manea (2011), Nguyen (2015), and Nguyen et al. (2016), after some players reach an agreement they must exit the game with what they have produced and they replaced by their *clones*. In our model, players can *buy out* others resources and rights by forming a coalition with a binding agreement.

8. A simple network is an unweighted and undirected network without self-loops or multiple edges. A network is connected if there is a path, or a sequence of links, between every pair of players.

9. A *circular network* (or a *circle*) should not be confused with a *cycle in a network*. A circular network is a network that consists of a single cycle.

10. We assume that $p_i > 0$ for any $i \in N$. That means, each player has a positive chance of being a proposer. It is easy to see that the equilibrium payoff to the players with zero recognition probability is always zero.

11. The result does not depend on the order of responses.

12. We assume that players have a full commitment power through upfront transfers. To clarify, they initially have money in their own pocket, allowing them to pay monetary transfers in order to buy other players *before* the surplus generates. Allowing buyout with upfront transfers enables players to form an intermediate coalition even though it generates nothing.

13. Without stationarity, due to a huge multiplicity of equilibria, a model fails to provide a meaningful prediction.
14. Krishna and Serrano (1996) consider intermediate bargaining steps to obtain a unique equilibrium of a multilateral bargaining game, in which only a unanimous agreement generates a surplus.
15. The main result depends only on the existence of an efficient equilibrium, but does not rely on its uniqueness.
16. To overcome the multiplicity of equilibria, the uniqueness of equilibrium payoffs has been studied in the random-proposer bargaining model. Eraslan (2002) shows the equilibrium payoff uniqueness for a weighted majority game and Eraslan and McLennan (2013) generalizes this result to a general simple game using fixed point index theorem. Unfortunately, those results cannot be applied to the model, in which a player has a buyout option, because a player can expect some partial payoff by forming an intermediate subcoalition and hence the actual characteristic function that the players play is not of a simple game. The uniqueness of stationary equilibrium payoffs is conjectured in a broader class of characteristic function form games, but it remains an open question. See Eraslan and McLennan (2013) for a discussion.
17. The equilibrium payoff vector is not unique even as $\delta \rightarrow 1$. There exists another class of equilibrium payoff vectors, which converge to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ or its permutations. However, there is no symmetric equilibrium.
18. This result is not trivial even for this simple example, as all the stationary subgame perfect equilibria for all discount factors should be considered.
19. The Myerson-Shapley value in any unanimity game assigns the same value to all the players no matter what the underlying network is. The reason for this is that only the grand coalition generates a positive surplus and all the players have the same marginal contribution in a unanimity game.

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Technical Proofs and Equilibrium Verifications

EC.1. Missing Proofs

EC.1.1. Proof of Lemma 3

If $|N(g)| = 2$, then the statement is obviously true. As an induction hypothesis, suppose the statement is true for any less-than- n -player game. Consider g with $|N(g)| = n$. For any $\pi \in \Pi(g)$, observe that summing (4) over N^π yields

$$\begin{aligned} \sum_{i \in N^\pi} u_i^\pi(\mathbf{x}, \mathbf{q}) &= \sum_{i \in N^\pi} p_i^\pi \sum_{S \subseteq N^\pi} q_i^\pi(S) \left[e_i^\pi(S, \mathbf{x}) + \delta \left(\sum_{j \in S} x_j^\pi + \sum_{j \notin S} x_j^{\pi(i,S)} \right) \right] \\ &= \sum_{i \in N^\pi} p_i^\pi \sum_{S \subseteq N^\pi} q_i^\pi(S) X^{\pi(i,S)}, \end{aligned} \quad (\text{EC.1})$$

where $X^{\pi(i, N^\pi)} = 1$ and $X^{\pi(i,S)} = \delta \sum_{j \in N^\pi(i,S)} x_j^{\pi(i,S)}$ for all $S \subsetneq N^\pi$.

Sufficiency: Let (\mathbf{x}, \mathbf{q}) is an efficient equilibrium. By the consistency condition, for all $S \subsetneq N^\pi$,

$$\sum_{j \in N^\pi(i,S)} x_j^{\pi(i,S)} = \sum_{j \in N^\pi(i,S)} u_j^{\pi(i,S)}(\mathbf{x}, \mathbf{q}).$$

Since (\mathbf{x}, \mathbf{q}) is efficient, the induction hypothesis and the definition of efficiency yield $X^{\pi(i,S)} = \delta \bar{u}(\Gamma^{\pi(i,S)})$ for all $S \subsetneq N^\pi$. Suppose for contradiction that there exists $\pi \in \Pi_{\mathbf{x}, \mathbf{q}}(g)$, $i \in N^\pi$, and $S, S' \subseteq N_i^\pi$ such that $q_i^\pi(S) > 0$ and $\bar{u}(\Gamma^{\pi(i,S)}) < \bar{u}(\Gamma^{\pi(i,S')})$. Then i can strictly improve the sum of the players' payoff by putting more weight on S' in his coalition formation strategy and hence q_i^π cannot be a part of an efficient equilibrium.

