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AN ANALYST'S TRAVELING SALESMAN THEOREM FOR SETS OF DIMENSION LARGER THAN ONE

JONAS AZZAM AND RAANAN SCHUL

ABSTRACT. In his 1990 Inventiones paper, P. Jones characterized subsets of rectifiable curves in the plane via a multiscale sum of β -numbers. These β -numbers are geometric quantities measuring how far a given set deviates from a best fitting line at each scale and location. Jones' result is a quantitative way of saying that a curve is rectifiable if and only if it has a tangent at almost every point. Moreover, computing this square sum for a curve returns the length of the curve up to multiplicative constant. K. Okikiolu extended his result from subsets of the plane to subsets of Euclidean space. G. David and S. Semmes extended the discussion to include sets of (integer) dimension larger than one, under the assumption of Ahlfors regularity and using a variant of Jones' β -numbers. This variant has since been used by others to give structure theorems for rectifiable sets and to give upper bounds for the measure of a set.

In this paper we give a version of P. Jones' theorem for sets of arbitrary (integer) dimension lying in Euclidean space. Our main result is a lower bound for the d -dimensional Hausdorff measure of a set in terms of an analogous sum of β -type numbers. We also show an upper bound of this type. The combination of these results gives a Jones theorem for higher dimensional sets. While there is no assumption of Ahlfors regularity, or of a measure on the underlying set, there is an assumption of a lower bound on the Hausdorff content. We adapt David and Semmes' version of Jones' β -numbers by redefining them using a Choquet integral, allowing them to be defined for arbitrary sets (and not just sets of locally finite measure). A key tool in the proof is G. David and T. Toro's parametrization of Reifenberg flat sets (with holes).

CONTENTS

1. Introduction	2
1.1. Background	2
1.2. Main Results	5
1.3. Examples and Motivation for using $\beta_E^{d,p}$	8
1.4. Motivation for $\Theta_E^{d,\Delta,\epsilon}$, and when it disappears	10
1.5. Outline of the paper	11
1.6. Constants	12
1.7. Acknowledgements	12

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2. Preliminaries	13
2.1. Hausdorff measure and content	13
2.2. Reifenberg Flat sets and the theorem of David and Toro	14
2.3. Generalized dyadic cubes	18
2.4. Preliminaries with $\beta_E^{d,p}$	19
3. The theorems we actually prove	28
4. Sketch of the proof of Theorems 3.6 and Theorem 3.2	30
5. The stopping-time	32
6. The sequence of approximating surfaces	38
7. The map between layers	46
8. The telescoping sum	49
9. Reducing β 's to Ω 's and the completion of the proof of Theorem 3.6 ..	55
10. Proof of Theorem 3.2	58
11. Proof of Theorem 3.3 - part 1	67
12. Proof of Theorem 3.3 - part 2	70
13. Appendix	73
References	77

1. INTRODUCTION

1.1. **Background.** We will begin by recalling the *Analyst's Traveling Salesman Theorem*. It will serve as a model for the two main results in this paper. For sets $E, B \subseteq \mathbb{R}^n$, define

$$\beta_{E,\infty}^d(B) = \frac{2}{\text{diam}(B)} \inf_L \sup \{\text{dist}(y, L) : y \in E \cap B\} \quad (1.1)$$

where L ranges over d -planes in \mathbb{R}^n . Thus, $\beta_{E,\infty}^d(B) \text{diam}(B)$ is the width of the smallest tube containing $E \cap B$. We will typically have B be a ball or cube. We will denote by Δ the collection of dyadic cubes in \mathbb{R}^n (see the beginning of Section 2).

Theorem 1.1. (Jones: \mathbb{R}^2 [Jon90]; Okikiolu: \mathbb{R}^n [Oki92]) *Let $n \geq 2$. There is a $C = C(n)$ such that the following holds. Let $E \subset \mathbb{R}^n$. Then there is a connected set $\Gamma \supseteq E$ such that*

$$\mathcal{H}^1(\Gamma) \lesssim_n \text{diam } E + \sum_{\substack{Q \in \Delta \\ Q \cap E \neq \emptyset}} \beta_{E,\infty}^1(3Q)^2 \text{diam}(Q). \quad (1.2)$$

Conversely, if Γ is connected and $\mathcal{H}^1(\Gamma) < \infty$, then

$$\text{diam } \Gamma + \sum_{\substack{Q \in \Delta \\ Q \cap \Gamma \neq \emptyset}} \beta_{\Gamma,\infty}^1(3Q)^2 \text{diam}(Q) \lesssim_n \mathcal{H}^1(\Gamma). \quad (1.3)$$

Here, \mathcal{H}^k is the k -dimensional Hausdorff measure. Given two functions a and b into \mathbb{R} we say $a \lesssim b$ with constant C , when there exists a constant $C = C_{a,b}$ such that $a \leq Cb$. We say that $a \sim b$ if $a \lesssim b$ and $b \lesssim a$.

We remark that a version of Theorem 1.1 holds for E in an infinite dimensional Hilbert spaces [Sch07], but that requires switching from Δ to a multi-resolution of balls centered on E . We will revisit this point in Section 3.

Remark 1.2. We remark for the expert that the main result of this paper is Theorem I, and is a version of (1.3). We also prove Theorem II, a version of (1.2), though the proof of Theorem I is about six times longer than that of Theorem II.

Intrinsically, both implications of the Theorem 1.1 are interesting: a sufficient condition for the existence of a short curve is very useful, as well as the ability to quantify how non-flat a curve can be at various scales and locations. The result has applications to harmonic measure [BJ90], Kleinian groups [BJ97], analytic capacity [Tol05], and brownian motion [BJPY97].

A d -dimensional analogue of the second half of Theorem 1.1 has been known to be false since the 90's: in Fang's thesis [Fan90] he gives an example of a Lipschitz graph where this sum is infinite. David and Semmes, however, realized that one could still develop a theory of β -numbers if instead one adjusted the definition of a β -number. We will state a theorem by them below (Theorem 1.4). Their inspiration seems like it could come from the following theorem of Dorrnsoro.

Theorem 1.3 (Dorrnsoro, [Dor85], Theorem 6). *Let $1 \leq p < p(d)$ where*

$$p(d) := \begin{cases} \frac{2d}{d-2} & \text{if } d > 2 \\ \infty & \text{if } d \leq 2 \end{cases} . \quad (1.4)$$

For $x \in \mathbb{R}^d$, $r > 0$, and $f \in W^{1,2}(\mathbb{R}^d)$, define

$$\Omega_{f,p}(x, r) = \inf_A \left(\int_{B(x,r)} \left(\frac{|f - A|}{r} \right)^p \right)^{\frac{1}{p}}$$

where the infimum is over all affine maps $A : \mathbb{R}^d \rightarrow \mathbb{R}$. Set

$$\Omega_p(f) := \int_{\mathbb{R}^d} \int_0^\infty \Omega_{f,p}(x, r)^2 \frac{dr}{r} dx.$$

Then

$$\Omega_p(f) \lesssim_{d,p} \|\nabla f\|_2^2.$$

The result is actually much stronger than stated here. We refer the reader to [Dor85] for more details.

Define now the following β -number analogue: Given a closed ball $B \subset \mathbb{R}^d$ with radius r_B , a measure μ , an integer $0 < d < D$, and $1 \leq p < \infty$, let

$$\beta_{\mu,p}^d(B, L) = \left(\frac{1}{r_B^d} \int_B \left(\frac{\text{dist}(y, L)}{r_B} \right)^p d\mu(y) \right)^{1/p}$$

and

$$\beta_{\mu,p}^d(B) = \inf\{\beta_{\mu,p}^d(B, L) : L \text{ is a } d\text{-plane in } \mathbb{R}^n\}.$$

When $\mu = \mathcal{H}_E^d$ for some set E , we will write $\beta_{E,p}^d$ instead of $\beta_{\mathcal{H}_E^d|_{E,p}}^d$. Recall σ is a Carleson measure on $E \times (0, \infty)$ if $\sigma(B(x, r) \times (0, r)) \leq Cr^d$.

Theorem 1.4 (David, Semmes [DS91]). *Let $E \subseteq \mathbb{R}^n$ be an Ahlfors d -regular set E , meaning there is $A > 0$ so that*

$$r^d/A \leq \mathcal{H}^d(B(\xi, r) \cap E) \leq Ar^d \text{ for } \xi \in E, r \in (0, \text{diam } E). \quad (1.5)$$

Then the following are equivalent:

- (1) *The set E has big pieces of Lipschitz images, i.e. there are constants $L, c > 0$ so for all $\xi \in E$ and $r \in (0, \text{diam } E)$, there is an L -Lipschitz map $f : \mathbb{R}^d \rightarrow \mathbb{R}^n$ such that $\mathcal{H}^d(f(\mathbb{R}^d) \cap B(\xi, r)) \geq cr^d$.*
- (2) *For $1 \leq p < p(d)$, $\beta_{E,1}^d(B(x, r))^2 \frac{dxdr}{r}$ is a Carleson measure on $E \times (0, \infty)$.*

A closed set $E \subseteq \mathbb{R}^n$ is said to be *d -uniformly rectifiable* if it is d -Ahlfors regular and has big pieces of Lipschitz images. We remark that constants coming out of this theorem (and similar theorems) depend on the constant of Ahlfors-regularity (denoted above by A).

The motivation behind this result (and in fact also Jones' original motivation of Theorem 1.1) was the study of singular integrals. In [DS91], David and Semmes show that the two conditions in Theorem 1.4 are in fact equivalent to five other characterizations, one of them being that all odd singular integrals whose kernels are reasonably behaved are bounded operators. In fact, $\beta_{\mu,p}^d$ has been more amenable in applications to singular integrals even in the plane, see for example [MMM96, Dav98, Leg99, AT15, GT16] and the references therein.

We remark that using the work of David and Toro [DT12], for certain kinds of sets E , one can obtain a higher dimensional analogue of the first half of Theorem 1.1—that is, sufficient conditions in terms of $\beta_{E,\infty}^d(3Q)^2 \text{diam}(Q)^d$ for when a set can be contained in a d -dimensional surface of finite area (see Theorem 2.5 and 11.2 below).

The quantity β_2 is closely related to the Singular Value Decomposition of a matrix. See [LW12, ACM12, LMR16] and references therein for applications to data analysis.

Recently, the first author and Tolsa have given a characterization of rectifiability for sets of finite measure beyond the Ahlfors regular category.

Theorem 1.5 (Azzam, Tolsa [AT15]; Tolsa [Tol15]). *Let $E \subseteq \mathbb{R}^n$ be a Borel set with $0 < \mathcal{H}^d(E) < \infty$ and $\mu \ll \mathcal{H}^d|_E$.*

- (1) *If*

$$\int_0^1 \beta_{\mu,2}^d(B(x, r))^2 \frac{dr}{r} < \infty \text{ for } \mathcal{H}^d\text{-a.e. } x \in E, \quad (1.6)$$

then E is d -rectifiable.

(2) If E is d -rectifiable and $p \in [1, 2]$, then $\int_0^1 \beta_{\mu,p}^d(B(x,r))^2 \frac{dr}{r} < \infty$ for \mathcal{H}^d -a.e. $x \in E$.

The first part of this result was first shown by Pajot under some stronger assumptions (see also [BS16a]). In [BS16b], Badger and the second author gave a similar characterization for general measures on Euclidean space (see also [BS15]), but with $d = 1$.

Addendum:(added 2017/02/15) Part (1) of the above result was very recently improved by Edelen, Naber, and Valtorta in [ENV16]. In their paper, they give perhaps the most general results to date about how well the size of μ is bounded from above by the $\beta_{\mu,p}^d$ -numbers and describes the support. As a corollary of their work, they show that if μ is any Radon measure such that (1.6) holds and

$$\limsup_{r \rightarrow 0} \mu(B(x,r))/r^d > 0 \text{ for } \mu\text{-a.e. } x,$$

then μ can be covered by Lipschitz images of \mathbb{R}^d . This implies the first part of Theorem 1.5. See also [NV17, Sections 3-6] for similar results and their connection to singular sets of non-linear PDEs. Their results form an analogue of the (1.2) half of Theorem 1.1 in higher dimension.

By comparison, our main result, Theorem I below, says that for a class of sets E , we can obtain *lower bounds* for the Hausdorff measure of E in terms of content- β -numbers (see Definition 1.6), that is, a version of the (1.3) half of Theorem 1.1. This is, to our knowledge, the first time this has been achieved for sets of dimension larger than one (even for Hausdorff measure, let alone general measures). Additionally, in Theorem II below we can also obtain *upper bounds* on the Hausdorff measure of a set E in terms of its β -numbers (that is, a version of the (1.2) half of Theorem 1.1). The proof of Theorem II is much less substantial than Theorem I, but the result is interesting for a few reasons: firstly, while the works of [ENV16] and [NV17] obtain upper bounds on more general measures μ than just the Hausdorff measure, we do not assume the existence of a locally finite measure on our set, and in particular, we don't assume our set E has finite Hausdorff measure a priori. Moreover, the β -numbers we use don't require a finite measure in order to be defined like $\beta_{\mu,p}^d$ does, and thus we have a measure-independent and purely geometric method for testing whether a set has finite Hausdorff measure. Finally, the combination of Theorems I and II gives an analogue to the full Theorem 1.1 for higher dimensional objects.

1.2. Main Results. Before stating our main results, we first define our β -number that, firstly, is in some sense an L^p -average of distances to a plane (rather than an L^∞ -norm), and secondly, doesn't rely on the underlying measure on the set in question. This seems somewhat self-contradictory, but we are able to achieve this by manipulating the definition of $\beta_{\mathcal{H}^d|_E,p}^d$ and "integrate" with respect to \mathcal{H}_∞^d (rather than \mathcal{H}^d) using a Choquet integral.

Definition 1.6. For arbitrary sets E , and B , a d -dimensional plane L , define

$$\beta_E^{d,p}(B, L) = \left(\frac{1}{r_B^d} \int_0^1 \mathcal{H}_\infty^d(\{x \in B \cap E : \text{dist}(x, L) > tr_B\}) t^{p-1} dt \right)^{\frac{1}{p}}$$

where $2r_B = \text{diam}(B)$, and set

$$\beta_E^{d,p}(B) = \inf\{\beta_E^{d,p}(B, L) : L \text{ is a } d\text{-dimensional plane in } \mathbb{R}^n\}.$$

We will typically take B to be a ball or cube.

If we assume E is Ahlfors d -regular, then this quantity is comparable to $\beta_{\mathcal{H}^d|_{E,P}}^d$. Indeed, for a ball B centered on E ,

$$\begin{aligned} \beta_{\mathcal{H}^d|_{E,P}}^d(B, L)^p &\sim \frac{1}{r_B^d} \int_B \left(\frac{\text{dist}(y, L)}{r_B} \right)^p d\mathcal{H}^d(y) \\ &= \frac{1}{r_B^d} \int_0^\infty \mathcal{H}^d(\{x \in B : \text{dist}(x, L) > tr_B\}) t^{p-1} dt \end{aligned}$$

and it is not hard to check that this is comparable to $\beta_E^{d,p}(B, L)$ using the Ahlfors regularity of E .

Definition 1.7. Given two closed sets E and F , and B a set we denote

$$d_B(E, F) = \frac{2}{\text{diam } B} \max \left\{ \sup_{y \in E \cap B} \text{dist}(y, F), \sup_{y \in F \cap B} \text{dist}(y, E) \right\}$$

and

$$\vartheta_E^d(B) = \inf\{d_B(E, L) : L \text{ is a } d\text{-dimensional plane in } \mathbb{R}^n\}.$$

We will typically have B be a ball or cube. If $B = B(x, r)$, we will write $d_B = d_{x,r}$. We say E is (ϵ, d) -Reifenberg flat (or just ϵ -Reifenberg flat when the dimension is given) if

$$\vartheta_E^d(B(x, r)) < \epsilon \text{ for all } x \in E \text{ and } r > 0.$$

Definition 1.8. Let $E, B \subseteq \mathbb{R}^n$ and $\epsilon, r > 0$. We define

$$\Theta_E^{d,\Delta,\epsilon}(B) := \sum \{\text{diam}(Q)^d : Q \in \Delta, Q \cap E \cap B \neq \emptyset \text{ and } \vartheta_E(3Q) \geq \epsilon\}.$$

The Δ in the superscript denotes that this is a quantity involving dyadic cubes.

Remark 1.9. If E is an ϵ -Reifenberg flat then $\Theta_E^{d,\Delta,\epsilon} = 0$.

Definition 1.10. A set $E \subseteq \mathbb{R}^n$ is said to be (c, d) -lower content regular in a ball B if

$$\mathcal{H}_\infty^d(E \cap B(x, r)) \geq cr^d \text{ for all } x \in E \cap B \text{ and } r \in (0, r_B).$$

Remark 1.11. Some examples of sets $E \subseteq \mathbb{R}^n$ which are (c, d) -lower content regular for some $c > 0$ and integer d are: Reifenberg flat sets (see 1.7) and sets satisfying Condition B (see definition 1.18). One may also remove subsets from such E in a controlled way to keep lower content regularity, with a smaller $c > 0$.

Our first main result is the following.

Theorem I. *Let $1 \leq d < n$, $C_0 > 1$. Let $1 \leq p < p(d)$ where $p(d)$ was defined in (1.4). Let $E \subseteq \mathbb{R}^n$ be a closed set containing 0. Suppose that E is (c, d) -lower content regular in $B(0, 1)$. There is $\epsilon_0 = \epsilon_0(n, p, c) > 0$ such that for $0 < \epsilon < \epsilon_0$. Then*

$$1 + \sum_{\substack{Q \in \Delta \\ Q \cap E \cap B(0,1) \neq \emptyset}} \beta_E^{d,p}(C_0 Q)^2 \text{diam}(Q)^d \lesssim_{C_0, n, \epsilon, p, c} \mathcal{H}^d(E \cap B(0, 1)) + \Theta_E^{d, \Delta, \epsilon}(B(0, 1)). \quad (1.7)$$

In the case that E is Reifenberg flat, it will automatically be lower content regular by (2.20) below. In this case, $\Theta_E^{d, \Delta, \epsilon}$ will be zero by definition, and so Theorem I gives a much cleaner result for this class of sets. See also Section 1.4 for more scenarios where $\Theta_E^{d, \Delta, \epsilon}$ disappears.

Remark 1.12. We use the lower content regularity crucially in two places. One is the proof of Lemma 2.21, and the other is the proof of Lemma 10.5.

There will be a longer discussion about $\Theta_E^{d, \Delta, \epsilon}$ later after Example 1.17.

A converse to Theorem I is also possible.

Theorem II. *Let $1 \leq d < n$, $C_0 > 1$, and $1 \leq p \leq \infty$ and $E \subseteq \mathbb{R}^n$ be (c, d) -lower content regular in $B(0, 1)$ such that $0 \in E$. Let $\epsilon > 0$ be given. Then*

$$\mathcal{H}^d(E \cap B(0, 1)) + \Theta_E^{d, \Delta, \epsilon}(0, 1) \lesssim_{n, c, C_0, \epsilon} 1 + \sum_{\substack{Q \in \Delta \\ Q \cap E \cap B(0,1) \neq \emptyset}} \beta_E^{d,p}(C_0 Q)^2 \text{diam}(Q)^d. \quad (1.8)$$

Furthermore, if the right hand side of (1.8) is finite, then E is d -rectifiable

As a corollary of the above two theorems, we have that for any (c, d) -lower content regular set and $1 \leq p < p(d)$ that for $\epsilon > 0$ small enough

$$\mathcal{H}^d(E \cap B(0, 1)) + \Theta_E^{d, \Delta, \epsilon}(0, 1) \sim_{n, c, C_0, p, \epsilon} 1 + \sum_{\substack{Q \in \Delta \\ Q \cap E \cap B(0,1) \neq \emptyset}} \beta_E^{d,p}(C_0 Q)^2 \text{diam}(Q)^d.$$

Remark 1.13. The proof of Theorem II contains more information than presented in its statement. The rectifiability of E comes about from the construction of a sequence of surfaces, which are bi-Lipschitz images of d -dimensional cubes. The bi-Lipschitz constant and sizes of cubes are controlled. For example, one may slightly modify the construction to yield a connected, rectifiable set Γ such that $\mathcal{H}^d(E \setminus \Gamma) = 0$ and for all $x \in \Gamma$, $r \leq 1$ we have $\mathcal{H}_\infty^d(\Gamma \cap B(x, r)) \geq c' r^d$ for some (explicit) $c' > 0$. This Γ will have

$$\mathcal{H}^d(\Gamma) \leq C(n, c) \left(1 + \sum_{\substack{Q \in \Delta \\ Q \cap E \cap B(0,1) \neq \emptyset}} \beta_E^{d,p}(C_0 Q)^2 \text{diam}(Q)^d \right).$$

In fact, one obtains a coronization similar to that in [DS91], but with bi-Lipschitz surfaces in place of Lipschitz graphs. We refer the reader to [DS91] for the definition of a coronization.

1.3. Examples and Motivation for using $\beta_E^{d,p}$. Having stated these results, we can further motivate our choice of $\beta_E^{d,p}$. There are a few other natural choices that one could try to make: using averages (w.r.t. \mathcal{H}^d) to define β , or using a supremum. These are addressed in Examples 1.14, 1.15 and 1.16)

and discuss the difference between studying the geometry of the set E and the geometry of the surface measure $\mathcal{H}^d|_E$. Note that there are other β -type numbers we could have considered. One option is

$$\beta_E^{d,p,*}(B) = \inf_L \left(\int_{E \cap B} \left(\frac{\text{dist}(y, L)}{r_B} \right)^p d\mathcal{H}^d(y) \right)^{\frac{1}{p}}.$$

This quantity in some sense measures how flat the measure $\mathcal{H}^d|_E$ is by measuring how close it is to lying on a d -dimensional plane. However, it does require the a priori assumption that E has locally finite \mathcal{H}^d -measure, and even so, it does not necessarily give useful *geometric* information if the surface measure is unstable, as the following example will show.

Example 1.14. Consider the following set in the complex plane, let $I_j = [0, 1] + 2^{-j^2}i \subseteq \mathbb{C}$ and set

$$E = \partial[0, 1]^2 \cup \bigcup_{j=10}^N I_j.$$

If Q is a dyadic cube contained in $[0, 1]^2$, and $Q \cap E \neq \emptyset$ but $3Q$ does not contain any of the endpoints of the I_j or points in $i + \mathbb{R}$ (call this set of cubes \mathcal{G}), then $Q \cap I_j \neq \emptyset$ for some j . If additionally $3Q \cap I_k \neq \emptyset$ for some $k > j$, then in fact $3Q \cap \mathbb{R} \neq \emptyset$ so in fact $3Q \cap I_\ell \neq \emptyset$ for all $\ell \geq j$. Thus, either $\beta_E^{1,2,*}(3Q) = 0$ or $\beta_E^{1,2,*}(3Q) > 0$ and $3Q \cap \mathbb{R} \neq \emptyset$, in which case, if $j(Q)$ be the smallest integer so that $2^{-j(Q)} < \ell(Q)$, then

$$\begin{aligned} \beta_E^{d,2,*}(3Q)^2 &\sim \frac{1}{\ell(Q)(N - j(Q))} \int_{3Q \cap E} \left(\frac{\text{dist}(x, \mathbb{R})}{\ell(Q)} \right)^2 d\mathcal{H}^1(x) \\ &= \frac{1}{\ell(Q)(N - j(Q))} \sum_{j=j(Q)-1}^N 2^{-j^2+2j(Q)} \ell(Q) \\ &\lesssim \frac{1}{N - j(Q)}. \end{aligned}$$

It is not hard to show that if \mathcal{B} are those cubes Q for which $3Q$ contains an endpoint of some I_j then $\sum_{Q \in \mathcal{B}} \ell(Q) \lesssim 1$. If cubes only intersect $i + \mathbb{R}$ then $\beta_E^{d,2,*} = 0$.

Thus,

$$\sum_{\substack{Q \subseteq Q_0 \\ Q \cap E \neq \emptyset}} \beta_E^{d,2,*} (3Q)^2 \ell(Q) \lesssim 1 + \sum_{Q \in \mathcal{G}} \frac{1}{N-j(Q)} \ell(Q) \lesssim 1 + \sum_{j=1}^{N-1} \frac{1}{N-j} \lesssim \log N.$$

However, $\mathcal{H}^1(E) \sim N$, and so it is not possible to obtain a theorem like Theorem II, even if we assumed E had finite measure. Additionally, even if a theorem like Theorem I held with $\beta_E^{1,p,*}$, this estimate above shows that the square sum of $\beta_E^{1,p,*}$ in place of $\beta_E^{1,p}$ gives a weaker lower bound for $\mathcal{H}^1(E) + \Theta_E^{1,\Delta,\epsilon}(0, \sqrt{2})$.

Example 1.15. Using $\beta_{\mathcal{H}^1|_{E,2}}^d$ won't work either: if we take E to be the union of boundaries of squares in $[0, 1]^2$ of side length 2^{-N} , then, for each dyadic cube Q that intersects E with $\ell(Q) = 2^{-j} \geq 2^{-N}$,

$$\beta_{\mathcal{H}^1|_{E,2}}^d (3Q)^2 \gtrsim \frac{2^{2(N-j)} 2^{-N}}{\ell(Q)} = 2^{N-j} = 2^N \ell(Q)$$

and so

$$\sum_{\substack{Q \subseteq [0,1]^2 \\ Q \cap E \neq \emptyset}} \beta_{\mathcal{H}^1|_{E,2}}^2 (3Q)^2 \ell(Q) \gtrsim \sum_{\substack{Q \subseteq [0,1]^2 \\ Q \cap E \neq \emptyset, \ell(Q) \geq 2^{-N}}} 2^N \ell(Q)^2 \gtrsim 2^N N.$$

However, $\mathcal{H}^1(E) \sim 2^{2N} \cdot 2^{-N} = 2^N \sim \Theta_E^{1,\Delta,\epsilon}(0, \sqrt{2})$, so again, a version of Theorem I cannot hold with these β -numbers.

Example 1.16. Using β_∞ does not work. The following example was constructed by X. Fang in his Ph.D. dissertation under the supervision of P. Jones [Fan90]. It was communicated to the authors by Guy David and Sean Li.

Let $Q_0 \subset \mathbb{R}^3$ be the unit cube. We will construct a sequence of functions $f_0, f_1, f_2, \dots : Q_0 \rightarrow \mathbb{R}$ which uniformly converge to a function f . All of these functions will be 1-Lipschitz. To this end, let $\epsilon_1, \epsilon_2, \dots < 1$ be a sequence of powers of 2, such that $\sum \epsilon_k^2$ diverges, and $\sum \epsilon_k^3 < \frac{1}{2}$. Write $\epsilon_k = 2^{-j_k}$. We will set \mathcal{D}_k to be dyadic subcubes of Q_0 of side $2^{-j_1 - \dots - j_k}$. Let $f_0 : Q_0 \rightarrow \mathbb{R}$ be the constant function. Subdivide Q_0 into cubes of side ϵ_1 , i.e. into the cubes of \mathcal{D}_1 . Choose one of these \mathcal{D}_1 cubes, and modify f_0 on it to obtain f_1 by adding a piecewise linear spike of height $\epsilon_1/2$. Take now all other \mathcal{D}_1 cubes, and in each one choose a \mathcal{D}_2 cube. Get f_2 from f_1 by adding a piecewise linear spike of height $\epsilon_2/2$ at each of these special \mathcal{D}_2 cubes. Continue this way. To get f_{i+1} from f_i choose in each \mathcal{D}_i cube on which f_i is constant, a single subcube from \mathcal{D}_{i+1} , and add a piecewise linear spike there of height $\epsilon_{i+1}/2$. We now make a few observations.

- If on f_i was modified on a cube $Q \in \mathcal{D}_{i+1}$, then for all $k > i$, f_k will never be modified inside Q .
- $\|f_{i+1} - f_i\|_\infty < \epsilon_i$

- The total volume of cubes from \mathcal{D}_{k+1} which was modified in going from f_k to f_{k+1} was

$$1 - \sum_1^k \epsilon_i^3 \geq \frac{1}{2}.$$

- At each of the cubes modified in going from f_k to f_{k+1} , the, β_∞ of the corresponding dyadic cube in \mathbb{R}^4 (for the graph of the function f_l for any $l > k$) is $\sim \epsilon_{k+1}$.

Thus we have that if we compute

$$\sum \beta_\infty^2(Q) \text{diam}(Q)^3$$

for the graph of the limiting function, the sum is proportional to $\sum \epsilon_k^2$ which diverges. The main point above is that the exponent of the β and that of the ϵ sum differ because of the dimension. By scaling this example we may get that it is Reifenberg flat.

Thus, the $\beta_E^{d,p}$ that we have are ideal in that none of the previously studied β -numbers can achieve the same results.

1.4. Motivation for $\Theta_E^{d,\Delta,\epsilon}$, and when it disappears. The presence of $\Theta_E^{d,\Delta,\epsilon}$ in our results may seem odd, but our next example shows that it cannot be ignored.

Example 1.17. Let $f_j(z) = z/4 + z_j$ where $z_0 = 0$, $z_1 = 3/4$, $z_2 = 3i/4$, and $z_3 = 3(1+i)/4$. These are the usual contractions used in defining the 4-corner Cantor set in the plane. Let $f(K) := \bigcup_{j=0}^3 f_j(K)$, $K_0 := \partial[0, 1]^2$, and $K_n = f(K_{n-1})$. Then it is not hard to show that $\mathcal{H}^1(K_n) \sim 1$ for all n while

$$\sum_{Q \cap K_n \neq \emptyset} \beta_E^{1,2}(3Q)^2 \ell(Q) \sim n.$$

Thus, by Theorems I and II, for large n , $\Theta_{K_n}^{1,\Delta,\epsilon}(B(0, 2)) \sim n \gg \mathcal{H}^1(K_n)$.

This example should be viewed in the following way. One may estimate the length of the shortest curve containing a set E by Theorem 1.1. This length, however, could be much bigger than the length of E . The set E can be seen as the curve Γ punctured by holes, which $\Theta_E^{d,\Delta,\epsilon}$ accounts for. This happens since ϑ_E^d also measures how far an optimal plane is from E , and hence, if ϑ_E^d is large yet $\beta_E^{d,p}$ is small, this means that E is very flat but contains a large d -dimensional hole.

Finally, we note that the $\Theta_E^{d,\Delta,\epsilon}$ quantity is subsumed by the \mathcal{H}^d quantity in some natural situations. Firstly, Theorem 1.1 implies that

$$\Theta_\Gamma^{1,\Delta,\epsilon}(x, r) \leq C\epsilon^{-2} \mathcal{H}^1(B(x, r) \cap \Gamma)$$

for any curve Γ . There are also some objects of higher topological dimension for which this holds.

Definition 1.18 (Condition B). We will say $E \subseteq \mathbb{R}^n$ satisfies Condition B for some $c > 0$ if for all $x \in E$ and $r > 0$, one can find two balls of radius rc contained in $B(x, r)$ in two different components of E^c .

Usually, this definition also assumes E is Ahlfors regular (see for example [Dav88], [DS93b], and [DJ90]), and there the authors give different proofs that, in this situation, E is uniformly rectifiable. If E has locally finite \mathcal{H}^{n-1} -measure, then one can show that, for any ball B centered on E , there is a Lipschitz graph Γ so that $\mathcal{H}^{n-1}(\Gamma \cap B \cap E) \geq r_B^{n-1}$ with Lipschitz constant depending on $\mathcal{H}^{n-1}(B \cap E)$ (see [JKV97] and [Bad12]).

Theorem 1.19 (David, Semmes [DS93b]). Let $E \subseteq \mathbb{R}^n$ satisfy Condition B for some $c > 0$. Then for all $x \in E$, and $\epsilon, r > 0$,

$$\Theta_E^{n-1, \Delta, \epsilon}(x, r) \leq C(\epsilon, n) \mathcal{H}^{n-1}(E \cap B(x, r)). \quad (1.9)$$

This is not stated in [DS93b], but it is implied by the proof of [DS93b, Theorem 1.20], since it only really depends on the lower regularity of E , that is,

$$\mathcal{H}^d(B(x, r) \cap E) \geq cr^d \text{ for all } x \in E \text{ and } 0 < r < \text{diam } E,$$

and such sets are always lower regular with constant depending on the Condition B constant.

Thus, by Theorems I and II, we have the following.

Corollary III. If E satisfies Condition B for some constant $c > 0$, then for $C_0 > 1$ and $1 \leq p < p(d)$,

$$\mathcal{H}^{n-1}(E \cap B(0, 1)) \sim_{C_0, n, c, p} 1 + \sum_{\substack{Q \in \Delta \\ Q \cap E \cap B(0, 1) \neq \emptyset}} \beta_E^{n-1, p}(C_0 Q)^2 \text{diam}(Q)^{n-1} \quad (1.10)$$

An interesting problem would be to determine what simple geometric criteria a set E has to have in order to satisfy

$$\Theta_E^{d, \Delta, \epsilon}(B(x, r)) \leq C(\epsilon, n) \mathcal{H}^d(E \cap B(x, r)). \quad (1.11)$$

much like the program that already exists for finding criteria for when Ahlfors regular sets are uniformly rectifiable (see [DS91, DS93a]).

1.5. Outline of the paper. Section 2 contains preliminaries, a discussion of a theorem of David and Toro (Theorem 2.5), as well as some basic properties of our Choquet-style definition of β . Other preliminaries regarding \mathcal{H}_∞^d “integration” (in the sense of Choquet) were pushed to the Appendix (Section 13).

In Section 3, we restate Theorem I differently as Theorem 3.2, using maximal nets and balls instead of cubes, which will be more natural to prove. We also have a version of Theorem II with nets and balls. We then introduce Theorem 3.6, which is a version of Theorem 3.2 for the Reifenberg flat case. Section 4 will contain a

loose description of the proofs of Theorems 3.6 and 3.2, the details of which will be carried out in Sections 5–10.

Sections 11 and 12 contain a proof of Theorem 3.3. The proof can be summarized as *a stopping time applied to Theorems 2.5 and 11.2* (both from [DT12]).

1.6. Constants. We list some important constants, where they appear, are fixed, and their dependencies.

- n : ambient dimension (as in \mathbb{R}^n). Given in main theorems.
- d : intrinsic dimension (as in \mathcal{H}^d). Given in main theorems.
- C_0 : given in main theorems.
- c_0 : fixed in Theorem 2.9 as $1/500$.
- r_k : fixed in Theorem 2.5 as $r_k = 10^{-k}$.
- A : ball dilation factor. Given in the reformulation of the main theorems which happens in Section 3.
- C_1 : fixed as $C_1 = 2$ for Lemma 5.11 and $C_1 = 2C_2$ for Lemma 6.10. In section 10, it will be fixed so that $20C_0 \ll C_1$ where C_0 is the constant appearing in Theorem 3.2.
- C_2 : a constant introduced in Lemma 6.9 and fixed in Section 9, see Remark 9.1.
- ρ : scale factor. See Theorem 2.9. The constant ρ is fixed following the statement of Theorem 2.9.
- α : Allowed angle of rotation between planes in stopping time region. See remark 5.2 and definition of S_Q which follows it. The constant α may be fixed in the proof of Lemma 8.2 and depends only on n , see Remark 8.3.
- ϵ : Reifenberg flatness constant. Depends on n , and α, τ , and should be small enough for Theorem 2.5 below to hold. See Remark 5.2. In relation to α we will require that $\epsilon \ll \alpha^4$.
- τ, τ_0, τ_1 : Constant used for extending Stopping time regions. $\tau < \tau_0, \tau_1, \frac{c_0}{4}$, where $\tau_0 = \tau_0(\rho)$, and $\tau_1 = \min\{\tau_1(2C_2), \tau_1(2), \tau_1(4)\}$. See Remark 7.1. The constant τ is fixed (for the proof of Theorem 3.6) at the start of Section 7.
- M : a dilation factor for balls. Used in a similar fashion to C_0 and A .
- θ : an angle of rotation between planes. First introduced in Lemma 6.10. One can take $\theta \lesssim \epsilon$.

