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The Complexity of Planar Boolean #CSP with Complex Weights

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Abstract

We prove a complexity dichotomy theorem for symmetric complex-weighted Boolean #CSP when the constraint graph of the input must be planar. The problems that are #P-hard over general graphs but tractable over planar graphs are precisely those with a holographic reduction to matchgates. This generalizes a theorem of Cai, Lu, and Xia for the case of real weights. We also obtain a dichotomy theorem for a symmetric arity 4 signature with complex weights in the planar Holant framework, which we use in the proof of our #CSP dichotomy. In particular, we reduce the problem of evaluating the Tutte polynomial at the point $(3, 3)$ to counting the number of Eulerian orientations over planar 4-regular graphs to show the latter is #P-hard. This strengthens a theorem by Huang and Lu to the planar setting. Our proof techniques combine new ideas with refinements and extensions of existing techniques. These include planar pairings, the recursive unary construction, the anti-gadget technique, and pinning in the Hadamard basis.

1 Introduction

In 1979, Valiant [39] defined the class #P to explain the apparent intractability of counting the number of perfect matchings in a graph. Yet over a decade earlier, Kasteleyn [28] gave a polynomial-time algorithm to compute this quantity for planar graphs. This was an important milestone in a decades-long research program by physicists in statistical mechanics to determine what problems the restriction to the planar setting renders tractable [26, 35, 49, 50, 32, 37, 27, 28, 1, 34, 48]. More recently, Valiant introduced matchgates [42, 41] and *holographic* algorithms [44, 43] that rely on Kasteleyn's algorithm to solve certain counting problems over planar graphs. In a series of papers [6, 7, 11, 12], Cai et al. characterized the local constraint functions (which define counting problems) that are representable by matchgates in a holographic algorithm.

In the field of computational complexity, we seek to understand exactly which intractable problems the planarity restriction enable us to efficiently compute. Partial answers to this question have been given in the context of various counting frameworks [47, 14, 9, 10]. In every case, the problems that are #P-hard over general graphs but tractable over planar graphs are essentially those characterized by Cai et al. In this paper, we give more evidence for this phenomenon by extending the results of [14] to the setting of complex-valued constraint functions. This provides the most natural setting to express holographic algorithms and transformations.

Our main result is a dichotomy theorem for the framework of Counting Constraint Satisfaction Problems (#CSP), but we prove it in a generalized framework called Holant problems [17, 18, 13, 15]. We explain the main advantage of working with Holant problems shortly. A set of functions $\mathcal{F}$ defines the problem Holant($\mathcal{F}$). An instance of this problem is a tuple $\Omega = (G, \mathcal{F}, \pi)$ called a *signature grid*, where $G = (V, E)$ is a graph, π labels each $v \in V$ with a function $f_v \in \mathcal{F}$, and

f_v maps $\{0,1\}^{\deg(v)}$ to $\mathbb{C}$. We also call the functions in $\mathcal{F}$ *signatures*. An assignment σ for every $e \in E$ gives an evaluation $\prod_{v\in V} f_v(\sigma\mid_{E(v)})$, where $E(v)$ denotes the incident edges of v and $\sigma\mid_{E(v)}$ denotes the restriction of σ to $E(v)$. The counting problem on the instance Ω is to compute

$$\text{Holant}_\Omega = \sum_{\sigma:E\to\{0,1\}} \prod_{v\in V} f_v\left(\sigma\mid_{E(v)}\right).$$

Counting the number of perfect matchings in G corresponds to attaching the Exact-One signature at every vertex of G. A function or signature is called *symmetric* if its output depends only on the Hamming weight of the input. We often denote a symmetric signature by the list of its outputs sorted by input Hamming weight in ascending order. For example, $[0,1,0,0]$ is the Exact-One function on three bits. The output is 1 if and only if the input is 001, 010, or 100, and 0 otherwise.

We consider #CSP, which are also parametrized by a set of functions $\mathcal{F}$. The problem #CSP($\mathcal{F}$) is equivalent to Holant($\mathcal{F}\cup\mathcal{EQ}$), where $\mathcal{EQ} = \{=_1, =_2, =_3, \dots\}$ and $(=_k) = [1,0,\dots,0,1]$ is the equality signature of arity k.

We often consider a Holant problem over bipartite graphs, which is denoted by Holant($\mathcal{F}\mid\mathcal{G}$), where the sets $\mathcal{F}$ and $\mathcal{G}$ contain the signatures available for assignment to the vertices in each partition. Considering the edge-vertex incident graph, one can see that Holant($\mathcal{F}$) $\equiv_T$ Holant($=_2\mid\mathcal{F}$). One powerful tool in this setting is the holographic transformation. Let T be a nonsingular 2-by-2 matrix and define $T\mathcal{F} = \{T^{\otimes\,\text{arity}(f)} f \mid f\in\mathcal{F}\}$. Here we view f as a column vector by listing its values in lexicographical order as in a truth table. Similarly $\mathcal{F}T$ is defined by viewing $f\in\mathcal{F}$ as a row vector. Valiant's Holant theorem [44] states that Holant($\mathcal{F}\mid\mathcal{G}$) $\equiv_T$ Holant($\mathcal{F}T^{-1}\mid T\mathcal{G}$).

Cai, Lu, and Xia gave a dichotomy for complex-weighted Boolean #CSP($\mathcal{F}$) in [13]. Let Pl-#CSP($\mathcal{F}$) (resp. Pl-Holant($\mathcal{F}$)) denote the #CSP (resp. Holant problem) defined by $\mathcal{F}$ when the inputs are restricted to planar graphs. In this paper, we investigate the complexity of Pl-#CSP($\mathcal{F}$) for a set $\mathcal{F}$ of symmetric complex-weighted functions. In particular, we would like to determine which sets become tractable under this planarity restriction. Holographic algorithms with matchgates provide planar tractable problems for sets that are matchgate realizable after a holographic transformation. From the Holant perspective, the signatures in $\mathcal{EQ}$ are always available in #CSP($\mathcal{F}$). By the signature theory of Cai and Lu [12], the Hadamard matrix $H = \left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]$ essentially defines the only[1] holographic transformation under which $\mathcal{EQ}$ becomes matchgate realizable. Let $\widehat{\mathcal{F}}$ denote $H\mathcal{F}$ for any set of signatures $\mathcal{F}$. Then $\widehat{\mathcal{EQ}}$ is $\{[1,0],[1,0,1],[1,0,1,0],\dots\}$ while $(=_2)(H^{-1})^{\otimes 2}$ is still $=_2$. Therefore #CSP($\mathcal{F}$) $\equiv_T$ Holant($\mathcal{F}\cup\mathcal{EQ}$) $\equiv_T$ Holant($\widehat{\mathcal{F}}\cup\widehat{\mathcal{EQ}}$) by Valiant's Holant theorem.

Our main dichotomy theorem is stated as follows.

Theorem 1.1. *Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then* Pl-#CSP($\mathcal{F}$) *is* #P*-hard unless $\mathcal{F}$ satisfies one of the following conditions, in which case it is tractable:*

1. #CSP($\mathcal{F}$) *is tractable (cf. [13]); or*
2. *$\widehat{\mathcal{F}}$ is realizable by matchgates (cf. [12]).*

In many previous dichotomy theorems for Boolean #CSP($\mathcal{F}$), the proof of hardness began by pinning, which is to realize the constant functions $[1,0]$ and $[0,1]$. Formally speaking, the concept of pinning is to prove #CSP($\mathcal{F}$) is #P-hard (or in P) if and only if #CSP($\mathcal{F}\cup\{[1,0],[0,1]\}$) is

[1] Up to transformations under which matchgates are closed.

#P-hard (or in P), which was always achieved by a *nonplanar* reduction. This does not imply the collapse of any complexity classes because the tractable sets for #CSP($\mathcal{F}$) include $[1,0]$ and $[0,1]$. However, $\mathcal{EQ}$ with $\{[1,0],[0,1]\}$ are not simultaneously realizable as matchgates. Therefore, according to our main result, if pinning were possible for Pl-#CSP($\mathcal{F}$), then #P collapses to P! Instead, apply the Hadamard transformation and consider Pl-Holant($\widehat{\mathcal{F}} \cup \widehat{\mathcal{EQ}}$) as mentioned earlier. In this Hadamard basis, pinning becomes possible again since $[1,0]$ and $[0,1]$ are included in every tractable set. Indeed, we prove our pinning result in this Hadamard basis in Section 8.

For Holant problems, it is often important to understand the complexity of the small arity cases first [14, 25, 8]. In [14], Cai, Lu, and Xia gave a dichotomy for Pl-Holant(f) when f is a symmetric arity 3 signature while a dichotomy for Holant(f) when f is a symmetric arity 4 signature was shown in [8]. In the proof of the latter result, most of the reductions were planar. However, the crucial starting point for hardness, namely counting Eulerian orientations (#EO) over 4-regular graphs, was not known to be #P-hard under the planarity restriction. Huang and Lu [25] had recently proved that #EO is #P-hard over 4-regular graphs but left open its complexity when the input is also planar. We show that #EO remains #P-hard over planar 4-regular graphs. The problem we reduce from is the evaluation of the Tutte polynomial at the point (3,3), which is naturally expressed in the Holant framework. In addition, we determine the complexity of counting complex-weighted matchings over planar 4-regular graphs. The problem is #P-hard except for the tractable case of counting perfect matchings. With these two ingredients, we obtain a dichotomy for Pl-Holant(f) when f is a symmetric arity 4 signature.

Our result is a generalization of the dichotomy by Cai, Lu, and Xia [14] for Pl-#CSP($\mathcal{F}$) when $\mathcal{F}$ contains symmetric real-weighted Boolean functions. It is natural to consider complex weights in the Holant framework because surprising equivalences between problems are often discovered via complex holographic transformations, sometimes even between problems using only rational weights. Extending the range from $\mathbb{R}$ to $\mathbb{C}$ also enlarges the set of problems that can be transformed into the framework.

However, a dichotomy for complex weights is more technically challenging. The proof technique of polynomial interpolation often has infinitely many failure cases in $\mathbb{C}$ corresponding to the infinitely many roots of unity, which prevents a brute force analysis of failure cases as was done in [14]. This increased difficulty requires us to develop new ideas to bypass previous interpolation proofs. In particular, we perform a planar interpolation with a rotationally invariant signature to prove the #P-hardness of #EO over planar 4-regular graphs. For the complexity of counting complex-weighted matchings over planar 4-regular graphs, we introduce the notion of planar pairings to build reductions. We show that every planar 3-regular graph has a planar pairing and that it can be efficiently computed. We also refine and extend existing techniques for application in the new setting, including the recursive unary construction, the anti-gadget technique, compressed matrix criteria, domain pairing, and pinning in the Hadamard basis.

This paper is organized as follows. In Section 2, we give a review of terminology and previous dichotomy theorems. In Section 3, we prove that counting Eulerian orientations is #P-hard for planar 4-regular graphs. In Section 4, we strengthen a popular interpolation technique that uses recursive constructions, which leads to simpler proofs. In Section 5, we obtain our dichotomy theorem for Pl-Holant(f) when f is a symmetric arity 4 signature with complex weights. In Section 6, we prove several useful lemmas about a technique we call *domain pairing* that essentially realizes an odd arity signature using only signatures of even arity. In Section 7, we show that the three known sets of tractable signatures become #P-hard when mixed. In Section 8, we use the pinning technique in a new planar proof to realize the constant functions $[1,0]$ and $[0,1]$. In Section 9, we obtain our dichotomy theorem for Pl-#CSP($\mathcal{F}$).

2 Preliminaries

2.1 Problems and Definitions

The framework of Holant problems is defined for functions mapping any $[q]^k \to \mathbb{F}$ for a finite q and some field $\mathbb{F}$. In this paper, we investigate the complex-weighted Boolean Holant problems, that is, all functions are $[2]^k \to \mathbb{C}$. Strictly speaking, for consideration of models of computation, functions take complex algebraic numbers.

A *signature grid* $\Omega = (G, \mathcal{F}, \pi)$ consists of a graph $G = (V, E)$, where each vertex is labeled by a function $f_v \in \mathcal{F}$, and $\pi : V \to \mathcal{F}$ is the labeling. If the graph G is planar, then we call Ω a *planar signature grid*. The Holant problem on instance Ω is to evaluate $\text{Holant}_\Omega = \sum_\sigma \prod_{v \in V} f_v(\sigma \mid_{E(v)})$, a sum over all edge assignments $\sigma : E \to \{0, 1\}$.

A function f_v can be represented by its truth table, which is a vector in $\mathbb{C}^{2^{\deg(v)}}$, or as a tensor in $(\mathbb{C}^2)^{\otimes \deg(v)}$. We also use f^α to denote the value $f(\alpha)$, where α is a binary string. A function $f \in \mathcal{F}$ is also called a *signature*. A symmetric signature f on k Boolean variables can be expressed as $[f_0, f_1, \ldots, f_k]$, where f_w is the value of f on inputs of Hamming weight w. In this paper, we consider symmetric signatures. Since a signature of arity k must be placed on a vertex of degree k, we can represent a signature of arity k by a labeled vertex with k ordered dangling edges. Throughout this paper, we do not distinguish between these two views.

A Holant problem is parametrized by a set of signatures.

Definition 2.1. *Given a set of signatures $\mathcal{F}$, we define the counting problem* Holant($\mathcal{F}$) *as:*
Input: A signature grid $\Omega = (G, \mathcal{F}, \pi)$*;*
Output: Holant_Ω.

The problem Pl-Holant($\mathcal{F}$) is defined similarly using a planar signature grid. The Holant^c framework is the special case of Holant problems when the constant signatures of the domain are freely available. In the Boolean domain, the constant signatures are $[1,0]$ and $[0,1]$.

Definition 2.2. *Given a set of signatures $\mathcal{F}$,* $\text{Holant}^c(\mathcal{F})$ *denotes* $\text{Holant}(\mathcal{F} \cup \{[0,1],[1,0]\})$.

The problem Pl-$\text{Holant}^c(\mathcal{F})$ is defined similarly. A symmetric signature f of arity n is *degenerate* if and only if there exists a unary signature u such that $f = u^{\otimes n}$. Replacing a signature $f \in \mathcal{F}$ by a constant multiple cf, where $c \neq 0$, does not change the complexity of Holant($\mathcal{F}$). It introduces a global factor to Holant_Ω. Hence, for two signatures f, g of the same arity, we use $f \neq g$ to mean that these signatures are not equal in the projective space sense, i.e. not equal up to any nonzero constant multiple. We follow the usual conventions about polynomial time Turing reduction $\leq_T$ and polynomial time Turing equivalence $\equiv_T$.

2.2 Holographic Reduction

To introduce the idea of holographic reductions, it is convenient to consider bipartite graphs. For a general graph, we can always transform it into a bipartite graph while preserving the Holant value, as follows. For each edge in the graph, we replace it by a path of length two. (This operation is called the *2-stretch* of the graph and yields the edge-vertex incident graph.) Each new vertex is assigned the binary EQUALITY signature $(=_2) = [1, 0, 1]$.

We use Holant$(\mathcal{F} \mid \mathcal{G})$ to denote the Holant problem on bipartite graphs $H = (U, V, E)$, where each signature for a vertex in U or V is from $\mathcal{F}$ or $\mathcal{G}$, respectively. An input instance for this bipartite Holant problem is a bipartite signature grid and is denoted by $\Omega = (H;\ \mathcal{F} \mid \mathcal{G};\ \pi)$. Signatures in $\mathcal{F}$ are considered as row vectors (or covariant tensors); signatures in $\mathcal{G}$ are considered as column vectors (or contravariant tensors) [19]. Similarly, Pl-Holant$(\mathcal{F} \mid \mathcal{G})$ denotes the Holant problem on planar bipartite graphs.

