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# Review on non-relativistic gravity

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This study reviews the history of Newton–Cartan (NC) gravity with an emphasis on recent developments, including the covariant, off-shell large speed of light expansion of general relativity. Depending on the matter content, this expansion leads to either NC geometry with absolute time or NC geometry with non-relativistic gravitational time dilation effects. The latter shows that non-relativistic gravity (NRG) includes a strong field regime and goes beyond Newtonian gravity. We start by reviewing early developments in NC geometry, including the covariant description of Newtonian gravity, mainly through the works of Trautman, Dautcourt, Künzle, and Ehlers. We then turn to more modern developments, such as the gauging of the Bargmann algebra and describe why the latter cannot be used to find an off-shell covariant description of Newtonian gravity. We review recent work on the  $1/c$  expansion of general relativity and show that this leads to an alternative “type II” notion of NC geometry. Finally, we discuss matter couplings, solutions, and odd powers in  $1/c$  and conclude with a brief summary of related topics.

## KEYWORDS

Newton–Cartan gravity, non-relativistic gravity, torsional Newton–Cartan geometry, expansions of general relativity, Bargmann symmetry, null reduction

## 1 Introduction

Although nature is Lorentz-invariant at the fundamental level, there are many instances where it appears effectively non-relativistic. This can happen in condensed matter or biological systems, but it can also be true for gravitational phenomena. Indeed, Newtonian gravity can be obtained as a limit of general relativity. Additionally, general relativity can often be approximated using non-relativistic descriptions of gravity, such as in the post-Newtonian approximation. This review of non-relativistic gravity (NRG) is based on a modern description of non-relativistic approximations in terms of Newton–Cartan (NC) geometries, their dynamics, and their interaction with matter.

The idea of geometrizing Newton’s law of gravitation dates back to Cartan’s introduction of NC geometry in 1923 [1, 2]. The deep insight into the heart of that work is that “gravity is geometry” is true independently regarding whether local observers in inertial frames see the laws of special relativity or Galilean relativity. Special relativity in inertial frames is a separate and essential ingredient in the formulation of Einstein’s theory of general relativity. Conversely, inertial frames with local Galilean relativity lead to NC gravity.

Recent years have witnessed a revival of research on NRG. In particular, two important novel insights have been obtained. First, we have learned that NRG is much richer than Newtonian gravity and goes beyond it by allowing for a strong field regime, including gravitational time dilation. One key ingredient for this was allowing for NC metric-compatible connections with non-zero torsion. Geometrically, non-zero torsion leads to spacetime manifolds without absolute time, which goes beyond the historical perspective on NC geometry, where time is always absolute. Another crucial development came from

considering the off-shell large speed of light expansions, as opposed to on-shell expansions or large speed of light limits. These off-shell expansions naturally led to a notion of NC geometry whose local symmetries are different from what was previously considered, which is necessary to provide an action principle for NRG and Newtonian gravity in particular.

The recent revival of NRG has been spurred on in large part by a deeper understanding of non-relativistic geometry and its multiple connections to field theory, holography, and string theory.<sup>1</sup> We highlight, in particular, Andringa et al.'s study [3], which showed how to obtain torsionless NC geometry by gauging the Bargmann algebra (the Galilei algebra with a central extension). This provided a modern understanding of the geometric fields of NC geometry. Another important advance was the discovery of a torsionful generalization of NC geometry, which was first observed as the boundary geometry in the context of non-AdS (Lifshitz) holography [4–6]. In a remarkable parallel development, Son [7] showed that the non-relativistic effective field theory describing certain aspects of the fractional quantum Hall effect naturally couples to NC geometry, highlighting its relevance for non-relativistic field theories. Finally, a systematic analysis of the large speed of light expansion of GR, considering the possibility of torsion in NC geometry, was performed in [8–11], leading to our present understanding of NRG.

We start this review in Section 2 by presenting an overview of the most pertinent advances in the historical development of NC geometry up to the recent revival of NC geometry 10 years ago. We will not be fully exhaustive, but we have attempted to collect the main references and milestones into a readable introduction to the subject of (torsion-free) NC geometry. Further details are provided in Section 3, where we present the notion of NC geometry that was developed in the work of Trautman, Dautcourt, Künzle, Ehlers, and others. In contrast to some of these original works, our discussion is fully covariant.

Then, in Section 4, we turn to the modern era, which we define as starting from [3], in which NC gravity with absolute time could be viewed as the dynamics of a geometry that can be obtained by gauging the Bargmann algebra. Details of this procedure, including its extension to non-zero torsion, are discussed in Section 5. This results in what we refer to as “type I” torsional Newton–Cartan (TNC) geometry. We obtain the action for a non-relativistic point particle coupled to this geometry and an action for its gravitational dynamics. On a technical level, these actions are obtained from two distinct routes, namely, a large speed of light limit and null reduction, whose results are equivalent. At the end of this section, we show that type I TNC is not the correct geometrical framework to get an off-shell action for Newtonian gravity.

Thus, we turn in Section 6 to the derivation of an action that encapsulates the Poisson equation of Newtonian gravity as an equation of motion. In fact, this action describes a more general notion of NRG, which includes a strong field regime. This action is obtained by carefully considering the large speed of light expansion

of GR, which naturally leads to a new notion of NC geometry, referred to as “type II” TNC geometry.

Section 7 briefly discusses various other aspects of NRG from the perspective of the large speed of light expansion, including the coupling to matter and the particular case of the strong field expansion of the Schwarzschild solution of GR, as well as some other examples of solutions. We also remark on the inclusion of odd powers of  $1/c$  in the expansion, as the analysis in Section 6 is restricted to even powers, which form a closed subsector.

Finally, Section 8 discusses various related applications and appearances of NRG in the fast-growing recent literature on these and related topics, including many additional references.

## 2 History

We start by giving an overview of some of the important historical developments. We will not attempt to be exhaustive, and our aim is not to present each study's contents as accurately as possible. Rather, we want to weave an accessible narrative through the early developments of the subject. This section and the following one can also be read as an introduction to NC geometry. We have attempted to provide clickable links to the relevant studies, which can be hard to find.

Historically, the subject was introduced by Cartan in [1, 2]; see [12] for an English translation. However, these studies are not easily digestible as they are not written in a modern geometrical language, so it would be difficult to incorporate them accurately into our narrative. For this reason and with much regret, we will not discuss these works.

Instead, the earliest historical reference for our current discussion and one of the pioneering works on the subject of  $1/c$  expansions of general relativity is [13]. Friedrichs introduced the NC metric  $\tau_\mu\tau_\nu$  and co-metric  $h^{\mu\nu}$ , which is symmetric, has a signature  $(0, 1, \dots, 1)$ , and writes Newton's second law in a covariant form in terms of an NC metric-compatible connection while realizing that the latter is not unique. Friedrichs then found these objects from general relativity (GR) via a  $1/c$  expansion. Similar comments about NC metrics and their properties can be found in [14].

For an overview of early aspects of NC geometry, we refer to [15]. The subject has also been covered in [16, 17].

### 2.1 Trautman, Dautcourt, Künzle, and Ehlers

Despite these important pioneering works, we consider the more modern notion of NC geometry to begin in earnest with the work<sup>2</sup> of [18–20], which provides an axiomatic definition of Newtonian gravity.

In all three articles, Trautman gave an axiomatic definition of NC gravity. The precise set of postulates changes slightly from one

1 In addition to this introduction, many more relevant and important works will be mentioned in the following sections and the final discussion section. We also refer the reader to the companion reviews on other topics in applications of non-relativistic (more generally non-Lorentzian) geometry [78, 129, 130].

2 The works of Trautman can be found at: <http://trautman.fuw.edu.pl/publications/scientific-articles.html>. The first reference [15] is in French. The second reference [16] is a chapter from a book (see in particular Section 5.3), and it basically reviews the contents of [15] with slightly more detail. The later paper [17] is essentially an English version of [15].

article to another. We follow [20], where Trautman introduced the following postulates:

- (1) Spacetime is a four-dimensional differentiable manifold endowed with a symmetric affine connection  $\Gamma_{\mu\nu}^\rho$ .
- (2) There is a nowhere-vanishing one-form  $\tau_\mu$  and a nowhere vanishing co-metric  $h^{\mu\nu}$  of signature (0,1,1,1) such that  $h^{\mu\nu}\tau_\mu = 0$ .
- (3) The symmetric affine connection  $\Gamma_{\mu\nu}^\rho$  is metric-compatible in the sense that

$$\nabla_\mu \tau_\nu = 0, \quad \nabla_\mu h^{\nu\rho} = 0, \tag{2.1}$$

where  $\nabla_\mu$  is the associated covariant derivative.

- (4) The Riemann curvature tensor  $R_{\mu\nu\rho}^\sigma$  associated with the affine connection obeys the following two conditions:

$$R_{\kappa\lambda}[\mu^\rho \tau_\nu] = 0, \quad R_{\mu\rho}^\nu{}^\sigma = R_{\rho\mu}^\sigma{}^\nu, \tag{2.2}$$

where the second index has been raised with  $h^{\mu\nu}$ , so  $R_{\mu\rho}^\nu{}^\sigma = h^{\nu\kappa}R_{\mu\kappa\rho}^\sigma$ . See [Supplementary Appendix SA](#) for our conventions for the Riemann tensor.

- (5) Particles in free fall follow geodesics of the affine connection, and gravity is described by an equation of the form (in 3 + 1 dimensions)

$$R_{\mu\nu} = 4\pi G\rho\tau_\mu\tau_\nu, \tag{2.3}$$

where  $\rho$  is the mass density,  $G$  is Newton’s constant, and  $R_{\mu\nu} = R_{\mu\rho\nu}{}^\rho$  is the Ricci tensor.

As the connection is symmetric, taking the antisymmetric part of  $\nabla_\mu\tau_\nu = 0$  implies that  $2\partial_{[\mu}\tau_{\nu]} = 0$ , or equivalently  $d\tau = 0$  in form notation. The first of the two conditions in (2.2) was later dropped in favor of a boundary condition, as discussed in the following equations. The second condition in (2.2) is often simply referred to as the Trautman condition. In various places in the literature, it is phrased as

$$h^{\lambda[\mu}R_{\lambda}{}^{\nu]} = 0. \tag{2.4}$$

The latter is equivalent to

$$R_{(\mu\rho)}^\nu{}^\sigma = R_{(\mu\rho)}^\sigma{}^\nu, \tag{2.5}$$

which appears to be weaker than the second condition of (2.2). However, using  $\nabla_\rho h^{\mu\nu} = 0$ , which implies  $[\nabla_\rho, \nabla_\sigma]h^{\mu\nu} = 0$  or equivalently  $h^{\kappa\nu}R_{\rho\sigma\kappa}{}^\mu + h^{\mu\kappa}R_{\rho\sigma\kappa}{}^\nu = 0$ , Eq. 2.4 can be equivalent to the second condition of (2.2).

Around the time of Trautman’s work, the mathematical framework underlying NC geometry was put on a firm foundation in [21] (see also the later work by [22]). We will not focus on the mathematical development of NC geometry in this review, so we will not comment further on these works.

Trautman’s work can be viewed as an intrinsic definition of Newtonian gravity without any reference to  $1/c$  expansions of GR. It was later improved by [22], who, among other things, worked out the class of torsion-free metric-compatible connections and realized that the first Trautman condition (the first equation in (2.2) is redundant if we assume asymptotic flatness. It was shown by [23, 24] that an appropriate covariant expansion of GR in powers of  $1/c$

reproduces Trautman’s formulation of NC gravity, with the caveat that the first Trautman condition is dropped. We refer to [Sections 6Sections 3.8](#) for more details. The first Trautman condition is equivalent to the following condition:  $h^{\rho\kappa}R_{\mu\nu\rho}{}^\sigma = 0$  that Dautcourt in [24] attributed to Dixon<sup>3</sup> [25]. Dautcourt wrote that this condition did not follow from the  $1/c$  expansion of GR. Conversely, the second equation of (2.2) does follow from the  $1/c$  expansion of the Riemann tensor of GR. More details will be given in the following paragraphs.

In [24], Dautcourt only discussed the expansion in terms of even powers of  $1/c$ . From the outset, the NC connection is considered symmetric and metric-compatible. As shown previously, this implies that  $d\tau = 0$  so that  $\tau = dT$  for a time coordinate  $T$ . Therefore, time is absolute in this notion of NC geometry. This is an important restriction from the  $1/c$  expansion perspective. Dautcourt knew that the latter could be formulated without this restriction but did not consider the general case.

Later, Dautcourt [26] included odd powers in  $1/c$ . Dautcourt more explicitly discussed the option of allowing for a non-trivial lapse function  $N$  in NC geometry so that  $\tau = NdT$ , which is now known as twistless torsional NC (TTNC) geometry. It is equivalent to the Frobenius condition  $\tau \wedge d\tau = 0$ , implying that equal  $T$  surfaces define spatial hypersurfaces.<sup>5</sup> In particular, Dautcourt’s concluded that the  $1/c$  expansion of GR leads to a theory that is more general than Newtonian gravity (on page 7) but then suggested that this more general case with  $N$  a priori arbitrary is not so interesting because insisting on global regularity of the NC lapse function  $N$  would reduce  $N$  to a constant and thus force time to be absolute. The argument here is that the NC lapse function is harmonic, and using Liouville’s theorem, it must be constant to be regular everywhere without appropriate sources. The current viewpoint is that the condition of global regularity of the NC lapse function is too restrictive because this rules out interesting non-relativistic approximations of GR (given in the following sections) and that one can find sources leading to singularities in  $N$ .

Ehlers [27, 28] developed “frame theory,” which is a geometrical formulation that treats Lorentzian and Galilean geometry on equal footing. The frame theory has a parameter  $\lambda = c^{-2}$  that can be either zero or positive, leading to Galilean or Lorentzian geometry. Furthermore, it has a second parameter, Newton’s constant  $G$ , that can also be zero or positive, leading to four different theories depending on whether  $\lambda$  and  $G$  are zero or positive. Setting  $G = 0$  means that one considers test particles in a fixed background that obeys some curvature constraints. The motivation behind frame theory is to put the  $1/c$  expansion on a firmer mathematical foundation, improving on the work of Dautcourt. For a review of the frame theory, we refer to [29].

3 In [22], it is shown that Newtonian gravity, as defined under point 5 of Trautman’s axiomatic definition (plus a cosmological constant), is unique in that it follows from points 1 to 3, as well as the second equation of (2.2) and some general assumptions about the structure of the equation and the matter content.

4 We do not allow for non-contractible closed timelike loops so that  $\tau$  is exact when it is closed.

5 As shown in [Section 6](#), the  $1/c$  expansion of Einstein’s equations rules out the possibility that  $\tau \wedge d\tau \neq 0$ .

Ehlers reviewed Trautman's work and noticed that the condition in the first equation of (2.2) could be written in other forms. In particular, it can be shown that the following equations are all equivalent formulations of this condition:

$$0 = R_{\mu\nu\sigma}{}^{\rho} R_{\alpha\beta\rho}{}^{\sigma} h^{\nu\beta}, \quad (2.6)$$

$$0 = R_{\mu\nu}[\sigma{}^{\rho} \tau_{\rho}{}^{\nu}], \quad (2.7)$$

$$0 = R_{\mu\nu\sigma}{}^{\rho} h^{\nu\sigma}. \quad (2.8)$$

These three conditions are now often referred to as the Ehlers conditions. The main point of these conditions is to restrict the geometry such that the NC gravity in Eq. 2.3 only contains the Newtonian potential (in appropriate coordinates). However, as we know from Dautcourt's work (and as was also known to Ehlers), these conditions are unnecessary, as this restriction can also be achieved by imposing asymptotic flatness. Ehlers [30] gave a short review of frame theory and examples of Newtonian limits of the well-known GR solutions.

## 2.2 Post-Newtonian corrections

Part of the motivation behind frame theory is the question of whether solutions to Newtonian gravity are extendable (e.g., in the sense of post-Newtonian corrections) to full relativistic solutions of GR. Rendall [31] showed that post-Newtonian corrections are incompatible with asymptotic flatness. The physical reason behind this is that these corrections do not correctly describe the far zone of some non-relativistic matter distribution where gravitational waves dominate. In other words, the post-Newtonian regime corresponding to, for example, a perfect fluid matter source has a finite radius of validity. This effect is noticeable when one goes beyond the first post-Newtonian correction. Lottemoser [32, 33] showed that the constraint equations of GR (in harmonic gauge) admitted a well-defined  $1/c$  expansion (in the sense of a convergent series).