Necessity: Given g , $\pi \in \Pi(g)$, and (\mathbf{x}, \mathbf{q}) , define a partial strategy profile $(\mathbf{x}_{|\pi}, \mathbf{q}_{|\pi}) = \{(x^{\pi'}, q^{\pi'})\}_{\pi' \in \Pi(g^\pi)}$. By induction hypothesis, for all $\pi \in \Pi_{\mathbf{x}, \mathbf{q}}(g) \setminus \{\pi^\circ\}$, $(\mathbf{x}_{|\pi}, \mathbf{q}_{|\pi})$ is an efficient equilibrium for a game with g^π . Consider the initial state. By (EC.1), in order to maximize $\sum_{i \in N} u_i(\mathbf{x}, \mathbf{q})$, each player i must maximize $\sum_{S \subseteq N} q_i(S) X^{(i,S)}$. Since, for all $i \in N$ and all $S \in \mathcal{E}_i$, $(\mathbf{x}_{|(i,S)}, \mathbf{q}_{|(i,S)})$ is an efficient equilibrium for a game with $g^{(i,S)}$, the condition $q_i \in \Delta(\mathcal{E}_i)$ maximizes $\sum_{i \in N} u_i(\mathbf{x}, \mathbf{q})$ and hence (\mathbf{x}, \mathbf{q}) is efficient. Q.E.D.

EC.1.2. Proof of Proposition 1

Let g be a complete network. We will use mathematical induction on the number of players to show that, for any $\Gamma = (g, p, \delta)$, there exists a unique cutoff strategy equilibrium $(\mathbf{p}, \bar{\mathbf{q}})$, where $\mathbf{p} = \{\{p_i^\pi\}_{i \in N^\pi}\}_{\pi \in \Pi}$ as in (2). The equilibrium payoff vector is then $\{u_i(\mathbf{p}, \bar{\mathbf{q}})\}_{i \in N} = p$. Furthermore, due to Lemma 1, the payoff vector is unique in the class of stationary subgame perfect equilibria.

Case 1: $|N(g)| = 2$.

First, we show that (p, \bar{q}) is an equilibrium by verifying the conditions in Lemma 2. For each $i \in N$, note that $N_i = \{\{i\}, N\}$. It is easy to verify the optimality condition as $e_i(N, p) = 1 - \delta$ is strictly greater than $e_i(\{i\}, p) = \delta p_i - \delta p_i = 0$. To verify the consistency condition, compute the expected payoff for player i :

$$u_i(p, \bar{q}) = p_i e_i(N, p) + \delta p_i = p_i(1 - \delta) + \delta p_i = p_i,$$

as desired.

Next, we show the uniqueness of the equilibrium. Suppose that there exists an equilibrium (x, q) with $q_i(N) < 1$ for any i . The optimality requires $e_i(\{i\}, x) \geq e_i(N, x)$, or $0 \geq 1 - \delta x_N$. By rearranging the terms, we have $x_N \geq 1/\delta$, which in turn contradicts that the coalition formation strategy profile is inefficient. Thus, (x, q) cannot be an equilibrium and the equilibrium must involve with \bar{q} . Now suppose (x, \bar{q}) is an equilibrium. Since this equilibrium is efficient, it must be $u_1(x, \bar{q}) + u_2(x, \bar{q}) = 1$ and hence $x_N = 1$. For each player i , the expected payoff is

$$u_i(x, \bar{q}) = p_i e_i(N, x) + \delta x_i = p_i(1 - \delta) + \delta x_i.$$

Due to the consistency, it follows that $x_i = p_i(1 - \delta) + \delta x_i$. Rearranging the terms, for any $\delta < 1$, it must be that $x_i = p_i$. Thus, (p, \bar{q}) is the unique cutoff strategy equilibrium.

Case 2: $|N(g)| > 2$.

Suppose that, for any game (g', p', δ) with $|N(g')| < |N(g)|$, $(\mathbf{p}', \bar{\mathbf{q}}')$ is the unique equilibrium, where $\mathbf{p}' = \{\{p_i'^\pi\}_{i \in N^\pi}\}_{\pi \in \Pi(g')}$ and $\bar{\mathbf{q}}' = \{\{\bar{q}_i'^\pi\}_{i \in N^\pi}\}_{\pi \in \Pi(g')}$. Note that, in such an equilibrium, for each $i \in N(g')$, player i 's expected payoff is p'_i . We show the inductive step: a cutoff strategy

profile $(\mathbf{p}, \bar{\mathbf{q}})$ is the unique equilibrium for (g, p, δ) . Due to the induction hypothesis, each player i expects $p_i^{(i,S)} = p_S$ in the subsequent game $(g^{(i,S)}, p^{(i,S)}, \delta)$ by forming $S \subsetneq N$. It is easy to verify the optimality condition as $e_i(N, \mathbf{p}) = 1 - \delta$ is strictly greater than $e_i(S, \mathbf{p}) = \delta p_i^{(i,S)} - \delta p_S = 0$ for all $S \subsetneq N$. The consistency condition can also be verified by computing each player's expected payoff as before, and hence $(\mathbf{p}, \bar{\mathbf{q}})$ is an equilibrium for (g, p, δ) .