For a 2-by-2 matrix T and a signature set $\mathcal{F}$, define $T\mathcal{F} = \{g \mid \exists f \in \mathcal{F} \text{ of arity } n,\ g = T^{\otimes n} f\}$, similarly for $\mathcal{F}T$. Whenever we write $T^{\otimes n} f$ or $T\mathcal{F}$, we view the signatures as column vectors; similarly for $fT^{\otimes n}$ or $\mathcal{F}T$ as row vectors. In the special case that $T = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$, we also define $T\mathcal{F} = \widehat{\mathcal{F}}$.

Let T be an invertible 2-by-2 matrix. The holographic transformation defined by T is the following operation: given a signature grid $\Omega = (H;\ \mathcal{F} \mid \mathcal{G};\ \pi)$, for the same graph H, we get a new grid $\Omega' = (H;\ \mathcal{F}T \mid T^{-1}\mathcal{G};\ \pi')$ by replacing each signature in $\mathcal{F}$ or $\mathcal{G}$ with the corresponding signature in $\mathcal{F}T$ or $T^{-1}\mathcal{G}$.

Theorem 2.3 (Valiant's Holant Theorem [44])**.** *If there is a holographic transformation mapping signature grid Ω to Ω', then* $\text{Holant}_\Omega = \text{Holant}_{\Omega'}$.

Therefore, an invertible holographic transformation does not change the complexity of the Holant problem in the bipartite setting. Furthermore, there is a special kind of holographic transformation, the orthogonal transformation, that preserves the binary equality and thus can be used freely in the standard setting.

Theorem 2.4 (Theorem 2.2 in [13])**.** *Suppose T is a 2-by-2 orthogonal matrix ($TT^{\mathsf{T}} = I_2$) and let $\Omega = (H, \mathcal{F}, \pi)$ be a signature grid. Under a holographic transformation by T, we get a new grid $\Omega' = (H, T\mathcal{F}, \pi')$ and* $\text{Holant}_\Omega = \text{Holant}_{\Omega'}$.

Since the complexity of signatures are equivalent up to a nonzero constant factor, we also call a transformation T such that $TT^{\mathsf{T}} = \lambda I$ for some $\lambda \neq 0$ an orthogonal transformation. Such transformations do not change the complexity of a problem.

2.3 Realization

One basic notion used throughout the paper is realization. We say a signature f is *realizable* or *constructible* from a signature set $\mathcal{F}$ if there is a gadget with some dangling edges such that each vertex is assigned a signature from $\mathcal{F}$, and the resulting graph, when viewed as a black-box signature with inputs on the dangling edges, is exactly f. If f is realizable from a set $\mathcal{F}$, then we can freely add f into $\mathcal{F}$ preserving the complexity.

Formally, such a notion is defined by an $\mathcal{F}$-gate [13, 14]. An $\mathcal{F}$-gate is similar to a signature grid $(H, \mathcal{F}, \pi)$ except that $H = (V, E, D)$ is a graph with some dangling edges D. The dangling edges define external variables for the $\mathcal{F}$-gate. (See Figure 1 for an example.) We denote the regular edges in E by $1, 2, \ldots, m$, and denote the dangling edges in D by $m+1, \ldots, m+n$. Then we can define a function Γ for this $\mathcal{F}$-gate as

$$\Gamma(y_1, y_2, \ldots, y_n) = \sum_{x_1, x_2, \ldots, x_m \in \{0,1\}} H(x_1, x_2, \ldots, x_m, y_1, \ldots, y_n),$$

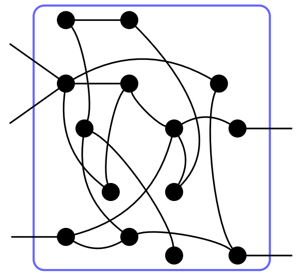

Figure 1: An $\mathcal{F}$-gate with 5 dangling edges.

where $(y_1, y_2, \ldots, y_n) \in \{0,1\}^n$ denotes an assignment on the dangling edges and $H(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n)$ denotes the value of the signature grid on an assignment of all edges, which is the product of evaluations at all internal vertices. We also call this function the signature Γ of the $\mathcal{F}$-gate. An $\mathcal{F}$-gate with underlying graph H is planar if the graph H', formed by introducing a new vertex v and attaching each dangling edge to v, is also planar. A planar $\mathcal{F}$-gate can be used in a planar signature grid as if it is just a single vertex with the particular signature.

Using the idea of planar $\mathcal{F}$-gates, we can reduce one planar Holant problem to another. Suppose g is the signature of some planar $\mathcal{F}$-gate. Then Pl-Holant($\mathcal{F} \cup \{g\}$) $\leq_T$ Pl-Holant($\mathcal{F}$). The reduction is quite simple. Given an instance of Pl-Holant($\mathcal{F} \cup \{g\}$), by replacing every appearance of g by the $\mathcal{F}$-gate, we get an instance of Pl-Holant($\mathcal{F}$). Since the signature of the $\mathcal{F}$-gate is g, the Holant values for these two signature grids are identical.

We note that even for a very simple signature set $\mathcal{F}$, the signatures for all planar $\mathcal{F}$-gates can be quite complicated and expressive.

2.4 #CSP and Its Tractable Signatures

An instance of #CSP($\mathcal{F}$) has the following bipartite view. We make a node for each variable and each constraint. Connect a variable node to a constraint node if the variable appears in the constraint function. This bipartite graph is also known as the *constraint graph.* Under this view, we can see that

$$\#\mathrm{CSP}(\mathcal{F}) \equiv_T \mathrm{Holant}(\mathcal{F} \mid \mathcal{EQ}) \equiv_T \mathrm{Holant}(\mathcal{F} \cup \mathcal{EQ}),$$

where $\mathcal{EQ} = \{=_1, =_2, =_3, \ldots\}$ is the set of equalities of all arities. This equivalence also holds for the planar versions of these frameworks.

For the #CSP framework, the following two sets of signatures are tractable [13].

Definition 2.5. *A k-ary function $f(x_1, \ldots, x_k)$ is* affine *if it has the form*

$$\lambda \chi_{Ax=0} \cdot \sqrt{-1}^{\sum_{j=1}^{n} \langle \alpha_j, x \rangle},$$

where $\lambda \in \mathbb{C}$, $x = (x_1, x_2, \ldots, x_k, 1)^T$, A is a matrix over $\mathbb{F}_2$, α_j is a vector over $\mathbb{F}_2$, and χ is a 0-1 indicator function such that $\chi_{Ax=0}$ is 1 iff $Ax = 0$. Note that the dot product $\langle \alpha_j, x \rangle$ is calculated over $\mathbb{F}_2$, while the summation $\sum_{j=1}^{n}$ on the exponent of $i = \sqrt{-1}$ is evaluated as a sum mod 4 of 0-1 terms. We use $\mathscr{A}$ to denote the set of all affine functions.

Definition 2.6. *A function is of* product type *if it can be expressed as a product of unary functions, binary equality functions* ($[1,0,1]$)*, and binary disequality functions* ($[0,1,0]$)*. We use $\mathscr{P}$ to denote the set of product type functions.*

An alternate definition for $\mathscr{P}$, implicit in [16], is the tensor closure of signatures with support on two entries of complement indices.

It is easy to see (cf. Lemma A.1 in the full version of [25]) that if f is a symmetric signature in $\mathscr{P}$, then f is either degenerate, binary disequality, or generalized equality (i.e. $[a, 0, \ldots, 0, b]$ for $a, b \in \mathbb{C}$). Since our main dichotomy theorem is for symmetric signatures, we use $\mathscr{A}$ (resp. $\mathscr{P}$) to refer to the set of *symmetric* affine (resp. product-type) signatures. It is known that the set of non-degenerate symmetric signatures in $\mathscr{A}$ is contained in $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$, where $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$ are three families of signatures defined as

$$\mathscr{F}_1 = \left\{\lambda \left([1,0]^{\otimes k} + i^r [0,1]^{\otimes k}\right) \mid \lambda \in \mathbb{C}, k = 1,2,\ldots, r = 0,1,2,3\right\},$$
$$\mathscr{F}_2 = \left\{\lambda \left([1,1]^{\otimes k} + i^r [1,-1]^{\otimes k}\right) \mid \lambda \in \mathbb{C}, k = 1,2,\ldots, r = 0,1,2,3\right\}, \text{ and}$$
$$\mathscr{F}_3 = \left\{\lambda \left([1,i]^{\otimes k} + i^r [1,-i]^{\otimes k}\right) \mid \lambda \in \mathbb{C}, k = 1,2,\ldots, r = 0,1,2,3\right\}.$$

We explicitly list all the signatures in $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$ up to an arbitrary constant multiple from $\mathbb{C}$:

1. $[1, 0, \ldots, 0, \pm 1]$; $(\mathscr{F}_1, r = 0, 2)$
2. $[1, 0, \ldots, 0, \pm i]$; $(\mathscr{F}_1, r = 1, 3)$
3. $[1, 0, 1, 0, \ldots, 0 \text{ or } 1]$; $(\mathscr{F}_2, r = 0)$
4. $[1, -i, 1, -i, \ldots, (-i) \text{ or } 1]$; $(\mathscr{F}_2, r = 1)$
5. $[0, 1, 0, 1, \ldots, 0 \text{ or } 1]$; $(\mathscr{F}_2, r = 2)$
6. $[1, i, 1, i, \ldots, i \text{ or } 1]$; $(\mathscr{F}_2, r = 3)$
7. $[1, 0, -1, 0, 1, 0, -1, 0, \ldots, 0 \text{ or } 1 \text{ or } (-1)]$; $(\mathscr{F}_3, r = 0)$
8. $[1, 1, -1, -1, 1, 1, -1, -1, \ldots, 1 \text{ or } (-1)]$; $(\mathscr{F}_3, r = 1)$
9. $[0, 1, 0, -1, 0, 1, 0, -1, \ldots, 0 \text{ or } 1 \text{ or } (-1)]$; $(\mathscr{F}_3, r = 2)$
10. $[1, -1, -1, 1, 1, -1, -1, 1, \ldots, 1 \text{ or } (-1)]$. $(\mathscr{F}_3, r = 3)$

In the Holant framework, there are two corresponding signature sets that are tractable. A signature f is $\mathscr{A}$-transformable if there exists a holographic transformation T such that $f \in T\mathscr{A}$ and $[1, 0, 1]T^{\otimes 2} \in \mathscr{A}$. Similarly, a signature f is $\mathscr{P}$-transformable if there exists a holographic transformation T such that $f \in T\mathscr{P}$ and $[1, 0, 1]T^{\otimes 2} \in \mathscr{P}$. These two families are tractable because after a transformation by T, it is a tractable #CSP instance.

2.5 Matchgate Signatures

Matchgates were introduced by Valiant [42, 41] and are combinatorial in nature. They encode computation as a sum of weighted perfect matchings, which has a polynomial-time algorithm by the work of Kasteleyn [28].

We say a signature is a *matchgate signature* if there is some matchgate that realizes this signature and use $\mathscr{M}$ to denote the set of all matchgate signatures. The parity of a matchgate signature is even (resp. odd) if its support is on entries of even (resp. odd) Hamming weight. Lemmas 6.2 and 6.3 in [6] (and the paragraph the follows them) characterize the symmetric signatures in $\mathscr{M}$. Instead of formally stating these two lemmas, we explicitly list all the symmetric signatures in $\mathscr{M}$: For any $\alpha, \beta \in \mathbb{C}$,

1. $[\alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n]$;
2. $[\alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n, 0]$;
3. $[0, \alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n]$;
4. $[0, \alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n, 0]$.

Roughly speaking, the symmetric matchgate signatures have 0 for every other entry (which is called the *parity condition*), and form a geometric progression with the remaining entries.

In the standard basis of the #CSP framework, the set of signatures $\left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]\mathscr{M} = \widehat{\mathscr{M}}$ is tractable and consists of signatures with the following expressions.[2]

Theorem 2.7 (Special case of Theorem 4 in [11])**.** *A symmetric signature* $[f_0, f_1, \ldots, f_n]$ *is realizable under the basis* $\left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]$ *iff it takes one of the following forms:*

1. *there exists constants* $\lambda, \alpha, \beta \in \mathbb{C}$ *and* $\varepsilon = \pm 1$*, such that for all* ℓ*,* $0 \le \ell \le n$*,*
$$f_\ell = \lambda[(\alpha+\beta)^{n-\ell}(\alpha-\beta)^{\ell} + \varepsilon(\alpha-\beta)^{n-\ell}(\alpha+\beta)^{\ell}];$$
2. *there exists a constant* $\lambda \in \mathbb{C}$*, such that for all* ℓ*,* $0 \le \ell \le n$*,*
$$f_\ell = \lambda(n-2\ell)(-1)^{\ell};$$
3. *there exists a constant* $\lambda \in \mathbb{C}$*, such that for all* ℓ*,* $0 \le \ell \le n$*,*
$$f_\ell = \lambda(n-2\ell).$$

We note that case 1 corresponds to the general case ($\varepsilon = +1$ for signatures with even parity and $\varepsilon = -1$ for signatures with odd parity) while case 3 corresponds to the perfect matching signatures $[0, 1, 0, \ldots, 0]$ and case 2 corresponds to their reversals.

We summarize the known tractability results for the Pl-#CSP framework in the following theorem, which is stated in the Hadamard basis with $[1, 0]$ and $[0, 1]$ present.

Theorem 2.8. *Let* $\mathcal{F}$ *be any set of symmetric, complex-valued signatures in Boolean variables. Then* Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is tractable if* $\mathcal{F} \subseteq \mathscr{A}$*,* $\mathcal{F} \subseteq \widehat{\mathscr{P}}$*, or* $\mathcal{F} \subseteq \mathscr{M}$*.*

We also say a signature f is $\mathscr{M}$-transformable if there exists a holographic transformation T such that $f \in T\mathscr{M}$ and $[1, 0, 1]T^{\otimes 2} \in \mathscr{M}$.

2.6 Some Known Dichotomies

We use the dichotomy for a single ternary signature in the Holant framework to prove the dichotomy for a single arity 4 signature. A signature is called *vanishing* if the Holant of any signature grid using only that signature is zero [8].

Theorem 2.9 (Special case of Theorem V.1 in [14])**.** *If* f *is a symmetric, non-degenerate, complex-valued signature, then* Pl-Holant(f) *is* #P*-hard unless* f *satisfies one of the following conditions, in which case the problem is in* P*:*

1. Holant(f) *is tractable (i.e.* f *is* $\mathscr{A}$*-transformable,* $\mathscr{P}$*-transformable, or vanishing);*
2. f *is* $\mathscr{M}$*-transformable.*

We use the following theorem about edge-weighted signatures on degree prescribed graphs in both of our dichotomy theorems.

[2]Even though Theorem 2.7 is technically about generator signatures, neither generators nor recognizers are mentioned because Theorems 3 and 4 in [11] coincide when the basis is an orthogonal transformation.

