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Carroll/fracton particles and their correspondence

José Figueroa-O’Farrill,^{a,1} Alfredo Pérez^{b,c,2} and Stefan Prohazka^{a,3}

^a*Maxwell Institute and School of Mathematics, The University of Edinburgh,
James Clerk Maxwell Building, Peter Guthrie Tait Road, Edinburgh EH9 3FD, Scotland, U.K.*

^b*Centro de Estudios Científicos (CECs),
Avenida Arturo Prat 514, Valdivia, Chile*

^c*Facultad de Ingeniería, Arquitectura y Diseño, Universidad San Sebastián,
sede Valdivia, General Lagos 1163, Valdivia 5110693, Chile*

E-mail: j.m.figueroa@ed.ac.uk, alfredo.perez@uss.cl,
stefan.prohazka@ed.ac.uk

ABSTRACT: We exploit the close relationship between the Carroll and fracton/dipole algebras, together with the method of coadjoint orbits, to define and classify classical Carroll and fracton particles. This approach establishes a Carroll/fracton correspondence and provides an answer to the question “What is a fracton?”

Under this correspondence, carrollian energy and center-of-mass correspond to the fracton electric charge and dipole moment, respectively. Then immobile massive Carroll particles correspond to the fracton monopoles, whereas certain mobile Carroll particles (“centrons”) correspond to fracton elementary dipoles. We uncover various new massless carrollian/neutral fractonic particles, provide an action in each case and relate them via a $GL(2, \mathbb{R})$ symmetry.

We also comment on the limit from Poincaré particles, the relation to (electric and magnetic) Carroll field theories, contrast Carroll boosts with dipole transformations and highlight a generalisation to curved space ((A)dS Carroll).

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¹0000-0002-9308-9360.

²0000-0003-0989-9959.

³0000-0002-3925-3983.

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1 Introduction

Carrollian [1, 2] and fractonic [3–6] theories both put mobility restrictions on their elementary particles and play a prominent rôle in exciting recent advances in high energy and condensed matter physics. But this unconventional feature and its connected exotic symmetries also mean that many of the properties we take for granted for Lorentz-invariant theories now need to be reconsidered. One of them is the very definition of an elementary system or particle with a specific symmetry.

In this work we define, classify, and analyse classical carrollions (i.e., Carroll particles) and fractons, which for the purposes of this paper we take to mean particles with conserved electric charge and dipole moment. We propose that fractons are elementary systems with fracton symmetry; in other words, homogeneous symplectic manifolds of the dipole group [7] which, since this group has vanishing symplectic cohomology, are nothing but its coadjoint orbits. This provides, at least at a classical level, one answer to the question “*What are fractons?*”. Indeed, we show that our proposed fractons indeed satisfy the expected properties. This definition follows Souriau [8], which may be interpreted as a classical analogue of Wigner’s classification of Poincaré particles [9].

Most of our discussion applies equally to both carrollions and fractons. The close relation between their underlying symmetries [10] (see also [11]) reveals an interesting correspondence between them. More precisely, they both possess conserved angular momentum \mathbf{j} and linear momentum \mathbf{p} , but Carroll energy E and center-of-mass charge \mathbf{k} can be reinterpreted as fracton electric charge q and dipole moment \mathbf{d} , respectively. At the heart of our discussion are the following commutation relations

$$[\mathbf{k}, \mathbf{p}] = E \quad (\text{Carroll}) \quad \Leftrightarrow \quad [\mathbf{d}, \mathbf{p}] = q \quad (\text{Fracton}) \quad (1.1)$$

from which many of the unusual properties emerge. We should remark that the symmetries do not precisely match: fracton energy has no counterpart in the Carroll world, but since it is central in the dipole algebra it does not affect the classification of coadjoint orbits and hence may be safely ignored in most of our discussion.

Classical Carroll/fracton particles (i.e., the coadjoint orbits of the Carroll/dipole group) fall broadly into two classes, depending on whether or not the carrollian energy (dually, the fracton charge) vanishes. If nonzero, we call these particles *massive carrollions* for reasons we will explain in the bulk of the paper and, similarly, if the carrollian energy is zero, we call the corresponding particles *massless*. Carroll/fracton correspondence relates massive carrollions to charged monopoles, both of which share the characteristic feature of being stuck to a point: (see figure 1)

$$\dot{\mathbf{x}} = \mathbf{0}. \quad (1.2)$$

Whereas for carrollions this feature is due to the conservation of center-of-mass $\mathbf{k} = E\mathbf{x}$, for monopoles it is due to the conservation of the dipole moment $\mathbf{d} = q\mathbf{x}$. This conclusion can be systematically derived from a phase space action associated with the relevant coadjoint orbit, which for the massive spinless carrollion is given by

$$S_{\text{massive}} = \int d\tau [-E\dot{t} + \boldsymbol{\pi} \cdot \dot{\mathbf{x}} - N(E - E_0)] \quad (1.3)$$

and for the spinless fracton monopole by

$$S_{\text{monopole}} = \int d\tau [-E\dot{t} + \boldsymbol{\pi} \cdot \dot{\mathbf{x}} + \Phi\dot{q} - N(E - E_0) - \eta(q - e)] . \quad (1.4)$$

In these particle actions, all quantities are varied except for the fixed values E_0 for the energy and e for the charge. We also see that once we fix the time $\tau = t$ and solve the constraints we arrive at equivalent actions on the reduced phase space

$$S_{\text{red}} = \int dt [\boldsymbol{\pi} \cdot \dot{\mathbf{x}} - E_0] . \quad (1.5)$$

In this sense they are intrinsically the same.

On the other hand, certain massless carrollions (which we tentatively call *centrons*) having vanishing energy and momentum but nonzero center-of-mass charge correspond to elementary fractonic dipoles. Their mobility, as expected, is not restricted as we can see from their phase space action

$$S_{\text{dipole}} = \int dt [\mathbf{d} \cdot \dot{\mathbf{v}} + \boldsymbol{\pi} \cdot \dot{\mathbf{x}} - \eta(\|\mathbf{d}\|^2 - d^2) - \mathbf{u} \cdot \boldsymbol{\pi}] , \quad (1.6)$$

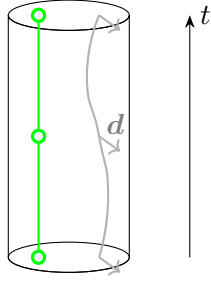


Figure 1. This figure is a sketch of the movement of fractons (and carrollions, dually) in space. When the dipole moment is conserved, monopoles are restricted to a point in space as pictured by the straight left line. On the other hand, the mobility of elementary dipoles is not restricted: dipole conservation $\dot{\mathbf{d}} = \mathbf{0}$ implies the dipole vector is inert (right). Dually, massive carrollions are stuck at a point since the center-of-mass charge is conserved as can be visualised by thinking of a massive Poincaré particle for which the light cone closes to a line. For the massless carrollian centrons again mobility is not restricted.

where only d is not varied (the centron action is given by $\mathbf{d} \mapsto \mathbf{k}$). In this action \mathbf{x} represents the position of the dipole and \mathbf{d} is a vector that represents the dipole moment, anchored at \mathbf{x} . They are independent degrees of freedom in the action. Variation with respect to $\boldsymbol{\pi}$ shows that $\dot{\mathbf{x}} = \mathbf{u}$, but since \mathbf{u} is an arbitrary Lagrange multiplier, \mathbf{x} is unconstrained, i.e., without coupling to external sources their time evolution is undetermined and not fixed by the symmetries. On the other hand varying with respect to \mathbf{v} leads to $\dot{\mathbf{d}} = \mathbf{0}$, which shows that the dipole moment is indeed conserved. The constraint $\|\mathbf{d}\|^2 = d^2$ fixes the norm of the dipole momentum, where d is an external parameter characterising the magnitude of the dipole moment. This leaves us with two remaining degrees of freedom, which are the two angles that determine the direction of the dipole moment vector with norm d .

We may also add spin for the massive carrollion/monopole and in this way also uncover various other massless carrollions/neutral fractons, which have not appeared in the literature, and show that the zoo of carrollions and fractons is more diverse than discussed so far (see table 1). The good news is that notwithstanding these particles being physically distinct, they are related by $\text{GL}(2, \mathbb{R})$ transformations arising as outer automorphisms of the Carroll/dipole group. This implies that at a technical level we may often reuse results for one type of particle and apply them to another type, simply by acting with a $\text{GL}(2, \mathbb{R})$ transformation.

We also comment on the connection to Poincaré particles (see in particular figure 2), the relation to (electric and magnetic) carrollian field theories and provide some remarks on Carroll boost versus dipole symmetry for field theories. We also highlight that there is a curved generalisation to (A)dS Carroll and fractons on curved space.

The rest of the paper is organised as follows.

We start in section 2 by briefly reviewing Carroll symmetries and describing the classification of carrollions, which is summarised in table 1. A more comprehensive discussion of Carroll/dipole symmetry (in arbitrary dimension) can be found in appendix A, whereas appendix B contains more details about the classification of the coadjoint orbits (in three spatial dimensions), including their structure as manifolds.

In section 3 we provide a systematic derivation of particle actions for carrollions.

In section 4 we discuss the limit from Poincaré particles to carrollions and highlight that for to each spinning massive Poincaré particle there corresponds a massive spinning carrollion. The massless carrollions mostly derive from Poincaré tachyons and the massless Poincaré particles seem to vanish in the limit.

In section 5 we introduce dipole symmetries and the Carroll/fracton correspondence. We then discuss the mobility restrictions of the monopoles and dipoles. As an instructive example we also show how the elementary dipoles emerge from two monopoles.

In section 6 we comment on the generalisation to field theories and curved space. In section 6.1 we discuss the relation of massive and massless carrollions to Carroll field theories. Massive Carroll particles are related to what is sometimes called “electric” theory. Massless carrollions with helicity seem to be related to “magnetic” theories, but the known actions have an additional source term, for which we propose a possible resolution. In section 6.2 we contrast Carroll boost symmetry with dipole symmetries. In section 6.3 we describe how the Carroll/fracton correspondence can be generalised to curved space, more precisely, (A)dS Carroll.

In section 7 we comment on various potential applications of our results such as to time-like symmetries, other exotic particles, flat holography and black holes.

2 Classical Carroll particles: coadjoint orbits

In this section we summarise the classification of classical carrollions (i.e., Carroll particles) in $3 + 1$ spacetime dimensions. In other words we classify coadjoint orbits of the $(3 + 1)$ -dimensional Carroll group. This summary should be read together with table 1. Readers interested in the mathematical details are encouraged to read the two appendices. In appendix A, whose point of departure is the Carroll Lie algebra in general dimension, we show how to view the Carroll group as a matrix group, allowing us to determine the adjoint and coadjoint actions explicitly. In appendix B we go through the systematic classification of coadjoint orbits as well as providing more details about their structure.

We may give Carroll spacetime global coordinates (t, \mathbf{x}) on which the Carroll group G [1, 2] acts via the following three kinds of Carroll transformations

- **rotations:** $(t, \mathbf{x}) \mapsto (t, R\mathbf{x})$, where $R \in SO(3)$;
- **(carrollian) boosts:** $(t, \mathbf{x}) \mapsto (t + \mathbf{v} \cdot \mathbf{x}, \mathbf{x})$;
- and **translations:** $(t, \mathbf{x}) \mapsto (t + s, \mathbf{x} + \mathbf{a})$.

The distinguishing feature of Carroll symmetries are the carrollian boosts which only act on time, rather than time and space, like Poincaré, or only space, like for Galilei. The general Carroll transformation $(R, \mathbf{v}, \mathbf{a}, s)$ is a composition of rotations, boosts and translations and is given by (see appendix A.1.1):

$$(t, \mathbf{x}) \mapsto (t + s + \mathbf{v} \cdot \mathbf{x}, R\mathbf{x} + \mathbf{a}). \tag{2.1}$$

In this section we will restrict to the $(3 + 1)$ -dimensional Carroll group, whose Lie algebra \mathfrak{g} is spanned by J_i, B_i, P_i, H with nonzero Lie brackets

$$[J_i, J_j] = \epsilon_{ijk} J_k \quad [J_i, B_j] = \epsilon_{ijk} B_k \quad [J_i, P_j] = \epsilon_{ijk} P_k \quad [B_i, P_j] = \delta_{ij} H, \quad (2.2)$$

where the Levi-Civita symbol ϵ_{ijk} is normalised so that $\epsilon_{123} = 1$. This defines the adjoint action of \mathfrak{g} on itself and by exponentiation also the adjoint representation which for a group element g acts as $\text{Ad}_g A = gAg^{-1}$ on $A \in \mathfrak{g}$. From that we define the coadjoint representation: if $\alpha \in \mathfrak{g}^*$ is a element in the dual of the Lie algebra, then

$$\langle \text{Ad}_g^* \alpha, A \rangle = \langle \alpha, \text{Ad}_{g^{-1}} A \rangle \quad (2.3)$$

Covectors $\alpha \in \mathfrak{g}^*$ may be interpreted as the conserved quantities of the elementary systems: $\alpha = (\mathbf{j}, \mathbf{k}, \mathbf{p}, E)$, where $\mathbf{j} = \langle \alpha, \mathbf{J} \rangle$ is the angular momentum, $\mathbf{k} = \langle \alpha, \mathbf{B} \rangle$ the centre of mass, $\mathbf{p} = \langle \alpha, \mathbf{P} \rangle$ the (linear) momentum and $E = \langle \alpha, H \rangle$ the energy. The coadjoint action of the Carroll group element $g = (R, \mathbf{v}, \mathbf{a}, s)$ on $\alpha = (\mathbf{j}, \mathbf{k}, \mathbf{p}, E)$ is given by $\text{Ad}_g^* \alpha = (\mathbf{j}', \mathbf{k}', \mathbf{p}', E')$ where [12]

$$\begin{aligned} \mathbf{j}' &= R\mathbf{j} + \mathbf{v} \times R\mathbf{k} + \mathbf{a} \times R\mathbf{p} + E\mathbf{v} \times \mathbf{a} \\ \mathbf{k}' &= R\mathbf{k} + E\mathbf{a} \\ \mathbf{p}' &= R\mathbf{p} - E\mathbf{v} \\ E' &= E. \end{aligned} \quad (2.4)$$

The intuition is that the coadjoint action characterises the transformation of the conserved quantities under the action of the Carroll group. Each coadjoint orbit defines a specific “particle”. Because coadjoint orbits are elementary systems which often have the interpretation of a particle, we use the two synonymously in this work. Of course, not every coadjoint orbit admits such an interpretation, e.g., the vacuum.

There are two obvious Casimirs of the Carroll group [1]: H and W^2 , which define a linear and a quartic function on \mathfrak{g}^* , respectively, evaluating on $\alpha = (\mathbf{j}, \mathbf{k}, \mathbf{p}, E)$ to

$$H(\alpha) = E \quad \text{and} \quad W^2(\alpha) = \|E\mathbf{j} + \mathbf{p} \times \mathbf{k}\|^2. \quad (2.5)$$

It is clear that E is invariant under the coadjoint representation and it is a short calculation using equation (2.4) to see that $E\mathbf{j} + \mathbf{p} \times \mathbf{k}$ transforms by a rotation, so its norm is invariant. Therefore these functions are constant on coadjoint orbits and therefore they are useful in their classification.

The energy E is analogous to the mass of Poincaré coadjoint orbits and we will therefore tentatively follow the convention to call the orbits *massive* when $E \neq 0$ and *massless* when $E = 0$.

2.1 Massive carrollions ($E \neq 0$)

As shown in appendix B.1, any $\alpha \in \mathfrak{g}^*$ with $E \neq 0$ can be boosted to the “rest frame” so that $\mathbf{p} = \mathbf{0}$, in analogy with the massive Galilei and Poincaré particles. In addition, we

may translate so that the centre of mass $\mathbf{k} = \mathbf{0}$. Doing so brings α to the form $(\mathbf{S}, \mathbf{0}, \mathbf{0}, E)$, where we have introduced the intrinsic spin vector

$$\mathbf{S} := \mathbf{j} + E^{-1} \mathbf{p} \times \mathbf{k}, \tag{2.6}$$

whose norm $S = \|\mathbf{S}\|$ is the intrinsic spin of the particle. We can use this quantity to differentiate between two kinds of massive particles: **spinless massive carrollions** where $S = 0$ and **massive carrollions with spin** $S > 0$.

Notice that we have nonzero angular momentum, even though \mathbf{p} vanishes, so it is justified to call S the spin. This is in contrast to the orbital angular momentum that even spinless carrollions have: $\mathbf{j} = E^{-1} \mathbf{k} \times \mathbf{p}$.

In energy-momentum space, the condition $E = E_0 \neq 0$ fixes a three-dimensional affine hyperplane (see figure 2) over which the coadjoint orbit fibres. The coadjoint orbits for spinless massive particles are then given by the cotangent bundle of these affine hyperplanes. For the massive particles with nonzero spin, the coadjoint orbits acquire two extra dimensions: to every point in the phase space of the spinless particle there is associated a 2-sphere of radius the intrinsic spin. This is explained in appendix B.5.

In summary, massive coadjoint orbits are in one-to-correspondence with pairs (E, S) , where the energy $E \neq 0$ and the spin $S \geq 0$.

2.2 Massless orbits ($E = 0$)

When $E = 0$ the coadjoint action reduces to

$$\begin{aligned} \mathbf{j}' &= R\mathbf{j} + \mathbf{v} \times R\mathbf{k} + \mathbf{a} \times R\mathbf{p} \\ \mathbf{k}' &= R\mathbf{k} \\ \mathbf{p}' &= R\mathbf{p} \end{aligned} \tag{2.7}$$

and the Casimir W^2 is still an invariant and on $\alpha = (\mathbf{j}, \mathbf{k}, \mathbf{p}, 0)$ it now takes the value

$$W^2(\alpha) = \|\mathbf{p} \times \mathbf{k}\|^2 = \|\mathbf{p}\|^2 \|\mathbf{k}\|^2 - (\mathbf{p} \cdot \mathbf{k})^2, \tag{2.8}$$

where we have used a standard vector identity. When $E = 0$, it is clear from equation (2.7) that $\|\mathbf{p}\|^2$, $\|\mathbf{k}\|^2$ and $\mathbf{p} \cdot \mathbf{k}$ are separately invariant [12], which we may use to further refine the classification of massless orbits. Before doing so, however, we may already highlight two interesting physical consequences:

- Since $\|\mathbf{p}\|^2$ is invariant, massless orbits cannot be boosted to a “rest frame” where $\mathbf{p} = \mathbf{0}$. This similarity to the massless Poincaré particles is a further justification to call these orbits massless (this is however also a property of tachyons).
- Another peculiar feature of the massless coadjoint orbits is the existence of the invariant $\|\mathbf{k}\|^2$. This differs from the Poincaré and Galilei (more precisely, Bargmann) case where we can always translate massive and massless particles in such a way that the centre of mass \mathbf{k} vanishes.

We now return to the classification of massless orbits. We will first focus on the orbits where $\mathbf{p} \times \mathbf{k} = \mathbf{0}$, so that the linear momentum \mathbf{p} and the centre of mass \mathbf{k} are parallel.

2.2.1 Vacuum sector ($\mathbf{p} = \mathbf{k} = \mathbf{0}$)

The vacuum sector is given by restricting to $E = 0$ and by additionally setting $\mathbf{p} = \mathbf{k} = \mathbf{0}$. The remaining nontrivial coadjoint action is a rotation of the angular momentum $\mathbf{j}' = R\mathbf{j}$. This gives rise to the quadratic invariant $\|\mathbf{j}\|^2$. The orbit with $\mathbf{j} = \mathbf{0}$ consists of the origin in \mathfrak{g}^* and we call it the **vacuum**. The orbits with $\mathbf{j} \neq \mathbf{0}$ are 2-spheres of radius $\|\mathbf{j}\|$ and we call the corresponding particles **spinning vacua**.

In summary, these orbits are parametrised by $j = \|\mathbf{j}\| \geq 0$ and they may be uniquely specified by the equations $E = 0$, $\mathbf{p} = \mathbf{0}$, $\mathbf{k} = \mathbf{0}$ and $\|\mathbf{j}\| = j \geq 0$.

2.2.2 Massless (parallel) carrollions

These are the orbits where $\mathbf{p} \times \mathbf{k} = \mathbf{0}$ but not both \mathbf{p} and \mathbf{k} are zero. These orbits are all related by the action of automorphisms, as explained in appendix B.4. The action of automorphisms on momenta is described by equation (B.15), where we see that automorphisms act on \mathbf{p}, \mathbf{k} via general linear transformations which act transitively on lines. Nevertheless, the orbits are described by different equations and have different physical interpretations, so it is worth looking at them separately.