Next, we show the uniqueness. Suppose that there exists an equilibrium (\mathbf{x}, \mathbf{q}) . Due to the induction hypothesis, any non-maximum coalition formation strategy must be played only in the non-initial state wherever possible. Suppose there exists $i \in N$ and $S \subsetneq N$ such that $q_i(S) > 0$. The optimality requires $e_i(S, \mathbf{x}) \geq e_i(N, \mathbf{x})$, or equivalently

$$\delta x_i^{(i,S)} - \delta x_S \geq 1 - \delta x_N > 1 - \delta,$$

where the second inequality is due to the inefficient coalition formation strategy. By the induction hypothesis, the inequality implies

$$\delta x_S + 1 < \delta p_S + \delta. \quad (\text{EC.2})$$

On the other hand, by letting $Q_j = \sum_{k \in N} p_k \sum_{S \subsetneq N} q_k(S) \mathbf{1}(j \in S)$, for each $j \in S$, we have

$$\begin{aligned} x_j = u_j(\mathbf{x}, \mathbf{q}) &\geq p_j (1 - \delta x_N) + \delta(Q_j x_j + (1 - Q_j)p_j) \\ &> p_j (1 - \delta) + \delta(Q_j x_j + (1 - Q_j)p_j) \\ &= p_j + \delta Q_j (x_j - p_j). \end{aligned} \quad (\text{EC.3})$$

Rearranging the terms, (EC.3) yields $x_j > p_j$ for all $j \in S$. However, this contradicts to (EC.2). Thus, any equilibrium must involve with $\bar{\mathbf{q}}$. As in the base case, one can confirm that for any cutoff equilibrium involving maximum coalition formation strategies $\bar{\mathbf{q}}$ must yield a equilibrium payoff vector p . Q.E.D.

EC.1.3. Proof of Proposition 2

Let g be a circular network. We will use mathematical induction on the diameter of networks to show that, for any $\Gamma = (g, p, \delta)$, there exists a unique cutoff strategy equilibrium $(\mathbf{x}, \bar{\mathbf{q}})$, where

$$\mathbf{x} = \{ \{ \delta^{\text{diam}(g^\pi)-1} p_i^\pi \}_{i \in N(g^\pi)} \}_{\pi \in \Pi(g)}.$$

Case 1: $\text{diam}(g) = 1$.

If g is circular and $\text{diam}(g) = 1$, then g must be a three-player circle, which is complete. Thus, Section EC.1.2 proves this case.

Case 2: $\text{diam}(g) > 2$.

As an induction hypothesis, suppose that, for all circular network g' such that $\text{diam}(g') < m$, a cutoff strategy profile $(\mathbf{x}', \bar{\mathbf{q}}')$ is an equilibrium for (g', p', δ) , where $\mathbf{x}' = \{ \{ \delta^{\text{diam}(g'^\pi)-1} p_i'^\pi \}_{i \in N(g'^\pi)} \}_{\pi \in \Pi(g')}$. Now we show that a cutoff strategy profile $(\mathbf{x}, \bar{\mathbf{q}})$ is an equilibrium for (g, p, δ) with a circular network g and $\text{diam}(g) = m$, where $\mathbf{x} = \{ \{ \delta^{\text{diam}(g^\pi)-1} p_i^\pi \}_{i \in N(g^\pi)} \}_{\pi \in \Pi(g)}$. Take any $i \in N$ and let $N_i = \{i, j, k\}$. We verify the equilibrium conditions for player i .

- i) *Optimality:* After i 's maximum coalition formation, the active players face a game with a circular network g' and $\text{diam}(g') = m - 1$. Due to the induction hypothesis, since $x_i^{(i, \{i, j, k\})} = \delta^{m-1}(p_i + p_j + p_k)$, we have

$$\begin{aligned} e_i(\{i, j, k\}, \mathbf{x}) &= \delta^m(p_i + p_j + p_k) - \delta(x_i + x_j + x_k) \\ &= \delta^m(p_i + p_j + p_k) - \delta(\delta^m p_i + \delta^m p_j + \delta^m p_k) \\ &= \delta^m(1 - \delta)(p_i + p_j + p_k). \end{aligned} \tag{EC.4}$$

Suppose i decline to make an offer, that is i forms $\{i\}$. Since $e_i(\{i\}, \mathbf{x}) = 0$ is strictly less than (EC.4), i 's $\{i\}$ -formation is not optimal. Suppose i forms $\{i, j\}$. Note that

$$x_i^{(i, \{i, j\})} = \begin{cases} \delta^{m-1}(p_i + p_j) & \text{if } |N(g)| \text{ is even,} \\ \delta^m(p_i + p_j) & \text{if } |N(g)| \text{ is odd.} \end{cases}$$

Thus, we have

$$e_i(\{i, j\}, \mathbf{x}) \leq \delta^m(p_i + p_j) - \delta(x_i + x_j) = \delta^m(1 - \delta)(p_i + p_j),$$

which is strictly less than (EC.4), and hence i 's S -formation with $|S| = 2$ is not optimal.

ii) *Consistency*: Since all the players play maximum coalition formation strategies, player i 's continuation payoff is:

$$\begin{aligned} u_i(\mathbf{x}, \mathbf{q}) &= p_i e_i(\{i, j, k\}, \mathbf{x}) + \delta \left((p_i + p_j + p_k)x_i + \sum_{\ell \in N \setminus \{i, j, k\}} p_\ell x_i^{(\ell, N_\ell)} \right) \\ &= p_i \delta^m (1 - \delta)(p_i + p_j + p_k) + \delta(p_i + p_j + p_k)x_i + \delta(1 - (p_i + p_j + p_k))\delta^{m-1}p_i. \end{aligned}$$

Since $u_i(\mathbf{x}, \mathbf{q}) = x_i$, rearranging the terms, we have

$$(1 - \delta(p_i + p_j + p_k))x_i = p_i \delta^m (1 - \delta)(p_i + p_j + p_k) + (1 - (p_i + p_j + p_k))\delta^m p_i,$$

which yields $x_i = \delta^m p_i$. Q.E.D.