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2. PRELIMINARIES

We will write $a \lesssim b$ if there is $C > 0$ such that $a \leq Cb$ and $a \lesssim_t b$ if the constant C depends on the parameter t . We also write $a \sim b$ to mean $a \lesssim b \lesssim a$ and define $a \sim_t b$ similarly.

For sets $A, B \subset \mathbb{R}^n$, let

$$\text{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\}, \quad \text{dist}(x, A) = \text{dist}(\{x\}, A),$$

and

$$\text{diam } A = \sup\{|x - y| : x, y \in A\}.$$

Recall the definition of $d_{x,r}(E, F)$ from Definition 1.7. It is not hard to show that, for sets E, F, G , while this does not satisfy the triangle inequality, we do have

$$d_{x,r}(E, G) \lesssim d_{x,2r}(E, F) + d_{x,2r}(F, G). \quad (2.1)$$

We will denote by Δ the standard dyadic grid on \mathbb{R}^n , that is

$$\Delta = \left\{ \left[\frac{j_1}{2^k}, \frac{j_1 + 1}{2^k} \right) \times \cdots \times \left[\frac{j_n}{2^k}, \frac{j_n + 1}{2^k} \right), \quad k, j_1, \dots, j_n \in \mathbb{Z} \right\}.$$

For any cube Q and $\lambda > 0$, we let λQ the cube which is a dilation by λ of Q , that is, λQ is concentric with Q , has sides parallel to Q and with $\text{diam}(\lambda Q) = \lambda \text{diam}(Q)$.

2.1. Hausdorff measure and content. For a subset $A \subset \mathbb{R}^{n+1}$, integer $d > 0$, and $0 < \delta \leq \infty$ one sets

$$\mathcal{H}_\delta^d(A) = \inf \left\{ \sum (\text{diam } A_i)^d : A \subset \bigcup A_i, \text{diam } A_i \leq \delta \right\}.$$

The d -dimensional Hausdorff measure of A is defined as

$$\mathcal{H}^d(A) = \lim_{\delta \downarrow 0} \mathcal{H}_\delta^d(A),$$

and $\mathcal{H}_\infty^d(A)$ is called the d -dimensional Hausdorff content of A . See [Mat95, Chapter 4] for more details. Note that \mathcal{H}_∞^d is not a measure. We do, however, want to use the notation of integration with respect to \mathcal{H}_∞^d . For $0 < p < \infty$ and $E \subseteq \mathbb{R}^n$ Borel, we will define the p -Choquet integral

$$\int_E f^p d\mathcal{H}_\infty^d := \int_0^\infty \mathcal{H}_\infty^d(\{x \in E : f(x) > t\}) t^{p-1} dt$$

and

$$\int f d\mathcal{H}_\infty^d = \int_{\mathbb{R}^n} f d\mathcal{H}_\infty^d.$$

The proofs of the following three lemmas may be found in the Appendix (Section 13).

Lemma 2.1. *Let $0 < p < \infty$. Let f_i be a countable collection of Borel functions in \mathbb{R}^n . If the sets $\text{supp } f_i = \{f_i > 0\}$ have bounded overlap, meaning there exists a $C < \infty$ such that*

$$\sum \mathbb{1}_{\text{supp } f_i} \leq C,$$

then

$$\int \left(\sum f_i \right)^p d\mathcal{H}_\infty^d \leq C^p \sum \int f_i^p d\mathcal{H}_\infty^d. \quad (2.2)$$

Lemma 2.2. *Let $E \subseteq \mathbb{R}^n$ be compact and f a continuous function on \mathbb{R}^n . Let E_j be a decreasing sequence of sets containing E and converging to E in the Hausdorff metric. Then*

$$\lim_{j \rightarrow \infty} \int_{E_j} f d\mathcal{H}_\infty^d \sim \int_E f d\mathcal{H}_\infty^d. \quad (2.3)$$

Lemma 2.3. *Let $E \subseteq \mathbb{R}^n$ be either compact or bounded and open so that $\mathcal{H}^d(E) > 0$, and let $f \geq 0$ be continuous on E . Then for $1 < p \leq \infty$,*

$$\frac{1}{\mathcal{H}_\infty^d(E)} \int_E f d\mathcal{H}_\infty^d \lesssim_n \left(\frac{1}{\mathcal{H}_\infty^d(E)} \int_E f^p d\mathcal{H}_\infty^d \right)^{\frac{1}{p}}. \quad (2.4)$$

2.2. Reifenberg Flat sets and the theorem of David and Toro. Our main tool will be the enhanced Reifenberg parametrization theorem of David and Toro. First, we recall Reifenberg's theorem.

Theorem 2.4. *For all $0 < d < n$ and $0 < s < 1/10$, we may find $\epsilon > 0$ such that the following holds. Let $E \subseteq \mathbb{R}^n$ be a closed set containing the origin that is ϵ -Reifenberg flat in $B(0, 10)$. Then there is a bijective mapping $g : \mathbb{R}^n \hookrightarrow \mathbb{R}^n$ such that*

$$\frac{1}{4}|x - y|^{1+s} \leq |g(x) - g(y)| \leq 10|x - y|^{1-s}$$

and

$$E \cap B(0, 1) = g(\mathbb{R}^d) \cap B(0, 1).$$

Moreover, $\Sigma := g(\mathbb{R}^n)$ is C^ϵ -Reifenberg flat.

This is the main result as stated, but there are special properties that are implicit in David and Toro's version that we shall employ. What we don't cite below is covered in Sections 2,3 and 4 of [DT12].

Theorem 2.5. [DT12] *For $k \in \mathbb{N} \cup \{0\}$, set $r_k = 10^{-k}$ and let $\{x_{jk}\}_{j \in J_k}$ be a collection of points so that for some d -plane P_0 ,*

$$\begin{aligned} \{x_{j0}\}_{j \in J_0} &\subset P_0, \\ |x_{ik} - x_{jk}| &\geq r_k, \end{aligned}$$

and, denoting $B_{jk} = B(x_{jk}, r_k)$,

$$x_{ik} \in V_{k-1}^2 \quad (2.5)$$

where

$$V_k^\lambda := \bigcup_{j \in J_k} \lambda B_{jk}.$$

To each point x_{jk} , associate a d -plane $P_{jk} \subset \mathbb{R}^n$ such that $P_{jk} \ni x_{jk}$ and set

$$\begin{aligned} \epsilon_k(x) &= \sup \{ d_{x, 10^4 r_l}(P_{jk}, P_{il}) : j \in J_k, |l - k| \leq 2, i \in J_l, \\ &\quad x \in 100B_{jk} \cap 100B_{il} \}. \end{aligned}$$

There is $\epsilon_0 > 0$ such that if $\epsilon \in (0, \epsilon_0)$ and

$$\epsilon_k(x_{jk}) < \epsilon \text{ for all } k \geq 0 \text{ and } j \in J_k, \quad (2.6)$$

then there is a bijection $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ so that the following hold

(1) We have

$$E_\infty := \bigcap_{K=1}^{\infty} \overline{\bigcup_{k=K}^{\infty} \{x_{jk}\}_{j \in J_k}} \subseteq \Sigma := g(\mathbb{R}^d). \quad (2.7)$$

(2) $g(z) = z$ when $\text{dist}(z, P_0) > 2$.

(3) For $x, y \in \mathbb{R}^n$,

$$\frac{1}{4}|x - y|^{1+\tau} \leq |g(x) - g(y)| \leq 10|x - y|^{1-\tau}.$$

(4) $|g(z) - z| \lesssim \epsilon$ for $z \in \mathbb{R}^n$

(5) For $x \in P_0$, $g(x) = \lim_k \sigma_k \circ \dots \circ \sigma_1(x)$ where

$$\sigma_k(y) = \psi_k(y)y + \sum_{j \in J_k} \theta_{j,k}(y)\pi_{j,k}(y). \quad (2.8)$$

Here, $\{x_{j,k}\}_{j \in L_k}$ is a maximal $\frac{r_k}{2}$ -separated set in $\mathbb{R}^n \setminus V_k^9$,

$$B_{j,k} = B(x_{j,k}, r_k/10) \text{ for } j \in L_k,$$

$\{\theta_{j,k}\}_{j \in L_k \cup J_k}$ is a partition of unity such that $\mathbb{1}_{9B_{j,k}} \leq \theta_{j,k} \leq \mathbb{1}_{10B_{j,k}}$ for all k and $j \in L_k \cup J_k$, and $\psi_k = \sum_{j \in L_k} \theta_{j,k}$.

(6) [DT12, Equation (4.5)] For $k \geq 0$,

$$\sigma_k(y) = y \text{ and } D_k(y) = I \text{ for } y \in \mathbb{R}^n \setminus V_k^{10}. \quad (2.9)$$

(7) [DT12, Proposition 5.1] Let $\Sigma_0 = P_0$ and

$$\Sigma_k = \sigma_k(\Sigma_{k-1}).$$

There is a function $A_{j,k} : P_{j,k} \cap 49B_{j,k} \rightarrow P_{j,k}^\perp$ of class C^2 such that $|A_{j,k}(x_{jk})| \lesssim \epsilon r_k$, $|DA_{j,k}| \lesssim \epsilon$ on $P_{j,k} \cap 49B_{j,k}$, and if $\Gamma_{j,k}$ is its graph over $P_{j,k}$, then

$$\Sigma_k \cap D(x_{jk}, P_{j,k}, 49r_k) = \Gamma_{j,k} \cap D(x_{jk}, P_{j,k}, 49r_k) \quad (2.10)$$

where

$$D(x, P, r) = \{z + w : z \in P \cap B(x, r), w \in P^\perp \cap B(0, r)\}. \quad (2.11)$$

(Above P^\perp is the plane perpendicular to P going through 0.) In particular,

$$d_{x_{jk}, 49r_{jk}}(\Sigma_k, P_{j,k}) \lesssim \epsilon. \quad (2.12)$$

(8) [DT12, Lemma 6.2] For $k \geq 0$ and $y \in \Sigma_k$, there is an affine d -plane P through y and a $C\epsilon$ -Lipschitz and C^2 function $A : P \rightarrow P^\perp$ so that if Γ is the graph of A over P , then

$$\Sigma_k \cap B(y, 19r_k) = \Gamma \cap B(y, 19r_k). \quad (2.13)$$

(9) [DT12, Proposition 6.3] $\Sigma = g(P_0)$ is $C\epsilon$ -Reifenberg flat in the sense that for all $z \in \Sigma$, and $t \in (0, 1)$, there is $P = P(z, t)$ so that $d_{z,t}(\Sigma, P) \lesssim \epsilon$.

(10) For all $y \in \Sigma_k$,

$$|\sigma_k(y) - y| \lesssim \epsilon_k(y)r_k \quad (2.14)$$

This is not stated as such in [DT12], but it follows from (7.13) in [DT12] and the definition of σ_k . In particular, it follows that

$$\text{dist}(y, \Sigma) \lesssim \epsilon r_k \text{ for } y \in \Sigma_k. \quad (2.15)$$

(11) [DT12, Lemma 7.2] For $k \geq 0$, $y \in \Sigma_j \cap V_k^8$, choose $i \in J_k$ such that $y \in 10B_{i,k}$. Then

$$|\sigma_k(y) - \pi_{i,k}(y)| \lesssim \epsilon_k(y)r_k \quad (2.16)$$

and

$$|D\sigma_k(y) - D\pi_{i,k}| \lesssim \epsilon_k(y) \quad (2.17)$$

If $T\Sigma_k(x)$ denotes the tangent space at $x \in \Sigma_k$, then

$$\angle(T\Sigma_{k+1}(\sigma_k(x)), P_{i,k}) \lesssim \epsilon_k(y) \text{ for } x \in \Sigma_k \cap B(x_{i,k}, 10r_k). \quad (2.18)$$

(12) [DT12, Lemma 9.1] For $x, y \in \Sigma_k$,

$$\angle(T\Sigma_k(x), T\Sigma_k(y)) \lesssim \epsilon \frac{|x - y|}{r_k}. \quad (2.19)$$

(13) [DT12, Lemma 13.2] For $x \in \Sigma$ and $r > 0$,

$$\mathcal{H}_\infty^d(B(x, r) \cap \Sigma) \geq (1 - C\epsilon)\omega_d r^d \quad (2.20)$$

where ω_d is the volume of the unit ball in \mathbb{R}^d . They prove this statement with \mathcal{H}^d in place of \mathcal{H}_∞^d , but the same proof works for \mathcal{H}_∞^d .

Lemma 2.6. *With the notation as in Theorem 2.5, there is $C > 0$ depending only on n so that for all $k \geq 0$, Σ_k is $C\epsilon$ -Reifenberg flat.*

Proof. Let $x \in \Sigma_k$ and $r > 0$. If $r \leq r_k$, then $\vartheta_{\Sigma_k}(x, r) \lesssim \epsilon$ by (2.13), so assume $r > r_k$. By (2.15), there is $x' \in \Sigma$ with $|x - x'| \lesssim \epsilon r_k$. By Theorem 2.5, Σ is $C\epsilon$ -Reifenberg flat for some $C > 0$, and so there is a d -plane P passing through x' so that $d_{x', 2r}(\Sigma, P') < C\epsilon$. Let P be the plane parallel to P' but containing x . Then it is not hard to show that, for $\epsilon > 0$ small, $d_{x', 2r}(\Sigma, P) \lesssim \epsilon$ as well.

Let $y \in B(x, r) \cap \Sigma_k$. Again, there is $y' \in \Sigma$ so that $|y - y'| \lesssim \epsilon r_k$. For $\epsilon > 0$ small enough, $y' \in B(x', 2r) \cap \Sigma$, and so $\text{dist}(y', P') \lesssim \epsilon r_k$. Since P is P' translated by no more than a constant times ϵr_k , we also have that $\text{dist}(y', P) \lesssim \epsilon r_k$. Thus,

$$\text{dist}(y, P) \leq |y - y'| + \text{dist}(y', P) \lesssim \epsilon r_k \quad (2.21)$$

and this holds for all $y \in B(x, r) \cap \Sigma_k$.

Now let $z \in B(x, r) \cap P$ and $z' = \pi_{P'}(z)$. Then $|z - z'| \lesssim \epsilon r_k$, and so for $\epsilon > 0$ small enough, $z' \in B(x', 2r) \cap P'$. Thus, there is $z'' \in \Sigma$ with $|z' - z''| \lesssim \epsilon r_k$. Recall that g is a homeomorphism, which thus forces every σ_j to be a homeomorphism (since g is the bi-Hölder composition of them all). Hence g_k has an inverse, and by (2.14), there is $z''' \in \Sigma_k$ with $|z''' - z''| \lesssim \epsilon r_k$. Combining these estimates, we have

$$\text{dist}(z, \Sigma_k) \leq |z - z''| \leq |z - z'| + |z' - z''| + |z'' - z''| \lesssim \epsilon r_k$$

and this holds for all $z \in B(x, r) \cap P$. Combining this with (2.21) gives $\vartheta_{\Sigma_k}(B(x, r)) \lesssim \epsilon$, which proves the lemma. \square

The following lemma will allow us to localize by finding a Reifenberg flat surface Σ that agrees with a Reifenberg flat surface E in a ball B but whose surface measure is controlled by the measure of E in B .

Lemma 2.7. *Let $E \subseteq \mathbb{R}^n$ be ϵ -Reifenberg flat containing 0. Then there is a $C\epsilon$ -Reifenberg flat surface Σ so that*

- (1) $E \cap \overline{B(0, 1)} \subseteq \Sigma \cap \overline{B(0, 1)}$,
- (2) $\Sigma \setminus B(0, 10) = P_0 \setminus B(0, 10)$ for some d -plane P_0 ,
- (3) for all $A > 0$,

$$\mathcal{H}^d(\Sigma \cap B(0, A)) \lesssim A^d \mathcal{H}^d(E \cap B(0, 1)). \quad (2.22)$$

Proof. Let

$$E_k = \{x \in E : \text{dist}(x, E \setminus B(0, 1)) \geq r_k\}$$

and $\{x_{jk}\}_{j \in J_k}$ be maximally separated r_k -nets in E_k so that $\{x_{j0}\}_{j \in J_0} = \{0\}$. Then, with the notation of Theorem 2.5,

$$\{x_{j,k+1}\}_{j \in J_{k+1}} \subseteq E \cap \overline{B(0, 1)} \subseteq V_k^2 \quad \text{for all } k \geq 0.$$

In this way, $E_\infty = E \cap \overline{B(0, 1)}$. If we let $P_{j,k}$ be the plane that infimizes $\vartheta_E(B(x_{jk}, 10^6 r_k))$ and P_0 the plane that infimizes $\vartheta_E(B(0, 10^6 r_k))$, then for $\epsilon > 0$ small enough, we may apply Theorem 2.5 to obtain a $C\epsilon$ -Reifenberg flat surface Σ .

Let $x \in \Sigma \cap B(0, 10) \setminus E_\infty$ and let $k(x)$ be the maximal k for which $x \in V_{k-1}^{11}$. Then $x \notin V_{k(x)}^{11}$, and so if we set $r(x) = r_{k(x)}$, we have $B(x, r(x)) \subseteq V_{k(x)}^{10}$. Thus,

$$\Sigma_k \cap B(x, r(x)) = \Sigma \cap B(x, r(x)).$$

By (2.13), $\Sigma \cap B(x, r(x))$ is a $C\epsilon$ -Lipschitz graph, so this and the above equation imply

$$\mathcal{H}^d(\Sigma \cap B(x, r(x))) \lesssim r^d. \quad (2.23)$$

By the Besicovitch covering theorem, we can find $x_i \in \Sigma \cap B(0, 10) \setminus E_\infty$ so that $B_i := B(x_i, r(x_i))$ cover $\Sigma \cap B(0, 10) \setminus E_\infty$ with bounded overlap. Since $x_i \in V_{k(x_i)-1}^{11}$, we may find $x'_i \in \{x_{j,k-1}\}_{j \in J_{k-1}}$ so that

$$x_i \in B(x'_i, 10r_{k(x_i)-1}) = B(x'_i, r_{k(x_i)}).$$

Since $x'_i \in \{x_{j,k-1}\}_{j \in J_{k-1}} \subseteq E_{k-1}$, we know

$$B(x'_i, r_{k(x_i)-1}) \cap E \subseteq B(0, 1). \quad (2.24)$$

By (2.20), we have

$$r(x_i)^d \lesssim \mathcal{H}^d(B(x'_i, r_{k(x_i)-1}) \cap E_\infty). \quad (2.25)$$

For $j \in J_k$, let

$$A_j = \{B_i : x'_i = x_{jk}\}.$$

Since the balls in A_j have bounded overlap, the same radius, and distance eat most r_k from x_{jk} , we know $\#A_j \lesssim_n 1$. Thus,

$$\begin{aligned}
\mathcal{H}^d(\Sigma \cap B(0, 10) \setminus E_\infty) &\leq \sum_i \mathcal{H}^d(B_i \cap \Sigma) \stackrel{(2.23)}{\lesssim} \sum_i r(x_i)^d \\
&\stackrel{(2.25)}{\lesssim} \sum_i \mathcal{H}^d(B(x'_i, r_{k(x_i)-1}) \cap E) \\
&\leq \sum_{k \geq 0} \sum_{j \in J_k} \#A_j \mathcal{H}^d(B(x_{jk}, r_{k-1}) \cap E) \\
&\lesssim \sum_{k \geq 0} \sum_{j \in J_k} \mathcal{H}^d(B(x_{jk}, r_{k-1}) \cap E) \\
&\stackrel{(2.24)}{\leq} \mathcal{H}^d(E \cap B(0, 1))
\end{aligned}$$

Recalling that $E_\infty = E \cap \overline{B(0, 1)}$,

$$\begin{aligned}
\mathcal{H}^d(\Sigma \cap B(0, 10)) &\leq \mathcal{H}^d(\Sigma \cap B(0, 10) \setminus E_\infty) + \mathcal{H}^d(E_\infty) \\
&\lesssim \mathcal{H}^d(E \cap \overline{B(0, 1)}).
\end{aligned}$$

Now note that $V_k^{10} \subseteq V_0^{10}$ for all k , and so by (2.8),

$$\begin{aligned}
\mathcal{H}^d(\Sigma \cap B(0, A) \setminus B(0, 10)) &= \mathcal{H}^d(P_0 \cap B(0, A) \setminus B(0, 10)) \\
&\lesssim A^d \stackrel{(2.20)}{\lesssim} A^d \mathcal{H}^d(E \cap B(0, 1)).
\end{aligned}$$

Combining these estimates completes the proof. \square

2.3. Generalized dyadic cubes. We recall the construction of cubes on a metric space, originally due to by David and Christ ([Dav88], [Chr90]), but the current formulation is from Hytönen and Martikainen [HM12]. A metric space X is *doubling* if there is N so that any ball can be covered by at most N balls of half the radius. In practice, the metric space X in the theorem will be a subset of Euclidean space and thus doubling.

Definition 2.8. We say that a set X is δ -*separated* or a δ -*net* if for all $x, y \in X$ we have $|x - y| \geq \delta$.

Theorem 2.9. *Let X be a doubling metric space. Let X_k be a nested sequence of maximal ρ^k -nets for X where $\rho < 1/1000$ and let $c_0 = 1/500$. For each $n \in \mathbb{Z}$ there is a collection \mathcal{D}_k of “cubes,” which are Borel subsets of X such that the following hold.*

- (1) For every integer k , $X = \bigcup_{Q \in \mathcal{D}_k} Q$.
- (2) If $Q, Q' \in \mathcal{D} = \bigcup \mathcal{D}_k$ and $Q \cap Q' \neq \emptyset$, then $Q \subseteq Q'$ or $Q' \subseteq Q$.
- (3) For $Q \in \mathcal{D}$, let $k(Q)$ be the unique integer so that $Q \in \mathcal{D}_k$ and set $\ell(Q) = 5\rho^{k(Q)}$. Then there is $\zeta_Q \in X_k$ so that

$$B_X(\zeta_Q, c_0 \ell(Q)) \subseteq Q \subseteq B_X(\zeta_Q, \ell(Q)) \quad (2.26)$$

and

$$X_k = \{\zeta_Q : Q \in \mathcal{D}_k\}.$$

From now on we will let \mathcal{D} denote the cubes from Theorem 2.9 for E and will write

$$B_Q = B(\zeta_Q, \ell(Q)).$$

Fix $\rho = 10^{-4}$, which we want to be a power of 10 for Section 10.

Lemma 2.10. *Let $Q, R \in \mathcal{D}$.*

$$\text{If } Q \subseteq R, \text{ then } CB_Q \subseteq CB_R \text{ for all } C > 1000/999. \quad (2.27)$$

Proof. We can assume $Q \neq R$, so $\ell(Q) < \ell(R)$. In particular, $\ell(Q) \leq \rho \ell(R)$. Thus, for $x \in CB_Q$, so long as $C\rho + 1 < C$ (which happens if $C \geq \frac{1}{1-\rho} > 1000/999$ by our choice of ρ in Theorem 2.9), we have

$$|x - x_R| \leq |x - x_Q| + |x_Q - x_R| < C\ell(Q) + \ell(R) \leq (C\rho + 1)\ell(R) \leq C\ell(R). \quad \square$$

Let $j \geq 0$. For $Q \in \mathcal{D}_{k+j}$, we will denote by $Q^{(k)}$ the (unique) cube in \mathcal{D}_j containing Q . For $R \in \mathcal{D}_{k+j}$ we denote by $\text{Child}_k(R)$ the collection of cubes $Q \in \mathcal{D}_j$ such that $Q \subset R$. We will call $Q^{(1)}$ the the *parent* of Q . If $Q' \in \text{Child}_1(Q^{(1)})$ and $Q' \neq Q$, then we will call Q' a *sibling* of Q .

2.4. Preliminaries with $\beta_E^{d,p}$. Again, we note that the Appendix (Section 13) contains preliminaries regarding the Choquet integral with respect to Hausdorff content, which is what is used for the definition of β .

For a ball B centered on E , we will denote by P_B^p the d -plane for which

$$\beta_E^{d,p}(B, P_B^p) = \beta_E^{d,p}(B).$$

Lemma 2.11. *Assume $0 < p < \infty$, $E \subseteq \mathbb{R}^n$, and B is centered on E . Then*

$$\beta_E^{d,p}(B) \leq \frac{2^{d/p}}{p^{1/p}} \beta_{E,\infty}^d(B). \quad (2.28)$$

In particular, for $p \geq 1$, we have

$$\beta_E^{d,p}(B) \leq 2^d \beta_{E,\infty}^d(B). \quad (2.29)$$

Proof. Let P be the minimal d -plane for $\beta_{E,\infty}^d(B)$. Since for any set A we have $\mathcal{H}_\infty^d(A) \leq (\text{diam } A)^d$, we know $\mathcal{H}_\infty^d(E \cap B) \leq (2r_B)^d$, and so

$$\begin{aligned} \beta_E^{d,p}(B)^p &\leq \frac{1}{r_B^d} \int_0^\infty \mathcal{H}_\infty^d\{x \in B \cap E : \text{dist}(x, P) > tr_B\} t^{p-1} dt \\ &\leq \frac{1}{r_B^d} \int_0^{\beta_{E,\infty}^d(B)} \mathcal{H}_\infty^d\{x \in B \cap E : \text{dist}(x, P) > tr_B\} t^{p-1} dt \\ &\leq \frac{\mathcal{H}_\infty^d(B \cap E)}{r_B^d} \int_0^{\beta_{E,\infty}^d(B)} t^{p-1} dt \leq \frac{2^d}{p} \beta_{E,\infty}^d(B)^p. \end{aligned}$$

□

Lemma 2.12. *Assume $E \subseteq \mathbb{R}^n$ and there is B centered on E so that for all $B' \subseteq B$ centered on E we have $\mathcal{H}_\infty^d(E \cap B') \geq cr_{B'}^d$. Then*

$$\beta_{E,\infty}^d \left(\frac{1}{2}B \right) \leq 2c^{-\frac{1}{d+1}} \beta_E^{d,1}(B)^{\frac{1}{d+1}} \quad (2.30)$$

Proof. We can assume E is closed and that $r_B = 1$. Let $P = P_B^1$ and let $y \in \frac{1}{2}B \cap E$ be farthest from P . Set $\tau = \text{dist}(y, P)$ and note

$$B(y, \tau/2) \subseteq \{x \in B \cap E : \text{dist}(x, P) > t\} \text{ for } t \leq \tau/2.$$

Thus,

$$\begin{aligned} 1 &\leq \frac{1}{c(\tau/2)^d} \mathcal{H}_\infty^d(B(y, \tau/2)) \\ &\leq \frac{2^d}{c\tau^d} \int_0^{\tau/2} \mathcal{H}_\infty^d(\{x \in B \cap E : \text{dist}(x, P) > t\}) dt \leq \frac{2^{d+1}}{c\tau^{d+1}} \beta_E^{d,1}(B) \end{aligned}$$

Hence,

$$\beta_{E,\infty}^d \left(\frac{1}{2}B \right) \leq 2\tau \leq 2c^{-\frac{1}{d+1}} \beta_E^{d,1}(B)^{\frac{1}{d+1}}.$$

□

Lemma 2.13. *Let $1 \leq p < \infty$, $E \subseteq \mathbb{R}^n$ be a closed set and B a ball centered on E so that $\mathcal{H}_\infty^d(E \cap B) > 0$. Then*

$$\beta_E^{d,1}(B) \lesssim_n \beta_E^{d,p}(B) \quad (2.31)$$

Proof. We can assume without loss of generality that $r_B = 1$. Let $P = P_B^p$ and $f(x) = \text{dist}(x, P)$. Let

$$E_j = E \cap \overline{(1 - j^{-1})B}.$$

Then E is compact, and so we may apply Lemma 2.3 to these sets. By Frostmann's Lemma [Mat95, Theorem 8.8], for each $t > 0$ there is a measure μ_t with

$$\text{supp } \mu_t \subseteq \{x \in E \cap B : f(x) > t\}$$

so that

$$\mu_t(B(x, r)) \leq r^d \text{ for all } x \in \mathbb{R}^n \text{ and } r > 0$$

and

$$\mu_t(\{x \in E : f(x) > t\}) \gtrsim_n \mathcal{H}_\infty^d(\{x \in E : f(x) > t\}).$$

For each $t > 0$,

$$\begin{aligned} \lim_j \mathcal{H}_\infty^d(\{x \in E_j \cap B : f(x) > t\}) &\geq \liminf_j \mu_t(\{x \in E_j : f(x) > t\}) \\ &= \mu_t(\{x \in E : f(x) > t\}) \gtrsim_n \mathcal{H}_\infty^d(\{x \in E : f(x) > t\}) \end{aligned}$$

where the first limit converges because it is monotone (since the E_j are nested). Hence, by the Monotone Convergence Theorem,

$$\begin{aligned}
\beta_E^{d,1}(B) &\leq \int_{E \cap B} f d\mathcal{H}_\infty^d = \int_0^\infty \mathcal{H}_\infty^d(\{x \in E : f(x) > t\}) dt \\
&\lesssim_n \lim_j \int_0^\infty \mathcal{H}_\infty^d(\{x \in E_j : f(x) > t\}) dt \\
&= \lim_j \int_{E_j} f d\mathcal{H}_\infty^d \\
&\stackrel{(2.4)}{\lesssim_n} \lim_j \mathcal{H}_\infty^d(E_j)^{1-\frac{1}{p}} \left(\int_{E_j} f^p d\mathcal{H}_\infty^d \right)^{\frac{1}{p}} \\
&\leq \mathcal{H}_\infty^d(E \cap B)^{1-\frac{1}{p}} \left(\int_{E \cap B} f^p d\mathcal{H}_\infty^d \right)^{\frac{1}{p}} \\
&\lesssim_d 1 \cdot \beta_E^{d,p}(B).
\end{aligned}$$

and this implies (2.31). \square

Lemma 2.14. *Let $1 \leq p < \infty$ and let $E \subseteq \mathbb{R}^n$. Then for balls $B' \subseteq B$ centered on E ,*

$$\beta_E^{d,p}(B') \leq \left(\frac{r_B}{r_{B'}} \right)^{d+p} \beta_E^{d,p}(B). \quad (2.32)$$

Proof. Let $P = P_B^d$. Using a change of variables, we get

$$\begin{aligned}
\beta_E^{d,p}(B')^p r_{B'}^d &\leq \int_0^\infty \mathcal{H}_\infty^d\{x \in B' \cap E : \text{dist}(x, P) > tr_{B'}\} t^{p-1} dt \\
&= \left(\frac{r_B}{r_{B'}} \right)^p \int_0^\infty \mathcal{H}_\infty^d\{x \in B' \cap E : \text{dist}(x, P) > tr_B\} t^{p-1} dt \\
&\leq \left(\frac{r_B}{r_{B'}} \right)^p \int_0^\infty \mathcal{H}_\infty^d\{x \in B \cap E : \text{dist}(x, P) > tr_B\} t^{p-1} dt \\
&\leq \left(\frac{r_B}{r_{B'}} \right)^p r_B^d \beta_E^{d,p}(B)^p.
\end{aligned}$$

\square

For two planes P, P' containing the origin, we define

$$\angle(P, P') = d_{B(0,1)}(P, P').$$

For general affine planes P, P' , let $x \in P$ and $y \in P'$. We set

$$\angle(P, P') = \angle(P - x, P' - y).$$

If P_1 and P_2 are both d -planes containing x , we clearly have

$$d_{x,ar}(P_1, P_2) = d_{x,r}(P_1, P_2) \text{ for all } a, r > 0 \quad (2.33)$$

and it is also not hard to show that if P_1, P_2 , and P_3 are all d -planes containing x ,

$$d_{x,r}(P_1, P_3) \leq d_{x,r}(P_1, P_2) + d_{x,r}(P_2, P_3). \quad (2.34)$$

Indeed, if $y \in P_1 \cap B(x, r)$, $y' = \pi_{P_2}(y)$, and $y'' = \pi_{P_3}(y')$, then $y, y', y'' \in B(x, r)$ and

$$\begin{aligned} r \operatorname{dist}(y, P_3) &\leq |y - y''| \leq |y - y'| + |y' - y''| = \operatorname{dist}(y, P_2) + \operatorname{dist}(y', P_3) \\ &\leq rd_{x,r}(P_1, P_2) + rd_{x,r}(P_2, P_3) \end{aligned}$$

and supremizing over all $y \in P_1 \cap B(x, r)$ gives (2.34).

Lemma 2.15. *Suppose $d_{z,r}(V_1, V_2) < \epsilon$ are d -planes. Then*

$$|\pi_{V_1^\perp}(x - y)| \leq \epsilon |x - y| \quad \text{for } x, y \in V_2. \quad (2.35)$$

Proof. Let $x, y \in V_2$. By subtracting y from V_1 and V_2 , we may assume without loss of generality that $y = 0$, so V_2 contains the origin. Recall that V_1^\perp is the $(n - d)$ -plane orthogonal to V_1 and containing the origin. Let V_1' be the translate of V_1 that contains the origin. Then

$$\begin{aligned} |\pi_{V_1^\perp}(x)| &= |x - \pi_{V_1'}(x)| \leq \operatorname{dist}(x, V_1') \leq |x| d_{B(0,|x|)}(V_2, V_1') \\ &\stackrel{(2.33)}{=} |x| d_{B(0,1)}(V_2, V_1') = \angle(V_1, V_2) < \epsilon |x|. \end{aligned}$$

□

Lemma 2.16. *Suppose $E \subseteq \mathbb{R}^n$ and there is B centered on E so that for all $B' \subseteq B$ centered on E we have $\mathcal{H}^d|_\infty(B') \geq cr_B^d$. Let P and P' be two d -planes. Then*

$$d_{B'}(P, P') \lesssim_{d,c} \left(\frac{r_B}{r_{B'}} \right)^{d+1} \beta_E^{d,1}(B, P) + \beta_E^{d,1}(B', P') \quad (2.36)$$

and in particular

$$\angle(P, P') \lesssim_{d,c} \left(\frac{r_B}{r_{B'}} \right)^{d+1} \beta_E^{d,1}(B, P) + \beta_E^{d,1}(B', P'). \quad (2.37)$$

In particular, it follows that

$$d_{B'}(P_{B'}^1, P_B^1) \lesssim_{d,c} \left(\frac{r_B}{r_{B'}} \right)^{d+1} \beta_E^{d,1}(B) \quad (2.38)$$

and

$$\angle(P_{B'}^1, P_B^1) \lesssim_{d,c} \left(\frac{r_B}{r_{B'}} \right)^{d+1} \beta_E^{d,1}(B). \quad (2.39)$$

The proof of Lemma 2.16 will use the following lemma.