In the current literature on post-Newtonian corrections, which will not be reviewed here (e.g., [34, 35] and references therein), the dominant approach does not consist of expanding the Einstein equations in powers of  $1/c$  and then solving them order by order, which is sometimes called the classic approach. Instead, one formally solves the Einstein equations in harmonic gauge (using what are known as the relaxed Einstein equations<sup>6</sup>) and imposes a boundary condition that leads to an integral equation using a retarded Green's function. In the Blanchet–Damour approach,<sup>7</sup> this integral equation is solved outside the source (i.e., in vacuum) as an expansion in  $G$ . In a region containing the non-relativistic source, the integral equation is solved as an expansion in  $1/c$ . The  $G$  and  $1/c$

expansions are then matched multipole by multipole in their overlap region using matched asymptotic expansion.

The recent work on the covariant  $1/c$  expansion of GR can be viewed as an attempt to revive the classic approach used in the early days of work on post-Newtonian expansions. It also goes beyond that because it can cover regimes of strong gravity<sup>8</sup> and is more flexible regarding the gauge choice one uses. In general, one will have to match the  $1/c$  expansion onto a  $G$  expansion and thus find some hybrid of the classical and more modern Blanchet–Damour or Will–Wiseman approaches, which is the aim of the upcoming work [36] (see also [37] for a covariant approach to the post-Newtonian expansion up to 1PN order).

## 2.3 Null reduction

Finally, even though it is not the focus of this review article, we would be remiss not to mention Duval et al.'s work [38]. So far, we have discussed NC geometry either intrinsically or as originating from the  $1/c$  expansion of GR. Duval et al. offered a third perspective by showing that NC geometry can also be obtained from the null reduction of a Lorentzian geometry with a null isometry. They [38] (see [39] for further developments and [40] for a review) considered null reduction of Lorentzian geometry and null uplifts of NC geometry to Lorentzian geometry with a null Killing vector. This is related to the Eisenhart lift [41] of Hamiltonian dynamics as shown in [42].<sup>9</sup> It is also related to the Bargmann algebra [43] because this algebra is the centralizer of the null isometry in the higher-dimensional Poincaré algebra [44], as shown in Eq. 5.4.

Duval et al. [38] discussed uplifts of NC geometries for which  $d\tau = 0$  to pp-waves in one dimension higher. This can be generalized to cases where  $\tau \wedge d\tau = 0$  [45] and even to cases where  $\tau \wedge d\tau \neq 0$  [46].

However, as reviewed in Section 5.5, Eq. 2.3 for NC gravity coupled to matter cannot be obtained from null reduction. Despite this shortcoming, it remains useful to adapt this higher-dimensional perspective for many aspects, such as particle motion in a fixed background and more geometrical questions.

## 3 Basics of torsion-free NC geometry

This section provides more details about the covariant formulation of NC geometry that started with the work of Trautman. This section does not follow a strict historical path and uses slightly more modern tools. For example, in contrast to much of the early literature, we will use a fully covariant approach and largely refrain from choosing special coordinates.

<sup>6</sup> The relaxed Einstein equations are equivalent to Einstein's equations but written in terms of a different variable  $k^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}$ , where  $\eta_{\mu\nu}$  is the Minkowski metric. Furthermore, this formulation only works in harmonic gauge for which  $\partial_{\mu}k^{\mu\nu} = 0$ .

<sup>7</sup> See [32] and references therein. An alternative approach to solving this integral equation is the Will–Wiseman approach, see [34] and references therein.

<sup>8</sup> By strong gravity, we mean a regime where the clock one-form  $\tau$  obeys  $\tau \wedge d\tau = 0$  but  $d\tau \neq 0$  so that there is a non-trivial NC lapse function (describing an NC geometry with gravitational time dilation).

<sup>9</sup> See, for example, the review paper in [131] for details about the Lorentzian Eisenhart lift.

### 3.1 NC metric data

The main objects of interest are the nowhere-vanishing “clock” one-form  $\tau_\mu$  and the nowhere-vanishing “spatial” co-metric  $h^{\mu\nu}$  that has a signature  $(0, 1, \dots, 1)$ . These objects obey the condition that  $\tau_\mu h^{\mu\nu} = 0$ . We take the dimension of the underlying spacetime manifold to be  $(d + 1)$ , so that the spatial co-metric has rank  $d$ . We can also define the inverse objects  $\nu^\mu$  and  $h_{\mu\nu}$  by demanding that the following relations hold:

$$h_{\mu\rho}h^{\rho\nu} - \tau_\mu\nu^\nu = \delta_\mu^\nu, \quad \nu^\mu h_{\mu\nu} = 0, \quad \tau_\mu\nu^\mu = -1, \quad \tau_\mu h^{\mu\nu} = 0. \tag{3.1}$$

Another way of phrasing these relations among the various objects is by saying that  $\tau_\mu\tau_\nu + h_{\mu\nu}$  is invertible with inverse  $\nu^\nu\nu^\rho + h^{\nu\rho}$ . The positive determinant of  $\tau_\mu\tau_\nu + h_{\mu\nu}$  is denoted by  $e^2$ , and we can use  $e$  as an integration measure. The objects  $\nu^\mu$  and  $h_{\mu\nu}$  are defined up to a local Galilean boost, which acts as

$$\delta\nu^\mu = h^{\mu\nu}\lambda_\nu, \quad \delta h_{\mu\nu} = \tau_\mu\lambda_\nu + \tau_\nu\lambda_\mu. \tag{3.2}$$

Here, we require  $\nu^\mu\lambda_\mu = 0$ , and  $\lambda_\mu$  transforms under a second Galilean boost transformation as  $\delta\lambda_\mu = \tau_\mu h^{\nu\rho}\lambda_\nu\lambda_\rho$  so that  $\nu^\mu\lambda_\mu = 0$  and  $\nu^\mu h_{\mu\nu} = 0$  are boost invariant under a second-order boost transformation. The exponentiation of the infinitesimal Galilean boost transformation terminates after the second order in  $\lambda_\mu$ . The reason why we call these transformations Galilean boosts<sup>10</sup> will become clear in Section 5. We note that the integration measure  $e$  is Galilean boost invariant. Finally, we can introduce frame fields  $e_\mu^a$  and  $e^{\mu a}$  for the degenerate spatial metric and co-metric, which satisfy

$$h_{\mu\nu} = \delta_{ab}e_\mu^a e_\nu^b, \quad h^{\mu\nu} = \delta^{ab}e^{\mu a} e^{\nu b}, \tag{3.3}$$

where  $a = 1, \dots, d$  are flat frame indices. The integration measure  $e = \det(\tau_\mu, e_\mu^a)$  is the determinant of this spatial vielbein together with the clock one-form.

### 3.2 Class of torsion-free metric-compatible connections

Consider the case of a symmetric connection that is metric-compatible in the sense that

$$\nabla_\mu\tau_\nu = 0, \quad \nabla_\mu h^{\nu\rho} = 0, \tag{3.4}$$

such that  $\Gamma^{\rho}_{[\mu\nu]} = 0$ . We can split such a connection as

$$\Gamma^{\rho}_{\mu\nu} = \check{\Gamma}^{\rho}_{\mu\nu} + C^{\rho}_{\mu\nu}, \tag{3.5}$$

where  $C^{\rho}_{\mu\nu}$  is symmetric in  $\mu$  and  $\nu$  and transforms as a tensor under coordinate transformations, and furthermore, we define

$$\check{\Gamma}^{\rho}_{\mu\nu} = -\nu^\rho\partial_\mu\tau_\nu + \frac{1}{2}h^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}). \tag{3.6}$$

As we have assumed  $\Gamma^{\rho}_{[\mu\nu]} = 0$  in Trautman’s postulate 3 mentioned previously, it follows that we have  $d\tau = 0$ . Demanding metric compatibility leads to

$$C^{\rho}_{\mu\nu}\tau_\rho = 0, \quad C^{\nu}_{\mu\lambda}h^{\lambda\rho} + C^{\rho}_{\mu\lambda}h^{\nu\lambda} = 0. \tag{3.7}$$

We can solve the first of these conditions by writing  $C^{\rho}_{\mu\nu} = h^{\rho\sigma}Y_{\sigma\mu\nu}$ , where  $Y_{\sigma\mu\nu} = Y_{\sigma\nu\mu}$ . The second condition tells us that

$$(h^{\nu\sigma}h^{\lambda\rho} + h^{\rho\sigma}h^{\nu\lambda})Y_{\sigma\mu\lambda} = 0. \tag{3.8}$$

Using completeness on the  $\mu$  index (i.e., invoking  $\delta^\kappa_\mu = -\tau_\mu\nu^\kappa + h_{\mu\rho}h^{\rho\kappa}$  following Eq. 3.1), we can write

$$Y_{\sigma\mu\nu} = -\frac{1}{2}\tau_\mu F_{\nu\sigma} - \frac{1}{2}\tau_\nu F_{\mu\sigma} + L_{\sigma\mu\nu}, \tag{3.9}$$

for some  $F_{\mu\nu}$  and some  $L_{\sigma\mu\nu} = L_{\sigma\nu\mu}$ , which is purely spatial (meaning that all contractions with  $\nu^\rho$  are zero). The factor of  $-1/2$  is there for later convenience. From Eq. 3.8, we then get two equations by contracting with  $\nu^\mu$  and  $h^{\mu\kappa}$ :

$$(h^{\nu\rho}h^{\lambda\sigma} + h^{\rho\sigma}h^{\nu\lambda})F_{\nu\sigma} = 0, \quad L_{\sigma\mu\nu} = -L_{\nu\mu\sigma}. \tag{3.10}$$

$L_{\sigma\mu\nu}$  must be antisymmetric in its first two indices because

$$L_{\sigma\mu\nu} = L_{\sigma\nu\mu} = -L_{\mu\nu\sigma} = -L_{\mu\sigma\nu}. \tag{3.11}$$

Therefore, we have a tensor  $L_{\sigma\mu\nu}$  that is antisymmetric in slots 1 and 2, as well as in slots 1 and 3, and it is symmetric in slots 2 and 3. Such a tensor is zero, as follows from

$$L_{\sigma\mu\nu} = -L_{\mu\sigma\nu} = L_{\nu\mu\sigma} = L_{\nu\sigma\mu}, \tag{3.12}$$

which shows that  $L$  is symmetric in slots 1 and 3, but that means it must be zero because it is also antisymmetric in slots 1 and 3. Hence, we have

$$C^{\rho}_{\mu\nu} = -\frac{1}{2}h^{\rho\sigma}(\tau_\mu F_{\nu\sigma} + \tau_\nu F_{\mu\sigma}), \tag{3.13}$$

where  $(h^{\nu\rho}h^{\lambda\sigma} + h^{\nu\lambda}h^{\rho\sigma})F_{\nu\sigma} = 0$  so that  $F_{\mu\nu}$  is of the form  $F_{\mu\nu} = \tau_\mu X_\nu + \tau_\nu Y_\mu + X_{\mu\nu}$ , where  $X_{\mu\nu}$  is purely spatial and antisymmetric. As only  $h^{\rho\sigma}F_{\nu\sigma}$  appears in the expression for  $C^{\rho}_{\mu\nu}$ , we can, without loss of generality, set  $Y_\mu = -X_\mu$  so that  $F_{\mu\nu}$  is antisymmetric. Therefore, we see that

$$\Gamma^{\rho}_{\mu\nu} = \check{\Gamma}^{\rho}_{\mu\nu} - \frac{1}{2}h^{\rho\sigma}(\tau_\mu F_{\nu\sigma} + \tau_\nu F_{\mu\sigma}), \tag{3.14}$$

where  $\check{\Gamma}^{\rho}_{\mu\nu}$  is given in (3.6). Equation 3.14 is the most general torsion-free NC metric-compatible connection. Unlike in Lorentzian geometry, we see here that this connection is not unique, and its freedom is parametrized by the antisymmetric tensor  $F_{\mu\nu}$  [21, 22].

We stress that  $\check{\Gamma}^{\rho}_{\mu\nu}$  is not invariant under Galilean boosts, and therefore, the covariant derivative whose connection coefficients are given by  $\check{\Gamma}^{\rho}_{\mu\nu}$  does not form a proper affine connection (although the covariant derivative transforms correctly under general coordinate transformations). For  $\Gamma^{\rho}_{\mu\nu}$ , as defined in (3.14), to be boost-invariant,  $F_{\mu\nu}$  must transform appropriately under Galilean boost transformations. This is possible for a symmetric connection, and we will reach this point in the following paragraphs. With this caveat in mind, we will still use a covariant derivative  $\check{\nabla}_\mu$  with connection coefficients  $\check{\Gamma}^{\rho}_{\mu\nu}$  and define an associated Riemann curvature  $\check{R}^{\rho}_{\mu\nu\sigma}{}^\rho$  in

10 They are also sometimes referred to as Milne boosts.

the usual way. We refer to [Supplementary Appendix SA](#) for our curvature tensor conventions.

### 3.3 Ehlers conditions

For completeness, we briefly comment on the first Trautman condition, corresponding to the first equation of (2.2), even though we will not use it in the following. First, we show that it is equivalent to the Ehlers conditions:

$$1) \quad R_{\mu\nu\sigma}{}^\rho R_{\lambda\kappa\rho}{}^\sigma h^{\lambda\kappa} = 0, \tag{3.15}$$

$$2) \quad R_{\mu\nu[\sigma}{}^\rho \tau_{\gamma]} = 0, \tag{3.16}$$

$$3) \quad R_{\mu\nu\sigma}{}^{[\rho} h^{\gamma]\sigma} = 0. \tag{3.17}$$

We will show that 1) is equivalent to 2) and that 2) is equivalent to 3).

First, we show that 1) implies 2). We project 1) with  $v^\mu v^\lambda$ . In terms of tangent space indices, this means that the condition becomes  $R_{0abc}R_{0abc} = 0$ , where  $R_{0abc} = -v^\mu e^\nu{}_a e^\sigma{}_b e^\rho{}_c R_{\mu\nu\sigma}{}^\rho$  and where we used  $R_{\mu\nu\sigma}{}^\rho \tau_\rho = 0$ , which follows from the covariant constancy of  $\tau_\mu$ . This is a sum of squares and hence  $R_{0abc}$ . Contracting condition 1) with  $h^{\mu\lambda}$ , we find  $R_{abcd}R_{abcd} = 0$  so that  $R_{abcd} = 0$ . From  $R_{0bcd} = 0$  and  $R_{abcd} = 0$ , it follows that  $R_{\mu\nu\sigma}{}^\rho h^{\sigma\gamma} = 0$ . The latter equation is equivalent to condition 2). This follows from  $0 = R_{\mu\nu\sigma}{}^\rho h^{\sigma\gamma} h_{\gamma\kappa} = R_{\mu\nu\kappa}{}^\rho + R_{\mu\nu\rho}{}^\sigma v^\rho \tau_{\kappa\sigma}$ , where we used completeness.

The other direction, that 2) implies 1), is more straightforward as 2) is equivalent to  $R_{\mu\nu\sigma}{}^\rho h^{\sigma\gamma} = 0$ , which implies 1). Here, we used again  $R_{\mu\nu\sigma}{}^\rho \tau_\rho = 0$ . To show that 2) is equivalent to 3), all we need is  $R_{\mu\nu\sigma}{}^{(\rho} h^{\gamma)\sigma} = 0$ , which follows from  $[\nabla_\mu, \nabla_\nu]h^{\rho\gamma} = 0$ . This result implies that  $R_{\mu\nu\sigma}{}^\rho h^{\sigma\gamma} = R_{\mu\nu\sigma}{}^{[\rho} h^{\gamma]\sigma} = 0$  from which the result follows.