Theorem 2.10 (Theorem 22 in [29])**.** *Let $S \subseteq \mathbb{Z}^+$ be nonempty, let $\mathcal{G} = \{=_k \mid k \in S\}$, and let $d = \gcd(S)$. Then* Pl-Holant$([f_0, f_1, f_2] \mid \mathcal{G})$ *is* #P*-hard for all $f_0, f_1, f_2 \in \mathbb{C}$ unless one of the following conditions hold, in which case the problem is in* P*:*

1. $\mathcal{G} \subseteq \{=_1, =_2\}$*;*
2. $f_0 f_2 = f_1^2$*;*
3. $f_1 = 0$*;*
4. $f_0 f_2 = -f_1^2 \wedge f_0^d = -f_2^d$*;*
5. $f_0^d = f_2^d$*.*

For the arity 4 dichotomy, we use Theorem 2.10 with $\mathcal{G} = \{=_4\}$. For the Pl-#CSP dichotomy, we use Theorem 2.10 with $\mathcal{G} = \mathcal{EQ}$, which is the special case of Pl-#CSP$(\mathcal{F})$ when $\mathcal{F}$ contains a single binary signature. Over general domains, this special case is also known as counting graph homomorphism from a planar input graph to a fixed target graph. Furthermore, we perform a holographic transformation by the Hadamard matrix $H = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$. Under this transformation, it is easy to see that the conditions $f_0 f_2 = f_1^2$ and $f_0 f_2 = -f_1^2 \wedge f_0 = -f_2$ are invariant while the conditions $f_1 = 0$ and $f_0 = f_2$ map to each other. Therefore, by an apparent coincidence, the tractability conditions remain the same. To be clear, we restate Theorem 2.10 both before and after a holographic transformation by H with $\mathcal{G} = \mathcal{EQ}$.

Theorem 2.11 (Special case of Theorem 2.10)**.** *For any $f_0, f_1, f_2 \in \mathbb{C}$, both* Pl-Holant$([f_0, f_1, f_2] \mid \mathcal{EQ})$ *and* Pl-Holant$([f_0, f_1, f_2] \mid \widehat{\mathcal{EQ}})$ *are* #P*-hard unless one of the following conditions hold, in which case both problems are in* P*:*

1. $f_0 f_2 = f_1^2$*;*
2. $f_1 = 0$*;*
3. $f_0 f_2 = -f_1^2$ *and* $f_0 = -f_2$*;*
4. $f_0 = f_2$*.*

3 The Complexity of Counting Eulerian Orientations

Recall the definition of an Eulerian orientation.

Definition 3.1. *Given a graph G, an orientation of its edges is an* Eulerian orientation *if for each vertex v of G, the number of incoming edges of v equals the number of outgoing edges of v.*

Counting the number of Eulerian orientations over 4-regular graphs was shown to be #P-hard in Theorem V.10 of [25]. We improve this result by showing that this problem remains #P-hard when the input is also planar. The reduction begins with the problem of evaluating the Tutte polynomial at the point (3,3), which is #P-hard even for planar graphs.

Theorem 3.2 (Theorem 5.1 in [47])**.** *For any $x, y \in \mathbb{C}$, the problem of computing the Tutte polynomial at (x, y) over planar graphs is* #P*-hard unless $(x-1)(y-1) \in \{1, 2\}$ or $(x, y) \in \{(1,1), (-1,-1), (j, j^2), (j^2, j)\}$, where $j = e^{2\pi i/3}$. In each of these exceptional cases, the computation can be done in polynomial time.*

The first step in the reduction concerns a sum of weighted Eulerian orientations on the medial graph of a planar graph. Recall the definition of a medial graph (cf. Section 2 in [45]).

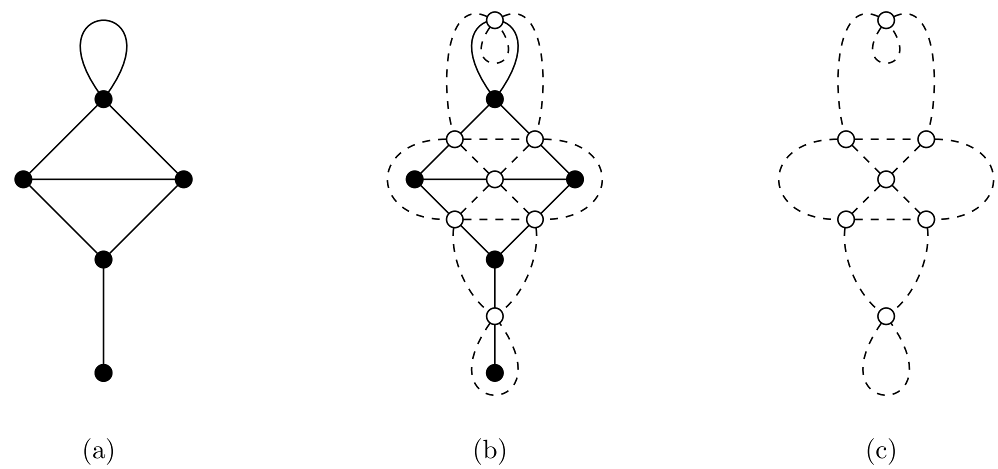

Figure 2: A planar graph (a), its medial graph (c), and the two graphs superimposed (b). This image is from [2].

Definition 3.3. *For a connected planar graph G, its* medial graph *H has a vertex for each edge of G and two vertices in H are adjacent if their corresponding edges in G are adjacent in some face of G.*

An example of a planar graph and its medial graph are given in Figure 2. Notice that the medial graph of a planar graph is always a planar 4-regular graph. Las Vergnas [46] connected the evaluation of the Tutte polynomial at the point (3,3) with a sum of weighted Eulerian orientations.

Theorem 3.4 (Theorem 2.1 in [46])**.** *Let G be a connected planar graph and let $\mathscr{O}(H)$ be the set of all Eulerian orientations of the medial graph H of G. Then*

$$2 \cdot \mathrm{T}(G; 3, 3) = \sum_{O \in \mathscr{O}(H)} 2^{\beta(O)}, \tag{1}$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. the number of vertices in which the edges are oriented "in, out, in, out" in cyclic order.

In addition to these two theorems, our proof also uses two definitions from [8].

Definition 3.5 (Definition 5.1 in [8])**.** *A 4-by-4 matrix is* redundant *if its middle two rows and middle two columns are the same.*

An example of a redundant matrix is the signature matrix of a symmetric arity 4 signature.

Definition 3.6 (Definition 5.2 in [8])**.** *The* signature matrix *of a symmetric arity 4 signature $f = [f_0, f_1, f_2, f_3, f_4]$ is*

$$M_f = \begin{bmatrix} f_0 & f_1 & f_1 & f_2 \\ f_1 & f_2 & f_2 & f_3 \\ f_1 & f_2 & f_2 & f_3 \\ f_2 & f_3 & f_3 & f_4 \end{bmatrix}.$$

This definition extends to an asymmetric signature g as

$$M_g = \begin{bmatrix} g^{0000} & g^{0010} & g^{0001} & g^{0011} \\ g^{0100} & g^{0110} & g^{0101} & g^{0111} \\ g^{1000} & g^{1010} & g^{1001} & g^{1011} \\ g^{1100} & g^{1110} & g^{1101} & g^{1111} \end{bmatrix}.$$

When we present g as an $\mathcal{F}$-gate, we order the four external edges ABCD counterclockwise. In M_g, the row index bits are ordered AB and the column index bits are ordered DC, in a reverse way. This is for convenience so that the signature matrix of the linking of two arity 4 $\mathcal{F}$-gates is the matrix product of the signature matrices of the two $\mathcal{F}$-gates.

If M_g is redundant, we also define the compressed signature matrix *of g as*

$$\widetilde{M_g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} M_g \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now we can prove our hardness result.

Theorem 3.7. #Eulerian-Orientations *is #P-hard for planar 4-regular graphs.*

Proof. We reduce calculating the right-hand side of Equation (1) to Pl-Holant($\neq_2 \mid [0,0,1,0,0]$), which denotes the problem of counting the number of Eulerian orientations over planar 4-regular graphs as bipartite Holant problem. Then by Theorems 3.2 and 3.4, we conclude that Pl-Holant($\neq_2 \mid [0,0,1,0,0]$) is #P-hard.

The right-hand side of Equation (1) is the bipartite Holant problem Pl-Holant($\neq_2 \mid f$), where the signature matrix of f is

$$M_f = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

We perform a holographic transformation by $Z = \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$ to get

$$\begin{aligned} \text{Pl-Holant}(\neq_2 \mid f) &\equiv_T \text{Pl-Holant}([0,1,0](Z^{-1})^{\otimes 2} \mid Z^{\otimes 4} f) \\ &\equiv_T \text{Pl-Holant}([1,0,1]/2 \mid 4\hat{f}) \\ &\equiv_T \text{Pl-Holant}(\hat{f}), \end{aligned}$$

where the signature matrix of $\hat{f}$ is

$$M_{\hat{f}} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

We also perform the same holographic transformation by Z on our target counting problem Pl-Holant($\neq_2 \mid [0,0,1,0,0]$) to get

$$\begin{aligned} \text{Pl-Holant}(\neq_2 \mid [0,0,1,0,0]) &\equiv_T \text{Pl-Holant}([0,1,0](Z^{-1})^{\otimes 2} \mid Z^{\otimes 4}[0,0,1,0,0]) \\ &\equiv_T \text{Pl-Holant}([1,0,1]/2 \mid 2[3,0,1,0,3]) \\ &\equiv_T \text{Pl-Holant}([3,0,1,0,3]). \end{aligned}$$

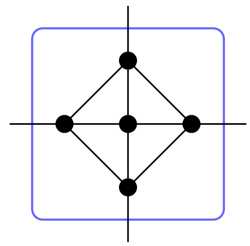

Figure 3: The planar tetrahedron gadget. Each vertex is assigned $[3, 0, 1, 0, 3]$.

Using the planar tetrahedron gadget in Figure 3, we assign $[3, 0, 1, 0, 3]$ to every vertex and obtain a gadget with signature $32\hat{g}$, where the signature matrix of $\hat{g}$ is

$$M_{\hat{g}} = \frac{1}{2}\begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}.$$

Now we show how to reduce Pl-Holant$(\hat{f})$ to Pl-Holant$(\hat{g})$ by interpolation. Consider an instance Ω of Pl-Holant$(\hat{f})$. Suppose that $\hat{f}$ appears n times in Ω. We construct from Ω a sequence of instances Ω_s of Holant$(\hat{g})$ indexed by $s \geq 1$. We obtain Ω_s from Ω by replacing each occurrence of $\hat{f}$ with the gadget N_s in Figure 4 with $\hat{g}$ assigned to all vertices. Although $\hat{f}$ and $\hat{g}$ are asymmetric signatures, they are invariant under a cyclic permutation of their inputs. Thus, it is unnecessary to specify which edge corresponds to which input. We call such signatures *rotationally symmetric*.

To obtain Ω_s from Ω, we effectively replace $M_{\hat{f}}$ with $M_{N_s} = (M_{\hat{g}})^s$, the sth power of the signature matrix $M_{\hat{g}}$. Let

$$T = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Then

$$M_{\hat{f}} = T\Lambda_{\hat{f}}T^{-1} = T\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}T^{-1} \qquad \text{and} \qquad M_{\hat{g}} = T\Lambda_{\hat{g}}T^{-1} = T\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}T^{-1}.$$

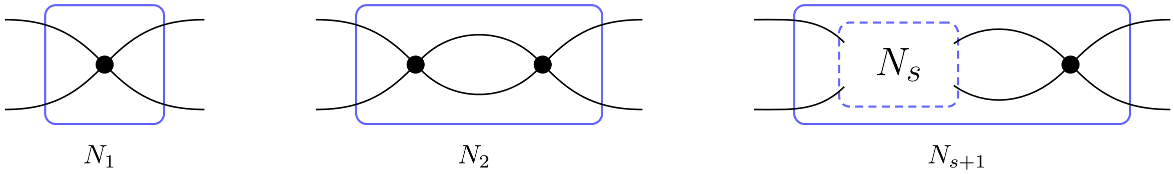

Figure 4: Recursive construction to interpolate $\hat{f}$. The vertices are assigned $\hat{g}$.

We can view our construction of Ω_s as first replacing each $M_{\hat{f}}$ by $T\Lambda_{\hat{f}}T^{-1}$, which does not change the Holant value, and then replacing each $\Lambda_{\hat{f}}$ with $\Lambda_{\hat{g}}^s$. We stratify the assignments in Ω based on the assignment to $\Lambda_{\hat{f}}$. We only need to consider the assignments to $\Lambda_{\hat{f}}$ that assign

- 0000 j many times,
- 0110 or 1001 k many times, and
- 1111 ℓ many times.

Let $c_{jk\ell}$ be the sum over all such assignments of the products of evaluations (including the contributions from T and T^{-1}) on Ω. Then

$$\text{Pl-Holant}_{\Omega} = \sum_{j+k+\ell=n} 3^{\ell} c_{jk\ell}$$

and the value of the Holant on Ω_s, for $s \geq 1$, is

$$\text{Pl-Holant}_{\Omega_s} = \sum_{j+k+\ell=n} (6^k 13^{\ell})^s c_{jk\ell}.$$

This coefficient matrix in the linear system involving $\text{Pl-Holant}_{\Omega_s}$ is Vandermonde and of full rank since for any $0 \leq k+\ell \leq n$ and $0 \leq k'+\ell' \leq n$ such that $(k,\ell) \neq (k',\ell')$, $6^k 13^{\ell} \neq 6^{k'} 13^{\ell'}$. Therefore, we can solve the linear system for the unknown $c_{jk\ell}$'s and obtain the value of Holant_{Ω}. □

One of our main results in this paper is a dichotomy for Pl-Holant(f) when f is a symmetric arity 4 signature with complex weights. This dichotomy uses the #P-hardness of counting Eulerian orientations over planar 4-regular graphs in a crucial way. In [8], it was shown that most arity 4 signatures define a #P-hard Holant problem by a reduction from counting Eulerian orientations over 4-regular graphs (see Lemmas 5.4 and 5.6 in [8]). Although the reductions were planar, #P-hardness over planar 4-regular graphs did not follow because the complexity of counting Eulerian orientations over such graphs was unknown. Theorem 3.7 shows that this problem is #P-hard. Therefore, we obtain the planar version of Corollary 5.7 in [8].

Corollary 3.8. *Let f be an arity 4 signature with complex weights. If M_f is redundant and $\widetilde{M_f}$ is nonsingular, then* Pl-Holant(f) *is* #P*-hard.*

There is a simpler corollary for symmetric signatures.

Corollary 3.9. *For a symmetric arity 4 signature $[f_0, f_1, f_2, f_3, f_4]$ with complex weights, if there does* not *exist $a, b, c \in \mathbb{C}$, not all zero, such that for all $k \in \{0, 1, 2\}$,*

$$af_k + bf_{k+1} + cf_{k+2} = 0,$$

then Pl-Holant(f) *is* #P*-hard.*

Proof. If the compressed signature matrix $\widetilde{M_f}$ is nonsingular, then Pl-Holant(f) is #P-hard by Corollary 3.8, so assume that the rank of $\widetilde{M_f}$ is at most 2. Then we have

$$a' \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} + 2b' \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} + c' \begin{pmatrix} f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

for some $a', b', c' \in \mathbb{C}$ that are not all zero. Thus, $a = a'$, $b = 2b'$, and $c = c'$ have the desired property. □

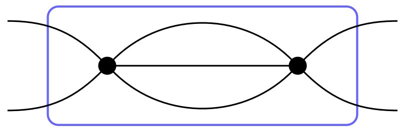

Figure 5: The circles are assigned $[a, 0, 0, 0, b, c]$.

We close this section with a simple application of Corollary 3.8 to an arity 5. We encounter signatures of this form in Sections 8 and 9.