Lemma 2.17. [AT15, Lemma 6.4] *Suppose P_1 and P_2 are d -planes in \mathbb{R}^n and $X = \{x_0, \dots, x_d\}$ are points so that*

(a) $\eta \in (0, 1)$ where

$$\eta = \eta(X) = \min\{\operatorname{dist}(x_i, \operatorname{span} X \setminus \{x_i\}) / \operatorname{diam} X \in (0, 1)\}$$

and

(b) $\operatorname{dist}(x_i, P_j) < \epsilon \operatorname{diam} X$ for $i = 0, \dots, d$ and $j = 1, 2$, where $\epsilon < \eta d^{-1}/2$.

Then

$$\text{dist}(y, P_1) \leq \epsilon \left(\frac{2d}{\eta} \text{dist}(y, X) + \text{diam } X \right) \quad \text{for all } y \in P_2. \quad (2.40)$$

Proof of Lemma 2.16. Let $X = \{x_0, \dots, x_d\}$ be vectors in $\frac{1}{2}B' \cap E$ such that

$$\text{dist}(x_{i+1}, \text{span}\{x_0, \dots, x_i\}) \gtrsim r_{B'} \text{ for } i = 1, \dots, d.$$

These can be found by induction using the fact that $\mathcal{H}_\infty^d(E \cap B') \geq cr_{B'}^d$ (see for example [DS91, Section 5]). Then $\eta(X) \sim 1$. Let $B_i = B(x_i, r_{B'}/100)$ and for $t > 0$, set

$$E_{t,i} = \{x \in E \cap B_i : \text{dist}(x, P') > tr_{B'} \text{ or } \text{dist}(x, P) > tr_{B'}\}.$$

Let $T > 0$. Suppose $E_{t,i} = B_i \cap E$ for all $t \leq T$. Observe that

$$\mathcal{H}_\infty^d(B_i \cap E) \geq cr_{B_i}^d = \frac{c}{100^d} r_{B'}^d.$$

Hence,

$$\begin{aligned} T &\leq \mathcal{H}_\infty^d(B_i \cap E)^{-1} \int_0^T \mathcal{H}_\infty^d(E_t) dt \lesssim_c r_{B_i}^{-d} \int_0^T \mathcal{H}_\infty^d(E_t) dt \\ &\lesssim r_{B'}^{-d} \int_0^T \mathcal{H}_\infty^d\{x \in E \cap B_i : \text{dist}(x, P') > tr_{B'}/2\} dt \\ &\quad + r_{B'}^{-d} \int_0^T \mathcal{H}_\infty^d\{x \in E \cap B_i : \text{dist}(x, P) > tr_{B'}/2\} dt \\ &\leq \frac{2}{r_{B'}^d} \int_0^\infty \mathcal{H}_\infty^d\{x \in E \cap B' : \text{dist}(x, P') > tr_{B'}\} dt \\ &\quad + \frac{2r_B^{d+1}}{r_{B'}^{d+1}} \frac{1}{r_B^d} \int_0^\infty \mathcal{H}_\infty^d\{x \in E \cap B : \text{dist}(x, P) > tr_B\} dt \\ &\lesssim \beta_E^{d,1}(B', P') + \left(\frac{r_B}{r_{B'}} \right)^{d+1} \beta_E^{d,1}(B, P) =: \lambda. \end{aligned}$$

Thus, there is $c' = c'(d, c)$ so that $T \leq c'\lambda$. Hence, for $t = 2c'\lambda$, $(B_i \cap E) \setminus E_{t,i} \neq \emptyset$, and so there are points $y_i \in B_i \setminus E_{c't,i}$. Since $y_i \in B_i$ and $\eta(X) \sim 1$, it is not hard to show that $\eta(\{y_0, \dots, y_d\}) \sim 1$ as well. By the definition of $E_{t,i}$, the lemma follows from Lemma 2.17. \square

Lemma 2.18. *Let $M > 1$, E be a Borel set so that $\mathcal{H}_\infty^d(E \cap B) \geq cr_B^d$ for all balls B centered on E with $0 < r < \text{diam } E$. Let \mathcal{D} be the cubes from Theorem 2.9 for E (or $E \cap B(0, 1)$), and $Q_0 \in \mathcal{D}$. For $Q \in \mathcal{D}$, let $P_Q = P_{MB_Q}^1$. Suppose that for all balls $B \subseteq B_{Q_0}$ centered on E that $\mathcal{H}_\infty^d(B \cap E) \geq cr^d$. Let $Q, R \in \mathcal{D}$, $Q, R \subseteq Q_0$, and suppose that for all cubes $S \subseteq Q_0$ so that S contains either Q or R that $\beta_E^{d,1}(MB_Q) < \epsilon$. Then for $\Lambda > 0$, if $\text{dist}(Q, R) \leq \Lambda \max\{\ell(Q), \ell(R)\} \leq \Lambda^2 \min\{\ell(Q), \ell(R)\}$, then*

$$\angle(P_Q, P_R) \lesssim_{M, \Lambda} \epsilon. \quad (2.41)$$

Proof. Suppose $\ell(Q) \leq \ell(R)$. Let Q' be the smallest ancestor of Q so that $3B_R \subseteq 3B_{Q'}$ (note that since both Q and R are contained in Q_0 and hence $3B_R \subseteq 3B_{Q_0}$ by (2.27), this is well defined). Then

$$\ell(R) \sim \ell(Q') \sim \text{dist}(Q, R) + \ell(Q) + \ell(R) \lesssim \Lambda^2 \ell(Q) \quad (2.42)$$

and so by Lemma 2.16,

$$\angle(P_Q, P_{3B_{Q'}}) \stackrel{(2.34)}{\leq} \angle(P_Q, P_{3B_Q}) + \angle(P_{3B_Q}, P_{3B_{Q'}}) \lesssim_{\Lambda, M} \epsilon.$$

Let R' be the largest ancestor of R so that $3B_{R'} \subseteq 6B_{Q'}$. Again,

$$\ell(R') \sim \ell(Q') \sim \ell(R)$$

and so $\ell(R') \lesssim \ell(Q)$ by (2.42). Hence, applying Lemma 2.16

$$\begin{aligned} \angle(P_R, P_{3B_{Q'}}) &\stackrel{(2.34)}{\leq} \angle(P_R, P_{3B_R}) + \angle(P_{3B_R}, P_{3B_{R'}}) + \angle(P_{3B_{R'}}, P_{6B_{Q'}}) \\ &\quad + \angle(P_{6B_{Q'}}, P_{3B_{Q'}}) \lesssim_{\Lambda, M} \epsilon. \end{aligned}$$

Combining these two chains of inequalities proves the lemma. \square

Lemma 2.19. *Suppose E is ϵ -Reifenberg flat, $0 < \epsilon < \epsilon_0$, B is a ball centered on E , and P is some d -plane. For ϵ_0 small enough, if $\beta_{E, \infty}^d(B, P) < 1/100$, then*

$$\pi_P(B \cap E) \supseteq B(\pi_P(x_B), r_B/2). \quad (2.43)$$

Proof. Without loss of generality, we will assume $B = \mathbb{B}$ and $P = \mathbb{R}^d + \pi_P(0)$. Then

$$\beta_{E, \infty}^d(\mathbb{B}, \mathbb{R}^d) \leq 2\beta_{E, \infty}^d(\mathbb{B}, P) < \frac{1}{50}.$$

Let $g : \mathbb{R}^n \hookrightarrow \mathbb{R}^d$ be the map from Theorem 2.5 with $P_0 = \mathbb{R}^d$, π the orthogonal projection onto \mathbb{R}^d , and $h = \pi \circ g$. Then $E \cap \mathbb{B} = g(\mathbb{R}^d) \cap \mathbb{B}$. For $x \in \frac{3}{4}\mathbb{B}$, $|g(x) - x| \leq C\epsilon$ for some $C = C(d)$ by Theorem 2.5 (4), and so $g(x) \in (\frac{3}{4} + C\epsilon)\mathbb{B} \cap E \subseteq \mathbb{B} \cap E$. Thus, for $x \in \mathbb{R}^d \cap \frac{3}{4}\mathbb{B}$, $\text{dist}(g(x), \mathbb{R}^d) \leq \beta_{E, \infty}^d(\mathbb{B}, \mathbb{R}^d)$. Combining these estimates, we get that for $x \in \frac{3}{4}\mathbb{B}_d$ and $\epsilon_0 > 0$ small enough,

$$|h(x) - x| \leq |\pi(g(x)) - g(x)| + |g(x) - x| \leq \frac{1}{50} + C\epsilon < \frac{1}{10}.$$

If there is $x_0 \in \frac{1}{2}\mathbb{B}_d \setminus h(\frac{3}{4}\mathbb{B}_d)$, then it is not hard to show that $h|_{\frac{3}{4}\mathbb{B}_d}$ is contractible in $\mathbb{R}^d \cap \frac{3}{4}\mathbb{B} \setminus \{x_0\}$ (since it is homotopic to a map of $\frac{3}{4}\mathbb{B}_d$ into $\frac{3}{4}\mathbb{S}^{d-1}$ and maps of the disk into the sphere are always contractible). Thus $h|_{\frac{3}{4}\mathbb{S}^{d-1}}$ is contractible, but $h(x, t) = (th(x) + (1-t)x)/|th(x) + (1-t)x|$ for $x \in \frac{3}{4}\mathbb{S}_d$ is homotopic in $\mathbb{R}^d \cap \frac{3}{4}\mathbb{B} \setminus \{x_0\}$ to the identity map on $\frac{3}{4}\mathbb{S}^d$, which is not contractible in $\mathbb{R}^d \cap \frac{3}{4}\mathbb{B} \setminus \{x_0\}$, and we get a contradiction. We have thus shown $\frac{1}{2}\mathbb{B}_d \subseteq h(\frac{3}{4}\mathbb{B}_d) \subseteq \pi(E \cap \mathbb{B})$. It then follows that

$$\pi_P(E \cap \mathbb{B}) = \pi_P(\pi_{\mathbb{R}^d}(E \cap \mathbb{B})) \supseteq \pi_P(\frac{1}{2}\mathbb{B}_d) = B(\pi_P(0), 1/2) \cap P. \quad (2.44)$$

\square

Lemma 2.20. *Suppose E is ϵ -Reifenberg flat, $0 < \epsilon < \epsilon_0$, B is a ball centered on E , and P is some d -plane. There is $\beta_0 > 0$ small so that if $\beta_{E,\infty}^d(B, P) < \beta_0$, then*

$$\begin{aligned} 2^{-d}\beta(\tfrac{1}{2}B, P) &\leq \beta_{E,\infty}^d(\tfrac{1}{2}B, P) \leq d_{\tfrac{1}{2}B}(E, P) \leq 2\beta_{E,\infty}^d(B, P) \\ &\leq \frac{4}{(1-C\epsilon)\omega_d} \beta_E^{d,1}(2B, P)^{\frac{1}{d+1}}. \end{aligned} \quad (2.45)$$

In particular,

$$\begin{aligned} 2^{-d}\beta(\tfrac{1}{2}B) &\leq \beta_{E,\infty}^d(\tfrac{1}{2}B) \leq \vartheta_E^d(\tfrac{1}{2}B) \leq 2\beta_{E,\infty}^d(B) \\ &\leq \frac{4}{(1-C\epsilon)\omega_d} \beta_E^{d,1}(2B)^{\frac{1}{d+1}}. \end{aligned} \quad (2.46)$$

Proof. By the previous lemma, (2.43) holds. In particular, any $x \in B(\pi_P(0), r_B/2) \cap P$ is the image of a point $y \in B \cap E$ under π_P , and thus $\text{dist}(x, E) \leq \beta_{E,\infty}^d(B, P)$. By definition, any $x \in E \cap \frac{1}{2}\mathbb{B}$ is at most $\beta_{E,\infty}^d(B, P)$ from P . We have thus shown that $d_{\frac{1}{2}\mathbb{B}}(E, P) \leq 2\beta_{E,\infty}^d(B, P)$. The other inequalities in (2.45) follow from Lemmas 2.11, 2.12, and the definitions. \square

The following lemma says that the $\beta_E^{d,p}$ -numbers for two sets are approximately the same, with error depending on the average distance between one and the other, where "average" is taken with respect to a Choquet integral.

Lemma 2.21. *Let $1 \leq p < \infty$ and $E_1, E_2 \subseteq \mathbb{R}^n$. Suppose B^1 is a ball centered on E_1 and B^2 is a ball of same radius but centered in E_2 so that $B^1 \subseteq 2B^2$. Suppose that for $i = 1, 2$ and all balls $B \subseteq 2B^i$ centered on E_i we have $\mathcal{H}^d|_{\infty}(B \cap E_i) \geq cr_B^d$ for some $c > 0$. Then*

$$\beta_{E_1}^{d,p}(B^1, P) \lesssim_{c,p} \beta_{E_2}^{d,p}(2B^2, P) + \left(\frac{1}{r_{B^1}^d} \int_{E_1 \cap 2B^1} \left(\frac{\text{dist}(y, E_2)}{r_{B^1}} \right)^p d\mathcal{H}_{\infty}^d(y) \right)^{\frac{1}{p}} \quad (2.47)$$

We will first need an intermediary lemma.

Lemma 2.22. *Let $F_1, F_2 \subseteq \mathbb{R}^n$. Suppose B_0 is a ball centered on F_2 and for all balls $B \subseteq 2B_0$ centered on F_2 we have $\mathcal{H}^d|_{\infty}(B \cap F_2) \geq cr_B^d$ for some $c > 0$. Finally, let $\alpha \in (0, 1)$, $\{B_j\}_{j \in \mathcal{X}}$ be a collection of disjoint balls with centers $z_j \in F_1$ such that $\text{dist}(z_j, F_2) < \alpha r_{B_j}$, f a non-negative function on $\{z_j\}$, $z'_j \in F_2 \cap B(z_j, \alpha r_{B_j})$, $B'_j = B(z'_j, r_{B_j})$. Then*

$$\begin{aligned} \int_{F_1} \sum_{j \in \mathcal{X}} \mathbb{1}_{B_j}(y) f(z_j) d\mathcal{H}_{\infty}^d(y) \\ \lesssim_{\alpha} \int_{F_2} \sum_{j \in \mathcal{X}} \mathbb{1}_{(1-\alpha)B'_j}(y) f(z_j) d\mathcal{H}_{\infty}^d(y). \end{aligned} \quad (2.48)$$

The proof of Lemma 2.22 may be found in the Appendix (Section 13).

Proof of Lemma 2.21. Without loss of generality, we may assume $B^1 = \mathbb{B}$, the unit ball, so $r_{B^1} = 1$. Let $\delta > 0$ be small and $\delta(z) = \text{dist}(z, E_2) + \delta$. We may assume without loss of generality that $\delta(z) < 1/16$ for all $z \in E_1 \cap \mathbb{B}$, for otherwise the inequalities are trivial by lower regularity. By the Besicovitch covering theorem, there are collections of points $\mathcal{X}_1, \dots, \mathcal{X}_N \subseteq E_1 \cap \mathbb{B}$ with $N = N(n)$ so that if

$$B_z = B(z, 4\delta(z)),$$

then the balls $\{B_z : z \in \mathcal{X}_k\}$ are pairwise disjoint and

$$\mathbb{1}_{E_1 \cap \mathbb{B}} \leq \sum_{k=1}^N \sum_{z \in \mathcal{X}_k} \mathbb{1}_{B_z} \lesssim_n 1. \quad (2.49)$$

Note that since $\delta(z) < 1/16$, $B_z \subseteq 2\mathbb{B}$ for all $z \in E_1 \cap \mathbb{B}$. Since $d(\cdot, P)$ is 1-Lipschitz, we have that for $y \in B_z$,

$$\text{dist}(y, P)^p \leq (\text{dist}(z, P) + 4\delta(z))^p \lesssim_p \text{dist}(z, P)^p + \delta(z)^p.$$

Hence,

$$\begin{aligned} \beta_{E_1}^{d,p}(\mathbb{B}, P)^p &= \int_{E_1 \cap \mathbb{B}} \text{dist}(y, P)^p d\mathcal{H}_\infty^d(y) \\ &\stackrel{(2.2)}{\lesssim} \sum_{k=1}^N \int_{E_1} \sum_{z \in \mathcal{X}_k} \mathbb{1}_{B_z}(y) \text{dist}(y, P)^p d\mathcal{H}_\infty^d(y) \\ &\stackrel{(2.2)}{\lesssim} \sum_{k=1}^N \int_{E_1} \sum_{z \in \mathcal{X}_k} \mathbb{1}_{B_z}(y) \text{dist}(z, P)^p d\mathcal{H}_\infty^d(y) \\ &\quad + \sum_{k=1}^N \int_{E_1} \sum_{z \in \mathcal{X}_k} \mathbb{1}_{B_z}(y) \delta(z)^p d\mathcal{H}_\infty^d(y) \\ &= I_1 + I_2. \end{aligned} \quad (2.50)$$

By Lemma 2.22 with $F_1 = F_2 = E_1$, $f(z) = \delta(z)^p$, and $\alpha = \frac{7}{8}$, we have that

$$I_2 \lesssim \sum_{k=1}^N \int_{E_1} \sum_{z \in \mathcal{X}_k} \mathbb{1}_{\frac{1}{8}B_z}(y) \delta(z)^p d\mathcal{H}_\infty^d(y).$$

Since δ is 1-Lipschitz, we have for each z and $y \in E_1 \cap \frac{1}{8}B_z$ where $\frac{1}{8}B_z = B(z, \delta(z)/2)$,

$$\delta(z) \leq \delta(y) + |z - y| < \delta(y) + \frac{\delta(z)}{2}.$$

Hence $\delta(z) < 2\delta(y)$, and so for $1 \leq k \leq N$,

$$\int_{E_1} \sum_{z \in \mathcal{X}_k} \mathbb{1}_{\frac{1}{8}B_z}(y) \delta(z)^p d\mathcal{H}_\infty^d(y) \leq 2^p \int_{E_1} \sum_{z \in \mathcal{X}_k} \mathbb{1}_{\frac{1}{8}B_z}(y) \delta(y)^p d\mathcal{H}_\infty^d(y).$$

Thus, since the B_z are disjoint for $z \in \mathcal{X}_k$, and since N depends only on n ,

$$\begin{aligned} I_2 &\lesssim \sum_{k=1}^N \int_{E_1} \sum_{z \in \mathcal{X}_k} \mathbb{1}_{\frac{1}{8}B_z}(y) \delta(y)^p d\mathcal{H}_\infty^d(y) \leq \sum_{k=1}^N \int_{E_1 \cap 2B^1} \delta(y)^p d\mathcal{H}_\infty^d(y) \\ &\lesssim_n \int_{E_1 \cap 2B^1} \delta(y)^p d\mathcal{H}_\infty^d(y). \end{aligned} \quad (2.51)$$

To bound I_1 , again we use Lemma 2.22. We let $F_1 = E_1$, $F_2 = E_2$, $f = \text{dist}(\cdot, P)^p$, and $\alpha = 1/3$. Also let $z' \in E_2 \cap \frac{1}{3}B_z = B(z, \frac{4}{3}\delta(z))$ and $B'_z := B(z', 4\delta(z))$. Because $\text{dist}(\cdot, P)^p$ is Lipschitz in $2\mathbb{B}$, we have

$$\begin{aligned} I_1 &\lesssim \sum_{k=1}^N \int_{E_2} \sum_{z \in \mathcal{X}_k} \mathbb{1}_{\frac{2}{3}B'_z}(y) \text{dist}(z, P)^p d\mathcal{H}_\infty^d(y) \\ &\stackrel{(2.2)}{\lesssim} \sum_{k=1}^N \int_{E_2} \sum_{z \in \mathcal{X}_k} \mathbb{1}_{\frac{2}{3}B'_z}(y) \text{dist}(y, P)^p d\mathcal{H}_\infty^d(y) \\ &\quad + \sum_{k=1}^N \int_{E_2} \sum_{z \in \mathcal{X}_k} \mathbb{1}_{\frac{2}{3}B'_z}(y) \delta(z)^p d\mathcal{H}_\infty^d(y) \\ &= I_{11} + I_{12} \end{aligned}$$

Note that

$$\frac{2}{3}B'_z = B\left(z', \frac{8\delta(z)}{3}\right) \subseteq B\left(z, \frac{8\delta(z)}{3} + \frac{4\delta(z)}{3}\right) = B(z, 4\delta(z)) = B_z$$

and so the $\{\frac{2}{3}B'_z\}_{z \in \mathcal{X}_k}$ are disjoint. Hence,

$$I_{11} \leq \sum_{k=1}^N \int_{E_2 \cap 2B_2} \text{dist}(z, P)^p d\mathcal{H}_\infty^d(y).$$

Next, by Lemma 2.22 with $F_1 = E_2$, $F_2 = E_1$, $\alpha = \frac{1}{2}$, $\{\frac{2}{3}B'_z\}_{z \in \mathcal{X}_k}$ as our collection of balls B_j with centers $z' \in F_1$, $f(z') = \delta(z)^p$, and $\frac{2}{3}B_z$ as our B'_j (noting that $z \in B(z', \frac{4}{3}\delta(z)) = \alpha(\frac{2}{3}B'_z)$),

$$I_{12} \lesssim \sum_{k=1}^N \int_{E_1} \sum_{z \in \mathcal{X}_k} \mathbb{1}_{\frac{1}{3}B_z}(y) \delta(z)^p d\mathcal{H}_\infty^d(y) \stackrel{(2.51)}{\lesssim} \int_{E_1 \cap 2B^1} \delta(y)^p d\mathcal{H}_\infty^d(y).$$

Combining our estimates for I_1 , I_2 , I_{12} , and I_{22} , we obtain

$$\beta_{E_1}^{d,p}(B^1, P)^p \lesssim_{c,p} \beta_{E_2}^{d,p}(2B^2, P)^p + \int_{E_1 \cap 2B^1} \delta(y)^p d\mathcal{H}_\infty^d(y).$$

Now using the fact that $(a+b)^{\frac{1}{p}} \leq 2^{\frac{1}{p}} \max\{a, b\}^{\frac{1}{p}} \leq 2^{\frac{1}{p}}(a^{\frac{1}{p}} + b^{\frac{1}{p}})$, we are done. \square

It is not hard to show that Theorem 1.3 can be rewritten in the following way. For a cube I in \mathbb{R}^d and $f : \mathbb{R}^d \rightarrow \mathbb{R}^{n-d}$ Lipschitz, we set

$$\Omega_{f,p}(I)^p = \inf_A \int_I \left| \frac{f - A}{r} \right|^p \quad (2.52)$$

Then

$$\sum_{I \subseteq \mathbb{R}^d} \Omega_{f,p}(3I)^2 \mathcal{H}^d(I) \lesssim_{p,n} \|\nabla f\|_2^2 \quad (2.53)$$

where the sum is taken over all dyadic cubes I in \mathbb{R}^d .

Lemma 2.23. *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^{n-d}$ be an L -Lipschitz function, $\Gamma = \{(x, f(x)) : x \in \mathbb{R}^d\}$ be its graph, $x \in \Gamma$, $r > 0$, and let I be a cube in \mathbb{R}^d that contains $\pi_{\mathbb{R}^d}(B(x, r))$. Then*

$$\beta_{\Gamma}^{d,p}(x, r) \lesssim_{d,L,p} \left(\frac{\mathcal{H}^d(I)}{r^d} \right)^{1/p} \Omega_{f,p}(I). \quad (2.54)$$

Proof. Let A be the plane that infimizes (2.52). Since f is L -Lipschitz, we have that Γ is C -Ahlfors regular with C depending on d and L . In particular, $\mathcal{H}^d(E) \sim_L \mathcal{H}_{\infty}^d(E)$ for all $E \subseteq \Gamma$. Set $F(x) = (x, f(x))$, which is $2L$ -Lipschitz.

$$\begin{aligned} \beta_{\Gamma,p}^d(x, r)^p &\leq r^{-d} \int \mathcal{H}_{\infty}^d(\{y \in \Gamma \cap B(x, r) : \text{dist}(y, A(\mathbb{R}^d)) > tr\}) t^{p-1} dt \\ &\sim_L r^{-d} \int \mathcal{H}^d(\{y \in \Gamma \cap B(x, r) : \text{dist}(y, A(\mathbb{R}^d)) > tr\}) t^{p-1} dt \\ &= r^{-d} \int_{B(x,r) \cap \Gamma} \left(\frac{\text{dist}(y, A(\mathbb{R}^d))}{r} \right)^p d\mathcal{H}^d|_{\Gamma}(y) \\ &= r^{-d} \int_{\pi_{\mathbb{R}^d}(B(x,r))} \left(\frac{\text{dist}(F(z), A(\mathbb{R}^d))}{r} \right)^p J_F(z) dz \\ &\lesssim_L r^{-d} \int_I \left(\frac{|F(z) - A(z)|}{r} \right)^p dz \\ &\leq \frac{\mathcal{H}^d(I)}{r^d} \Omega_{f,p}(I)^p \end{aligned}$$

□

3. THE THEOREMS WE ACTUALLY PROVE

We first reformulate Theorem I with an equivalent version that uses balls and nets rather than dyadic cubes. This is slightly more technical to state, but will be more natural to prove. First, we define $\Theta_E^{d,A,\epsilon}$ in terms of a net (as opposed to $\Theta_E^{d,\Delta,\epsilon}$ from definition 1.8).

Definition 3.1. Let $E \subseteq \mathbb{R}^n$ be a Borel set, $0 < \rho < 1/1000$. For $k \in \mathbb{Z}$, let X_k be a sequence of maximal ρ^{-k} -separated sets of points for E and $\mathcal{X} = \{B(x, \rho^{-k}) :$

$k \in \mathbb{Z}$, $x \in X_k$ }. For $\epsilon, A, r > 0$ and $x \in E$, we define

$$\Theta_E^{d,A,\epsilon}(x, r) := \sum \{r_B^d : B \in \mathcal{X}, x_B \in B(x, r), \\ 0 < r_B < r, \text{ and } \vartheta_E(AB) \geq \epsilon\}.$$

Theorem 3.2. *Let $1 \leq d < n$, $C_0 > 1$, and $A > \max\{C_0, 10^5\}$. Let $1 \leq p < p(d)$. Let $E \subseteq \mathbb{R}^n$ be a closed set containing 0. Suppose that E is (c, d) -lower content regular in $B(0, 1)$. There is $\epsilon_0 = \epsilon_0(n, A, p, c) > 0$ such that the following holds. Let $0 < \epsilon < \epsilon_0$. For integer $k \geq 0$, let $X_k \subset X_{k+1}$ be a maximal 2^{-k} -separated set of points in $E \cap B(0, 1)$. Suppose further that for each k we have $X_k \subset X_{k+1}$. Let $\mathcal{X}_k = \{B(x, 2^{-k}) : x \in X_k\}$. Then*

$$1 + \sum_{k \geq 0} \sum_{B \in \mathcal{X}_k} \beta_E^{d,p}(C_0 B)^2 r_B^d \\ \leq C(A, C_0, n, \epsilon, p, c) \left(\mathcal{H}^d(E \cap B(0, 1)) + \Theta_E^{d,A,\epsilon}(0, 1) \right). \quad (3.1)$$

Similarly, Theorem II has a version for nets as follows.

Theorem 3.3. *Let $1 \leq d < n$. Let $1 \leq p \leq \infty$ and $E \subseteq \mathbb{R}^n$ be such that $0 \in E$. Suppose that E is (c, d) -lower content regular in $B(0, 1)$. Let X_k be a nested sequence of maximal 2^{-k} -separated points in E and $\mathcal{X}_k = \{B(x, 2^{-k}) : x \in X_k\}$. Let $A > 1$ and $\epsilon > 0$ be given as well. Then for C_0 sufficiently large (depending only on n and A),*

$$\mathcal{H}^d(E \cap B(0, 1)) + \Theta_E^{d,A,\epsilon}(0, 1) \\ \leq C(A, n, c, \epsilon) \left(1 + \sum_{k \geq 0} \sum_{\substack{B \in \mathcal{X}_k \\ x_B \in B(0, 1)}} \beta_E^{d,p}(C_0 B)^2 r_B^d \right). \quad (3.2)$$

Furthermore, if the right hand side of (3.2) is finite, then E is d -rectifiable

It is not hard to show that Theorem 3.2 implies Theorem I. To do this, and for what follows, we will need to relate the two versions of Θ . Indeed

$$\Theta_E^{d,\Delta,\epsilon}(B(0, 1)) \lesssim_{A,n,\epsilon} 1 + \Theta_E^{d,A,\epsilon}(0, 1),$$

and

$$\Theta_E^{d,A,\epsilon}(B(0, 1)) \lesssim_{A,n,\epsilon} \Theta_E^{d,\Delta,\epsilon}(0, 1).$$

The sketch of the argument for the first inequality is that cubes of diameter less than one are contained in balls from some \mathcal{X}_k , so we can use monotonicity of ϑ_E ; the cubes of diameter larger than one have geometrically decaying ϑ , and so only $\sim \log(\epsilon)$ generations need to be counted. The second inequality follows in a similar manner (but note that we have cubes of all scales). Moreover, it is not hard to show that $\beta_E^{d,p}(Q) \lesssim \beta_E^{d,p}(B)$ if Q is a cube and $B \supseteq Q$ is a ball of comparable size. Thus, we have

$$\sum_{\substack{Q \in \Delta \\ Q \cap E \cap B(0,1) \neq \emptyset}} \beta_E^{d,p}(C_0 Q)^2 \text{diam}(Q)^d \lesssim 1 + \sum_{k \geq 0} \sum_{B \in \mathcal{X}_k} \beta_E^{d,p}(C_0 B)^2 r_B^d.$$

Again, this follows since the cubes of side length much less than one are contained in a ball $C_0 B$ from the sum on the right with comparable size and the number of cubes associated to each ball can be chosen to be uniformly bounded. For the cubes of larger scale, we bound the beta numbers trivially by one and then the sum over them is a geometric sum and totals at most a universal constant. This proves the above inequality, and thus we can reduce from the dyadic version Theorem I to the ball version, i.e. Theorem 3.2. A similar argument shows Theorem 3.3 implies Theorem II. The main thing to note is that any ball B has a dyadic cube Q so that $B \subseteq C_0 Q$ and $r_B \sim_{C_0, n} \text{diam}(Q)$.

Remark 3.4. We will from now on focus on showing Theorems 3.2 and 3.3, as they will imply our main results.

Remark 3.5. One open question is whether one can generalize these techniques to give a version of our main results that hold for Hilbert spaces. We believe this is possible after suitably proving some of the results from [DT12] for this setting. Such a result would take on this form of statement, using nets and balls instead of dyadic cubes.

Theorem 3.6 just below is a version of Theorem 3.2 specialized to Reifenberg flat surfaces. Theorem 3.6 will be used in the proof of Theorem 3.2.

Theorem 3.6. *Let $1 \leq d < n$, $C_0 > 1$, $1 \leq p < \frac{2d}{d-2}$ if $d > 2$ and $1 \leq p < \infty$ if $d \leq 2$. There is $\epsilon_0 = \epsilon_0(n, C_0, p) > 0$ such that the following holds. Let $0 < \epsilon < \epsilon_0$, Σ be a (ϵ, d) -Reifenberg flat surface so that for some d -plane P_0 ,*

$$\Sigma \setminus V_0^1 = \Sigma \setminus B(0, 1) = P_0 \setminus B(0, 1). \quad (3.3)$$

Let \mathcal{X}_k be a nested sequence of maximally separated ρ^k -nets in Σ so that $0 \in \mathcal{X}_0$. For $C_0 > 0$,

$$\sum_{k \geq 0} \sum_{x \in \mathcal{X}_k} \beta_\Sigma^{d,p}(B(x, C_0 \rho^k)) \rho^{kd} \lesssim_{p, C_0, n, \epsilon} \mathcal{H}^d(\Sigma \cap B(0, 1)).$$

4. SKETCH OF THE PROOF OF THEOREMS 3.6 AND THEOREM 3.2

This section will contain a loose description of the proofs of Theorems 3.6 and 3.2. The proof of Theorem 3.6 is carried out in Sections 5–9. Theorem 3.2 is then proven in Section 10 by building a stopping time argument on top of Theorem 3.6. We assume below that the reader is fluent with the language and results of the previous sections. We expect that some readers may not gain much from this section, but hope that others will appreciate it.

The fundamental tool one would like to use to upper bound a sum of $\beta_E^{d,p}$ -numbers as in the left-hand-side of Theorem 3.6 is Dorrnsoro's theorem (see Theorem 1.3). That requires having a graph. Thus, we would like to describe the

surface Σ as the limit of a progression of graphs in which, locally, each successive surface is a graph over the previous surface, in the same spirit as the progression of curves that leads to the von Koch snowflake. Indeed, Theorem 2.5 supplies us with Σ as a limit of (constructed) surfaces Σ_k which are locally C^2 graphs (with control on what ‘locally’ means) but these are not enough for the accounting we need to do. Instead we need to stop the construction of the Σ_k ’s in different places, and let it keep running in others. We’ll be more specific below:

Consider a family of Christ-like ‘cubes’ for Σ as assured by Theorem 2.9. To each cube Q , associate an affine d -plane P_Q which well-approximates Σ inside a ball B_Q , where $B_Q \supset Q$ and is not too big. Now we generate layers of cubes as follows. Our zeroth layer are the cubes of side length one. Now, run a stopping time on each cube Q from this family to construct a new layer by adding smaller and smaller cubes R in Q and stop if P_R changes its angle too much from the initial plane P_Q . Let S_Q be the resulting stopping time region (i.e. those cubes for which you didn’t stop). We will restart on the stopped cubes to get more stopping time regions. Our first layer is the union of all S_Q over all stopped cubes Q from the zeroth layer. Now look at the minimal cubes of this layer and run the same stopping-time process to create a new collection of stopping times that form the 2nd layer, and so on.

These layers are *almost* good enough to work with, but not quite and we will now describe why. Take a layer and apply the construction of David and Toro (Theorem 2.5) to it, in the sense that you use the centers of the cubes Q and the planes P_Q (really, you first have to shift them to go through the centers). The resulting surface is in fact a graph around each maximal cube in the layer. Thus you can use Dorronsoro’s theorem on each of these pieces to bound the $\beta_E^{d,p}$ -numbers by a portion of E around that maximal cube. However, the sizes of these maximal cubes can vary wildly, and so they may not have bounded overlap. This is a problem, which is why we said the layers are *almost* good enough. To solve this, when constructing our layers, we add an extra step that extends them so that adjacent minimal cubes have comparable sizes (Lemma 5.10) and then we repeat our construction on the minimal cubes of these extended layers and so on. Note that they will still give rise to graphs over the planes corresponding to the maximal cubes in each layer (Lemma 5.11).

Section 5 describes these regions and their properties. Section 6 describes building a progression of surfaces Σ^N corresponding to a progression of stopping times (N here corresponds to how many generations from the root our extended stopping time is) and records some properties. Each Surface there is its own application of Theorem 2.5.