Let us first set $\mathbf{k} = \mathbf{0}$, but $\mathbf{p} \neq \mathbf{0}$. The coadjoint action reduces to

$$\begin{aligned} \mathbf{j}' &= R\mathbf{j} + \mathbf{a} \times R\mathbf{p} \\ \mathbf{p}' &= R\mathbf{p}. \end{aligned} \tag{2.9}$$

We see that not just is $\|\mathbf{p}\|$ invariant, but also $\mathbf{j} \cdot \mathbf{p}$. We define the *helicity* h by

$$\mathbf{j} \cdot \mathbf{p} = h\|\mathbf{p}\|. \tag{2.10}$$

The helicity can be any real number. As shown in appendix B.2.3, we may always bring such a covector to the form $(\mathbf{j}, \mathbf{0}, \mathbf{p}, 0)$, where $\mathbf{j} = h\mathbf{p}/\|\mathbf{p}\|$, where \mathbf{p} can be rotated into any desired direction. For example, we can take $\mathbf{p} = (0, 0, p)$ and hence $\mathbf{j} = (0, 0, h)$. While we will call these particles **massless carrollions with helicity** h it would be equally justified to call them **aristotelions** which emerge from the (flat) aristotelian space (see, e.g., [13]) and which have no boost symmetries and thus no center-of-mass conservation and precisely the coadjoint action (2.9) with $E' = E$.

These orbits are cut out by the equations $E = 0$, $\mathbf{k} = \mathbf{0}$, $\|\mathbf{p}\| = p \neq 0$ and $\mathbf{j} \cdot \mathbf{p} = hp$, depending on the two parameters $p > 0$ and $h \in \mathbb{R}$.

Everything we just discussed applies mutatis mutandis to the case where $\mathbf{p} = \mathbf{0}$ and $\mathbf{k} \neq \mathbf{0}$. We may always bring a covector in such an orbit to the form $(\mathbf{j}, \mathbf{k}, \mathbf{0}, 0)$, where $\mathbf{j} = h\mathbf{k}/\|\mathbf{k}\|$ and $h \in \mathbb{R}$. Such orbits are characterised by the equations $E = 0$, $\mathbf{p} = \mathbf{0}$, $\|\mathbf{k}\| = k \neq 0$ and $\mathbf{j} \cdot \mathbf{k} = hk$. They characterise the center of mass and we therefore tentatively call them **centrons**.

Finally, we discuss the case of $\mathbf{p} \times \mathbf{k} = \mathbf{0}$, but neither \mathbf{p} nor \mathbf{k} are zero. This breaks up into two cases depending on whether \mathbf{p} and \mathbf{k} are parallel or antiparallel. In this case, since $E = 0$, the inner products $\mathbf{j} \cdot \mathbf{p}$ and $\mathbf{j} \cdot \mathbf{k}$ are constant on the orbit. Also $\mathbf{p} \cdot \mathbf{k}$ is constant, but since $\mathbf{p} \times \mathbf{k} = \mathbf{0}$, it follows that $\mathbf{p} \cdot \mathbf{k} = \pm\|\mathbf{p}\|\|\mathbf{k}\|$, where the plus sign says the angle between them is 0 (parallel) and the minus sign says the angle between them is π (antiparallel).

It is also the case that $\mathbf{j} \cdot \mathbf{p}$ and $\mathbf{j} \cdot \mathbf{k}$ are not independent, so that if we know $\|\mathbf{p}\|$, $\|\mathbf{k}\|$, $\mathbf{p} \cdot \mathbf{k}$ and one of $\mathbf{j} \cdot \mathbf{p}$ or $\mathbf{j} \cdot \mathbf{k}$, we know them all. We may impose these conditions in order. We start with the 10-dimensional \mathfrak{g}^* and impose $E = 0$ to drop down to a 9-dimensional hyperplane with coordinates $(\mathbf{j}, \mathbf{k}, \mathbf{p})$. We impose $\|\mathbf{p}\| = p > 0$ and $\|\mathbf{k}\| = k > 0$ and we get a 7-dimensional manifold diffeomorphic to $\mathbb{R}^3 \times S^2 \times S^2$. Next we impose $\mathbf{p} \times \mathbf{k} = \mathbf{0}$. This results in two disconnected 5-dimensional submanifolds $\mathbb{R}^3 \times M_{\pm}^2$ one for each sign in $\mathbf{p} \cdot \mathbf{k} = \pm pk$. The two-dimensional manifolds M_{\pm}^2 are submanifolds of $S^2 \times S^2$ and consist of points $(\pm k\mathbf{u}, p\mathbf{u})$, where \mathbf{u} is a unit vector in the direction of \mathbf{p} . Notice that both M_{\pm}^2 are diffeomorphic to S^2 , so that each of the five-dimensional manifolds is diffeomorphic to $\mathbb{R}^3 \times S^2$. Finally, we impose $\mathbf{j} \cdot \mathbf{p} = hp$, which cuts the dimension by one, resulting in a 4-dimensional orbit. In summary, the condition $\mathbf{p} \times \mathbf{k} = \mathbf{0}$ does not impose three relations as one might naively suspect, but only two sets of two relations, distinguished by a sign.

2.2.3 Generic massless carrollions

It remains to discuss the orbits where $E = 0$ but $\mathbf{p} \times \mathbf{k} \neq \mathbf{0}$, which is the generic case and motivated us to call them **generic massless carrollions**. This implies that $\|\mathbf{p}\| = p > 0$ and $\|\mathbf{k}\| = k > 0$ and hence $\mathbf{p} \cdot \mathbf{k} = pk \cos \theta$ for some angle $\theta \in (0, \pi)$. Let us write $\mathbf{p} = p\mathbf{u}$, for \mathbf{u} a unit vector in the direction of \mathbf{p} . Then $\mathbf{k} - k \cos \theta \mathbf{u}$ is perpendicular to \mathbf{p} and hence to \mathbf{u} . Let us write it as $\mathbf{k} - k \cos \theta \mathbf{u} = k \sin \theta \mathbf{u}_{\perp}$, where \mathbf{u}_{\perp} is a second unit vector that is perpendicular to \mathbf{u} . Therefore we may write $\mathbf{k} = k(\cos \theta \mathbf{u} + \sin \theta \mathbf{u}_{\perp})$.

Since $\mathbf{p} \times \mathbf{k} \neq \mathbf{0}$ we may use boosts and translations to set $\mathbf{j} = \mathbf{0}$. This should be contrasted with the orbits of sections 2.2.1 and 2.2.2 where this is not possible when $h \neq 0$. Related to this observation is the absence of a notion of spin for the generic massless orbits.

As expected for $\theta = 0$, where \mathbf{p} and \mathbf{k} are parallel and $\theta = \pi$, where they are antiparallel, we are back to the earlier case of section 2.2.2, except that the orbit drops dimension from 6 to 4.

3 Particle actions

Given a coadjoint orbit there is a systematic way to associate with it a particle action. This is fundamental to the symplectic approach to dynamical systems pioneered by Souriau [8]. Since the actions provide information concerning the mobility of the particle and are useful for many applications, e.g., for path integral quantisation [14], we will provide them in this section and analyse their (classical) properties. For each carrollion we follow the method explained below and appendix A.4, but see also [15, section 4.4.5], [16, section 2,3] and [17, section 5] for further useful details and references.

Let $\tau \mapsto g(\tau)$ be a curve in the Carroll group G . The action corresponding to the coadjoint orbit of $\alpha \in \mathfrak{g}^*$ is given by

$$S = \int L d\tau = \int \langle \alpha, g^{-1} \dot{g} \rangle d\tau. \tag{3.1}$$

The term $g^{-1} \dot{g}$, where $\dot{g} = \frac{dg}{d\tau}$, is the pull-back of the left-invariant Maurer-Cartan form (A.28). For the case at hand the lagrangian is then given by

$$L[R(\varphi), \mathbf{v}, \mathbf{x}, t] = \frac{1}{2} \text{Tr} \left(J^T R^T \dot{R} \right) + (R\mathbf{k}) \cdot \dot{\mathbf{v}} + (R\mathbf{p}) \cdot \dot{\mathbf{x}} + E \left(\dot{t} + \frac{1}{2} \mathbf{x} \cdot \dot{\mathbf{v}} - \frac{1}{2} \mathbf{v} \cdot \dot{\mathbf{x}} \right) \tag{3.2}$$

$$= \frac{1}{2} \text{Tr} \left(J^T R^T \dot{R} \right) + (R\mathbf{k} + \frac{1}{2} E \mathbf{x}) \cdot \dot{\mathbf{v}} + (R\mathbf{p} - \frac{1}{2} E \mathbf{v}) \cdot \dot{\mathbf{x}} + E \dot{t} \tag{3.3}$$

#	Particle description	Orbit representative $\alpha = (\mathbf{j}, \mathbf{k}, \mathbf{p}, E)$	$\dim \mathcal{O}_\alpha$	Equations for orbits
1	Massive spinless	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, E_0)$	6	$E = E_0 \neq 0, E_0 \mathbf{j} + \mathbf{p} \times \mathbf{k} = \mathbf{0}$
2	Massive with spin S	$(S\mathbf{u}, \mathbf{0}, \mathbf{0}, E_0)$	8	$E = E_0 \neq 0, \ \mathbf{j} + E_0^{-1} \mathbf{p} \times \mathbf{k}\ = S > 0$
3	Vacuum	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, 0)$	0	$E = 0, \mathbf{p} = \mathbf{0}, \mathbf{k} = \mathbf{0}, \mathbf{j} = \mathbf{0}$
4	Spinning vacuum	$(\mathbf{j}\mathbf{u}, \mathbf{0}, \mathbf{0}, 0)$	2	$E = 0, \mathbf{p} = \mathbf{0}, \mathbf{k} = \mathbf{0}, \ \mathbf{j}\ = j > 0$
5	Centrons	$(h\mathbf{u}, k\mathbf{u}, \mathbf{0}, 0)$	4	$E = 0, \mathbf{p} = \mathbf{0}, \ \mathbf{k}\ = k > 0, \mathbf{j} \cdot \mathbf{k} = h\ \mathbf{k}\ \in \mathbb{R}$
6	Massless with helicity h	$(h\mathbf{u}, \mathbf{0}, p\mathbf{u}, 0)$	4	$E = 0, \mathbf{k} = \mathbf{0}, \ \mathbf{p}\ = p > 0, \mathbf{j} \cdot \mathbf{p} = h\ \mathbf{p}\ \in \mathbb{R}$
7 ₊	Parallel massless	$(h\mathbf{u}, k\mathbf{u}, p\mathbf{u}, 0)$	4	$E = 0, \ \mathbf{p}\ = p > 0, \ \mathbf{k}\ = k > 0, \mathbf{p} \cdot \mathbf{k} = pk, \mathbf{j} \cdot \mathbf{p} = h\ \mathbf{p}\ \in \mathbb{R}$
7 ₋	Antiparallel massless	$(h\mathbf{u}, -k\mathbf{u}, p\mathbf{u}, 0)$	4	$E = 0, \ \mathbf{p}\ = p > 0, \ \mathbf{k}\ = k > 0, \mathbf{p} \cdot \mathbf{k} = -pk, \mathbf{j} \cdot \mathbf{p} = h\ \mathbf{p}\ \in \mathbb{R}$
8	Generic massless	$(\mathbf{0}, k \cos \theta \mathbf{u} + k \sin \theta \mathbf{u}_\perp, p\mathbf{u}, 0)$	6	$E = 0, \ \mathbf{p}\ = p > 0, \ \mathbf{k}\ = k > 0, \mathbf{p} \cdot \mathbf{k} = pk \cos \theta, \theta \in (0, \pi)$

Table 1. *Overview of carrollions.* This table provides an overview of the carrollions (= Carroll particles) which are summarised in section 2. As indicated by the horizontal line they are roughly separated into massive ($E \neq 0$) and massless ($E = 0$) orbits. The shorter horizontal line separates the vacuum sector from the massless particles. The second column provides a tentative descriptive name (if one exists). The third column displays an orbit representative: the notation is such that $\mathbf{u} \in \mathbb{R}^3$ represents a fixed unit-norm vector and in the last row $\mathbf{u}_\perp \in \mathbb{R}^3$ is a second unit-norm vector perpendicular to \mathbf{u} . The last column provides the equations which define the orbits. One can easily check that $10 - \#\text{equations} = \dim \mathcal{O}_\alpha$ in all cases but the (anti)parallel massless, which might seem to be under-constrained but as discussed in the text they are not.

The terms in square brackets denote the quantities that are varied (in this case, $\varphi, \mathbf{v}, \mathbf{x}, t$) while the remaining quantities are not, i.e., $J = \varepsilon(\mathbf{j}), \mathbf{k}, \mathbf{p}, E$ are fixed.

The action has a global Carroll symmetry as can be seen from $g^{-1}\dot{g}$ which is invariant under τ -independent left multiplication $g \mapsto hg$. Infinitesimally these transformations can be parametrised by $(\boldsymbol{\lambda}, \boldsymbol{\beta}, \mathbf{a}, s)$ and act as

$$\delta R = \varepsilon(\boldsymbol{\lambda})R \quad \delta \mathbf{v} = \boldsymbol{\lambda} \times \mathbf{v} + \boldsymbol{\beta} \quad \delta \mathbf{x} = \boldsymbol{\lambda} \times \mathbf{x} + \mathbf{a} \quad \delta t = s + \frac{1}{2}(\boldsymbol{\beta} \cdot \mathbf{x} - \mathbf{a} \cdot \mathbf{v}), \quad (3.4)$$

where $\varepsilon(\boldsymbol{\lambda})_{ab} = -\epsilon_{abc}\lambda^c$ (see appendix A.5). This leads to the following Noether charges, which we denote with a subscript Q :

$$\begin{aligned} \mathbf{j}_Q &= R\mathbf{j} + \mathbf{v} \times R\mathbf{k} + \mathbf{x} \times R\mathbf{p} + E\mathbf{v} \times \mathbf{x} \\ \mathbf{k}_Q &= R\mathbf{k} + E\mathbf{x} \\ \mathbf{p}_Q &= R\mathbf{p} - E\mathbf{v} \\ E_Q &= E. \end{aligned} \quad (3.5)$$

This shows that the Noether charges are given by a coadjoint action on α .

These actions have gauge symmetry given by a right action $g \mapsto gh(\tau)$, where $h(\tau)$ has to be in the stabiliser of α . Since the stabiliser, and consequently the physical degrees of freedom and constraints, depends on the specific particle, we will now analyse them case by case. As shown in appendix A.4 (but see, e.g., also [16]), particle actions only depend on the coadjoint orbit of α and not on α itself. Therefore we will feel free to choose a convenient representative in the following in order to simplify our computations. We will also neglect subtleties related to quantisation, like, e.g., boundary terms.

3.1 Massive spinless carrollion action

Massive spinless carrollions can always be brought into the “rest frame”

$$\alpha = (0, \mathbf{0}, \mathbf{0}, -E_0), \tag{3.6}$$

where E_0 is a constant. The lagrangian (3.2) is then up to a total derivative of the form

$$L[\mathbf{v}, \mathbf{x}, t] = E_0 (\mathbf{v} \cdot \dot{\mathbf{x}} - \dot{t}). \tag{3.7}$$

We have used the fact that the rotations are part of the stabiliser to go to a reduced phase space without φ . As discussed, this action is by construction invariant under Carroll symmetries and the Noether charges are given by restricting (3.5) to our choice of representative (3.6).

To express the lagrangian in Hamiltonian form, it is necessary to introduce the canonical momenta

$$\boldsymbol{\pi} := \frac{\partial L}{\partial \dot{\mathbf{x}}} = E_0 \mathbf{v}, \quad E := -\frac{\partial L}{\partial \dot{t}} = E_0. \tag{3.8}$$

The field \mathbf{v} is associated with the canonical momentum $\boldsymbol{\pi}$ conjugate to \mathbf{x} . In other words, the first term in (3.7) is already in Hamiltonian form when $\boldsymbol{\pi} = E_0 \mathbf{b}$. The lagrangian of the massive Carroll particle in a canonical form can then be written as

$$L_{\text{can}}[\mathbf{x}, t, \boldsymbol{\pi}, E, N] = -E\dot{t} + \boldsymbol{\pi} \cdot \dot{\mathbf{x}} - N(E - E_0), \tag{3.9}$$

where N is the Lagrange multiplier that enforces the constraint $E - E_0 = 0$. This restricts to a unique orbit and is the very same constraint that we also found in section 2.1, cf., the planes in figure 2.

A related action was previously found in [18], where the lagrangian is invariant under time reversal, i.e., the constraint was chosen to be of the form $E^2 - E_0^2 = 0$, describing both carrollian particles and anti-particles simultaneously, i.e., it does not restrict to just one orbit. On the other hand, the lagrangian (3.9) describes a particle, or an antiparticle, depending on the sign of E_0 . This case was also discussed in [12, appendix A] and [19].

Solving the constraint $E - E_0 = 0$, and imposing the gauge condition $t = \tau$, one finds the lagrangian describing the dynamics in the reduced phase space

$$L_{\text{red}}[\mathbf{x}, \boldsymbol{\pi}] = \boldsymbol{\pi} \cdot \dot{\mathbf{x}} - E_0. \tag{3.10}$$

It depends on 6 independent canonical variables, which is precisely the dimension of the coadjoint orbit (see table 1, orbit #1), as it must.

The just discussed actions imply that

$$\dot{\mathbf{x}} = \mathbf{0} \qquad \dot{\boldsymbol{\pi}} = \mathbf{0} \qquad E = E_0, \tag{3.11}$$

i.e., an isolated massive Carroll particle does not move, has constant conjugate momentum $\boldsymbol{\pi}$ and fixed energy E_0 . Let us emphasise that the velocity $\dot{\mathbf{x}}$, which is bound to be zero, is not connected to the momentum $\boldsymbol{\pi}$ which, although constant, can be nonzero. We will have more to say about these particles in section 5, where we also highlight the relation to fractons (more precisely, monopoles).

3.1.1 Infinite symmetries of massive carrollions

The action functional of the massive carrollion is invariant not only under Carroll transformations but actually under an infinite-dimensional symmetry. The lagrangian in (3.9) is invariant (up to boundary terms) under canonical transformations generated by an arbitrary function $F = F(E, \mathbf{x}, \boldsymbol{\pi})$. For a canonical variable z , its transformation law is given by $\delta z = \{z, F\}$. Explicitly, they read

$$\delta t = -\frac{\partial F}{\partial E} \quad \delta E = 0 \quad \delta \mathbf{x} = \frac{\partial F}{\partial \boldsymbol{\pi}} \quad \delta \boldsymbol{\pi} = -\frac{\partial F}{\partial \mathbf{x}} \quad (3.12)$$

together with $\delta N = 0$.

The conserved charge associated to this symmetry is the function $F(E, \mathbf{x}, \boldsymbol{\pi})$ itself. Indeed, using the equations of motion $\dot{E} = \dot{\mathbf{x}} = \dot{\boldsymbol{\pi}} = 0$, one can directly show that

$$\dot{F}(E, \mathbf{x}, \boldsymbol{\pi}) = 0, \quad (3.13)$$

i.e., it is conserved under τ evolution. This discussion also generalises for the massive spinning carrollions.

There are some interesting particular cases. For example, the Carroll transformations are obtained from the following generator

$$F = \boldsymbol{\lambda} \cdot (\mathbf{x} \times \boldsymbol{\pi}) - E\boldsymbol{\beta} \cdot \mathbf{x} + \mathbf{a} \cdot \boldsymbol{\pi} - sE. \quad (3.14)$$

In addition, the weak carrollian structure of Carroll spacetime [20, 21] has symmetries that include non-linear transformations, which can be obtained using the following generator

$$F = -Ef(\mathbf{x}), \quad (3.15)$$

with $f(\mathbf{x})$ being an arbitrary function of the coordinates which generalises the linear Carroll boosts $f(\mathbf{x}) = \boldsymbol{\beta} \cdot \mathbf{x}$. Thus, using (3.12) one finds

$$\delta t = f(\mathbf{x}) \quad \delta E = 0 \quad \delta \mathbf{x} = 0 \quad \delta \boldsymbol{\pi} = E \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}. \quad (3.16)$$

This symmetry is also present in the action of the massive carrollian scalar field (“electric”) which we discuss in section 6.1.

3.2 Massive spinning carrollion action

In order to incorporate a non-vanishing spin, we choose the following representative of the coadjoint orbit

$$\alpha = (J, \mathbf{0}, \mathbf{0}, -E_0). \quad (3.17)$$

Here, E_0 is a constant and J is a generic angular momentum matrix (see appendix A). Both of these quantities are considered as fixed parameters in the action.