EC.1.4. Proof of Proposition 4

Let g be an incomplete and non-circular network. In addition, assume that g is not pre-complete. By definition of a pre-complete network, there exists $i \in N(g)$ such that, for all $S \subseteq N_i(g)$, $g^{(i, S)}$ is incomplete. If, for all $S \subseteq N_i(g)$, $g^{(i, S)}$ is non-circular, there is nothing to prove. Now suppose there exists $S \subseteq N_i(g)$ such that $g^{(i, S)}$ is circular (but it is still incomplete). Then S must be directly connected to only two distinct players, say a and b , and $g_{|N(g) \setminus S}$ must be a chain in which a and b are the two ends of the chain. As $g^{(i, S)}$ is incomplete, $|N(g) \setminus S| \geq 3$. Take $j \in N(g) \setminus (S \cup \{a\} \cup \{b\})$. For any $S' \in N_j(g)$, $g^{(j, S')}$ is non-circular, otherwise g must be circular. If $|N(g) \setminus S| \geq 4$, then $g^{(j, S')}$ is incomplete for any $S' \in N_j(g)$, because a and b are not directly connected in $g^{(j, S')}$. Suppose $|N(g) \setminus S| = 3$ and there exists $S' \in N_j(g)$ such that $g^{(j, S')}$ is complete. Then $g_{|S}$ must be complete and all the players in S must be directly connected either a or b . Hence, for any $k \in N(g) \setminus \{j\}$, $g^{(k, N_k(g))}$ is a three-player complete network. This implies that g is pre-complete, which causes a contradiction. For all $S' \in N_j(g)$, therefore, $g^{(j, S')}$ is neither complete nor circular, as desired. Q.E.D.

EC.1.5. Proof of Lemma 4.

Step 1: Consider a three-person chain, that is, $J_1 = \{j_1\}$ and $J_2 = \{j_2\}$. Since $x_d^{(j_1, J_1)} = x_d^{(j_2, J_2)} = x_d$ and $u_N(\mathbf{x}, \mathbf{q}) \leq \bar{u}(\Gamma) = p_d + \delta(1 - p_d)$, player i 's expected payoff is

$$\begin{aligned} x_d &\geq p_d e_d(N, \mathbf{x}) + \sum_{k \in N} p_k \sum_{S \ni i} q_k(S) \delta x_d + \delta \sum_{k \in N} p_k \sum_{S \not\ni i} q_k(S) x_d \\ &\geq p_d (1 - \delta(p_d + \delta(1 - p_d))) + \delta x_d. \end{aligned}$$

Rearranging the terms, we have the desired result.

Step 2: As an induction hypothesis, assume that for any pre-complete network g' with $D(g') = \{d\}$, $1 \leq |J_1(g')| \leq a$, and $1 \leq |J_2(g')| \leq b$, $x'_d \geq p'_d + p'_d(1 - p'_d)\delta$. Now we consider a pre-complete network g with $D(g) = \{i\}$, $|J_1(g)| = a$, and $|J_2(g)| = b + 1$. Player i 's expected payoff is

$$x_d \geq p_d e_d(N, \mathbf{x}) + \sum_{k \in N} p_k \sum_{S \ni i} q_k(S) \delta x_d + \delta \sum_{k \in N} p_k \sum_{S \not\ni i} q_k(S) x_d^{(k,S)}. \quad (\text{EC.5})$$

For any $k \in N$ and $S \subseteq N$ such that $i \notin S$, the induction hypothesis implies $x_d^{(k,S)} \geq p_d + p_d(1 - p_d)\delta$.

Suppose by way of contradiction that $p_d + p_d(1 - p_d)\delta > x_d$. Then, (EC.5) can be written as $x_d \geq p_d(1 - \delta(p_d + \delta(1 - p_d))) + \delta x_d$, or equivalently, $x_d > p_d + p_d(1 - p_d)\delta$, which yields a contradiction.

Induction argument completes the proof. Q.E.D.

EC.1.6. Proof of Lemma 5.

First, we define some graph-theoretic definitions. A *complete cover* of g is a collection \mathcal{M} of subsets of $N(g)$, such that, $\cup \mathcal{M} = N(g)$ and $g|_M$ is a complete network for all $M \in \mathcal{M}$. A *complete covering number* of g is the minimum cardinality of a complete cover of g . A *minimal complete cover* is a complete cover of which cardinality is minimum.

Since g is pre-complete non-circular, its complete covering number is 2. Let \mathcal{M} be a minimal complete cover of g . Since $D(g) = \emptyset$, \mathcal{M} must be disjoint. Given $i \in N$, then let $M_i \in \mathcal{M}$ such that $i \in M_i$. Since $D(g) = \emptyset$, for all $k \in N$, there exists at least one $k' \in M_k^c$ such that $kk' \notin E(g)$, that is, it must be $|M_k^c \setminus N_k(g)| \geq 1$. We will show that there exists $i \in N$ and $j \in M_i^c$ such that $i \in D(g^{(i, \{i, j\})}) \subsetneq N(g^{(i, \{i, j\})})$, by constructing such a pair of i and j in the following two cases. First, suppose there exists $k \in N$ such that $|M_k^c \setminus N_k(g)| \geq 2$. Take $i \in M_k^c \setminus N_k(g)$ and $j \in M_i^c$ with $ij \in E(g)$. Take $i' \in M_k^c \setminus N_k(g)$ with $i' \neq i$. Since $g|_{M_i}$ and $g|_{M_i^c}$ are complete, $i \in D(g^{(i, \{i, j\})})$. Since $d(k, i'; g) = d(k, i'; g^{(i, \{i, j\})}) = 2$, $k \notin N(g^{(i, \{i, j\})})$, as desired. Second, suppose, for all $k \in N$, $|M_k^c \setminus N_k(g)| = 1$. Take any $i \in N$ and $j \in M_i^c$ such that $ij \in E(g)$. Take $k \in M_i \setminus \{i\}$ and $k' \in M_i^c$ such that $d(k, k'; g) = 2$. Again we have $i \in D(g^{(i, \{i, j\})})$ and $d(k, k'; g) = d(k, k'; g^{(i, \{i, j\})}) = 2$, as desired. Q.E.D.