Therefore, what is the purpose of the first Trautman condition? To answer this, we look closer at  $R_{\mu\nu\sigma}{}^\rho h^{\sigma\gamma} = 0$ . Expressing this in terms of  $\check{R}_{\mu\nu\sigma}{}^\rho$  and projecting by  $h^{\alpha\mu}h^{\nu\beta}$  and  $v^\gamma h^{\mu\kappa}$ , we find

$$0 = h^{\alpha\mu}h^{\nu\beta}\check{R}_{\mu\nu\sigma}{}^\rho h^{\sigma\gamma}, \tag{3.18}$$

$$0 = h^{\rho\alpha}h^{\gamma\lambda}h^{\beta\kappa}\left(\check{\nabla}_\alpha K_{\lambda\beta} - \check{\nabla}_\lambda K_{\alpha\beta} + \frac{1}{2}\check{\nabla}_\beta F_{\alpha\lambda}\right), \tag{3.19}$$

where  $K_{\alpha\beta} = -\frac{1}{2}\mathcal{L}_v h_{\alpha\beta}$  with  $\mathcal{L}_v$ , the Lie derivative along  $v^\mu$ . The first equation states that we must always have a flat geometry on constant time slices (these are the surfaces to which  $\tau$  is the normal one-form). The second equation is a condition on the magnetic part of the field strength  $F_{\mu\nu}$ . However, these conditions are too strong. It should only be the equivalent of the Einstein equation that decides what the allowed spaces are. In addition, the first Trautman condition does not follow from the  $1/c$  expansion of Lorentzian geometry.

### 3.4 Trautman condition

The relation between the Riemann tensors associated with two connections related *via* (3.5) is given by

$$R_{\mu\nu\sigma}{}^\rho = \check{R}_{\mu\nu\sigma}{}^\rho - \check{\nabla}_\mu C_{\nu\sigma}{}^\rho + \check{\nabla}_\nu C_{\mu\sigma}{}^\rho - C_{\mu\lambda}{}^\rho C_{\nu\sigma}{}^\lambda + C_{\nu\lambda}{}^\rho C_{\mu\sigma}{}^\lambda, \tag{3.20}$$

where on the left-hand side, we have the Riemann tensor associated with  $\nabla_\mu$ , and on the right-hand side, we have the

Riemann tensor associated with  $\check{\nabla}_\mu$ . This gives, for the Trautman condition (2.4), that

$$0 = h^\mu{}_{[\gamma} R_{\mu(\nu\sigma)}{}^\rho] = h^\mu{}_{[\gamma} \check{R}_{\mu(\nu\sigma)}{}^\rho] + \frac{3}{4}h^{\mu\gamma}h^{\rho\kappa}\left(\tau_\nu \check{\nabla}_{[\mu} F_{\sigma\kappa]} + \tau_\sigma \check{\nabla}_{[\mu} F_{\nu\kappa]}\right). \tag{3.21}$$

It can be shown that  $h^{\mu[\gamma} \check{R}_{\mu(\nu\sigma)}{}^\rho] = 0$ . The details can be found in [Supplementary Appendix SB](#). For now, let us see what its consequences are

$$0 = h^{\mu\gamma}h^{\rho\kappa}\left(\tau_\nu \check{\nabla}_{[\mu} F_{\sigma\kappa]} + \tau_\sigma \check{\nabla}_{[\mu} F_{\nu\kappa]}\right). \tag{3.22}$$

Contracting this equation with  $v^\nu v^\sigma$  and  $v^\gamma h^{\sigma\lambda}$ , we see that this is true if and only if we have

$$0 = \check{\nabla}_{[\mu} F_{\sigma\kappa]} = \partial_{[\mu} F_{\sigma\kappa]}, \tag{3.23}$$

where the torsion is zero. In other words, the Trautman condition (2.4) leads to a Bianchi identity for  $F_{\mu\nu}$ .

We point out that as an alternative to the Trautman condition (2.4), one could also impose

$$h_{\rho[\kappa} R_{\mu\nu]\sigma}{}^\rho v^\sigma = 0, \tag{3.24}$$

whenever  $\Gamma_{\mu\nu}^\rho$  is symmetric because of the following identity:

$$h_{\rho[\kappa} R_{\mu\nu]\sigma}{}^\rho v^\sigma = -\check{\nabla}_{[\mu} F_{\nu\kappa]} = -\partial_{[\mu} F_{\nu\kappa]}, \tag{3.25}$$

where we used (3.14).

From Eq. 3.23, we see that the second Trautman condition (2.4) is obeyed if we take  $F_{\mu\nu} = \partial_\mu m_\nu - \partial_\nu m_\mu$  with  $m_\mu$  defined up to a gauge transformation of the form  $\delta m_\mu = \partial_\mu \sigma$ . For the connection (3.14) to be invariant under Galilean boosts, we need  $m_\mu$  to transform as  $\delta m_\mu = \lambda_\mu$ , where we remind the reader that  $v^\mu \lambda_\mu = 0$ . Furthermore, in contrast to the first Trautman condition, the second Trautman condition does follow from the  $1/c$  expansion of the following identity when working with a Lorentzian metric and the Levi-Civita connection:

$$g^\mu{}_{[\gamma} [{}^\sigma] R_{\mu\rho\sigma]} = 0, \tag{3.26}$$

where in this expression  $R_{\mu\nu\rho}{}^\sigma$  is the curvature of the Levi-Civita connection. The  $1/c$  expansion of (3.26), assuming that  $d\tau = 0$ , leads to (2.4), which is equally identically satisfied in NC geometry because the  $1/c$  expansion also tells us that  $F_{\mu\nu} = \partial_\mu m_\nu - \partial_\nu m_\mu$  (see further in [Section 3.8](#)).

### 3.5 Field content of torsion-free NC geometry

From now on, we will always take  $F_{\mu\nu}$  to be given by  $\partial_\mu m_\nu - \partial_\nu m_\mu$ . For this choice of  $F_{\mu\nu}$ , we will denote the connection coefficients by  $\bar{\Gamma}_{\mu\nu}^\rho$ , that is,

$$\bar{\Gamma}_{\mu\nu}^\rho = \check{\Gamma}_{\mu\nu}^\rho - \frac{1}{2}h^{\rho\sigma}\left(\tau_\mu F_{\nu\sigma} + \tau_\nu F_{\mu\sigma}\right), \tag{3.27}$$

where  $F_{\mu\nu} = \partial_\mu m_\nu - \partial_\nu m_\mu$ . The associated covariant derivatives and curvatures will be denoted by barred quantities.

The total field content of NC geometry is given by the gauge potential  $m_\mu$ , as well as  $\tau_\mu$  and  $h_{\mu\nu}$ . These fields transform under the gauge transformations as

$$\delta\tau_\mu = 0, \quad \delta h_{\mu\nu} = \tau_\mu\lambda_\nu + \tau_\nu\lambda_\mu, \quad \delta m_\mu = \partial_\mu\sigma + \lambda_\mu, \quad (3.28)$$

where  $\nu^\mu\lambda_\mu = 0$  and where we left out the transformation under diffeomorphisms. The connection (3.14) is inert under these transformations. The symmetry of the connection implies that  $d\tau = 0$ , and thus, time is absolute.

### 3.6 NC gravity

So far, we have made no statements about dynamics. We now review how standard Newtonian gravity can be formulated in the framework of torsionless NC geometry. Test particles will follow the geodesics of the connection (3.14). In other words, they are described by

$$\ddot{x}^\mu + \check{\Gamma}_{\nu\rho}^\mu \dot{x}^\nu \dot{x}^\rho = \ddot{x}^\mu + \check{\Gamma}_{\nu\rho}^{\check{\mu}} \dot{x}^\nu \dot{x}^\rho - h^{\mu\sigma} \dot{x}^\rho F_{\rho\sigma} = 0, \quad (3.29)$$

where the dots denote derivatives with respect to  $\lambda$ , the affine parameter along the geodesic. We set  $\tau_\mu \dot{x}^\mu = 1$ , which implies that the  $\tau_\mu$  contraction of this equation is trivially satisfied. This geodesic equation follows from the action

$$S = m \int d\lambda \left[ \frac{h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{2\tau_\rho \dot{x}^\rho} - m_\mu \dot{x}^\mu \right]. \quad (3.30)$$

This action has worldline reparametrization invariance  $\delta\lambda = \xi(\lambda)$  and  $\delta x^\mu = \xi(\lambda)\dot{x}^\mu$ , which can be used to fix  $\tau_\mu \dot{x}^\mu = 1$ . The fact that  $m$  appears as an overall parameter in this particle action, so that inertial and gravitational masses are equal, manifests the equivalence principle. The time component of  $m_\mu$  is the Newtonian potential.

Following Trautman’s postulate 5, the equations corresponding to Newtonian gravity should take the following covariant form in terms of the NC geometry:

$$\bar{R}_{\mu\nu} = 8\pi G \frac{d-2}{d-1} \rho \tau_\mu \tau_\nu, \quad d\tau = 0, \quad (3.31)$$

where  $\bar{R}_{\mu\nu} = \bar{R}_{\mu\sigma\nu}{}^\sigma$  is the Ricci curvature associated with (3.27), where  $\rho$  is the mass density of a non-relativistic matter source and  $(d+1)$  is the total spacetime dimension. This corresponds to the Poisson equation of Newtonian gravity written in an arbitrary frame. Eq. 3.31 follows from an action principle obtained from the  $1/c^2$  expansion of the Einstein–Hilbert (EH) action coupled to the action of a massive point particle [10], as shown in Section 5.5.

Rewriting the equations for NC gravity (3.31), which are expressed in terms of the  $\bar{\Gamma}_{\mu\nu}^\rho$  connection to a set of equations that are expressed in terms of the  $\check{\Gamma}_{\mu\nu}^\rho$  connection, the equations for NC gravity can be written as (where we have contracted (3.31) with  $\nu^\mu$  and  $h^{\mu\nu}$ )

$$h^{\mu\rho} h^{\nu\sigma} \check{R}_{\rho\sigma} = 0, \quad (3.32)$$

$$h^{\mu\rho} \nu^\sigma \check{R}_{\rho\sigma} = -\frac{1}{2} e^{-1} \partial_\lambda (e h^{\mu\rho} h^{\lambda\sigma} F_{\rho\sigma}), \quad (3.33)$$

$$\nu^\rho \nu^\sigma \check{R}_{\rho\sigma} = -e^{-1} \partial_\rho (e \nu^\nu h^{\rho\sigma} F_{\nu\sigma}) - \frac{1}{4} h^{\mu\nu} h^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + 8\pi G \frac{d-2}{d-1} \rho. \quad (3.34)$$

In NC gravity, this should be supplemented with the condition  $d\tau = 0$ . The left-hand side is pure geometric data, and the right-hand side depends entirely on the “electric” and “magnetic” field strength components of  $F_{\mu\nu}$ . We will see in the following sections that the divergence of the electric field strength in the third equation, that is,  $e^{-1} \partial_\rho (e \nu^\nu h^{\rho\sigma} F_{\nu\sigma})$ , is what gives rise to Newton’s law of gravity. To arrive at Newtonian gravity, we need to somehow get rid of the magnetic field strength term,  $h^{\mu\nu} h^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}$  in Eq. 3.34. This is the rationale behind the first Trautman condition. However, it was later realized that this condition is not necessary as one can argue that the magnetic field strength has to be zero as a result of a boundary condition that states that  $h^{\mu\nu} h^{\rho\sigma} F_{\mu\rho}$  has to vanish at infinity. This latter condition follows from the  $1/c$  expansion of an asymptotically flat metric. Therefore, this is a more minimal approach to recovering Newtonian gravity.

### 3.7 Gauge fixing

In order to recover Newtonian gravity in its usual form, it is unavoidable to talk about gauge fixing the NC gauge symmetries (3.28) and diffeomorphisms. A covariant definition of a locally flat NC geometry could be to simply require  $\bar{R}_{\mu\nu\sigma}{}^\rho = 0$ . This condition is invariant under the Galilean boost and  $\sigma$ -gauge transformations of Eq. 3.28. However, in practice, only  $\tau_\mu$  and  $h_{\mu\nu}$  are often treated as geometric fields, whereas  $m_\mu$  is interpreted as a force field. This is not a covariant distinction and largely results from historical bias. For example, one can always use (3.28) to gauge away  $m_\mu$  entirely, as shown in [47]; see [48] for some examples.

A common gauge fixing that can always be made when  $d\tau = 0$  is as follows: first, we partially fix diffeomorphisms so that  $\tau = dt$ , where  $t$  is our time coordinate. In this class of coordinate systems, the boosts act as  $\delta h_{ti} = \lambda_i$ , where the latter is an arbitrary one-form in  $d$  spatial dimensions. We can thus completely fix the Galilean boost symmetry by demanding that  $h_{ti} = 0$  in these coordinates. We then find from  $\nu^\mu h_{\mu\nu} = 0$  and  $\nu^\mu \tau_\mu = -1$  that  $\nu^t = -1$  and  $h_{tt} = 0$  and that  $\nu^i = 0$ , where  $h_{ij}$  is invertible with inverse  $h^{ij}$ . Using  $\tau_\mu h^{\mu\nu} = 0$ , we find  $h^{tt} = 0 = h^{ti}$ . To summarize, we can always go to a gauge in which

$$\tau = dt, \quad h_{\mu\nu} dx^\mu dx^\nu = h_{ij} dx^i dx^j, \quad \nu = -\partial_t, \quad h^{\mu\nu} \partial_\mu \partial_\nu = h^{ij} \partial_i \partial_j, \quad (3.35)$$

and where  $m_\mu$  is completely arbitrary. This gauge choice still has residual gauge transformations acting on it. Demanding that the geometry on constant time slices is flat leads to the condition that  $h^{\alpha\mu} h^{\beta\nu} h^{\gamma\sigma} \check{R}_{\mu\nu\sigma}{}^\rho = 0$  (which is invariant under Galilean boosts). In our partially fixed gauge (3.35), this amounts to demanding that the Riemann tensor of the Riemannian metric  $h_{ij}$  is zero. We can fix diffeomorphisms further so that locally  $h_{ij} = a^2(t) \delta_{ij}$  for some function  $a(t)$ . This function plays an important role in Newtonian cosmology. We can perform a coordinate transformation of the form  $x'^i = a(t)x^i$  and  $t' = t$  followed by a finite Galilean boost so that in the primed



coordinate system  $h'_{\mu\nu} dx'^{\mu} dx'^{\nu} = \delta_{ij} dx'^i dx'^j$  and  $\tau = dt'$ . This transformation will affect the  $m_{\mu}$  connection, but the point here is to fix  $\tau_{\mu}$  and  $h_{\mu\nu}$  as much as possible. Thus, without loss of generality, we can set  $a(t) = 1$ . Hence, we will continue by working with

$$\tau = dt, \quad h_{\mu\nu} dx^{\mu} dx^{\nu} = \delta_{ij} dx^i dx^j, \quad v = -\partial_t, \\ h^{\mu\nu} \partial_{\mu} \partial_{\nu} = \delta^{ij} \partial_i \partial_j. \tag{3.36}$$

In this case, all the non-trivial information about the NC geometry is in  $m_{\mu}$ , which remains fully arbitrary at this point. In this gauge and with these choices, Eqs (3.32)–(3.34) reduce to

$$0 = \partial_i F_{ij}, \tag{3.37}$$

$$0 = \partial_i F_{ii} - \frac{1}{4} F_{ij} F_{ij} + 8\pi G \frac{d-2}{d-1} \rho. \tag{3.38}$$

The first of these two equations is solved by  $F_{ij} = \epsilon_{ijk} \partial_k F$ . The Bianchi identity  $\partial_{[i} F_{jk]} = 0$  (which follows from the Trautman condition) then tells us that  $F$  is harmonic on flat space (without any sources). By using Liouville’s theorem, this function must be constant to be regular everywhere (including infinity); hence, we conclude that  $F_{ij} = 0$ . We can gauge fix  $m_i = 0$ , using the freedom to transform  $m_{\mu}$  as  $\delta m_{\mu} = \partial_{\mu} \sigma$ . Now, the NC gravity equations simplify to the well-known Poisson equation:

$$\nabla^2 \Phi_N = 8\pi G \frac{d-2}{d-1} \rho, \tag{3.39}$$

where we defined  $m_t = \Phi_N$ , the Newtonian potential. Therefore, we see that the covariant Eq. 3.31 reproduces the usual form of Newtonian gravity in an appropriate coordinate system.