From (3.2) we obtain the lagrangian

$$L[R(\boldsymbol{\varphi}), \mathbf{v}, \mathbf{x}, t] = \frac{1}{2} \text{Tr} \left(J^T R^T \dot{R} \right) + E_0 (\mathbf{v} \cdot \dot{\mathbf{x}} - \dot{t}). \quad (3.18)$$

Here, $R = R(\boldsymbol{\varphi})$ is a rotation matrix that depends on three independent angles φ^a . We have not yet specified the particular dependence of R in terms of these angles. Consequently, the lagrangian can be rewritten as

$$L[R(\boldsymbol{\varphi}), \mathbf{v}, \mathbf{x}, t] = \frac{1}{2} \text{Tr} \left(J^T R^T \frac{\partial R}{\partial \varphi^a} \right) \dot{\varphi}^a + E_0 (\mathbf{v} \cdot \dot{\mathbf{x}} - \dot{t}). \quad (3.19)$$

By comparing this lagrangian with that of the spinless case (3.7), it is evident that there is an additional term present which takes into account the spin degrees of freedom, φ^a . The presence of additional degrees of freedom due to spin is also a feature of actions describing Poincaré invariant particles (see, e.g., [22–25] and [26] for a useful summary).

This action is again invariant under Carroll symmetry with Noether charges given by (3.5) restricted to (3.17). In particular since

$$\mathbf{j}_Q - E_Q^{-1} \mathbf{p}_Q \times \mathbf{k}_Q = R\mathbf{j} \quad (3.20)$$

we see that these particles have spin (cf. (2.6))

$$\mathbf{S} \cdot \boldsymbol{\lambda} = \frac{1}{2} \text{Tr} \left(J^T R^T \varepsilon(\boldsymbol{\lambda}) R \right) = R\mathbf{j} \cdot \boldsymbol{\lambda} \quad (3.21)$$

which describes the non-orbital part of the total angular momentum with $\|\mathbf{S}\|^2 = S^2$ an invariant quantity, as expected.

We shall use the following parametrisation of the rotation matrix

$$R(\boldsymbol{\varphi}) = e^{\varphi_1 \varepsilon_1} e^{\varphi_2 \varepsilon_2} e^{\varphi_3 \varepsilon_3}. \quad (3.22)$$

(where explicitly $(\varepsilon_a)_{bc} = -\varepsilon_{abc}$, see (A.32)). In addition, by an appropriate rotation we can always choose to align J with the z -axis, i.e., $J = S\varepsilon_3$ for a constant $S > 0$. Using this parametrisation, the lagrangian (3.19) takes the form

$$L[\boldsymbol{\varphi}, \mathbf{v}, t, \mathbf{x}] = S(\dot{\varphi}_3 + \sin(\varphi_2)\dot{\varphi}_1) + E_0(\mathbf{v} \cdot \dot{\mathbf{x}} - \dot{t}), \quad (3.23)$$

and the spin vector becomes

$$\mathbf{S} = S\hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is a unit vector given by

$$\hat{\mathbf{n}} = (\sin \varphi_2, -\sin \varphi_1 \cos \varphi_2, -\cos \varphi_1 \cos \varphi_2). \quad (3.24)$$

Therefore, $\|\mathbf{S}\|^2 = S^2$ is an invariant quantity, as expected.

The canonical momenta associated with the spin degrees of freedom are given by

$$\Pi_1 = \frac{\partial L}{\partial \dot{\varphi}_1} = S \sin \varphi_2, \quad \Pi_2 = \frac{\partial L}{\partial \dot{\varphi}_2} = 0, \quad \Pi_3 = \frac{\partial L}{\partial \dot{\varphi}_3} = S. \quad (3.25)$$

The last two terms define constraint equations. Therefore, if $\boldsymbol{\pi} = E_0 \mathbf{v}$, the lagrangian (3.23) can be written in canonical form as

$$L_{\text{can}} = \boldsymbol{\Pi} \cdot \dot{\boldsymbol{\varphi}} - E\dot{t} + \boldsymbol{\pi} \cdot \dot{\mathbf{x}} - N(E - E_0) - \eta_2 \Pi_2 - \eta_3 (\Pi_3 - S) \quad (3.26)$$

where $L_{\text{can}}[\boldsymbol{\varphi}, \mathbf{x}, t, \boldsymbol{\Pi}, \boldsymbol{\pi}, E, N, \eta_2, \eta_3]$.

To highlight the physical degrees of freedom we solve the constraints, impose the gauge condition $t = \tau$ and neglect boundary terms. The lagrangian in the reduced phase space is given by

$$L_{\text{red}} [\varphi^1, \mathbf{x}, \Pi_1, \boldsymbol{\pi}] = \Pi_1 \dot{\varphi}^1 + \boldsymbol{\pi} \cdot \dot{\mathbf{x}} - E_0. \quad (3.27)$$

Here φ^1 , together with its conjugate Π_1 , describe the spin degrees of freedom. Note that the reduced action depends on 8 independent canonical variables, which coincides with the dimension of the coadjoint orbit (see table 1, orbit #2).

3.3 Spinning vacuum action

While the vacuum has a trivial action the spinning vacuum can be obtained using the following representative of the coadjoint orbit

$$\alpha = (J, \mathbf{0}, \mathbf{0}, 0). \quad (3.28)$$

The resulting particle action can be obtained from the results of the preceding section by setting $E_0 = 0$ in (3.19), so we will not repeat the computation. Let us note that the spinning vacuum is described by the geometric action for SO(3) [14].

3.4 Massless carrollion action

Let us choose the following representative of the coadjoint orbit

$$\alpha = (J, \mathbf{0}, \mathbf{p}, 0), \quad (3.29)$$

where

$$\|\mathbf{p}\|^2 = p^2, \quad (3.30)$$

with constant $p > 0$. Using (3.2) we obtain the lagrangian

$$L[\boldsymbol{\varphi}, \mathbf{x}] = \frac{1}{2} \text{Tr} \left[J^T R^T \frac{\partial R}{\partial \varphi^a} \right] \dot{\varphi}^a + (R\mathbf{p})^T \dot{\mathbf{x}} \quad (3.31)$$

with nonzero Noether charges

$$\mathbf{j}_Q = R\mathbf{j} + \mathbf{x} \times R\mathbf{p} \qquad \mathbf{p}_Q = R\mathbf{p}. \quad (3.32)$$

Let us parametrise the rotation matrix as in (3.22) and align J and \mathbf{p} with the z -axis, i.e., $J = h\varepsilon_3$ with helicity h constant and $\mathbf{p} = (0, 0, p)$. The lagrangian is then given by

$$L[\mathbf{x}, \boldsymbol{\varphi}] = h(\dot{\varphi}_3 + \sin(\varphi_2)\dot{\varphi}_1) + p\hat{\mathbf{n}} \cdot \dot{\mathbf{x}}, \quad (3.33)$$

where the unit vector $\hat{\mathbf{n}}$ is defined in (3.24), and the constants h and p are kept fixed in the action. Using this parametrisation it is straightforward to show that

$$\|\mathbf{p}_Q\|^2 = p^2 \qquad \mathbf{j}_Q \cdot \mathbf{p}_Q = h\|\mathbf{p}_Q\|, \quad (3.34)$$

are invariant quantities, in perfect agreement with our analysis in section 2.2.2.

To write the massless carrollion action in canonical form we introduce the canonical momenta

$$\boldsymbol{\pi} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = p \hat{\boldsymbol{n}} \quad \Pi_1 = \frac{\partial L}{\partial \dot{\varphi}_1} = h \sin \varphi_2 \quad \Pi_2 = \frac{\partial L}{\partial \dot{\varphi}_2} = 0 \quad \Pi_3 = \frac{\partial L}{\partial \dot{\varphi}_3} = h \quad (3.35)$$

which satisfy the following constraints

$$\|\boldsymbol{\pi}\|^2 - p^2 = 0 \quad p\Pi_1 - h\pi_1 = 0 \quad \Pi_2 = 0 \quad \Pi_3 - h = 0. \quad (3.36)$$

Therefore, the lagrangian in canonical form becomes

$$L_{\text{can}}[\boldsymbol{\varphi}, \boldsymbol{x}, t, \boldsymbol{\Pi}, \boldsymbol{\pi}, E, N, \eta_i] = \quad (3.37)$$

$$\boldsymbol{\Pi} \cdot \dot{\boldsymbol{\varphi}} + \boldsymbol{\pi} \cdot \dot{\boldsymbol{x}} - Et - NE - \eta_0 (\|\boldsymbol{\pi}\|^2 - p^2) - \eta_1 (p\Pi_1 - h\pi_1) - \eta_2 \Pi_2 - \eta_3 (\Pi_3 - h).$$

Implementing the constraints $E = 0$, $\Pi_2 = 0$ and $\Pi_3 = h$, up to boundary terms, the lagrangian takes the form

$$L_{\text{can}}[\varphi^1, \boldsymbol{x}, \Pi_1, \boldsymbol{\pi}, \eta_0, \eta_1] = \Pi_1 \dot{\varphi}^1 + \boldsymbol{\pi} \cdot \dot{\boldsymbol{x}} - \eta_0 (\|\boldsymbol{\pi}\|^2 - p^2) - \eta_1 (p\Pi_1 - h\pi_1).$$

In the particular case when the spin vanishes, $h = 0$, the lagrangian (3.33) simplifies to

$$L[\boldsymbol{x}] = p \hat{\boldsymbol{n}} \cdot \dot{\boldsymbol{x}}, \quad (3.38)$$

while the lagrangian in canonical form becomes

$$L_{\text{can}}[\boldsymbol{x}, t, \boldsymbol{\pi}, E, N, \eta_0] = \boldsymbol{\pi} \cdot \dot{\boldsymbol{x}} - Et - NE - \eta_0 (\|\boldsymbol{\pi}\|^2 - p^2). \quad (3.39)$$

The action for the case $h = 0$ was previously studied in [19].

As discussed in appendix B.4, under the outer automorphisms of the Carroll algebra discussed in appendix A.2, the orbit discussed in this section is mapped to those of cases #5 and #7 (see table 1). As a consequence, the action for all these cases takes precisely the same intrinsic form. However, the physical interpretation may differ in general. We shall revisit this point in section 5 where we will provide a useful physical interpretation in the context of fractons and also discuss the mobility restrictions.

3.5 Generic massless carrollion action

The action for the generic massless case can be constructed using the following representative element of the coadjoint orbit

$$\alpha = (0, \boldsymbol{k}, \boldsymbol{p}, 0), \quad (3.40)$$

where \boldsymbol{k} and \boldsymbol{p} are nonzero and not parallel.

From (3.2) we obtain the lagrangian

$$L[\boldsymbol{\varphi}, \boldsymbol{x}, \boldsymbol{v}] = (R\boldsymbol{p}) \cdot \dot{\boldsymbol{x}} + (R\boldsymbol{k}) \cdot \dot{\boldsymbol{v}}.$$

The non-vanishing Carroll conserved charges of this action are given by

$$\mathbf{j}_Q = \mathbf{v} \times R\mathbf{k} + \mathbf{x} \times R\mathbf{p} \qquad \mathbf{k}_Q = R\mathbf{k} \qquad \mathbf{p}_Q = R\mathbf{p} \qquad (3.41)$$

From these expressions it is clear that we indeed recover

$$\|\mathbf{p}_Q\|^2 = p^2 \qquad \|\mathbf{v}_Q\|^2 = k^2 \qquad \mathbf{p}_Q \cdot \mathbf{v}_Q = pk \cos \theta, \qquad (3.42)$$

as discussed in section 2.2.3.

Since the action is more transparent in canonical form we introduce the canonical momenta are given by

$$\boldsymbol{\pi} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = R\mathbf{p} \qquad \boldsymbol{\pi}_v = \frac{\partial L}{\partial \dot{\mathbf{v}}} = R\mathbf{k} \qquad (3.43)$$

which are restricted to obey the constraints

$$\|\boldsymbol{\pi}\|^2 - p^2 = 0 \qquad \|\boldsymbol{\pi}_v\|^2 - k^2 = 0 \qquad \boldsymbol{\pi} \cdot \boldsymbol{\pi}_v - pk \cos \theta = 0. \qquad (3.44)$$

Therefore, the lagrangian in canonical form can be written as

$$L_{\text{can}} = -E\dot{t} + \boldsymbol{\pi} \cdot \dot{\mathbf{x}} + \boldsymbol{\pi}_v \cdot \dot{\mathbf{v}} - NE - \eta_1 \left(\|\boldsymbol{\pi}\|^2 - p^2 \right) - \eta_2 \left(\|\boldsymbol{\pi}_v\|^2 - k^2 \right) - \eta_3 \left(\boldsymbol{\pi} \cdot \boldsymbol{\pi}_v - pk \cos \theta \right), \qquad (3.45)$$

where $L_{\text{can}}[\mathbf{v}, \mathbf{x}, t, \boldsymbol{\pi}_v, \boldsymbol{\pi}, E, N, \eta_i]$. In addition to the position \mathbf{x} and its conjugate momentum $\boldsymbol{\pi}$, the state of a generic massless Carroll particle is also described by a vector field \mathbf{v} and its conjugate momentum $\boldsymbol{\pi}_v$, which can be considered as an additional internal degree of freedom. In section 5, we shall provide a physical interpretation of this case in the context of fractons and provide a discussion of the solutions. The counting of degrees of freedom shows that there are 6 independent canonical variables, in agreement with the dimension of the orbit.

4 From Poincaré to Carroll particles

In this section we study the carrollian limit of Poincaré particles. We intend to provide a useful orientation and not a comprehensive study of the possible limits. In short, there is a clear relation between massive spinning Poincaré and massive spinning Carroll particles. Massless carrollions seem to derive from a limit of Poincaré tachyons, and the limit of massless Poincaré particles seems to trivialise, see figure 2.

4.1 Poincaré particles

We first review some well-known facts about Poincaré particles. The coadjoint action of a Poincaré transformation (Λ, a) on the four-momenta $p^\mu = (E, \mathbf{p})$ is given by $p' = \Lambda p$, where we can see that the translations act trivially. It follows that $p_\mu p^\mu$ is an invariant on the coadjoint orbits. Using the rest mass M this foliates the momentum space into disjoint orbits

$$-p_\mu p^\mu = E^2 - \|\mathbf{p}\|^2 = \begin{cases} M^2 & \text{massive} \\ 0 & \text{massless} \\ -M^2 & \text{tachyonic} \end{cases} . \qquad (4.1)$$

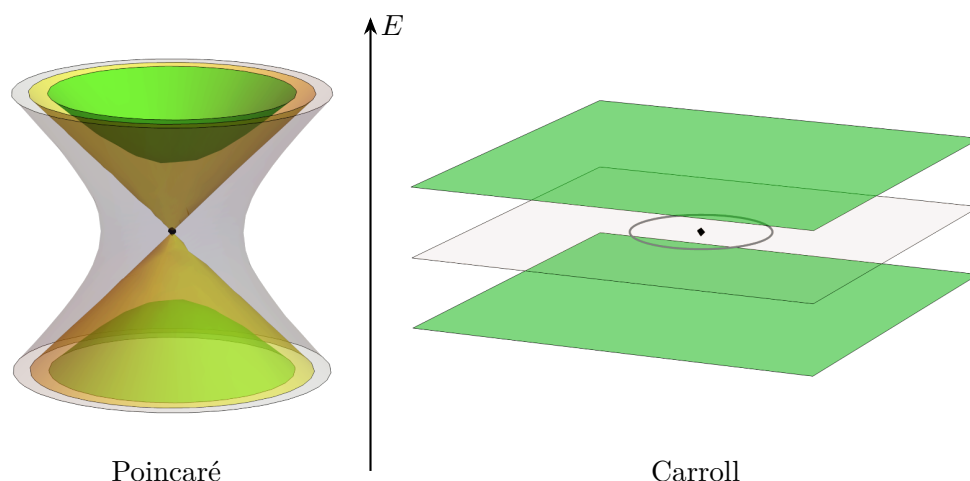


Figure 2. This figure shows the coadjoint orbits of the Poincaré and Carroll group in momentum space (E, \mathbf{p}) . The Poincaré orbits are given by two families of massive particles (green), two massless orbits (yellow), one family of tachyonic orbits (gray) and the vacuum (black dot).

In the Carroll limit the green massive Poincaré orbits (hyperboloids) flatten out and lead to the planes (see section 4.2.1). The lightcone would lead to a the $E = 0$ plane, but since $E = \|\mathbf{p}\|$ this orbit actually vanishes in the limit. The sphere ($\mathbf{p}^2 = \text{const.}$), represented as a circle in the $E = 0$ plane on the right-hand side, can be seen to arise as a limit of the tachyonic orbits (see section 4.2.3 for details). The whole carrollian $E = 0$ plane is foliated by such spheres.

Let us emphasise that this figure only represents the (E, \mathbf{p}) part of the full dual space $(\mathbf{j}, \mathbf{v}, \mathbf{p}, E)$ and the complete structure of the orbits is more intricate and involves spin degrees of freedom.

They are given by (see figure 2): a family of two-sheeted hyperboloids, with each sheet being an orbit, corresponding to the massive momenta, one family of one-sheeted hyperboloids, corresponding to tachyonic momenta and the future and past deleted lightcone, corresponding to massless momenta, and the origin corresponding to the vacuum.

In the following we will also use the second Poincaré invariant, the square of the Pauli-Lubański vector

$$W^2 = -(\mathbf{j} \cdot \mathbf{p})^2 + \|E\mathbf{j} + \mathbf{p} \times \mathbf{k}\|^2, \tag{4.2}$$

which should be contrasted with its carrollian analog (2.5).

4.2 Carroll limits

Let us now discuss the Carroll limit of the Poincaré coadjoint orbits. We do not claim that these are the unique limits and we will call the contraction parameters uniformly by c (which will not necessarily have the units of velocity).

4.2.1 Massive Poincaré to massive Carroll particles

Upon suitable rescalings the carrollian limit of the invariants of the Poincaré group lead to

$$\lim_{c \rightarrow 0} (-p_\mu p^\mu) = \lim_{c \rightarrow 0} (E^2 - c^2 \|\mathbf{p}\|^2) = E^2 = M^2 \tag{4.3}$$

and

$$\lim_{c \rightarrow 0} W^2 = \lim_{c \rightarrow 0} \left[-c^2(\mathbf{j} \cdot \mathbf{p})^2 + \|E\mathbf{j} + \mathbf{p} \times \mathbf{k}\|^2 \right] = \|E\mathbf{j} + \mathbf{p} \times \mathbf{k}\|^2 = E^2 S^2. \quad (4.4)$$

When we identify $M \rightarrow E_0$ we recover the invariants of the massive carrollions that were already discussed in section 2.1. Since they uniquely characterise the massive orbits of Poincaré and Carroll we can map for any spin S the massive Poincaré particles to massive Carroll particles and vice versa.

Geometrically the Carroll limit of (4.3) implies that in momentum space (E, \mathbf{p}) the hyperbolic mass-shells get flattened to planes with $E = \pm E_0$, cf., the green surfaces in figure 2. The stabiliser of the momentum orbit is in both cases given by $\text{SO}(3)$, which implies a close relation between the induced representations.

4.2.2 Carroll limit of the vacuum sector

If we restrict to the Poincaré vacuum sector $p^\mu = 0$ the action of the translations on the coadjoint orbits trivialises and we are left with the coadjoint orbits of the Lorentz group $\text{SO}(3,1)$. In this sector we have the invariants $-\|\mathbf{k}\|^2 + \|\mathbf{j}\|^2$ and $\mathbf{j} \cdot \mathbf{k}$ (they derive from $j^{\mu\nu} j_{\mu\nu}$ and $\epsilon^{\mu\nu\rho\xi} j_{\mu\nu} j_{\rho\xi}$, respectively).

Taking the limit

$$\lim_{c \rightarrow 0} (\|\mathbf{k}\|^2 - c^2 \|\mathbf{j}\|^2) = \|\mathbf{k}\|^2, \quad (4.5)$$

and leaving the other invariant unaltered provides a limit to what we called centrons in table 1. So they derive from the Poincaré vacuum sector.

Taking the dual limit of (4.5) leads to the invariants of the spinning vacua $\|\mathbf{j}\|^2 = j^2$.

4.2.3 Carroll limit of massless and tachyonic Poincaré particles

For the remaining orbits we restrict the discussion to energy and momentum space. One can then take the dual limit of (4.3), upon suitable rescaling, given by

$$\lim_{c \rightarrow 0} (p_\mu p^\mu) = \lim_{c \rightarrow 0} (-c^2 E^2 + \|\mathbf{p}\|^2) = \|\mathbf{p}\|^2 = M^2. \quad (4.6)$$

When we identify $M \rightarrow p$ we recover the relation $\|\mathbf{p}\| = p$ which was already discussed in section 2. This limit is just the restriction of the three dimensional de Sitter space to the two spheres of radius p in the $E = 0$ plane, see the gray hyperboloid and circle in figure 2. This suggests that the massless carrollions emerge from tachyons.