In Section 7 we will introduce a bi-Lipschitz map $F_N : \Sigma^N \rightarrow \Sigma^{N+1}$, whose image has (locally) a graph structure (see Figure 1). Then, in Section 8, we show that the sum of the area of the cubes where we stopped in our construction of the N th layer, is bounded by the total area of the limit surface $\Sigma = \lim \Sigma^N$. Here the area of Σ is really thought of as the limit of a telescoping sum, where, the N th summand bounds (with controllable errors) the area of the stopped cubes in N th layer. (Note that this is really not a precise statement, since some summands in the

telescoping sum could even be negative!) Finally, in Section 9, we reduce upper bound for $\beta_E^{d,p}$ -numbers for a single stopping time to the upper bound given by Dorronsoro's theorem Theorem 1.3, which is bounded by the area of the root cube. These are exactly the cubes whose total area was bounded in Section 8. This will complete the proof of Theorem 3.6.

Theorem 3.2 follows in a similar fashion, except we also need to stop every time we cease to be Reifenberg flat. The onerous details of this are carried out in Section 10.

5. THE STOPPING-TIME

Sections 5, 6, 7, 8, and 9 will together give Theorem 3.6.

Note that (2.20) means we can use the lemmas from Section 2.4. Fix $M > 1$. Let \mathcal{D} be the cubes for Σ from Theorem 2.9 using our maximal nets \mathcal{X}_k and set P'_Q be such that

$$\beta_{\Sigma, \infty}^d(P'_Q, MB_Q) = \beta_{\Sigma, \infty}^d(MB_Q) < \epsilon$$

and set $P_Q \ni x_Q$ be the plane parallel to P'_Q , and so

$$\beta_{\Sigma, \infty}^d(P_Q, MB_Q) \leq 2\beta_{\Sigma, \infty}^d(MB_Q) < 2\epsilon \quad (5.1)$$

We can assume there is $Q_0 \in \mathcal{D}_0$ with center 0. Without loss of generality, we may assume $P_{Q_0} = P_0$

Definition 5.1. ([DS93a, I.3.2]) A *stopping-time region* $S \subseteq \mathcal{D}$ is a collection of cubes such that the following hold.

- (1) All cubes $Q \in S$ are contained in a maximal cube $Q(S) \in S$.
- (2) S is *coherent*, meaning that if $Q \in S$ and $Q \subseteq R \subseteq Q(S)$ then $R \in S$.
- (3) For all $Q \in S$, each of its siblings of Q are also in S .

Let $\alpha > \epsilon > 0$.

Remark 5.2. We will adjust the value of ϵ (the Reifenberg flatness parameter) as we go along, but it shall be much smaller than α (more precisely, $\epsilon \ll \alpha^4$), and will also depend on a parameter τ . See Remark 8.3 below as well as Section 8. The constant α may be fixed in the proof of Lemma 8.2.

Let $Q \in \mathcal{D}$ and let S_Q be the stopping time region constructed by adding cubes R to S if both of the following are satisfied.

- (1) If $T \in \mathcal{D}$ and $R \subseteq T \subseteq Q$ then $T \in S_Q$.
- (2) We have $\angle(P_T, P_Q) < \alpha$ for any sibling T of R (including R itself).

In this way, $Q(S_Q) = Q$.

Remark 5.3. A property which S_Q does not enjoy is that if two minimal cubes are 'near' each other, then they have 'similar' sizes. Because of this, we will define 'extensions' of these regions, S'_Q later on (see (5.9) and its preceding Lemma 5.10, as well as Lemma 5.11). The reason we care about this is that we will have a sequence of surfaces Σ_N , where Σ_N comes from cubes stopped N times. Had we

not had ‘near by cubes have similar sizes’, we could not control well enough the relationship between Σ_N and Σ_{N+1} . This happens in Section 6.

For a collection of cubes \mathcal{C} , we define a distance function

$$d_{\mathcal{C}}(x) = \inf\{\ell(Q) + \text{dist}(x, Q) : Q \in \mathcal{C}\}$$

and for $Q \in \mathcal{D}$, set

$$d_{\mathcal{C}}(Q) = \inf_{x \in Q} d_{\mathcal{C}}(x) = \inf\{\ell(R) + \text{dist}(Q, R) : R \in \mathcal{C}\}$$

Let $m(S)$ be the set of minimal cubes of S , i.e. those $Q \in S$ for which there are no cubes $R \in S$ properly contained in Q , and define

$$z(S) = Q(S) \setminus \bigcup\{Q : Q \in m(S)\}. \quad (5.2)$$

We will define a sequence of collections of cubes $\text{Layer}(N)$, $\text{Up}(N)$, $\text{Stop}(N)$, and a sequence of collections of stopping-times \mathcal{F}^N , $N = 0, 1, 2, \dots$ as follows.

First, set

$$\text{Stop}(-1) = \mathcal{D}_0.$$

Let $\tau \in (0, 1)$ be small.

Remark 5.4. The constant τ will be upper bounded in Lemmas 5.7, 5.10, and 6.10, see Remark 7.1 below.

Suppose we have defined $\text{Stop}(N-1)$ for some integer $N \geq 0$. Let

$$\text{Layer}(N) = \bigcup\{S_Q : Q \in \text{Stop}(N-1)\}.$$

Clearly

$$\bigcup\{Q \in \text{Layer}(N)\} = \bigcup\{Q \in \text{Stop}(N-1)\}. \quad (5.3)$$

Let $\text{Stop}(N)$ be the set of maximal cubes Q which have a sibling Q' (possibly themselves) such that $\ell(Q') < \tau d_{\text{Layer}(N)}(Q')$, i.e. (recalling that $\text{Child}_{(1)}Q^{(1)}$ is the collection of children of Q 's parent)

$$\text{Stop}(N) = \{Q \in \mathcal{D} : Q \text{ max. such that there is } Q' \in \text{Child}_{(1)}Q^{(1)} \text{ with } \ell(Q') < \tau d_{\text{Layer}(N)}(Q')\}. \quad (5.4)$$

See Corollary 5.8 below, which may elucidate why we use the terminology ‘‘stop’’.

Set $\text{Up}(-1) = \emptyset$ and

$$\text{Up}(N) = \text{Up}(N-1) \cup \{Q \in \mathcal{D} : Q \supseteq R \text{ for some } R \in \text{Layer}(N) \cup \text{Stop}(N)\} \quad (5.5)$$

Lemma 5.5.

$$\text{Up}(N) = \{Q \in \mathcal{D} : Q \not\subset R \text{ for any } R \in \text{Stop}(N)\} \quad (5.6)$$

Proof. Denote by U the right hand side of (5.6). First suppose $Q \in \text{Up}(N)$. If we have $Q \notin U$, then there is an R such that $Q \subsetneq R \in \text{Stop}(N)$. By (5.4), we then have that R cannot contain a $\text{Layer}(N)$ cube, and so the same holds for Q . Thus, Q cannot contain a cube from either $\text{Layer}(N)$ or $\text{Stop}(N)$, which implies $Q \notin \text{Up}(N)$.

If $Q \in U$ then either $Q \supset T \in \text{Stop}(N)$ (in which case $Q \in \text{Up}(N)$) or Q is disjoint from $\cup \text{Stop}(N)$. Hence, if $x \in c_0 B_Q \cap \Sigma$, then $\tau d_{\text{Layer}(N)}(x) \geq \ell(Q')$ for all cubes Q' containing x , so in particular, $d_{\text{Layer}(N)}(x) = 0$, so we may find $R \in \text{Layer}(N)$ so that $R \subseteq c_0 B_Q \cap \Sigma \subseteq Q$, and thus $Q \in \text{Up}(N)$ as well. \square

Lemma 5.6. *Let $\mathcal{C} \subseteq \mathcal{D}$ and $Q, Q' \in \mathcal{D}$. Then*

$$d_{\mathcal{C}}(Q) \leq 2\ell(Q) + \text{dist}(Q, Q') + 2\ell(Q') + d_{\mathcal{C}}(Q'). \quad (5.7)$$

Proof. Let $x, w \in Q$, $y, z \in Q'$, and $Q'' \in \mathcal{C}$. Since $Q' \subseteq B_{Q'}$, $\text{diam } Q' \leq 2\ell(Q')$, and so

$$\begin{aligned} d_{\mathcal{C}}(x) &\leq \text{dist}(x, Q'') + \ell(Q'') \leq |x - y| + \text{dist}(y, Q'') + \ell(Q'') \\ &\leq |x - w| + |w - z| + |y - z| + \text{dist}(y, Q'') + \ell(Q'') \\ &\leq 2\ell(Q) + |w - z| + 2\ell(Q') + \text{dist}(y, Q'') + \ell(Q''). \end{aligned}$$

Now we infimize $\text{dist}(y, Q'') + \ell(Q'')$ over all $Q'' \in \mathcal{C}$, then over all $y \in Q'$, and then over all $w \in Q$ and $z \in Q'$ and we get (5.7). \square

Below, the constant ρ comes from the statement of Theorem 2.9.

Lemma 5.7. *Let $\tau_0 = \frac{1}{4(2+\rho^{-1})}$. For $0 < \tau < \tau_0$,*

$$\rho\tau d_{\text{Layer}(N)}(Q) \leq \ell(Q) \leq 2\tau d_{\text{Layer}(N)}(Q) \text{ for all } Q \in \text{Stop}(N). \quad (5.8)$$

Proof. Indeed, if Q' is the sibling of Q (or Q itself) satisfying (5.4), then

$$\text{dist}(Q, Q') \leq \text{diam } Q^{(1)} \leq \text{diam } B_{Q^{(1)}} = \frac{2}{\rho}\ell(Q)$$

and so

$$\begin{aligned} \ell(Q) &= \ell(Q') < \tau d_{\text{Layer}(N)}(Q') \\ &\stackrel{(5.7)}{\leq} \tau(2\ell(Q') + \text{dist}(Q, Q') + 2\ell(Q) + d_{\text{Layer}(N)}(Q)) \\ &\leq \tau(2\ell(Q) + 2\rho^{-1}\ell(Q) + 2\ell(Q) + d_{\text{Layer}(N)}(Q)) \\ &= 2\tau(2 + \rho^{-1})\ell(Q) + \tau d_{\text{Layer}(N)}(Q) < \frac{\ell(Q)}{2} + \tau d_{\text{Layer}(N)}(Q) \end{aligned}$$

since $\tau < \tau_0$, and so we may regroup the terms and get $\ell(Q) < 2\tau d_{\text{Layer}(N)}(Q)$. On the other hand, by the maximality of Q ,

$$\rho^{-1}\ell(Q) = \ell(Q^{(1)}) \geq \tau d_{\text{Layer}(N)}(Q^{(1)}) \geq \tau d_{\text{Layer}(N)}(Q)$$

which proves (5.8). \square

Corollary 5.8. *Let τ be as in Lemma 5.7. Let*

$$Z = \{x : d_{\text{Layer}(N)}(x) = 0\}.$$

Then

$$\bigcup\{Q \in \text{Stop}(N-1)\} \setminus Z = \bigcup\{Q \in \text{Stop}(N)\},$$

and $Q \in \text{Stop}(N)$ implies $d_{\text{Layer}(N)}(Q) \sim \tau \ell(Q)$.

Proof. By Lemma 5.7, it remains to see that $\bigcup\text{Stop}(N-1) \setminus Z = \bigcup\text{Stop}(N)$. Let $x \in \bigcup\{Q \in \text{Stop}(N-1)\}$ be such that $d_{\text{Layer}(N)}(x) > 0$. By (5.3) and because $x \notin Z$, we have a minimal (with respect to containment) Q_N such that $x \in Q_N \in \text{Layer}(N)$. Let $Q_{N,k}$ be such that $x \in Q_{N,k} \in \text{Child}_k(Q_N)$. We have that

$$0 < d_{\text{Layer}(N)}(x) \leq \ell(Q_{N,k}) = \rho^k \ell(Q_N).$$

Thus, there is a k such that $x \in Q_{N,k} \in \text{Stop}(N)$.

Conversely, let $x \in Q \in \text{Stop}(N)$. Then

$$d_{\text{Layer}(N)}(x) \geq d_{\text{Layer}(N)}(Q) \gtrsim \tau \ell(Q)$$

and so $x \notin Z$. Moreover, $Q \subseteq R$ for some $R \in \text{Stop}(N-1)$, and this finishes the proof. \square

Corollary 5.9. *Suppose $d_{\text{Up}(N)}(x) > 0$. Then there is a $Q \ni x$ such that $Q \in \text{Stop}(N)$.*

Proof. We have for all $n \leq N$ that

$$d_{\text{Layer}(N)}(x) \geq d_{\text{Up}(n)}(x) \geq d_{\text{Up}(N)}(x) > 0.$$

The result now follows by the first part of Corollary 5.8. \square

Lemma 5.10. *Let $C_1 > 1$, $\tau_1(C_1) = \min\{\tau_0, (16+8C_1)^{-1}\}$, and assume $0 < \tau < \tau_1(C_1)$. If $\epsilon < \alpha$, $Q, Q' \in \text{Stop}(N)$ and $C_1 B_Q \cap C_1 B_{Q'} \neq \emptyset$, then $\ell(Q) \sim \ell(Q')$ (with constants independent of C_1), and $\angle(P_Q, P_{Q'}) \lesssim_{C_1} \epsilon$.*

Proof. Assume $\ell(Q') \leq \ell(Q)$, we will first show $\ell(Q) \lesssim \ell(Q')$. Since $C_1 B_Q \cap C_1 B_{Q'} \neq \emptyset$, we know

$$\text{dist}(Q, Q') \leq C_1 \ell(Q) + C_1 \ell(Q') \leq 2C_1 \ell(Q).$$

Thus, since $Q, Q' \in \text{Stop}(N)$,

$$\begin{aligned} \ell(Q) &\stackrel{(5.8)}{\leq} 2\tau d_{\text{Layer}(N)}(Q) \stackrel{(5.7)}{\leq} 2\tau d_{\text{Layer}(N)}(Q') \\ &\quad + 2\tau(2\ell(Q) + \text{dist}(Q, Q') + 2\ell(Q')) \\ &\stackrel{(5.8)}{\leq} \frac{2}{\rho} \ell(Q') + 2\tau(2\ell(Q) + 2C_1 \ell(Q) + 2\ell(Q)) \\ &= \frac{2}{\rho} \ell(Q') + (8 + 4C_1)\tau \ell(Q) < \frac{2}{\rho} \ell(Q') + \frac{\ell(Q)}{2} \end{aligned}$$

where in the last inequality we used $\tau < \frac{1}{16+8C_1}$. This then gives $\ell(Q') \leq \ell(Q) \leq \frac{4}{\rho} \ell(Q')$. The lemma now follows from Lemma 2.18 and the fact that Σ is lower regular. \square

For $Q \in \text{Stop}(N-1)$, let

$$S'_Q = \{R \in \text{Up}(N) : R \subseteq Q\} \supseteq S_Q. \quad (5.9)$$

Observe that this is again a stopping-time region by construction (this is why we defined $\text{Stop}(N)$ using siblings).

Lemma 5.11. *For $0 < \tau < \tau_1(3)$ and $Q \in \text{Stop}(N-1)$*

$$\angle(P_R, P_Q) \lesssim \alpha \text{ for all } R \in S'_Q. \quad (5.10)$$

Proof. Let $R \in S'_Q$ and $T \in \text{Layer}(N)$ be such that

$$\ell(T) + \text{dist}(T, R) \leq 2d_{\text{Layer}(N)}(R). \quad (5.11)$$

Then $T \in S_{Q'}$ for some $Q' \in \text{Stop}(N-1)$. Hence,

$$\begin{aligned} \text{dist}(Q, Q') &\leq \text{dist}(R, T) \stackrel{(5.11)}{\leq} 2d_{\text{Layer}(N)}(R) \\ &\leq 2(\ell(Q) + \text{dist}(R, Q)) = 2\ell(Q). \end{aligned} \quad (5.12)$$

In particular, $3B_{Q'} \cap 3B_Q \neq \emptyset$, and since $Q, Q' \in \text{Stop}(N-1)$, we have by Lemma 5.10 with $C_1 = 2$ if $\tau < \tau_1(3)$ that

$$\ell(Q) \sim \ell(Q') \text{ and } \angle(P_Q, P_{Q'}) \lesssim_{C_1} \epsilon. \quad (5.13)$$

We split into two cases.

(1) Suppose first that $\ell(Q') \leq \frac{2}{\rho\tau}\ell(R)$. Then

$$\ell(R) \leq \ell(Q) \stackrel{(5.13)}{\sim} \ell(Q') \leq \frac{2}{\rho\tau}\ell(R),$$

and so $\ell(R) \sim_\tau \ell(Q')$. Moreover,

$$\text{dist}(Q, Q') \stackrel{(5.12)}{\leq} 2\ell(Q) \stackrel{(5.13)}{\lesssim} \ell(Q'),$$

hence Lemma 2.18 implies $\angle(P_R, P_{Q'}) \lesssim_\tau \epsilon$, and so

$$\angle(P_R, P_Q) \leq \angle(P_R, P_{Q'}) + \angle(P_{Q'}, P_Q) \stackrel{(5.13)}{\lesssim}_\tau \epsilon.$$

(2) Now suppose $\ell(Q') > \frac{2}{\rho\tau}\ell(R)$. Note that

$$\ell(T) \stackrel{(5.11)}{\leq} 2d_{\text{Layer}(N)}(R) \stackrel{(5.8)}{\leq} \frac{2}{\rho\tau}\ell(R) < \ell(Q').$$

Let $Q'' \in S_{Q'}$ be the largest parent of T for which

$$\ell(Q'') \leq \frac{2}{\rho\tau}\ell(R).$$

By the above inequality, this is well defined and $\ell(Q'') \sim \tau^{-1}\ell(R)$. Moreover,

$$\text{dist}(R, Q'') \leq \text{dist}(R, T) \stackrel{(5.11)}{\leq} 2d_{\text{Layer}(N)}(R) \stackrel{(5.8)}{\lesssim} \tau^{-1}\ell(R).$$

Thus, Lemma 2.18 implies

$$\angle(P_R, P_{Q''}) \lesssim_{\tau} \epsilon. \quad (5.14)$$

Also note that since $Q'' \in S_{Q'}$,

$$\angle(P_{Q''}, P_{Q'}) \lesssim \alpha. \quad (5.15)$$

Thus, combining (5.13), (5.14), and (5.15), we get

$$\begin{aligned} \angle(P_R, P_Q) &\leq \angle(P_R, P_{Q''}) + \angle(P_{Q''}, P_{Q'}) + \angle(P_{Q'}, P_Q) \\ &\lesssim_{\tau} \epsilon + \alpha + \epsilon \lesssim \alpha. \end{aligned} \quad (5.16)$$

□

Lemma 5.12. *Let $0 < \tau < \min\{\tau_0, c_0/4\}$. For $\epsilon > 0$ small enough depending on τ and α , the following holds. Suppose $Q \in m(S)$ where $S = S_{Q(S)}$, with $Q(S) \in \text{Stop}(N-1)$ (so $S \subset \text{Layer}(N)$). Then there is R such that*

- (1) $R \in \text{Stop}(N)$,
- (2) $R \subseteq Q$,
- (3) $\ell(R) \sim \tau \ell(Q)$, and
- (4) $\angle(R, Q(S)) \gtrsim_{\tau} \alpha$.

Proof. If $Q \in \mathcal{D}_k$, let $R \in \text{Stop}(N)$ be the cube with the same center as Q , so $x_R = x_Q$. Then $R \in \mathcal{D}_{k+k_1}$ for some $k_1 \geq 0$. Observe that by Lemma 5.7

$$5\rho^{k+k_1} = \ell(R) < 2\tau d_{\text{Layer}(N)}(R) \leq 2\tau \ell(Q) = 10\tau \rho^k \quad (5.17)$$

and so

$$\rho^{k_1} < 2\tau. \quad (5.18)$$

We claim that

$$d_{\text{Layer}(N)}(R) \geq \frac{c_0}{2} \ell(Q). \quad (5.19)$$

First note that $Q \in \text{Layer}(N)$ and

$$\text{dist}(R, Q) + \ell(Q) = \ell(Q).$$

Thus when considering $d_{\text{Layer}(N)}(R)$ we may look at the quantity $\text{dist}(R, T) + \ell(T)$ where $T \in \text{Layer}(N)$, and we assume without loss of generality $\ell(T) < \ell(Q)$ and $\text{dist}(T, R) < \ell(Q)$. Since $Q \in m(S)$, we must have that $T \subseteq Q^c$, hence $T \subseteq c_0 B_Q^c$. By (5.17) and assuming $\tau < c_0/4$, since $x_R = x_Q$,

$$R \subseteq 2\tau B_Q \subseteq \frac{c_0}{2} B_Q,$$

and thus $\text{dist}(T, R) \geq \frac{c_0}{2} \ell(Q)$. Infimizing $\text{dist}(R, T) + \ell(T)$ over all such T gives (5.19).

Thus,

$$\frac{1}{\tau \rho} \ell(R) \stackrel{(5.8)}{\geq} d_{\text{Layer}(N)}(R) \stackrel{(5.19)}{\geq} \frac{c_0}{2} \ell(Q)$$

and so $\ell(R) \gtrsim \tau \ell(Q)$. This and (5.18) imply $\ell(R) \sim \tau \ell(Q)$.

Let Q' be a child of Q such that $\angle(Q', Q(S)) \geq \alpha$ (which exists by minimality of Q in S). Then $\angle(P_{Q'}, P_Q) \lesssim \epsilon$ by Lemma 2.18. By the same lemma,

$\angle(P_Q, P_R) \lesssim \epsilon$. Thus, for some constant $C > 0$ and $\epsilon > 0$ small enough depending on α and τ .

$$\angle(P_R, P_{Q(S)}) \geq \angle(P_{Q'}, P_{Q(S)}) - \angle(P_Q, P_R) - \angle(P_Q, P_{Q'}) \geq \alpha - C\epsilon \gtrsim \alpha.$$

□

Recall the definition of $z(S)$ (for a collection of cubes S) given by equation (5.2). Define

$$\mathcal{F}_N = \{S'_Q : Q \in \text{Stop}(N-1)\}.$$

Lemma 5.13. *If $S'_1 \in \mathcal{F}_N$ and $S'_2 \in \mathcal{F}_M$ are distinct, then $z(S'_1) \cap z(S'_2) = \emptyset$.*

Proof. First note that for $i = 1, 2$, S'_i are stopping times, and by construction, if they are distinct, then $S'_1 \cap S'_2 \subseteq \{Q(S'_1), Q(S'_2)\}$. However, if $x \in z(S'_1) \cap z(S'_2)$, then every cube Q containing x with $\ell(Q) < \min\{\ell(Q(S'_1)), \ell(Q(S'_2))\}$ is in $S'_1 \cap S'_2$, which is a contradiction. □

Lemma 5.14. *Let $0 < \tau < \tau_0$. For $N \geq 0$,*

$$d_{\text{Layer}(N)}(x) \sim d_{\text{Up}(N)}(x) \text{ for all } x \in \mathbb{R}^n \quad (5.20)$$

and in particular

$$d_{\text{Up}(N)}(Q) \sim \tau \ell(Q) \text{ for all } Q \in \text{Stop}(N). \quad (5.21)$$

Proof. Since $\text{Layer}(N) \subseteq \text{Up}(N)$, we have $d_{\text{Layer}(N)}(x) \geq d_{\text{Up}(N)}(x)$, so we just have to verify the opposite inequality.

Let $T \in \text{Up}(N)$ be such that $\ell(T) + \text{dist}(T, Q) < 2d_{\text{Up}(N)}(x)$. We consider two cases.

(1) If $T \supset R \in \text{Layer}(N)$, then

$$\ell(R) + \text{dist}(x, R) \leq \ell(T) + \text{dist}(x, R) \leq 3\ell(T) + \text{dist}(x, T) \leq 6d_{\text{Up}(N)}(x),$$

and so $d_{\text{Layer}(N)}(x) \lesssim d_{\text{Up}(N)}(x)$, and we are done with this case.

(2) For the second case, suppose $T \supset R \in \text{Stop}(N)$. Without loss of generality, we can assume $T \notin \text{Layer}(N)$. Then there is $Q \in \text{Layer}(N)$ containing R , and by our assumption, $T \supset Q$. Hence, just as in the previous case,

$$\begin{aligned} \ell(R) + \text{dist}(x, R) &\leq 3(\ell(Q) + \text{dist}(x, Q)) \leq 9(\ell(T) + \text{dist}(x, T)) \\ &\leq 18d_{\text{Layer}(N)}(x). \end{aligned}$$

The last part of the lemma now follows from Lemma 5.7. □

6. THE SEQUENCE OF APPROXIMATING SURFACES

For $k \geq 0$ an integer, let $s(k)$ be such that $5\rho^{s(k)} \leq r_k < 5\rho^{s(k)-1}$. Set

$$\text{Up}(N)_k = \mathcal{D}_{s(k)} \cap \text{Up}(N)$$

and let $\mathcal{X}_k^N = \{x_{j,k}\}_{j \in J_k^N}$ be a maximal r_k -separated set of points for the set

$$\mathcal{C}_k^N = \{x_Q : Q \in \text{Up}(N)_k\}.$$

For $j \in J_k^N$, let $Q_{j,k} \in \text{Up}(N)_k$ be such that $x_{Q_{j,k}} = x_{j,k}$ and let $P_{j,k} = P_{Q_{j,k}}$. Note that in this way $x_{j,k} \in P_{j,k}$ and

$$\ell(Q_{j,k}) \leq r_k < \rho^{-1} \ell(Q_{j,k}). \quad (6.1)$$

Lemma 6.1. *For each N , $\{x_{j,k}\}_{j \in J_k^N}$ satisfies the conditions of Theorem 2.5.*

Proof. Let $j \in J_k^N$. If $s(k) = s(k-1)$, then there is $i \in J_{k-1}$ so that $x_{j,k} = x_{Q_{j,k}} \in B_{i,k-1}$ since $\{x_{i,k-1}\}_{i \in J_{k-1}}$ is a maximal net for $\mathcal{C}_{k-1}^N = \mathcal{C}_k^N$. Otherwise, if $s(k) > s(k-1)$, then there is $i \in k-1$ so that $x_{Q_{j,k}}^{(1)} \in B_{i,k-1}$. Since

$$\ell(Q_{j,k}^{(1)}) = 5\rho^{s(k)-1} \leq 5\rho^{s(k-1)} \leq r_{k-1},$$

we have

$$\begin{aligned} x_{j,k} \in Q_{j,k} \subseteq Q_{j,k}^{(1)} &\subseteq B\left(x_{Q_{j,k}^{(1)}}, \ell(Q_{j,k}^{(1)})\right) \subseteq B\left(x_{i,k-1}, r_{k-1} + \ell(Q_{j,k}^{(1)})\right) \\ &\subseteq B(x_{i,k-1}, 2r_{k-1}) = 2B_{i,k-1}. \end{aligned}$$

Furthermore, $\epsilon_k(x_{j,k}) \lesssim \epsilon$ by Lemma 2.18, so for $\epsilon > 0$ small enough, the lemma follows. \square

Let $P_0 = P_{Q_0}$ and let σ_k^N , Σ_k^N , Σ^N , and σ^N be the functions and surfaces obtained from Theorem 2.5 with the nets \mathcal{X}_k^N . In this way, $\Sigma_0^N = P_0$ for all N .

Lemma 6.2. *If $d_{\text{Up}(N)}(x) = 0$ then $x \in \Sigma^N \cap \Sigma$*

Proof. Suppose $d_{\text{Up}(N)}(x) = 0$. This guarantees a sequence of cubes $Q_i \in \text{Up}(N)$ such that $\ell(Q_i) + \text{dist}(x, Q_i) \rightarrow 0$. From the definition of \mathcal{X}_k^N as a net, and Theorem 2.5, equation (2.7), we have $x \in \Sigma^N$. It remains to see that $x \in \Sigma$. To this end, note that $\text{dist}(x, \Sigma) \leq \text{dist}(x, Q_i) \downarrow 0$, and so $x \in \Sigma$. \square

For $x \in \Sigma^N$, we define $k_N(x)$ to be the maximal integer such that $x \in V_{k-1}^{11}$.

Lemma 6.3. *For $x \in \Sigma^N$, let $k = k_N(x)$. Then we have*

$$\text{dist}(x, \Sigma) \lesssim \epsilon r_k \sim \epsilon d_{\text{Up}(N)}(x) \quad (6.2)$$

and

$$B(x, r_k) \cap \Sigma^N = B(x, r_k) \cap \Sigma_k^N. \quad (6.3)$$

Proof. We first prove the left-hand-side inequality (6.2). There is nothing to show if $x \in \Sigma$, so assume $x \notin \Sigma$. If $x \notin B(0, 1)$, then $x \in P_{Q_0}$ by (2.9) and so $x \in \Sigma$ by (3.3), so we may assume $x \in B(0, 1)$. Hence, there is a maximal $k \geq 1$ for which $x \in V_{k-1}^{11}$. Let $x' \in \Sigma_{k-1}^N$ be such that $x = \lim_{K \rightarrow \infty} \sigma_{k+K}^N \circ \cdots \circ \sigma_{k-1}^N(x')$. Then by (2.15),

$$|x' - x| \lesssim \epsilon r_{k-1} \lesssim \epsilon r_k. \quad (6.4)$$

Hence, for ϵ small enough, $x' \in V_{k-1}^{12}$. Thus, there is $j \in J_{k-1}$ so that $x' \in 12B_{j,k-1}$. By (2.12), $\text{dist}(x', P_{j,k-1}) \lesssim \epsilon r_k$ and $\pi_{j,k-1}(x') \in 13B_{j,k-1}$. By our choice of $P_{j,k-1}$, $\text{dist}(\pi_{j,k-1}(x'), \Sigma) \lesssim \epsilon r_{k-1}$. Combining these inequalities, we get that $\text{dist}(x, \Sigma) \lesssim \epsilon r_k$. It now remains to show the right-hand-side inequality (6.2), i.e. that $r_k \sim d_{\text{Up}(N)}(x)$. First, since $x' \in 12B_{j,k-1}$,

$$\begin{aligned} d_{\text{Up}(N)}(x) &\leq \ell(Q_{j,k-1}) + \text{dist}(x, Q_{j,k-1}) \\ &\stackrel{(6.1)}{\lesssim} r_{k-1} + |x - x'| + \text{dist}(x', Q_{j,k-1}) \\ &\stackrel{(6.4)}{\lesssim} r_{k-1} + \epsilon r_{k-1} + |x' - x_{j,k-1}| \stackrel{x' \in 12B_{j,k-1}}{\lesssim} r_{k-1} \lesssim r_k. \end{aligned}$$

Next, let $Q \in \text{Up}(N)$ be such that $d_{\text{Up}(N)}(x) \sim \ell(Q) + \text{dist}(x, Q)$. Suppose $d_{\text{Up}(N)}(x) \sim r_\ell$ for some $\ell \geq k$. Since $|x - x_Q| \lesssim d_{\text{Up}(N)}(x)$ we know $|x - x_Q| \leq Cr_\ell$ for some universal constant C , and if $\ell - k \gtrsim \log(C)$, $|x - x_Q| < r_k$. Then $x_Q \in B_{i_\ell}$ for some $i \in J_\ell^N$ (since \mathcal{X}_ℓ^N is a maximal net), so in particular, $x_Q \in V_\ell^1$. Thus, $x_Q \in V_k^2$, and hence $x \in V_k^{11}$, contradicting our choice of k . Thus, $|\ell - k|$ is bounded by a universal constant, implying $d_{\text{Up}(N)}(x) \gtrsim r_k$, we have (6.2).