### 3.8 Large speed of light expansion of GR

We very briefly review how Trautman’s definition of NC gravity follows from the  $1/c$  expansion of GR. We will have much more to say about the  $1/c$  expansion in Section 6, so we will keep it brief. The following is essentially Dautcourt’s work [23, 24]. We will only consider even powers of  $1/c^2$  and assume analyticity in  $1/c^2$ . The metric expands as

$$g_{\mu\nu} = -c^2 \tau_{\mu} \tau_{\nu} + h_{\mu\nu} - \tau_{\mu} m_{\nu} - \tau_{\nu} m_{\mu} + \mathcal{O}(c^{-2}), \tag{3.40}$$

where  $h_{\mu\nu}$  has signature  $(0, 1, \dots, 1)$  and  $\tau_{\mu} \tau_{\nu} + h_{\mu\nu}$  is invertible. The Christoffel connection expands as

$$\Gamma^{\rho}_{\mu\nu} = \bar{\Gamma}^{\rho}_{\mu\nu} + \mathcal{O}(c^{-2}), \tag{3.41}$$

where  $\bar{\Gamma}^{\rho}_{\mu\nu}$  is given in Eq. 3.27, but only provided we set  $d\tau = 0$  by hand. If we also expand the diffeomorphisms parameter  $\Xi^{\mu} = \xi^{\mu} + c^{-2} \zeta^{\mu} + \mathcal{O}(c^{-4})$ , then  $m_{\mu}$  transforms as  $\delta m_{\mu} = \partial_{\mu} \sigma$  with  $\sigma = \tau_{\mu} \zeta^{\mu}$  under the sleading diffeomorphisms with parameter  $\zeta^{\mu}$ , again only provided that  $d\tau = 0$ . Finally, the combination  $h_{\mu\nu} - \tau_{\mu} m_{\nu} - \tau_{\nu} m_{\mu}$  is invariant under the Galilean boost transformation with parameter  $\lambda_{\mu}$  discussed previously. We recover the NC fields and their gauge properties as discussed previously but only under the assumption that  $d\tau = 0$ . The  $1/c$  expansion of matter will be discussed in later sections, but an important observation is that one can only obtain the NC geodesic Equation 3.29 from a  $1/c$  expansion if  $d\tau = 0$ . Then, the  $1/c$  expansion of the Einstein equations coupled

to the energy-momentum tensor of a massive point particle leads to (3.31). If  $g_{\mu\nu}$  is asymptotically flat, it follows that  $m_{\mu}$  (in the limit  $r \rightarrow \infty$ ) is at most pure gauge at spatial infinity. This is why, in (3.38),  $F_{ij}$  is non-singular at infinity.

## 4 Recent history: Revival and new developments

The last decade has seen a surge of interest in the topic of non-Lorentzian geometries, particularly NC geometry. To separate this new development from the older work reviewed previously, we treat [3] as the beginning of this new development. This work played an important role in later developments. It showed that for  $d\tau = 0$ , it is possible to view NC gravity as the dynamics of a geometry that can be obtained by gauging the Bargmann algebra subject to appropriate curvature constraints. We will review this approach further in the following paragraphs. This approach made conditions such as the Trautman condition (2.4) obsolete, as the latter now follows trivially from a Bianchi identity associated with the field strength that appears in the gauging procedure.

Another major step forward was [8], where it was realized that the  $1/c$  expansion of GR can be performed in full generality without imposing by hand (or *via* boundary conditions and assumptions about sources) the requirement that  $d\tau = 0$ . This led to the realization that a non-relativistic approximation can describe gravitational fields that are, for example, strong on the scale of the Schwarzschild radius.

Textbook non-relativistic approximations of GR often assume a weak field limit. In this case, GR time dilation effects are (in part) described by the Newtonian potential. In the case of a strong field non-relativistic regime, GR time dilation is (in part) described by an NC lapse function  $N$  such that  $\tau = NdT$  for scalars  $N$  and  $T$ . In other words, gravitational time dilation could be incorporated into the framework of NC geometry by allowing for  $d\tau \neq 0$ , which corresponds to non-zero torsion in the NC connection. Not all torsion is allowed, and at least on the shell, we still need to impose the requirement that  $\tau \wedge d\tau = 0$ , which guarantees that spacetime can be consistently decomposed into spatial submanifolds.

It seemed natural that to describe NC geometry with  $d\tau \neq 0$ , all one had to carry out was to extend the gauging methods of [3] to this more general case with non-zero torsion. However, it turned out that this was not the right thing to do. One cannot consistently couple NC geometry with local Bargmann symmetry to matter sources without turning on torsion, which is incompatible with the usual description of Newtonian gravity, as discussed in Section 3. Furthermore, the resulting torsion even violates the condition  $\tau \wedge d\tau = 0$ , as we demonstrate explicitly in Section 5.5. Instead, the relevant algebra is different, and its gauging leads to the correct extension of NC geometry when the torsion is non-zero. To distinguish this new framework of torsional NC geometry from the one obtained by gauging the Bargmann algebra, we refer to the latter as type I (gauging of Bargmann) and the former as type II TNC geometry ( $1/c^2$  expansion). These notions of geometry coincide if the torsion vanishes.

We now focus on reviewing type I TNC geometry, its relation to the gauging of the Bargmann algebra, and its gravitational action in Section 5. Once we have seen the incompatibility of the latter with the standard formulation of Newtonian gravity, we introduce type II TNC geometry in Section 6.

## 5 Type I TNC geometry

As is well known, one can obtain Lorentzian geometry through a gauging of the Poincaré algebra. After quickly reviewing this construction, we show how one can obtain type I TNC geometry from a gauging of the Bargmann algebra. We subsequently discuss how a  $(d + 1)$ -dimensional TNC geometry can be obtained from the following two constructions:

- A null reduction of a  $(d + 2)$ -dimensional Lorentzian geometry.
- A large speed of light limit of a  $(d + 1)$ -dimensional Lorentzian geometry with an electromagnetic background field.

Next, we show how the action equivalent of these constructions allows one to find the action of a non-relativistic particle probe and the dynamics of type I TNC spacetime itself. We then demonstrate that the resulting gravity actions for dynamical type I TNC geometry lead to non-zero torsion in the presence of mass sources. In this sense, type I TNC geometry is not appropriate to describe the zero-torsion limit corresponding to Newtonian gravity in the presence of matter. Finally, we show that the same arguments still apply if one only works on the shell by null-reducing Einstein’s equations.

### 5.1 The Bargmann algebra

As we will review momentarily, one can obtain Lorentzian geometry by applying a gauging procedure to the Poincaré algebra. Similarly, type I TNC geometry can be obtained from a gauging of the Bargmann algebra, which encodes its local symmetries. In this section, we first show how the Bargmann algebra can be obtained from a contraction and a null reduction of the Poincaré algebra. These two derivations will be mirrored at the level of the point particle and gravity actions later.

First, we consider an İnönü–Wigner contraction of the Poincaré algebra trivially extended with a  $U(1)$  generator that commutes with the entire Poincaré algebra, which we denote by  $Q$ . As shown in the following, this generator is associated with electromagnetic coupling. We work in  $(d + 1)$  spacetime dimensions and use the following conventions for the Poincaré algebra:

$$[M_{AB}, M_{CD}] = \eta_{AC}M_{BD} - \eta_{BC}M_{AD} + \eta_{BD}M_{AC} - \eta_{AD}M_{BC}, \quad (5.1a)$$

$$[M_{AB}, P_C] = \eta_{AC}P_B - \eta_{BC}P_A. \quad (5.1b)$$

Here,  $P_A$  are translation generators and  $M_{AB} = -M_{BA}$  are the Lorentz and rotation generators. Now, we consider a spacetime split of the  $(d + 1)$ -dimensional algebra indices  $A = (0, a)$ , where  $a = 1, \dots, d$  is a spatial index, and we define the generators

$$H = cP_0 + Q, \quad N = \frac{1}{c}P_0, \quad G_a = \frac{1}{c}M_{0a}, \quad J_{ab} = M_{ab}. \quad (5.2)$$

So far, this is just a change of basis. However, if we now take the limit  $c \rightarrow \infty$ , we end up with an inequivalent algebra whose non-zero commutation relations are

$$[J_{ab}, J_{cd}] = \delta_{ac}J_{bd} - \delta_{bc}J_{ad} + \delta_{bd}J_{ac} - \delta_{ad}J_{bc}, \quad (5.3a)$$

$$[J_{ab}, P_c] = \delta_{ac}P_b - \delta_{bc}P_a, \quad [J_{ab}, G_c] = \delta_{ac}G_b - \delta_{bc}G_a, \quad (5.3b)$$

$$[G_a, H] = -P_a, \quad [G_a, P_b] = -\delta_{ab}N. \quad (5.3c)$$

This is the Bargmann algebra. For  $N = 0$ , we recover the Galilei algebra, which contains time translations  $H$ , spatial translations  $P_a$ , and rotations  $J_{ab}$ , as well as the Galilean boosts  $G_a$ . For  $N \neq 0$ , we obtain the centrally extended Galilei algebra known as the Bargmann algebra. The central extension  $N$ , as shown later, corresponds to the gauge potential  $m_\mu$  that was introduced previously.

Another way to obtain the algebra (5.3) is via a null reduction of the Poincaré algebra in  $(d + 2)$  dimensions [43]. If we consider all generators of this algebra that commute with the null translation generator  $N = P_+ = (P_0 + P_{d+1})/\sqrt{2}$ , we get

$$\begin{aligned} P_a, \quad H = P_- = (P_0 - P_{d+1})/\sqrt{2}, \\ J_{ab} = M_{ab}, \quad G_a = M_{+a} = (M_{0a} + M_{(d+1)a})/\sqrt{2}, \end{aligned} \quad (5.4)$$

which forms a subalgebra corresponding to the Bargmann algebra (5.3a). Note that the higher-dimensional Lorentz boosts  $M_{-a}$  and  $M_{+-}$  do not commute with  $H$  and therefore do not enter this subalgebra.

### 5.2 Type I TNC geometry

As was first realized in [3], type I TNC geometry can be obtained by gauging the Bargmann algebra (5.3b). This section reviews this gauging construction, including its generalization to non-zero torsion [49].<sup>11</sup> Before that, we briefly review the well-known procedure for obtaining Lorentzian geometry from a gauging of the Poincaré algebra. Finally, we also show how type I TNC geometry can be obtained from a null reduction of higher-dimensional Lorentzian geometry.

#### 5.2.1 Gauging Poincaré

The gauging procedure starts from a connection valued in the algebra in question,<sup>12</sup>

$$\mathcal{A}_\mu = E_\mu^A P_A + \frac{1}{2}\Omega_\mu^{AB} M_{AB}, \quad (5.5)$$

where  $A, B = 0, 1, \dots, d$  and whose coefficients  $E_\mu^A$  and  $\Omega_\mu^{AB}$  can be interpreted as frame fields or vielbein and a “spin” connection for the frames. Indices are raised and lowered using the Minkowski metric  $\eta_{AB}$  on the frame bundle. The curvature of the total connection  $\mathcal{A}$  is then

<sup>11</sup> Type I Newton–Cartan geometry with torsion can also be derived from the Noether procedure [132].

<sup>12</sup> See also the recent reviews in [67] for more background on non-Lorentzian geometries, in general, and [133] for a more mathematical perspective on the gauging procedure.

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \\ &= R(P)_{\mu\nu}{}^A P_A + \frac{1}{2} R(M)_{\mu\nu}{}^{AB} M_{AB}, \end{aligned} \tag{5.6}$$

whose components correspond to the torsion and curvature of the frame connection. The connection  $\mathcal{A}_\mu$  transforms in the adjoint of the Poincaré algebra:

$$\delta \mathcal{A}_\mu = \partial_\mu \Lambda + [\mathcal{A}_\mu, \Lambda], \tag{5.7}$$

where the transformation parameter  $\Lambda$  can likewise be decomposed in  $P_A$  and  $M_{AB}$  components. However, given a vector field  $\xi^\mu$  on the base manifold, it is useful to parametrize  $\Lambda$  using

$$\Lambda = \xi^\mu \mathcal{A}_\mu + \Sigma, \quad \Sigma = \frac{1}{2} \Lambda^{AB} M_{AB}, \tag{5.8}$$

which can be carried out without loss of generality. We can then define a distinct transformation  $\bar{\delta}$  acting on  $\mathcal{A}_\mu$  as follows:

$$\bar{\delta} \mathcal{A}_\mu = \delta \mathcal{A}_\mu - \xi^\nu \mathcal{F}_{\mu\nu} = \mathcal{L}_\xi \mathcal{A}_\mu + \partial_\mu \Sigma + [\mathcal{A}_\mu, \Sigma]. \tag{5.9}$$

At this point, we recover the Lie derivative  $\mathcal{L}_\xi$  along  $\xi^\mu$ , and we have effectively exchanged the gauge transformations along  $P_A$  for diffeomorphisms.<sup>13</sup>

Studies on gauging spacetime symmetry groups often suggested that diffeomorphisms can only be obtained once specific curvature constraints are imposed.<sup>14</sup> We emphasize that the transformation  $\bar{\delta} \mathcal{A}_\mu$  can be considered for any value of the total curvature  $\mathcal{F}_{\mu\nu}$ , including non-zero torsion  $R(P)_{\mu\nu}{}^A$ . Although this extension is often not of immediate interest in Lorentzian geometry, it is crucial in non-Lorentzian geometry.

Expanding (5.9) in components, we obtain the transformation rules.

$$\bar{\delta} E_\mu{}^A = \mathcal{L}_\xi E_\mu{}^A + \Lambda^A{}_B E_\mu{}^B, \tag{5.10a}$$

$$\bar{\delta} \Omega_\mu{}^A{}_B = \mathcal{L}_\xi \Omega_\mu{}^A{}_B + \partial_\mu \Lambda^A{}_B + \Lambda^A{}_C \Omega_\mu{}^C{}_B - \Lambda^C{}_B \Omega_\mu{}^A{}_C, \tag{5.10b}$$

corresponding to diffeomorphisms and local Lorentz transformations. Then, we can introduce a set of inverse vielbeins  $E^\mu{}_A$  such that

$$E_\mu{}^A E^\mu{}_B = \delta^A{}_B, \quad E_\mu{}^A E^\nu{}_A = \delta^\nu_\mu. \tag{5.11}$$

Using these properties, the vielbein transformations (5.10a) imply

$$\bar{\delta} E^\mu{}_A = \mathcal{L}_\xi E^\mu{}_A - \Lambda^B{}_A E^\mu{}_B. \tag{5.12}$$

Then, we can construct the Lorentzian metric and its inverse:

$$g_{\mu\nu} = \eta_{AB} E_\mu{}^A E_\nu{}^B, \quad g^{\mu\nu} = \eta^{AB} E^\mu{}_A E^\nu{}_B, \tag{5.13}$$

which are invariant under local Lorentz transformations.

Finally, we can translate between the connection  $\Omega_\mu{}^A{}_B$  in the frame bundle and the affine connection  $\Gamma_{\mu\nu}^\rho$  using the vielbein postulate:

$$\partial_\mu E_\nu{}^A + \Omega_\mu{}^A{}_B E_\nu{}^B - \Gamma_{\mu\nu}^\rho E_\rho{}^A = 0. \tag{5.14}$$

As  $\Omega_\mu{}^{AB}$  is antisymmetric in its frame indices  $AB$  due to its definition in (5.5), the covariant derivative  $\nabla_\mu$  of the corresponding affine connection is always compatible with the Lorentzian metric (5.13). However, since the torsion  $2\Gamma_{[\mu\nu]}^\rho = E^{\rho}{}_{AR}(P)_{\mu\nu}{}^A$  is not necessarily zero, this connection is not necessarily equal to the usual Levi-Civita connection. In the following, we will mainly work directly with affine connections instead of the frame bundle connections. However, for the metric variables, the gauging procedure outlined previously is a useful method for obtaining the transformations of geometric quantities under local symmetries, even in the presence of torsion.