From this perspective is it also clear that the massless Poincaré particles lead to $E = 0$ and $\mathbf{p} = \mathbf{0}$. We start by setting $M = 0$ in (4.1), which leads to the lightcone

$$E^2 = \|\mathbf{p}\|^2. \quad (4.7)$$

If we want to keep rotational invariance we can send either side to zero, but that implies that both sides vanish. Geometrically we can think about it as light cone that opens up with the limit being the $E = 0$ plane. But due to the relation (4.7) now \mathbf{p} also vanishes.

5 Carroll/fracton correspondence

In this section we explore the relationship between the Carroll and dipole symmetries. In particular, we shall focus on models with a dipole symmetry. As it was explained, their symmetries are closely related, and therefore one can map carrollions to fractons using the Carroll/fracton correspondence. Fracton monopoles are related to massive carrollions, both of which cannot move. On the other hand, the massless carrollions are related to fractonic dipoles. Some of the unusual properties of carrollions and fractons will be elucidated by an analysis based on the use of two fundamental monopoles, together with the application of appropriate constraints on the phase space.

5.1 Fracton symmetry and Carroll/fracton correspondence

From a purely theoretical perspective, fractons are very puzzling objects, as they seem to defy the standard methods of quantum field theory (see the reviews [27–29] for (original) references, further applications and details). In particular, for the complex scalar field theory introduced in [30], there exists a conserved dipole charge in addition to the electric charge. Suppose these symmetries act as

$$\phi(t, \mathbf{x}) \rightarrow e^{i(\alpha + \mathbf{v} \cdot \mathbf{x})} \phi(t, \mathbf{x}), \quad (5.1)$$

where α and \mathbf{v} parametrise the charge and dipole transformations, respectively. When we then insist on spatial derivatives in the action, terms like $\partial_i \phi^* \partial_i \phi$ are forbidden and we are led to non-gaussian and non-lorentzian theories [10] (see (6.11) for an explicit action). An immediate consequence is that the conventional expansion of the fields in terms of oscillators, commonly used in quantum field theory to define particle states, is not directly applicable to fractonic models. Therefore, a natural unanswered question is how to determine the elementary excitations or particles of fractonic theories.

To answer this question we can use a remarkable fact [10] (see also [11]): *the symmetries of fractonic theories with a conserved electric and dipole charges coincide with the Carroll symmetries, up to the inclusion of an additional central element.* Thus, the analysis of the coadjoint orbits developed for carrollian theories in the previous sections can be easily extended to characterise the elementary excitations of fractonic models with dipole charges. While they have different interpretations it is interesting to think about carrollions from the fracton perspective and vice versa.

The generators of the dipole algebra [7], spanned by the fracton energy H_F , angular momentum L_{ab} , linear momentum P_a , dipole charge D_a , and electric charge Q , have the following non-vanishing commutation relations

$$\begin{aligned} [L_{ab}, L_{cd}] &= \delta_{bc} L_{ad} - \delta_{ac} L_{bd} - \delta_{bd} L_{ac} + \delta_{ad} L_{bc} \\ [L_{ab}, P_c] &= \delta_{bc} P_a - \delta_{ac} P_b \\ [L_{ab}, D_c] &= \delta_{bc} D_a - \delta_{ac} D_b \\ [D_a, P_b] &= Q \delta_{ab} \end{aligned} \quad (5.2)$$

This is precisely the Carroll algebra (A.1), when we apply $Q \mapsto H$ and $D_a \mapsto B_a$ and ignore the additional generator H_F that commutes with all the generators of the algebra. This

Carroll particles	Fractonic particles
angular momentum \mathbf{j}	angular momentum \mathbf{j}
center-of-mass \mathbf{k}	dipole moment \mathbf{d}
momentum \mathbf{p}	momentum \mathbf{p}
energy E	charge q
—	energy E

Table 2. Correspondence of conserved quantities between carrollian and fractonic theories.

implies that in a Carroll/fracton correspondence the Carroll energy is in correspondence with the fractonic electric charge, and the carrollian center-of-mass is related to the dipole charge. The fracton energy, which commutes with all the generators, does not have a counterpart in the Carroll algebra. Table 2 exhibits the precise correspondence between the conserved quantities of carrollian and fractonic theories. In [31] we will provide a study of other fracton-like models and provide their correspondence with different non-relativistic systems.

While the Carroll/fracton correspondence might be useful in general we will now apply it on the level of the elementary particles. In what follows, we shall use the results of the previous sections to identify the fracton particles. Like carrollions, fractons fall into two categories: when the charge q vanishes they correspond to (immobile) monopoles, but when the charge is nonzero they are given by neutral fractons, in particular dipoles which are mobile.

This can be seen by applying the correspondence of table 2 to the coadjoint action (2.4). The coadjoint action for fractons acts therefore on $\alpha = (\mathbf{j}, \mathbf{d}, \mathbf{p}, q, E)$, where the conserved quantities are described in table 2, via the coadjoint $\text{Ad}_g^* \alpha = (\mathbf{j}', \mathbf{d}', \mathbf{p}', q', E')$ as follows

$$\begin{aligned}
 \mathbf{j}' &= R\mathbf{j} + \mathbf{v} \times R\mathbf{d} + \mathbf{a} \times R\mathbf{p} + q\mathbf{v} \times \mathbf{a} \\
 \mathbf{d}' &= R\mathbf{d} + q\mathbf{a} \\
 \mathbf{p}' &= R\mathbf{p} - q\mathbf{v} \\
 q' &= q \\
 E' &= E
 \end{aligned} \tag{5.3}$$

The group element $g = (R, \mathbf{v}, \mathbf{a}, \lambda, s)$ is given by rotations R , dipole transformations \mathbf{v} , spatial translations \mathbf{a} , charge rotations λ and temporal translations s (where the last two act trivially). A few remarks concerning the fracton coadjoint orbits are in order:

- By inspecting (5.3) and by our discussions of the Carroll symmetries it is clear that E , q and $W^2 = \|q\mathbf{j} + \mathbf{p} \times \mathbf{d}\|^2$ are invariants.
- The coadjoint action indeed agrees with the intuition of the transformation behaviour of the well-known dipole moment $\mathbf{d} = \int \mathbf{x}\rho(\mathbf{x})d^3x$ of electrodynamics (ρ is the charge density). Suppose we have a particle with nonzero charge q at the origin, then $\rho(\mathbf{x}) = q\delta(\mathbf{x})$ and $\mathbf{d} = \mathbf{0}$. If we now shift the particle to the point \mathbf{a} we obtain $\rho(\mathbf{x}) = q\delta(\mathbf{x} - \mathbf{a})$ and as expected a non-vanishing dipole moment given by $\mathbf{d}' = q\mathbf{a}$.

This is just a manifestation of the well-known fact that the dipole moment for nonzero charge depends on the choice of origin, see, e.g, [32].

This shift is also precisely the one we obtain by the coadjoint action of a pure translation \mathbf{a} , with group element $g = (R = 1, \mathbf{v} = \mathbf{0}, \mathbf{a}, \lambda = 0, s = 0)$, on a charged particle with zero dipole moment $\alpha = (\mathbf{j} = \mathbf{0}, \mathbf{d} = \mathbf{0}, \mathbf{p} = 0, q, E)$, see (5.3).

- A subtle difference between Carroll and fracton particles is the global group structure. While we assumed that the action of Carroll time translations results in noncompact orbits (so that the subgroup generated by H is isomorphic to \mathbb{R}), the group of charge phase rotations is expected to be $U(1)$. This generator is however central and therefore its coadjoint action is trivial and hence, at the level of the coadjoint orbits, the topology of the subgroup it generates is immaterial.

The elementary fractonic particles are physically interpreted as charged monopoles and different classes of neutral fractons, in particular dipoles. Their dynamics is described by actions that are in a one-to-one correspondence with those of section 3. In particular, the analysis of the dynamics of these systems will reveal the restricted mobility of fractonic particles. For simplicity in the presentation, we shall only consider the cases of spinless particles. The actions for spinning fractons can be directly derived by using the results of section 3. So the fractonic dual of (3.2) and the starting point for our analysis of the actions $S = \int L d\tau$ is

$$\begin{aligned}
 L[R(\varphi), \mathbf{v}, \mathbf{x}, \phi, t] &= \frac{1}{2} \text{Tr} \left(J^T R^T \dot{R} \right) + (R\mathbf{d}) \cdot \dot{\mathbf{v}} + (R\mathbf{p}) \cdot \dot{\mathbf{x}} + q \left(\dot{\Phi} + \frac{1}{2} \mathbf{x} \cdot \dot{\mathbf{v}} - \frac{1}{2} \mathbf{v} \cdot \dot{\mathbf{x}} \right) - Et \\
 &= \frac{1}{2} \text{Tr} \left(J^T R^T \dot{R} \right) + (R\mathbf{d} + \frac{1}{2} q \mathbf{x}) \cdot \dot{\mathbf{v}} + (R\mathbf{p} - \frac{1}{2} q \mathbf{v}) \cdot \dot{\mathbf{x}} + q \dot{\Phi} - Et \quad (5.4)
 \end{aligned}$$

where the canonical pair (q, Φ) represents the total electric charge q and its canonical conjugate Φ .

5.2 Fractonic monopole ($q \neq 0$)

Let us consider a fractonic system with non-vanishing total electric charge. This means an element of the dual space of the form $(\mathbf{j}, \mathbf{d}, \mathbf{p}, E, q)$, where everything vanishes except $E = E_0$ and $q = e$, for constants $E_0 \in \mathbb{R}$ and $e \neq 0$ (this corresponds to massive spinless carrollions).

The particle lagrangian associated to this orbit can be directly obtained from the results in section 3.1. For fractons this case can be physically interpreted as the description of the dynamics of an elementary monopole carrying a total electric charge of magnitude e . The lagrangian in canonical form can be written as

$$L_{\text{can}}[\mathbf{x}, q, t, \boldsymbol{\pi}, \Phi, E, N, \eta] = -Et + \boldsymbol{\pi} \cdot \dot{\mathbf{x}} + \Phi \dot{q} - N(E - E_0) - \eta(q - e). \quad (5.5)$$

While the Lagrange multipliers N and η enforce that the energy is E_0 and the charge e , the variation with respect to $\boldsymbol{\pi}$ leads to

$$\dot{\mathbf{x}} = \mathbf{0}. \quad (5.6)$$

Therefore we recover the characteristic feature that a single fractonic monopole cannot move, which is now a consequence of the action (5.5).

The non-vanishing conserved charges that realise the fractonic algebra as Poisson brackets are given by the charges

$$\mathbf{j}_Q = \mathbf{x} \times \boldsymbol{\pi} \quad \mathbf{d}_Q = e\mathbf{x} \quad \mathbf{p}_Q = \boldsymbol{\pi} \quad q_Q = e \quad E_Q = E_0 \quad (5.7)$$

where \mathbf{j}_Q is the total angular momentum, \mathbf{p}_Q the linear momentum, q_Q the total electric charge, \mathbf{d}_Q the dipole charge and E_Q the fractonic energy. As expected and due to (5.6) the dipole charge is indeed conserved $\dot{\mathbf{d}} = \mathbf{0}$. We also recover the characteristic commutation relations

$$\{\mathbf{d}_Q, \mathbf{p}_Q\} = q_Q \mathbf{1}. \quad (5.8)$$

The lagrangian defined on the reduced phase space can be derived by solving the constraints and imposing the gauge fixing $t = \tau$. Neglecting boundary terms one finds

$$L_{\text{red}}[\mathbf{x}, \boldsymbol{\pi}] = \boldsymbol{\pi} \cdot \dot{\mathbf{x}} - E_0 = \frac{1}{e} \mathbf{p}_Q \cdot \dot{\mathbf{d}}_Q - E_0 \quad (5.9)$$

which again manifests the canonical relation between linear momentum and dipole moment (5.8) and shows the equivalence to the massless carrollion upon correspondence.

5.3 Fractonic dipole ($q = 0$, $\mathbf{d} \neq \mathbf{0}$, $\mathbf{p} = \mathbf{0}$)

In analogy to the electromagnetic theory, an elementary dipole has a vanishing electric charge, and a nonzero dipole moment. Consequently, this fractonic excitation can be described by massless carrollions (with zero helicity for simplicity, see table 1).

The lagrangian can be written as

$$L_{\text{can}}[\mathbf{d}, t, \mathbf{v}, E, N, \eta] = -Et + \mathbf{d} \cdot \dot{\mathbf{v}} - N(E - E_0) - \eta \left(\|\mathbf{d}\|^2 - d^2 \right). \quad (5.10)$$

Here \mathbf{d} is the dipole moment, and can be considered as a fundamental degree of freedom which is varied in the action. The constraint enforced by η tells us that the constant d (which is not varied) fixes the magnitude of the dipole moment. Solving the constraint $E - E_0$ and imposing the gauge condition $t = \tau$, the lagrangian can be rewritten as

$$L_{\text{can}}[\mathbf{d}, \mathbf{v}, \eta] = \mathbf{d} \cdot \dot{\mathbf{v}} - E_0 - \eta \left(\|\mathbf{d}\|^2 - d^2 \right). \quad (5.11)$$

The position of the dipole is not a dynamical variable in the lagrangian due to the vanishing of its conjugate momentum, the linear momentum. Consequently, the action does not specify the position of the dipole in space, and as a result, there are no mobility restrictions for this pure dipole. This action should be compared with the massless carrollions in section 3.4 to which they correspond to.

To make this even more manifest it is useful to remember that the original action indeed had a dependence on \mathbf{x} which dropped out since we looked at orbits with $\mathbf{p} = \mathbf{0}$. If we do not integrate it out, as we did earlier, but calculate its canonical momentum

$\boldsymbol{\pi} = \partial L / \partial \dot{\boldsymbol{x}} = 0$ we find the constraint that the canonical momentum vanishes. The action is then given by

$$L_{\text{can}}[\mathbf{d}, \mathbf{x}, \mathbf{v}, \boldsymbol{\pi}, \eta, \mathbf{u}] = \mathbf{d} \cdot \dot{\mathbf{v}} + \boldsymbol{\pi} \cdot \dot{\mathbf{x}} - E_0 - \eta \left(\|\mathbf{d}\|^2 - d^2 \right) - \mathbf{u} \cdot \boldsymbol{\pi} \quad (5.12)$$

where \mathbf{u} are Lagrange multipliers that enforce the constraint that $\boldsymbol{\pi}$ vanishes. The corresponding gauge transformations tell us that \mathbf{x} is arbitrary, i.e., the position of the dipoles is not restricted. We can also see this from the equation of motion coming from the variation of $\boldsymbol{\pi}$ which leads to

$$\dot{\mathbf{x}} = \mathbf{u} \quad (5.13)$$

where the Lagrange multiplier \mathbf{u} is an arbitrary function of time. So again there is no restriction on the position, in drastic contradistinction to the monopoles.

5.4 Generic fractonic dipole ($q = 0, \mathbf{d} \neq 0, \mathbf{p} \neq 0$)

Let us consider the lagrangian associated with generic dipoles (corresponding to generic massless carrollions in table 1)

$$\begin{aligned} L_{\text{can}} = & -E\dot{t} + \mathbf{p} \cdot \dot{\mathbf{x}} + \mathbf{d} \cdot \dot{\mathbf{v}} + \Phi\dot{q} - N(E - E_0) - \eta_0 q - \eta_1 \left(\|\mathbf{p}\|^2 - p^2 \right) \\ & - \eta_2 \left(\|\mathbf{d}\|^2 - d^2 \right) - \eta_3 (\mathbf{p} \cdot \mathbf{d} - pd \cos \theta) , \end{aligned} \quad (5.14)$$

where $L_{\text{can}}[\mathbf{v}, \mathbf{x}, q, t, \mathbf{d}, \mathbf{p}, \Phi, E, N, \eta_i]$.

The total electric charge vanishes for this orbit. Therefore, this case can be interpreted as a dipole \mathbf{d} with an additional degree of freedom that describes its position in space \mathbf{x} .

The non-vanishing charges of the dipole algebra are $\mathbf{p}_Q = \mathbf{p}$, $\mathbf{d}_Q = \mathbf{d}$ together with

$$\mathbf{j}_Q = \mathbf{x} \times \mathbf{p} + \mathbf{v} \times \mathbf{d} \quad E_Q = E_0 . \quad (5.15)$$

After solving the trivial constraints $q = 0$, $E - E_0 = 0$, and imposing the gauge fixing condition $t = \tau$, the lagrangian can be rewritten in the following form

$$L_{\text{red}} = \mathbf{p} \cdot \dot{\mathbf{x}} + \mathbf{d} \cdot \dot{\mathbf{v}} - E_0 - \eta_1 \left(\|\mathbf{p}\|^2 - p^2 \right) - \eta_2 \left(\|\mathbf{d}\|^2 - d^2 \right) - \eta_3 (\mathbf{p} \cdot \mathbf{d} - pd \cos \theta) , \quad (5.16)$$

where $L_{\text{red}}[\mathbf{v}, \mathbf{x}, \mathbf{d}, \mathbf{p}, N, \eta_i]$.

Before we discuss solutions of this model, let us provide a physical interpretation of the action (5.16). To that end it is useful to construct a dipole by considering two monopoles with opposite charges separated by a small distance. This precisely corresponds to the notion of an ideal dipole in electrodynamics. This system, by definition, would not be considered an elementary object, but rather a composite one. However, once the constraints

$$\|\mathbf{p}\|^2 = p^2 \quad \|\mathbf{d}\|^2 = d^2 \quad \mathbf{p} \cdot \mathbf{d} = pd \cos \theta \quad (5.17)$$

are imposed, the system can be regarded as a fundamental one.

5.5 Dipoles from two monopoles

To obtain a clear physical interpretation of the above systems it is instructive to construct a dipole in terms of two monopoles.¹ Thus, let us consider two monopoles with opposite charges of equal magnitude e and $-e$. The action for the generic dipole is constructed by taking the sum of the actions for each individual monopole and subsequently imposing the constraints (5.17).

As a starting point, let us consider the lagrangian (5.5), describing two monopoles

$$L[\mathbf{x}_i, \boldsymbol{\pi}_i] = \boldsymbol{\pi}_1 \cdot \dot{\mathbf{x}}_1 + \boldsymbol{\pi}_2 \cdot \dot{\mathbf{x}}_2 - (E_1 + E_2). \quad (5.18)$$

The total dipole moment, linear momentum and energy of the system are given by

$$\mathbf{d} = e(\mathbf{x}_1 - \mathbf{x}_2) \quad \mathbf{p} = \boldsymbol{\pi}_1 + \boldsymbol{\pi}_2 \quad E = E_1 + E_2. \quad (5.19)$$

As discussed previously, this system cannot be considered as fundamental without imposing the constraints (5.17). This system is at this stage not elementary; that is, it is analogous to a reducible, rather than to an irreducible, representation.

Introducing the following quantities

$$\mathbf{x} := \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2) \quad \mathbf{v} := \frac{1}{2e}(\boldsymbol{\pi}_2 - \boldsymbol{\pi}_1), \quad (5.20)$$

and imposing the constraints (5.17), we obtain precisely the lagrangian of the generic fractonic dipole (5.16). The degrees of freedom of this system are characterised by the position \mathbf{x} of the dipole, which is given by the average of the positions of each monopole, and the total dipole moment \mathbf{d} , as well as their respective canonical conjugate momenta \mathbf{p} and \mathbf{v} . These variables are subject to the constraints in (5.17). Out of two elementary monopoles, which together are reducible, we have thus created an elementary fundamental system, the generic fractonic dipole.

The equations of motion imply the conservation of the momentum and dipole moment, $\dot{\mathbf{p}} = \dot{\mathbf{d}} = \mathbf{0}$. In addition, one has

$$\dot{\mathbf{x}} = 2\eta_1\mathbf{p} + \eta_3\mathbf{d} \quad \dot{\mathbf{v}} = 2\eta_2\mathbf{d} + \eta_3\mathbf{p}. \quad (5.21)$$

In analogy with the gauge fixing used in [19], let us impose the following conditions:

$$\|\dot{\mathbf{x}}\| - p = 0 \quad \|\dot{\mathbf{v}}\| - d = 0. \quad (5.22)$$

The equations of motion lead to $\eta_1 = \eta_2 = 1/2$ and $\eta_3 = 0$. Therefore, they can be integrated as follows

$$\mathbf{x}(t) = \mathbf{p}t + \mathbf{x}_0, \quad \mathbf{v}(t) = \mathbf{d}t + \mathbf{v}_0. \quad (5.23)$$

Again, it is evident that the elementary dipole is not constrained to remain static, in contrast to the fractonic monopole. However, in contradistinction to the dipole of section 5.3, it can be seen from (5.23) that the spatial evolution is not completely undetermined anymore.

This realisation of the generic dipole in terms of two monopoles also elucidates some of the particles we have already discussed.