To get (6.3), notice that since $x \notin V_k^{11}$ and by the maximality of k , $B(x, r_k) \subseteq (V_\ell^{10})^c$ for all $\ell \geq k$, and so σ_ℓ^N is the identity on $B(x, r_k)$ for all $\ell \geq k$. This and (2.13) imply (6.3). \square

Lemma 6.4. *For $\epsilon > 0$ small enough (depending on τ) and $x \in \Sigma^N \setminus \Sigma$, there is $Q_x^N \in \text{Stop}(N)$ for which $x \in 2B_{Q_x^N}$ and $\ell(Q_x^N) \sim d_{\text{Up}(N)}(x)$. If $x \in c_0 B_Q$ for some $Q \in \text{Stop}(N)$ (where c_0 is as in Theorem 2.9), we may set $Q_x^N = Q$.*

Proof. As in the statement, if $x \in c_0 B_Q$ for some $Q \in \text{Stop}(N)$, we just set $Q_x^N = Q$, and it is easy to check $\ell(Q_x^N) \sim d_{\text{Up}(N)}(x)$. Otherwise, let $x' \in \Sigma$ be such that

$$|x - x'| = \text{dist}(x, \Sigma) \stackrel{(6.2)}{\lesssim} \epsilon d_{\text{Up}(N)}(x). \quad (6.5)$$

Note that if $d_{\text{Up}(N)}(x) = 0$, then $x \in \Sigma$ by Lemma 6.2, which contradicts our choice of x . Thus, $d_{\text{Up}(N)}(x) > 0$. Since $d_{\text{Up}(N)}$ is 1-Lipschitz, (6.5) implies

$$\frac{1}{2}d_{\text{Up}(N)}(x) < d_{\text{Up}(N)}(x') < 2d_{\text{Up}(N)}(x) \quad (6.6)$$

if $\epsilon > 0$ is small enough. Thus, by Corollary 5.9 $x' \in Q$ for some $Q \in \text{Stop}(N)$, and since

$$|x - x'| \stackrel{(6.5)}{\lesssim} \epsilon d_{\text{Up}(N)}(x) \stackrel{(6.6)}{\lesssim} \epsilon d_{\text{Up}(N)}(x') \stackrel{(5.6)}{\lesssim} \epsilon d_{\text{Up}(N)}(Q) \stackrel{(5.21)}{\lesssim} \epsilon \tau \ell(Q),$$

we have that $x \in 2B_Q$ if $\epsilon > 0$ small (depending on τ). \square

Lemma 6.5. *Let $M' = M + 11$. Then for all $k \geq 0$,*

$$\Sigma_k^N \setminus B(0, 1 + M' r_k) = P_0 \setminus B(0, 1 + M' r_k). \quad (6.7)$$

In particular, for all $N \geq 0$,

$$\Sigma^N \setminus B(0, 1) = P_0 \setminus B(0, 1) \quad (6.8)$$

Proof. Recall by (2.9) that $\sigma_k^N(x) = x$ for $x \notin V_k^{10}$ and $k \geq 0$. Let $x_{j,k} \in \Sigma \setminus B(0, 1 + Mr_k)$. Then $x_{j,k} = x_{Q_{j,k}}$ and

$$|x_{Q_{j,k}}| \geq 1 + Mr_k \stackrel{(6.1)}{\geq} 1 + M\ell(Q_{j,k})$$

which implies $MB_{Q_{j,k}} \cap B(0, 1) = \emptyset$. By (3.3) we have

$$MB_{Q_{j,k}} \cap \Sigma = MB_{Q_{j,k}} \cap P_0$$

and thus $P_{j,k} = P_{MB_{Q_{j,k}}} = P_0$. If $x \in \Sigma_{k-1}^N \setminus B(0, 1 + (M + 10)r_k)$, then for each $j \in J_k$ with $x \in 10B_{j,k}$, we must have $x_{j,k} \in \Sigma \setminus B(0, 1 + Mr_k)$, and so

$$\pi_{j,k} = \pi_{P_0}. \quad (6.9)$$

Moreover, by (2.12), if $x \in 10B_{j,k}$ for some $j \in L_k$, then

$$|x - \pi_{j,k}(x)| \lesssim \epsilon r_k, \quad (6.10)$$

and so for $\epsilon > 0$ small enough,

$$\pi_{j,k}(x) \in 11B_{j,k}. \quad (6.11)$$

Since Σ is ϵ -Reifenberg flat, there is a plane P passing through $x_{j,k}$ so that $d_{x_{j,k}, 11r_k}(\Sigma, P) < \epsilon$, thus

$$\begin{aligned} d_{x_{j,k}, 11r_k}(P_{j,k}, P) &\stackrel{(2.33)}{=} d_{x_{j,k}, r_k/2}(P_{j,k}, P) \stackrel{(2.1)}{\lesssim} d_{x_{j,k}, r_k}(P_{j,k}, \Sigma) + d_{x_{j,k}, r_k}(\Sigma, P) \\ &\lesssim \epsilon + 11d_{x_{j,k}, 11r_k}(\Sigma, P) \lesssim \epsilon \end{aligned} \quad (6.12)$$

Thus,

$$d_{x_{j,k}, 11r_k}(P_{j,k}, \Sigma) \leq d_{x_{j,k}, 11r_k}(P_{j,k}, P) + d_{x_{j,k}, 11r_k}(P, \Sigma) \stackrel{(6.12)}{\lesssim} \epsilon$$

This and (6.11) imply $\text{dist}(\pi_{j,k}(x), \Sigma) \lesssim \epsilon r_k$, and along with (6.10) gives $\text{dist}(x, \Sigma) \lesssim \epsilon r_k$. For $\epsilon > 0$ small enough, this implies $\text{dist}(x, \Sigma) < r_k$ and hence $x \in V_k^2$. Recall the notation from Theorem 2.5 that $B_{j,k} = B(x_{j,k}, r_{j,k}/10)$ for $j \in L_k$, and that since $\{x_{j,k}\}_{j \in L_k} \subseteq \mathbb{R}^n \setminus V_k^9$,

$$\text{supp } \psi_k \subseteq \bigcup_{j \in L_k} 10B_{j,k} = \bigcup_{j \in L_k} B(x_{j,k}, r_k) \subseteq \mathbb{R}^n \setminus V_k^7,$$

and thus $\psi_k = 0$ on V_k^2 ; in particular, $\psi_k(x) = 0$. Hence

$$\sigma_k^N(x) \stackrel{(2.8)}{=} \psi_k(x)x + \sum_{j \in J_k} \theta_{j,k}(x)\pi_{j,k}(x) \stackrel{(6.9)}{=} 0 + \pi_{P_0}(x) \sum_{j \in J_k} \theta_{j,k}(x) = \pi_{P_0}(x).$$

Thus, $\sigma_k^N = \pi_{P_0}$ on $\Sigma_{k-1}^N \setminus B(0, 1 + (M + 10)r_k)$. Recall that $\Sigma_k^N = \sigma_k^N(\Sigma_{k-1}^N)$. By (2.14), $|\sigma_k^N(y) - y| \lesssim \epsilon r_k$, and so for $\epsilon > 0$ small enough,

$$\Sigma_k^N \setminus B(0, 1 + (M + 11)r_k) \subseteq \sigma_k^N(\Sigma_{k-1}^N \setminus B(0, 1 + (M + 10)r_k)) \subseteq P_0.$$

Let $M' = M + 11$ and, contrary to (6.7), assume that

$$P_0 \setminus (\Sigma_k^N \cup B(0, 1 + M'r_k)) \neq \emptyset.$$

Note that $\Sigma_k^N \setminus \overline{B(0, 1 + M'r_k)} \neq \emptyset$ since Σ_k^N is $C\epsilon$ -Reifenberg flat by Lemma 2.6. Thus, it is possible to find $x \in P_0$ and $r > 0$ so that

$$B(x, 8r) \cap \overline{B(0, 1 + M'r_k)} = \emptyset, \quad B(x, r) \cap \Sigma_k^N = \emptyset,$$

and so that there is

$$y \in \partial B(x, r) \cap \Sigma_k^N \setminus \overline{B(0, 1 + M'r_k)}.$$

As Σ_k^N is $C\epsilon$ -Reifenberg flat, there is a d -plane P' containing y so that $d_{y, 4r}(\Sigma_k^N, P') \lesssim \epsilon$. By (2.20), for $\epsilon > 0$ small enough,

$$\mathcal{H}_\infty^d(\Sigma_k^N \cap B(y, r)) > \frac{\omega_d}{2} r^d$$

and so we may find points $X = \{x_0, \dots, x_d\} \subseteq \Sigma_k^N \cap B(y, r)$ for which $\eta(X) \gtrsim_d 1$, where η is as in Lemma 2.17; by this same lemma, we can conclude that $d_{y, 4r}(P', P_0) = d_{y, r}(P', P_0) \lesssim \epsilon$. Hence,

$$d_{y, 2r}(\Sigma_k, P_0) \stackrel{(2.1)}{\lesssim} d_{y, 4r}(\Sigma_k, P') + d_{y, 4r}(P_0, P') \lesssim \epsilon.$$

Hence, there is $z \in \Sigma_k^N$ so that $|z - x| \lesssim \epsilon r$, and $z \in B(x, r)$ for $\epsilon > 0$ small enough. Since $\Sigma_k^N \setminus \overline{B(0, 1 + M'r_k)} \subseteq P_0$ and $B(x, r) \subseteq \overline{B(0, 1 + M'r_k)}^c$, this means $z \in P_0 \cap B(x, r) \subseteq (\Sigma_k^N)^c$, a contradiction.

Now (6.8) follows since Σ^N is the limit of the Σ_k^N . □

Lemma 6.6. *If $Q \in \text{Stop}(N)$ for some $N \geq 0$, then $Q \subseteq B(0, 1)$.*

Proof. Recall \mathcal{D}_0 as defined in Theorem 2.9. Let $x \in \Sigma \setminus B(0, 1)$, and $Q \subseteq R \in \text{Stop}(-1) = \mathcal{D}_0$ be such that $x \in Q$. Then

$$\mathcal{H}_\infty^d \left(\frac{M}{2} B_Q \cap \Sigma \cap P_0 \setminus B(0, 1) \right) \gtrsim \ell(Q)^d$$

and so we may find points $X = \{x_0, \dots, x_d\} \in \frac{M}{2} B_Q \cap \Sigma \cap P_0 \setminus B(0, 1)$ so that $\eta(X) \gtrsim 1$. By our choice of P_Q (see (5.1)) and since $X \subseteq P_0$, we have by Lemma 2.17 that $\angle(P_0, P_Q) \lesssim \epsilon$. In particular, this also holds if $Q = R$, and so for $\epsilon > 0$ small enough $\angle(P_R, P_Q) < \alpha$. Thus, there are no cubes from $\text{Stop}(0)$ containing x , which implies that every Q with $x \in Q \subseteq R \in \mathcal{D}_0$ is in $\text{Up}(0) \subseteq \text{Up}(N)$ for all $N \geq 0$. In particular, $Q \notin \text{Stop}(N)$ for any $N \geq 0$. □

Lemma 6.7. *For $x \in \Sigma$,*

$$\text{dist}(x, \Sigma^N) \lesssim \frac{\epsilon}{\tau} d_{\text{Up}(N)}(x). \quad (6.13)$$

Proof. If $d_{\text{Up}(N)}(x) = 0$, then $x \in \Sigma^N$ by Lemma 6.2 and there is nothing to show, so assume $d_{\text{Up}(N)}(x) > 0$. Then, by Corollary 5.9, $x \in Q$ for some $Q \in \text{Stop}(N)$. Let k be such that $Q \in \text{Up}(N)_k$, then $x_Q \in B_{jk}$ for some $j \in J_k$. Hence

$$\text{dist}(x, P_{jk}) \lesssim \epsilon r_k \stackrel{(6.1)}{\sim} \epsilon \ell(Q) \stackrel{(5.21)}{\sim} \frac{\epsilon}{\tau} d_{\text{Up}(Q)}(x).$$

By (2.12), there is $y \in \Sigma_k^N$ so that $|\pi_{jk}(x) - y| \lesssim \epsilon r_k$, and (2.15) implies $\text{dist}(y, \Sigma^N) \lesssim \epsilon r_k$. Combining these estimates gives

$$\text{dist}(x, \Sigma^N) \lesssim \epsilon r_k \sim \frac{\epsilon}{\tau} d_{\text{Up}(Q)}(x).$$

□

Lemma 6.8. For $x \in \Sigma^N$,

$$\text{dist}(x, \Sigma^{N+1}) \lesssim \frac{\epsilon}{\tau} d_{\text{Up}(N)}(x). \quad (6.14)$$

Proof. This follows from Lemmas 6.3 and 6.7. □

Let $C_2 > 1$.

Lemma 6.9. Let $0 < \tau < \tau_1(4)$. There is $M_0 = M_0(C_2) > 0$ and $\epsilon_0 = \epsilon_0(C_2) > 0$ so that for $M > M_0$ and $0 < \epsilon < \epsilon_0$, the following holds. For $N \geq 0$, $x \in \Sigma^N$, and $r > 0$, there are planes $P_{x,r}^N$ that satisfy the following.

(1) For all $x \in \Sigma^N$ and $r > 0$,

$$d_{x,r}(\Sigma^N, P_{x,r}^N) \lesssim \epsilon. \quad (6.15)$$

(2) Suppose $x \in Q \in \text{Stop}(N-1)$. For $0 < r < 2C_2\ell(Q)$,

$$\angle(P_{x,r}^N, P_Q) \lesssim_{C_2} \alpha. \quad (6.16)$$

(3) Suppose $x \notin \Sigma$ (so that $d_{\text{Up}(N)}(x) > 0$), and let Q_x^N be as in Lemma 6.4. For $k = k_N(x)$ as in Lemma 6.3 and $0 < r \leq r_k$,

$$\angle(P_{x,r}^N, P_{Q_x^N}) \lesssim_{C_2} \epsilon \quad (6.17)$$

and there is a twice-differentiable $C\epsilon$ -Lipschitz graph Γ_x^N over P_{x,r_k}^N so that

$$B(x, r_k) \cap \Sigma^N = B(x, r_k) \cap \Sigma_k^N = B(x, r_k) \cap \Gamma_x^N \quad (6.18)$$

Proof. Let $x \in \Sigma^N$ and $k = k_N(x)$. For $0 < r \leq r_k$, let Γ_x and $P_{x,r}^N = P_{x,r_k}^N$ be the graph and plane given by Theorem 2.5 (8), then (6.15) and (6.18) follow from (2.13) and (6.3). For $r > r_k$, Lemma 2.6 implies the existence of a plane $P_{x,r}^N$ satisfying (6.15) again. Thus, we just need to verify (6.16) and (6.17).

- (6.16) for $r \geq r_k$: Suppose $r \geq r_k$. Let $x' \in \Sigma$ be the closest point to x in Σ , so $x' \in Q_x^N$ by construction (of Lemma 6.4). Let $R \in \mathcal{D}$ be the largest parent such that $\ell(R) < r$. We have that for M large that $B(x', r/2) \subseteq MB_R$, and hence

$$\beta_{E,\infty}(B(x', r/2), P_R) \lesssim \epsilon. \quad (6.19)$$

Let $z \in B(x', r/2) \cap \Sigma \subseteq B(x, r)$. Then by Lemma 6.13 there is $z' \in \Sigma^N$ with

$$\begin{aligned} |z - z'| &\lesssim \epsilon d_{\text{Up}(N)}(z) \leq \epsilon(|x - x'| + |x' - z| + d_{\text{Up}(N)}(x)) \\ &\stackrel{(6.2)}{\lesssim} \epsilon(\epsilon r_k + r + r_k) \lesssim \epsilon r. \end{aligned}$$

In particular, $z \in B(x, r) \cap \Sigma^N$ for ϵ small enough. By the definition of $P_{x,r}^N$, $\text{dist}(z', P_{x,r}^N) \lesssim \epsilon r$, and so $\text{dist}(z, P_{x,r}^N) \lesssim \epsilon r$ as well. Since this holds for all $z \in B(x', r/2) \cap \Sigma$, this implies

$$\beta_{\Sigma, \infty}^d(B(x', r/2), P_{x,r}^N) \lesssim \epsilon. \quad (6.20)$$

Equations (6.19) and (6.20) and Lemmas 2.16 and 2.11 imply $\angle(P_R, P_{x,r}^N) \lesssim \epsilon$, and since $\angle(P_R, P_Q) \lesssim_{C_2} \epsilon$ (by (5.10) and $r < 2C_2\ell(Q)$), we have $\angle(P_{x,r}^N, P_Q) \lesssim_{C_2} \alpha$.

- (6.17) ($0 < r \leq r_k$): Let $0 < r \leq r_k$. Since

$$\ell(Q_x^N) \sim d_{\text{Up}(N)}(x) \sim r_k,$$

we have that for $M > 1$ large enough, $MB_{Q_x^N} \supseteq B(x, r_k)$. Note that (6.2) and (6.13) imply that $d_{x, r_k}(\Sigma, \Sigma^N) \lesssim \epsilon$ and so

$$\begin{aligned} \angle(P_{x,r}^N, P_{Q_x^N}) &= \angle(P_{x, r_k}^N, P_{Q_x^N}) \stackrel{(2.33)}{=} d_{x, r_k/4}(P_{x, r_k}^N, P_{Q_x^N}) \\ &\stackrel{(2.1)}{\lesssim} d_{x, r_k}(P_{x, r_k}^N, \Sigma^N) + d_{x, r_k}(\Sigma^N, \Sigma) + d_{x, r_k}(\Sigma, P_{Q_x^N}) \\ &\stackrel{(6.15)}{\lesssim} \epsilon + \epsilon + d_{MB_{Q_x^N}}(\Sigma, P_{Q_x^N}) \lesssim \epsilon \end{aligned}$$

- (6.16) for $r < r_k$: By the two previous cases,

$$\angle(P_{x,r}^N, P_Q) \leq \angle(P_{x,r}^N, P_{Q_x^N}) + \angle(P_{Q_x^N}, P_{x, r_k}^N) + \angle(P_{x, r_k}^N, P_Q) \lesssim_{C_2} \alpha.$$

□

Recall the definition of \mathcal{F}_N of extended stopping times before Lemma 5.13. Recall also that $D(\cdot, \cdot, \cdot)$ is a cylinder, as in equation (2.11).

Lemma 6.10. *Let $M > M_1(C_2) := \max\{M_0(2C_2), 4C_2\}$, $\tau < \tau_1(2C_2)$, $Q \in S \in \mathcal{F}_N$ and P be a d -plane such that $d_{B_Q}(P, P_Q) < \theta$. If α, θ , and ϵ are small enough (depending on C_2), then there is a $C(\alpha + \theta)$ -Lipschitz map $A_{P,Q}^N : P \rightarrow P^\perp$ that is zero outside of $P \cap B(\pi(x_Q), 2C_2\ell(Q))$ such that if $\Gamma_{P,Q}^N$ is the graph of $A_{P,Q}^N$ along P , then*

$$\Sigma^N \cap D(x_Q, P, C_2\ell(Q)) = \Gamma_{P,Q}^N \cap D(\pi_P(x_Q), P, C_2\ell(Q)). \quad (6.21)$$

If $Q = Q(S)$, we will set $A_S = A_{P,Q(S)}^N$ and $\Gamma_S = \Gamma_{P,Q(S)}^N$. If $Q = Q_x^N$ for some $x \in \Sigma^N$, then $A_{P,Q}^N$ is $C(\epsilon + \theta)$ -Lipschitz.

Proof. First, we claim that

$$\angle(P_{Q_y^N}, P_Q) \lesssim \alpha \text{ for } y \in 2C_2B_Q \cap \Sigma^N. \quad (6.22)$$

Since $Q_y^N \in \text{Stop}(N)$, there is $S' \in \mathcal{F}_N$ such that $Q_y^N \in S'$. Since $y \in 2B_{Q_y^N}$, we know $y \in 2B_{Q(S')}$ by (2.27). Since we also have $y \in 2C_2B_Q \subseteq 2C_2B_{Q(S')}$,

this implies $2C_2B_{Q(S)} \cap 2C_2B_{Q(S')} \neq \emptyset$. Hence, by Lemma 5.10 with $C_1 = 2C_2$,

$$\begin{aligned} \angle(P_Q, P_{Q_y^N}) &\leq \angle(P_Q, P_{Q(S)}) + \angle(P_{Q(S)}, P_{Q(S')}) + \angle(P_{Q(S')}, P_{Q_y^N}) \\ &\stackrel{(5.10)}{\lesssim} C_2 \alpha + \epsilon + \alpha \lesssim \alpha. \end{aligned}$$

This proves the claim.

Let $x, y \in \Sigma^N \cap B(x_Q, C_2\ell(Q))$, $r = |x - y|$, and $P_{x,r}^N$ be from Lemma 6.9. Note that

$$\angle(P_{x,r}^N, P_Q) \leq \angle(P_{x,r}^N, P_{Q_x^N}) + \angle(P_{Q_x^N}, P_Q) \stackrel{(6.17)}{\stackrel{(6.22)}{\lesssim}} C_2 \alpha. \quad (6.23)$$

Let x', y' be the projections of x and y into $P_{x,r}^N$. Then

$$|x' - x| + |y' - y| \leq c'\epsilon r \quad (6.24)$$

for some universal constant $c' > 0$. Moreover,

$$\angle(P, P_{x,r}^N) \leq \angle(P, P_Q) + \angle(P_Q, P_{x,r}^N) \stackrel{(6.23)}{<} \theta + c\alpha =: t.$$

Thus, by a bit of trigonometry, setting $t' = \sqrt{1 - (t/2)^2}$, for ϵ small enough (depending on M, α , and θ),

$$\begin{aligned} |\pi_P(x - y)| &\stackrel{(6.24)}{\geq} |\pi_P(x' - y')| - 2c'\epsilon r \geq t'|x' - y'| - 2c'\epsilon|x - y| \\ &\stackrel{(6.24)}{\geq} t'|x - y| - (1 + t')2c'\epsilon|x - y| \\ &\geq (t' - (t/2)^2)|x - y| \geq (1 - t^2/2)|x - y| \end{aligned}$$

Assume $x_Q \in P$. For $x \in \pi_P(\Sigma^N \cap B(x_Q, C_2\ell(Q)))$, we now set $A_{P,Q}^N(x) = y$ where $y \in \Sigma^N \cap B(x_Q, C_2\ell(Q))$ is such that $x = \pi_P(y)$. For $x \notin P \cap B(\pi(x_Q, 2C_2\ell(Q)))$, we set $A_{P,Q}^N = 0$. For α and θ small enough, the resulting map is $C'(\alpha + \theta)$ -Lipschitz where it is defined for some universal constant C' , and we can extend to the rest of P so that it is still $C'(\alpha + \theta)$ -Lipschitz and

$$\Sigma^N \cap D(x_Q, 2C_1\ell(Q)) = \Gamma_{P,Q}^N \cap D(x_Q, 2C_1\ell(Q)).$$

If $x_Q \notin P$, then

$$\text{dist}(x_Q, P) \leq r_{B_Q} d_{B_Q}(P, P_Q) < \theta\ell(Q)$$

and so we can apply our previous work to the plane $P - x_Q + \pi_P(x_Q)$ in place of P , then translating the resulting graph by $x_Q - \pi_P(x_Q)$. For $\theta > 0$ small enough (depending on C_2), this gives the result.

If $Q = Q_x^N$, we run the same proof, only instead of (6.23) we use (6.17) to bound $\angle(P_{x,r}^N, P_Q) \lesssim \epsilon$, and then replace each instance of α in the proof with ϵ . \square

7. THE MAP BETWEEN LAYERS

Remark 7.1. We fix $\tau > 0$ so that the results of the previous two sections hold, that is, we pick $\tau < \min\{\tau_0, \tau_1(4), \tau_2(3C_2), c_0/4\}$, where C_2 can be freely fixed to be any constant larger than $4C_0$. We explicitly fix C_2 in Remark 9.1 where it is relevant.

In this section, we will construct a map $F_N : \Sigma^N \rightarrow \Sigma^{N+1}$. First, set

$$F_N(x) = x \text{ for } x \in \Sigma^N \cap \Sigma. \quad (7.1)$$

For $x \in \Sigma^N \setminus \Sigma$, let $k = k_N(x)$, so by Lemma 6.3, $r_k \sim d_{\text{Up}(N)}(x)$ and

$$B(x, r_k) \cap \Sigma^N = B(x, r_k) \cap \Sigma_k^N.$$

In particular, $B(x, r_k) \cap \Sigma^N$ is a smooth surface. Let V_x^N be the d -dimensional tangent plane to Σ^N at x , and $W_x^N = (V_x^N)^\perp + x$.

Lemma 7.2. For $x \in \Sigma^N \setminus \Sigma$, let $k = k_N(x)$, W be a $(n-d)$ -plane passing through x with $\angle(W, W_x^N) \leq \theta$. Then for θ small enough, $W \cap B(x, 10r_k) \cap \Sigma^{N+1}$ contains exactly one point z with $|x - z| \lesssim \epsilon d_{\text{Up}(N)}(x)$.

Proof. Let V the d -plane perpendicular to W passing through x and $\Gamma = \Gamma_{V, Q_x^N}^{N+1}$. Recalling Lemma 6.9, as $x \in \Sigma^N \setminus \Sigma$, V_x^N is tangent to Γ_x^N which is a $C\epsilon$ -Lipschitz graph over P_{x, r_k}^N , we have $\angle(V_x^N, P_{x, r_k}^N) \lesssim \epsilon$. Thus,

$$\angle(P_{Q_x^N}, V_x^N) \leq \angle(P_{Q_x^N}, P_{x, r}^N) + \angle(P_{x, r}^N, V_x^N) \stackrel{(6.17)}{\lesssim} \epsilon.$$

Moreover, $\angle(V, V_x^N) = \angle(W, W_x^N) < \theta$. By Lemma 6.10, for ϵ and θ small enough, we have

$$\Sigma^{N+1} \cap D(x_{Q_x^N}, V, C_2\ell(Q_x^N)) = \Gamma_{V, Q_x^N}^{N+1} \cap D(x_{Q_x^N}, V, C_2\ell(Q_x^N)).$$

In particular,

$$\Sigma^{N+1} \cap D(x_{Q_x^N}, V, C_2\ell(Q_x^N)) \cap W = \{A_{V, Q_x^N}^{N+1}(x) + x\}.$$

Let $z = A_{V, Q_x^N}^{N+1}(x) + x$. Since $A_{V, Q_x^N}^{N+1}$ is $C(\alpha + \theta)$ -Lipschitz and vanishes outside $V \cap B(\pi_V(x_{Q_x^N}), 2C_2\ell(Q_x^N))$ by Lemma 6.10

$$|z - x| = |A_{V, Q_x^N}^{N+1}(x)| \lesssim (\alpha + \theta)\ell(Q_x^N) \sim (\alpha + \theta)d_{\text{Up}(N)}(x) \sim (\alpha + \theta)r_k$$

For α and δ small, we know that $|z - x| < \ell(Q_x^N)$, and since $x \in 2B_{Q_x^N}$ by Lemma 6.4, we have $z \in 3B_{Q_x^N}$. Then

$$\begin{aligned} |z - x| &\leq \text{dist}(z, V_x^N) \leq 3\ell(Q_x^N)d_{3B_{Q_x^N}}(\Sigma^{N+1}, V_x^N) \\ &\lesssim \ell(Q_x^N) \left(d_{6B_{Q_x^N}}(\Sigma^{N+1}, \Sigma) + d_{6B_{Q_x^N}}(\Sigma, V_x^N) \right) \\ &\lesssim \ell(Q_x^N) \left(d_{6B_{Q_x^N}}(\Sigma^{N+1}, \Sigma) + d_{12B_{Q_x^N}}(\Sigma, P_{Q_x^N}^N) + d_{12B_{Q_x^N}}(P_{Q_x^N}^N, V_x^N) \right) \\ &= \ell(Q_x^N)(I_1 + I_2 + I_3). \end{aligned}$$

Frist off, using (6.2) and (6.13) and the fact that $x \in 2B_{Q_x}^N$ and $Q_x^N \in \text{Up}(N) \subseteq \text{Up}(N+1)$, we have

$$\begin{aligned} I_1 &\leq \frac{\sup_{y \in 6B_{Q_x^N} \cap \Sigma^{N+1}} \text{dist}(y, \Sigma) + \sup_{y \in 6B_{Q_x^N} \cap \Sigma} \text{dist}(y, \Sigma^{N+1})}{6\ell(B_{Q_x^N}^N)} \\ &\lesssim \frac{\sup_{y \in 6B_{Q_x^N} \cap (\Sigma^{N+1} \cup \Sigma)} \epsilon d_{\text{Up}(N+1)}(y)}{6\ell(B_{Q_x^N}^N)} \\ &\leq \epsilon \frac{d_{\text{Up}(N+1)}(x) + 6\ell(B_{Q_x^N}^N)}{6\ell(B_{Q_x^N}^N)} \leq \epsilon \frac{2\ell(Q_x^N) + 6\ell(B_{Q_x^N}^N)}{6\ell(B_{Q_x^N}^N)} \lesssim \epsilon \end{aligned}$$

Next, by our choice of $P_{Q_x^N}$,

$$I_2 = d_{12B_{Q_x^N}^N}(\Sigma, P_{Q_x^N}^N) \lesssim d_{MB_{Q_x^N}^N}(\Sigma, P_{Q_x^N}^N) \lesssim \epsilon.$$

Finally, note that by (6.2), there is $y \in \Sigma$ with $|x - y| \lesssim \epsilon d_{\text{Up}(N)}(x) \sim \epsilon \ell(Q_x^N)$. For $\epsilon > 0$ small, enough, $y \in \Sigma \cap MB_{Q_x^N}$, and so by our choice of plane $P_{Q_x^N}$,

$$\text{dist}(x, P_{Q_x^N}^N) \leq |x - y| + \text{dist}(y, P_{Q_x^N}^N) \lesssim \epsilon \ell(Q_x^N) + \epsilon M \ell(Q_x^N) \lesssim \epsilon \ell(Q_x^N).$$

Hence, since $12B_{Q_x^N}^N \subseteq B(x, 14\ell(Q_x^N))$,

$$\begin{aligned} I_3 &\leq d_{12B_{Q_x^N}^N}(P_{Q_x^N}^N, P_{Q_x^N}^N - \pi_{P_{Q_x^N}^N}(x)) + d_{12B_{Q_x^N}^N}(P_{Q_x^N}^N - \pi_{P_{Q_x^N}^N}(x) + x, V_x^N) \\ &\lesssim \epsilon + d_{x, 14\ell(Q_x^N)}(P_{Q_x^N}^N - \pi_{P_{Q_x^N}^N}(x) + x, V_x^N) = \epsilon + \angle(P_{Q_x^N}^N, V_x^N) \lesssim \epsilon. \end{aligned}$$

Combining our estimate for I_1 , I_2 , and I_3 gives $|x - z| \lesssim \epsilon \ell(Q_x^N) \sim \epsilon d_{\text{Up}(N)}(x)$, which finishes the lemma. \square

Now we set

$$A_x^{N+1} := A_{V_x, Q_x^N}^{N+1} \quad \text{and} \quad \Gamma_x^N = \Gamma_{V_x, Q_x^N}^{N+1}.$$

By the previous lemma, applied to $W = V_x^\perp$ (and so $\theta = 0$), and using the fact that $Q_x^N \in \text{Stop}(N)$, we know A_x is $(1 + C\alpha)$ -Lipschitz by Lemma 6.10. We also set

$$F_N(x) = A_x(x) + x \in \Sigma^{N+1} \quad \text{for } x \in \Sigma^N \setminus \Sigma. \quad (7.2)$$

By (6.8),

$$F_N(x) = x \quad \text{for } x \in \Sigma^N \setminus B(0, 1) = P_0 \setminus B(0, 1) \quad (7.3)$$

and by Lemma 7.2,

$$|F_N(x) - x| \lesssim \epsilon d_{\text{Up}(N)}(x). \quad (7.4)$$

Lemma 7.3. $F_N : \Sigma^N \rightarrow \Sigma^{N+1}$ is $(1 + C\alpha)$ -bi-Lipschitz. Moreover, (and recalling that $\epsilon \ll \alpha$) there is $C > 0$ so that

$$(1 - C\epsilon)|x - y| \leq |F_N(x) - F_N(y)| \leq (1 + C\alpha)|x - y| \quad \text{for all } x, y \in \Sigma^N, \quad (7.5)$$

Proof. Let $x, y \in \Sigma^N$ and let $k = k_N(x)$. If $y \notin B(x, r_k/2)$, then since d_{Up} is 1-Lipschitz,

$$\begin{aligned}
| |F_N(x) - F_N(y)| - |x - y| | &\leq |F_N(x) - x| + |F_N(y) - y| \\
&\stackrel{(7.4)}{\lesssim} \epsilon(d_{\text{Up}(N)}(x) + d_{\text{Up}(N)}(y)) \leq \epsilon(2d_{\text{Up}(N)}(x) + |x - y|) \\
&\stackrel{(6.2)}{\lesssim} \epsilon(r_k + |x - y|) \lesssim \epsilon|x - y|
\end{aligned} \tag{7.6}$$

Thus, it suffices to consider the case that $y \in B(x, r_k/2)$. Let $x' = F_N(x)$ and $y' = F_N(y)$. See Figure 1.

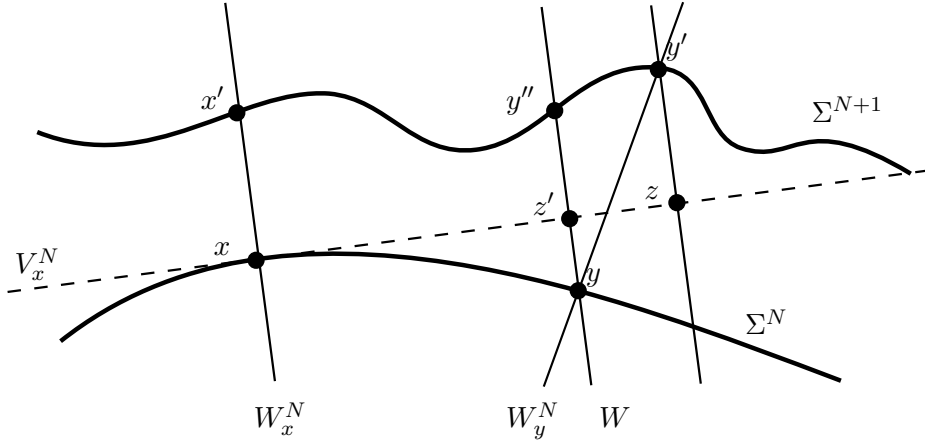


FIGURE 1. The diagram for the proof of Lemma 7.3

Let $W = W_x^N - x + y$. By Lemma 7.2, there is a (unique) $y'' \in W \cap \Sigma^{N+1}$ with

$$|y - y''| \lesssim \epsilon d_{\text{Up}(N)}(y) \leq \epsilon(|x - y| + d_{\text{Up}(N)}(x)) \lesssim \epsilon r_k.$$

Let $z = \pi_{V_x^N}(y') \in V_x^N$ and

$$z' = \pi_{V_x^N}(y) = \pi_{V_x^N}(y'') \in W \cap V_x^N.$$

Since $y \in B(x, r_k)$ and by Lemma 6.10, $B(x, r_k) \cap \Sigma^N$ is contained in the graph of some $C\epsilon$ -Lipschitz function A along V_x^N , we have

$$\begin{aligned}
|y - z'| &= |A(z')| \leq |A(x)| + \epsilon|x - z'| = 0 + \epsilon|x - z'| \\
&\leq \epsilon|x - y| + \epsilon|y - z'|
\end{aligned}$$

and hence for ϵ small

$$|y - z'| \lesssim \epsilon|x - y|. \tag{7.7}$$

Let $y''' = \pi_W(y') \in W \cap \overline{B(y, C\epsilon r_k)}$. Then

$$\begin{aligned}
|z - z'| &= |\pi_{V_x^N}(y'' - y')| = |\pi_{V_x^N}(y''' - y')| \leq |y''' - y'| \\
&\leq d_{y, C\epsilon r_k}(W, W_y^N) C\epsilon r_k = \angle(W, W_y^N) C\epsilon r_k = \angle(V_x^N, V_y^N) C\epsilon r_k \\
&\stackrel{(2.19)}{\lesssim} \frac{\epsilon}{r_k} |x - y| C\epsilon r_k = C\epsilon^2 |x - y|. \tag{7.8}
\end{aligned}$$

Hence,

$$\|x - z\| - \|x - y\| \leq \|y - z\| \leq \|y - z'\| + \|z' - z\| \stackrel{(7.7)}{\lesssim} \epsilon \|x - y\|. \tag{7.9}$$

Thus, recalling that A_x^{N+1} is $(1 + C\alpha)$ -Lipschitz,

$$\begin{aligned}
|x' - y'| &= \sqrt{|x - z|^2 + |A_x(x) - A_x(z)|^2} \leq |x - z| + |A_x(x) - A_x(z)| \\
&\leq (1 + C\alpha) |x - z| \stackrel{(7.9)}{\leq} (1 + C\alpha)(1 + C\epsilon) |x - y| \\
&\leq (1 + C\alpha) |x - y|
\end{aligned}$$

for some perhaps larger value of C in the last inequality and $\epsilon \ll \alpha$. Moreover, since $\pi_{V_x^N}(x') = x$ and $\pi_{V_x^N}(y') = z$ and projections are 1-Lipschitz

$$|x' - y'| \geq |x - z| \stackrel{(7.9)}{\geq} (1 - C\epsilon) |x - y|.$$

Since $F_N(x) = x'$ and $F_N(y) = y'$, these two inequalities finish the proof of (7.5). \square

8. THE TELESCOPING SUM

The proof Theorem 3.6 will be completed in Section 8 and 9. The main objective of this section is to prove the following proposition.

Proposition 8.1.

$$\sum_{N \geq 0} \sum_{Q \in \text{Stop}(N)} \ell(Q)^d \lesssim_{d, \epsilon, \alpha} \mathcal{H}^d(\Sigma \cap B(0, 1)). \tag{8.1}$$

This will follow from the lemmas below. For $Q \in \text{Stop}(N)$, set

$$\begin{aligned}
a_Q &= \mathcal{H}^d(\Sigma^{N+1} \cap D(x_Q, P_Q, C_2\ell(Q))) - |B(x_Q, C_2\ell(Q)) \cap P_Q| \\
&= \mathcal{H}^d(\Gamma_Q^{N+1} \cap D(x_Q, P_Q, C_2\ell(Q))) - |B(x_Q, C_2\ell(Q)) \cap P_Q|
\end{aligned}$$

Lemma 8.2. For $0 < \epsilon \ll \alpha^4$,

$$\sum_{N \geq 0} \sum \{\ell(Q)^d : Q \in \text{Stop}(N), \sqrt{\epsilon} \ell(Q)^d > a_Q\} \lesssim_d \mathcal{H}^d(\Sigma \cap B(0, 1)). \tag{8.2}$$

Proof. Let $Q \in \text{Stop}(N)$, and assume $a_Q < \sqrt{\epsilon} \ell(Q)^d$.