### 5.2.2 Gauging Bargmann

Now, we repeat the gauging procedure for the Bargmann algebra (5.3c) and show that this leads to type I TNC geometry with torsion. Our starting point is now

$$\mathcal{A}_\mu = H\tau_\mu + e_\mu{}^a P_a + G_a \Omega_\mu{}^a + \frac{1}{2} J_{ab} \Omega_\mu{}^{ab} + N m_\mu, \tag{5.15}$$

where we use  $\delta_{ab}$  to raise and lower spatial indices and  $\tau_\mu$  is the clock one-form.  $e_\mu{}^a$  are spatial vielbein  $\Omega_\mu{}^a$ ,  $\Omega_\mu{}^a{}_b$  are frame connections, and  $m_\mu$  is the Bargmann gauge potential associated with the central element  $N$ . Following (5.8), we parametrize the total gauge parameter as

$$\Lambda = \xi^\mu \mathcal{A}_\mu + \Sigma, \quad \Sigma = G_a \lambda^a + \frac{1}{2} J_{ab} \lambda^{ab} + N \sigma. \tag{5.16}$$

Then, the  $\bar{\delta}$  transformations defined in (5.9) lead to

$$\bar{\delta} \tau_\mu = \mathcal{L}_\xi \tau_\mu, \tag{5.17a}$$

$$\bar{\delta} e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a + \lambda^a{}_b e_\mu{}^b + \lambda^a \tau_\mu, \tag{5.17b}$$

$$\bar{\delta} \Omega_\mu{}^a = \mathcal{L}_\xi \Omega_\mu{}^a + \partial_\mu \lambda^a + \lambda^a{}_b \Omega_\mu{}^b + \lambda^b \Omega_{\mu b}{}^a, \tag{5.17c}$$

$$\bar{\delta} \Omega_\mu{}^{ab} = \mathcal{L}_\xi \Omega_\mu{}^{ab} + \partial_\mu \lambda^{ab} + 2\lambda_c^{[a} \Omega_\mu{}^{c|b]}, \tag{5.17d}$$

$$\bar{\delta} m_\mu = \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + e_\mu{}^a \lambda_a, \tag{5.17e}$$

where  $\lambda^a$ ,  $\lambda^{ab}$ , and  $\sigma$  are the parameters of the local Galilean boosts, local rotations, and local  $U(1)$  Bargmann transformations, respectively. Likewise, following (5.11), we can define a set of inverse vielbeins  $(\tau^\mu, e_\mu{}^a)$  that satisfy

$$v^\mu \tau_\mu = -1, \quad v^\mu e_\mu{}^a = 0, \quad e_\mu{}^a \tau_\mu = 0, \quad e_\mu{}^a e_\mu{}^b = \delta_a^b. \tag{5.18}$$

Using 5.17a and 5.17b, their transformations are

$$\bar{\delta} v^\mu = \mathcal{L}_\xi v^\mu - \lambda^a e_\mu{}^a, \quad \bar{\delta} e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a - \lambda^b e_\mu{}^b. \tag{5.19}$$

Accordingly, we can define the rotation-invariant spatial tensors as

$$h_{\mu\nu} = \delta_{ab} e_\mu{}^a e_\nu{}^b, \quad h^{\mu\nu} = \delta^{ab} e_\mu{}^a e_\nu{}^b, \tag{5.20}$$

which satisfy the orthonormality relations:

$$v^\mu \tau_\mu = -1, \quad v^\mu h_{\mu\nu} = 0, \quad h^{\mu\nu} \tau_\nu = 0, \quad h^{\mu\rho} h_{\rho\nu} = \delta^\mu_\nu + v^\mu \tau_\nu. \tag{5.21}$$

13 Strictly speaking, once we work with the  $\bar{\delta}$  transformations, we are no longer gauging the Poincaré algebra, but we are passing to the Cartan geometry modeled on the Klein pair consisting of the Poincaré algebra and the Lorentz algebra generated by  $M_{AB}$ . See [133] for more details.

14 This is because setting to zero some of the curvatures in  $\mathcal{F}_{\mu\nu}$  identifies  $\bar{\delta}$  with  $\delta$  in (5.9) for those fields that are not fixed by the curvature constraints. There is no need for the  $\delta$  and  $\bar{\delta}$  transformations to coincide.

Hence, we obtain the full field content of type I TNC geometry, which consists of a timelike one-form  $\tau_\mu$ , a spatial symmetric tensor  $h_{\mu\nu}$  of signature  $(0, 1, \dots, 1)$ , and a  $U(1)$  gauge field  $m_\mu$  associated with the central Bargmann mass generator  $N$ . All these objects are spacetime tensors, as they transform by a Lie derivative under diffeomorphisms  $\xi^\mu$ . In addition, they transform as

$$\bar{\delta}\tau_\mu = 0, \quad \bar{\delta}h_{\mu\nu} = \lambda_\mu\tau_\nu + \lambda_\nu\tau_\mu, \quad \bar{\delta}m_\mu = \partial_\mu\sigma + \lambda_\mu, \quad (5.22)$$

under local Galilean boosts  $\lambda_\mu = e_\mu^a\lambda^a$  and  $U(1)$  gauge transformations  $\sigma$ . Thus, we recover the transformations (3.28) given before. Similarly, the transformations of the inverse timelike vielbein and spatial co-metric are

$$\bar{\delta}v^\mu = h^{\mu\nu}\lambda_\nu, \quad \bar{\delta}h^{\mu\nu} = 0. \quad (5.23)$$

Hence, both  $\tau_\mu$  and  $h^{\mu\nu}$  are invariant under Galilean boost. Additionally, using the  $U(1)$  gauge field  $m_\mu$ , we can construct the following boost-invariant combinations:

$$\hat{v}^\mu = v^\mu - h^{\mu\nu}m_\nu, \quad (5.24a)$$

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tau_\mu m_\nu - \tau_\nu m_\mu, \quad (5.24b)$$

$$\hat{\Phi} = -v^\mu m_\mu + \frac{1}{2}h^{\mu\nu}m_\mu m_\nu. \quad (5.24c)$$

Given an appropriate gauge choice, we will see later that the first term in  $\hat{\Phi}$  essentially plays the role of the Newtonian gravitational potential.

### 5.2.3 Affine connection, torsion, and curvature

Given the metric variables for type I TNC geometry and their transformations obtained from the aforementioned gauging procedure, we would like to construct the closest analog of the Levi-Civita connection for NC geometry.<sup>15</sup> Our starting point is the TNC analog of metric compatibility:

$$\nabla_\mu\tau_\nu = 0, \quad \nabla_\mu h^{\nu\rho} = 0. \quad (5.25)$$

These conditions do not uniquely specify the connection in terms of geometry. Following Eq. 3.5, the general solution takes the form

$$\Gamma_{\mu\nu}^\rho = \check{\Gamma}_{\mu\nu}^\rho + C_{\mu\nu}^\rho, \quad (5.26)$$

$$\check{\Gamma}_{\mu\nu}^\rho = -v^\rho\partial_\mu\tau_\nu + \frac{1}{2}h^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}), \quad (5.27)$$

$$C_{\mu\nu}^\rho = \frac{1}{2}h^{\rho\sigma}(\tau_\mu K_{\sigma\nu} + \tau_\nu K_{\sigma\mu} + L_{\sigma\mu\nu}). \quad (5.28)$$

where  $K_{\mu\nu}$  and  $L_{\sigma\mu\nu}$  satisfy additional constraints (see [49]).

First, we no longer require the connection to be torsionless. As a result,  $d\tau$  is no longer necessarily zero, as we can see, for example, from the torsion of the connection  $\check{\Gamma}_{\mu\nu}^\rho$  in (5.27), that

$$2\check{\Gamma}_{[\mu\nu]}^\rho = -v^\rho(\partial_\mu\tau_\nu - \partial_\nu\tau_\mu). \quad (5.29)$$

Following [4, 5, 50], we distinguish three cases:

- (1) Zero torsion  $\partial_{[\mu}\tau_{\nu]} = 0$  corresponds to “regular” NC geometry, which is the case considered in Section 3.
- (2) “Twistless” torsion is defined by

$$\tau_{[\mu}\partial_\nu\tau_{\rho]} = 0 \iff h^{\mu\rho}h^{\nu\sigma}(\partial_\rho\tau_\sigma - \partial_\sigma\tau_\rho) = 0, \quad (5.30)$$

which implies that  $\tau_\mu$  can be used to define spatial hypersurfaces of the co-dimension one, where  $h_{\mu\nu}$  pulls back to a non-degenerate Riemannian metric. The corresponding geometry is known as twistless torsional Newton-Cartan (TTNC) geometry, which will be our main focus.

- (3) No constraint exists on  $\partial_{[\mu}\tau_{\nu]}$  and general TNC geometry.

The notion of TTNC geometry goes back to [45]. However, Julia and Nicolai performed a conformal rescaling to create a frame of no torsion. The benefit of adding torsion to formalism, including the general TNC case, was first considered in [4, 5]. One can also obtain TTNC geometries by gauging the Schrödinger algebra [51, 52].

In the case of TTNC geometry, we have the following useful identities:

$$h^{\mu\rho}h^{\nu\sigma}(\partial_\rho a_\sigma - \partial_\sigma a_\rho) = h^{\mu\rho}h^{\nu\sigma}(\nabla_\rho a_\sigma - \nabla_\sigma a_\rho) = 0, \quad (5.31)$$

$$\partial_\mu\tau_\nu - \partial_\nu\tau_\mu = a_\mu\tau_\nu - a_\nu\tau_\mu, \quad (5.32)$$

in terms of the “acceleration” vector  $a_\mu = \mathcal{L}_v\tau_\mu$  of the foliation. The second identity tells us that  $h^{\mu\nu}a_\mu$  describes the TTNC torsion. That is why the latter is sometimes known as the torsion vector. Equation 5.31 shows that the twist tensor vanishes. That is why we refer to the geometry as twistless torsional NC geometry.

Additionally, recall that the connection  $\check{\Gamma}_{\mu\nu}^\rho$  is not invariant under Galilean boosts. Using the Bargmann  $U(1)$  field  $m_\mu$ , we can choose  $K_{\mu\nu}$  and  $L_{\sigma\mu\nu}$  such that the affine connection is invariant under Galilean boosts:  $\delta_G\check{\Gamma}_{\mu\nu}^\rho = 0$ ,

$$K_{\mu\nu} = \partial_\mu m_\nu - \partial_\nu m_\mu, \quad (5.33)$$

$$L_{\sigma\mu\nu} = m_\sigma(\partial_\mu\tau_\nu - \partial_\nu\tau_\mu) - m_\mu(\partial_\nu\tau_\sigma - \partial_\sigma\tau_\nu) - m_\nu(\partial_\mu\tau_\sigma - \partial_\sigma\tau_\mu). \quad (5.34)$$

The resulting connection takes the form

$$\bar{\Gamma}_{\mu\nu}^\rho = -v^\rho\partial_\mu\tau_\nu + \frac{1}{2}h^{\rho\sigma}(\partial_\mu\bar{h}_{\nu\sigma} + \partial_\nu\bar{h}_{\mu\sigma} - \partial_\sigma\bar{h}_{\mu\nu}). \quad (5.35)$$

which is the generalization of connection (3.27) to non-zero torsion. This is not the unique boost-invariant connection, but it is one of the more natural choices and is commonly used in the literature. However, connection (5.35) is no longer invariant under  $U(1)$  transformations in the presence of torsion.<sup>16</sup> With torsion, one can show that it is no longer possible to build a connection that is invariant under  $U(1)$  transformations and Galilean boosts.

<sup>15</sup> We refer the reader to Sections 2–7 in [45] for further details relevant to this subsection.

<sup>16</sup> As before, the connection (5.35) is not invariant under all local symmetries, so it is strictly speaking not an affine connection in the regular sense. In particular, not all of its curvature tensors will be invariant under local symmetries. However, this potential problem is circumvented by explicitly checking that all actions we consider are invariant under non-manifest gauge symmetries. Often, this is guaranteed by their origin as a limit or reduction of a Lorentz-invariant action.

Finally, following [Supplementary Appendix SA](#), we can define the Riemann curvature tensor associated with, for example, the TNC connection (5.27) that is not invariant under boosts (but is invariant under  $U(1)$  gauge transformations) or the TNC connection (5.35), which is boost-invariant (but not invariant under  $U(1)$  gauge transformations) through

$$\check{R}_{\mu\nu\sigma}{}^\rho = -\partial_\mu \check{\Gamma}_{\nu\sigma}^\rho + \partial_\nu \check{\Gamma}_{\mu\sigma}^\rho - \check{\Gamma}_{\mu\lambda}^\rho \check{\Gamma}_{\nu\sigma}^\lambda + \check{\Gamma}_{\nu\lambda}^\rho \check{\Gamma}_{\mu\sigma}^\lambda, \tag{5.36}$$

$$\bar{R}_{\mu\nu\sigma}{}^\rho = -\partial_\mu \bar{\Gamma}_{\nu\sigma}^\rho + \partial_\nu \bar{\Gamma}_{\mu\sigma}^\rho - \bar{\Gamma}_{\mu\lambda}^\rho \bar{\Gamma}_{\nu\sigma}^\lambda + \bar{\Gamma}_{\nu\lambda}^\rho \bar{\Gamma}_{\mu\sigma}^\lambda, \tag{5.37}$$

from which one can compute the Ricci tensor and curvature scalars in the usual way.

### 5.2.4 Type I TNC geometry from null reduction

Another way to obtain  $(d + 1)$ -dimensional type I TNC geometry is through null reduction of a  $(d + 2)$ -dimensional Lorentzian geometry (see also [Section 2.3](#)). Choosing coordinates  $(u, x^\mu)$  such that the null isometry is generated by  $\partial_u$ , we can parametrize such a metric as

$$ds^2 = g_{MN} dx^M dx^N = 2\tau_\mu (du - m_\mu dx^\mu) + h_{\mu\nu} dx^\mu dx^\nu, \tag{5.38}$$

where none of the metric components depend on  $u$ .  $x^\mu$  are  $(d + 1)$ -dimensional coordinates. The splitting of the  $g_{\mu\nu}$  components into  $(\tau_\mu, h_{\mu\nu}, m_\mu)$  is ambiguous, which is the origin of the local Galilean boost transformation of  $h_{\mu\nu}$  and  $m_\mu$  in (5.22). Alternatively, the metric and its inverse can be naturally decomposed in terms of the boost-invariant quantities defined in (5.20), (5.24a), (5.24b), and (5.24c):

$$g_{MN} = \begin{pmatrix} 0 & \tau_\nu \\ \tau_\mu & \bar{h}_{\mu\nu} \end{pmatrix}, \quad g^{MN} = \begin{pmatrix} 2\hat{\Phi} & -\hat{v}^\nu \\ -\hat{v}^\nu & \bar{h}^{\mu\nu} \end{pmatrix}. \tag{5.39}$$

Furthermore, as there is no restriction on  $\tau_\mu$ , the resulting geometry will generically be torsionful. The fact that the null reduction of a Lorentzian geometry leads to type I TNC geometry with local Bargmann symmetries can be understood because the Bargmann algebra can be obtained from a null reduction of the Poincaré algebra, as mentioned in [Eq. 5.4](#).

## 5.3 Particle action with type I TNC background

We have obtained type I TNC geometry from a gauging of the Bargmann algebra and from null reduction. We will then consider a third way of obtaining it, from a limit procedure. In the process, we will construct the TNC analog of the Lorentzian point particle action:

$$S = -mc \int d\lambda \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} + q \int A_\mu \dot{x}^\mu. \tag{5.40}$$

First, we will consider the analog of the İnönü–Wigner-type contraction (5.2) using a background Maxwell potential, following [53–55]. Then, we will obtain the same TNC action from a null reduction of a massless particle coupled to a Lorentzian background with a null isometry.

### 5.3.1 From a contraction with Maxwell background

In terms of the Poincaré algebra trivially extended with a  $U(1)$  generator  $Q$ , connection (5.5) becomes

$$\begin{aligned} A_\mu &= E_\mu{}^A P_A + \frac{1}{2} \Omega_\mu{}^{AB} M_{AB} + A_\mu Q \\ &= \tau_\mu H + e_\mu{}^a P_a + \Omega_\mu{}^a G_a + \frac{1}{2} \Omega_\mu{}^{ab} J_{ab} + m_\mu N, \end{aligned} \tag{5.41}$$

where we used redefinition (5.2) to obtain the second line and where we have defined

$$E_\mu{}^0 = c\tau_\mu + \frac{1}{c}m_\mu, \quad E_\mu{}^a = e_\mu{}^a, \quad A_\mu = \tau_\mu. \tag{5.42}$$

The second line of (5.41) corresponds to the Bargmann connection (5.15) upon taking the  $c \rightarrow \infty$  contraction.