Lemma 3.10. *Let $a, b, c \in \mathbb{C}$. If $ab \neq 0$, then for any set $\mathcal{F}$ of complex-weighted symmetric signatures containing $[a, 0, 0, 0, b, c]$, Pl-Holant$(\mathcal{F})$ is #P-hard.*

Proof. Let f be the signature of the gadget in Figure 5 with $[a, 0, 0, 0, b, c]$ assigned to both vertices. The signature matrix of f is

$$\begin{bmatrix} a^2 & 0 & 0 & 0 \\ 0 & b^2 & b^2 & bc \\ 0 & b^2 & b^2 & bc \\ 0 & bc & bc & 3b^2 + c^2 \end{bmatrix},$$

which is redundant. Its compressed form is nonsingular since its determinant is $6a^2b^4 \neq 0$. Thus, Pl-Holant(f) is #P-hard by Corollary 3.8, so Pl-Holant$(\mathcal{F})$ is also #P-hard. □

4 An Improved Interpolation Technique

In the previous section, we used interpolation to show that counting the number of Eulerian orientations is #P-hard over planar 4-regular graphs. Polynomial interpolation is a powerful tool in the study of counting problems that was initiated by Valiant [40]. In this section, we discuss a common interpolation method called the *recursive unary construction* and obtain a tight characterization of when it succeeds. The goal of this construction is to interpolate a unary signature and is based on work by Vadhan [38] and further developed by others [18, 15].

There are two gadgets in the recursive unary construction: a *starter* gadget of arity 1 and a *recursive* gadget of arity 2. The signature of the starter gadget is represented by a two-dimensional column vectors s and the signature of the recursive gadget is represented by a 2-by-2 matrix M. The construction begins with the starter gadget and proceeds by connecting $k \geq 0$ recursive gadgets, one at a time, to the only available edge (see Figure 6). The signature of this gadget can be expressed as $M^k s$. This construction is denoted by (M, s).

The essential difficulty in using polynomial interpolation is constructing an infinite set of signatures that are pairwise linearly independent. The pairwise linear independence of signatures translates into distinct evaluation points for the polynomial being interpolated. Thus, the essence of this interpolation technique can be stated as follows.

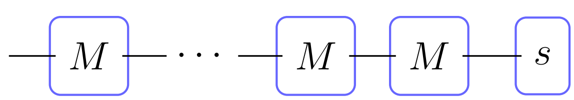

Figure 6: Recursive unary construction (M, s).

Lemma 4.1 (Lemma 5.2 in [15])**.** *Suppose $M \in \mathbb{C}^{2\times 2}$ and $s \in \mathbb{C}^{2\times 1}$. If the following three conditions are satisfied,*

1. $\det(M) \neq 0$*;*
2. *s is not a column eigenvector of M (nor the zero vector);*
3. *the ratio of the eigenvalues of M is not a root of unity;*

then the vectors in the set $V = \{M^k s\}_{k\geq 0}$ are pairwise linearly independent.

It is easy to see that the first condition is necessary. The second condition is equivalent to $\det([s\ Ms]) \neq 0$, which is necessary since it checks the linear dependence of the first two vectors in V.

The recursive unary construction can be generalized to larger dimensions, where the starter gadget has arity d and the recursive gadget has arity $2d$ [30]. In this generalized construction, the starter gadget is represented by a column vector in $\mathbb{C}^{2^d}$ and the recursive gadget is represented by a matrix in $\mathbb{C}^{2^d\times 2^d}$.

For dimensions larger than one, the second condition in Lemma 4.1 must be replaced by a stronger assumption, such as "*s is not orthogonal to any row eigenvector of M*" [18]. Previous work (Lemma 4.10 in [31], the full version of [30]) satisfied this stronger condition by showing that it follows from $\det([s\ Ms\ \dots\ M^{n-1}s]) \neq 0$. For completeness, we show that these two conditions are equivalent. We note that the use of n instead of 2^d in the next two lemmas is not overly general. Sometimes degeneracies or redundancies in the starter and recursive gadgets warrant the consideration of such cases.

Lemma 4.2. *Suppose $M \in \mathbb{C}^{n\times n}$ and $s \in \mathbb{C}^{n\times 1}$. Then $\det([s\ Ms\ \dots\ M^{n-1}s]) \neq 0$ iff s is not orthogonal to any row eigenvector of M.*

Proof. Suppose $\det([s\ Ms\ \dots\ M^{n-1}s]) \neq 0$ and assume for a contradiction that s is orthogonal to some row eigenvector v of M with eigenvalue λ. Then $v[s\ Ms\ \dots\ M^{n-1}s] = \mathbf{0}$ is the zero vector because $vM^i s = \lambda^i vs = 0$. Since $v \neq \mathbf{0}$, this a contradiction.

Now suppose that s is not orthogonal to any row eigenvector of M and assume for a contradiction that $\det([s\ Ms\ \dots\ M^{n-1}s]) = 0$. Then there is a nonzero row vector v such that $v[s\ Ms\ \dots\ M^{n-1}s] = \mathbf{0}$ is the zero vector. Consider the linear span S by row vectors in the set $\{v, vM, \dots, vM^{n-1}\}$. We claim that S is an invariant subspace of row vectors under the action of multiplication by M from the right.

By Caylay-Hamilton theorem, M satisfies its own characteristic polynomial, which is a monic polynomial of degree n. Thus, M^n is a linear combination of $I_n, M, \dots, M^{n-1}$. This shows that for any $u \in S$, uM still belongs to S.

Thus, there exists a $u \in S$ such that u is a row eigenvector of M. By the definition of S, this u is orthogonal to s, which is a contradiction. □

Another necessary condition, even for the d-dimensional case, is that M has infinite order modulo a scalar. Otherwise, $M^k = \beta I_n$ for some k and any vector of the form $M^\ell s$ for $\ell \geq k$ is some multiple of a vector in the set $\{M^i s\}_{0\leq i<k}$. We improve the d-dimensional version of Lemma 4.1 by replacing the third condition with this necessary condition.

Lemma 4.3. *Suppose $M \in \mathbb{C}^{n\times n}$ and $s \in \mathbb{C}^{n\times 1}$. If the following three conditions are satisfied,*

1. $\det(M) \neq 0$*;*
2. *s is not orthogonal to any row eigenvector of M;*

3. M has infinite order modulo a scalar;
then the vectors in the set $V = \{M^k s\}_{k \geq 0}$ are pairwise linearly independent.

Proof. Since $\det(M) \neq 0$, M is nonsingular and the eigenvalues λ_i of M, for $1 \leq i \leq n$, are nonzero. Let $M = P^{-1} J P$ be the Jordan decomposition of M and let $p = Ps \in \mathbb{C}^{n \times 1}$. Suppose for a contradiction that the vectors in V are not pairwise linearly independent. This means that there exists integers $k > \ell \geq 0$ such that $M^k s = \beta M^\ell s$ for some nonzero complex value β. Let $t = k - \ell > 0$. Then we have $P^{-1} J^t P s = M^t s = \beta s$ and $J^t p = \beta p$.

Suppose that J contains some nontrivial Jordan block and consider the 2-by-2 submatrix in the bottom right corner of this block. From this portion of J, the two equations given by $J^t p = \beta p$ are $\lambda_i^t p_{i-1} + t \lambda_i^{t-1} p_i = \beta p_{i-1}$ and $\lambda_i^t p_i = \beta p_i$. Since s is not orthogonal to any row eigenvector of M, $p_i \neq 0$. But then these equations imply that $t \lambda_i^{t-1} p_i = 0$, a contradiction.

Otherwise, J contains only trivial Jordan blocks. From $J^t p = \beta p$, we get the equations $\lambda_i p_i = \beta p_i$ for $1 \leq i \leq n$. Since s is not orthogonal to any row eigenvector of M, $p_i \neq 0$ for $1 \leq i \leq n$. But then $M^t = \beta I_n$, which contradicts that fact that M has infinite order modulo a scalar. □

With this lemma, we obtain a tight characterization for the success of interpolation by a recursive unary construction. For example, the construction using a recursive gadget with signature matrix $M = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and a starter gadget with signature $s = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ is successful because M and s satisfy our conditions but do not satisfy previous sufficient conditions.

Lemma 4.4. *Suppose $M \in \mathbb{C}^{2 \times 2}$ and $s \in \mathbb{C}^{2 \times 1}$ and let $\mathcal{F}$ be a set of signatures. If there exists an $\mathcal{F}$-gate with signature matrix M and an $\mathcal{F}$-gate with signature s satisfying the following conditions,*
1. $\det(M) \neq 0$;
2. $\det([s\ Ms]) \neq 0$;
3. M has infinite order modulo a scalar;
then Pl-Holant$(\mathcal{F} \cup \{[a, b]\}) \leq_T$ Pl-Holant$(\mathcal{F})$ *for any $a, b \in \mathbb{C}$.*

Proof. Consider an instance Ω of Pl-Holant$(\mathcal{F} \cup \{[a, b]\})$. Suppose that $[a, b]$ appears n times in Ω. We construct from Ω a sequence of instances Ω_k of Pl-Holant$(\mathcal{F})$ indexed by $k \geq 1$. We obtain Ω_k from Ω by replacing each occurrence of $[a, b]$ with the recursive unary construction (M, s) in Figure 6 containing k copies of the recursive gadget. This recursive unary construction has signature $[x_k, y_k] = M^k s$.

By applying our assumptions to Lemmas 4.2 and 4.3, we know that the signatures in the set $V = \{[x_k, y_k] \mid 0 \leq k \leq n + 1\}$ are pairwise linearly independent. In particular, at most one y_k can be 0, so we may assume that $y_k \neq 0$ for $0 \leq k \leq n$, renaming variables if necessary.

We stratify the assignments in Ω based on the assignment to $[a, b]$. Let c_ℓ be the sum over all assignments of products of evaluations on Ω such that exactly ℓ occurrences of $[a, b]$ have their incident edge assigned 0 (and $n - \ell$ have their incident edge assigned 1). Then

$$\text{Pl-Holant}_\Omega = \sum_{0 \leq \ell \leq n} a^\ell b^{n-\ell} c_\ell$$

and the value of the Holant on Ω_k, for $k \geq 1$, is

$$\begin{aligned} \text{Pl-Holant}_{\Omega_k} &= \sum_{0 \leq \ell \leq n} x_k^\ell y_k^{n-\ell} c_\ell \\ &= y_k^n \sum_{0 \leq \ell \leq n} \left(\frac{x_k}{y_k}\right)^\ell c_\ell. \end{aligned}$$

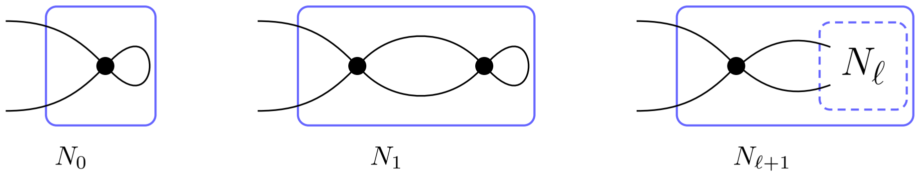

Figure 7: Binary recursive construction to interpolate $[1, 0, 0]$. The vertices are assigned $[v, 1, 0, 0, 0]$.

The coefficient matrix of this linear system is Vandermonde. Since the signatures in V are pairwise linearly independent, the ratios x_k/y_k are distinct (and well-defined since $y_k \neq 0$), which means that the Vandermonde matrix has full rank. Therefore, we can solve the linear system for the unknown c_ℓ's and obtain the value of Holant_Ω. □

The first two conditions of Lemma 4.4 are easy to check. The third condition holds in one of the two cases: either the eigenvalues are the same but M is not a multiple of the identity matrix, or the eigenvalues are different but their ratio is not a root of unity.

Our refined conditions work well with the anti-gadget technique [10]. The power of this lemma is that when the third condition fails to hold, there exists an integer k such that $M^k = I_2$, where I_2 is the 2-by-2 identity matrix. Therefore we can construct $M^{k-1} = M^{-1}$ and use this in other gadget constructions.

The anti-gadget technique is used in combination with Lemma 4.4 to give a succinct proof of Lemma 5.1. The construction in this proof is actually not a recursive unary construction, but a recursive binary construction. However, degeneracies in the starter and recursive gadgets permit analysis equivalent to that of the recursive unary construction. We also use the anti-gadget technique and the power of Lemma 4.4 (via Lemma 6.4) in the proof of Theorem 9.1 to handle the most difficult case.

5 Pl-Holant Dichotomy for a Symmetric Arity 4 Signature

With Corollary 3.9 in hand, only one obstacle remains in proving a dichotomy for a symmetric arity 4 signature in the Pl-Holant framework: the case $[v, 1, 0, 0, 0]$ when v is different from 0. Over the next two lemmas, we prove that this problem is #P-hard by a reduction from Pl-Holant($[v, 1, 0, 0]$). These problems are the weighted versions of counting matchings over planar k-regular graphs for $k = 4$ and $k = 3$ respectively.

In the first lemma, we show how to use either the anti-gadget technique from [10] or interpolation by our tight characterization of the recursive unary construction from Section 4 to effectively obtain $[1, 0, 0]$.

Lemma 5.1. *For any $v \in \mathbb{C}$ and signature set $\mathcal{F}$ containing $[v, 1, 0, 0, 0]$,*

$$\text{Pl-Holant}(\mathcal{F} \cup \{[1, 0, 0]\}) \leq_T \text{Pl-Holant}(\mathcal{F}).$$

Proof. Consider the gadget construction in Figure 7. For $k \geq 0$, the signature of N_k is of the form $[a_k, b_k, 0]$, and $N_0 = [v, 1, 0]$. Since N_k is symmetric and always ends with 0, we can analyze this construction as though it were a recursive unary construction. Let $s_k = [a_k, b_k]^{\mathtt{T}}$, so $s_0 = [v, 1]^{\mathtt{T}}$. It is clear that $s_k = M^k s_0$, where $M = \left[\begin{smallmatrix} v & 2 \\ 1 & 0 \end{smallmatrix}\right]$.

(a) The neighborhood around u and v in G.   (b) The same neighborhood in H.

Figure 8: The neighborhood around u and v both before and after they are removed.

Since $\det(M) = -2$, M is nonsingular. If M has finite order modulo a scalar, then $M^\ell = \beta I_2$ for some positive integer ℓ and some nonzero complex value β. Thus, the signature of $N_{\ell-1}$, which contains the anti-gadget of M, is $M^{\ell-1} s_0 = \beta M^{-1} s_0 = \beta [1, 0]^{\mathsf{T}}$. After normalizing, we directly realize $[1, 0, 0]$.

Now assume that M has infinite order modulo a scalar. Since $\det([s_0 \ M s_0]) = -2$, we can interpolate any signature of the form $[x, y, 0]$ by Lemma 4.4, including $[1, 0, 0]$. $\square$

For the next lemma, we use a well-known and easy generalization of a classic result of Petersen [36]. Petersen's theorem considers 3-regular, bridgeless, simple graphs (i.e. graphs without self-loops or parallel edges) and concludes that there exists a perfect matching. The same conclusion holds even if the the graphs are not simple. For completeness, the cited work provides a proof of this generalization.

Theorem 5.2 (Corollary 1 in [33])**.** *Any 3-regular bridgeless graph has a perfect matching.*

We use this result to show the existence of what we call a *planar pairing* for any planar 3-regular graph, which we use in our proof of #P-hardness.

Definition 5.3 (Planar pairing)**.** *A* planar pairing *in a graph $G = (V, E)$ is a set of edges $P \subset V \times V$ such that P is a perfect matching in the graph $(V, V \times V)$, and the graph $(V, E \cup P)$ is planar.*

Obviously, a perfect matching in the original graph is a planar pairing.