Let $R \in m(S_Q)$. By Lemma 5.12, there is $R' \in \text{Stop}(N + 1)$ with $R' \subseteq R$, $\ell(R) \sim \ell(R')$, and $\angle(P_{R'}, P_Q) \gtrsim \alpha$. By Lemma 6.7, there is $x \in \Sigma^{N+1}$ so that $|x_{R'} - x| \lesssim \epsilon \ell(R')$, so in particular, $x \in c_0 B_{R'}$ for ϵ small enough. By Lemma

6.4. $Q_x^{N+1} = R'$. Let $k = k_{N+1}(x)$. By Lemma 6.9, $B(x, r_k) \cap \Sigma^{N+1} = B(x, r_k) \cap \Gamma_x^{N+1}$ for some Lipschitz function along P_{x, r_k}^{N+1} where

$$\angle(P_{x, r_k}^{N+1}, P_{Q_x^N}) = \angle(P_{x, r_k}^{N+1}, P_{R'}) \lesssim \epsilon.$$

Thus, for $\epsilon \ll \alpha$,

$$\angle(P_{x, r_k}^{N+1}, P_Q) \geq \angle(P_{R'}, P_Q) - \angle(P_{R'}, P_{x, r_k}^{N+1}) \gtrsim \alpha - \epsilon \gtrsim \alpha.$$

Since Γ_x^{N+1} is $C\epsilon$ -Lipschitz and C^2 , the tangent plane at any $y \in B(x, r_k) \cap \Sigma^{N+1}$ to Σ^{N+1} has angle ϵ from $P_{x, r}^{N+1}$, and thus $\gtrsim \alpha$ from $P_{R'}$ for ϵ small. Thus, $|DA_{Q, P_Q}^{N+1}(\pi_Q(y))| \gtrsim \alpha$ for all $y \in B(x, r_k) \cap \Sigma^{N+1}$, so in particular, for all $y \in \frac{c_0}{2}B_{R'} \cap \Sigma^{N+1}$. For α small enough, we have that $\{\pi_Q(\frac{c_0}{2}B_{R'} \cap \Sigma^{N+1}) : R \in m(S_Q)\}$ are disjoint sets contained in $2B_Q \cap P_Q$. If we set

$$f_Q = x + A_{Q, P_Q}^{N+1}(x),$$

then

$$\begin{aligned} \sum_{R \in m(S_Q)} \ell(R)^d &\sim \sum_{R \in m(S)} \left| \pi_Q \left(\frac{c_0}{2} B_R \cap \Sigma^{N+1} \right) \right| \\ &\lesssim \alpha^{-2} \int_{2B_Q \cap P_Q} |DA_{Q, P_Q}^{N+1}|^2 \\ &\lesssim \alpha^{-2} \int_{2B_Q \cap P_Q} (\sqrt{1 + |DA_{Q, P_Q}^{N+1}|^2} - 1) \\ &= \alpha^{-2} \int_{2B_Q \cap P_S} (J_{f_Q} - 1) \\ &= \alpha^{-2} \left(\mathcal{H}^d(\Gamma_{Q, P_Q}^{N+1} \cap D(x_Q, P_Q, C_2\ell(Q))) \right. \\ &\quad \left. - |B(x_Q, C_2\ell(Q)) \cap P_Q| \right) \\ &\leq \alpha^{-2} a_Q \lesssim \frac{\sqrt{\epsilon}}{\alpha^2} \ell(Q)^d \end{aligned}$$

Thus, since $S_Q \subseteq S'_Q$, for $\epsilon \ll \alpha^4$, (and recalling that $z(S)$ which was defined in (5.2))

$$\begin{aligned} \mathcal{H}^d(z(S'_Q)) &\geq \mathcal{H}^d(z(S_Q)) \geq \mathcal{H}_\infty^d(Q) - \mathcal{H}_\infty^d \left(\bigcup_{R \in m(S_Q)} R \right) \\ &\gtrsim \ell(Q)^d - C \frac{\sqrt{\epsilon}}{\alpha^2} \ell(Q)^d \gtrsim \ell(Q)^d. \end{aligned}$$

Since the $\{z(S) : S \in \mathcal{F}_N, N \geq 0\}$ are mutually disjoint by Lemma 5.13 and $z(S(Q)) \subseteq Q \subseteq B(0, 1)$ for each $Q \in \text{Stop}(N)$ by Lemma 6.6,

$$\sum_{N \geq 0} \sum_{\substack{Q \in \text{Stop}(N) \\ \sqrt{\epsilon} \ell(Q)^d > a_Q}} \ell(Q)^d \lesssim \sum_{N \geq 0} \sum_{\substack{Q \in \text{Stop}(N) \\ \sqrt{\epsilon} \ell(Q)^d > a_Q}} \mathcal{H}^d(z'(S_Q)) \leq \mathcal{H}^d(\Sigma \cap B(0, 1))$$

□

Remark 8.3. We now fix $\alpha > 0$ so that the previous lemma holds.

While a_Q was the difference in area from a flat disk, we will need another quantity, t_Q , which is the local difference in area between Σ^{N+1} and Σ^N . Specifically,

$$\begin{aligned} t_Q &= \left(\mathcal{H}^d(F_N(C_1 B_Q \cap \Sigma^N)) - \mathcal{H}^d(C_1 B_Q \cap \Sigma^N) \right) \\ &= \int_{C_1 B_Q \cap \Sigma^N} (J_{F_N}(x) - 1) d\mathcal{H}^d(x). \end{aligned}$$

Lemma 8.4. For $Q \in \text{Stop}(N)$,

$$a_Q \lesssim t_Q + \epsilon \ell(Q)^d. \quad (8.3)$$

Proof. For $x \in \Sigma^N \cap B(x_Q, C_1 \ell(Q))$, let

$$f(x) = \pi_{P_Q} \circ F_N(x).$$

Our goal is to show that

$$|Df(x) - I| \lesssim \epsilon \quad (8.4)$$

since then $|J_{f^{-1}}(x) - 1| \lesssim \epsilon$ and so

$$\begin{aligned} a_Q &= \mathcal{H}^d(\Sigma^{N+1} \cap D(x_Q, P_Q, C_2 \ell(Q))) - |B(x_Q, C_2 \ell(Q)) \cap P_Q| \\ &= \mathcal{H}^d(\Sigma^{N+1} \cap D(x_Q, P_Q, C_2 \ell(Q))) \\ &\quad - |F_N^{-1}(\Sigma^{N+1} \cap D(x_Q, P_Q, C_2 \ell(Q)))| \\ &\quad + |F_N^{-1}(\Sigma^{N+1} \cap D(x_Q, P_S, C_2 \ell(Q)))| \\ &\quad - |B(x_Q, C_2 \ell(Q)) \cap P_Q| \\ &\leq t_Q + \int_{B(x_Q, C_2 \ell(Q)) \cap \Sigma^N} (J_{f^{-1}}(x) - 1) d\mathcal{H}^d(x) \\ &\lesssim t_Q + \epsilon \mathcal{H}^d(B(x_Q, C_2 \ell(Q)) \cap \Sigma^N) \\ &\stackrel{(6.21)}{\leq} t_Q + \epsilon \mathcal{H}^d(\Gamma_{P_Q, Q}^N \cap D(x_Q, P_Q, C_2 \ell(Q))) \\ &\lesssim t_Q + \epsilon \ell(Q)^d. \end{aligned}$$

Thus, it remains to show (8.4). We will again use Figure 1. Let $y \in B(x, r_k)$, $x' = F_N(x)$, $y' = F_N(y)$, and $z = \pi_{V_x^N}(y') \in V_x^N$. Since $x - z = \pi_{V_x^N}(x' - y')$,

$\angle(V_x^N, P_Q) \lesssim \epsilon$, and A_x is $C\alpha$ -Lipschitz, we have

$$\begin{aligned}
|(f(x) - f(y)) - (x - y)| &= |\pi_{P_Q}(x' - y') - (x - y)| \\
&\leq |\pi_{P_Q}(x' - y') - (x - z)| + |z - y| \\
&\leq |\pi_{P_Q}(x' - y') - \pi_{P_Q}(\pi_{V_x^N}(x' - y'))| \\
&\quad + |\pi_{P_Q}(x - z) - (x - z)| + |z - y| \\
&= |\pi_{P_Q}(\pi_{W_x^N}(x' - y'))| + |\pi_{P_Q^\perp}(x - z)| + |z - y| \\
&\stackrel{(2.35)}{\lesssim} \epsilon |\pi_{W_x^N}(x' - y')| + \epsilon |x - z| + |z - y| \\
&\leq \epsilon |x' - y'| + \epsilon |x - z| + |z - y| \\
&\stackrel{(7.9)}{\lesssim} \epsilon |F_N(x) - F_N(y)| + \epsilon |x - y| \stackrel{(7.6)}{\lesssim} \epsilon |x - y|
\end{aligned}$$

which implies (8.4). □

Lemma 8.5.

$$\sum_{N \geq 0} \sum \{\ell(Q)^d : Q \in \text{Stop}(N), a_Q \geq \sqrt{\epsilon} \ell(Q)^d\} \lesssim \mathcal{H}^d(B(0, 1) \cap \Sigma).$$

Proof. Note that by (8.3) that $a_Q \geq \sqrt{\epsilon} \ell(Q)^d$ implies

$$\begin{aligned}
\sqrt{\epsilon} \ell(Q)^d &\lesssim t_Q = \left(\mathcal{H}^d(F_N(C_1 B_Q \cap \Sigma^N)) - \mathcal{H}^d(C_1 B_Q \cap \Sigma^N) \right) \\
&= \int_{C_1 B_Q \cap \Sigma^N} (J_{F_N}(x) - 1) d\mathcal{H}^d(x).
\end{aligned}$$

By Lemma 5.10, The balls $C_1 B_Q$ have bounded overlap with constant depending on n and are contained in $C_1 B_{Q_0}$ by (2.27). Also, by (6.13) and (5.21), $\{C_1 B_Q : Q \in \text{Stop}(N)\}$ form a cover of Σ_N . Finally, recalling (7.3), $F_N(x) = x$ and hence

$J_{F_N}(x) = 1$ for $x \notin B(0, 1)$. These facts imply

$$\begin{aligned}
\sum_{\substack{Q \in \text{Stop}(N) \\ a_Q \geq \sqrt{\epsilon} \ell(Q)^d}} \sqrt{\epsilon} \ell(Q)^d &\lesssim \sum_{Q \in \text{Stop}(N)} \int_{C_1 B_Q \cap \Sigma^N} (J_{F_N}(x) - 1) d\mathcal{H}^d(x) \\
&\leq \sum_{Q \in \text{Stop}(N)} \int_{C_1 B_Q \cap \Sigma^N} (J_{F_N}(x) - 1)_+ d\mathcal{H}^d(x) \\
&\lesssim \int_{\bigcup_{Q \in \text{Stop}(N)} C_1 B_Q \cap \Sigma^N} (J_{F_N}(x) - 1)_+ d\mathcal{H}^d(x) \\
&= \int_{B(0,1) \cap \Sigma^N} (J_{F_N}(x) - 1)_+ d\mathcal{H}^d(x) \\
&\leq \int_{B(0,1) \cap \Sigma^N} (J_{F_N}(x) - 1) d\mathcal{H}^d(x) \\
&\quad + \int_{\bigcup_{Q \in \text{Stop}(N)} C_1 B_Q \cap \Sigma^N} (J_{F_N}(x) - 1)_- d\mathcal{H}^d(x) \\
&\leq \int_{B(0,1) \cap \Sigma^N} (J_{F_N}(x) - 1) d\mathcal{H}^d(x) \\
&\quad + \sum_{Q \in \text{Stop}(N)} \int_{C_1 B_Q \cap \Sigma^N} (J_{F_N}(x) - 1)_- d\mathcal{H}^d(x)
\end{aligned}$$

Note that by (7.5) that $J_{F_N} \geq 1 - C\epsilon$, so in particular, $(J_{F_N} - 1)_- \lesssim \epsilon$. Hence, we have

$$\begin{aligned}
\sum_{\substack{Q \in \text{Stop}(N) \\ a_Q \geq \sqrt{\epsilon} \ell(Q)^d}} \sqrt{\epsilon} \ell(Q)^d &\lesssim \int_{B(0,1) \cap \Sigma^{N+1}} (J_{F_N}(x) - 1) d\mathcal{H}^d(x) \\
&\quad + \sum_{Q \in \text{Stop}(N)} \int_{C_1 B_Q \cap \Sigma^N} (J_{F_N}(x) - 1)_- d\mathcal{H}^d(x) \\
&\lesssim \mathcal{H}^d(B(0, 1) \cap \Sigma^{N+1}) - \mathcal{H}^d(B(0, 1) \cap \Sigma^N) \\
&\quad + \sum_{Q \in \text{Stop}(N)} \epsilon \ell(Q)^d \\
&\leq \mathcal{H}^d(B(0, 1) \cap \Sigma^{N+1}) - \mathcal{H}^d(B(0, 1) \cap \Sigma^N) \\
&\quad + \sum_{\substack{Q \in \text{Stop}(N) \\ a_Q \geq \sqrt{\epsilon} \ell(Q)^d}} \epsilon \ell(Q)^d + \sum_{\substack{Q \in \text{Stop}(N) \\ a_Q < \sqrt{\epsilon} \ell(Q)^d}} \epsilon \ell(Q)^d.
\end{aligned}$$

Thus, for $\epsilon > 0$ small enough, and summing over N , this implies

$$\begin{aligned} \sum_{N \geq 0} \sum_{\substack{Q \in \text{Stop}(N) \\ a_Q \geq \sqrt{\epsilon} \ell(Q)^d}} \sqrt{\epsilon} \ell(Q)^d &\lesssim \sum_{N \geq 0} \mathcal{H}^d(B(0, 1) \cap \Sigma^{N+1}) - \mathcal{H}^d(B(0, 1) \cap \Sigma^N) \\ &\quad + \sum_{N \geq 0} \sum_{\substack{Q \in \text{Stop}(N) \\ a_Q < \sqrt{\epsilon} \ell(Q)^d}} \epsilon \ell(Q)^d \end{aligned}$$

The second sum is controlled by (8.2), so we just need to control the first. To this end, observe that for $L \in \mathbb{N}$,

$$\begin{aligned} \sum_{N=0}^{L-1} \left(\mathcal{H}^d(F_N(B(0, 1) \cap \Sigma^N)) - \mathcal{H}^d(B(0, 1) \cap \Sigma^N) \right) \\ \lesssim_d \mathcal{H}^d(B(0, 1) \cap \Sigma^L) - \mathcal{H}^d(B(0, 1)) \end{aligned}$$

It now suffices to show

$$\limsup_{L \rightarrow \infty} \mathcal{H}^d(B(0, 1) \cap \Sigma^L) \lesssim \mathcal{H}^d(B(0, 1) \cap \Sigma). \quad (8.5)$$

Let $x \in \Sigma^L \setminus \Sigma \cap B(0, 1)$. By Lemma 6.9, $B(x, r_{k_N(x)}) \cap \Sigma^L$ is contained in a $C\epsilon$ -Lipschitz graph, and thus $\mathcal{H}^d(B(x, r_{k_N(x)}) \cap \Sigma^L) \lesssim r_{k_N(x)}^d$. By the Vitali covering theorem, there is a collection of disjoint balls $B_j := B(x_j, r_{k_N(x_j)}/5)$ so that $5B_j$ cover $\Sigma^N \setminus \Sigma$. Moreover, as $\text{dist}(x_j, \Sigma) \lesssim \epsilon r_{k_N(x_j)}$, for ϵ small enough there is $x'_j \in \Sigma$ so that $B(x'_j, r_{k_N(x_j)}/10) \subseteq B_j$. Since $r_{k_N(x)} \leq 1$ for all $x \in \Sigma^N$, and $x_j \in B(0, 1)$,

$$B_j \subseteq B\left(0, 1 + \frac{1}{5}\right) \subseteq B(0, 2)$$

and so

$$\begin{aligned} \mathcal{H}^d(B(0, 1) \cap \Sigma^L \setminus \Sigma) &\leq \sum \mathcal{H}^d(5B_j \cap \Sigma^L) \lesssim \sum r_{k(x_j)}^d \\ &\lesssim \sum_j \mathcal{H}^d(B(x'_j, r_{k(x_j)}/10) \cap \Sigma) \\ &\leq \mathcal{H}^d\left(\bigcup B_j \cap \Sigma\right) \leq \mathcal{H}^d(B(0, 2) \cap \Sigma) \end{aligned} \quad (8.6)$$

Also,

$$\begin{aligned} \mathcal{H}^d(\Sigma \cap (B(0, 2) \setminus B(0, 1))) &= \mathcal{H}^d(P_0 \cap (B(0, 2) \setminus B(0, 1))) \\ &\lesssim 1 \stackrel{(2.20)}{\lesssim} \mathcal{H}^d(B(0, 1) \cap \Sigma) \end{aligned} \quad (8.7)$$

Thus,

$$\begin{aligned} \mathcal{H}^d(B(0, 1) \cap \Sigma^L) &\leq \mathcal{H}^d(B(0, 1) \cap \Sigma) + \mathcal{H}^d(B(0, 1) \cap \Sigma^L \setminus \Sigma) \\ &\stackrel{(8.6)}{\lesssim} \mathcal{H}^d(B(0, 2) \cap \Sigma) \stackrel{(8.7)}{\lesssim} \mathcal{H}^d(B(0, 1) \cap \Sigma). \end{aligned}$$

□

Proof of Proposition 8.1. The proposition now follows from Lemmas 8.2 and 8.5. \square

9. REDUCING β 'S TO Ω 'S AND THE COMPLETION OF THE PROOF OF THEOREM 3.6

Let $1 \leq p < p(d)$, where $p(d)$ is as in (1.4). Recall the definition of $\beta_{\Sigma}^{d,p}$ in Definition 1.6 (as well as the content of subsection 2.1). We will suppress the superscript d from our notation in this section. The estimates here are modelled on those in [DS91, Chapter 15].

Let C_0 as in Theorem 3.6.

Let $S \in \mathcal{F}^N$. For $Q \in S$, let $x'_Q \in \Sigma^N$ be closest to x_Q and $B'_Q = B(x'_Q, \ell(Q))$. Then

$$\begin{aligned} \sum_{Q \in S} \beta_{\Sigma}^{d,p} (C_0 B_Q)^2 \ell(Q)^d &\stackrel{(2.47)}{\lesssim} \sum_{Q \in S} \beta_{\Sigma^N}^{d,p} (2C_0 B'_Q)^2 \ell(Q)^d \\ &+ \sum_{Q \in S} \left(\frac{1}{\ell(Q)^d} \int_{2C_0 B_Q \cap \Sigma} \left(\frac{\text{dist}(y, \Sigma^N)}{\ell(Q)} \right)^p d_{\mathcal{H}_{\infty}^d}(y) \right)^{\frac{2}{p}} \ell(Q)^d \\ &= I_1 + I_2. \end{aligned}$$

We begin by bounding I_2 . Note that if $y \in R$ for some $R \in \text{Stop}(N)$, then

$$\text{dist}(y, \Sigma^N) \stackrel{(6.13)}{\lesssim} \epsilon d_{U_p(N)}(y) \leq \epsilon d_{U_p(N)}(x_R) + 2\epsilon \ell(R) \lesssim \epsilon \ell(R) \quad (9.1)$$

Moreover, if $y \notin R$ for any $R \in \text{Stop}(N)$, then $d_{U_p(N)}(y) = 0$ by Corollary 5.9, and (6.13), we know $\text{dist}(y, \Sigma^N) = 0$. Combined with the fact that the cubes in $\text{Stop}(N)$ are disjoint, and by Jensen's inequality (i.e. Lemma 2.4), we know $\beta_E^{d,p} \lesssim \beta_E^{d,2}$ for $p < 2$, and so we can assume without loss of generality that $p \geq 2$. Then

$$I_2 \lesssim \epsilon \sum_{Q \in S} \ell(Q)^d \left(\sum_{\substack{R \in \text{Stop}(N) \\ R \cap 2C_0 B_Q \neq \emptyset}} \frac{\ell(R)^{d+p}}{\ell(Q)^{d+p}} \right)^{\frac{2}{p}} \leq \epsilon \sum_{Q \in S} \sum_{\substack{R \in \text{Stop}(N) \\ R \cap 2C_0 B_Q \neq \emptyset}} \frac{\ell(R)^{d\frac{2}{p}+2}}{\ell(Q)^{d(\frac{2}{p}-1)+2}} \quad (9.2)$$

Next, for a fixed $R \in \text{Stop}(N)$, if $R \cap 2C_0 B_Q \neq \emptyset$ for some $Q \in S$, then $\text{dist}(x_Q, R) \leq 2C_0 \ell(Q)$ and

$$\ell(R) \sim d_{U_p(N)}(R) \leq \ell(Q) + \text{dist}(R, Q) \leq \ell(Q) + 2C_0 \ell(Q) \leq 3C_0 \ell(Q) \quad (9.3)$$

hence, for each k ,

$$\#\{Q \in \mathcal{D}_k \cap S : 2C_0 B_Q \cap R \neq \emptyset\} \lesssim_{C_0, n} 1 \quad (9.4)$$

and this is non-zero only if $\ell(Q) \gtrsim_{C_0} \ell(R)$. Thus, by summing a geometric series, and using the fact that $p < \frac{2d}{d-2}$ implies $d(2/p - 1) + 2 > 0$ (and that this holds for

all $p \geq 1$ if $d = 1, 2$),

$$I_2 \lesssim \epsilon \sum_{\substack{R \in \text{Stop}(N) \\ R \cap 2C_0 B_{Q(S)} \neq \emptyset}} \sum_{\substack{Q \in S \\ R \cap 2C_0 B_Q \neq \emptyset}} \frac{\ell(R)^{d\frac{2}{p}+2}}{\ell(Q)^{d(\frac{2}{p}-1)+2}} \lesssim \epsilon \sum_{\substack{R \in \text{Stop}(N) \\ R \cap 2C_0 B_{Q(S)} \neq \emptyset}} \ell(R)^d. \quad (9.5)$$

Now, suppose $R \in \text{Stop}(N)$ is such that $R \cap 2C_0 B_{Q(S)} \neq \emptyset$. Let $x'_R \in \Sigma^N$ be closest to x_R , hence

$$|x'_R - x_R| \stackrel{(6.13)}{\lesssim} \epsilon d_{\text{Up}(N)}(x_R) \sim \epsilon \ell(R) \stackrel{(9.3)}{\lesssim} C_0 \ell(Q(S))$$

and so for ϵ small enough (in relation to C_0) and if $C_2 > 2$,

$$B(x'_R, \ell(R)) \subseteq D(x_{Q(S)}, P_S, C_2 \ell(Q(S)))$$

this, Lemma 6.10, and the fact that the balls $\{C_1 B_R : R \in \text{Stop}(N)\}$ have bounded overlap by Lemma 5.10 imply

$$\begin{aligned} \sum_{\substack{R \in \text{Stop}(N) \\ R \cap 2C_0 B_{Q(S)} \neq \emptyset}} \epsilon \ell(R)^d &\lesssim \sum_{\substack{R \in \text{Stop}(N) \\ R \cap 2C_0 B_{Q(S)} \neq \emptyset}} \epsilon \mathcal{H}^d(B(x'_R, \ell(R)) \cap \Gamma_S) \\ &\lesssim \epsilon \mathcal{H}^d(\Gamma_S \cap D(x_Q, P_S, C_2 \ell(Q(S)))) \lesssim \epsilon \ell(Q(S))^d. \end{aligned}$$

This combined with (9.2) and (9.5) show that

$$I_2 \lesssim \epsilon \ell(Q(S))^d. \quad (9.6)$$

We now turn to bounding I_1 . Choose a family of dyadic cubes for P_S and let $I_Q \ni \pi_{P_S}(x_Q)$ be a minimal dyadic cubes for which $\ell(I_Q) \geq 8C_0 \ell(Q)$. Note that

$$|\pi_{P_S}(x_Q) - \pi_{P_S}(x'_Q)| \leq |x_Q - x'_Q| \stackrel{(6.13)}{\lesssim} \epsilon d_{\text{Up}(N)}(x_Q) \leq \epsilon \ell(Q)$$

and so for ϵ small enough,

$$\pi_{P_S}(2C_0 B'_Q) \subseteq 3I_Q.$$

If $4C_0 < C_2$ and $\epsilon > 0$ is small, then $2C_0 B'_Q \subseteq C_2 B_{Q(S)}$, and so

$$\begin{aligned} \sum_{Q \in S} \beta_{\Sigma^N}^{d,p}(2C_0 B'_Q)^2 \ell(Q)^d &\lesssim_d \sum_{Q \in S} \beta_{\Sigma^N}^{d,p}(2C_0 B'_Q)^2 \ell(Q)^d \\ &\stackrel{(2.54)}{\lesssim_d} \sum_{Q \in S} \Omega_{A_S}^2 (3I_Q)^2 \ell(I_Q)^d \\ &= \sum_{I \subseteq P_S} \Omega_{A_S}^2 (3I)^2 \ell(I)^d \#\{Q \in S : I_Q = I\}. \end{aligned}$$

Remark 9.1. We now fix $C_2 = 1 + 4C_0$.

In the second inequality, we used the fact that A_S is $(1 + C\alpha)$ -Lipschitz. We will now show $\#\{Q \in S : I_Q = I\}$ is bounded for each dyadic cube I . Suppose

$Q, R \in S$ are such that $I_Q = I_R = I$. This implies $\ell(Q) \sim_n \ell(R)$. Then by Lemma 6.10, since π_S is $(1 + C\alpha)$ -bi-Lipschitz on $D(x_{Q(S)}, P_S, C_2\ell(Q))$,

$$\begin{aligned}
|x_Q - x_R| &\leq |x'_Q - x'_R| + \text{dist}(x_Q, \Sigma^N) + \text{dist}(x_R, \Sigma^N) \\
&\lesssim |\pi_S(x'_Q) - \pi_S(x'_R)| + \text{dist}(x_Q, \Sigma^N) + \text{dist}(x_R, \Sigma^N) \\
&\leq |\pi_S(x_Q) - \pi_S(x_R)| + 2 \text{dist}(x_Q, \Sigma^N) + 2 \text{dist}(x_R, \Sigma^N) \\
&\stackrel{(6.13)}{\lesssim} \ell(I_Q) + \epsilon d_{\text{Up}(N)}(x_Q) + \epsilon d_{\text{Up}(N)}(x_R) \\
&\lesssim \ell(I_Q) + \epsilon \ell(Q) + \epsilon \ell(R) \lesssim \ell(I).
\end{aligned}$$

Moreover, since $c_0 B_Q \cap c_0 B_R = \emptyset$ by Theorem 2.9, $|x_Q - x_R| \gtrsim \ell(Q) \sim \ell(I)$. Thus, $|x_Q - x_R| \sim \ell(I)$ for any two $Q, R \in S$ for which $I_Q = I_R = I$, and this implies $\#\{Q \in S : I_Q = I\} \lesssim_n 1$. Thus, by Theorem 1.3 and Lemma 6.10

$$\begin{aligned}
\sum_{Q \in S} \beta_{\Sigma^N}^{d,p} (2C_0 B'_Q)^2 \ell(Q)^d &\lesssim \sum_{I \subseteq P_S} \Omega_{A_S}^2 (3I)^2 \ell(I)^d \stackrel{(2.53)}{\lesssim} \|\nabla A_S\|_{L^2(P_S)}^2 \\
&\lesssim \alpha^2 \ell(Q(S))^d.
\end{aligned}$$

Thus, if $R \in \mathcal{D}_0$ and $R \cap B(0, 1) \neq \emptyset$,

$$\begin{aligned}
\sum_{Q \subseteq R} \beta_{\Sigma}^{d,p} (C_0 B_Q)^2 \ell(Q)^d &\lesssim \sum_{N \geq 0} \sum_{\substack{S \in \mathcal{F}^N \\ Q(S) \subseteq R}} \sum_{Q \in S} \beta_{\Sigma}^{d,p} (C_0 B_Q)^2 \ell(Q)^d \\
&\lesssim \sum_{N \geq 0} \sum_{\substack{S \in \mathcal{F}^N \\ Q(S) \subseteq R}} (\epsilon + \alpha^2) \ell(Q(S))^d \\
&= (\epsilon + \alpha^2) \left(\ell(R)^d + \sum_{N \geq 0} \sum_{Q \in \text{Stop}(N)} \ell(Q)^d \right) \\
&\stackrel{(8.1)}{\lesssim} 1 + \mathcal{H}^d(\Sigma \cap B(0, 1)) \lesssim \mathcal{H}^d(\Sigma \cap B(0, 1)).
\end{aligned}$$

If $R \in \mathcal{D}_0$ and $R \cap B(0, 1) = \emptyset$, then it is not hard to show that, since $\Sigma \setminus B(0, 1)$ is a plane,

$$\begin{aligned}
\sum_{Q \subseteq R} \beta_{\Sigma}^{d,p} (C_0 B_Q)^2 \ell(Q)^d &\leq \sum \{ \ell(Q)^d : Q \subseteq R, C_0 B_Q \cap B(0, 1) \neq \emptyset \} \\
&\lesssim_{C_0} 1 \lesssim \mathcal{H}^d(\Sigma \cap B(0, 1)).
\end{aligned}$$

Here we used the fact that, since $\Sigma \setminus B(0, 1)$ is a plane, $\beta_E^{d,p}(C_0 B_Q) \neq 0$ only when $C_0 B_Q \cap B(0, 1) \neq \emptyset$. For the same reason, there are at most boundedly many $R \in \mathcal{D}_0$ for which $R \cap B(0, 1) = \emptyset$ and $\sum_{Q \subseteq R} \beta_E^{d,p}(C_0 B_Q)^2 \ell(Q)^d \neq 0$ (with constant depending on C_0). Combining this with the sum in Q_0 above, we get that

$$\begin{aligned} \sum_{k \geq 0} \sum_{x \in \mathcal{X}_k} \beta_{\Sigma}^{d,p}(B(x, C_0 \rho^{-k}))^2 \rho^{-kd} &\lesssim_d \sum_{R \in \mathcal{D}_0} \sum_{Q \subseteq R} \beta_{\Sigma}^{d,p}(C_0 B_Q)^2 \ell(Q)^d \\ &\lesssim_{C_0} \mathcal{H}^d(\Sigma \cap B(0, 1)). \end{aligned}$$

This completes the proof of Theorem 3.6.

10. PROOF OF THEOREM 3.2

We first collect some notation and lemmas, which will be quite similar to that found in previous sections.

Let E satisfy the conditions of Theorem 3.2. Let \mathcal{D} be the cubes from Theorem 2.9 for $E \cap B(0, 1)$ with ρ^k -nets such that $\mathcal{X}_0 = \{0\}$. In this way, $\mathcal{D}_0 = \{Q_0\}$ where $Q_0 = B(0, 1) \cap E$ and all cubes in \mathcal{D} are contained in and partition $E \cap B(0, 1)$. Let

$$\mathcal{G} = \{Q \in \mathcal{D} : \vartheta_E(AB_Q) < \epsilon\}.$$

For a cube $Q \in \mathcal{G}$, we will construct an approximating surface for E relative to Q called Σ_Q .

Define a stopping-time region $S(Q)$ by inductively adding cubes R to $S(Q)$ if either $R = Q$ or

- (1) $R^{(1)} \in S(Q)$
- (2) $\vartheta_E(AB_{R'}) < \epsilon$ for each sibling R' of R , including R itself.

For $x \in \mathbb{R}^n$, define

$$d_Q(x) = d_{S(Q)}(x) \quad \text{and} \quad d_Q(R) = \inf_{x \in R} d_Q(x).$$

Let $C_1 > 10$. We will adjust its value as we go along but we will assume that $10 < C_1 \ll A$.

Again, for $k \geq 0$ an integer, let $s(k)$ be such that $5\rho^{s(k)} \leq r_k < 5\rho^{s(k)-1}$. We can assume that ρ is 5 times a power of 10 and so there is k_Q so that $5\rho^{s(k_Q)} = r_{k_Q}$. Set

$$S(Q)_k = \mathcal{D}_{s(k)} \cap S(Q)$$

and let $\mathcal{X}_k^Q = \{x_{j,k}\}_{j \in J_k^Q}$ be a maximal r_k -separated set of points for the set

$$\mathcal{C}_k^Q = \{x_Q : Q \in S(Q)_k\}.$$

For $j \in J_k^Q$, let $Q_{j,k} \in S(Q)_k$ be such that $x_{Q_{j,k}} = x_{j,k}$. In this way, $\mathcal{X}_k^Q = \{x_Q\}$. For each $R \in \mathcal{G}$, there is P'_R so that

$$\vartheta_E(AB_R, P'_R) < \epsilon.$$

Let P_R be the translate of P'_R so that $x_R \in P_R$, then it is not hard to show that, for $\epsilon > 0$ small enough,

$$\vartheta_E(AB_R, P_R) \lesssim \epsilon. \tag{10.1}$$

Let $P_{j,k} = P_{AB_{Q_{j,k}}}$.

The following has the same proof as Lemma 6.1.

Lemma 10.1. *For each N and $A > 10^5$, $\{x_{j,k}\}_{j \in J_k^Q}$ satisfies the conditions of Theorem 2.5.*

Let P_Q be defined as before and let σ_k^Q , Σ_k^Q , Σ^Q , and σ^Q be the functions and surfaces obtained from Theorem 2.5 with the nets \mathcal{X}_k^Q .

The following has a similar proof as Lemma 6.2.

Lemma 10.2. *If $d_Q(x) = 0$ then $x \in \Sigma^Q \cap \Sigma$*

For $x \in \Sigma^Q$, we define $k_Q(x)$ to be the maximal integer k such that $x \in V_{k-1}^{11}$.

Lemma 10.3. *For $x \in \Sigma^Q \cap C_1 B_Q$, let $k = k_Q(x)$. Then we have*

$$\text{dist}(x, E) \lesssim \epsilon r_k \sim \epsilon d_Q(x) \quad (10.2)$$

and there is a twice-differentiable $C\epsilon$ -Lipschitz graph Γ_x^Q over P_{x,r_k}^N so that

$$B(x, r_k) \cap \Sigma^Q = B(x, r_k) \cap \Sigma_k^Q = B(x, r_k) \cap \Gamma_x^Q \quad (10.3)$$

Proof. We first prove the left-hand-side inequality (10.2). There is nothing to show if $x \in E$, so assume $x \notin E$.

Suppose first that $x \notin 10B_Q$. Let $k = k_Q(x)$. Note that Σ^Q is $C\epsilon$ -flat for some constant C , thus there is a d -plane P for which $d_{C_1 B_Q}(\Sigma^Q, P) < C\epsilon$. As $\vartheta_E(C_1 B_Q) < \epsilon$, we then have $d_{C_1 B_Q}(P, P_Q) \lesssim \epsilon$, and since $C_1 > 10$, we have that $x \in C_1 B_Q$ and thus

$$\text{dist}(x, E) \lesssim_{C_1} \epsilon \ell(Q).$$

In this case, $r_k \sim_{C_1} \ell(Q)$, and we are done.