Recalling that  $g_{\mu\nu} = -E_\mu{}^0 E_\nu{}^0 + \delta_{ab} E_\mu{}^a E_\nu{}^b$  and subsequently applying the aforementioned parametrization to the Lorentzian action (5.40), we get

$$S = (q - mc^2) \int d\lambda \tau_\mu \dot{x}^\mu + \frac{m}{2} \int d\lambda \frac{\bar{h}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{\tau_\rho \dot{x}^\rho} + \mathcal{O}(1/c^2), \tag{5.43}$$

where  $\bar{h}_{\mu\nu}$  is defined in (5.24b), and we recall that  $h_{\mu\nu} = \delta_{ab} e_\mu{}^a e_\nu{}^b$ . We can cancel the leading-order term by setting  $q = mc^2$ , corresponding to an extremal charge. In the limit  $c \rightarrow \infty$ , the remaining action describes the coupling of a point particle of mass  $m$  to type I TNC geometry. By construction, it is invariant under the Bargmann transformations (5.17a), as can also be checked explicitly. Additionally, it agrees with the action given in (3.30).

### 5.3.2 From null reduction

Finally, we can obtain the same action from a null reduction of the Lorentzian action for a massless particle without the Maxwell coupling. By using the parametrization in (5.38) for a  $(d + 2)$ -dimensional Lorentzian metric  $g_{MN}$  with a null isometry, the action for a massless particle is given by

$$S = \int \frac{1}{2e} g_{MN} \dot{X}^M \dot{X}^N d\lambda = \int \left( \frac{1}{e} \dot{u} \tau_\mu \dot{X}^\mu + \frac{1}{2e} \bar{h}_{\mu\nu} \dot{X}^\mu \dot{X}^\nu \right) d\lambda. \tag{5.44}$$

The momentum associated with the null direction  $p_u = \partial\mathcal{L}/\partial\dot{u} = \tau_\mu \dot{X}^\mu/e$  is conserved due to the isometry, which allows us to solve for the worldline “einbein”  $e = \tau_\mu \dot{X}^\mu/p_u$ . After setting  $p_u = m$ , action (5.44) reproduces

$$S = \frac{m}{2} \int \frac{\bar{h}_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}{\tau_\rho \dot{X}^\rho} d\lambda, \tag{5.45}$$

which is the same action we obtained from (5.43) after canceling the leading-order term.

## 5.4 Gravity action for type I TNC geometry

Similar to the point particle action, we can construct a gravitational action for dynamical type I TNC geometry in two ways. First, we will consider an İnönü–Wigner-type contraction of Einstein gravity coupled to a Maxwell action. Then, we will perform the null reduction of the EH action on a Lorentzian background with a null isometry, which also gives an action that is invariant under Bargmann symmetries. Afterward, we show that the two actions obtained in this way are, in fact, equal. As far as we know, this relation has not been

explicitly identified yet in the literature in this form, although a similar discussion appears in frame language in [56].

### 5.4.1 From a contraction of Einstein–Maxwell

In our first approach to constructing an action for dynamical type I TNC geometry, we start from the Einstein–Maxwell gravity, whose Lagrangian is<sup>17</sup>

$$\mathcal{L}_{\text{EM}} = \frac{c^3}{16\pi G} \sqrt{-g} R - \frac{1}{4c k^2} \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}. \quad (5.46)$$

As we will explain in more detail in Section 6.4, the Lorentzian metric  $g_{\mu\nu}$  and the Levi–Civita Ricci scalar  $R$  can be covariantly expanded in powers of  $c^2$ . In particular, we will see that the leading-order term in the expansion of  $R$  is not related to the curvature of an NC connection, but instead, it depends on

$$\tau_{\mu\nu} = 2\partial_{[\mu} \tau_{\nu]}, \quad (5.47)$$

which parametrizes the torsion of such a connection. Specifically, we get

$$\mathcal{L}_{\text{EM}} = e h^{\mu\rho} h^{\nu\sigma} \left( \frac{c^6}{64\pi G} \tau_{\mu\nu} \tau_{\rho\sigma} - \frac{1}{4k^2} F_{\mu\nu} F_{\rho\sigma} \right) + \mathcal{O}(c^4), \quad (5.48)$$

where  $e = \det(\tau_\mu, e_\mu^a)$  is the NC vielbein determinant. In analogy with the expansion of the point particle action in (5.43), we see that we can cancel this leading-order term by setting

$$\frac{1}{k^2} = \frac{c^6}{16\pi G}, \quad A_\mu = \tau_\mu. \quad (5.49)$$

Once the leading-order terms are canceled, the  $c \rightarrow \infty$  limit yields

$$\mathcal{L} = \frac{e}{16\pi G} \left( h^{\mu\nu} \check{R}_{\mu\nu} + \frac{1}{2} h^{\mu\nu} a_\mu a_\nu + \frac{1}{2} h^{\mu\rho} h^{\nu\sigma} \tau_{\mu\nu} m_{\rho\sigma} \right), \quad (5.50)$$

where we have defined

$$m_{\mu\nu} = 2\partial_{[\mu} m_{\nu]}, \quad (5.51)$$

and where  $\check{R}_{\mu\nu}$  is the Ricci tensor associated with the connection  $\check{\Gamma}_{\mu\nu}^\rho$  in (5.27), and we have rescaled  $G \rightarrow Gc^4$ .

### 5.4.2 From the null reduction of EH

Conversely, we can obtain an action for type I TNC gravity using a null reduction of the EH action. Rewriting the  $(d + 2)$ -dimensional Levi–Civita connection denoted by  $\hat{\Gamma}_{MN}^R$  using decomposition (5.39) of the metric, we find  $\hat{\Gamma}_{uu}^u = \hat{\Gamma}_{uu}^\rho = 0$  and

$$\hat{\Gamma}_{\mu u}^\mu = \frac{1}{2} \hat{a}_\mu, \quad (5.52a)$$

$$\hat{\Gamma}_{\mu\nu}^\mu = -2\tau_{(\mu} \partial_{\nu)} \hat{\Phi} - \bar{K}_{\mu\nu}, \quad (5.52b)$$

$$\hat{\Gamma}_{\mu\nu}^\rho = \frac{1}{2} \tau_{\mu\lambda} h^{\lambda\rho}, \quad (5.52c)$$

$$\hat{\Gamma}_{\mu\nu}^\rho = -\hat{v}^\rho \partial_{(\mu} \tau_{\nu)} + \frac{1}{2} h^{\rho\lambda} \left[ \partial_\mu \bar{h}_{\lambda\nu} + \partial_\nu \bar{h}_{\lambda\mu} - \partial_\lambda \bar{h}_{\mu\nu} \right] = \bar{\Gamma}_{(\mu\nu)}^\rho. \quad (5.52d)$$

Here, the boost-invariant objects  $\hat{a}_\mu$  and  $\bar{K}_{\mu\nu}$  are

$$\hat{a}_\mu = \mathcal{L}_{\hat{v}} \tau_\mu = \hat{v}^\rho \tau_{\rho\mu}, \quad \bar{K}_{\mu\nu} = -\frac{1}{2} \mathcal{L}_{\hat{v}} \bar{h}_{\mu\nu}, \quad (5.53)$$

and  $\bar{\Gamma}_{\mu\nu}^\rho$  is the boost-invariant connection defined in (5.35). Similarly, we can decompose the Ricci tensor  $\hat{R}_{MN}$  into lower-dimensional components. For example, we get

$$\hat{R}_{uu} = \frac{1}{4} h^{\mu\rho} h^{\nu\sigma} \tau_{\mu\nu} \tau_{\rho\sigma}, \quad (5.54)$$

which we will use later. Using the decomposition of the  $(d + 2)$ -dimensional Ricci scalar, the EH action becomes

$$\begin{aligned} \mathcal{L} &= \frac{\hat{E}}{16\pi G} g^{MN} R_{MN} \\ &= \frac{e}{16\pi G} \left( h^{\mu\nu} \bar{R}_{\mu\nu} + \frac{1}{2} h^{\mu\nu} \hat{a}_\mu \hat{a}_\nu - \frac{1}{2} \hat{\Phi} h^{\mu\rho} h^{\nu\sigma} \tau_{\mu\rho} \tau_{\nu\sigma} \right). \end{aligned} \quad (5.55)$$

As all of its components are independent of  $u$ , we can interpret this as an action for  $(d + 1)$ -dimensional type I TNC geometry. Indeed, one can check that it is also invariant under Bargmann transformations.

This action is not the same as the action (5.50) we obtained from a limit. However, as in the case of the point particle, the two actions are, in fact, equal. To see this, one can use relation (3.20) to rewrite

$$\begin{aligned} h^{\mu\nu} \check{R}_{\mu\nu} &= h^{\mu\nu} \bar{R}_{\mu\nu} - \frac{1}{2} \hat{\Phi} h^{\mu\rho} h^{\nu\sigma} \tau_{\mu\nu} \tau_{\rho\sigma} - \frac{1}{2} m_\mu h^{\mu\nu} \tau_{\nu\rho} h^{\rho\sigma} \tau_{\sigma\alpha} h^{\alpha\beta} m_\beta, \\ &\quad - \frac{1}{2} h^{\mu\rho} h^{\nu\sigma} \tau_{\mu\nu} m_{\rho\sigma} + h^{\mu\nu} a^\rho \tau_{\rho\mu} m_\nu, \end{aligned} \quad (5.56a)$$

$$\frac{1}{2} h^{\mu\nu} a_\mu a_\nu = \frac{1}{2} h^{\mu\nu} \hat{a}_\mu \hat{a}_\nu - h^{\mu\nu} a^\rho \tau_{\rho\mu} m_\nu + \frac{1}{2} m_\mu h^{\mu\nu} \tau_{\nu\rho} h^{\rho\sigma} \tau_{\sigma\alpha} h^{\alpha\beta} m_\beta, \quad (5.56b)$$

which means

$$\begin{aligned} h^{\mu\nu} \check{R}_{\mu\nu} + \frac{1}{2} h^{\mu\nu} a_\mu a_\nu + \frac{1}{2} h^{\mu\rho} h^{\nu\sigma} \tau_{\mu\nu} m_{\rho\sigma} \\ = h^{\mu\nu} \bar{R}_{\mu\nu} + \frac{1}{2} h^{\mu\nu} \hat{a}_\mu \hat{a}_\nu - \frac{1}{2} \hat{\Phi} h^{\mu\rho} h^{\nu\sigma} \tau_{\mu\nu} \tau_{\rho\sigma}. \end{aligned} \quad (5.57)$$

This identity equates the action (5.55) obtained from the null reduction of EH to the action (5.50) obtained from a contraction of the Einstein–Maxwell action.

Finally, note that we could also have performed the null reduction at the level of the  $(d + 2)$ -dimensional Einstein equations:

$$\hat{G}_{MN} = 8\pi \hat{G} \hat{T}_{MN}, \quad (5.58)$$

where  $\hat{G}_{MN}$  and  $\hat{T}_{MN}$  are the higher-dimensional Einstein tensor and Lorentzian energy-momentum tensor, respectively. In this case, we would obtain one additional equation of motion that does not follow from the variations of the null-reduced action (5.55):

$$\hat{G}^{uu} = 8\pi \hat{G} T^{uu}. \quad (5.59)$$

This is because  $g_{uu} = 0$  was fixed off-shell to obtain the action (5.45). However, the fact that this equation is missing does not make the reduction inconsistent because it turns out that the remaining equations of motion agree with the null reduction of the higher-dimensional Einstein equation. They form a closed set under Bargmann transformations [57]. Therefore,

<sup>17</sup> The powers of  $c$  follow from the fact that in our  $c$ -expansion, we have  $\sqrt{-g} = ce(1 + \mathcal{O}(c^{-2}))$ .

we can add the missing equation of motion (5.59) by hand to the equations obtained by null reduction of the action.

### 5.5 No mass coupling to torsionless type I TNC gravity

We now demonstrate that the type I TNC gravity actions we constructed previously cannot be coupled to mass sources without turning on torsion. This leads us to conclude that type I Bargmann symmetry is inappropriate for reproducing Newtonian gravity, which contains mass coupling but requires vanishing torsion to ensure that time is absolute.

First, let us consider the formulation of the action in (5.50), which contains the NC variables  $(\tau_\mu, h_{\mu\nu})$  and the  $U(1)$  Bargmann potential  $m_\mu$  as dynamical fields. We define the following energy-momentum and mass currents from the coupling of a matter Lagrangian:

$$\delta\mathcal{L}_{\text{mat}} = e\left(T_\tau^\mu\delta\tau_\mu + \frac{1}{2}T_h^{\mu\nu}\delta h_{\mu\nu} + T_m^\mu\delta m_\mu\right), \tag{5.60}$$

where  $T_\tau^\mu$  is the energy current,  $T_h^{\mu\nu}$  is the mass-momentum tensor,<sup>18</sup> and  $T_m^\mu$  is the mass current. We will show that having a non-zero mass density  $\rho = -\tau_\mu T_m^\mu$  is incompatible with the Newtonian requirement of vanishing torsion. Consider the variation of the type I gravity action in (5.50) with respect to  $m_\mu$ :

$$\delta_m\mathcal{L} = -\frac{e}{8\pi G}G_m^\mu\delta m_\mu \Rightarrow G_m^\mu = \frac{1}{2e}\partial_\nu(eh^{\nu\rho}h^{\mu\sigma}\tau_{\rho\sigma}). \tag{5.61}$$

We can rewrite the contraction of this vacuum equation of motion with  $\tau_\mu$  as follows:

$$e\tau_\mu G_m^\mu = -\frac{e}{2}(\partial_\nu\tau_\mu)h^{\nu\rho}h^{\mu\sigma}\tau_{\rho\sigma} = -\frac{e}{4}h^{\mu\rho}h^{\nu\sigma}\tau_{\mu\nu}\tau_{\rho\sigma}. \tag{5.62}$$

As a result, we see that the  $\tau_\mu$  projection of the  $m_\mu$  equation of motion for type I TNC gravity yields

$$\tau_\mu G_m^\mu = -\frac{1}{4}h^{\mu\rho}h^{\nu\sigma}\tau_{\mu\nu}\tau_{\rho\sigma} = 8\pi G\tau_\mu T_m^\mu = -8\pi G\rho. \tag{5.63}$$

This implies that the Newtonian zero torsion requirement  $d\tau = 0$  is incompatible with non-zero mass density  $\rho \neq 0$ . In fact, the situation remains worse: having non-zero mass density breaks the twistless torsion condition and no longer allows defining spatial hypersurfaces.

The same result can be obtained from the null reduction at the level of the equations of motion (5.58). From the reduction of the energy-momentum tensor  $\hat{T}_{MN}$ , we can identify the type I TNC mass current as follows:

$$T_m^\mu = -\hat{T}^\mu{}_{\text{u}}, \tag{5.64}$$

following, for example, [Supplementary Appendix SA2](#) of [58]. After null reduction, we get

$$\tau_\mu\hat{G}^\mu{}_{\text{u}} = \hat{G}_{\text{uu}} = \hat{R}_{\text{uu}} = \frac{1}{4}h^{\mu\rho}h^{\nu\sigma}\tau_{\mu\nu}\tau_{\rho\sigma} = \tau_\mu\hat{T}^\mu{}_{\text{u}} = -\rho, \tag{5.65}$$

where we used Eq. 5.54. This reproduces (5.63).

<sup>18</sup> This is defined up to terms proportional to  $v^\mu v^\nu$  because  $v^\mu v^\nu \delta h_{\mu\nu} = 0$  as a result of  $v^\mu h_{\mu\nu} = 0$ .

Either way, we conclude that a different notion of dynamical NC geometry is necessary to reproduce Newtonian gravity. We address this problem in the following section.

## 6 Non-relativistic expansion of general relativity

We will now derive an action whose equations of motion contain the Poisson equation of Newtonian gravity. This construction requires a new notion of TNC geometry based on an underlying symmetry algebra that differs from the usual Bargmann algebra. This geometry naturally arises in a covariant  $1/c$  expansion of general relativity, with  $c$  being the speed of light. The truncation of this expansion at subleading order provides the fields and transformation rules of “type II” TNC geometry. The corresponding action and equations of motion include the Poisson equation when sourced with the non-relativistic matter. Generally, they go beyond Newtonian gravity as they allow for the effect of gravitational time dilation due to strong gravitational fields. The following is mainly based on [10, 59], built on earlier work by Dautcourt [26] and crucially on the recent work by Van den Bleeken [8].

### 6.1 Pre-non-relativistic form of general relativity

In Lorentzian geometry, the slope of the light cone is  $1/c$ , with  $c$  denoting the speed of light. The distinguishing feature of non-relativistic geometry is that this light cone is flattened out completely. This means that we need to perform an expansion around  $c = \infty$  to relate Lorentzian geometry to non-relativistic geometry.