¹In the carrollian case, a similar construction can be performed, although instead of a dipole created from two monopoles of opposite charges, one must instead consider a particle/antiparticle pair with energies of the same magnitude but opposite sign.

5.5.1 Dipole moment as a degree of freedom

Let us consider the particular case when $\mathbf{p} = \mathbf{0}$. According to (5.19), this implies that $\pi_1 = -\pi_2$. In turn, this implies that the generic fractonic dipole (5.16) reduces to (5.10), which is the action that describes exclusively the degree of freedom that is associated with the dipole moment.

5.5.2 Neutral carrollian particle from dipoles

In the case when $q = 0$ and $\mathbf{d} = \mathbf{0}$, from (5.19) one finds $\mathbf{x} = \mathbf{x}_1 = \mathbf{x}_2$. Therefore, both monopoles are located at exactly the same position. This should not be confused with the definition of an elementary dipole, where the separation between the monopoles approaches zero while the electric charge tends to infinity, resulting in a finite dipole moment $\mathbf{d} \neq \mathbf{0}$. In this case, the electric charge of each monopole is opposite but finite, resulting in an elementary particle that is electrically neutral and described by the following lagrangian

$$L[\mathbf{x}, \mathbf{p}, \eta_1] = \mathbf{p} \cdot \dot{\mathbf{x}} - E_0 - \eta_1 \left(\|\mathbf{p}\|^2 - p^2 \right). \tag{5.24}$$

The equations of motion derived from this lagrangian are equivalent to those of a massless carrollian particle with zero helicity which also has vanishing total electric charge.

Fundamentally this relation is rooted in the fact that the distinguishing feature, centre-of-mass and dipole charge, vanish. The Carroll and fracton particles are not just dual to each other, but in this case even physically equivalent and given by an aristotelion as already discussed in section 2.2.2.

6 Field theories and generalisation to curved space

In this section we provide some remarks concerning the relation to known “electric” and “magnetic” carrollian field theories, the difference between Carroll boost and dipole symmetry and comment on the generalisation of the Carroll/fracton correspondence to curved space.

6.1 Massive and massless Carroll field theories

Given our understanding of the elementary particles it is natural to ask if there are field theory realisations for which they can be seen as excitations. We will elaborate this point in detail in our future work [33], but let us nevertheless provide some remarks. To that end we restrict for simplicity to massive spin zero and massless carrollions with zero helicity.

Using Dirac quantisation, see, e.g., [34, section 13], for the constraint $E - E_0 = 0$ and using $E \mapsto i\partial_t$ we obtain the following equation for a massive carrollian spin 0 field $\phi(t, \mathbf{x})$

$$(i\partial_t - E_0)\phi = 0. \tag{6.1}$$

This equation can be derived from an action of the form

$$S_{E_0 \neq 0}[\phi, \phi^*] = \int dt d^3x \left(i\phi^* \dot{\phi} - E_0 \phi^* \phi \right). \tag{6.2}$$

If we want to consider particles and antiparticles at once (in which case we can for simplicity restrict to a real scalar) we obtain

$$(\partial_t^2 + E_0^2)\phi = 0, \tag{6.3}$$

which we can derive from

$$S_{E_0^2 \neq 0}[\phi] = \frac{1}{2} \int dt d^3x (\dot{\phi}^2 - E_0^2 \phi^2). \tag{6.4}$$

This action agrees with the ultralocal or “electric” field theories considered in [19, 35–37]. Similarly to the particles, see section 3.1.1, these actions also admit a symmetry enhancement and are not only invariant under linear Carroll boosts but under $\delta\phi = -f(\mathbf{x})\partial_t\phi$ where $f(\mathbf{x})$ is a free function (these symmetries were called spacetime subsymmetries in [38]).

For the massless carrollion with helicity $h = 0$ we obtain using the constraints $E = 0$ and $\|\mathbf{p}\|^2 - p^2 = 0$ the following equations

$$\partial_t\phi = 0 \qquad (\partial_i\partial^i + p^2)\phi = 0. \tag{6.5}$$

Let us contrast these equations with what is sometimes called “magnetic” carrollian theory in the literature [19, 37] (similar actions have also appeared in the context of flat space holography and deformations in lower dimensions [39, 40])

$$S_{\text{magnetic}}[\phi, \pi] = \int dt d^3x \left(\pi\dot{\phi} - \frac{1}{2}\partial_i\phi\partial^i\phi + \frac{1}{2}p^2\phi^2 \right). \tag{6.6}$$

The variation of π indeed leads to the first equation in (6.5), the variation of ϕ however provides

$$(\partial_i\partial^i + p^2)\phi = \dot{\pi}, \tag{6.7}$$

which has a source term with respect to (6.5). One could try to remedy this by considering an action of the form

$$S_{E_0=0}[\phi, \pi, u] = \int dt d^3x \left(\pi\dot{\phi} - u(\Delta + p^2)\phi \right), \tag{6.8}$$

which leads indeed to the equations (6.5), we leave however further explorations to our future work [33].

6.2 Carroll boost versus dipole symmetry for field theories

Let us provide some cautionary remarks concerning the Carroll/fracton correspondence, for simplicity we restrict to scalar fields.

Following the action of Carroll symmetry on spacetime (2.1) the Carroll boosts act as

$$\phi(t, \mathbf{x}) \mapsto \phi(t - \mathbf{v}_C \cdot \mathbf{x}, \mathbf{x}) \tag{6.9}$$

on complex scalar fields, while linear dipole transformations act as [30]

$$\phi(t, \mathbf{x}) \mapsto e^{i\mathbf{v}_F \cdot \mathbf{x}} \phi(t, \mathbf{x}). \tag{6.10}$$

In this case it is clear that these symmetries are inequivalent, while Carroll boosts are spacetime symmetries the dipole symmetries are internal symmetries and do not act on the geometry.

There also exist theories that admit, both, either or none of these symmetries, e.g., let us consider the complex scalar $\phi(t, \mathbf{x})$ theory [30]

$$S[\phi] = \int dt d^3x \left(\dot{\phi}\dot{\phi}^* - \lambda X_{ij} X_{ij}^* \right) \tag{6.11}$$

where $X_{ij} = \partial_i \phi \partial_j \phi - \phi \partial_i \partial_j \phi$. For $\lambda = 0$ this theory has Carroll boost symmetry $\delta\phi = -(\mathbf{v}_C \cdot \mathbf{x}) \partial_t \phi$ and dipole symmetry $\delta\phi = i(\mathbf{v}_F \cdot \mathbf{x})\phi$.² When $\lambda \neq 0$ only the dipole symmetry remains and the theory has no Carroll boost symmetry. On the other hand the real scalar field $L[\phi] = \frac{1}{2}\dot{\phi}^2$ has only Carroll boost symmetry and when the gradient term $\frac{1}{2}\partial_i \phi \partial^i \phi$ is added it has neither.

From this perspective dipole symmetry is an internal symmetry, similar to, e.g., internal spin degrees of freedom or internal $SU(n)$ symmetries. In particular for models of the type (6.11) dipole conservation is not related to a spacetime symmetry and is therefore different from Carroll boosts [10]. For this reason it was argued in [10, 41] that the geometry to which fracton theories of the type (6.11) are coupled is aristotelian, and therefore it does not admit boosts. See also [42] for the (asymptotic) analysis of the gauge theory sector which also finds aristotelian symmetries.

What gives rise to the correspondence on the level of the particle is the fact that coadjoint orbits and therefore the intrinsic definition of elementary particles are based on the group structure and not necessarily the underlying spacetime geometry. However it would be interesting to see if there is more to be learned about this correspondence between internal and external symmetries, see, e.g., the interesting recent works [38, 43, 44].

6.3 (A)dS Carroll and fractons on curved space

Our discussions so far were focused on flat Carroll space, but we would like to mention a possible generalisation to curved space. More precisely to (A)dS Carroll which can be thought of the carrollian analogs of (anti-)de Sitter space [13] from which they arise as a limit [45].

On the level of the symmetries this means that the flat Carroll symmetries (A.1) have the following additional commutation relations

$$[P_a, P_b] = -\Lambda J_{ab} \qquad [P_a, H] = \Lambda B_a \tag{6.12}$$

where the cosmological constant $\Lambda < 0$ leads to AdS Carroll and $\Lambda > 0$ to dS Carroll. They share similarities with their lorentzian counter parts and are therefore interesting candidates for holography and cosmology. But rather than lorentzian they also have carrollian boosts. Another interesting property of AdS Carroll is that upon the exchange of boosts and translations the symmetries are isomorphic to Poincaré symmetries. This

²For $\lambda = 0$ this theory has even more “supertranslation-like” symmetries (see, e.g., [10, section 2.4 and 2.6]), but this is not important for the argument.

means that for this case the particles should be describable in terms of well known Poincaré particles. Furthermore, AdS Carroll is closely related to time-like infinity of asymptotically flat spacetimes [46].

Using our correspondence means that the dipole algebra (5.2) obtains the following additional commutation relations

$$[P_a, P_b] = -\Lambda J_{ab} \qquad [P_a, Q] = \Lambda D_a. \qquad (6.13)$$

The first commutation relation implies that they are now living on hyperbolic space ($\Lambda < 0$) or on a sphere ($\Lambda > 0$). The second commutation relation implies that the charge is no longer central and that it is related to the dipole moment via the Casimir $\Lambda Q^2 + \|\mathbf{D}\|^2$. For $\Lambda < 0$ we see an emergent Lorentz symmetry, of course related to the underlying Poincaré symmetry. It would be interesting to further explore the Carroll/fracton correspondence in these curved spaces.

7 Discussion and outlook

This work provides a definition and classification of classical Carroll particles and fractons in $3 + 1$ dimensions, summarised in table 1. Based on the known relation between Carroll and dipole symmetries and their free theories [10] (see also [11]) we propose a correspondence on the level of the elementary particles, which is summarised in table 2, and show that while their physical interpretations differ they are indeed equivalent (at least classically on the reduced phase space).

The Carroll/fracton correspondence is indeed useful to obtain physical insights. For instance, isolated massive carrollions are stuck to a point due to the conservation of the center-of-mass charge for the very same reason that fracton monopoles are stuck to a point due to dipole conservation. A carrollian way to think about this property would be to think about the closing of the Minkowskian light cone which also implies immobility in space. On the other hand it might be useful to think about massless carrollions as moving dipoles.

Given that both of these subjects connect to various interesting areas of current research it is clear that many things can be said. Let us now relate our results to various other interesting topics and provide some areas for future exploration.

Quantum Carroll/fracton particles.

It is well-known that coadjoint orbits [8, 47, 48] and particles actions [14] provide a fruitful starting point for quantisation. In a future work [33] we will look at the quantum particles to which this correspondence generalises.

Field theories.

It is natural to try to systematically connect these particles to field theories on Carroll spacetime. We will show [33] that of the unitary irreducible representations of the Carroll group, there are two classes of representations which can be realised as (finite-component) fields on Carroll spacetime. The first class are the massive carrollions (except that the spin is quantised) and the second class as the massless

carrollions with helicity (which is also quantised). The former are related to electric field theories, whereas the latter to magnetic field theories, as discussed in section 6.1.

Relation to time-like symmetries.

In [49] fractons were described using “time-like” higher-form global symmetries. It might be interesting to understand the relation between these generalised symmetries and our results.

Planons, lineons and other exotic particles.

This work highlights the applicability of the orbit method beyond the conventional framework and we will show that this also generalises to other exotic particles with restricted mobility [31].

In particular the symmetries of planons are isomorphic to the Bargmann (=centrally extended Galilei) algebra [7]. Using the methods described in this work we can relate the respective particles, e.g., planons are related to massless galilean particles [31].

The worldline description has also been applied to fractonic theories with subsystem symmetries [50].

Other dimensions.

Much of what has been said should be generalisable to generic dimension. Let us however remark that in $2 + 1$ dimensions there are nontrivial central extensions which would make this correspondence more involved, cf., [51].

Fractons in flat holography and black holes.

Given that carrollian symmetries have emerged in flat holography (see, e.g., [46, 52–57]) and for black holes (see, e.g., [11, 58]) it is intriguing to try understand them from a fractonic perspective, see also our comments in section 6.3.

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A Carroll symmetry

In this appendix we define the Carroll algebra, the Carroll group, discuss the adjoint and coadjoint actions, automorphisms and their effect on coadjoint orbits and the Maurer-Cartan one-forms. We do this in general dimension before specialising to dimension $3 + 1$.

A.1 The Carroll group and its Lie algebra

The Carroll group acts via affine transformations on Carroll spacetime, the ultra-relativistic limit of Minkowski spacetime [1, 2]. The $(n + 1)$ -dimensional Carroll algebra \mathfrak{g} , by which we mean the Carroll algebra acting on $(n + 1)$ -dimensional Carroll spacetime, is spanned by $J_{ab} = -J_{ba}, B_a, P_a, H$, with $a, b = 1, \dots, n$, with nonzero Lie brackets

$$\begin{aligned} [J_{ab}, J_{cd}] &= \delta_{bc}J_{ad} - \delta_{ac}J_{bd} - \delta_{bd}J_{ac} + \delta_{ad}J_{bc} \\ [J_{ab}, B_c] &= \delta_{bc}B_a - \delta_{ac}B_b \\ [J_{ab}, P_c] &= \delta_{bc}P_a - \delta_{ac}P_b \\ [B_a, P_b] &= \delta_{ab}H. \end{aligned} \tag{A.1}$$

We may embed the Carroll Lie algebra \mathfrak{g} in $\mathfrak{gl}(n + 2, \mathbb{R})$ as follows:

$$\frac{1}{2}X^{ab}J_{ab} + v^a B_a + a^a P_a + sH \mapsto \begin{pmatrix} X & \mathbf{0} & \mathbf{a} \\ \mathbf{v}^T & 0 & s \\ \mathbf{0}^T & 0 & 0 \end{pmatrix}, \tag{A.2}$$

where $X^T = -X \in \mathfrak{so}(n)$. We may parametrise the (connected) Carroll group as follows:

$$g(R, \mathbf{v}, \mathbf{a}, s) = \exp(sH) \exp(v^a B_a + a^a P_a)R, \tag{A.3}$$

where $R \in \text{SO}(n)$. This non-standard parametrisation of the Carroll group has the advantage that it puts B_a and P_a on equal footing, reflecting the fact that they can be mapped into each other under automorphisms, as we discuss in section A.2. This leads to more symmetric equations that make this symmetry manifest, at the cost that some equations are more complicated. We discuss in the next subsection A.1.1 parametrisations which are more economical for other aspects, e.g., when acting on the spacetime.

The resulting group is seen to be the subgroup of $\text{GL}(n + 2, \mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} R & \mathbf{0} & \mathbf{a} \\ \mathbf{v}^T R & 1 & s + \frac{1}{2}\mathbf{v}^T \mathbf{a} \\ \mathbf{0}^T & 0 & 1 \end{pmatrix}, \tag{A.4}$$

with $R \in \text{SO}(n)$, from where we can work out the group multiplication

$$\begin{pmatrix} R_1 & \mathbf{0} & \mathbf{a}_1 \\ \mathbf{v}_1^T R_1 & 1 & s_1 + \frac{1}{2}\mathbf{v}_1^T \mathbf{a}_1 \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} \begin{pmatrix} R_2 & \mathbf{0} & \mathbf{a}_2 \\ \mathbf{v}_2^T R_2 & 1 & s_2 + \frac{1}{2}\mathbf{v}_2^T \mathbf{a}_2 \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} = \begin{pmatrix} R_3 & \mathbf{0} & \mathbf{a}_3 \\ \mathbf{v}_3^T R_3 & 1 & s_3 + \frac{1}{2}\mathbf{v}_3^T \mathbf{a}_3 \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} \tag{A.5}$$

where

$$\begin{aligned} R_3 &= R_1 R_2 \\ \mathbf{v}_3 &= \mathbf{v}_1 + R_1 \mathbf{v}_2 \\ \mathbf{a}_3 &= \mathbf{a}_1 + R_1 \mathbf{a}_2 \\ s_3 &= s_1 + s_2 + \frac{1}{2}\mathbf{v}_1^T R_1 \mathbf{a}_2 - \frac{1}{2}\mathbf{a}_1^T R_1 \mathbf{v}_2. \end{aligned} \tag{A.6}$$

It is then straightforward to work out the inverse of the generic element:

$$\begin{pmatrix} R & \mathbf{0} & \mathbf{a} \\ \mathbf{v}^T R & 1 & s + \frac{1}{2}\mathbf{v}^T \mathbf{a} \\ \mathbf{0}^T & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} R^T & \mathbf{0} & -R^T \mathbf{a} \\ -\mathbf{v}^T & 1 & -s + \frac{1}{2}\mathbf{a}^T R \mathbf{v} \\ \mathbf{0}^T & 0 & 1 \end{pmatrix}, \quad (\text{A.7})$$

where we have used that $R^T R = \mathbb{1}$.

Finally, identifying Carroll spacetime with the affine hyperplane of \mathbb{R}^{n+2} consisting of those vectors whose last entry is equal to 1, we work out the action of the Carroll group on Carroll spacetime:

$$\begin{pmatrix} R & \mathbf{0} & \mathbf{a} \\ \mathbf{v}^T R & 1 & s + \frac{1}{2}\mathbf{v}^T \mathbf{a} \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} R\mathbf{x} + \mathbf{a} \\ t + s + \frac{1}{2}\mathbf{v}^T \mathbf{a} + \mathbf{v}^T R\mathbf{x} \\ 1 \end{pmatrix} \quad (\text{A.8})$$

from where we see that the action (in this non-standard parametrisation) of the Carroll group consists of a rotation $(\mathbf{x}, t) \mapsto (R\mathbf{x}, t)$, followed by a Carroll boost $(R\mathbf{x}, t) \mapsto (R\mathbf{x}, t + \mathbf{v}^T R\mathbf{x})$ followed in turn by a translation $(R\mathbf{x}, t + \mathbf{v}^T R\mathbf{x}) \mapsto (R\mathbf{x} + \mathbf{a}, t + \mathbf{v}^T R\mathbf{x} + s + \frac{1}{2}\mathbf{v}^T \mathbf{a})$. In the next subsection we will discuss a parametrisation of the group element that is more economical.

A.1.1 Non-symmetric group parametrisations

When we do not insist on a parametrisation that puts the two vectors B_a and P_a on equal footing there are other useful choices, in particular when we are interested in the group action on the spacetime. As a first step we parametrise the group element as

$$g(R, \mathbf{v}, \mathbf{a}, s') = \exp(s'H) \exp(a^a P_a) \exp(v^a B_a) R. \quad (\text{A.9})$$

Using $e^{a^a P_a + v^a B_a} = e^{\frac{1}{2}\mathbf{v} \cdot \mathbf{a} H} e^{a^a P_a} e^{v^a B_a}$ we can relate this parametrisation to the one above (A.3) by $s' = s + \frac{1}{2}\mathbf{v} \cdot \mathbf{a}$ which leads to the matrix representation

$$\begin{pmatrix} R & \mathbf{0} & \mathbf{a} \\ \mathbf{v}^T R & 1 & s' \\ \mathbf{0}^T & 0 & 1 \end{pmatrix}. \quad (\text{A.10})$$

When our main concern is the action on the underlying spacetime the following parametrisation is particularly useful

$$g(R, \mathbf{v}', \mathbf{a}, s') = \exp(s'H) \exp(a^a P_a) R \exp(v'^a B_a). \quad (\text{A.11})$$

Using $R^{-1} e^{v \cdot B} R = e^{(R^{-1}v) \cdot B}$ it can be related to (A.9) via $\mathbf{v} = R^{-1}\mathbf{v}'$ and consequently we can write the group element as

$$\begin{pmatrix} R & \mathbf{0} & \mathbf{a} \\ \mathbf{v}'^T & 1 & s' \\ \mathbf{0}^T & 0 & 1 \end{pmatrix}. \quad (\text{A.12})$$

The group law may look unconventional

$$\begin{aligned}
 R_3 &= R_1 R_2 \\
 \mathbf{v}'_3 &= R_2^{-1} \mathbf{v}'_1 + \mathbf{v}'_2 \\
 \mathbf{a}_3 &= \mathbf{a}_1 + R_1 \mathbf{a}_2 \\
 s'_3 &= s'_1 + s'_2 + \mathbf{v}'_1 \cdot \mathbf{a}_2,
 \end{aligned}
 \tag{A.13}$$

but acting with the group element (A.12) on the spacetime, as in (A.8), leads precisely to the simple transformation of the spacetime given in (2.1) (where we have dropped the primes).