Now suppose $x \in 10B_Q$. Again, set $k = k_Q(x)$. Let $x' \in \Sigma_{k-1}^Q$ be such that $x = \lim_{K \rightarrow \infty} \sigma_{k+K}^Q \circ \cdots \circ \sigma_{k-1}^Q(x')$. Then by (2.15),

$$|x' - x| \lesssim \epsilon r_{k-1} \lesssim \epsilon r_k.$$

Hence, for ϵ small enough, $x' \in V_{k-1}^{12}$. Thus, there is $j \in J_{k-1}^Q$ so that $x' \in 12B_{j,k-1}$. By (2.12), $\text{dist}(x', P_{j,k-1}) \lesssim \epsilon r_k$ and $\pi_{j,k-1}(x') \in 13B_{j,k-1}$. By our choice of $P_{j,k-1}$, $\text{dist}(\pi_{j,k-1}(x'), E) \lesssim \epsilon r_{k-1}$. Combining these inequalities, we get that $\text{dist}(x, E) \lesssim \epsilon r_k$. It now remains to show the right-hand-side inequality (10.2), i.e. that $r_k \sim d_Q(x)$. First,

$$\begin{aligned} d_Q(x) &\leq \ell(Q_{j,k-1}) + \text{dist}(x, Q_{j,k-1}) \lesssim r_{k-1} + |x - x'| + \text{dist}(x', Q_{j,k-1}) \\ &\leq r_{k-1} + \epsilon r_{k-1} + |x' - x_{j,k-1}| \lesssim r_{k-1} \lesssim r_k. \end{aligned}$$

Next, let $Q \in S(Q)$ be such that $d_Q(x) \sim \ell(Q) + \text{dist}(x, Q)$. Suppose $d_Q(x) \sim r_\ell$ for some $\ell \geq k$. Since $|x - x_Q| \lesssim d_Q$ we know $|x - x_Q| \leq Cr_\ell$ for some universal constant C , and if $\ell - k \gtrsim \log(C)$, $|x - x_Q| < r_k$. Then $x_Q \in B_{i\ell}$ for some $i \in J_\ell^Q$ (since \mathcal{X}_ℓ^Q is a maximal net), so in particular, $x_Q \in V_\ell^1$. Thus, $x_Q \in V_k^2$, and hence $x \in V_k^{11}$, contradicting our choice of k . Thus, $|\ell - k|$ is bounded by a universal constant, implying $d_Q(x) \gtrsim r_k$, we have (10.2).

To get (10.3), notice that since $x \notin V_k^{11}$ and by the maximality of k , $B(x, r_k) \subseteq (V_\ell^{10})^c$ for all $\ell \geq k$, and so σ_ℓ^Q is the identity on $B(x, r_k)$ for all $\ell \geq k$. This and (2.13) imply (10.3). \square

Lemma 10.4. For $Q \in \mathcal{G}$,

$$\mathcal{H}^d(C_1 B_Q \cap \Sigma^Q) \lesssim \sum_{R \in m(S(Q))} \ell(R)^d + \mathcal{H}^d(\{x \in \bar{Q} : d_Q(x) = 0\}). \quad (10.4)$$

Proof. Let $\delta > 0$ (to be decided later) and

$$E_1 = \{x \in \Sigma^Q \cap C_1 B_Q : d_Q(x) > 0 \text{ and } \text{dist}(x, Q) < \delta r_{k_Q(x)}\} \quad (10.5)$$

By Besicovitch's covering theorem and Lemma 10.3, we may find balls $\{B_j = B(x_j, r_{k_Q(x_j)})\}_{j \in J}$ of bounded overlap with centers in E_1 so that

$$E_1 \subseteq \bigcup_{j \in J} B_j.$$

Let $k_j = k_Q(x_j)$ and $x'_j \in E$ be closest to x_j . Again, it's not hard to show using (10.2) that

$$d_Q(x_j) \sim d_Q(x'_j) \quad (10.6)$$

For $j \in J$, let $z_j \in Q$ be a closest point to x_j so that

$$\text{dist}(x_j, Q) = |x_j - z_j| \stackrel{(10.5)}{<} \delta r_{k_j}. \quad (10.7)$$

Then for any cube $R \in S(Q)$ containing z_j , since $x_j \in E_1$,

$$\ell(R) + \delta r_{k_j} \stackrel{(10.5)}{>} \ell(R) + \text{dist}(x_j, Q) \geq \ell(R) + \text{dist}(x_j, R) \geq d_Q(x_j) \stackrel{(10.2)}{\sim} r_{k_j}$$

and so for δ small,

$$\ell(R) \gtrsim r_{k_j},$$

thus there is $Q_j \in m(S(Q))$ containing z_j with $\ell(Q_j) \gtrsim r_{k_j}$. Moreover,

$$r_{k_j} \stackrel{(10.2)}{\sim} d_Q(x_j) \leq |x_j - z_j| + d_Q(z_j) \stackrel{(10.7)}{<} \delta r_{k_j} + \ell(Q_j)$$

which implies for $\delta > 0$ small enough that

$$r_{k_j} \sim \ell(Q_j). \quad (10.8)$$

For $R \in m(S(Q))$, consider

$$\mathcal{C}_R = \{j \in J : Q_j = R\}.$$

We claim that

$$\#\mathcal{C}_R \lesssim 1. \quad (10.9)$$

We have already seen that $\ell(Q_j) \sim r_{k_j}$, and thus $r_{k_j} \sim \ell(R)$ for all $j \in \mathcal{C}_R$. Moreover,

$$\text{dist}(x_j, R) = \text{dist}(x_j, Q_j) \leq |x_j - z_j| \stackrel{(10.7)}{<} \delta r_{k_j} \sim \delta \ell(R).$$

Since the balls $\{B_j\}_{j \in \mathcal{C}_R}$ have bounded overlap, these facts imply (10.9), which proves the claim.

These facts and (10.3) imply that

$$\begin{aligned} \mathcal{H}^d(E_1) &\leq \sum_{j \in J} \mathcal{H}^d(E_1 \cap B_j) \stackrel{(10.3)}{=} \sum_{j \in J} \mathcal{H}^d(\Sigma^Q \cap B_j) \stackrel{(10.3)}{\lesssim} \sum_{j \in J} r_{k_j}^d \\ &\leq \sum_{j \in J} \ell(Q_j)^d = \sum_{R \in m(S(Q))} \sum_{j \in \mathcal{C}_R} \ell(R)^d \stackrel{(10.9)}{\lesssim} \sum_{R \in m(S(Q))} \ell(R)^d. \end{aligned} \quad (10.10)$$

Now set

$$E_2 = \{x \in \Sigma^Q \cap C_1 \ell(Q) : d_Q(x) > 0 \text{ and } \text{dist}(x, Q) \geq \delta r_{k_Q(x)}\}. \quad (10.11)$$

Again by Besicovitch's theorem, we may find a collection of balls $\{B_j = B(x_j, r_{k_Q(x_j)})\}_{j \in I}$ with centers in E_2 of bounded overlap. Then there is $Q_j \in S(Q)$ so that $\ell(Q_j) + \text{dist}(x_j, Q_j) < 2d_Q(x_j)$. Let R_j be the maximal ancestor of Q_j so that $\ell(R_j) + \text{dist}(x_j, R_j) < 2d_Q(x_j)$, then

$$\ell(R_j) \sim d_Q(x_j) \sim r_{k_j} \text{ and } \text{dist}(x_j, R_j) < 2d_Q(x_j). \quad (10.12)$$

Let

$$B'_j = \frac{c_0}{2} \rho B_{R_j}.$$

We claim that the balls $\{B'_j\}_{j \in I}$ have bounded overlap on Σ^Q . Let $x \in \Sigma^Q$ and

$$I(x) = \{j \in I : x \in B'_j\}.$$

Let $x' \in E$ be closest to x , so for any $j \in I(x)$,

$$\begin{aligned} |x - x'| &= \text{dist}(x, E) \stackrel{(10.2)}{\lesssim} \epsilon d_Q(x) \leq \epsilon(\text{dist}(x, R_j) + \ell(R_j)) \\ &< \epsilon(r_{B'_j} + \ell(R_j)) \lesssim \epsilon \ell(R_j). \end{aligned} \quad (10.13)$$

Also, if $x'_j \in E$ is closest to x_j , then

$$|x_j - x'_j| = \text{dist}(x_j, E) \stackrel{(10.2)}{\lesssim} \epsilon r_{k_j} \stackrel{(10.12)}{\sim} \ell(R_j) \quad (10.14)$$

and so for $\epsilon > 0$ small enough depending on δ ,

$$\text{dist}(x'_j, Q) \geq \text{dist}(x_j, Q) - |x_j - x'_j| \stackrel{(10.11)}{\stackrel{(10.14)}{\geq}} \delta r_{k_j} - C\epsilon r_{k_j} \gtrsim \delta r_{k_j}. \quad (10.15)$$

Suppose $i, j \in I(x)$. Note that since $x \in B'_i \cap \Sigma^Q$,

$$\text{dist}(x, E) \stackrel{(10.2)}{\lesssim} \epsilon d_Q(x) \lesssim \epsilon \ell(R_j)$$

For $\epsilon > 0$ small enough, the closest point to x in E must be in R_j , that is, $x' \in R_j$. Then as

$$B'_j \cap E \subseteq c_0 B_Q \cap E \subseteq Q \quad (10.16)$$

and $x'_i \in E \setminus Q$ by (10.15), we have

$$\begin{aligned} \frac{c_0}{2} \ell(R_j) &\leq \text{dist}(x'_i, R_j) < \text{dist}(x'_i, B'_j \cap E) \leq |x'_i - x| \\ &\leq |x'_i - x'| + |x' - x| \stackrel{(10.13)}{\lesssim} \text{dist}(x'_i, R_i) + \ell(R_i) \stackrel{(10.12)}{\lesssim} \ell(R_i). \end{aligned}$$

Interchanging the roles of i and j , we see that $\ell(R_i) \sim \ell(R_j)$ for all $i, j \in I(x)$. Let $r(x) = \sup_{j \in I(x)} \ell(R_j)$. Then for each $j \in I(x)$, $\ell(R_j) \sim r(x)$ and

$$\text{dist}(x, R_j) \leq \text{dist}(B'_j, R_j) + 2r_{B'_j} \lesssim 0 + \ell(R_j) \leq r(x)$$

and these facts imply that $\#I(x) \lesssim 1$, and thus the balls $\{B'_j\}_{j \in I}$ have bounded overlap on Σ^Q and proves the claim.

Thus, since Σ^Q is lower regular by virtue of being Reifenberg flat (see (2.20)), this bounded overlap implies

$$\sum_{j \in I} r_{k_j}^d \lesssim \sum_{j \in I} \ell(R_j)^d \lesssim \sum_{j \in I} \mathcal{H}^d(B'_j \cap \Sigma^Q) \lesssim \mathcal{H}^d\left(\bigcup_j B'_j \cap \Sigma^Q\right) \quad (10.17)$$

Now, recall that if $x \in B'_j \cap \Sigma^Q$, then $x' \in R_j$, and hence in Q , thus

$$\text{dist}(x, E) = \text{dist}(x, Q) \lesssim \epsilon d_Q(x) \sim \epsilon r_{k_Q(x)}$$

and so for $\epsilon > 0$ small enough, this implies $\text{dist}(x, E) < \delta r_{k_Q(x)}$, and so either $x \in E_1$ or $x \in E_0$ where

$$E_0 = \{x \in C_1 B_Q \cap E : d_Q(x) = 0\}.$$

We then have

$$\bigcup_j B'_j \cap \Sigma^Q \subseteq E_0 \cup E_1 \quad (10.18)$$

Now we estimate $\mathcal{H}^d(E_2)$:

$$\begin{aligned} \mathcal{H}^d(E_2) &\leq \sum \mathcal{H}^d(E_2 \cap B_j) = \sum \mathcal{H}^d(\Sigma^Q \cap B_j) \stackrel{(10.3)}{\lesssim} \sum_{j \in I} r_{k_j}^d \\ &\stackrel{(10.17)}{\lesssim} \mathcal{H}^d\left(\bigcup_j B'_j \cap \Sigma^Q\right) \stackrel{(10.18)}{\leq} \mathcal{H}^d(E_0 \cup E_1) \\ &\stackrel{(10.10)}{\lesssim} \sum_{R \in m(S(Q))} \ell(R)^d + \mathcal{H}^d(E_0) \end{aligned}$$

Thus, combining this with (10.10), we have

$$\begin{aligned} \mathcal{H}^d(\{x \in C_1 Q \cap \Sigma^Q : d_Q(x) > 0\}) &\leq \mathcal{H}^d(E_0) + \mathcal{H}^d(E_1) + \mathcal{H}^d(E_2) \\ &\lesssim \mathcal{H}^d(E_0) + \sum_{R \in m(S(Q))} \ell(R)^d. \end{aligned}$$

Note that we actually have $E_0 \subseteq \overline{Q}$ by definition. This observation and the above inequality finish the lemma. \square

Lemma 10.5. *For $Q \subseteq Q_0$, $\mathcal{H}^d(\partial Q) = 0$, where*

$$\partial Q = \{x \in \overline{Q} : B(x, r) \cap E \setminus Q \neq \emptyset \text{ for all } r > 0\}.$$

Proof. Let $x \in \partial Q$ and R be any cube containing x with $\ell(R) \leq \ell(Q)$. Then $c_0 B_R \cap Q = \emptyset$. In particular, $\frac{c_0}{2} B_R \cap \overline{Q} = \emptyset$. Thus,

$$\mathcal{H}^d(E \cap B(x, 2\ell(R)) \setminus \partial Q) \geq \mathcal{H}^d\left(\frac{c_0}{2} B_R \cap E\right) \gtrsim \ell(R)^d.$$

Since there are such cubes R of arbitrarily small size,

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}^d(E \cap B(x, r) \setminus \partial Q)}{r^d} > 0.$$

However, by [Mat95, Corollary 2.14], this limit must be zero for \mathcal{H}^d -a.e. $x \in \partial Q$, hence $\mathcal{H}^d(\partial Q) = 0$. \square

Let

$$\mathcal{B} = \mathcal{D} \setminus \mathcal{G} = \{Q \subseteq Q_0 : \vartheta_E^d(AB_Q) \geq \epsilon\}.$$

Let Q_j be a collection of maximal cubes in \mathcal{G} . Note that since $\Theta_E^{d,A,\epsilon}(B(0,1)) < \infty$, Q_j covers almost all of Q_0 (up to \mathcal{H}^d measure zero). Let $\mathcal{F}_0 = \{S(Q_j)\}$. Assume we have defined \mathcal{F}_N and let $S \in \mathcal{F}_N$. Let $m'(S)$ be the collection of maximal cubes in \mathcal{G} contained in cubes in $m(S)$. Let

$$\mathcal{F}_{N+1} = \bigcup_{S \in \mathcal{F}_N} \{S(Q) : Q \in m'(S)\} \text{ and } \mathcal{F} = \bigcup \mathcal{F}_N.$$

For $S \in \mathcal{F}$, write $\Sigma^S = \Sigma^{Q(S)}$.

Lemma 10.6.

$$\sum_{S \in \mathcal{F}} \sum_{R \in m(S) \cap \mathcal{B}} \ell(R)^d \lesssim \Theta_E^{d,A,\epsilon}(0,1) \quad (10.19)$$

and

$$\sum_{S \in \mathcal{F}} \mathcal{H}^d(C_1 B_{Q(S)} \cap \Sigma^S) \lesssim \mathcal{H}^d(E \cap B(0,1)) + \Theta_E^{d,A,\epsilon}(0,1). \quad (10.20)$$

Proof. Write

$$d(S) = \{x \in \overline{Q(S)} : d_{Q(S)}(x) = 0\}.$$

Then

$$\begin{aligned} \sum_{S \in \mathcal{F}} \mathcal{H}^d(C_1 B_{Q(S)} \cap \Sigma^S) &\stackrel{(10.4)}{\lesssim} \sum_{S \in \mathcal{F}} \sum_{R \in m(S)} \ell(R)^d + \sum_{S \in \mathcal{F}} \mathcal{H}^d(d(S)) \\ &= I_1 + I_2. \end{aligned}$$

To bound the first term, note that for $S, S' \in \mathcal{F}$, if $S \cap S' \neq \emptyset$, then either $Q(S) \in m(S')$ or $Q(S') \in m(S)$ by construction.

Also observe also that if $Q \in m(S)$ for some $S \in \mathcal{F}$, then one of its siblings is in \mathcal{B} , and so by definition of $\Theta_E^{d,A,\epsilon}(0,1)$,

$$I_1 \lesssim \sum_{S \in \mathcal{F}} \sum_{R \in m(S) \cap \mathcal{B}} \ell(R)^d \leq \sum_{Q \in \mathcal{B}} \ell(Q)^d \leq \Theta_E^{d,A,\epsilon}(0,1).$$

Note that this proves (10.19). Now we bound the second term. Let $Q^\circ = Q \setminus \partial Q$ and note that if $S, S' \in \mathcal{F}$ and $Q(S) \subseteq T \in m(S')$, then we have $d(S) \subseteq \overline{Q(S)}$. Hence

$$d(S) \cap Q(S)^\circ \subseteq T^\circ \subseteq Q(S') \setminus d(S'),$$

and thus, for $S, S' \in \mathcal{F}$, by Lemma 10.5, $\mathcal{H}^d(d(S) \cap d(S')) = 0$.

Hence,

$$I_2 = \sum_{S \in \mathcal{F}} \mathcal{H}^d(d(S)) \leq \mathcal{H}^d(Q_0) = E \cap B(0,1).$$

□

Proposition 10.7 (Theorem A Part 1). *With the assumptions of Theorem 3.2,*

$$\begin{aligned} \sum_{k \geq 0} \sum_{B \in \mathcal{X}_k} \beta_E^{d,p}(C_0 B)^2 r_B^d \\ \lesssim_{A,C_0,n,\epsilon,p} \left(\mathcal{H}^d(E \cap B(0,1)) + \Theta_E^{d,A,\epsilon}(0,1) \right). \end{aligned} \quad (10.21)$$

Proof. Let $1 \leq p < \frac{2d}{2-d}$ if $d > 2$ and $1 \leq p < \infty$ if $1 \leq d \leq 2$. Note that

$$\sum_{Q \subseteq Q_0} \beta_E^{d,p}(C_0 B_Q)^2 \ell(Q)^d \lesssim \Theta_E^{d,A,\epsilon}(B_{Q_0}) + \sum_{S \in \mathcal{F}} \sum_{Q \in S} \beta_E^{d,p}(C_0 B_Q)^2 \ell(Q)^d$$

and so we just need to bound the second sum. For $S \in \mathcal{F}$, set

$$\Sigma^S = \Sigma^{Q(S)}.$$

If $Q \in S$, we will let x'_Q denote the closest point in Σ^S to x_Q and $B'_Q = B(x'_Q, \ell(Q))$. Then

$$\begin{aligned} \sum_{S \in \mathcal{F}} \sum_{Q \in S} \beta_E^{d,p}(C_0 B_Q)^2 \ell(Q)^d &\stackrel{(2.47)}{\lesssim} \sum_{S \in \mathcal{F}} \sum_{Q \in S} \beta_{\Sigma^S}^{d,p}(2C_0 B'_Q)^2 \ell(Q)^d \\ &\quad + \sum_{S \in \mathcal{F}} \sum_{Q \in S} \int_{2C_0 B_Q \cap E} \frac{\text{dist}(x, \Sigma^S)}{\ell(Q)} d\mathcal{H}_\infty^d(x) \\ &= I_1 + I_2. \end{aligned}$$

We first bound the second term. Let $Q \in S$. Let k be so that $\ell(Q) \leq r_k < \rho^{-1}\ell(Q)$, and let $j \in J_k$ be so that $x_Q \in B_{jk}$. By (2.12), since $x_{jk} \in P_{jk}$, $\text{dist}(x_{jk}, \Sigma_k^S) \lesssim \epsilon r_k$. If $x'_{jk} \in \Sigma_k^S$ is closest to x_{jk} , then this means $x'_{jk} \in B_{jk}$. By (2.14), $\text{dist}(x'_{jk}, \Sigma^S) \lesssim \epsilon r_k$, and so $\text{dist}(x_{jk}, \Sigma^S) \lesssim \epsilon r_k$. Thus, $\text{dist}(x_Q, \Sigma^S) \lesssim r_k \sim \ell(Q)$, and so if $x \in E \cap 2C_0 B_Q$, then $\text{dist}(x, \Sigma^S) \lesssim C_0 \ell(Q)$.

Let B_j be a Besicovitch sub-cover of the collection

$$\{B(x, 2d(x, \Sigma^S)) : x \in 2C_0B_{Q(S)} \cap E\}$$

and write $B_j = B(x_j, s_j)$. Hence, if $x'_j \in \Sigma^S$ is closest to x_j , then

$$s_j^d \lesssim \mathcal{H}^d(B(x'_j, s_j/2) \cap \Sigma^S) \leq \mathcal{H}^d(B_j \cap \Sigma^S)$$

If $Q \in S$ is such that $2C_0B_Q \cap B_j \neq \emptyset$, then $s_j = d_S(x_j) \lesssim \ell(Q)$, and so

$$\begin{aligned} \sum_{Q \in S} \int_{2C_0B_Q \cap E} \frac{\text{dist}(x, \Sigma^S)}{\ell(Q)} d\mathcal{H}_\infty^d &\stackrel{(2.2)}{\lesssim} \sum_{Q \in S} \sum_j \int_{2C_0B_Q \cap E \cap B_j} \frac{\text{dist}(x, \Sigma^S)}{\ell(Q)} d\mathcal{H}_\infty^d \\ &\lesssim \sum_{Q \in S} \sum_j \int_{2C_0B_Q \cap E \cap B_j} \frac{s_j}{\ell(Q)} d\mathcal{H}_\infty^d \lesssim \sum_{Q \in S} \sum_{2C_0B_Q \cap E \cap B_j \neq \emptyset} \frac{s_j^{d+1}}{\ell(Q)} \\ &= \sum_j \sum_{\substack{Q \in S \\ 2C_0B_Q \cap E \cap B_j \neq \emptyset}} \frac{s_j^{d+1}}{\ell(Q)} \lesssim \sum_j s_j^d \\ &\quad \sum_j \mathcal{H}^d(B_j \cap \Sigma^S) \leq \mathcal{H}^d(C_1B_{Q(S)} \cap \Sigma^S). \end{aligned}$$

Thus,

$$\begin{aligned} I_2 &= \sum_{S \in \mathcal{F}} \sum_{Q \in S} \int_{2C_0B_Q \cap E} \frac{\text{dist}(x, \Sigma^S)}{\ell(Q)} d\mathcal{H}_\infty^d(x) \lesssim \sum_{S \in \mathcal{F}} \mathcal{H}^d(C_1B_{Q(S)} \cap \Sigma^S) \\ &\stackrel{(10.20)}{\lesssim} \mathcal{H}^d(E \cap B(0, 1)) + \Theta_E^{d, A, \epsilon}(0, 1). \end{aligned}$$

To bound I_1 , fix $S \in \mathcal{F}$ and suppose $Q(S) \in \mathcal{D}_N$. For $n \geq N$, let \mathcal{X}_n be a collection of maximally separated ρ^n nets for Σ^Q . Then for $n \geq N$ and $Q \in S \cap \mathcal{D}_n$, there is $x^Q \in X_n$ so that $x'_Q \in B^Q := B(x^Q, \rho^n)$, and so $2C_0B'_Q \subseteq 20C_0B^Q$. Moreover, since the centers of cubes in \mathcal{D}_n are maximally ρ^n -separated, we know that for any $x \in \mathcal{X}_n$ that $\#\{Q \in S \cap \mathcal{D}_n : x^Q = x\} \lesssim 1$. Thus, if $20C_0 \ll C_1$, Theorem 3.6 implies

$$\begin{aligned} \sum_{Q \in S} \beta_{\Sigma^S}^{d, p} (2C_0B'_Q)^2 \ell(Q)^d &\lesssim \sum_{n \geq N} \sum_{x \in \mathcal{X}_n} \beta_{\Sigma^S}^{d, p} (B(x, 20C_0\rho^n)^2 \rho^{nd}) \\ &\lesssim \mathcal{H}^d(C_1B_{Q(S)} \cap \Sigma^S). \end{aligned}$$

Thus,

$$I_1 \lesssim \sum_{S \in \mathcal{F}} \mathcal{H}^d(C_1B_{Q(S)} \cap \Sigma^S) \stackrel{(10.20)}{\lesssim} \mathcal{H}^d(Q_0) + \Theta_E^{d, A, \epsilon}(B_{Q_0}).$$

Combining the estimates for I_1 and I_2 gives (10.21). \square

Proposition 10.8 (Theorem A Part 2). *Let $E \subseteq \mathbb{R}^d$ be closed. Then*

$$1 \lesssim_d \mathcal{H}^d(E \cap B(0, 1)) + \Theta_E^{d,A,\epsilon}(0, 1). \quad (10.22)$$

Proof. If $Q_0 \notin \mathcal{G}$, then $\Theta_E^{d,A,\epsilon}(0, 1) \geq 1$ and there is nothing to show, so we may assume $Q_0 \in \mathcal{G}$.

Suppose that $\Theta_E^{d,A,\epsilon}(0, 1) < \delta \ell(Q_0)^d$, where $\delta > 0$ will be decided shortly. Let $x \in (\Sigma^{Q_0} \setminus E) \cap \frac{c_0}{2} B_{Q_0}$. Let $x' \in E$ be closest to x , so (10.2) implies

$$|x - x'| \lesssim \epsilon d_{Q_0}(x). \quad (10.23)$$

For $\epsilon > 0$ small enough, this implies $x' \in c_0 B_{Q_0} \cap E \subseteq Q_0$. Hence, there is $R \in m(S(Q_0))$ that contains x' and $\ell(R) \geq d_{Q_0}(x)$ by definition of d_{Q_0} . Since $R \in m(S(Q_0))$, it has a sibling R' for which $\vartheta_E(AB_{R'}) \geq \epsilon$. Since $\ell(R) = \ell(R')$ and $|x_R - x_{R'}| \leq \frac{2}{\rho} \ell(R)$, we have that

$$\begin{aligned} |x_{R'} - x| &\leq |x_{R'} - x_R| + |x_R - x'| + |x' - x| < \frac{2}{\rho} \ell(R) + \ell(R') + \epsilon d_{Q_0}(x) \\ &< \left(\frac{2}{\rho} + 1 + \epsilon \right) \ell(R') \leq \frac{4}{\rho} \ell(R'). \end{aligned}$$

Thus,

$$(\Sigma^{Q_0} \setminus E) \cap \frac{c_0}{2} B_{Q_0} \subseteq \bigcup_{R \in m(S(Q_0))} \frac{4}{\rho} B_{R'}$$

and thus

$$\begin{aligned} \mathcal{H}_\infty^d \left((\Sigma^{Q_0} \setminus E) \cap \frac{c_0}{2} B_{Q_0} \right) \\ \lesssim \sum_{R \in m(S(Q_0))} \ell(R')^d \lesssim \Theta_E^{d,A,\epsilon}(B_{Q_0}) < \delta \ell(Q_0)^d. \end{aligned} \quad (10.24)$$

Since $\mathcal{X}_0 = \{0\}$, $\Sigma_0^{Q_0} = P_{Q_0}$. By (2.14), $\text{dist}(y, \Sigma^{Q_0}) \lesssim \epsilon$ for all $y \in P_{Q_0}$, and since $0 = x_{Q_0} \in P_{Q_0}$, this means $\text{dist}(0, \Sigma^{Q_0}) \lesssim \epsilon < \frac{\epsilon}{\rho} \ell(Q_0) \lesssim \epsilon$. Thus, for $\epsilon > 0$ small enough, if $x'_{Q_0} \in \Sigma^{Q_0}$ is closest to 0,

$$\mathcal{H}_\infty^d(\Sigma^{Q_0} \cap c_0 B_{Q_0}) \geq \mathcal{H}_\infty^d \left(\Sigma^{Q_0} \cap B \left(x'_{Q_0}, \frac{c_0}{2} \ell(Q_0) \right) \right) \stackrel{(2.20)}{\gtrsim} \ell(Q_0)^d \gtrsim 1.$$

so for $\epsilon > 0$ small, this and (10.24) imply

$$\mathcal{H}^d(E \cap B(0, 1)) = \mathcal{H}^d(Q_0) \geq \mathcal{H}_\infty^d(\Sigma^{Q_0} \cap E \cap c_0 B_{Q_0}) \gtrsim 1.$$

□

Combining Propositions 10.7 and 10.8 gives Theorem 3.2.

11. PROOF OF THEOREM 3.3 - PART 1

In this Section we prove half of Theorem 3.3, which is summarized in the following proposition.

Proposition 11.1. *With the assumptions of Theorem 3.3, we have*

$$\mathcal{H}^d(E \cap B(0, 1)) \leq C(A, d, n, c) \left(1 + \sum_{k \geq 0} \sum_{\substack{B \in \mathcal{X}_k \\ x_B \in B(0, 1)}} \beta_E^{d, 1}(AB)^2 r_B^d \right). \quad (11.1)$$

Furthermore, if the right hand side of (11.1) is finite then we have that E is d -rectifiable.

Let ϵ be sufficiently small for the application of Theorem 2.5. Let \mathcal{D} be the cubes from Theorem 2.9 for $E \cap B(0, 1)$ so that $\mathcal{X}_0 = \{0\}$. Again, in this way, all cubes are contained in $B(0, 1)$ and $Q_0 = B(0, 1) \cap E$.

We wish to show that for M sufficiently large (we may fix $M = 10^5$ to avoid ambiguity),

$$\mathcal{H}^d(Q_0) \leq \mathcal{H}^d\left(\bigcup_i \Sigma_{S^i}\right) \leq C_{\epsilon, n} \left(1 + \sum_{Q \subseteq Q_0} \beta_E^{d, 1}(MB_Q)^2 \ell(Q)^d \right), \quad (11.2)$$

which A large enough, gives (3.2).

We recall a result from [DT12].

Theorem 11.2 (David, Toro [DT12, Theorem 2.5]). *With the notation and assumptions of Theorem 2.5, assume additionally that for some $M < \infty$ that*

$$\sum_{k \geq 0} \epsilon'_k (f_k(z))^2 \leq M_1 \text{ for } z \in \Sigma_0$$

where

$$\epsilon'_k(z) = \sup \{ d_{x_{i,l}, 100r_l}(P_{jk}, P_{il}) : j \in J_k, |l - k| \leq 2, i \in J_l, \\ x \in 10B_{jk} \cap 11B_{il} \}.$$

Then $f = \lim f_N = \lim_{N \rightarrow \infty} \sigma_N \circ \dots \circ \sigma_0 : \Sigma_0 \rightarrow \Sigma$ is $C(M_1, n)$ -bi-Lipschitz.

For each $Q \subseteq Q_0$, we define stopping-times S^Q as follows: we starting by adding Q to S^Q and inductively on each descendant R from largest to small, we add R to S^Q if

- (1) $R^{(1)} \in S^Q$ or $R = Q$,
- (2) every sibling R' satisfies

$$\sum_{R' \subset T \subseteq Q} \beta_E^{d, 1}(MB_Q)^2 < \epsilon^2.$$

Now, using the definition of S^Q , we break up

$$\mathcal{D}(Q_0) := \{Q \in \mathcal{D} : Q \subseteq Q_0\}$$

into a collection of stopping-times \mathcal{S} with $\bigcup \mathcal{S} = \mathcal{D}(Q_0)$. Start with $\mathcal{S} = \emptyset$. First place S^{Q_0} in \mathcal{S} . Now, if S has been added to \mathcal{S} , and Q is a child of a cube in $m(S)$, add S^Q to \mathcal{S} . Repeat this indefinitely. This gives us \mathcal{S} with $\bigcup \mathcal{S} = \mathcal{D}$. Note that if $\beta_E^{d,1}(MB_Q) \geq \epsilon$, then $S^Q = \{Q\}$. We enumerate the regions $S \in \mathcal{S}$ which are *not* singletons by $\{S^i\}$.

Our plan is that each S^i will correspond to a surface Σ_{S^i} , which will be obtained from Theorem 2.5 and the will have the bi-Lipschitz estimates of [DT12, Theorem 2.5] (Theorem 11.2 above). This will use the lemmas from Section 2.4.

Let us fix such an S^i .

Lemma 11.3. *There is a surface Σ_{S^i} such that*

$$\text{dist}(R, \Sigma_{S^i}) \lesssim_n \ell(R) \quad (11.3)$$

for each $R \in S^i$, and

$$\mathcal{H}^d(\Sigma_{S^i}) \lesssim_{\epsilon, n} \ell(Q(S^i))^d$$

Proof. We will use Theorem 11.2 stated above, which is building upon Theorem 2.5. Fix $M = \Lambda = 10^5$. For $k \geq 0$ an integer, let $s(k)$ be such that $5\rho^{s(k)} \leq r_k < 5\rho^{s(k)-1}$. Let L_Q be a plane so that $\beta_E^{d,1}(MB_Q, L_Q) = \beta_E^{d,1}(MB_Q)$. Let $x'_Q \in L_Q \cap MB_Q$ be the closest point to x_Q . Associate L_Q to x'_Q and the scale r_k .

Consider a maximal r_k net for $\mathcal{C}_k = \{x'_Q : Q \in \mathcal{D}_{s(k)} \cap S^i\}$. For these maximal nets, Lemma 2.18 guarantees that the planes L_Q satisfy the assumptions of Theorem 2.5. with $\epsilon_k(x) \lesssim \beta_E^{d,1}(10^5 B_Q)$, when $x \in Q \in \mathcal{C}_k$.

We now wish to apply Theorem 11.2. To this end, let $x = f(z) \in \Sigma_{S^i}$. By (2.14), $x_N := f_N(z)$ is a Cauchy sequence in N and

$$|x - x_N| \lesssim \sum_{k \geq N} \epsilon_k(x_k) r_k \lesssim \epsilon \sum_{k \geq N} r_k \lesssim \epsilon r_N.$$

Thus, for $\epsilon > 0$ small enough, $x_N \in B(x, 2r_k)$. We then have $\epsilon'_k(x) \lesssim \beta_E^{d,1}(10^5 B_Q)$, when $x \in Q \in \mathcal{C}_k$.

Theorem 11.2 now assures that Σ_{S^i} is a bi-Lipschitz surface with constant depending on n, ϵ . Finally, note that (11.3) follows from (2.15) coupled with (2.16) and $\epsilon_k \lesssim_n \epsilon$. \square

Let $\text{min}\mathcal{S}$ be the minimal cubes of all $S \in \mathcal{S}$. In particular, for any $R \in \text{min}\mathcal{S}$, there is $S(R) \in \mathcal{S}$ such that R is a minimal cube in $S(R)$. Furthermore, any $S \in \mathcal{S}$ has $Q(S)$ either equal to Q_0 , or a cube in $\text{min}\mathcal{S}$.