EC.1.7. Proof of Lemma 6.

It requires two additional lemmas. Lemma EC.1 shows that for any dominating player, her expected payoff is strictly greater than her recognition probability.

LEMMA EC.1. *Let g be a pre-complete network with $\emptyset \subsetneq D(g) \subsetneq N(g)$ and (\mathbf{x}, \mathbf{q}) be an equilibrium of (g, p, δ) with $\delta < 1$. If $i \in D(g)$ then $x_i > p_i$.*

Proof of Lemma EC.1. If $|N(g)| = 3$, due to Lemma 4, then $x_i \geq p_i + p_i(1 - p_i)\delta > p_i$ for any $i \in D$. As an induction hypothesis, suppose the statement is true for any g' with $|N(g')| < n$. Now consider g with $|N(g)| = n$. Take any $i \in D(g)$. For any $k \in N$ and any S such that $i \notin S$, if $g^{(k,S)}$ is complete then $x_i^{(k,S)} = p_i$; and if $g^{(k,S)}$ is incomplete then $x_i^{(k,S)} > p_i$ by the induction hypothesis. Thus, letting $Q_i = \sum_{k \in N} p_k (\sum_{S \ni i} q_i(S) + q_k(\{k\}))$, we have $x_i \geq p_i(1 - \delta x_N) + Q_i \delta x_i + \delta(1 - Q_i)p_i$, and hence $x_i \geq p_i + \frac{\delta(1-\delta)}{1-\delta Q_i} p_i > p_i$. Q.E.D.

However, we need a stronger result: the difference between the expected payoff and the recognition probability is bounded away from zero. Lemma EC.2 below shows that there exists such a dominating player before proving it for all dominating players. For notational convenience, denote $\Delta_i = x_i - p_i$ and $\Delta_i^{(j,S)} = x_i^{(j,S)} - p_i^{(j,S)}$. If $g^{(i,S)}$ is complete with $|N(g^{(i,S)})| \geq 2$, by Proposition 1, note that $e_i(S, \mathbf{x}) = \delta(x_i^{(i,S)} - x_S) = \delta(p_S - x_S) = -\delta\Delta_S$.

LEMMA EC.2. *Let g be a pre-complete network with $\emptyset \subsetneq D(g) \subsetneq N(g)$ and (\mathbf{x}, \mathbf{q}) be an equilibrium of (g, p, δ) . There exists $h \in D(g)$ such that*

$$x_h - p_h \geq \frac{p_h(p_D(1 - p_D)\delta^2 - (1 - \delta))}{1 + (|D| - 1)p_h\delta}.$$

Furthermore, $\lim_{\delta \rightarrow 1} (x_h - p_h) \geq \frac{p_h p_D(1 - p_D)}{1 + (|D| - 1)p_h} > 0$

Proof of Lemma EC.2. Take any $h \in \arg \max_{i \in N} \Delta_i$ and let $Q_h = \sum_{i \in N} \sum_{S \ni h} p_i q_i(S)$. For any $i \in N$ and $S \subseteq N$ such that $h \notin S$, since $h \in D(g^{(i,S)})$, Lemma EC.1 implies $x_h^{(i,S)} \geq p_h$, and hence we have

$$\begin{aligned} x_h &\geq p_h e_h(D, \mathbf{x}) + Q_h \delta x_h + \delta(1 - Q_h)p_h \\ &\geq p_h p_D(1 - p_D)\delta^2 - p_h \Delta_D \delta + Q_h \delta(p_h + \Delta_h) + (1 - Q_h)\delta p_h \\ &\geq p_h p_D(1 - p_D)\delta^2 - p_h |D| \Delta_h \delta + \delta p_h + p_h \Delta_h \delta, \end{aligned} \tag{EC.6}$$

where the second inequality is due to Lemma 4, which implies

$$e_h(D, \mathbf{x}) = \delta \left(x_h^{(h,D)} - x_D \right) \geq \delta(p_D + p_D(1 - p_D)\delta - x_D) = p_D(1 - p_D)\delta^2 - \Delta_D\delta,$$

and the last inequality comes from $p_h \leq Q_h$, $p_h \geq x_h$, and $\Delta_D \leq |D|\Delta_h$. Subtracting p_h from both sides of (EC.6), we have $\Delta_h \geq \frac{p_h(p_D(1-p_D)\delta^2 - (1-\delta))}{1+(|D|-1)p_h\delta}$, as desired. Since $D \subsetneq N$, it must be $p_D < 1$ and hence $\lim_{\delta \rightarrow 1} \Delta_h \geq \frac{p_h p_D(1-p_D)}{1+(|D|-1)p_h} > 0$. Q.E.D.