A.2 Automorphisms

The group of automorphisms of the Carroll Lie algebra (for any $n \geq 3$) which fix the rotational subalgebra is isomorphic to $\text{GL}(2, \mathbb{R})$, with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R})$ acting as

$$J_{ab} \mapsto J_{ab}, \quad B_a \mapsto aB_a + bP_a, \quad P_a \mapsto cB_a + dP_a \quad \text{and} \quad H \mapsto (ad - bc)H.
 \tag{A.14}$$

These automorphisms are not inner: they do not arise by conjugation in the Carroll group. These automorphisms act on the dual \mathfrak{g}^* of the Lie algebra as follows. If we let $\lambda^{ab}, \beta^a, \pi^a, \eta$ denote the canonical dual basis to J_{ab}, B_a, P_a, H , we see that

$$\lambda^{ab} \mapsto \lambda^{ab}, \quad \beta^a \mapsto \frac{1}{ad-bc}(d\beta^a - c\pi^a), \quad \pi^a \mapsto \frac{1}{ad-bc}(-b\beta^a + a\pi^a) \quad \text{and} \quad \eta \mapsto \frac{1}{ad-bc}\eta.
 \tag{A.15}$$

We may use these automorphisms to relate coadjoint orbits which otherwise might seem unrelated. In appendix A.6 we show how group automorphisms act on coadjoint orbits and then in appendix B.4 we will see how this allows us to simplify the classification of coadjoint orbits for $n = 3$.

A.3 The adjoint and coadjoint actions

We now work out the adjoint and coadjoint actions of the (connected) Carroll group on its Lie algebra and its dual. Having embedded the Carroll group inside $\text{GL}(n+2, \mathbb{R})$, the adjoint action is simply conjugation.

Consider the following matrix $A \in \mathfrak{g}$,

$$A = \begin{pmatrix} X & \mathbf{0} & \mathbf{a} \\ \mathbf{b}^T & 0 & c \\ \mathbf{0}^T & 0 & 0 \end{pmatrix}
 \tag{A.16}$$

and let us conjugate by a generic group element $g \in G$, given by (A.4), whose inverse is given by (A.7). We obtain

$$\text{Ad}_g A = \begin{pmatrix} R & \mathbf{0} & \mathbf{a} \\ \mathbf{v}^T R & 1 & s + \frac{1}{2} \mathbf{v}^T \mathbf{a} \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} \begin{pmatrix} X & \mathbf{0} & \mathbf{a} \\ \mathbf{b}^T & 0 & c \\ \mathbf{0}^T & 0 & 0 \end{pmatrix} \begin{pmatrix} R^T & \mathbf{0} & -R^T \mathbf{a} \\ -\mathbf{v}^T & 1 & -s + \frac{1}{2} \mathbf{a}^T R \mathbf{v} \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} = \begin{pmatrix} X' & \mathbf{0} & \mathbf{a}' \\ \mathbf{b}'^T & 0 & c' \\ \mathbf{0}^T & 0 & 0 \end{pmatrix},
 \tag{A.17}$$

where

$$\begin{aligned}
 X' &= R X R^T \\
 \mathbf{a}' &= R \mathbf{a} - R X R^T \mathbf{a} \\
 \mathbf{b}' &= R \mathbf{b} - R X R^T \mathbf{v} \\
 c' &= c + \mathbf{v}^T R \mathbf{a} - \mathbf{a}^T R \mathbf{b} - \mathbf{v}^T R X R^T \mathbf{a}.
 \end{aligned}
 \tag{A.18}$$

We now define an inner product on \mathfrak{g} by

$$\left\langle \begin{pmatrix} X_1 & \mathbf{0} & \mathbf{a}_1 \\ \mathbf{b}_1^T & 0 & c_1 \\ \mathbf{0}^T & 0 & 0 \end{pmatrix}, \begin{pmatrix} X_2 & \mathbf{0} & \mathbf{a}_2 \\ \mathbf{b}_2^T & 0 & c_2 \\ \mathbf{0}^T & 0 & 0 \end{pmatrix} \right\rangle = \frac{1}{2} \text{Tr} \left(X_1^T X_2 \right) + \mathbf{b}_1^T \mathbf{b}_2 + \mathbf{a}_1^T \mathbf{a}_2 + c_1 c_2.
 \tag{A.19}$$

In this way we may identify \mathfrak{g}^* as a vector space with \mathfrak{g} with the dual pairing being the above inner product. Let $\alpha \in \mathfrak{g}^*$, then

$$\langle \text{Ad}_g^* \alpha, A \rangle = \langle \alpha, \text{Ad}_{g^{-1}} A \rangle.
 \tag{A.20}$$

We may calculate $\text{Ad}_{g^{-1}} A$ as follows

$$\begin{aligned}
 \text{Ad}_{g^{-1}} \begin{pmatrix} X & \mathbf{0} & \mathbf{a} \\ \mathbf{b}^T & 0 & c \\ \mathbf{0}^T & 0 & 0 \end{pmatrix} &= \begin{pmatrix} R^T & \mathbf{0} & -R^T \mathbf{a} \\ -\mathbf{v}^T & 1 & -s + \frac{1}{2} \mathbf{a}^T R \mathbf{v} \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} \begin{pmatrix} X & \mathbf{0} & \mathbf{a} \\ \mathbf{b}^T & 0 & c \\ \mathbf{0}^T & 0 & 0 \end{pmatrix} \begin{pmatrix} R & \mathbf{0} & \mathbf{a} \\ \mathbf{v}^T R & 1 & s + \frac{1}{2} \mathbf{v}^T \mathbf{a} \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} X' & \mathbf{0} & \mathbf{a}' \\ \mathbf{b}'^T & 0 & c' \\ \mathbf{0}^T & 0 & 0 \end{pmatrix},
 \end{aligned}
 \tag{A.21}$$

where

$$\begin{aligned}
 X' &= R^T X R \\
 \mathbf{a}' &= R^T (\mathbf{a} + X \mathbf{a}) \\
 \mathbf{b}' &= R^T (\mathbf{b} + X \mathbf{v}) \\
 c' &= c + \mathbf{a}^T \mathbf{b} - \mathbf{v}^T (\mathbf{a} + X \mathbf{a}).
 \end{aligned}
 \tag{A.22}$$

Let $\alpha \in \mathfrak{g}^*$ be given by

$$\alpha = \begin{pmatrix} J & \mathbf{0} & \mathbf{p} \\ \mathbf{k}^T & 0 & E \\ \mathbf{0}^T & 0 & 0 \end{pmatrix},
 \tag{A.23}$$

with $J^T = -J$, and $g \in G$ the generic element in (A.4). Then we have that

$$\text{Ad}_g^* \alpha = \begin{pmatrix} J' & \mathbf{0} & \mathbf{p}' \\ \mathbf{k}'^T & 0 & E' \\ \mathbf{0}^T & 0 & 0 \end{pmatrix},
 \tag{A.24}$$

where

$$\frac{1}{2} \text{Tr}(J'^T X) + \mathbf{b}'^T \mathbf{k}' + \mathbf{a}'^T \mathbf{p}' + c' E' = \frac{1}{2} \text{Tr}(J^T X) + \mathbf{b}^T \mathbf{k} + \mathbf{a}^T \mathbf{p} + c E,
 \tag{A.25}$$

from where we can read off

$$\begin{aligned}
 J' &= RJR^T + (Rk)v^T - v(Rk)^T + Rpa^T - a(Rp)^T + E(av^T - va^T) \\
 k' &= Rk + Ea \\
 p' &= Rp - Ev \\
 E' &= E.
 \end{aligned} \tag{A.26}$$

This coadjoint action was already discussed in [12, appendix A].

A.4 Maurer-Cartan one-form and particle actions

Let $\alpha \in \mathfrak{g}^*$ and let ω_{KKS} denote the KKS invariant symplectic form on the coadjoint orbit \mathcal{O}_α of α . This coadjoint orbit is by definition a homogeneous symplectic manifold of the Carroll group and choosing α as the base point, we can define an orbit map $\pi : G \rightarrow \mathcal{O}_\alpha$ by $g \mapsto \text{Ad}_g^* \alpha$. Pulling back ω_{KKS} via π we see that it is not just closed, but actually exact

$$\pi^* \omega_{\text{KKS}} = -d \langle \alpha, \theta^L \rangle, \tag{A.27}$$

where θ^L is the left-invariant \mathfrak{g} -valued Maurer-Cartan one-form. The one-form $\langle \alpha, \theta^L \rangle$ is the main ingredient in the construction of particle actions associated to the coadjoint orbit and hence it is convenient to record here the one-form relative to our chosen group parametrisation.

Parametrising the generic group element as in (A.4), we find that the pull-back of the left-invariant Maurer-Cartan one-form is given by

$$\begin{aligned}
 g^{-1} dg &= \begin{pmatrix} R^T & \mathbf{0} & -R^T \mathbf{a} \\ -\mathbf{v}^T & 1 & -s + \frac{1}{2} \mathbf{a}^T R \mathbf{v} \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} \begin{pmatrix} dR & \mathbf{0} & d\mathbf{a} \\ d\mathbf{v}^T R + \mathbf{v}^T dR & 0 & ds + \frac{1}{2} d\mathbf{v}^T \mathbf{v} + \frac{1}{2} \mathbf{v}^T d\mathbf{a} \\ \mathbf{0}^T & 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} R^T dR & \mathbf{0} & R^T d\mathbf{a} \\ d\mathbf{v}^T R & 0 & ds + \frac{1}{2} \mathbf{a}^T d\mathbf{v} - \frac{1}{2} \mathbf{v}^T d\mathbf{a} \\ \mathbf{0}^T & 0 & 0 \end{pmatrix}.
 \end{aligned} \tag{A.28}$$

Therefore for $\alpha \in \mathfrak{g}^*$ given by equation (A.23),

$$\langle \alpha, g^{-1} dg \rangle = \frac{1}{2} \text{Tr} J^T R^T dR + (Rp - \frac{1}{2} Ev)^T d\mathbf{a} + (Rk + \frac{1}{2} Ea)^T d\mathbf{v} + E ds. \tag{A.29}$$

The particle action associated with the coadjoint orbit of $\alpha \in \mathfrak{g}^*$ is defined as follows. Let $I \subset \mathbb{R}$ be a real interval parametrising a curve $g : I \rightarrow G$ in the group. We define the action functional

$$S[g] := \int_I \langle \alpha, g^* \theta^L \rangle = \int_I \langle \alpha, g^{-1} \dot{g} \rangle d\tau \tag{A.30}$$

where τ denotes the coordinate on the interval. Varying the action, we obtain

$$\begin{aligned}
 \delta S &= \int_I \langle \alpha, -g^{-1} \delta g g^{-1} \dot{g} + g^{-1} \delta \dot{g} \rangle d\tau \\
 &= \int_I \langle \alpha, [g^{-1} \dot{g}, g^{-1} \delta g] \rangle d\tau + \int_{\partial I} \langle \alpha, g^{-1} \delta g \rangle \quad (\text{integrating by parts}) \\
 &= - \int_I \langle \text{ad}_{g^{-1} \dot{g}}^* \alpha, g^{-1} \delta g \rangle d\tau,
 \end{aligned}$$

where we have discarded the boundary term $\int_{\partial I} \langle \alpha, g^{-1} \delta g \rangle$, assuming an endpoint-fixed variational problem. The variation of the action vanishes for all δg if and only if $g^{-1} \dot{g}$ takes values in the stabiliser subalgebra \mathfrak{g}_α of α . One might be tempted to think that this requires $g(\tau)$ to be in the stabiliser subgroup G_α , but recall that $g^{-1} \dot{g}$ is the pull-back of a left-invariant one-form, so the solution is actually $g(\tau) = g_0 h(\tau)$ for some $h : I \rightarrow G_\alpha$ and where $g_0 \in G$ is a constant element of the group. Pushing down this curve via the orbit map $\pi : G \rightarrow \mathcal{O}_\alpha$, produces $\text{Ad}_{g_0}^* \alpha \in \mathcal{O}_\alpha$. So the extremals of the action do not necessarily have momenta α : all we can say is that their momenta lie in the coadjoint orbit of α . This merely highlights the fact that the action is indeed associated with the coadjoint orbit and not with any orbit representative α .

In Souriau’s language, but going back to Lagrange, the coadjoint orbit is the space of motions: a point in the coadjoint orbit represents a trajectory. Particle actions whose extremals are curves live in an evolution space fibering over the coadjoint orbit. The difficulty in describing the evolution space intrinsically can be circumvented by lifting the trajectories to the group as we have done above; even though doing so, as mentioned in the bulk of the paper, results in a redundancy in the description; in effect, in gauge invariance.

A.5 The case $n = 3$

In $n = 3$ the adjoint and vector representations of $\mathfrak{so}(3)$ are isomorphic. We can therefore trade 3×3 skewsymmetric matrices for vectors. Let us define the linear map $\varepsilon : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ by

$$\varepsilon(\mathbf{a})\mathbf{b} = \mathbf{a} \times \mathbf{b}. \tag{A.31}$$

This belongs to $\mathfrak{so}(3)$ because $\varepsilon(\mathbf{a})\mathbf{b} \cdot \mathbf{b} = 0$, so the endomorphism $\varepsilon(\mathbf{a})$ is skew-symmetric. Explicitly,

$$\varepsilon(\mathbf{e}_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \varepsilon(\mathbf{e}_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \varepsilon(\mathbf{e}_3) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.32}$$

so that $\varepsilon(\mathbf{e}_i)_{jk} = -\epsilon_{ijk}$. It follows from the standard vector identities that ε is a Lie algebra isomorphism provided that we use the cross product to define the Lie algebra structure on \mathbb{R}^3 :

$$[\varepsilon(\mathbf{a}), \varepsilon(\mathbf{b})] = \varepsilon(\mathbf{a} \times \mathbf{b}). \tag{A.33}$$

Moreover, a standard calculation shows that ε is an isometry provided that we use the standard euclidean inner product on \mathbb{R}^3 and half the trace in the defining representation on $\mathfrak{so}(3)$:

$$\frac{1}{2} \text{Tr} \varepsilon(\mathbf{a})^T \varepsilon(\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}. \tag{A.34}$$

If $R \in \text{SO}(3)$, it follows that

$$R(\mathbf{a} \times \mathbf{b}) = (R\mathbf{a}) \times (R\mathbf{b}), \tag{A.35}$$

which implies that

$$\varepsilon(R\mathbf{a}) = R\varepsilon(\mathbf{a})R^T. \tag{A.36}$$

Finally, it follows from a straightforward calculation that

$$\mathbf{a}\mathbf{b}^T - \mathbf{b}\mathbf{a}^T = \varepsilon(\mathbf{b} \times \mathbf{a}). \quad (\text{A.37})$$

Taking these formula into account and letting $J = \varepsilon(\mathbf{j})$, we may write the coadjoint action of the group element $g \in G$ given in (A.4) on

$$\alpha = \begin{pmatrix} \varepsilon(\mathbf{j}) & \mathbf{0} & \mathbf{p} \\ \mathbf{k}^T & 0 & E \\ \mathbf{0}^T & 0 & 0 \end{pmatrix} \in \mathfrak{g}^* \quad (\text{A.38})$$

as

$$\text{Ad}_g^* \alpha = \begin{pmatrix} \varepsilon(\mathbf{j}') & \mathbf{0} & \mathbf{p}' \\ \mathbf{k}'^T & 0 & E' \\ \mathbf{0}^T & 0 & 0 \end{pmatrix}, \quad (\text{A.39})$$

where

$$\begin{aligned} \mathbf{j}' &= R\mathbf{j} + \mathbf{v} \times R\mathbf{k} + \mathbf{a} \times R\mathbf{p} + E\mathbf{v} \times \mathbf{a} \\ \mathbf{k}' &= R\mathbf{k} + E\mathbf{a} \\ \mathbf{p}' &= R\mathbf{p} - E\mathbf{v} \\ E' &= E. \end{aligned} \quad (\text{A.40})$$

A.6 Action of automorphisms on coadjoint orbits

In this section we will show that group automorphisms map coadjoint orbits to coadjoint orbits symplectomorphically. Later in appendix B.4 we apply this to further simplify the classification of coadjoint orbits of the Carroll group.

Let G be a Lie group and $\tau : G \rightarrow G$ an automorphism; that is, a diffeomorphism which is also a group homomorphism $\tau(e) = e$ and $\tau(ab) = \tau(a)\tau(b)$ for all $a, b \in G$. Differentiating at the identity we get $\tau_* : \mathfrak{g} \rightarrow \mathfrak{g}$, which is an automorphism of the Lie algebra and moreover $\tau(\exp X) = \exp \tau_* X$ for all $X \in \mathfrak{g}$. The invertible linear transformation $\tau_* \in \text{GL}(\mathfrak{g})$ induces an invertible linear transformation $\tau^* \in \text{GL}(\mathfrak{g}^*)$ by $\tau^* \alpha = \alpha \circ \tau_*^{-1}$ for $\alpha \in \mathfrak{g}^*$. A natural question is how the coadjoint orbits of α and $\tau^* \alpha$ are related.

Lemma 1. *Let \mathcal{O}_α denote the coadjoint orbit of $\alpha \in \mathfrak{g}^*$. Then $\mathcal{O}_{\tau^* \alpha} = \tau^* \mathcal{O}_\alpha$.*

Proof. Let $g \in G$, $\alpha \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. Then

$$\begin{aligned} \langle \text{Ad}_g^* \tau^* \alpha, X \rangle &= \langle \tau^* \alpha, \text{Ad}_{g^{-1}} X \rangle \\ &= \langle \alpha, \tau_*^{-1} \text{Ad}_{g^{-1}} X \rangle. \end{aligned}$$

But now

$$\begin{aligned} \tau_*^{-1} \text{Ad}_{g^{-1}} X &= \left. \frac{d}{dt} \tau^{-1}(\exp(t \text{Ad}_{g^{-1}} X)) \right|_{t=0} && (\tau_*^{-1} = (\tau^{-1})_*) \\ &= \left. \frac{d}{dt} \tau^{-1}(g^{-1} \exp(tX) g) \right|_{t=0} \\ &= \left. \frac{d}{dt} (\tau^{-1}(g^{-1}) \tau^{-1}(\exp(tX)) \tau^{-1}(g)) \right|_{t=0} && (\tau^{-1} \text{ is an automorphism of } G) \\ &= \text{Ad}_{\tau^{-1}(g^{-1})} \tau_*^{-1} X. \end{aligned}$$

Therefore,

$$\begin{aligned} \langle \text{Ad}_g^* \tau^* \alpha, X \rangle &= \langle \alpha, \text{Ad}_{\tau^{-1}(g^{-1})} \tau_*^{-1} X \rangle \\ &= \langle \text{Ad}_{\tau^{-1}(g)}^* \alpha, \tau_*^{-1} X \rangle && (\tau^{-1}(g^{-1}) = \tau^{-1}(g)^{-1}) \\ &= \langle \tau^* \text{Ad}_{\tau^{-1}(g)}^* \alpha, X \rangle, \end{aligned}$$

so that

$$\text{Ad}_g^* \tau^* \alpha = \tau^* \text{Ad}_{\tau^{-1}(g)}^* \alpha. \tag{A.41}$$

This finally implies that

$$\begin{aligned} \mathcal{O}_{\tau^* \alpha} &= \{ \text{Ad}_g^* \tau^* \alpha \mid g \in G \} \\ &= \{ \tau^* \text{Ad}_{\tau^{-1}(g)}^* \alpha \mid g \in G \} \\ &= \tau^* \{ \text{Ad}_{\tau^{-1}(g)}^* \alpha \mid g \in G \} \\ &= \tau^* \mathcal{O}_\alpha, \end{aligned}$$

since $g = \tau^{-1}(\tau(g))$ for all $g \in G$. □

Of course, if τ is an inner automorphism, so that $\tau(g) = hgh^{-1}$ for some $h \in G$, it follows that $\tau^* \alpha = \text{Ad}_h^* \alpha$ and hence $\mathcal{O}_{\tau^* \alpha} = \mathcal{O}_\alpha$. Hence only outer automorphisms relate different coadjoint orbits.

More is true and the diffeomorphism τ^* of \mathfrak{g}^* relates the KKS symplectic forms on \mathcal{O}_α and on $\mathcal{O}_{\tau^* \alpha}$.