The constant  $c$  is dimensionful, but if we assume analyticity in  $1/c$  of an appropriate set of variables, then there must exist some other characteristic velocity that is small compared to  $c$ . However, the nature of this velocity is context-dependent and can only be identified on the shell for a specific problem. Nevertheless, it is possible to formulate the general theory of the  $1/c$  expansion without knowing the dimensionless ratio(s) in which we expand the Einstein equation. In this section, we focus on expanding in even powers of  $1/c$ , but we briefly discuss some features of the full expansion, including odd powers in [Section 7.3](#).

A convenient starting point for this expansion is the pre-non-relativistic (PNR) rewriting of GR, where we make the way in which the speed of light enters in a Lorentzian metric manifest:

$$g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}, \quad g^{\mu\nu} = -\frac{1}{c^2} V^\mu V^\nu + \Pi^{\mu\nu}. \tag{6.1}$$

In this parametrization, the Lorentzian metric and its inverse are split into a component involving a “timelike” one-form or vectors  $T_\mu$  and  $V^\mu$ , respectively, and the “spatial” symmetric tensor  $\Pi_{\mu\nu}$  or  $\Pi^{\mu\nu}$ . The latter two can be written in terms of space-like vielbein as  $\Pi^{\mu\nu} = \delta^{ab} E_\mu^a E_\nu^b$  and  $\Pi_{\mu\nu} = \delta_{ab} E_\mu^a E_\nu^b$ , where  $a, b = 1 \dots d$  are spacelike indices in the tangent space, and the total spacetime dimension is  $d + 1$ .

These PNR variables satisfy the following orthonormality relations:

$$T_\mu V^\mu = -1, \quad T_\mu \Pi^{\mu\nu} = 0, \quad \Pi_{\mu\nu} V^\nu = 0, \quad \delta_\nu^\mu = -V^\mu T_\nu + \Pi^{\mu\rho} \Pi_{\rho\nu}. \tag{6.2}$$

One can view the PNR parametrization as a split of the tangent bundle in “temporal” and “spatial” components, which also makes the factors of  $c^2$  that appear in the Lorentzian metric explicit. Then, the new PNR tensor variables can be expanded uniformly in  $\sigma = 1/c^2$ , with the leading components being order one.

The following step is to obtain the form of the EH Lagrangian in terms of the PNR variables. After some algebra, this (up to total derivatives) takes the form <sup>19</sup>

$$\mathcal{L}_{\text{EH}} = \frac{c^6}{16\pi G} E \left[ \frac{1}{4} \Pi^{\mu\nu} \Pi^{\rho\sigma} T_{\mu\rho} T_{\nu\sigma} + \sigma \Pi^{\mu\nu} R_{\mu\nu}^{(C)} - \sigma^2 V^\mu V^\nu R_{\mu\nu}^{(C)} \right], \tag{6.3}$$

where  $T_{\mu\nu} = 2\partial_{[\mu} T_{\nu]}$  and  $E = \det(T_\mu, E_\mu^a)$ . The Ricci tensor  $R_{\mu\nu}^{(C)}$  is defined in the usual way from the Riemann tensor  $R_{\mu\nu\rho}^{(C)\sigma}$ ; see Appendix A for our conventions. However, the latter is now constructed from a PNR covariant derivative  $\nabla_\mu^{(C)}$  corresponding to a PNR connection  $C_{\mu\nu}^\rho$ , instead of the usual Levi–Civita connection. The PNR connection is given by

$$C_{\mu\nu}^\rho = -V^\rho \partial_\mu T_\nu + \frac{1}{2} \Pi^{\rho\sigma} (\partial_\mu \Pi_{\nu\sigma} + \partial_\nu \Pi_{\mu\sigma} - \partial_\sigma \Pi_{\mu\nu}). \tag{6.4}$$

This connection satisfies  $\nabla_\mu^{(C)} T_\nu = 0$  and  $\nabla_\mu^{(C)} \Pi^{\rho\sigma} = 0$ , which are the PNR analogs of NC metric compatibility conditions (3.4). In addition, it satisfies

$$\nabla_\mu^{(C)} T^\nu = \frac{1}{2} \Pi^{\nu\rho} \mathcal{L}_V \Pi_{\rho\mu}, \quad \nabla_\mu^{(C)} \Pi_{\nu\rho} = T_{(\nu} \mathcal{L}_V \Pi_{\rho)\mu}, \tag{6.5}$$

where  $\mathcal{L}_V$  is the Lie derivative with respect to  $V^\mu$ . This new connection is, in general, torsionful because  $C_{[\mu\nu]}^\rho = -V^\rho \partial_{[\mu} T_{\nu]}$  is not necessarily zero.

From a Lorentzian point of view, the parametrization (6.3) of the EH Lagrangian may seem odd, even though it remains invariant under local Lorentz symmetry. Instead, we have written it in terms of variables adapted to the local Galilei symmetry that arises in the large speed of light expansion.<sup>20</sup> For more details on this, we refer the reader to [60], where a novel Palatini-type formulation of GR is obtained, which provides a natural starting point for a first-order non-relativistic expansion. This involves a reformulation of the Lorentzian Palatini action in terms of moving frames that exhibit local Galilean covariance in a large speed of light expansion. A comparison between Lorentzian and NC metric compatibility gives another explanation of the generic appearance of torsion in the non-relativistic expansion.

We now wish to consider the large speed of light expansion of the EH action. Therefore, it is useful to first consider the expansion of a general Lagrangian.

## 6.2 Large speed of light expansion of general Lagrangians

Consider a Lagrangian  $\mathcal{L} = \mathcal{L}(\sigma, \phi^I, \partial_\mu \phi^I)$  that is a function of some set of field  $\phi^I(x; \sigma)$  and its derivatives, where we also allow for an explicit dependence on the speed of light. The starting assumption is that, up to an overall power of  $c$ , which will be factored out, any field  $\phi^I(x; \sigma)$  is analytic in  $\sigma$  such that it admits a Taylor expansion around  $\sigma = 0$ :

$$\phi^I(x; \sigma) = \phi_{(0)}^I(x) + \sigma \phi_{(2)}^I(x) + \sigma^2 \phi_{(4)}^I(x) + \mathcal{O}(\sigma^3), \tag{6.6}$$

where  $\phi_{(n)}^I(x)$  is used to denote the coefficient of  $c^{-n}$  in the expansion and  $I$  a shorthand for any spacetime and/or internal indices. We confine ourselves here to the case of even powers in  $c$  only. Given this expansion of the fields, we want to expand the Lagrangian in powers of  $\sigma$ . The  $\sigma$  dependence can come from the expansion of the background metric or matter fields and from the parameters in the kinetic or potential terms. The result [9, 10, 59] is that

$$\mathcal{L}(c^2, \phi, \partial_\mu \phi) = c^N \overset{(-N)}{\mathcal{L}}_{\text{LO}} + c^{N-2} \overset{(2-N)}{\mathcal{L}}_{\text{NLO}} + c^{N-4} \overset{(4-N)}{\mathcal{L}}_{\text{NNLO}} + \mathcal{O}(c^{N-6}), \tag{6.7}$$

where we have taken the overall power of the Lagrangian to be  $\sigma^{-N/2} = c^N$  for some  $N$ .

Restricting for simplicity to the structure of the expanded Lagrangian for a single field, one finds the following LO and NLO terms:

$$\overset{(-N)}{\mathcal{L}}_{\text{LO}} = \tilde{\mathcal{L}}(0) = \overset{(-N)}{\mathcal{L}}_{\text{LO}}(\phi_{(0)}, \partial_\mu \phi_{(0)}), \tag{6.8}$$

$$\begin{aligned} \overset{(2-N)}{\mathcal{L}}_{\text{NLO}} = \tilde{\mathcal{L}}'(0) &= \left. \frac{\partial \tilde{\mathcal{L}}}{\partial \sigma} \right|_{\sigma=0} + \phi_{(2)} \frac{\partial}{\partial \phi_{(0)}} \overset{(-N)}{\mathcal{L}}_{\text{LO}} + \partial_\mu \phi_{(2)} \frac{\partial}{\partial \partial_\mu \phi_{(0)}} \overset{(-N)}{\mathcal{L}}_{\text{LO}} \\ &= \left. \frac{\partial \tilde{\mathcal{L}}}{\partial \sigma} \right|_{\sigma=0} + \phi_{(2)} \frac{\delta}{\delta \phi_{(0)}} \overset{(-N)}{\mathcal{L}}_{\text{LO}}. \end{aligned} \tag{6.9}$$

The variation of the NLO action with respect to the NLO field yields the equations of motion of the LO field in the LO action. A similar property holds at any order. The corresponding expression for the NNLO action can be found in [10, 59]. Following the general principle discussed previously, its EOM satisfies the relations

$$\frac{\delta \overset{(4-N)}{\mathcal{L}}_{\text{NNLO}}}{\delta \phi_{(2)}} = \frac{\delta \overset{(2-N)}{\mathcal{L}}_{\text{NLO}}}{\delta \phi_{(0)}}, \quad \frac{\delta \overset{(4-N)}{\mathcal{L}}_{\text{NNLO}}}{\delta \phi_{(4)}} = \frac{\delta \overset{(2-N)}{\mathcal{L}}_{\text{NLO}}}{\delta \phi_{(2)}} = \frac{\delta \overset{(-N)}{\mathcal{L}}_{\text{LO}}}{\delta \phi_{(0)}}. \tag{6.10}$$

This general expansion can be applied to the spacetime fields of Lorentzian gravity and other types of (bosonic) Lorentzian fields that couple to Lorentzian geometry.

## 6.3 Type II TNC geometry

We now show how this general procedure can expand the fields describing a  $(d + 1)$ -dimensional Lorentzian manifold. We focus on the metric description and will briefly comment on the vielbein

19 The overall factor of  $c^6$  arises from a combination of a factor of  $c^2$  from the rewriting of the Levi–Civita Ricci scalar to the PNR Ricci scalar and a factor of  $c^4$  from the dimensional prefactors of the action and the square root of the metric determinant.

20 Similarly, a variant of this rewriting of the EH action with a slightly different adapted connection [134, 135] can be used to perform a small speed of light expansion [75], leading to Carroll or ultra-local gravity.



description. By assumption, the PNR fields  $T_\mu$  and  $\Pi_{\mu\nu}$  in the parametrization (6.1) of the Lorentzian metric are considered analytic in  $\sigma$ , and thus, they admit a Taylor expansion. This means we can write them as follows:

$$T_\mu = \tau_\mu + c^{-2}m_\mu + c^{-4}B_\mu + \mathcal{O}(c^{-6}), \tag{6.11a}$$

$$\Pi_{\mu\nu} = h_{\mu\nu} + c^{-2}\Phi_{\mu\nu} + c^{-4}\Psi_{\mu\nu} + \mathcal{O}(c^{-4}), \tag{6.11b}$$

and similarly for the inverse fields

$$V^\mu = v^\mu + c^{-2}(v^\mu v^\rho m_\rho - h^{\mu\rho} v^\sigma \Phi_{\rho\sigma}) + \mathcal{O}(c^{-4}), \tag{6.12a}$$

$$\Pi^{\mu\nu} = h^{\mu\nu} + c^{-2}(2h^{\mu\rho} v^\nu m_\rho - h^{\mu\rho} h^{\nu\sigma} \Phi_{\rho\sigma}) + \mathcal{O}(c^{-4}). \tag{6.12b}$$

The various tensors in this expansion satisfy orthonormality conditions that follow from expanding (6.2), which can be found in [10]. In particular, we can use these relations to solve for the subleading fields in (6.11) in terms of the fields in (6.12).

Up to next-to-leading order, the expansion features the following fields:

$$\text{LO fields: } \tau_\mu, h_{\mu\nu}, \quad \text{NLO fields: } m_\mu, \Phi_{\mu\nu}, \tag{6.13}$$

along with their inverses. As we will show momentarily, the LO fields precisely exhibit the Galilean transformation rules of the corresponding fields with the same names in type I TNC geometry. In a slight abuse of notation, we have also used the same name  $m_\mu$  as in type I for one of the NLO fields, but this field will generally transform differently in the current notion of TNC geometry. Finally, we now have an extra field  $\Phi_{\mu\nu}$  at NLO.

Along with their transformations, the collection of these four fields defines type II TNC geometry. As shown in the following equations, this NLO geometric structure allows us to describe the NLO and NNLO actions and equations of motion of NRG owing to the simple form of the LO contributions.

We can find the transformation rules of the expanded fields by performing a similar large  $c$  expansion in the transformations of the Lorentzian metric and vielbein under diffeomorphisms and local Lorentz transformations. This leads to the following transformations of the LO and NLO fields:

$$\bar{\delta}\tau_\mu = \mathcal{L}_\xi \tau_\mu, \tag{6.14a}$$

$$\delta h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu} + \tau_\mu \lambda_\nu + \tau_\nu \lambda_\mu, \tag{6.14b}$$

$$\bar{\delta}m_\mu = \mathcal{L}_\xi m_\mu + \lambda_\mu + \partial_\mu \Lambda - \Lambda a_\mu + h^{\rho\sigma} \zeta_\sigma (\partial_\rho \tau_\mu - \partial_\mu \tau_\rho), \tag{6.14c}$$

$$\begin{aligned} \bar{\delta}\Phi_{\mu\nu} = & \mathcal{L}_\xi \Phi_{\mu\nu} + 2\lambda_a \left( \tau_{(\mu} \pi_{\nu)}^a + m_{(\mu} e_{\nu)}^a \right) + 2\eta_a e^a_{(\mu} \tau_{\nu)} + 2\Lambda K_{\mu\nu} \\ & + 2\bar{\nabla}_{(\mu} \zeta_{\nu)}, \end{aligned} \tag{6.14d}$$

where  $\lambda_\mu = e^a_\mu \lambda_a$  is the Galilean boost parameter that obeys  $v^\mu \lambda_\mu = 0$  and the parameter  $\eta_a$  arises from subleading boost. Additionally, the notation  $\bar{\delta}$  foreshadows a link to the transformations (5.9) in the gauging procedure, and the fields  $e_\mu^a$  and  $\pi_\mu^a$  are the leading and subleading terms in the expansion of the spatial vielbein  $E_\mu^a$ , respectively. We similarly expanded the diffeomorphisms so that  $\xi^\mu$  is the LO part, whereas the subleading diffeomorphisms are  $\zeta^\mu$ , which we have decomposed previously as

$$\zeta^\mu = -\Lambda v^\mu + h^{\mu\nu} \zeta_\nu. \tag{6.15}$$

In these expressions, we used the acceleration and extrinsic curvature tensors:

$$a_\mu = \mathcal{L}_v \tau_\mu = v^\nu \tau_{\mu\nu}, \quad K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_v h_{\mu\nu}, \tag{6.16}$$

which will appear in the expanded EH action. It can be easily checked that the transformations of the three fields  $\tau_\mu$ ,  $h_{\mu\nu}$ , and  $m_\mu$  in (6.14) reduce to those of the corresponding type I TNC fields in (5.22) when  $\tau_\mu$  is closed.

To write the expanded EH action, we should introduce a connection on type II TNC geometry. At leading order in the expansion of  $C_{\mu\nu}^\rho$  in (6.4), we recover the torsionful connection  $\check{\Gamma}_{\mu\nu}^\rho$  from (5.27):

$$\check{\Gamma}_{\mu\nu}^\rho = C_{\mu\nu}^\rho \Big|_{\sigma=0} = -v^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} (\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}). \tag{6.17}$$

This combination is, in some sense, the minimal collection of terms that transforms as an affine connection under diffeomorphisms. Moreover, it follows from the metric compatibility conditions on the PNR connection  $C_{\mu\nu}^\rho$  that  $\check{\Gamma}_{\mu\nu}^\rho$  is an NC metric-compatible connection satisfying the properties  $\nabla_\mu \tau_\nu = 0$  and  $\nabla_\mu h^{\nu\rho} = 0$ . It transforms under local Galilei boosts; therefore, the corresponding curvature tensors will also generically transform under boosts. However, if we start from an action that is invariant under Lorentz boosts, such as the EH action, the expansion will only produce scalar combinations that are invariant under Galilean boosts.