Now we are ready to prove Lemma 6: For *any* dominating player i , her advantage Δ_i is bounded away from zero. We will show that $\lim_{\delta \rightarrow 1} \min_{i \in D} \Delta_i > 0$. Let $L = \arg \min_{i \in D} \Delta_i$. Since g is a pre-complete, as before there exists j_1 and j_2 such that $d(j_1, j_2; g) = 2$, and let $J_1(g) = N_{j_1}(g) \setminus D(g)$, $J_2(g) = N_{j_2}(g) \setminus D(g)$, and $J(g) = J_1(g) \cup J_2(g)$. Recall Lemma EC.1, which implies $(\forall i \in D) \Delta_i > 0$. Thus, for any $j \in J_1$ and $S \subsetneq N$, if $q_j(S) > 0$ then either $S \subseteq J_1$ or $S \cap D = \{\ell\}$ for some $\ell \in L$.

Case 1: Suppose $|J_1| = |J_2| = 1$. Then, for each $j \in J$, $q_j(\{j\}) + \sum_{\ell \in L} q_j(\{j, \ell\}) = 1$, and hence there exists $\ell \in L$ such that $\sum_{j \in J} p_j (q_j(\{j\}) + q_j(\{j, \ell\})) \geq \frac{p_J}{|L|}$. Let $Q_\ell = \sum_{j \in J} p_j (q_j(\{j\}) + q_j(\{j, \ell\})) + \sum_{i \in D} \sum_{S \ni \ell} p_i q_i(S)$, then $Q_\ell \geq \frac{p_J}{|L|} + p_\ell$. Since $x_\ell \geq p_\ell e_\ell(J \cup \{\ell\}, \mathbf{x}) + Q_\ell \delta x_\ell + (1 - Q_\ell) \delta p_\ell$, it follows

$$\Delta_\ell \geq \delta p_\ell (\Delta_\ell + \Delta_J) + \delta \left(\frac{p_J}{|L|} + p_\ell \right) \Delta_\ell - (1 - \delta) p_\ell,$$

which implies $\Delta_\ell \geq \frac{-\delta p_\ell \Delta_J - (1-\delta)p_\ell}{1-\delta \frac{p_J}{|L|}}$. Since $x_N - p_N = \Delta_N < 0$, we have $-\Delta_J \geq \Delta_D \geq \Delta_h$. Thus, by Lemma EC.2, we have the desired result,

$$\lim_{\delta \rightarrow 1} \Delta_\ell \geq -\frac{|L|p_\ell}{|L| - p_J} \Delta_J \geq \frac{|L|p_\ell}{|L| - p_J} \Delta_h \geq \frac{p_\ell p_h p_D (1 - p_D) |L|}{(|L| - p_J)(1 + (|D| - 1)p_h)} > 0.$$

Case 2: As an induction hypothesis, for any pre-complete network g' with $\emptyset \subsetneq D(g') \subsetneq N(g')$ and $1 \leq |J_1(g')| \leq a$ and $1 \leq |J_2(g')| \leq b$ and any equilibrium $(\mathbf{x}', \mathbf{q}')$ of (g', p', δ) , assume that $\lim_{\delta \rightarrow 1} \min_{i \in D(g')} (x'_i - p'_i) > 0$. Now we consider a pre-complete network g with $\emptyset \subsetneq D(g) \subsetneq N(g)$ and $|J_1(g)| = a$ and $|J_2(g)| = b + 1$. Due to the induction hypothesis, there exists $\Delta'_\ell > 0$ such that $\Delta'_\ell \geq \lim_{\delta \rightarrow 1} (x_\ell^{(j, J')} - p_\ell)$ for all $\alpha \in \{1, 2\}$, $j \in J_\alpha$, and $J' \subseteq J_\alpha$. Then, we have

$$\begin{aligned} x_\ell \geq p_\ell e_\ell(J \cup \{\ell\}, \mathbf{x}) + & \left(p_\ell + \sum_{\alpha \in \{1, 2\}} \sum_{j \in J_\alpha} p_j (q_j(\{j\}) + q_j(J_\alpha \cup \{\ell\})) \right) \delta (p_\ell + \Delta_\ell) \\ & + \left(\sum_{\alpha \in \{1, 2\}} \sum_{j \in J_\alpha} \sum_{J' \subseteq J_\alpha} p_j q_j(J') \right) \delta (p_\ell + \Delta'_\ell) + p_{D \setminus \{\ell\}} \delta p_\ell. \quad (\text{EC.7}) \end{aligned}$$

If $\lim_{\delta \rightarrow 1} \Delta_\ell \geq \Delta'_\ell$, then there is nothing to prove. Suppose that $\lim_{\delta \rightarrow 1} \Delta_\ell \leq \Delta'_\ell$. As $\delta \rightarrow 1$, then (EC.7) yields $x_\ell \geq -p_\ell \delta \Delta_J + \delta p_\ell + (1 - p_D) \delta \Delta_\ell$, or equivalently, $(1 - (1 - p_D) \delta) \Delta_\ell \geq -\delta p_\ell \Delta_J - (1 - \delta) p_\ell$. Take any $h \in \arg \max_{i \in D} \Delta_i$. Since $-\Delta_J > \Delta_D > \Delta_h$, it follows that

$$(1 - (1 - p_D) \delta) \Delta_\ell > \delta p_\ell \Delta_h - (1 - \delta) p_\ell.$$

By Lemma EC.1, we have the desired result, $\lim_{\delta \rightarrow 1} \Delta_\ell \geq \frac{p_\ell p_h (1 - p_D)}{1 + (|D| - 1) p_h} > 0$. Q.E.D.