Lemma 2. *Let $\omega \in \Omega^2(\mathcal{O}_{\tau^* \alpha})$ denote the KKS symplectic form on $\mathcal{O}_{\tau^* \alpha}$. Then $(\tau^*)^* \omega$ agrees with the KKS symplectic form on \mathcal{O}_α .*

Proof. Let us introduce the notation $\phi := \tau^*$ to denote the diffeomorphism of \mathfrak{g}^* given by the action of the automorphism τ . (This declutters the notation somewhat.) Then we wish to show that $\phi^* \omega$ is the KKS symplectic form on \mathcal{O}_α . Let $X \in \mathfrak{g}$ and let $\text{ad}_X^*|_\alpha \in T_\alpha \mathfrak{g}^*$ be defined by

$$\text{ad}_X^*|_\alpha := \left. \frac{d}{dt} \text{Ad}_{\exp(tX)}^* \alpha \right|_{t=0}. \tag{A.42}$$

Then the KKS symplectic structure ω_{KKS} on \mathcal{O}_α is defined by at $\alpha \in \mathfrak{g}^*$ by

$$\omega_{\text{KKS}}(\text{ad}_X^*|_\alpha, \text{ad}_Y^*|_\alpha) = \langle \alpha, [X, Y] \rangle. \tag{A.43}$$

On the other hand,

$$(\phi^* \omega)_\alpha(\text{ad}_X^*|_\alpha, \text{ad}_Y^*|_\alpha) = \omega_{\phi(\alpha)}(\phi_* \text{ad}_X^*|_\alpha, \phi_* \text{ad}_Y^*|_\alpha), \tag{A.44}$$

where

$$\begin{aligned}
 \phi_* \text{ad}_X^*|_\alpha &= \left. \frac{d}{dt} \phi \left(\text{Ad}_{\exp(tX)}^* \alpha \right) \right|_{t=0} \\
 &= \left. \frac{d}{dt} \tau^* \left(\text{Ad}_{\exp(tX)}^* \alpha \right) \right|_{t=0} \\
 &= \left. \frac{d}{dt} \text{Ad}_{\tau(\exp(tX))}^* \tau^* \alpha \right|_{t=0} && \text{(by equation (A.41))} \\
 &= \left. \frac{d}{dt} \text{Ad}_{\exp(t\tau_* X)}^* \tau^* \alpha \right|_{t=0} \\
 &= \text{ad}_{\tau_* X}^*|_{\tau^* \alpha}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 (\phi^* \omega)_\alpha(\text{ad}_X^*|_\alpha, \text{ad}_Y^*|_\alpha) &= \omega_{\tau^* \alpha}(\text{ad}_{\tau_* X}^*|_{\tau^* \alpha}, \text{ad}_{\tau_* Y}^*|_{\tau^* \alpha}) \\
 &= \langle \tau^* \alpha, [\tau_* X, \tau_* Y] \rangle && \text{(by definition of KKS 2-form)} \\
 &= \langle \tau^* \alpha, \tau_* [X, Y] \rangle && \text{(since } \tau_* \text{ is a Lie algebra automorphism)} \\
 &= \langle \alpha, [X, Y] \rangle \\
 &= \omega_{\text{KKS}}(\text{ad}_X^*|_\alpha, \text{ad}_Y^*|_\alpha),
 \end{aligned}$$

so that $\phi^* \omega = \omega_{\text{KKS}}$. □

B Coadjoint orbits ($n = 3$)

In this appendix we classify the coadjoint orbits of the connected Carroll group with $n = 3$. This Carroll group has (at least) two Casimir elements [1]. The generator H is central and hence itself a Casimir. It defines a linear function on \mathfrak{g}^* taking the value E on $\alpha = (\mathbf{j}, \mathbf{k}, \mathbf{p}, E)$, which is a constant on coadjoint orbits. There is also a second Casimir, namely the euclidean norm W^2 of $W_a = HJ_a + \epsilon_{abc} P_b B_c$, which is a quartic symmetric tensor of \mathfrak{g} . This also defines a quartic polynomial function on \mathfrak{g}^* , taking the value $\|E\mathbf{j} + \mathbf{p} \times \mathbf{k}\|^2$ on $\alpha = (\mathbf{j}, \mathbf{k}, \mathbf{p}, E)$, which is again constant on coadjoint orbits.

We focus first on the linear Casimir H . There are two main classes of coadjoint orbits: those for which $E \neq 0$ and those for which $E = 0$.

B.1 Coadjoint orbits with $E \neq 0$

Let $\alpha \in \mathfrak{g}^*$ be given by $(\mathbf{j}, \mathbf{k}, \mathbf{p}, E)$ with $E \neq 0$. We can act with $g(\mathbf{1}, \mathbf{v}, \mathbf{a}, 0)$ with $\mathbf{a} = -E^{-1}\mathbf{k}$ and $\mathbf{v} = E^{-1}\mathbf{p}$ so bring α to $(\mathbf{j}', \mathbf{0}, \mathbf{0}, E)$, where

$$\mathbf{j}' = \mathbf{j} + E^{-1}\mathbf{p} \times \mathbf{k}. \tag{B.1}$$

We distinguish two cases.

B.1.1 Spinless orbits

This results in the coadjoint orbit of $\alpha = (\mathbf{0}, \mathbf{0}, \mathbf{0}, E)$. The stabiliser subgroup consists of matrices of the form

$$\begin{pmatrix} R & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & 1 & s \\ \mathbf{0}^T & \mathbf{0} & 1 \end{pmatrix} \tag{B.2}$$

which is isomorphic to $\text{SO}(3) \times \mathbb{R}$. The coadjoint orbit is therefore six-dimensional.

B.1.2 Orbits with nonzero spin

We may use rotations to bring $\mathbf{j} + E^{-1}\mathbf{p} \times \mathbf{k}$ to any desired direction. This results in the orbit of $\alpha = (\mathbf{j}, \mathbf{0}, \mathbf{0}, E)$ with $\mathbf{j} = \begin{pmatrix} 0 \\ 0 \\ j \end{pmatrix}$ with $j > 0$. The stabiliser of α consists of matrices of the form

$$\begin{pmatrix} R & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & 1 & s \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} \quad \text{with } R\mathbf{j} = \mathbf{j}, \tag{B.3}$$

so it is isomorphic to $\text{SO}(2) \times \mathbb{R}$. The coadjoint orbit is therefore eight-dimensional.

B.2 Coadjoint orbits with $E = 0$

If $E = 0$, the coadjoint action of the generic element $g(R, \mathbf{v}, \mathbf{a}, s)$ on $\alpha = (\mathbf{j}, \mathbf{k}, \mathbf{p}, 0)$ becomes $(\mathbf{j}', \mathbf{k}', \mathbf{p}', 0)$ where

$$\begin{aligned} \mathbf{j}' &= R\mathbf{j} + \mathbf{v} \times R\mathbf{k} + \mathbf{a} \times R\mathbf{p} \\ \mathbf{k}' &= R\mathbf{k} \\ \mathbf{p}' &= R\mathbf{p}. \end{aligned} \tag{B.4}$$

The value of the quartic Casimir on α is given by $\|\mathbf{p} \times \mathbf{k}\|^2$ which is clearly seen to be invariant, since $\mathbf{p}' \times \mathbf{k}' = R\mathbf{p} \times R\mathbf{k} = R(\mathbf{p} \times \mathbf{k})$ for $R \in \text{SO}(3)$, so that $\|\mathbf{p}' \times \mathbf{k}'\|^2 = \|\mathbf{p} \times \mathbf{k}\|^2$. We can therefore distinguish between two cases: those with $\mathbf{p} \times \mathbf{k} \neq \mathbf{0}$ and those with $\mathbf{p} \times \mathbf{k} = \mathbf{0}$. This latter case says that \mathbf{p} and \mathbf{k} are collinear. There are several cases here, depending on whether \mathbf{p} or \mathbf{k} are zero or not.

B.2.1 $\mathbf{p} = \mathbf{k} = \mathbf{0}$

In this case, $(\mathbf{j}, \mathbf{0}, \mathbf{0}, 0)$ is mapped to $(R\mathbf{j}, \mathbf{0}, \mathbf{0}, 0)$, so we have two possibilities:

- If $\mathbf{j} = \mathbf{0}$, the orbit consists of the point $(\mathbf{0}, \mathbf{0}, \mathbf{0}, 0)$.
- If $\mathbf{j} \neq \mathbf{0}$, the orbit is a sphere of radius $\|\mathbf{j}\|$ and we may choose $\left(\begin{pmatrix} 0 \\ 0 \\ j \end{pmatrix}, \mathbf{0}, \mathbf{0}, 0 \right)$, with $j > 0$, as representative point in the orbit.

B.2.2 $\mathbf{k} \neq \mathbf{0}, \mathbf{p} = \mathbf{0}$

Here $(\mathbf{j}, \mathbf{k}, \mathbf{0}, 0)$ is sent to $(R\mathbf{j} + \mathbf{v} \times R\mathbf{k}, R\mathbf{k}, \mathbf{0}, 0)$. We may bring \mathbf{k} to $\begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}$, with $k > 0$, leaving still the possibility of acting with any R in the stabiliser of \mathbf{k} . Choosing \mathbf{v} suitably we may bring \mathbf{j} to $\begin{pmatrix} 0 \\ 0 \\ j \end{pmatrix}$, where $j \in \mathbb{R}$. In other words, we may take as a representative of

the orbit,

$$\left(\begin{pmatrix} 0 \\ 0 \\ j \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}, \mathbf{0}, 0 \right) \quad \text{where } j \in \mathbb{R} \text{ and } k > 0, \quad (\text{B.5})$$

whose stabiliser is the subgroup consisting of matrices of the form

$$\begin{pmatrix} R & \mathbf{0} & \mathbf{a} \\ \frac{v}{k} \mathbf{k}^T & 1 & s + \frac{1}{2} \frac{v}{k} \mathbf{a}^T \mathbf{k} \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} \quad \text{with } R\mathbf{k} = \mathbf{k}. \quad (\text{B.6})$$

The stabiliser is six-dimensional, so the orbit is four-dimensional.

B.2.3 $\mathbf{p} \neq \mathbf{0}, \mathbf{k} = \mathbf{0}$

The story here is very similar to the previous case and, indeed, the orbits are related via outer automorphisms. Now $(\mathbf{j}, \mathbf{0}, \mathbf{p}, 0)$ is sent to $(R\mathbf{j} + \mathbf{a} \times R\mathbf{p}, \mathbf{0}, R\mathbf{p}, 0)$. Hence we may

bring \mathbf{p} to $\begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix}$, with $p > 0$, leaving still the possibility of acting with any R in the stabiliser

of \mathbf{p} . By choosing \mathbf{a} suitably, we may bring \mathbf{j} to $\begin{pmatrix} 0 \\ 0 \\ j \end{pmatrix}$, where $j \in \mathbb{R}$. In other words, we may take as a representative of the orbit,

$$\left(\begin{pmatrix} 0 \\ 0 \\ j \end{pmatrix}, \mathbf{0}, \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix}, 0 \right) \quad \text{where } j \in \mathbb{R} \text{ and } p > 0, \quad (\text{B.7})$$

whose stabiliser is the subgroup consisting of matrices of the form

$$\begin{pmatrix} R & \mathbf{0} & \frac{a}{p} \mathbf{p} \\ \mathbf{v}^T & 1 & s + \frac{1}{2} \frac{a}{p} \mathbf{v}^T \mathbf{p} \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} \quad \text{with } R\mathbf{p} = \mathbf{p}. \quad (\text{B.8})$$

The stabiliser is six-dimensional, so the orbit is four-dimensional.

B.2.4 $\mathbf{p} \neq \mathbf{0}, \mathbf{k} \neq \mathbf{0}, \mathbf{p} \times \mathbf{k} = \mathbf{0}$

This case is also related to the previous two by automorphisms. Since $\mathbf{p} \times \mathbf{k} = \mathbf{0}$, they are collinear: either parallel or antiparallel. This means that, letting $\|\mathbf{p}\| = p > 0$ and $\|\mathbf{k}\| = k > 0$, $\mathbf{p} \cdot \mathbf{k} = \pm pk$, so the angle between them is 0 (for the plus sign) or π for the minus sign. Hence $(\mathbf{j}, \mathbf{k}, \mathbf{p}, 0)$ is sent to $(R\mathbf{j} + (\mathbf{a} \pm \frac{k}{p} \mathbf{v}) \times R\mathbf{p}, R\mathbf{k}, R\mathbf{p}, 0)$. We may bring

\mathbf{p} to $\begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix}$, with $p > 0$ and hence \mathbf{k} to $\begin{pmatrix} 0 \\ 0 \\ \pm k \end{pmatrix}$. With R such that $R\mathbf{p} = \mathbf{p}$ (and hence also

$R\mathbf{k} = \mathbf{k}$), we have that $(\mathbf{j}, \mathbf{k}, \mathbf{p}, 0)$ is sent to $(R\mathbf{j} + (\mathbf{a} \pm \frac{k}{p} \mathbf{v}) \times \mathbf{p}, \mathbf{k}, \mathbf{p}, 0)$ and we may use

$\mathbf{a} \pm \frac{k}{p}\mathbf{v}$ suitably in order to make $\mathbf{j} = \begin{pmatrix} 0 \\ 0 \\ j \end{pmatrix}$, for some $j \in \mathbb{R}$. In summary, as representative of the orbit we may take

$$\left(\begin{pmatrix} 0 \\ 0 \\ j \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \pm k \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix}, 0 \right) \quad \text{where } p > 0, k > 0 \text{ and } j \in \mathbb{R}, \quad (\text{B.9})$$

whose stabiliser is the subgroup consisting of matrices of the form

$$\begin{pmatrix} R & \mathbf{0} & \mathbf{a} \\ \mathbf{v}^T & 1 & s + \frac{1}{2}\mathbf{a}^T \mathbf{v} \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} \quad \text{with } R\mathbf{p} = \mathbf{p} \text{ and } (\mathbf{a} \pm \frac{k}{p}\mathbf{v}) \times \mathbf{p} = \mathbf{0}. \quad (\text{B.10})$$

The stabiliser is six-dimensional, so the orbit is four-dimensional.

B.2.5 $\mathbf{p} \times \mathbf{k} \neq \mathbf{0}$

Here \mathbf{p} and \mathbf{k} span a plane, which we can choose to be the plane of vectors whose first entry is zero. In other words, under

$$(\mathbf{j}, \mathbf{k}, \mathbf{p}, 0) \mapsto (R\mathbf{j} + \mathbf{a} \times R\mathbf{p} + \mathbf{v} \times R\mathbf{k}, R\mathbf{k}, R\mathbf{p}, 0) \quad (\text{B.11})$$

we may bring \mathbf{p} to $\begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix}$ with $p > 0$ and then with R such that $R\mathbf{p} = \mathbf{p}$, we may bring \mathbf{k}

to $\begin{pmatrix} 0 \\ k_2 \\ k_3 \end{pmatrix}$, with $k_2 \neq 0$. This fixes the rotational symmetry completely and we now have

$(\mathbf{j}, \mathbf{k}, \mathbf{p}, 0)$ is sent to $(\mathbf{j} + \mathbf{a} \times \mathbf{p} + \mathbf{v} \times \mathbf{k}, \mathbf{k}, \mathbf{p}, 0)$. It is not hard to see that by choosing \mathbf{a} and \mathbf{v} suitably, we can set $\mathbf{j} = \mathbf{0}$. In other words, as a representative of the orbit we may take the covector

$$\left(\mathbf{0}, \begin{pmatrix} 0 \\ k_2 \\ k_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix}, 0 \right) \quad \text{where } p > 0, k_3 \in \mathbb{R} \text{ and } k_2 \neq 0, \quad (\text{B.12})$$

whose stabiliser is the four-dimensional subgroup of the Carroll group consisting of matrices of the form

$$\begin{pmatrix} \mathbb{1} & \mathbf{0} & \mathbf{a} \\ \mathbf{v}^T & 1 & s + \frac{1}{2}\mathbf{v}^T \mathbf{a} \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} \quad \text{with } \mathbf{a} \times \mathbf{p} + \mathbf{v} \times \mathbf{k} = \mathbf{0}. \quad (\text{B.13})$$

The orbit is therefore six-dimensional.

These orbits are listed (listed using the names of the corresponding Carroll particles) in table 1.

B.3 Coadjoint orbits of the full Carroll group

The full Carroll group has two connected components, since now $R \in O(3)$. The group $O(3)$ is generated by $SO(3)$ and parity P , which we can think of as space inversion $P\mathbf{x} = -\mathbf{x}$ while leaving t inert. We expect that under parity, some coadjoint orbits are mapped to themselves whereas some other orbits are paired. However it is clear from the explicit form of the orbit representatives, that we can always undo the effect of parity

$$(\mathbf{j}, \mathbf{k}, \mathbf{p}, E) \mapsto (-\mathbf{j}, -\mathbf{k}, -\mathbf{p}, E) \quad (\text{B.14})$$

with a rotation, since always $\mathbf{j}, \mathbf{k}, \mathbf{p}$ have at least one zero component. Therefore, the above classification is also the classification of coadjoint orbits of the full Carroll group.

B.4 Coadjoint orbits up to automorphisms

Let us now see how automorphisms of the Carroll group relate different coadjoint orbits. As we saw in section A.2, the automorphisms of the Carroll algebra which fix the rotational subalgebra are given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R})$ acting via equation (A.14). The action on $\alpha \in \mathfrak{g}^*$ is via the transpose inverse, so that $(\mathbf{j}, \mathbf{k}, \mathbf{p}, E) \mapsto (\mathbf{j}', \mathbf{k}', \mathbf{p}', E')$ with

$$\begin{aligned} \mathbf{j}' &= \mathbf{j} \\ \mathbf{k}' &= \frac{d\mathbf{k} - b\mathbf{p}}{ad - bc} \\ \mathbf{p}' &= \frac{a\mathbf{p} - c\mathbf{k}}{ad - bc} \\ E' &= \frac{E}{ad - bc}. \end{aligned} \quad (\text{B.15})$$

In particular, we see that both E and $\mathbf{p} \times \mathbf{k}$ transform as densities of weight -1 ; that is, via multiplication by the reciprocal of the determinant.

It is now a simple matter to go through the coadjoint orbits listed in table 1 and see how the automorphisms act:

- (1) Since automorphisms allow us to rescale the energy by a nonzero number, all orbits are equivalent under automorphisms, so we can choose $(\mathbf{0}, \mathbf{0}, \mathbf{0}, 1)$ as orbit representative. The collection of all orbits of nonzero energy and $\mathbf{j} = \mathbf{0}$ can be obtained by the combined action of the Carroll group (via the coadjoint representation) and the automorphisms from $(\mathbf{0}, \mathbf{0}, \mathbf{0}, 1)$.
- (2) We may again rescale the energy to any desired (nonzero) value, but we cannot change the spin $S > 0$. So we can take $(S\mathbf{u}, \mathbf{0}, \mathbf{0}, 1)$ as the orbit representative.
- (3) There is only one orbit of this type.
- (4) Orbits of this type belong to a one-parameter family labelled by $j > 0$: automorphisms cannot change j .

#	Orbit representative $\alpha = (\mathbf{j}, \mathbf{k}, \mathbf{p}, E)$	$\dim \mathcal{O}_\alpha$
[1]	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, 1)$	6
[2] _S	$(S\mathbf{u}, \mathbf{0}, \mathbf{0}, 1)$	8
[3]	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, 0)$	0
[4] _j	$(j\mathbf{u}, \mathbf{0}, \mathbf{0}, 0)$	2
[5, 6, 7 _±] _h	$(h\mathbf{u}, \mathbf{0}, \mathbf{u}, 0)$	4
[8]	$(\mathbf{0}, \mathbf{u}, \mathbf{u}_\perp, 0)$	6

Table 3. *Coadjoint orbits up to automorphisms.* This table lists the equivalence classes of coadjoint orbits of the Carroll group up to the action of automorphisms. Some of the parameters in table 1 become ineffective (up to automorphisms), whereas the greatest simplifications come from the fact that all the four-dimensional orbits (with the same value of the “helicity” h) are related by automorphisms. As in table 1, \mathbf{u} stands for a fixed, but arbitrary, unit vector and \mathbf{u}_\perp a second unit vector perpendicular to \mathbf{u} . We see that the dimension of the orbit almost determines the class (up to automorphisms), except that we have two distinct classes of six-dimensional orbits: one with $E \neq 0$ and one with $E = 0$.

(5, 6, 7_±) The parameter h cannot be changed, but the automorphisms act transitively on the other parameters. Therefore up to automorphisms, we have a one-parameter ($h \in \mathbb{R}$) family of orbits.

(8) All orbits of this type are equivalent under automorphisms: \mathbf{p} and \mathbf{k} span a plane and $\text{GL}(2, \mathbb{R})$ acts transitively on the bases of that plane.

In summary, we may list the equivalence classes of coadjoint orbits of the Carroll group up to the action of automorphisms as in table 3.