Lemma 5.4. *For any planar 3-regular graph G, there exists a planar pairing that can be computed in polynomial time.*

Proof. We efficiently find a planar pairing in G by induction on the number of vertices in G. Since G is a 3-regular graph, it must have an even number of vertices. If there are no vertices in G, then there is nothing to do. Suppose that G has $n = 2k$ vertices and that we can efficiently find a planar pairing in graphs containing fewer vertices. If G is not connected, then we can already apply our inductive hypothesis on each connected component of G. The union of planar pairings in each connected component of G is a planar pairing in G, so we are done. Otherwise assume that G is connected.

Suppose that G contains a bridge (u, v). Let the three (though not necessarily distinct) neighbors of u be v, u_1, and u_2, and let the three (though not necessarily distinct) neighbors of v be u, v_1, and v_2 (see Figure 8a). Consider the induced subgraph H of G after removing u and v and adding the edges $e_u = (u_1, u_2)$ and $e_v = (v_1, v_2)$ (which might be self-loops). Then H contains

$n-2 = 2(k-1)$ vertices and is the disjoint union of two 3-regular graphs: H_u, which contains e_u, and H_v, which contains e_v (see Figure 8b).

By induction on both H_u and H_v, we have planar pairings P_u and P_v in H_u and H_v respectively. Let H' be the graph H along with the edges $P_u \cup P_v$. Embed H' in the plane so that both e_u and e_v (if both exist) are adjacent to the outer face. Then the graph G along with the edges in $P_u \cup P_v$ is also planar, so $P_u \cup P_v \cup \{(u,v)\}$ is a planar pairing in G.

Otherwise, G is bridgeless. Then by Theorem 5.2, G has a perfect matching, which is also a planar pairing in G. Since a perfect matching can be found in polynomial time by Edmond's blossom algorithm [22], the whole procedure is in polynomial time. □

Lemma 5.5. *If $v \in \mathbb{C} - \{0\}$, then* Pl-Holant($[v,1,0,0,0]$) *is* #P*-hard.*

Proof. We reduce from Pl-Holant($[v,1,0,0]$) to Pl-Holant($[v,1,0,0,0]$). Since Pl-Holant($[v,1,0,0]$) is #P-hard when $v \neq 0$ by Theorem 2.9, this shows that Pl-Holant($[v,1,0,0,0]$) is also #P-hard when $v \neq 0$.

An instance of Pl-Holant($[v,1,0,0]$) is a signature grid Ω with underlying graph $G = (V,E)$ that is planar and 3-regular. By Lemma 5.4, there exists a planar pairing P in G and it can be found in polynomial time. Then the graph $G' = (V, E \cup P)$ is planar and 4-regular. We assign $[v,1,0,0,0]$ to every vertex in G'. By Lemma 5.1, we can assume that we have $[1,0,0]$. We replace each edge in P with a path of length 2 to form a graph G'' and assign $[1,0,0] = [1,0]^{\otimes 2}$ to each of the new vertices. Then the signature grid Ω'' with underlying graph G'' has the same Holant value as the original signature grid Ω. □

Note that our proof of Lemma 5.5 reduces Pl-Holant($[v,1,0,0]$) to Pl-Holant($[v,1,0,0,0]$) for all $v \in \mathbb{C}$. Neither Lemma 5.1 nor Lemma 5.5 ever considers the value of v. This is consistent because both signatures are in $\mathscr{M}$ when $v = 0$, thus tractable, and both signatures are #P-hard when v is different from zero.

Now we are ready to prove our Pl-Holant dichotomy for a symmetric arity 4 signature. A signature is called *vanishing* if the Holant of any signature grid using only that signature is zero [8].

Theorem 5.6. *If f is a non-degenerate, symmetric, complex-valued signature of arity 4 in Boolean variables, then* Pl-Holant(f) *is* #P*-hard unless f is $\mathscr{A}$-transformable or $\mathscr{P}$-transformable or vanishing or $\mathscr{M}$-transformable, in which case the problem is in* P.

Proof. Let $f = [f_0, f_1, f_2, f_3, f_4]$. If there do not exist $a,b,c \in \mathbb{C}$, not all zero, such that for all $k \in \{0,1,2\}$, $af_k + bf_{k+1} + cf_{k+2} = 0$, then Pl-Holant($f$) is #P-hard by Corollary 3.9. Otherwise, there do exist such a,b,c. If $a = c = 0$, then $b \neq 0$, so $f_1 = f_2 = f_3 = 0$. In this case, $f \in \mathscr{P}$ is a generalized equality signature, so f is $\mathscr{P}$-transformable. Now suppose a and c are not both 0. Then f satisfies a second order recurrence relation. If the roots of the characteristic polynomial of the recurrence relation are distinct, then $f_k = \alpha_1^{4-k}\alpha_2^k + \beta_1^{4-k}\beta_2^k$, where $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. A holographic transformation by $\left[\begin{smallmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{smallmatrix}\right]$ transforms f to $=_4$ and we can use Theorem 2.10 to show that f is either $\mathscr{A}$-, $\mathscr{P}$-, or $\mathscr{M}$-transformable. Otherwise, the characteristic polynomial has a double root α and there are two cases. In the first, for any $0 \leq k \leq 2$, $f_k = ck\alpha^{k-1} + d\alpha^k$, where $c \neq 0$. In the second, for any $0 \leq k \leq 2$, $f_k = c(4-k)\alpha^3 + d\alpha^{4-k}$, where $c \neq 0$. These cases map between each other under a holographic transformation by $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, so assume that we are in the first case. If $\alpha = \pm i$, then f is vanishing. Otherwise, a further holographic transformation by $\frac{1}{\sqrt{1+\alpha^2}}\left[\begin{smallmatrix} 1 & \alpha \\ \alpha & -1 \end{smallmatrix}\right]$ transforms f to $\hat{f} = [v,1,0,0,0]$ for some $v \in \mathbb{C}$ after normalizing the second entry. (See Appendix B in [8] for details.) If $v = 0$, then the problem is counting prefect matching over planar 4-regular graphs, so $f \in \mathscr{M}$ is $\mathscr{M}$-transformable. Otherwise, $v \neq 0$ and we are done by Lemma 5.5. □

6 Domain Pairing

Now we turn our attention to our main result, a dichotomy for the Pl-#CSP framework. In this section, we discuss a technique called *domain pairing*, which pairs input variables to simulate a problem on a domain of size four and then reduces a problem in the Boolean domain to it. As explained in the introduction, we work in the Hadamard basis instead of the standard basis. The target then becomes a dichotomy for Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$).

In [8], a simple interpolation lemma for non-degenerate, generalized equality signatures of arity at least 3 was proved. Although the lemma was only for general graphs, it was mentioned that it also holds for planar graphs.

Lemma 6.1 (Lemma A.2 in [8]). *Let $a, b \in \mathbb{C}$. If $ab \neq 0$, then for any set $\mathcal{F}$ of complex-weighted signatures containing $[a, 0, \ldots, 0, b]$ of arity at least 3,*

$$\text{Pl-Holant}(\mathcal{F} \cup \{=_4\}) \leq_T \text{Pl-Holant}(\mathcal{F}).$$

By a simple parity argument, gadgets constructed with signatures of even arity can only realize other signatures of even arity. In particular, this means that $=_4$ cannot by itself be used to construct $=_3$. Nevertheless, there is clever argument that can realize $=_3$ using $=_4$. The catch is the domain changes from individual elements to pairs of elements. Thus, we call this reduction technique *domain pairing*. This technique was first used in the proof of Lemma III.2 in [14] with real weights and was also used in the proof of Lemma 2 in [24], which can be found in the full version of that paper. We prove a generalization of the domain pairing lemma for complex weights.

Lemma 6.2 (Domain pairing). *Let $a, b, x, y \in \mathbb{C}$. If $aby \neq 0$ and $x^2 \neq y^2$, then for any set $\mathcal{F}$ of complex-valued symmetric signatures containing $[x, 0, y, 0]$ and $[a, 0, \ldots, 0, b]$ of arity at least 3,* Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) *is* #P*-hard.*

Proof. We reduce from Pl-Holant($[x, y, y] \mid \mathcal{EQ}$) to Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$). Since Pl-Holant($[x, y, y] \mid \mathcal{EQ}$) is #P-hard when $y \neq 0$ and $x^2 \neq y^2$ by Theorem 2.11, this shows that Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) is also #P-hard.

An instance of Pl-Holant($[x, y, y] \mid \mathcal{EQ}$) is a signature grid Ω with underlying graph $G = (U, V, E)$. In addition to G being bipartite and planar, every vertex is U has degree 2. We replace every vertex in V of degree k (which is assigned $=_k \in \mathcal{EQ}$) with a vertex of degree $2k$, and bundle two adjacent variables to form k bundles of 2 edges each. The k bundles correspond to the k incident edges of the original vertex with degree k. By Lemma 6.1, we have $=_4$, which we use to construct $=_{2k}$ for any k. Then we assign $=_{2k}$ to the new vertices of degree $2k$.

If the inputs to these equality signatures are restricted to $\{(0, 0), (1, 1)\}$ on each bundle, then these equality signatures take value 1 on $((0, 0), \ldots, (0, 0))$ and $((1, 1), \ldots, (1, 1))$ and take value 0 elsewhere. Thus, if we restrict the domain to $\{(0, 0), (1, 1)\}$, it is the equality signature $=_k$ once again.

To simulate $[x, y, y]$, we connect $f = [x, 0, y, 0]$ to $g = [1, 0, 1, 0] \in \widehat{\mathcal{EQ}}$ by a single edge as shown in Figure 9 to form a gadget with signature

$$h(a_1, a_2, b_1, b_2) = \sum_{c=0,1} f(a_1, b_1, c) g(a_2, b_2, c).$$

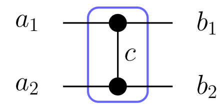

Figure 9: Gadget designed for the paired domain. One vertex is assigned $[1, 0, 1, 0]$ and the other is assigned $[x, 0, y, 0]$.

We replace every (degree 2) vertex in U (which is assigned $[x, y, y]$) by a degree 4 vertex assigned h, where the variables of h are bundled as (a_1, a_2) and (b_1, b_2).

The vertices in this new graph G' are connected as in the original graph G, except that every original edge is replaced by two parallel edges. Notice that h is only connected by (a_1, a_2) and (b_1, b_2) to some bundle of two incident edges of an equality signature. Since this equality signature enforces that the value on each bundle is either $(0, 0)$ or $(1, 1)$, we only need to consider the restriction of h to the domain $\{(0, 0), (1, 1)\}$. On this domain, $h = [x, y, y]$ is a *symmetric* signature of arity 2. Therefore, the signature grid Ω' with underlying graph G' has the same Holant value as the original signature grid Ω. □

There are two scenarios that lead to Lemma 6.2 but require stronger assumptions. The first is immediate.

Corollary 6.3. *Let $a, b \in \mathbb{C}$. If $abxy \neq 0$ and $x^4 \neq y^4$, then for any set $\mathcal{F}$ of complex-weighted symmetric signatures containing $[x, 0, y]$ and $[a, 0, \ldots, 0, b]$ of arity at least 3,* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard.*

Proof. Connect three copies of $[x, 0, y]$ to $[1, 0, 1, 0]$, with one on each edge, to get $x[x^2, 0, y^2, 0]$ and apply Lemma 6.2. □

The second scenario reaches Lemma 6.2 through Corollary 6.3 by interpolating a unary signature in one of two ways. The next lemma considers one of those ways.

Lemma 6.4. *Suppose $x \in \mathbb{C}$ and let $f = [1, x, 1]$. If $x \notin \{0, \pm 1\}$ and M_f has infinite order modulo a scalar, then for any set $\mathcal{F}$ of complex-weighted symmetric signatures containing f, we have*

$$\text{Pl-Holant}(\mathcal{F} \cup \{[a, b]\} \cup \widehat{\mathcal{EQ}}) \leq_T \text{Pl-Holant}(\mathcal{F} \cup \widehat{\mathcal{EQ}})$$

for any $a, b \in \mathbb{C}$.

Proof. Consider the recursive unary construction (M_f, s), where $s = [1, 0]^{\mathsf{T}}$. The determinant of M_f is $1 - x^2 \neq 0$. The determinant of $[s\ M_f s]$ is $x \neq 0$. By assumption, M_f has infinite order modulo a scalar. Therefore, we can interpolate any unary signature by Lemma 4.4. □

Lemma 6.5. *Let $a, b \in \mathbb{C}$. If $ab \neq 0$ and $a^4 \neq b^4$, then for any set $\mathcal{F}$ of complex-weighted symmetric signatures containing $f = [a, 0, \ldots, 0, b]$ of arity at least 3,* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard.*

Proof. Since $a \neq 0$, we normalize f to $[1, 0, \ldots, 0, x]$, where $x \neq 0$ and $x^4 \neq 1$. If the arity of f is even, then after some number of self-loops, we have $[1, 0, x]$ and are done by Corollary 6.3. Otherwise, the arity of f is odd. After some number of self-loops, we have $g = [1, 0, 0, x]$. If we had the signature $[1, 1]$, then we could connect this to g to get $[1, 0, x]$ and be done by Corollary 6.3. We now show how to interpolate $[1, 1]$ in one of two ways.

Suppose $\Re(x)$, the real part of x, is not 0. One more self-loop on g gives $[1, x]$ and connecting this to $[1, 0, 1, 0]$ gives $h = [1, x, 1]$. The eigenvalues of M_f are $\lambda_\pm = 1 \pm x$. Since $\Re(x) \neq 0$ iff $|\frac{\lambda_+}{\lambda_-}| \neq 1$, the ratio of the eigenvalues is not a root of unity, so M_h has infinite order modulo a scalar. Therefore, we can interpolate $[1, 1]$ by Lemma 6.4.

Otherwise, $\Re(x) = 0$ but x is not a root of unity since $x \neq \pm i$. Connecting $[1, x]$ to g gives $h = [1, 0, x^2]$. Consider the recursive unary construction (M_h, s), where $s = [1, x]^{\mathsf{T}}$. The determinant of M_h is $x^2 \neq 0$, so its eigenvalues are nonzero. Also, the determinant of $[s\ M_h s]$ is $x(x^2 - 1) \neq 0$. The ratio of the eigenvalues of M_h is x^2, which is not a root of unity since x is not a root of unity. Therefore M_h has infinite order modulo a scalar and we can interpolate $[1, 1]$ by Lemma 4.4. □

7 Mixing of Tractable Signatures

In this section, we determine which tractable signatures combine to give #P-hardness. To help understand the various cases considered in the lemmas, there is a Venn diagram of the signatures in $\mathscr{A}$, $\widehat{\mathscr{P}}$, and $\mathscr{M}$ in Figure 12 of Appendix A.

The first two lemmas consider the simplest case, which is when one of the signatures is a unary signature.

Lemma 7.1. *Suppose $f \in \mathscr{A} - \widehat{\mathscr{P}}$. If $ab \neq 0$ and $a^4 \neq b^4$, then for any set $\mathcal{F}$ of complex-weighted symmetric signatures containing f and $[a, b]$,* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard.*

Proof. Up to a nonzero scalar, the possibilities for f are

- $[1, 0, \pm i]$;
- $[1, 0, \ldots, 0, x]$ of arity at least 3 with $x^4 = 1$;
- $[1, \pm 1, -1, \mp 1, 1, \pm 1, -1, \mp 1, \ldots]$ of arity at least 2;
- $[1, 0, -1, 0, 1, 0, -1, 0, \ldots, 0 \text{ or } 1 \text{ or } (-1)]$ of arity at least 3;
- $[0, 1, 0, -1, 0, 1, 0, -1, \ldots, 0 \text{ or } 1 \text{ or } (-1)]$ of arity at least 3.

We handle these cases below.