EC.1.8. Proof of Lemma 7.

Since g is pre-complete and (\mathbf{x}, \mathbf{q}) is efficient, for all $j \in N$, $q_j(S) > 0$ implies $g^{(j, S)}$ is complete. Thus, each player i can expect p_i in the next period by rejecting any offer. Suppose player i gets an offer with $y_i < \delta^2 p_i$. By rejecting y_i , i can be strictly better since the stationary strategy profile guarantees δp_i in the next period. Hence, $x_i \geq \delta p_i$ for all $i \in N$. If there exists $i \in N$ such that $x_i > \delta p_i$, then it must be $x_N > \delta p_N = \delta$, which is infeasible. Q.E.D.

EC.2. Equilibrium Verifications

EC.2.1. Equilibrium in Example 6

We verify the equilibrium (\mathbf{x}, \mathbf{q}) with

- $x_1 = x_3 = \frac{(6 - \delta)\delta}{4(4 - \delta)(2 - \delta)}$; $x_2 = x_4 = \frac{(6 - 6\delta + \delta^2)\delta}{4(4 - \delta)(2 - \delta)}$;
- $q_1(\{1, 3\}) = q_3(\{1, 3\}) = 1$; $q_2(\{1, 2\}) = q_2(\{2, 3\}) = q_4(\{1, 4\}) = q_4(\{3, 4\}) = \frac{1}{2}$,

for $\delta > \bar{\delta}$, where $\bar{\delta} \approx 0.91$ is a solution to $\delta(8 - 8\delta + \delta^2) = (4 - \delta)(1 - \delta)(4 + 2\delta - \delta^2)$. Due to Lemma 2, it suffices to confirm the following conditions.

Optimality:

i) *Odd Players' Optimality:* Since $\delta > \frac{3}{4}$, Example 5 implies that $x_1^{(1, \{1, 3\})} = \frac{p_1 + p_3}{1 - (1 - p_1 - p_3)\delta} = \frac{1}{2 - \delta}$

and $x_1^{(1, \{1, 3\})} = x_4^{(1, \{1, 3\})} = 0$. Given \mathbf{x} , calculate player 1's excess surpluses:

- $e_1(\{1, 2\}, \mathbf{x}) = \delta x_1^{(1, \{1, 2\})} - \delta(x_1 + x_2) = \frac{\delta(1 - \delta)(4 - \delta)}{4(2 - \delta)}$
- $e_1(\{1, 3\}, \mathbf{x}) = \delta x_1^{(1, \{1, 3\})} - \delta(x_1 + x_3) = \frac{\delta(8 - 8\delta + \delta^2)}{2(2 - \delta)(4 - \delta)}$
- $e_1(\{1, 2, 4\}, \mathbf{x}) = \delta x_1^{(1, \{1, 2, 4\})} - \delta(x_1 + x_2 + x_4) = \frac{\delta(6 - 6\delta + \delta^2)}{2(4 - \delta)}$
- $e_1(N, \mathbf{x}) = 1 - \delta x_N = \frac{(1 - \delta)(4 + 2\delta - \delta^2)}{2(2 - \delta)}$

Given $e_1(S, \mathbf{x})$ for all $S \subseteq N_1$, it is routine to see that, $\delta > \bar{\delta} \implies \arg \max_{S \subseteq N_1} e_1(S, \mathbf{x}) = \{\{1, 3\}\}$. Similarly, we also have $\delta > \bar{\delta} \implies \arg \max_{S \subseteq N_3} e_3(S, \mathbf{x}) = \{\{1, 3\}\}$.

ii) *Even Players' Optimality:* For any $\{2\} \subsetneq S \subseteq N_2$, player 2's S -formation induces a complete network. Thus, given \mathbf{x} , one can compute player 2's excess surpluses:

- $e_2(\{1, 2\}, \mathbf{x}) = e_2(\{2, 3\}, \mathbf{x}) = \delta x_2^{(2, \{1, 2\})} - \delta(x_1 + x_2) = \frac{\delta(1-\delta)(4-\delta)}{4(2-\delta)}$
- $e(\{1, 2, 3\}, \mathbf{x}) = \delta x_2^{(2, \{1, 2, 3\})} - \delta(x_1 + x_2 + x_3) = \frac{\delta(24-36\delta+11\delta^2-\delta^3)}{4(2-\delta)(4-\delta)}$

Observe that $e_2(\{1, 2\}, \mathbf{x}) = e_2(\{2, 3\}, \mathbf{x}) > 0$ for all δ ; while $e(\{1, 2, 3\}, \mathbf{x})$ is strictly negative if $\delta > \bar{\delta}$. Thus, for any $\delta > \bar{\delta}$, we have $\arg \max_{S \subseteq N_2} e_2(S, \mathbf{x}) = \{\{1, 2\}, \{2, 3\}\}$ and similarly $\arg \max_{S \subseteq N_4} e_4(S, \mathbf{x}) = \{\{1, 4\}, \{3, 4\}\}$.