Lemma 11.4.

$$\sum_{R \in \text{min}\mathcal{S}} \ell(R)^d \lesssim_{\epsilon, n} \sum_{Q \in \mathcal{D}} \beta_E^{d,1}(MB_Q)^2 \ell(Q)^d \quad (11.4)$$

Proof. Each $R \in \text{min}\mathcal{S}$ is a minimal cube in an $S \in \mathcal{S}$, which we call $S(R)$. These are disjoint for different R . Let R' be a child of R for which

$$\sum_{Q \in S(R), Q \supset R'} \beta_E^{d,1}(MB_Q)^2 \geq \epsilon^2.$$

If $\beta_E^{d,1}(MB_{R'})^2 < \epsilon^2/2$, then this implies

$$\sum_{Q \in S(R), Q \supset R} \beta_E^{d,1}(MB_Q)^2 \geq \epsilon^2/2.$$

Otherwise, if $\beta_E^{d,1}(MB_{R'})^2 \geq \epsilon^2/2$, by monotonicity this implies

$$\sum_{\substack{Q \in S(R) \\ Q \supseteq R}} \beta_E^{d,1}(MB_Q)^2 \geq \beta_E^{d,1}(MB_{R'})^2 \gtrsim \beta_E^{d,1}(MB_{R'})^2 \geq \epsilon^2/2.$$

In any case, we have

$$\sum_{\substack{Q \in S(R) \\ Q \supseteq R}} \beta_E^{d,1}(MB_Q)^2 \gtrsim \epsilon^2 \text{ for all } R \in m(S).$$

Thus

$$\begin{aligned} \sum_{R \in \min \mathcal{S}} \ell(R)^d &\leq \frac{1}{\epsilon^2} \sum_{R \in \min \mathcal{S}} \ell(R)^d \sum_{Q \in S(R), Q \supset R} \beta_E^{d,1}(MB_Q)^2 \\ &\leq \frac{1}{\epsilon^2} \sum_{S \in \mathcal{S}} \sum_{Q \in S} \beta_E^{d,1}(MB_Q)^2 \sum_{\substack{R \in \min \mathcal{S} \\ Q \in S(R), Q \supset R}} \ell(R)^d \end{aligned}$$

Now observe that for a fixed cube $Q \in \mathcal{D}$, the cubes $R \in \min \mathcal{S}$ such that $Q \in S(R)$ are disjoint, and $\mathcal{H}^d(c_0 B_R \cap \Sigma_{S(R)}) \gtrsim \ell(R)^d$ for $\epsilon > 0$ small enough, since for $\epsilon > 0$ small, $c_0 B_R$ will intersect a large portion of $\Sigma_{S(R)}$. (If S is a singleton, then $\Sigma_{S(R)}$ was not defined, but then $S = \{Q\} = \{R\}$). Thus,

$$\begin{aligned} \sum_{\substack{R \in \min \mathcal{S} \\ Q \in S(R), Q \supset R}} \ell(R)^d &\lesssim_d \sum_{\substack{R \in \min \mathcal{S} \\ Q \in S(R), Q \supset R}} \mathcal{H}^d(c_0 B_R \cap \Sigma_{S(R)}) \\ &\leq \mathcal{H}^d(B_Q \cap \Sigma_{S(R)}) \lesssim_{\epsilon, n} \ell(Q)^d \end{aligned}$$

where the constant comes from the constant in [DT12, Theorem 2.5]. Putting this all together, we have

$$\begin{aligned} \sum_{R \in \min \mathcal{S}} \ell(R)^d &\leq \frac{1}{\epsilon^2} \sum_{S \in \mathcal{S}} \sum_{Q \in S} \beta_E^{d,1}(MB_Q)^2 \sum_{\substack{R \in \min \mathcal{S} \\ Q \in S(R), Q \supset R}} \ell(R)^d \\ &\lesssim_{\epsilon, d, n} \sum_{Q \in S} \beta_E^{d,1}(MB_Q)^2 \ell(Q)^d \end{aligned}$$

as desired. \square

Let G be the points in E which are either stopped a finite number of times, or never stopped, i.e.

$$G := \{x \in E \cap B(0, 1) : \sum \beta_E^{d,1}(MB_Q)^2 \chi_Q(x) < \infty\}.$$

Then, by (11.3), $G \subset \cup_i \Sigma_{S_i}$, and we have from the above lemmas the following corollary.

Corollary 11.5.

$$\mathcal{H}^d(G) \leq \mathcal{H}^d\left(\bigcup_i \Sigma_{S^i}\right) \leq C_{\epsilon,n} \left(1 + \sum_{Q \in \mathcal{D}} \beta_E^{d,1}(MB_Q)^2 \ell(Q)^d\right).$$

Let $E' = E \cap B(0,1) \setminus G$. Then $x \in E'$ implies $x = \lim x_{i_j}$ with $x_{i_j} \in S^{i_j}$, and $Q(S^{i_{j+1}})$. More is true:

Lemma 11.6. *Let $x \in E'$. For $r > 0$, there is a cube $Q \in \min \mathcal{S}$ with $\ell(Q) < r$ such that $x \in Q$.*

Proof. This lemma is simply a restatement of the fact that E' is the set of points that were stopped infinitely many times, and the scale must decrease to 0 by the properties of $\{Q \in \mathcal{D} : Q \ni x\}$. \square

Corollary 11.7. *If the right hand side of (11.2) is finite, then $\mathcal{H}^d(E') = 0$.*

Proof. Lemma 11.4 together with the previous lemma first give that

$$\mathcal{H}^d(E') \lesssim_{\epsilon,n} \sum_{Q \in \mathcal{D}} \beta_E^{d,1}(MB_Q)^2 \ell(Q)^d.$$

However, since each point $x \in E'$ is covered by an infinite number of cubes in $\min \mathcal{S}$ we have that Lemma 11.4 gives $\mathcal{H}^d(E') = 0$. \square

We now note that $E \cap B(0,1) \subset E' \cup \bigcup_i \Sigma_{S^i}$.

Corollary 11.8. *If the right hand side of (11.2) is finite then E is d -rectifiable.*

Proposition 11.1 follows from Corollary 11.5, Corollary 11.7 and Corollary 11.8.

12. PROOF OF THEOREM 3.3 - PART 2

We continue with the same notation of Section 11. In particular, $E \subset \mathbb{R}^n$, and \mathcal{D} be the cubes from Theorem 2.9 for $E \cap B(0,1)$ so that $\mathcal{X}_0 = \{0\}$. (again, in this way, all cubes are contained in $B(0,1)$ and $Q_0 = B(0,1)$.)

Proposition 12.1. *Suppose that for all $x \in E$ and $0 < r \leq 1$ we have $\mathcal{H}_\infty^d(B(x,r) \cap E) > tr^d$. Then*

$$\sum \{\ell(Q)^d : Q \in \mathcal{D}, \vartheta_E(MB_Q) \geq \epsilon\} \lesssim_{\epsilon,n,t} 1 + \sum_{\mathcal{D}} \beta_E^{d,1}(2MB_Q)^2 \ell(Q)^d$$

The purpose of this Section is to prove the Proposition 12.1, which together with Proposition 11.1, the properties of the cubes in Theorem 2.9, and Lemma 12.2 gives Theorem 3.3.

Lemma 12.2. *If $B_1 \subset B_2$ and r_i is the radius of B_i , then $\vartheta_E(B_1)r_1 \leq \vartheta_E(B_2)r_2$.*

The previous lemma is not hard to show using definitions.

The following lemma is a simple corollary of Lemma 2.12.

Lemma 12.3. *Suppose E is such that for all $x \in E$ and $r \in (0, R)$ we have $\mathcal{H}_\infty^d(B(x, r) \cap E) > tr^d$. Then there is a $c_t > 0$ such that if $\beta_{\infty, E}(2B) > \delta$ then $\beta_E^{d,1}(B)^2 \geq c_t \delta^{\frac{1}{d+1}}$*

Lemma 12.3 immediately gives the following.

Corollary 12.4. *Under the same assumptions of Lemma 12.3, there is a $c_t > 0$ such that*

$$\begin{aligned} \sum \{ \ell(Q)^d : Q \in \mathcal{D}, \beta_{\infty, E}(MB_Q) \geq \delta \} \\ \leq \sum \{ \ell(Q)^d : Q \in \mathcal{D}, \beta_E^{d,1}(2MB_Q) \geq c_t \delta^{\frac{1}{d+1}} \} \\ \leq \frac{1}{(c_t \delta^{\frac{1}{d+1}})^2} \sum_{Q \in \mathcal{D}} \beta_E^{d,1}(2MB_Q)^2 \ell(Q)^d \end{aligned}$$

Recall that for $S \in \mathcal{S}$, Σ_S is a bi-Lipschitz topological d -plane with bi-Lipschitz constant depending only on n, ϵ .

The following lemma is a special case of [DS93b, Theorem 2.4, page.32], which holds more generally for uniformly rectifiable sets, but we will apply it to the case of bi-Lipschitz surfaces (namely, the Σ_S). We note that it is possible to make use of the fact Σ_S is a bi-Lipschitz image of a plane to prove it directly, but omit the proof here. In [DS93b], the notation $b\beta$ (bilateral β) is used for ϑ .

Lemma 12.5.

$$\sum \{ \ell(Q)^d : Q \in S, \vartheta_{\Sigma_S}(MB_Q) \geq \delta \} \leq C_\delta \mathcal{H}^d(\Sigma_S) \lesssim_{\delta, \epsilon, n} \ell(Q(S))^d$$

For a ball B centered on E and d -plane L define

$$\eta_E(B, L) := \frac{1}{r_B} \sup_{x \in L \cap B} \text{dist}(x, E)$$

and $\eta_E(B) := \inf_L \eta_E(B, L)$ where L ranges over all affine d -planes. Thus $\eta_E(\cdot) \leq \vartheta_E(\cdot)$.

Lemma 12.6. *There is a constant $C_\vartheta > 0$ (independent of dimension) such that the following holds. Let B be a ball. Suppose $\beta_{E, \infty}(B) < \delta$, and $\eta_E(B) < \delta$. Then $\vartheta_E(B) < C_\vartheta \delta$.*

Proof. Without loss of generality suppose B is the unit ball.

Let L_η a d -plane giving $\eta_E(B)$ and Let L_β a d -plane giving $\beta_{\infty, E}(B)$. Let $\mathcal{N}_\delta(F)$ denote the δ neighborhood of a set F . Then

$$\mathcal{N}_\delta(E \cap B) \supset L_\eta \cap \frac{1}{2}B. \quad (12.1)$$

Let π_{L_β} be the projection to L_β . We have that $|\pi_{L_\beta}(x) - x| \leq \delta r_B$ for $x \in E \cap B$, and so

$$\mathcal{N}_\delta(E \cap B) \subset \mathcal{N}_{2\delta}(L_\beta \cap \frac{1}{2}B). \quad (12.2)$$

Combining (12.1) and (12.2), one can show (using Lemma 2.17, for example) that $d_B(L_\eta, L_\beta) \lesssim \delta$. Hence, for $x \in E \cap B$, $\pi_{L_\beta}(x) \in B \cap L_\beta$

$$\text{dist}(x, L_\eta) \leq |x - \pi_{L_\beta}(x)| + \text{dist}(\pi_{L_\beta}(x), L_\eta) \lesssim \delta r_B.$$

Thus, $\beta_{E, \infty}(B, L_\eta) \lesssim \delta$, and since we already have $\eta_E(B, L_\eta) < \delta$, the lemma follows. \square

Proposition 12.7. *Let $S \in \mathcal{S}$ be given. Set*

$$S_\delta = \{Q \in S, \eta_E(MB_Q) > 3\delta, \eta_{\Sigma_S}(MB_Q) < \delta\}.$$

Then

$$\sum_{Q \in S_\delta} \ell(Q)^d \lesssim_{\delta, \epsilon, n} \ell(Q(S))^d.$$

Proof. Assume without loss of generality that S is not a singleton, and thus we have defined Σ_S . Let $Lip = Lip(n, \epsilon)$ be the bi-Lipschitz constant of the implicit map from \mathbb{R}^d to Σ_S . We first note that $Q \in S_\delta$ satisfies

$$\frac{1}{M\ell(Q)} \sup_{x \in \Sigma_S \cap MB_Q} \text{dist}(x, E) > \delta. \quad (12.3)$$

Indeed suppose instead that all $x \in \Sigma_S \cap MB_Q$ have $\text{dist}(x, E) \leq \delta M\ell(Q)$. Then take L_Q such that $\delta > \eta_{\Sigma_S}(MB_Q) = \eta_{\Sigma_S}(MB_Q, L_Q)$ and $p \in L_Q \cap MB_Q$. Now, by our contrapositive assumption,

$$\begin{aligned} \text{dist}(p, E) &\leq \text{dist}(p, \Sigma_S \cap MB_Q) + \delta M\ell(Q) \leq \\ &(\eta_{\Sigma_S}(MB_Q) + \delta)M\ell(Q) \leq 2\delta M\ell(Q). \end{aligned}$$

This is a contradiction to $\eta_E(MB_Q) > 3\delta$.

Now that we have (12.3) for all $Q \in S_\delta$, there is $y_Q \in MB_Q$ so that $\text{dist}(y_Q, E) \geq \delta M\ell(Q)$. Let

$$b_Q = B(y_Q, \delta M\ell(Q)/2).$$

We claim that for $Q \in S_\delta$, the balls b_Q have bounded overlap, that is,

$$\#\{Q \in S_\delta : b_Q \ni x\} \leq C(\delta, d, Lip).$$

Indeed, suppose $x \in b_Q \cap b_R$ for some $Q, R \in S_\delta$. Then

$$2M\ell(Q) \geq |y_Q - x| \geq \text{dist}(x, E) \geq \text{dist}(b_R, E) \geq \delta M\ell(R)/2$$

and so $\ell(Q) \gtrsim \ell(R)$, and reversing the rolls of Q and R gives $\ell(Q) \sim \ell(R)$. Thus, all cubes Q with $x \in b_Q$ have comparable side lengths, so in particular, if Q is another such cube and $Q \in \mathcal{D}_k$ and $R \in \mathcal{D}_\ell$, then $|k - \ell| \lesssim 1$. Since each such Q is distance at most $M\ell(Q)$ from x and the centers of each \mathcal{D}_k are ρ^k -separated, these facts imply there are boundedly many such cubes (with constant depending on d and M). This proves the claim.

Thus, with implicit constants depending on δ, d, Lip

$$\sum_{Q \in S_\delta} \ell(Q)^d \lesssim \sum_{Q \in S_\delta} \mathcal{H}^d(b_Q) \lesssim \mathcal{H}^d(\Sigma_S) \lesssim \ell(Q(S))^d$$

giving the proposition. \square

Proof of Proposition 12.1. We use Lemma 12.6 to write

$$\begin{aligned} & \{Q \in \mathcal{D} : \vartheta_E(MB_Q) \geq \epsilon\} \subset \\ & \quad \{Q \in \mathcal{D} : \eta_E(MB_Q) \geq \epsilon/C_\vartheta\} \cup \{Q \in \mathcal{D} : \beta_{\infty,E}(MB_Q) \geq \epsilon/C_\vartheta\} \\ & \quad = \mathcal{D}_\eta \cup \mathcal{D}_\beta, \end{aligned}$$

where \mathcal{D}_η and \mathcal{D}_β are the two collections of cubes on the penultimate line. We use Corollary 12.4 with $\delta = \epsilon/C_\vartheta$ for the sum over \mathcal{D}_β i.e.

$$\sum_{Q \in \mathcal{D}_\beta} \ell(Q) \lesssim_{t,\epsilon} \sum_{\mathcal{D}} \beta_E(2MB_Q)^2 \ell(Q)^d.$$

We write $\mathcal{D}_\eta = \cup_{S \in \mathcal{S}} S \cap \mathcal{D}_\eta$. For each $S \in \mathcal{S}$, Proposition 12.7 and Lemma 12.5, both with $\delta = \epsilon/(3C_\vartheta)$, give control over the sum for $S \cap \mathcal{D}_\eta$ i.e.

$$\sum_{Q \in S \cap \mathcal{D}_\eta} \ell(Q)^d \lesssim_{\epsilon,n} \ell(Q(S))^d.$$

Summing over all $S \in \mathcal{S}$ is then controlled by Lemma 11.4. This completes the proof of Proposition 12.1. \square

13. APPENDIX

Proof of Lemma 2.1. Let

$$S_j = \{x : f_j(x) = \max\{f_i(x)\} > 0\}.$$

Then $\cup \text{supp } f_i = \cup S_j$. If $x \in S_j$, then there are at most C many indices i for which $f_i > 0$, and so $\sum f_i(x) \leq C f_j(x)$. Therefore,

$$\begin{aligned} \int \left(\sum f_i\right)^p d\mathcal{H}_\infty^d &= \int \mathcal{H}_\infty^d \left(\left\{\sum f_i > \lambda\right\}\right) \lambda^{p-1} d\lambda \\ &\leq \sum_j \int \mathcal{H}_\infty^d \left(S_j \cap \left\{\sum f_i > \lambda\right\}\right) \lambda^{p-1} d\lambda \\ &\leq \sum_j \int \mathcal{H}_\infty^d (\{C f_j > \lambda\}) \lambda^{p-1} d\lambda \\ &= C^p \sum_j \int f d\mathcal{H}_\infty^d. \end{aligned}$$

\square

Proof of Lemma 2.2. Extend f to be continuous on all of \mathbb{R}^n . Since E is compact and f is continuous, the set

$$E^t := \{x \in E : f(x) \geq t\}$$

is also compact for each $t > 0$. It is not hard to show (using compactness) that if

$$E_j^t := \{x \in E_j : f(x) \geq t\}$$

then, since $\bigcap E_j^t = E^t$, we have

$$\lim_{j \rightarrow \infty} \mathcal{H}_\infty^d(E_j^t) = \mathcal{H}_\infty^d(E^t) \text{ for } t > 0.$$

Hence, by the monotone convergence theorem,

$$\int_0^\infty \mathcal{H}_\infty^d(E^t) dt = \lim_{j \rightarrow \infty} \int_0^\infty \mathcal{H}_\infty^d(E_j^t) dt. \quad (13.1)$$

Now we observe that, for any function g and F any set,

$$\begin{aligned} \int_F g d\mathcal{H}_\infty^d &= \int_F \mathcal{H}_\infty^d(\{x \in F : g(x) > t\}) dt \\ &\leq \int_F \mathcal{H}_\infty^d(\{x \in F : g(x) \geq t\}) dt \end{aligned} \quad (13.2)$$

and

$$\begin{aligned} &\int_0^\infty \mathcal{H}_\infty^d(\{x \in F : g(x) \geq t\}) dt \\ &\leq \inf_{\alpha \in (0,1)} \int_0^\infty \mathcal{H}_\infty^d(\{x \in F : g(x) > \alpha t\}) dt \\ &= \inf_{\alpha \in (0,1)} \alpha^{-1} \int_0^\infty \mathcal{H}_\infty^d(\{x \in F : g(x) > t\}) dt = \int g d\mathcal{H}_\infty^d \end{aligned} \quad (13.3)$$

Combining (13.2) and (13.3) gives

$$\int_F g d\mathcal{H}_\infty^d = \int_0^\infty \mathcal{H}_\infty^d(\{x \in F : g(x) \geq t\}) dt \quad (13.4)$$

Thus, applying this to $g = f$ and F equal to either E or E_j ,

$$\begin{aligned} \int_E f d\mathcal{H}_\infty^d &\stackrel{(13.4)}{=} \int_0^\infty \mathcal{H}_\infty^d(E^t) dt \stackrel{(2.3)}{=} \lim_{j \rightarrow \infty} \int_0^\infty \mathcal{H}_\infty^d(E_j^t) dt \\ &\stackrel{(13.4)}{\sim} \lim_{j \rightarrow \infty} \int_{E_j} f d\mathcal{H}_\infty^d \end{aligned} \quad \square$$

Proof of Lemma 2.3. First assume that E is open. Without loss of generality, we may assume $\mathcal{H}_\infty^d(E) = 1$. Note that $f\mathbb{1}_E$ is still lower semicontinuous since E is open. By the corollary on page 118 of [Ad88], for $f \geq 0$ lower semicontinuous,

$$\int f d\mathcal{H}_\infty^d \sim_n \sup \left\{ \int f d\mu : \mu \in L^{1,d}(\mathbb{R}^n), \|\mu\| = 1 \right\}. \quad (13.5)$$

Where $L^{1,d}(\mathbb{R}^n)$ is the Morrey space of Radon measures with the norm

$$\|\mu\| = \sup_{\substack{x \in \mathbb{R}^n \\ r > 0}} |\mu|(B(x,r)) r^{-d}.$$

Note that if A_i is a cover of E , then each A_i is contained in a ball of radius $\text{diam } A_i$, and so

$$\mu(E) \leq \sum \mu(A_i) \leq \sum \|\mu\| (\text{diam } A_i)^d$$

and infimizing over all such covers gives

$$\mu(E) \leq \|\mu\| \mathcal{H}_\infty^d(E) = \|\mu\| = 1.$$

Thus, if $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} \int f d\mu &\leq \left(\int f^p d\mu \right)^{\frac{1}{p}} \left(\int_E d\mu \right)^{\frac{1}{q}} \leq \left(\|\mu\| \int f^p d\mathcal{H}_\infty^d \right)^{\frac{1}{p}} \mu(E)^{\frac{1}{q}} \\ &= \left(\int f^p d\mathcal{H}_\infty^d \right)^{\frac{1}{p}}. \end{aligned}$$

Supremizing over all μ and using (13.5) once more gives (2.4).

Now assume E be a compact set and f a continuous function on E . Again, we may assume $\mathcal{H}_\infty^d(E) = 1$. Extend f to a continuous function on all of \mathbb{R}^n and let

$$E_j = \{x \in \mathbb{R}^n : \text{dist}(x, E) < j^{-1}\}.$$

Since we know (2.4) for open sets, we may apply it to the sets E_j . Since f and f^p are continuous, and since the E_j are open, contain E , and converge to E in the Hausdorff metric, we may use Lemma 2.2 and get

$$\begin{aligned} \int_E f d\mathcal{H}_\infty^d &\stackrel{(2.3)}{=} \lim_{j \rightarrow \infty} \int_{E_j} f d\mathcal{H}_\infty^d \stackrel{(2.4)}{\lesssim} \lim_{j \rightarrow \infty} \left(\int_{E_j} f^p d\mathcal{H}_\infty^d \right)^{\frac{1}{p}} \\ &\stackrel{(2.3)}{=} \left(\int_E f^p d\mathcal{H}_\infty^d \right)^{\frac{1}{p}}. \end{aligned}$$

□

Proof of Lemma 2.22. Let $\alpha' = 1 - \alpha$. We will first prove that for all $\lambda > 0$

$$\begin{aligned} \mathcal{H}_\infty^d \left(\left\{ x \in F_1 : \sum_{j \in \mathcal{J}} \mathbb{1}_{B'_j} f(z_j) > \lambda \right\} \right) \\ \lesssim \mathcal{H}_\infty^d \left(\left\{ x \in F_1 : \sum_{j \in \mathcal{J}} \mathbb{1}_{\alpha' B'_j} f(z_j) > \lambda \right\} \right). \end{aligned} \quad (13.6)$$

Let

$$A = \left\{ x \in F_2 : \sum_{j \in \mathcal{J}} \mathbb{1}_{\alpha' B'_j} f(z_j) > \lambda \right\}.$$

Let \mathcal{J} be a collection of balls covering A so that

$$\mathcal{H}_\infty^d(A) \sim_d \sum_{B \in \mathcal{J}} (2r_B)^d. \quad (13.7)$$

Let

$$\mathcal{X}^\lambda = \{j \in \mathcal{X} : f(z_j) > \lambda\}.$$

Note that as $|z_j - z'_j| < \alpha r_{B_j}$,

$$\alpha' B'_j = B(z'_j, \alpha' r_{B_j}) \subseteq B(z_j, (\alpha' + \alpha)r_{B_j}) = B_j.$$

Thus, since the balls B_j are disjoint for $j \in \mathcal{X}$, so are the balls B'_j , and hence we have

$$A = F_2 \cap \bigcup_{j \in \mathcal{X}^\lambda} \alpha' B'_j.$$

We will define a new collection of balls as the limit of a sequence of collections $\mathcal{J}(j)$ which we define inductively as follows. Assume $\mathcal{X}^\lambda = \mathbb{N}$ and set $\mathcal{J}(0) = \emptyset$. Now assume that for some $j > 0$, $\mathcal{J}(j-1)$ has already been defined. Let

$$\mathcal{J}_j = \{B \in \mathcal{J} : B \cap \alpha' B'_j \neq \emptyset\}.$$

(1) If there is $B \in \mathcal{J}_j$ for which $r_B \geq \frac{\alpha}{2} r_{B_j}$, we let

$$\mathcal{J}(j) = \mathcal{J}(j-1) \cup \left\{ \frac{4}{\alpha} B \right\}$$

and note that $\frac{4}{\alpha} B \supseteq B_j$ since $B \cap B_j \supseteq B \cap \alpha' B'_j \neq \emptyset$ and $r_{B_j} \leq \frac{2}{\alpha} r_B$ by assumption.

(2) If $r_B < \frac{\alpha}{2} r_{B_j}$ for all $B \in \mathcal{J}_j$, we let

$$\mathcal{J}(j) = \mathcal{J}(j-1) \cup \{B_j\}.$$

Note that in this case, $B \subseteq B_j$ for all $B \in \mathcal{J}_j$.

We let $\mathcal{J}' = \bigcup \mathcal{J}(j)$. In this way, every B_j is contained in a ball from \mathcal{J}' , that is,

$$\bigcup_{j \in \mathcal{X}^\lambda} B_j \subseteq \bigcup_{B \in \mathcal{J}'} B.$$

For $i = 1, 2$, let \mathcal{J}'_i be those balls in \mathcal{J}'_i added in case i and let $\mathcal{X}^{\lambda,2}$ be those j for which case 2 happened. Since the B_j are disjoint, and $B_j \supseteq \alpha' B'_j \cap F_2$,

$$\begin{aligned} \sum_{B \in \mathcal{J}} r_B^d &\geq \sum_{j \in \mathcal{X}^{\lambda,2}} \sum_{B \in \mathcal{J}(j)} r_B^d \gtrsim \sum_{j \in \mathcal{X}^{\lambda,2}} \mathcal{H}_\infty^d(\alpha' B'_j \cap F_2) \gtrsim_\alpha \sum_{j \in \mathcal{X}^{\lambda,2}} r_{B_j}^d \\ &\gtrsim \sum_{B \in \mathcal{J}'_2} r_B^d. \end{aligned} \quad (13.8)$$

Also,

$$\sum_{B \in \mathcal{J}} r_B^d \geq \left(\frac{\alpha}{4}\right)^d \sum_{B \in \mathcal{J}'_1} r_B^d \quad (13.9)$$

and hence

$$\mathcal{H}_\infty^d(A) \stackrel{(13.7)}{\gtrsim} \sum_{B \in \mathcal{I}} r_B^d \stackrel{(13.8)}{\gtrsim} \sum_{B \in \mathcal{I}'} r_B^d \geq \mathcal{H}_\infty^d \left(\bigcup_{j \in \mathcal{X}^\lambda} B_j \right). \quad (13.10)$$

Let

$$A' = \left\{ x \in F_1 : \sum_{j \in \mathcal{X}} \mathbb{1}_{B_j} f(z_j) > \lambda \right\}.$$

If $x \in A'$, since the B_j are disjoint for $j \in \mathcal{X}$, there is $x \in B_j$ for some $j \in \mathcal{X}^\lambda$, and so $x \in \bigcup_{j \in \mathcal{X}^\lambda} B_j$, hence

$$\mathcal{H}_\infty^d(A') \leq \mathcal{H}_\infty^d \left(\bigcup_{j \in \mathcal{X}^\lambda} B_j \right) \stackrel{(13.10)}{\lesssim} \mathcal{H}_\infty^d(A)$$

and this finishes the proof of (13.6). Hence (2.48) follows by integrating (13.6). \square

REFERENCES

- [Ad88] David R. Adams. A note on Choquet integrals with respect to Hausdorff capacity. In *Function spaces and applications (Lund, 1986)*, volume 1302 of *Lecture Notes in Math.*, pages 115–124. Springer, Berlin, 1988. [74](#)
- [ACM12] Allard, W.; Chen, G.; Maggioni, M. Multi-scale geometric methods for data sets II: Geometric Multi-Resolution Analysis. *Applied and Computational Harmonic Analysis* **32** (2012), no. 3, 435–462. [4](#)
- [AT15] Azzam, J.; Tolsa, X. Characterization of n -rectifiability in terms of Jones' square function: Part II. *Geometric and Functional Analysis* **25** (2015), no. 5, 1371–1412. [4](#), [22](#)
- [Bad12] Badger, M. Null sets of harmonic measure on NTA domains: Lipschitz approximation revisited. *Math. Z.* **270** (2012), no. 1-2, 241–262. Available at: <http://dx.doi.org/10.1007/s00209-010-0795-1> [11](#)
- [BS15] Badger, M.; Schul, R. Multiscale analysis of 1-rectifiable measures: necessary conditions. *Math. Ann.* **361** (2015), no. 3-4, 1055–1072. [5](#)
- [BS16a] Badger, M.; Schul, R. Two sufficient conditions for rectifiable measures. *Proc. Amer. Math. Soc.* **144** (2016), no. 6, 2445–2454. [5](#)
- [BS16b] Badger, M.; Schul, R. Multiscale analysis of 1-rectifiable measures II: characterizations. *arXiv preprint arXiv:1602.03823* (2016). [5](#)
- [BJ90] Bishop, C. J.; Jones, P. W. Harmonic measure and arclength. *Ann. of Math. (2)* **132** (1990), no. 3, 511–547. Available at: <http://dx.doi.org/10.2307/1971428> [3](#)
- [BJ97] Bishop, C.; Jones, P. Wiggly sets and limit sets. *Ark. Mat.* **35** (1997), no. 2, 201–224. Available at: <http://dx.doi.org.offcampus.lib.washington.edu/10.1007/BF02559967> [3](#)
- [BJPY97] Bishop, C. J.; Jones, P. W.; Pemantle, R.; Peres, Y. The dimension of the Brownian frontier is greater than 1. *J. Funct. Anal.* **143** (1997), no. 2, 309–336. Available at: <http://dx.doi.org/10.1006/jfan.1996.2928> [3](#)
- [Chr90] Christ, M. A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. *Colloq. Math.* **60/61** (1990), no. 2, 601–628. [18](#)
- [Dav88] David, G. Morceaux de graphes lipschitziens et intégrales singulières sur une surface. *Rev. Mat. Iberoamericana* **4** (1988), no. 1, 73–114. Available at: <http://dx.doi.org/10.4171/RMI/64> [11](#), [18](#)

- [Dav98] David, G. Unrectifiable 1-sets have vanishing analytic capacity. *Rev. Mat. Iberoamericana* **14** (1998), no. 2, 369–479. [4](#)
- [DJ90] David, G.; Jerison, D. Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals. *Indiana Univ. Math. J.* **39** (1990), no. 3, 831–845. Available at: <http://dx.doi.org/10.1512/iumj.1990.39.39040> [11](#)
- [DS91] David, G.; Semmes, S. W. Singular integrals and rectifiable sets in \mathbf{R}^n : Beyond Lipschitz graphs. *Astérisque* (1991), no. 193, 152. [4](#), [8](#), [11](#), [23](#), [55](#)
- [DS93a] David, G.; Semmes, S. W. *Analysis of and on uniformly rectifiable sets*, *Mathematical Surveys and Monographs*, vol. 38, American Mathematical Society, Providence, RI, 1993. [11](#), [32](#)
- [DS93b] David, G.; Semmes, S. W. Quantitative rectifiability and Lipschitz mappings. *Trans. Amer. Math. Soc.* **337** (1993), no. 2, 855–889. Available at: <http://dx.doi.org/10.2307/2154247> [11](#), [71](#)
- [DT12] David, G.; Toro, T. *Reifenberg parameterizations for sets with holes*, American Mathematical Soc., 2012. [4](#), [12](#), [14](#), [15](#), [16](#), [30](#), [67](#), [68](#), [69](#)
- [Dor85] Dorronsoro, J. R. A characterization of potential spaces. *Proc. Amer. Math. Soc.* **95** (1985), no. 1, 21–31. Available at: <http://dx.doi.org/10.2307/2045567> [3](#)
- [ENV16] Edelen, N.; Naber, A.; Valtorta, D. Quantitative reifenberg theorem for measures. *arXiv preprint arXiv:1612.08052* (2016). [5](#)
- [Fan90] Fang, X., The Cauchy Integral of Calderon and Analytic Capacity . *Ph.D. dissertation* (1990). [3](#), [9](#)
- [GT16] Girela-Sarrión, D.; Tolsa, X. The Riesz transform and quantitative rectifiability for general Radon measures. *arXiv preprint arXiv:1601.08079* (2016). [4](#)
- [HM12] Hytönen, T.; Martikainen, H. Non-homogeneous Tb theorem and random dyadic cubes on metric measure spaces. *J. Geom. Anal.* **22** (2012), no. 4, 1071–1107. Available at: <http://dx.doi.org/10.1007/s12220-011-9230-z> [18](#)
- [Jon90] Jones, P. W. Rectifiable sets and the traveling salesman problem. *Invent. Math.* **102** (1990), no. 1, 1–15. Available at: <http://dx.doi.org/10.1007/BF01233418> [2](#)
- [Leg99] Léger, J. C. Menger curvature and rectifiability. *Ann. of Math. (2)* **149** (1999), no. 3, 831–869. Available at: <http://dx.doi.org/10.2307/121074> [4](#)
- [LW12] Lerman, G.; Whitehouse, J. T. Least squares approximations of measures via geometric condition numbers. *Mathematika* **58** (2012), no. 1, 45–70. Available at: <http://dx.doi.org/10.1112/S0025579311001720> [4](#)
- [LMR16] Little, A.; Maggioni, M.; Rosasco, L. Multi-scale geometric methods for data sets I: Multiscale SVD, noise and curvature. *Applied and Computational Harmonic Analysis* To appear. [4](#)
- [Mat95] Mattila, P. *Geometry of sets and measures in Euclidean spaces*, *Cambridge Studies in Advanced Mathematics*, vol. 44, Cambridge University Press, Cambridge, 1995. Fractals and rectifiability. [13](#), [20](#), [63](#)
- [MMM96] Mattila, P.; Melnikov, M. S.; Verdera, J. The Cauchy integral, analytic capacity, and uniform rectifiability. *Ann. of Math. (2)* **144** (1996), no. 1, 127–136. Available at: <http://dx.doi.org/10.2307/2118585> [4](#)
- [NV17] Naber, A.; Valtorta, D. Rectifiable-Reifenberg and the regularity of stationary and minimizing harmonic maps. *Ann. of Math. (2)* **185** (2017), no. 1, 131–227. Available at: <http://dx.doi.org/10.4007/annals.2017.185.1.3> [5](#)
- [Oki92] Okikiolu, K. Characterization of subsets of rectifiable curves in \mathbf{R}^n . *J. London Math. Soc. (2)* **46** (1992), no. 2, 336–348. Available at: <http://dx.doi.org/10.1112/jlms/s2-46.2.336> [2](#)
- [JKV97] Rey, A. V.; Katz, N.; Jones, P. Checkerboards, lipschitz functions and uniform rectifiability. *Revista matemática iberoamericana* **13** (1997), no. 1, 189–210. [11](#)

- [Sch07] Schul, R. Subsets of rectifiable curves in Hilbert space—the analyst’s TSP. *J. Anal. Math.* **103** (2007), 331–375. Available at: <http://dx.doi.org/10.1007/s11854-008-0011-y> **3**
- [Tol05] Tolsa, X. Bilipschitz maps, analytic capacity, and the Cauchy integral. *Ann. of Math.* (2) **162** (2005), no. 3, 1243–1304. Available at: <http://dx.doi.org/10.4007/annals.2005.162.1243> **3**
- [Tol15] Tolsa, X. Characterization of n -rectifiability in terms of Jones’ square function: Part I. *Calculus of Variations and Partial Differential Equations* **54** (2015), no. 4, 3643–3665. **4**

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