1. Suppose $f = [1, 0, \pm i]$. Connecting $[a, b]$ to $[1, 0, 1, 0]$ gives $[a, b, a]$ and connecting two copies of $[1, 0, \pm i]$ to $[a, b, a]$, one on each edge, gives $g = [a, \pm ib, -a]$. Since $a^4 \neq b^4$, Pl-Holant$(g \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 2.11, so Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.
2. Suppose $f = [1, 0, \ldots, 0, x]$ of arity at least 3 with $x^4 = 1$. Connecting $[a, b]$ to f gives $g = [a, 0, \ldots, 0, bx]$ of arity at least 2. Note that $(bx)^4 = b^4 \neq a^4$. If the arity of g is exactly 2, then Pl-Holant$(\{f, g\} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Corollary 6.3, so Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is also #P-hard. Otherwise, the arity of g is at least 3 and Pl-Holant$(\{g\} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 6.5, so Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.
3. Suppose $f = [1, \pm 1, -1, \ldots]$ of arity at least 2. Connecting some number of $[1, 0]$ gives $[1, \pm 1, -1]$ of arity exactly 2. Connecting $[a, b]$ to $[1, 0, 1, 0]$ gives $[a, b, a]$ and connecting two copies of $[a, b, a]$ to $[1, \pm 1, -1]$, one on each edge, gives $g = [a^2 \pm 2ab - b^2, \pm(a^2 + b^2), -a^2 \pm 2ab + b^2]$. This is easily verified by
$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} 1 & \pm 1 \\ \pm 1 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2 \pm 2ab - b^2 & \pm(a^2 + b^2) \\ \pm(a^2 + b^2) & -a^2 \pm 2ab + b^2 \end{bmatrix}.$$
Since $a^4 \neq b^4$, Pl-Holant$(g \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 2.11, so Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.

4. Suppose $f = [1, 0, -1, 0, \dots]$ of arity at least 3. Connecting some number of $[1, 0]$ gives $[1, 0, -1, 0]$ of arity exactly 3. Connecting $[a, b]$ to $[1, 0, -1, 0]$ gives $g = [a, -b, -a]$. Since $a^4 \neq b^4$, Pl-Holant$(g \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 2.11, so Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.
5. The argument for $f = [0, 1, 0, -1, \dots]$ is similar to the previous case. □

Lemma 7.2. *Suppose $f \in \mathscr{M} - \mathscr{A}$. If $ab \neq 0$, then for any set $\mathcal{F}$ of complex-weighted symmetric signatures containing f and $[a, b]$,* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard.*

Proof. Up to a nonzero scalar, the possibilities for f are

- $[1, 0, r]$ with $r \neq 0$ and $r^4 \neq 1$;
- $[1, 0, r, 0, r^2, 0, \dots]$ of arity at least 3 with $r \neq 0$ and $r^2 \neq 1$;
- $[0, 1, 0, r, 0, r^2, \dots]$ of arity at least 3 with $r \neq 0$ and $r^2 \neq 1$;
- $[0, 1, 0, \dots, 0]$ of arity at least 3;
- $[0, \dots, 0, 1, 0]$ of arity at least 3.

We handle these cases below.

1. Suppose $f = [1, 0, r]$ with $r^4 \neq 1$ and $r \neq 0$. Connecting $[a, b]$ to $[1, 0, 1, 0]$ gives $[a, b, a]$ and connecting two copies of $[1, 0, r]$ to $[a, b, a]$, one on each edge, gives $g = [a, br, ar^2]$. If $a^2 \neq b^2$, then Pl-Holant$(g \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 2.11, so Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.

   Otherwise, $a^2 = b^2$ and we begin by connecting $[a, b]$ to $[1, 0, r]$ to get $[a, br]$. Then by the same construction, we have $g = [a, br^2, ar^2]$ and Pl-Holant$(g \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 2.11, so Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.
2. Suppose $f = [1, 0, r, 0, \dots]$ of arity at least 3 with $r^2 \neq 1$ and $r \neq 0$. Connecting some number of $[1, 0]$ gives $[1, 0, r, 0]$ of arity exactly 3. Connecting $[a, b]$ to $[1, 0, r, 0]$ gives $g = [a, br, a]$. If $a^2 \neq b^2 r$, then Pl-Holant$(g \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 2.11, so Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.

   Otherwise, $a^2 = b^2 r$ and we begin by connecting $[1, 0]$ and $[a, b]$ to $[1, 0, r, 0]$ to get $[a, br]$. Then by the same construction, we have $g = [a, br^2, ar]$ and Pl-Holant$(g \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 2.11, so Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.
3. The argument for $f = [0, 1, 0, r, \dots]$ is similar to the previous case.
4. Suppose $f = [0, 1, 0, \dots, 0]$ of arity $k \geq 3$. Connecting $k - 2$ copies of $[a, b]$ to f gives $g = a^{k-3}[(k - 2)b, a, 0]$. Since $ab \neq 0$, Pl-Holant$(g \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 2.11, so Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.
5. The argument for $f = [0, \dots, 0, 1, 0]$ is similar to the previous case. □

Now we consider the general case of two signatures from two different tractable sets. The three tractable sets give rise to three pairs of tractable sets to consider, each of which is covered in one of the next three lemmas.

Lemma 7.3. *If $f \in \mathscr{A} - \widehat{\mathscr{P}}$ and $g \in \widehat{\mathscr{P}} - \mathscr{A}$, then for any set $\mathcal{F}$ of complex-weighted symmetric signatures containing f and g,* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard.*

Proof. The only possibility for g is $[a, b, a, b, \dots]$, where $ab \neq 0$ and $a^4 \neq b^4$. Connecting some number of $[1, 0]$ to g gives $[a, b]$ and we are done by Lemma 7.1. □

Lemma 7.4. *If $f \in \mathscr{A} - \mathscr{M}$ and $g \in \mathscr{M} - \mathscr{A}$, then for any set $\mathcal{F}$ of complex-weighted symmetric signatures containing f and g,* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard.*

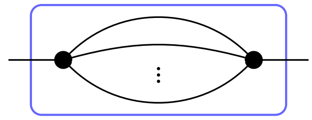

Figure 10: The vertices are assigned $g = [0, 1, 0, \dots, 0]$.

Proof. If f does not contain a 0 entry, then after connecting some number of $[1, 0]$ to f, we have a unary signature $[a, b]$ with $ab \neq 0$. Then we are done by Lemma 7.2.

Otherwise, f contains a 0 entry. Then $f = [x, 0, \dots, 0, y]$ of arity at least 3 with $xy \neq 0$ (and $x^4 = y^4$). Up to a nonzero scalar, the possibilities for g are

- $[1, 0, r]$ with $r \neq 0$ and $r^4 \neq 1$;
- $[1, 0, r, 0, r^2, 0, \dots]$ of arity at least 3 with $r \neq 0$ and $r^2 \neq 1$;
- $[0, 1, 0, r, 0, r^2, \dots]$ of arity at least 3 with $r \neq 0$ and $r^2 \neq 1$;
- $[0, 1, 0, 0, \dots, 0]$ of arity at least 3;
- $[0, \dots, 0, 0, 1, 0]$ of arity at least 3.

We handle these cases below.

1. Suppose $g = [1, 0, r]$ with $r \neq 0$ and $r^4 \neq 1$. Then we are done by Corollary 6.3.
2. Suppose $g = [1, 0, r, 0, \dots]$ of arity at least 3 with $r \neq 0$ and $r^2 \neq 1$. After connecting some number of $[1, 0]$ to g, we have $h = [1, 0, r, 0]$ of arity exactly 3. Then Pl-Holant($\{f, h\} \cup \widehat{\mathcal{EQ}}$) is #P-hard by Lemma 6.2, so Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) is also #P-hard.
3. Suppose $g = [0, 1, 0, r, \dots]$ of arity at least 3 with $r \neq 0$ and $r^2 \neq 1$. After connecting some number of $[1, 0]$ to g, we have $h = [0, 1, 0, r]$ of arity exactly 3. Connecting two more copies of $[1, 0]$ to h gives $[0, 1]$. Then we apply a holographic transformation by $T = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, so f is transformed to $\hat{f} = [y, 0, \dots, 0, x]$ and h is transformed to $\hat{h} = [r, 0, 1, 0]$. Every even arity signature in $\widehat{\mathcal{EQ}}$ remains unchanged after a holographic transformation by T. By attaching $[0, 1]T = [1, 0]$ to every even arity signature in $T\widehat{\mathcal{EQ}}$, we obtain all of the odd arity signatures in $\widehat{\mathcal{EQ}}$ again. Then Pl-Holant($\{\hat{f}, \hat{h}\} \cup \widehat{\mathcal{EQ}}$) is #P-hard by Lemma 6.2, so Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) is also #P-hard.
4. Suppose $g = [0, 1, 0, \dots, 0]$ of arity $k \geq 3$. The gadget in Figure 10 with g assigned to both vertices has signature $h = [k - 1, 0, 1]$. Then Pl-Holant($\{f, h\} \cup \widehat{\mathcal{EQ}}$) is #P-hard by Corollary 6.3, so Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) is also #P-hard.
5. The argument for $g = [0, \dots, 0, 1, 0]$ is similar to the previous case. □

Lemma 7.5. *Suppose $f \in \mathscr{M} - \widehat{\mathscr{P}}$ and $g \in \widehat{\mathscr{P}} - \mathscr{M}$ such that $\{f, g\} \not\subseteq \mathscr{A}$. Then for any set $\mathcal{F}$ of complex-weighted symmetric signatures containing f and g,* Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) *is* #P*-hard.*

Proof. The only possibility for g is $[a, b, a, b, \dots]$, where $ab \neq 0$. Connecting some number of $[1, 0]$ to g gives $h = [a, b]$. If $f \notin \mathscr{A}$, then Pl-Holant($\{f, h\} \cup \widehat{\mathcal{EQ}}$) is #P-hard by Lemma 7.2, so Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) is also #P-hard.

Otherwise, $f \in \mathscr{A}$, so $g \notin \mathscr{A}$. Then Pl-Holant($\{g, h\} \cup \widehat{\mathcal{EQ}}$) is #P-hard by Lemma 7.3, so Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) is also #P-hard. □

We summarize this section with the following theorem, which says that the tractable signature sets cannot mix. More formally, signatures from different tractable sets, when put together, lead to #P-hardness.

Theorem 7.6 (Mixing)**.** *Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. If $\mathcal{F} \subseteq \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$, then* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard unless $\mathcal{F} \subseteq \mathscr{A}$, $\mathcal{F} \subseteq \widehat{\mathscr{P}}$, or $\mathcal{F} \subseteq \mathscr{M}$, in which case* Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is tractable.*

Proof. If $\mathcal{F}$ is a subset of $\mathscr{A}$, $\widehat{\mathscr{P}}$, or $\mathscr{M}$, then the tractability is given in Theorem 2.8. Otherwise $\mathcal{F}$ is not a subset of $\mathscr{A}$, $\widehat{\mathscr{P}}$, or $\mathscr{M}$. Then $\mathcal{F}$ contains a signature $g \in (\widehat{\mathscr{P}} \cup \mathscr{M}) - \mathscr{A}$ since $\mathcal{F} \not\subseteq \mathscr{A}$. Suppose $\mathcal{F}$ contains a signature $f \in \mathscr{A} - \widehat{\mathscr{P}} - \mathscr{M}$. If $g \in \widehat{\mathscr{P}} - \mathscr{A}$, then Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 7.3. Otherwise, $g \in \mathscr{M} - \mathscr{A}$ and Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 7.4.

Now assume that $\mathcal{F} \subseteq \widehat{\mathscr{P}} \cup \mathscr{M}$. Since $(\widehat{\mathscr{P}} \cap \mathscr{M}) - \mathscr{A}$ is empty (see Figure 12 in Appendix A), either $g \in \widehat{\mathscr{P}} - \mathscr{M} - \mathscr{A}$ or $g \in \mathscr{M} - \widehat{\mathscr{P}} - \mathscr{A}$ because $\mathcal{F}$ is not a subset of either $\mathscr{M}$ or $\widehat{\mathscr{P}}$. If $g \in \widehat{\mathscr{P}} - \mathscr{M} - \mathscr{A}$, then there exists a signature $f \in \mathscr{M} - \widehat{\mathscr{P}}$ since $\mathcal{F} \not\subseteq \widehat{\mathscr{P}}$. In which case, Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 7.5. Otherwise, $g \in \mathscr{M} - \widehat{\mathscr{P}} - \mathscr{A}$ and there exists a signature $f \in \widehat{\mathscr{P}} - \mathscr{M}$ since $\mathcal{F} \not\subseteq \mathscr{M}$. In which case, Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 7.5. □

8 Pinning for Planar Graphs

The idea of "pinning" is a common reduction technique between counting problems. For the #CSP framework, pinning fixes some variables to specific values of the domain by means of the constant functions [4, 20, 25]. In particular, for counting graph homomorphisms, pinning is used when the input graph is connected and the target graph is disconnected. In this case, pinning a vertex of the input graph to a vertex of the target graph forces all the vertices of the input graph to map to the same connected component of the target graph [21, 3, 23, 5]. For the Boolean domain, the constant 0 and constant 1 functions are the signatures $[1, 0]$ and $[0, 1]$ respectively.

From these works, the most relevant pinning lemma for the Pl-#CSP framework is by Dyer, Goldberg, and Jerrum in [20], where they show how to pin in the #CSP framework. However, the proof of this pinning lemma is highly nonplanar. Cai, Lu, and Xia [14] overcame this difficultly in the proof of their dichotomy theorem for the real-weighted Pl-#CSP framework by first undergoing a holographic transformation by the Hadamard matrix $H = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ and then pinning in this Hadamard basis.[3] We stress that this holographic transformation is necessary. Indeed, if one were able to pin in the standard basis of the Pl-#CSP framework, then P = #P would follow since Pl-#CSP$(\widehat{\mathscr{M}})$ is tractable but Pl-#CSP$(\widehat{\mathscr{M}} \cup \{[1, 0], [0, 1]\})$ is #P-hard by our main dichotomy in Theorem 9.3 (or, more specifically, by Lemma 7.2).

Since Pl-#CSP$(\mathcal{F})$ is Turing equivalent to Pl-Holant$(\mathcal{F} \cup \mathcal{EQ})$, the expression of Pl-#CSP$(\mathcal{F})$ in the Hadamard basis is Pl-Holant$(H\mathcal{F} \cup \widehat{\mathcal{EQ}})$. Then we already have $[1, 0] \in \widehat{\mathcal{EQ}}$, so pinning in the Hadamard basis of Pl-#CSP$(\mathcal{F})$ amounts to obtaining the missing signature $[0, 1]$.

8.1 The Road to Pinning

We begin the road to pinning with a lemma that assumes the presence of $[0, 0, 1] = [0, 1]^{\otimes 2}$, which is the tensor product of two copies of $[0, 1]$. In our pursuit to realize $[0, 1]$, this may be as close as we can get, such as when every signature has even arity. Another roadblock to realizing $[0, 1]$ is when every signature has even parity. Recall that a signature has even parity if its support is on entries of even Hamming weight. By a simple parity argument, gadgets constructed with signatures of

[3] The pinning in [14], which is accomplished in Section IV, is not summarized in a single statement but is implied by the combination of all the results in that section.

even parity can only realize signatures of even parity. However, if every signature has even parity and $[0, 0, 1]$ is present, then we can already prove a dichotomy.

Lemma 8.1. *Suppose $\mathcal{F}$ is a set of symmetric signatures with complex weights containing $[0, 0, 1]$. If every signature in $\mathcal{F}$ has even parity, then either* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard or $\mathcal{F}$ is a subset of $\mathscr{A}$, $\widehat{\mathscr{P}}$, or $\mathscr{M}$, in which case* Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is tractable.*

Proof. The tractability is given in Theorem 2.8. If every non-degenerate signature in $\mathcal{F}$ is of arity at most 3, then $\mathcal{F} \subseteq \mathscr{M}$ since all signatures in $\mathcal{F}$ satisfy the (even) parity condition.