B.5 Structure of the coadjoint orbits

The Carroll group is a semidirect product $K \ltimes T$ where K is the subgroup generated by J_a, B_a and is isomorphic to the three-dimensional euclidean group, and T is the abelian normal subgroup generated by P_a, H . In terms of our explicit matrix parametrisations,

$$K = \left\{ \left(\begin{array}{ccc|c} R & \mathbf{0} & \mathbf{0} & \\ \mathbf{v}^T R & 1 & 0 & \\ \mathbf{0}^T & 0 & 1 & \end{array} \right) \middle| R \in \text{SO}(3), \mathbf{v} \in \mathbb{R}^3 \right\} \tag{B.16}$$

and

$$T = \left\{ \left(\begin{array}{ccc|c} \mathbb{1} & \mathbf{0} & \mathbf{a} & \\ \mathbf{0}^T & 1 & s & \\ \mathbf{0}^T & 0 & 1 & \end{array} \right) \middle| s \in \mathbb{R}, \mathbf{a} \in \mathbb{R}^3 \right\}. \tag{B.17}$$

B.5.1 Coadjoint orbits of semi-direct products

Coadjoint orbits of such a semidirect product $K \ltimes T$ are easy to describe geometrically. (See, e.g., Oblak's thesis [17].) We recall here the main points. Every $\alpha \in \mathfrak{g}^*$, with $\mathfrak{g} = \mathfrak{k} \ltimes \mathfrak{t}$, decomposes into $\alpha = (\kappa, \tau)$ with $\tau \in \mathfrak{t}^*$ and $\kappa \in \mathfrak{k}^*$. The G -coadjoint orbit of $\alpha \in \mathfrak{g}^*$ fibers over the K -orbit \mathcal{O}_τ of $\tau \in \mathfrak{t}^*$ under the K -action given by the semidirect product structure.

The action of K on T is by matrix conjugation in G :

$$\begin{pmatrix} R & \mathbf{0} & \mathbf{0} \\ \mathbf{v}^T R & 1 & 0 \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{1} & \mathbf{0} & \mathbf{a} \\ \mathbf{0}^T & 1 & s \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} \begin{pmatrix} R^T & \mathbf{0} & \mathbf{0} \\ -\mathbf{v}^T & 1 & 0 \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & \mathbf{0} & R\mathbf{a} \\ \mathbf{0}^T & 1 & s + \mathbf{v}^T R\mathbf{a} \\ \mathbf{0}^T & 0 & 1 \end{pmatrix}, \quad (\text{B.18})$$

from where we can read off the adjoint action on the Lie algebra \mathfrak{t} of T :

$$\text{Ad} \begin{pmatrix} R & \mathbf{0} & \mathbf{0} \\ \mathbf{v}^T R & 1 & 0 \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{0} & \mathbf{a} \\ \mathbf{0}^T & 0 & c \\ \mathbf{0}^T & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{0} & R\mathbf{a} \\ \mathbf{0}^T & 0 & c + \mathbf{v}^T R\mathbf{a} \\ \mathbf{0}^T & 0 & 0 \end{pmatrix}, \quad (\text{B.19})$$

or in abbreviated form

$$\text{Ad}_{(R, \mathbf{v})}(\mathbf{a}, c) = (R\mathbf{a}, c + \mathbf{v}^T R\mathbf{a}). \quad (\text{B.20})$$

The inverse of (R, \mathbf{v}) is given by $(R^T, -R^T \mathbf{v})$, and hence

$$\text{Ad}_{(R, \mathbf{v})^{-1}}(\mathbf{a}, c) = (R^T \mathbf{a}, c - \mathbf{v}^T \mathbf{a}). \quad (\text{B.21})$$

From this we can work out the action on \mathfrak{t}^* :

$$\text{Ad}_{(R, \mathbf{v})}(\mathbf{p}, E) = (R\mathbf{p} - E\mathbf{v}, E). \quad (\text{B.22})$$

Inspecting the coadjoint orbits in sections B.1 and B.2, we see that the representative $\tau = (\mathbf{p}, E) \in \mathfrak{t}^*$ are of three kinds:

- $\tau = (\mathbf{0}, E_0)$, $E_0 \neq 0$, whose orbit under K is the affine hyperplane \mathbb{A}^3 defined by $E = E_0$ in \mathfrak{t}^* ;
- $\tau = (\mathbf{0}, 0)$, whose orbit is only that point; and
- $\tau = (\mathbf{p} \neq \mathbf{0}, 0)$, whose orbit is the sphere $S_{\|\mathbf{p}\|}^2$ of radius $\|\mathbf{p}\|$ in the hyperplane $E = 0$ in \mathfrak{t}^* .

Let $K_\tau \subset K$ denote the stabiliser of $\tau \in \mathfrak{t}^*$ and \mathfrak{k}_τ denote its Lie algebra. Let $\kappa_\tau \in \mathfrak{k}_\tau^*$ denote the restriction of κ to \mathfrak{k}_τ . The other ingredient in the G -coadjoint orbit of α is the K_τ -coadjoint orbit of κ_τ , which we denote $\mathcal{O}_{\kappa_\tau}$.

Now, the K -orbit \mathcal{O}_τ of $\tau \in \mathfrak{g}^*$ is K -equivariantly diffeomorphic to K/K_τ and any space on which K_τ acts, e.g., $\mathcal{O}_{\kappa_\tau}$, defines an associated fibre bundle $K \times_{K_\tau} \mathcal{O}_{\kappa_\tau} \rightarrow \mathcal{O}_\tau$.

Finally, the G -coadjoint orbit \mathcal{O}_α of $\alpha = (\kappa, \tau)$ is the fibred product of $K \times_{K_\tau} \mathcal{O}_{\kappa_\tau} \rightarrow \mathcal{O}_\tau$ with the cotangent bundle $T^*\mathcal{O}_\tau \rightarrow \mathcal{O}_\tau$:

$$\begin{array}{ccc} \mathcal{O}_\alpha & \longrightarrow & T^*\mathcal{O}_\tau \\ \downarrow & & \downarrow \\ K \times_{K_\tau} \mathcal{O}_{\kappa_\tau} & \longrightarrow & \mathcal{O}_\tau \end{array} \tag{B.23}$$

The standard notation for this fibred product is

$$\mathcal{O}_\alpha = T^*\mathcal{O}_\tau \times_{\mathcal{O}_\tau} (K \times_{K_\tau} \mathcal{O}_{\kappa_\tau}). \tag{B.24}$$

Let us show how to calculate the dimension from this expression. The idea is to add the dimensions of the spaces that appear “above” (here $T^*\mathcal{O}_\tau$, K , $\mathcal{O}_{\kappa_\tau}$) and subtract the dimensions of the spaces which appear “below” (here \mathcal{O}_τ and K_τ). Doing so we obtain

$$\begin{aligned} \dim \mathcal{O}_\alpha &= \dim T^*\mathcal{O}_\tau - \dim \mathcal{O}_\tau + \dim K - \dim K_\tau + \dim \mathcal{O}_{\kappa_\tau} \\ &= 2 \dim \mathcal{O}_\tau + \dim \mathcal{O}_{\kappa_\tau}. \end{aligned} \tag{B.25}$$

The cotangent bundle $T^*\mathcal{O}_\tau$ is itself isomorphic to an associated vector bundle $T^*\mathcal{O}_\tau \cong K \times_{K_\tau} \mathfrak{k}_\tau^0$, where the annihilator $\mathfrak{k}_\tau^0 \subset \mathfrak{k}$ is isomorphic to the dual $(\mathfrak{k}/\mathfrak{k}_\tau)^*$. Hence the coadjoint orbit \mathcal{O}_α can also be written as

$$\mathcal{O}_\alpha = K \times_{K_\tau} (\mathfrak{k}_\tau^0 \times \mathcal{O}_{\kappa_\tau}). \tag{B.26}$$

Again we can calculate the dimension as we did above

$$\begin{aligned} \dim \mathcal{O}_\alpha &= \dim K - \dim K_\tau + \dim \mathfrak{k}_\tau^0 + \dim \mathcal{O}_{\kappa_\tau} \\ &= \dim K - \dim K_\tau + (\dim \mathfrak{k} - \dim \mathfrak{k}_\tau) + \dim \mathcal{O}_{\kappa_\tau} \\ &= 2 \dim \mathcal{O}_\tau + \dim \mathcal{O}_{\kappa_\tau}, \end{aligned} \tag{B.27}$$

resulting (of course) in the same dimension.

B.5.2 Coadjoint orbits of the euclidean groups

To see how this works in practice and because we shall need these results below, let us discuss the coadjoint orbits of the euclidean groups $\text{ISO}(n)$. We are particularly interested in $n = 2, 3$ as those appear as stabilisers K_τ in our discussion of Carroll orbits.

The euclidean group $\text{ISO}(n)$ is a subgroup of the affine group $\text{Aff}(n, \mathbb{R})$ and hence we may embed it as a subgroup of the general linear group $\text{GL}(n+1, \mathbb{R})$. We choose the following embedding

$$\text{ISO}(n) = \left\{ \begin{pmatrix} R & \mathbf{v} \\ \mathbf{0}^T & 1 \end{pmatrix} \mid R \in \text{SO}(n), \mathbf{v} \in \mathbb{R}^n \right\}. \tag{B.28}$$

We shall use the abbreviated notation (R, \mathbf{v}) for the generic element of $\text{ISO}(n)$. Then the group law in this parametrisation is given by

$$(R_1, \mathbf{v}_1)(R_2, \mathbf{v}_2) = (R_1 R_2, \mathbf{v}_1 + R_1 \mathbf{v}_2), \tag{B.29}$$

from where we read off the inverse

$$(R, \mathbf{v})^{-1} = (R^T, -R^T \mathbf{v}), \tag{B.30}$$

where we have used that $R^{-1} = R^T$ for $R \in \text{SO}(n)$. Let us introduce the following abbreviated notation for the Lie algebra $\mathfrak{iso}(n)$:

$$(X, \mathbf{b}) = \begin{pmatrix} X & \mathbf{b} \\ \mathbf{0}^T & 0 \end{pmatrix}. \tag{B.31}$$

The adjoint representation is easy to work out explicitly and one finds

$$\text{Ad}_{(R, \mathbf{v})^{-1}}(X, \mathbf{b}) = (R^T X R, R^T (\mathbf{b} + X \mathbf{v})). \tag{B.32}$$

This allows us to write the coadjoint action. Write $(J, \mathbf{k}) \in \mathfrak{iso}(n)^*$ with dual pairing

$$\langle (J, \mathbf{k}), (X, \mathbf{b}) \rangle = \frac{1}{2} \text{Tr } J^T X + \mathbf{k}^T \mathbf{b}. \tag{B.33}$$

Then it follows from a simple calculation that

$$\text{Ad}_{(R, \mathbf{v})}^*(J, \mathbf{k}) = (R J R^T + (R \mathbf{k})^T \mathbf{v} - \mathbf{v} (R \mathbf{k})^T, R \mathbf{k}). \tag{B.34}$$

Let us now specialise to $n = 2$. Since $\text{SO}(2)$ is abelian, we have now that $R J R^T = J$, so that

$$\text{Ad}_{(R, \mathbf{v})}^*(J, \mathbf{k}) = (J + (R \mathbf{k})^T \mathbf{v} - \mathbf{v} (R \mathbf{k})^T, R \mathbf{k}). \tag{B.35}$$

We see that the norm $\|\mathbf{k}\|$ is an invariant of the orbit. If $\mathbf{k} = \mathbf{0}$, the orbit of $(J, \mathbf{0})$ is a point $\{(J, \mathbf{0})\}$.

If $\mathbf{k} \neq \mathbf{0}$, we may bring it to the form $\mathbf{k} = \begin{pmatrix} 0 \\ k \end{pmatrix}$, where $k = \|\mathbf{k}\| > 0$ and then using \mathbf{v} we may set $J = 0$. Therefore the orbit representative is $(0, (0, k)^T)$ and the stabiliser consists of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix} \tag{B.36}$$

which is abelian and hence has point-like coadjoint orbits. The $\text{SO}(2)$ -orbit of \mathbf{k} is a circle of radius $\|\mathbf{k}\|$ and hence the $\text{ISO}(2)$ -coadjoint orbit of $(0, \mathbf{k})$ is the cotangent bundle $T^* S_{\|\mathbf{k}\|}^1 \cong S_{\|\mathbf{k}\|}^1 \times \mathbb{R}$.

In other words, the $\text{ISO}(2)$ -coadjoint orbits are either cylinders of some positive radius $\|\mathbf{k}\|$, or every point on the line $\mathbf{k} = \mathbf{0}$.

Finally, let us specialise to $n = 3$. Now a generic element of $\mathfrak{iso}(3)^*$ is (\mathbf{j}, \mathbf{k}) which is sent to $(R \mathbf{j} + \mathbf{v} \times R \mathbf{k}, R \mathbf{k})$ under the coadjoint action of (R, \mathbf{v}) . Again $\|\mathbf{k}\|$ is invariant.

If $\mathbf{k} = \mathbf{0}$, $(\mathbf{j}, \mathbf{0})$ is sent to $(R \mathbf{j}, \mathbf{0})$, so that $\|\mathbf{j}\|$ is invariant. If $\mathbf{j} = \mathbf{0}$, we have a point-like orbit $\{(\mathbf{0}, \mathbf{0})\}$; otherwise we have the sphere $S_{\|\mathbf{j}\|}^2$ of radius $\|\mathbf{j}\|$.

#	$\alpha \in \mathfrak{g}^*$	$\tau \in \mathfrak{t}^*$	\mathcal{O}_τ	K_τ	$\kappa \in \mathfrak{k}^*$	$\kappa_\tau \in \mathfrak{k}_\tau^*$	$\mathcal{O}_{\kappa_\tau}$	\mathcal{O}_α
1	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, E_0 \neq 0)$	$(\mathbf{0}, E_0)$	$\mathbb{A}_{E=E_0}^3$	SO(3)	$(\mathbf{0}, \mathbf{0})$	$\mathbf{0}$	$\{\mathbf{0}\}$	$T^*\mathbb{A}^3$
2	$(\mathbf{j} \neq \mathbf{0}, \mathbf{0}, \mathbf{0}, E_0 \neq 0)$	$(\mathbf{0}, E_0)$	$\mathbb{A}_{E=E_0}^3$	SO(3)	$(\mathbf{j}, \mathbf{0})$	\mathbf{j}	$S_{\ \mathbf{j}\ }^2$	$T^*\mathbb{A}^3 \times_{\mathbb{A}^3} (K \times_{K_\tau} S^2)$
3	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, 0)$	$(\mathbf{0}, 0)$	$\{(\mathbf{0}, 0)\}$	K	$(\mathbf{0}, \mathbf{0})$	$(\mathbf{0}, \mathbf{0})$	$\{(\mathbf{0}, 0)\}$	$\{(\mathbf{0}, \mathbf{0}, \mathbf{0}, 0)\}$
4	$(\mathbf{j} \neq \mathbf{0}, \mathbf{0}, \mathbf{0}, 0)$	$(\mathbf{0}, 0)$	$\{(\mathbf{0}, 0)\}$	K	$(\mathbf{j}, \mathbf{0})$	$(\mathbf{j}, \mathbf{0})$	$S_{\ \mathbf{j}\ }^2$	S^2
5	$(\mathbf{j}, \mathbf{k} \neq \mathbf{0}, \mathbf{0}, 0)_{\mathbf{j} \times \mathbf{k} = \mathbf{0}}$	$(\mathbf{0}, 0)$	$\{(\mathbf{0}, 0)\}$	K	(\mathbf{j}, \mathbf{k})	(\mathbf{j}, \mathbf{k})	$\{\mathbf{j}\} \times T^*S_{\ \mathbf{k}\ }^2$	T^*S^2
6	$(\mathbf{j}, \mathbf{0}, \mathbf{p} \neq \mathbf{0}, 0)_{\mathbf{j} \times \mathbf{p} = \mathbf{0}}$	$(\mathbf{p}, 0)$	$S_{\ \mathbf{p}\ }^2$	$\text{SO}(2) \times \mathbb{R}^3$	$(\mathbf{j}, \mathbf{0})$	$(\mathbf{j}, \mathbf{0})$	$\{(\mathbf{j}, \mathbf{0})\}$	T^*S^2
7	$(\mathbf{j}, \mathbf{k} \neq \mathbf{0}, \mathbf{p} \neq \mathbf{0}, 0)_{\mathbf{k} \times \mathbf{p} = \mathbf{j} \times \mathbf{k} = \mathbf{0}}$	$(\mathbf{p}, 0)$	$S_{\ \mathbf{p}\ }^2$	$\text{SO}(2) \times \mathbb{R}^3$	(\mathbf{j}, \mathbf{k})	(\mathbf{j}, \mathbf{k})	$\{(\mathbf{j}, \mathbf{k})\}$	T^*S^2
8	$(\mathbf{0}, \mathbf{k}, \mathbf{p}, 0)_{\mathbf{k} \times \mathbf{p} \neq \mathbf{0}}$	$(\mathbf{p}, 0)$	$S_{\ \mathbf{p}\ }^2$	$\text{SO}(2) \times \mathbb{R}^3$	$(\mathbf{0}, \mathbf{k})$	$(\mathbf{0}, \mathbf{k})$	$T^*S_{\ \mathbf{k}\ }^1$	$T^*S^2 \times_{S^2} (K \times_{K_\tau} T^*S^1)$

Table 4. *Deconstructing the coadjoint orbits.* The stabilisers K_τ in cases #6,7,8 consists of elements (R, \mathbf{v}) where $\mathbf{v} \in \mathbb{R}^3$ is arbitrary and $R\mathbf{p} = \mathbf{p}$. Since $\mathbf{p} \neq \mathbf{0}$, these are rotations about the axis defined by \mathbf{p} and hence isomorphic to $\text{SO}(2)$, so that the stabiliser is isomorphic to $\text{SO}(2) \times \mathbb{R}^3$, but actually $\text{SO}(2)$ only acts nontrivially on a plane in \mathbb{R}^3 , hence the stabiliser is more properly written as $\text{ISO}(2) \times \mathbb{R}$ or, even more invariantly, as $\text{ISO}(\mathbf{p}^\perp) \times \mathbb{R}\mathbf{p}$, as a subgroup of $\text{ISO}(3)$. The K_τ -coadjoint orbits should be self-explanatory. In cases #3,4,5, the stabiliser is $K \cong \text{ISO}(3)$, whose coadjoint orbits were determined in section B.5.2 above: in case #3 we have the point-like orbit $\{\mathbf{0}, \mathbf{0}\}$, in case #4 we have the 2-sphere of radius $\|\mathbf{j}\|$, and in case #5 we have the cotangent bundle of the sphere of radius $\|\mathbf{k}\|$. In cases #6,7,8, the stabiliser K_τ is isomorphic to $\text{ISO}(2) \times \mathbb{R}$ and the coadjoint orbits have been determined in section B.5.2 above. In case #6 we have the point-like orbit $\{(\mathbf{j}, \mathbf{0})\}$ and in case #8 we have the cylinder T^*S^1 . Only case #7 needs some explanation. In this case, all of $\mathbf{j}, \mathbf{k}, \mathbf{p}$ are collinear. The K_τ -coadjoint action of (R, \mathbf{v}) on $\kappa_\tau = (\mathbf{j}, \mathbf{k})$ gives $(\mathbf{j} + \mathbf{v} \times \mathbf{k}, \mathbf{k})$, but $\mathbf{j} + \mathbf{v} \times \mathbf{k} \in \mathfrak{so}(3)$ and we need to project to $\mathfrak{so}(2)$. This projection is along \mathbf{j} and since $(\mathbf{v} \times \mathbf{k}) \cdot \mathbf{j} = 0$, we see that the coadjoint orbit is only the point $\{(\mathbf{j}, \mathbf{k})\}$.

If $\mathbf{k} \neq 0$, we may rotate $\mathbf{k} = (0, 0, k)^T$, where $k = \|\mathbf{k}\| > 0$. Rotating in such a way that we leave \mathbf{k} fixed, we can bring $\mathbf{j} = (0, j_2, j_3)^T$, but then by choosing \mathbf{v} suitably, we can bring $\mathbf{j} = (0, 0, j)$ for some $j \in \mathbb{R}$. In other words, the orbit representative is

$$\left(\begin{pmatrix} 0 \\ 0 \\ j \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix} \right) \quad \text{with } j \in \mathbb{R} \text{ and } k > 0. \tag{B.37}$$

The stabiliser of this (\mathbf{j}, \mathbf{k}) is the two-dimensional abelian group consisting of elements (R, \mathbf{v}) with $R\mathbf{k} = \mathbf{k}$ and $\mathbf{v} \times \mathbf{k} = \mathbf{0}$. Since it is abelian, its coadjoint orbits are points. In summary, the coadjoint orbits of (\mathbf{j}, \mathbf{k}) with $\mathbf{k} \neq \mathbf{0}$ are of the form

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ j \end{pmatrix} \right\} \times T^*S_{\|\mathbf{k}\|}^2. \tag{B.38}$$

B.5.3 Structure of the coadjoint orbits of the Carroll group

We now determine the structure of the Carroll coadjoint orbits. It is now a simple matter to go through each of the coadjoint orbits and identify κ, τ, K_τ and κ_τ and the K_τ -coadjoint orbit $\mathcal{O}_{\kappa_\tau}$ of κ_τ . Table 4 summarises these results, with explanations following the table.

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