Consistency: Given (\mathbf{x}, \mathbf{q}) , compute each players' expected payoffs:

- $u_1(\mathbf{x}, \mathbf{q}) = p_1 e(\{1, 3\}, \mathbf{x}) + \delta \left[(p_1 + p_3 + \frac{1}{2}(p_2 + p_4)) x_1 + \frac{p_2}{2} x_1^{(2, \{2, 3\})} + \frac{p_4}{2} x_1^{(4, \{3, 4\})} \right]$
 $= \frac{1}{4} \cdot \frac{\delta(8-8\delta+\delta^2)}{2(2-\delta)(4-\delta)} + \delta \left[\frac{3}{4} \cdot \frac{(6-\delta)\delta}{4(4-\delta)(2-\delta)} + \frac{1}{2} \cdot \frac{1}{8} \right]$
 $= \frac{(6-\delta)\delta}{4(4-\delta)(2-\delta)} = x_1$
- $u_2(\mathbf{x}, \mathbf{q}) = p_2 e(\{1, 2\}, \mathbf{x}) + \delta [p_2 x_2 + p_4 p_2 + (p_1 + p_3) \cdot 0]$
 $= \frac{1}{4} \cdot \frac{\delta(1-\delta)(4-\delta)}{4(2-\delta)} + \delta \left[\frac{1}{4} \cdot \frac{(6-6\delta+\delta^2)\delta}{4(4-\delta)(2-\delta)} + \frac{1}{4} \cdot \frac{1}{4} \right]$
 $= \frac{(6-6\delta+\delta^2)\delta}{4(4-\delta)(2-\delta)} = x_2$

and similarly $u_3(\mathbf{x}, \mathbf{q}) = x_3$ and $u_4(\mathbf{x}, \mathbf{q}) = x_4$, and hence the consistency is verified. \square

EC.2.2. Equilibrium in Example 8

We verify the equilibrium (\mathbf{x}, \mathbf{q}) with

- $x_1 = x_2 = \frac{\delta}{3(2-\delta)}$; $x_3 = x_4 = \frac{\delta}{6(2-\delta)}$;
- $q_1(\{1\}) = q_2(\{2\}) = 1$; $q_3(\{3, 4\}) = q_4(\{3, 4\}) = 1$.

Again due to Lemma 2, it suffices to confirm the following conditions.

Optimality: Since recognition probabilities are not transferable, note that $x_1^{(1, \{1, 2\})} = x_1^{(1, \{1, 4\})} = \frac{1}{3}$ and $x_1^{(1, \{1, 2, 4\})} = \frac{1}{2}$. Given \mathbf{x} , player 1's excess surpluses are:

- $e_1(\{1, 2\}, \mathbf{x}) = \frac{1}{3}\delta - \delta(x_1 + x_2) = \frac{\delta(2-3\delta)}{3(2-\delta)}$
- $e_1(\{1, 4\}, \mathbf{x}) = \frac{1}{3}\delta - \delta(x_1 + x_4) = \frac{\delta(1-2\delta)}{6(2-\delta)}$

- $e_1(\{1, 2, 3\}, \mathbf{x}) = \frac{1}{2}\delta - \delta(x_1 + x_2 + x_3) = \frac{\delta(1-3\delta)}{6(2-\delta)}$.

Provided $\delta > \frac{2}{3}$, for all $\{1\} \subsetneq S \subseteq N_1$, we have $e_1(S, \mathbf{x}) < 0 = e_1(\{1\}, \mathbf{x})$ and hence $q_1(\{1\}) = 1$.

Similarly, we have $q_2(\{2\}) = 1$. Now calculate player 3's excess surpluses:

- $e_3(\{2, 3\}, \mathbf{x}) = \frac{1}{3}\delta - \delta(x_2 + x_3) = \frac{\delta(2-3\delta)}{3(2-\delta)}$
- $e_3(\{3, 4\}, \mathbf{x}) = \frac{1}{3}\delta - \delta(x_3 + x_4) = \frac{\delta(1-\delta)}{3(2-\delta)}$
- $e_3(\{2, 3, 4\}, \mathbf{x}) = \frac{1}{2}\delta - \delta(x_2 + x_3 + x_4) = \frac{\delta(2-3\delta)}{6(2-\delta)}$.

Provided $\delta > \frac{2}{3}$, since $e_3(\{3, 4\}, \mathbf{x}) > 0$ and $e_1(S, \mathbf{x}) < 0$ for any other $\{1\} \subseteq S \subseteq N_1$, we have $q_3(\{3, 4\}) = 1$, and similarly $q_4(\{3, 4\}) = 1$.

Consistency: Given (\mathbf{x}, \mathbf{q}) , calculate each player's expected payoff:

- $u_1(\mathbf{x}, \mathbf{q}) = p_1 e_1(\{1\}, \mathbf{x}) + \delta \left((p_1 + p_2)x_1 + (p_3 + p_4)\frac{1}{3} \right)$
 $= \frac{1}{4} \cdot 0 + \delta \left(\frac{1}{2} \frac{\delta}{3(2-\delta)} + \frac{1}{2} \frac{1}{3} \right) = \frac{\delta}{3(2-\delta)} = x_1,$
- $u_3(\mathbf{x}, \mathbf{q}) = p_3 e_3(\{3, 4\}, \mathbf{x}) + \delta x_3$
 $= \frac{1}{4} \left(\frac{1}{3}\delta - 2 \cdot \frac{\delta^2}{6(2-\delta)} \right) + \frac{\delta^2}{6(2-\delta)} = \frac{\delta}{6(2-\delta)} = x_3,$

and similarly, $u_2(\mathbf{x}, \mathbf{q}) = x_2$ and $u_4(\mathbf{x}, \mathbf{q}) = x_4$. □