Otherwise $\mathcal{F}$ contains some non-degenerate signature of arity at least 4. For every signature $f \in \mathcal{F}$ with $f = [f_0, f_1, \ldots, f_m]$ and $m \geq 4$, using $[0, 0, 1]$ and $[1, 0]$, we can obtain all subsignatures of the form $[f_{k-2}, 0, f_k, 0, f_{k+2}]$ for any even k such that $2 \leq k \leq m - 2$. If any subsignature g of this form satisfies $f_{k-2} f_{k+2} \neq f_k^2$ and $f_k \neq 0$, then Pl-Holant(g) is #P-hard by Corollary 3.8, so Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.

Otherwise all subsignatures of signatures in $\mathcal{F}$ of the above form satisfy $f_{k-2} f_{k+2} = f_k^2$ or $f_k = 0$. There are two types of signatures with this property. In the first type, the signature entries of even Hamming weight form a geometric progression. More specifically, the signatures of the first type have the form

$$[\alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n] \qquad \text{or} \qquad [\alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n, 0]$$

for some $\alpha, \beta \in \mathbb{C}$, which are in $\mathscr{M}$. In the second type, the signatures have arity at least 4 or 5 and are of the form $[x, 0, \ldots, 0, y]$ or $[x, 0, \ldots, 0, y, 0]$ respectively, with $xy \neq 0$ and an odd number of 0's between x and y (since they have even parity). If all of the signatures in $\mathcal{F}$ are of the first type, then $\mathcal{F} \subseteq \mathscr{M}$.

Otherwise $\mathcal{F}$ contains a signature f of the second type. Suppose $f = [x, 0, \ldots, 0, y, 0]$ of arity at least 5 with $xy \neq 0$. After some number of self-loops, we have $g = [x, 0, 0, 0, y, 0]$ of arity exactly 5. Then Pl-Holant(g) is #P-hard by Lemma 3.10, so Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.

Otherwise $f = [x, 0, \ldots, 0, y]$ of arity at least 4 with $xy \neq 0$. If $x^4 \neq y^4$, then Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 6.5.

Otherwise $x^4 = y^4$. This puts every signature of the second type in $\mathscr{A}$. Therefore $\mathcal{F} \subseteq \mathscr{A} \cup \mathscr{M}$ and we are done by Theorem 7.6. □

The conclusion of every result in the rest of this section states that we are able to pin (under various assumptions on $\mathcal{F}$). Formally speaking, we repeatedly prove that Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is #P-hard (or in P) if and only if Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is #P-hard (or in P). The difference between these two counting problems is the presence of $[0, 1]$ in Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$. We always prove this statement in one of three ways:

1. either we show that Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is tractable (so Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is as well);
2. or we show that Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is #P-hard (so Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is as well);
3. or we show how to reduce Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ to Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ by realizing $[0, 1]$ using signatures in $\mathcal{F} \cup \widehat{\mathcal{EQ}}$.

Lemma 8.2. *Let $\mathcal{F}$ be any set of complex-weighted symmetric signatures containing $[0, 0, 1]$. Then* Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard (or in* P*) iff* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard (or in* P*).*

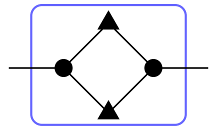

Figure 11: The circles are assigned $[1, 0, 1, 0]$ and the triangles are assigned $[1, 0, x]$.

Proof. If we had a unary signature $[a, b]$ where $b \neq 0$, then connecting $[a, b]$ to $[0, 0, 1]$ gives the signature $[0, b]$, which is $[0, 1]$ after normalizing. Thus, in order to reduce Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ to Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ by constructing $[0, 1]$, it suffices to construct a unary signature $[a, b]$ with $b \neq 0$.

For every signature $f \in \mathcal{F}$ with $f = [f_0, f_1, \ldots, f_m]$, using $[0, 0, 1]$ and $[1, 0]$, we can obtain all subsignatures of the form $[f_{k-1}, f_k]$ for any odd k such that $1 \leq k \leq m$. If any subsignature satisfies $f_k \neq 0$, then we can construct $[0, 1]$.

Otherwise all signatures in $\mathcal{F}$ have even parity and we are done by Lemma 8.1. □

There are two scenarios that lead to Lemma 8.2, which are the focus of the next two lemmas.

Lemma 8.3. *For $x \in \mathbb{C}$, let $\mathcal{F}$ be any set of complex-weighted symmetric signatures containing $[1, 0, x]$ such that $x \notin \{0, \pm 1\}$. Then* Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard (or in* P*) iff* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard (or in* P*).*

Proof. There are two cases to consider. In either case, we realize $[0, 0, 1]$ and finish by applying Lemma 8.2.

First we claim that the conclusion holds provided $|x| \neq 0, 1$. Combining k copies of $[1, 0, x]$ gives $[1, 0, x^k]$. Since $|x| \notin \{0, 1\}$, x is neither zero nor a root of unity, so we can use polynomial interpolation to realize $[a, 0, b]$ for any $a, b \in \mathbb{C}$, including $[0, 0, 1]$.

Otherwise $|x| = 1$. The gadget in Figure 11 has signature $[f_0, f_1, f_2] = [1 + x^2, 0, 2x]$. If $x = \pm i$, then we have $[0, 0, \pm 2i]$, which is $[0, 0, 1]$ after normalizing.

Otherwise $x \neq \pm i$, so $f_0 \neq 0$. Since $x \neq 0$, $f_2 \neq 0$. Since $x \neq \pm 1$, $|f_0| < 2$. However, $|f_2| = 2$. Therefore, after normalizing, the signature $[1, 0, y]$ with $y = \frac{2x}{1+x^2}$ has $|y| > 1$, so it can interpolate $[0, 0, 1]$ by our initial claim since $|y| \notin \{0, 1\}$. □

Lemma 8.4. *Let $\mathcal{F}$ be any set of complex-weighted symmetric signatures containing a signature $[f_0, f_1, \ldots, f_n]$ that is not identically 0 but has $f_0 = 0$. Then* Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard (or in* P*) iff* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard (or in* P*).*

Proof. If $f_1 \neq 0$, then we connect $n - 1$ copies of $[1, 0]$ to f to get $[0, f_1]$, which is $[0, 1]$ after normalizing. If $f_1 = 0$, then $n \geq 2$. If $f_2 \neq 0$, then we connect $n - 2$ copies of $[1, 0]$ to f to get $[0, 0, f_2]$, which is $[0, 0, 1]$ after normalizing. Then we are done by Lemma 8.2. If $f_1 = f_2 = 0$, then $n \geq 3$. With some number of self-loops, we get a signature with exactly one or two initial zeros, which is one of the above scenarios. □

As a significant step toward pinning for any signature set $\mathcal{F}$, we show how to pin given any binary signature. Some cases resist pinning and are excluded.

Lemma 8.5. *Let $\mathcal{F}$ be any set of complex-weighted symmetric signatures containing $f = [f_0, f_1, f_2]$. Then* Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard (or in* P*) iff* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard (or in* P*) unless $f \in \{[0, 0, 0], [1, 0, -1], [1, r, r^2], [1, b, 1]\}$, up to a nonzero scalar, for any $b, r \in \mathbb{C}$.*

Proof. If $f_0 = 0$ and either $f_1 \neq 0$ or $f_2 \neq 0$, then we are done by Lemma 8.4. Otherwise, $f = [0,0,0]$ or $f_0 \neq 0$, in which case we normalize f_0 to 1. If Pl-Holant($f \mid \widehat{\mathcal{EQ}}$) is #P-hard by Theorem 2.11, then Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) is also #P-hard. Otherwise, f is one of the tractable cases, which implies that

$$f \in \{[0,0,0], [1,r,r^2], [1,0,x], [1,\pm 1,-1], [1,b,1]\}.$$

If $f = [1,\pm 1,-1]$, then we connect f to $[1,0,1,0]$ to get $[0,\pm 2]$, which is $[0,1]$ after normalizing. If $f = [1,0,x]$, then we are done by Lemma 8.3 unless $x \in \{0,\pm 1\}$. The remaining cases are all excluded by assumption, so we are done. □

8.2 Pinning in the Hadamard Basis

Before we show how to pin in the Hadamard basis, we handle two simple cases.

Lemma 8.6. *For any set $\mathcal{F}$ of complex-weighted symmetric signatures containing $[1,\pm i]$, we have* Pl-Holant^c($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) $\leq_T$ Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$).

Proof. Connect two copies of $[1,\pm i]$ to $[1,0,1,0]$ to get $[0,\pm 2i]$, which is $[0,1]$ after normalizing. □

The next lemma considers the signature $[1,b,1,b^{-1}]$, which we also encounter in Theorem 9.1, the single signature dichotomy.

Lemma 8.7. *Let $b \in \mathbb{C}$. If $b \notin \{0,\pm 1\}$, then for any set $\mathcal{F}$ of complex-weighted symmetric signatures containing $f = [1,b,1,b^{-1}]$,* Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) *is* #P*-hard.*

Proof. Connect two copies of $[1,0]$ to f to get $[1,b]$. Connecting this back to f gives $g = [1+b^2, 2b, 2]$. Then Pl-Holant($g \mid \widehat{\mathcal{EQ}}$) is #P-hard by Theorem 2.11, so Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) is also #P-hard. □

Now we are ready to prove our pinning result.

Theorem 8.8 (Pinning)**.** *Let $\mathcal{F}$ be any set of complex-weighted symmetric signatures. Then* Pl-Holant^c($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) *is* #P*-hard (or in* P*) iff* Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) *is* #P*-hard (or in* P*).*

Proof. For simplicity, we normalize the first nonzero entry of every signature in $\mathcal{F}$ to 1, and we replace any degenerate signature in $\mathcal{F}$ by its unary version using $[1,0]$. This does not change the complexity of the problem. If $\mathcal{F}$ contains $[0,1]$, then we are done, so assume this is not the case.

Suppose $\mathcal{F}$ contains only unary signatures. Then $\mathcal{F} \subseteq \mathscr{P}$ and Pl-Holant^c($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) is tractable by Theorem 2.8.

Otherwise $\mathcal{F}$ contains a signature f of arity at least two. We connect some number of $[1,0]$ to f until we obtain a signature with arity exactly two. We call the resulting signature the binary prefix of f. If this binary prefix is not one of the exceptional forms in Lemma 8.5, then we are done, so assume that it is one of the exceptional forms.

Now we perform case analysis according to the exceptional forms in Lemma 8.5. There are five cases below because we consider $[1,r,r^2]$ as $[1,0,0]$ and $[1,r,r^2]$ with $r \neq 0$ as separate cases. In each case, we either show that the conclusion of the theorem holds or that $f \in \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$. After the case analysis, we then handle all of these tractable f together.

1. Suppose the binary prefix of f is $[0,0,0]$. If f is not identically 0, then we are done by Lemma 8.4.

   Thus, in this case, we may assume that $f = [0,0,\ldots,0]$ is identically 0.

2. Suppose the binary prefix of f is $[1, 0, -1]$. If f is not of the form

$$[1, 0, -1, 0, 1, 0, -1, 0, \ldots, 0 \text{ or } 1 \text{ or } (-1)], \tag{2}$$

then after one self-loop, we have a signature of arity at least one with 0 as its first entry but is not identically 0, so we are done by Lemma 8.4.

Thus, in this case, we may assume that f has the form given in (2).

3. Suppose the binary prefix of f is $[1, 0, 0]$. If f is not of the form $[1, 0, \ldots, 0]$, then after connecting some number of $[1, 0]$, we have $[1, 0, \ldots, 0, x]$ of arity at least 3, where $x \neq 0$. If $x^4 \neq 1$, then Pl-Holant($\{f\} \cup \widehat{\mathcal{EQ}}$) is #P-hard by Lemma 6.5, so Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) is also #P-hard.

Otherwise, $x^4 = 1$. Suppose that x is not the last entry in f. Then connecting one fewer $[1, 0]$ than before, we have $g = [1, 0, \ldots, 0, x, y]$ and there are two cases to consider. If the index of x in g is odd, then after some number of self-loops, we have $h = [1, 0, 0, x, y]$. The determinant of the compressed signature matrix of h is $-2x^2 \neq 0$. Thus, Holant(h) is #P-hard by Corollary 3.8, so Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) is also #P-hard.

Otherwise, the index of x in g is even. After some number of self-loops, we have $h = [1, 0, 0, 0, x, y]$. Then by Lemma 3.10, Holant(h) is #P-hard, so Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) is also #P-hard.

Thus, in this case, we may assume that either $f = [1, 0, \ldots, 0]$ or $f = [1, 0, \ldots, 0, x]$, where $x^4 = 1$.

4. Suppose the binary prefix of f is $[1, r, r^2]$, where $r \neq 0$. If f is not of the form $[1, r, \ldots, r^n]$, then after connecting some number of $[1, 0]$, we have $[1, r, \ldots, r^m, y]$, where $y \neq r^{m+1}$ and $m \geq 2$. Using $[1, 0]$, we can get $[1, r]$. If $r = \pm i$, then we are done by Lemma 8.6, so assume that $r \neq \pm i$. Then we can attach $[1, r]$ back to the initial signature some number of times to get $g = [1, r, r^2, x]$ after normalizing, where $x \neq r^3$. We connect $[1, r]$ once more to get $h = [1 + r^2, r(1 + r^2), r^2 + rx]$. If h does not have one of the exceptional forms in Lemma 8.5, then we are done, so assume that it does.

Since the second entry of h is not 0 and $x \neq r^3$, the only possibility is that h has the form $[1, b, 1]$ up to a scalar. This gives $x = r^{-1}$. Note that $r \neq \pm 1$ since $x \neq r^3$. A self-loop on $g = [1, r, r^2, r^{-1}]$ gives $[1 + r^2, r + r^{-1}]$, which is $[1, r^{-1}]$ after normalizing. Connecting this back to g gives $h = [2, 2r, r^2 + r^{-2}]$. We assume that h has one of the exceptional forms in Lemma 8.5 since we are done otherwise. If h has the form $[1, r, r^2]$ up to a scalar, then $r^4 = 1$, a contradiction, so it must have the form $[1, b, 1]$ up to a scalar. But then $r^2 = 1$, which is also a contradiction.

Thus, in this case, we may assume that $f = [1, r, \ldots, r^n]$.

5. Suppose the binary prefix of f is $[1, b, 1]$. If $b = \pm 1$, then this binary prefix is degenerate and was considered in the previous case, so assume that $b \neq \pm 1$. If f is not of the form $[1, b, 1, b, \ldots]$, then suppose that the index of the first entry in f to break the pattern is even. Then after connecting some number of $[1, 0]$, we have $[1, b, 1, \ldots, b, y]$, where $y \neq 1$. Then after some number of self-loops and normalizing, we have $g = [1, b, 1, b, x]$, where $x \neq 1$. The determinant of its compressed signature matrix is $(b^2 - 1)(1 - x) \neq 0$. Thus, Holant(g) is #P-hard by Corollary 3.8, so Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) is also #P-hard.

Otherwise, the index of the first entry in f to break the pattern is odd. Then after connecting some number of $[1, 0]$, we have $[1, b, 1, \ldots, 1, y]$, where $y \neq b$. Then after some number of self-loops and normalizing, we have $[1, b, 1, x]$, where $x \neq b$. We do a self-loop to get $g = [2, b+x]$. If $b = 0$, then connecting g to $[1,0,1,x]$ gives $h = [2, x, 2+x^2]$. We assume that h has one of the exceptional forms in Lemma 8.5 since we are done otherwise. Because $x \neq 0$, the only possibility is that h has the form $[1, r, r^2]$ up to a scalar. Then we get $x^2 = -4$, so $g = [2, x] = 2[1, \pm i]$ and we are done by Lemma 8.6. We use the signature g again below.

Otherwise, $b \neq 0$. Using $[1, 0]$, we can get $h = [1, b, 1]$. If the signature matrix M_h of h has finite order modulo a scalar, then $M_h^\ell = \beta I_2$ for some positive integer ℓ and some nonzero complex value β. Thus after normalizing, we can construct the anti-gadget $[1, -b, 1]$ by connecting $\ell - 1$ copies of h together. Connecting $[1, 0]$ to $[1, -b, 1]$ gives $[1, -b]$ and connecting this to $[1, b, 1, x]$ gives $[1 - b^2, 0, 1 - bx]$. If $\frac{1-bx}{1-b^2} \notin \{0, \pm 1\}$, then we are done by Lemma 8.3.

Otherwise, $y = \frac{1-bx}{1-b^2} \in \{0, \pm 1\}$. For $y = 0$, we get $x = b^{-1}$ and are done by Lemma 8.7 since $b \notin \{0, \pm 1\}$. For $y = 1$, we get $b = x$, a contradiction. For $y = -1$, we get $2 - b^2 - bx = 0$. Then connecting $[1, -b, 1]$ to $g = [2, b + x]$ gives $[2 - b^2 - bx, x - b] = [0, x - b]$, which is $[0, 1]$ after normalizing.

Otherwise, M_h has infinite order modulo a scalar. Then we can interpolate $[0, 1]$ by Lemma 6.4 since $b \notin \{0, \pm 1\}$.

Thus, in this case, we may assume that $f = [1, b, 1, b, \ldots]$.

At this point, every signature in $\mathcal{F}$ (including the unary signatures) must be of one of the following forms:

- $[0, \ldots, 0]$, which is in $\mathscr{A} \cap \widehat{\mathscr{P}} \cap \mathscr{M}$;
- $[1, 0, -1, 0, 1, 0, -1, 0, \ldots, 0 \text{ or } 1 \text{ or } (-1)]$, which is in $\mathscr{A} \cap \mathscr{M}$;
- $[1, 0, \ldots, 0, x]$, where $x^4 = 1$, which is in $\mathscr{A}$;
- $[1, b, 1, b, \ldots, 1 \text{ or } b]$, which is in $\widehat{\mathscr{P}}$.

In particular, every possible unary signature either fits into the first case or the last case. Therefore $\mathcal{F} \subseteq \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$ and we are done by Theorem 7.6. □

9 Main Dichotomy

In this section, we prove our main dichotomy theorem. We begin with a dichotomy for a single signature.

Theorem 9.1. *If f is a non-degenerate symmetric signature of arity at least 2 with complex weights in Boolean variables, then* Pl-Holant$(\{f\} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard unless $f \in \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$, in which case the problem is in* P.

Proof. When $f \in \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$, the problem is tractable by Theorem 2.8. When $f \notin \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$, we prove that Pl-Holant$^c(\{f\} \cup \widehat{\mathcal{EQ}})$ is #P-hard, which is sufficient because of pinning (Theorem 8.8). Using $[1, 0]$ and $[0, 1]$, we can obtain any subsignature of f. The possibilities for f can be divided into three cases:

- f satisfies the parity condition;
- f does not satisfy the parity condition but does contain a 0 entry;
- f does not contain a 0 entry.

We handle these cases below.

1. Suppose that f satisfies the parity condition. If f has even parity, then we are done by Lemma 8.1.

Otherwise, f has odd parity. If f has odd arity, then under a holographic transformation by $T = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, f is transformed to $\hat{f}$, which has even parity. Every even arity signature in $\widehat{\mathcal{EQ}}$ remains unchanged after a holographic transformation by T. By attaching $[0,1]T = [1,0]$ to every even arity signature in $T\widehat{\mathcal{EQ}}$, we obtain all of the odd arity signatures in $\widehat{\mathcal{EQ}}$ again. Then either Pl-Holant$^c(\{\hat{f}\} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 8.1 (and thus Pl-Holant$^c(\{f\} \cup \widehat{\mathcal{EQ}})$ is also #P-hard), or $\hat{f} \in \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$. In the latter case, we also have $f \in \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$ since $\mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$ is closed under T.

Otherwise, the arity of f is even. Connect $[0,1]$ to f to get a signature g with even parity and odd arity. Then either Pl-Holant$^c(\{g\} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 8.1 (and thus Pl-Holant$^c(\{f\} \cup \widehat{\mathcal{EQ}})$ is also #P-hard), or $g \in \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$. In the latter case, it must be that $g \in \mathscr{M}$ since non-degenerate generalized equality signatures cannot have both even parity and odd arity. (See Figure 12 at the end of the Appendix, which contains a Venn diagram of the signatures in $\mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$, up to constant factors.) In particular, the even parity entries of g form a geometric progression p. Therefore $f \in \mathscr{M}$ since f has odd parity and the same geometric progression p among its odd parity entries.

2. Suppose that f contains a 0 entry but does not satisfy the parity condition. Since f does not satisfy the parity condition, there must be at least two nonzero entries separated by an even number of 0 entries. Thus, f contains a subsignature $g = [a, 0, \ldots, 0, b]$ of arity $n = 2k+1$, where $ab \neq 0$. If $k = 0$, then $n = 1$ and we can shift either to the right or to the left and find the 0 entry in f and obtain a binary subsignature h of the form $[0, c, d]$ or $[c, d, 0]$, where $cd \neq 0$. Then Pl-Holant$(h \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 2.11, so Pl-Holant$(\{f\} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.

   Otherwise $k \geq 1$, so $n \geq 3$. If $a^4 \neq b^4$, then Pl-Holant$(\{g\} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 6.5, so Pl-Holant$(\{f\} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.

   Otherwise, $a^4 = b^4$, so $g \in \mathscr{A}$. If $f = g$, then we are done, so assume that $f \neq g$, which implies that there is another entry just before a or just after b. If this entry is nonzero, then f has a subsignature h of the form $[0, c, d]$ or $[c, d, 0]$, where $cd \neq 0$. Then Pl-Holant$(h \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 2.11, so Pl-Holant$(\{f\} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.

   Otherwise, this entry is 0 and f has a subsignature h of the form $[0, a, 0, \ldots, 0, b]$ or $[a, 0, \ldots, 0, b, 0]$ of arity at least 4. If the arity of h is even, then after some number of self-loops, we have a signature h' of the form $[0, a, 0, 0, b]$ or $[a, 0, 0, b, 0]$ of arity exactly 4. Then Pl-Holant(h') is #P-hard by Corollary 3.8 since $ab \neq 0$, so Pl-Holant$(\{f\} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.

   Otherwise, the arity of h is odd. After some number of self-loops, we have a signature h' of the form $[0, a, 0, 0, 0, b]$ or $[a, 0, 0, 0, b, 0]$ of arity exactly 5. Then Pl-Holant(h') is #P-hard by Lemma 3.10 since $ab \neq 0$, so Pl-Holant$(\{f\} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.

3. Suppose f contains no 0 entry. If f has a binary subsignature g such that Pl-Holant$(g \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 2.11, then Pl-Holant$(\{f\} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.

   Otherwise every binary subsignature of f satisfies the conditions of some tractable case in Theorem 2.11. The three possible tractable cases are degenerate with condition $ac = b^2$ (case 1), affine $\mathscr{A}$ with condition $ac = -b^2 \wedge a = -c$ (case 3), and a Hadamard-transformed product type $\widehat{\mathscr{P}}$ with condition $a = c$ (case 4). If every binary subsignature $[a, b, c]$ of f satisfies $ac = b^2$, then f is degenerate, a contradiction. If every binary subsignature $[a, b, c]$ of f satisfies $ac = -b^2 \wedge a = -c$, then $f = [1, \pm 1, -1, \mp 1, 1, \pm -1, -1, \mp 1, \ldots] \in \mathscr{A}$ (up to a scalar) and we are done. If every binary subsignature $[a, b, c]$ of f satisfies $a = c$, then $f \in \widehat{\mathscr{P}}$ and we are done.

Otherwise, there exists two binary subsignatures of f that exclusively satisfy the conditions of different tractable cases in Theorem 2.11. More specifically, f has arity at least 3 and there exists a ternary subsignature $g = [a, b, c, d]$ such that $h = [a, b, c]$ and $h' = [b, c, d]$ exclusively satisfy the conditions of different tractable cases in Theorem 2.11. By symmetry under a holographic transformation by $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, we can choose the order of these tractable cases. Suppose f contains a binary subsignature that satisfies the condition of the affine case, and let h be that subsignature. Then for either case of h', we have $g = [1, \pm 1, -1, \pm 1]$ after normalizing. Connecting two copies of $[0, 1]$ to g gives $[-1, \pm 1]$. Connecting this back to g gives $g' = [0, \mp 2, 2]$. Then Pl-Holant$(g' \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 2.11, so Pl-Holant$(\{f\} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.

Otherwise, no binary subsignature of f satisfies the condition of the affine case. Then there exists two binary subsignatures of f that exclusively satisfy the degenerate and product type conditions. Let h satisfy the product type condition (but not the degenerate condition) and h' satisfy the degenerate condition. Then $g = [1, b, 1, b^{-1}]$ after normalizing, where $b^2 \neq 1$. Then Pl-Holant$(g \mid \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 8.7, so Pl-Holant$(\{f\} \cup \widehat{\mathcal{EQ}})$ is also #P-hard. □

Now we are ready to prove our main dichotomy theorem.

Theorem 9.2. *Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard unless $\mathcal{F} \subseteq \mathscr{A}$, $\mathcal{F} \subseteq \widehat{\mathscr{P}}$, or $\mathcal{F} \subseteq \mathscr{M}$, in which case the problem is in* P.

Proof. The tractability is given in Theorem 2.8. When $\mathcal{F}$ is not a subset of $\mathscr{A}$, $\widehat{\mathscr{P}}$, or $\mathscr{M}$, we prove that Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is #P-hard, which is sufficient because of pinning (Theorem 8.8).

For any degenerate signature $f \in \mathcal{F}$, we connect some number of $[1, 0]$ to f to get its corresponding unary signature. We replace f by this unary signature, which does not change the complexity. Thus, assume that the only degenerate signatures in $\mathcal{F}$ are unary signatures.

If $\mathcal{F} \not\subseteq \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$, then the problem is #P-hard by Theorem 9.1. Otherwise, $\mathcal{F} \subseteq \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$ and we are done by Theorem 7.6. □

We also have the corresponding theorem for the Pl-#CSP framework in the standard basis, which is equivalent to Theorem 1.1.

Theorem 9.3. *Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then* Pl-#CSP$(\mathcal{F})$ *is* #P*-hard unless $\mathcal{F} \subseteq \mathscr{A}$, $\mathcal{F} \subseteq \mathscr{P}$, or $\mathcal{F} \subseteq \widehat{\mathscr{M}}$, in which case the problem is in* P.

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A Venn Diagram of the Tractable Signatures

This section contains a Venn diagram of the tractable Pl-#CSP signature sets in the Hadamard basis. Each signature may also take an arbitrary constant multiple from $\mathbb{C}$. This figure is particularly useful in Section 7, where we consider the complexity of multiple signatures from different tractable sets. The definition of each tractable signature set is given in Section 2.

For a signature f, the notation $f \geq k$ is short for $\operatorname{arity}(f) \geq k$. Notice that $\mathscr{M} \cap \widehat{\mathscr{P}} - \mathscr{A}$ is empty.

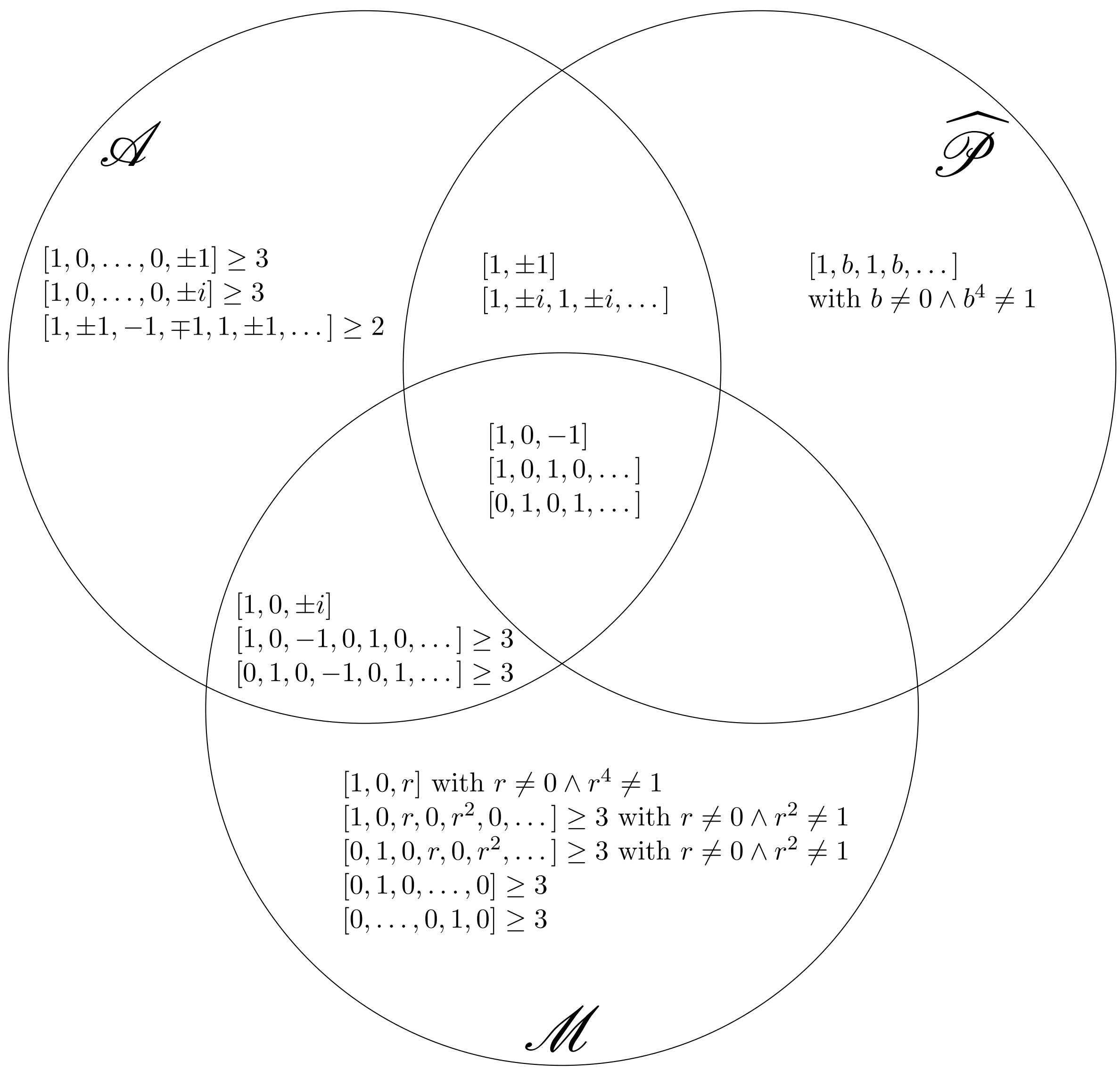

Figure 12: Venn diagram of the tractable Pl-#CSP signature sets in the Hadamard basis. Each signature has been normalized for simplicity of presentation.