### 6.3.1 Type II TNC symmetry algebra and Lie algebra expansion

In Section 5.2.2, we derived the transformation properties (5.17a) of the type I TNC fields from the gauging of the Bargmann algebra. It turns out that the transformations (6.14d) of the type II TNC fields can likewise be obtained from the gauging of an algebra. The corresponding symmetry algebra follows from an expansion of the Poincaré algebra using the general method of Lie algebra expansions.<sup>21</sup>

Applying the PNR decomposition to the Poincaré-valued Cartan connection (5.5), we obtain

$$\mathcal{A}_\mu = HT_\mu + P_a E_\mu^a + B_a \Omega_\mu^a + \frac{1}{2} J_{ab} \Omega_\mu^{ab}, \tag{6.18}$$

which contains the relativistic vielbein  $T_\mu$  and  $E_\mu^a$  along with the boost connection  $\Omega_\mu^a$  and the rotation connection  $\Omega_\mu^{ab}$ . If we schematically write this Cartan connection as  $\mathcal{A}_\mu = T_I \mathcal{A}_\mu^I$  and expand its components as  $\mathcal{A}_\mu^I = \sum_{n=0}^{\infty} \sigma^n \mathcal{A}_\mu^{I(2n)}$ , we will obtain the new generators  $T_I^{(n)} = T_I \otimes \sigma^n$ , where  $n \geq 0$  will be referred to as the level.

Using this expansion of the generators  $T_I^{(n)}$ , one obtains an algebra whose non-zero commutation relations are [61]

$$\begin{aligned} [H^{(m)}, B_a^{(n)}] &= P_a^{(m+n)}, & [P_a^{(m)}, B_b^{(n)}] &= \delta_{ab} H^{(m+n+1)}, \\ [B_a^{(m)}, B_b^{(n)}] &= -J_{ab}^{(m+n+1)}, & [J_{ab}^{(m)}, P_c^{(n)}] &= \delta_{ac} P_b^{(m+n)} - \delta_{bc} P_a^{(m+n)}, \end{aligned}$$

<sup>21</sup> This method has been considered in, for example, [136–138] and was applied to the  $1/c^2$  expansion of the Poincaré algebra in [10, 61] and [139–142].

$$[J_{ab}^{(m)}, B_c^{(n)}] = \delta_{ac} B_b^{(m+n)} - \delta_{bc} B_a^{(m+n)},$$

$$[J_{ab}^{(m)}, J_{cd}^{(n)}] = \delta_{ac} J_{bd}^{(m+n)} - \delta_{bc} J_{ad}^{(m+n)} - \delta_{ad} J_{bc}^{(m+n)} + \delta_{bd} J_{ac}^{(m+n)}. \quad (6.19)$$

We can quotient out all generators with level  $n > L$  for some  $L$ , which amounts to truncating the  $1/c^2$  expansion. At the lowest level  $L = 0$ , the algebra is isomorphic to the Galilei algebra, and the gauging of the algebra can be shown to generate the transformation rules of  $\tau_\mu$  and  $h_{\mu\nu}$ . At the following level  $L = 1$ , the number of generators doubles, and we get a novel algebra that can be shown to lead to the full set of symmetries in (6.14), which now acts on all of the LO and NLO fields.

The  $L = 1$  truncation of the algebra (6.19) does not have the Bargmann algebra as a subalgebra because the would-be Bargmann extension  $H^{(1)}$  is not central as it has non-zero commutator with the Galilean boosts  $B_a^{(1)}$ . We can obtain the Bargmann algebra as a quotient of the  $L = 1$  algebra by the ideal spanned by  $J_{ab}^{(1)}, B_a^{(1)}, P_a^{(1)}$ . Although the Bargmann algebra encodes the local symmetries of type I TNC geometry, this algebra can be considered the “type II Bargmann algebra” associated with type II TNC geometry. As discussed in (6.14c), the corresponding transformations agree on torsionless geometries, but they are generically distinct.

### 6.4 Expanding the EH action

We now have all ingredients in place to find the LO, NLO, and NNLO terms in the expansion of the EH action following the methods outlined in Section 6.2. Following the PNR parametrization (6.1) and the expansion (6.11), we obtain a theory that is expressed in terms of the LO and NLO fields:

$$\phi_{(0)}^I = \{\tau_\mu, h_{\mu\nu}\}, \quad \phi_{(2)}^I = \{m_\mu, \Phi_{\mu\nu}\}. \quad (6.20)$$

The NNLO fields  $\phi_{(4)}^I = \{B_\mu, \Psi_{\mu\nu}\}$  also enter in the NNLO EH action, but as shown in the following sections, they only play a limited role, implementing a particular constraint.

Following the general form (6.7), the  $1/c^2$  expansion of the EH Lagrangian reads

$$\mathcal{L}_{EH} = c^6 \left( \mathcal{L}_{LO}^{(-6)} + \sigma \mathcal{L}_{NLO}^{(-4)} + \sigma^2 \mathcal{L}_{NNLO}^{(-2)} + \mathcal{O}(\sigma^3) \right). \quad (6.21)$$

To compute the first terms in this expansion, we use the PNR form (6.3) of the EH action, the large  $c$  expansions of the metric variables in (6.11), and the choice of connection in (6.17). We review the main results in the following paragraphs, referring to [10] for more detail.

The LO action is

$$\mathcal{L}_{LO}^{(-6)} = \frac{e}{64\pi G} h^{\mu\nu} h^{\rho\sigma} \tau_{\mu\rho} \tau_{\nu\sigma}, \quad (6.22)$$

where  $e = \det(\tau_\mu, e_\mu^a)$  and  $\tau_{\mu\nu} = \partial_\mu \tau_\nu - \partial_\nu \tau_\mu$  as usual. This LO action is manifestly invariant under Galilean boosts. Its variation takes the following form:

$$\delta \mathcal{L}_{LO}^{(-6)} = -\frac{1}{8\pi G} e \left( G_\tau^\alpha \delta \tau_\alpha + \frac{1}{2} G_h^{\alpha\beta} \delta h_{\alpha\beta} \right), \quad (6.23)$$

where the leading order equations of motion are

$$G_h^{\alpha\beta} = -\frac{1}{8} h^{\mu\nu} h^{\rho\sigma} \tau_{\mu\rho} \tau_{\nu\sigma} h^{\alpha\beta} + \frac{1}{2} h^{\mu\alpha} h^{\nu\beta} h^{\rho\sigma} \tau_{\mu\rho} \tau_{\nu\sigma}, \quad (6.24a)$$

$$G_\tau^\alpha = \frac{1}{8} h^{\mu\nu} h^{\rho\sigma} \tau_{\mu\rho} \tau_{\nu\sigma} v^\alpha + \frac{1}{2} a_\mu h^{\mu\nu} h^{\rho\alpha} \tau_{\nu\rho} + \frac{1}{2} e^{-1} \partial_\mu (e h^{\mu\nu} h^{\rho\alpha} \tau_{\nu\rho}). \quad (6.24b)$$

These equations imply  $h^{\mu\nu} h^{\rho\sigma} \tau_{\mu\rho} \tau_{\nu\sigma} = 0$ . As the latter is a sum of squares, it implies  $h^{\mu\nu} h^{\rho\sigma} \tau_{\mu\rho} = 0$  or equivalently  $\tau \wedge d\tau = 0$ , which is the TTNC condition discussed in Section 5.2.3. Hence, the on-shell geometry arising from the expansion is a TTNC geometry [4]. The LO equations of motion (6.24) vanish identically once the TTNC condition is imposed.

The NLO Lagrangian then takes the form

$$\mathcal{L}_{NLO}^{(-4)} = -\frac{e}{8\pi G} \left( -\frac{1}{2} h^{\mu\nu} \check{R}_{\mu\nu} + G_\tau^{(-6)\mu} m_\mu + \frac{1}{2} G_h^{(-6)\mu\nu} \Phi_{\mu\nu} \right), \quad (6.25)$$

where  $G_\tau^{(-6)\mu}$  and  $G_h^{(-6)\mu\nu}$  are the LO EOMs given in (6.24). As the latter is equivalent to the TTNC condition, we can write the NLO action as the first term in (6.25) together with a Lagrange multiplier term  $\chi_{\rho\sigma} h^{\rho\mu} h^{\sigma\nu} \tau_{\mu\nu}$ , enforcing the TTNC condition.<sup>22</sup> The resulting NLO Lagrangian also has Galilean symmetries, and we will refer to its dynamics as Galilean gravity. This theory was also studied in [62] using first-order formalism. Equation 6.25 can be related to the Lagrangian appearing in that work by a specific choice of the undetermined Lagrange multipliers.

Following the general observation around Eq. 6.9, the leading order equations of motion are included in the NLO Lagrangian as the equations of motion of the NLO fields  $m_\mu$  and  $\Phi_{\mu\nu}$ . The NLO equations of motion of the LO fields  $\tau_\mu$  and  $h_{\mu\nu}$  are

$$G_\tau^{(-4)\nu} = \frac{1}{2} \left[ 2(h^{\mu\rho} h^{\nu\sigma} - h^{\mu\nu} h^{\rho\sigma}) \check{\nabla}_\mu K_{\rho\sigma} + v^\nu h^{\rho\sigma} \check{R}_{\rho\sigma} + (\check{\nabla}_\mu + 2a_\mu) h^{\mu\rho} h^{\nu\sigma} F_{\rho\sigma} \right] + \dots, \quad (6.26a)$$

$$G_h^{(-4)\rho\sigma} = h^{\mu\rho} h^{\nu\sigma} \left( \check{R}_{\mu\nu} - \frac{1}{2} h_{\mu\nu} h^{\kappa\lambda} \check{R}_{\kappa\lambda} - (\check{\nabla}_\mu + a_\mu) a_\nu + h_{\mu\nu} h^{\kappa\lambda} (\check{\nabla}_\kappa + a_\kappa) a_\lambda \right) + \dots, \quad (6.26b)$$

where the dots denote terms that vanish on the shell upon using the  $m_\mu$  and  $\Phi_{\mu\nu}$  equations of motion or equivalently upon imposing the TTNC condition.

Finally, the NNLO Lagrangian requires considerably further algebra. Its form is the simplest when we add a Lagrange multiplier to enforce the TTNC condition.<sup>23</sup> We refer to the result as the NRG Lagrangian:

$$\begin{aligned} \mathcal{L}_{NRG} &\equiv \mathcal{L}_{NNLO}^{(-2)} \Big|_{\tau \wedge d\tau = 0} + \frac{e}{16\pi G} \frac{1}{2} \check{\chi}_{\rho\sigma} h^{\mu\rho} h^{\nu\sigma} (\partial_\mu \tau_\nu - \partial_\nu \tau_\mu) \\ &= \frac{e}{16\pi G} \left[ h^{\mu\rho} h^{\nu\sigma} K_{\mu\nu} K_{\rho\sigma} - (h^{\mu\nu} K_{\mu\nu})^2 - 2m_\nu (h^{\mu\rho} h^{\nu\sigma} - h^{\mu\nu} h^{\rho\sigma}) \check{\nabla}_\mu K_{\rho\sigma} \right. \\ &\quad \left. + \Phi h^{\mu\nu} \check{R}_{\mu\nu} + \frac{1}{4} h^{\mu\rho} h^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + \frac{1}{2} \check{\chi}_{\rho\sigma} h^{\mu\rho} h^{\nu\sigma} (\partial_\mu \tau_\nu - \partial_\nu \tau_\mu) \right. \\ &\quad \left. - \Phi_{\rho\sigma} h^{\mu\rho} h^{\nu\sigma} \left( \check{R}_{\mu\nu} - \check{\nabla}_\mu a_\nu - a_\mu a_\nu - \frac{1}{2} h_{\mu\nu} h^{\kappa\lambda} \check{R}_{\kappa\lambda} + h_{\mu\nu} e^{-1} \partial_\kappa (e h^{\kappa\lambda} a_\lambda) \right) \right], \end{aligned} \quad (6.27)$$

22 Equivalently, one can obtain the resulting action by rewriting the first term in the PNR form of the EH action in terms of an auxiliary field  $\chi_{\mu\nu}$  with appropriate quadratic and linear terms, and then taking the large  $c$  limit in the same way that the magnetic Carroll limit of GR was obtained in [75].

23 The full result at NNLO without the TTNC condition can be found in [118].

where  $\Phi \equiv -v^\mu m_\mu$  is the Newtonian potential. The resulting EOMs can be found in [10].

We remark that the NRG Lagrangian can also be obtained *via* another method. This alternative route employs the type II TNC gauge symmetries and constructs the unique (up to a cosmological constant) two-derivative action respecting this symmetry, starting with the correct kinetic term required for Newton’s law of gravitation and then completing the full action. This was first carried out in [9] and elaborated on in [10], and we refer to these studies for details, including the form of the EOMs. As expected, the two Lagrangians are identical. The main difference in their appearances arises from the slightly different geometric variables used in each because the gauging construction naturally leads to manifestly Galilean boost invariant quantities. Depending on taste and type of application, one can work with either one of them.

### 6.5 Newton’s Poisson equation

We will now show how the NRG action (6.27) yields the Poisson equation of Newtonian gravity when it is coupled to a massive non-relativistic particle. In this way, we show that type II TNC geometry, particularly action (6.27), provides an off-shell formulation of Newtonian gravity.

We start by considering the large  $c$  expansion of the Lorentzian particle Lagrangian:

$$\mathcal{L} = -mc(-g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu)^{1/2}, \tag{6.28}$$

where  $X^\mu(\lambda)$  are embedding scalars and  $\lambda$  is the geodesic parameter. We expand the metric according to (6.1) and (6.11a), as performed previously. In addition, we now need to expand the embedding scalars according to the general expression (6.6), which gives

$$X^\mu = x^\mu + \frac{1}{c^2}y^\mu + \mathcal{O}(c^{-4}). \tag{6.29}$$

This is necessary because the equations of motion for  $X^\mu$  would, otherwise, be overconstrained. Then, the resulting LO Lagrangian is just

$$\overset{(-2)}{\mathcal{L}}_{\text{LO}} = -m\tau_\mu\dot{x}^\mu, \tag{6.30}$$

whereas the NLO Lagrangian becomes (up to a boundary term)

$$\overset{(0)}{\mathcal{L}}_{\text{NLO}} = m\left(\left(\partial_\nu\tau_\mu - \partial_\mu\tau_\nu\right)\dot{x}^\nu y^\mu + \frac{1}{2}\frac{\bar{h}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{\tau_\rho\dot{x}^\rho}\right). \tag{6.31}$$

The latter is the Lagrangian of a particle on a type II TNC geometry. Again, it can be checked that the LO EOM is correctly reproduced by the EOMs of  $y^\mu$  in the subleading Lagrangian.

On a fixed torsionless NC background, the action (6.31) is the same as the standard point particle action (5.45) on a torsionless type I TNC geometry [3, 54, 63]. We also emphasize that the LO term in the expansion of the particle action (6.28) is of order  $c^2$  so that it couples to the NNLO gravity action. Likewise, the NLO particle action only couples to the  $N^3\text{LO}$  gravity action, where it will source NNLO fields. This means we can solve the geodesic equation at a given order in  $1/c^2$  on a background whose fields were determined at the previous order. For example, at order  $c^2$  of the

geodesic equation, the equation of motion of  $x^\mu$ , shows that  $d\tau = 0$ . The coupling of  $\overset{(-2)}{\mathcal{L}}_{\text{LO}}$  to NRG leads to the equation of NC gravity as originally formulated by Trautman. We solve this equation for the NC variables. Then, at the following order  $c^0$ , we solve for the embedding scalars in the geodesic equation (Newton’s law) obtained by varying  $\overset{(0)}{\mathcal{L}}_{\text{NLO}}$  with respect to  $x^\mu$ . Then, this solution sources the Einstein equations at order  $c^0$ .

Thus, we consider the NRG Lagrangian (6.27) coupled to the LO point particle Lagrangian (6.30). The  $x^\mu$  equation enforces absolute time  $d\tau = 0$ . In general, the equations of motion of the NRG part of the Lagrangian can be shown [10] as follows:

$$\bar{R}_{\mu\nu} = \frac{8\pi G}{d-1}\left(-\left(d-2\right)\tau_\rho\mathcal{T}_\tau^\rho + h_{\rho\sigma}\mathcal{T}_h^{\rho\sigma}\right)\tau_\mu\tau_\nu, \tag{6.32}$$

where  $\mathcal{T}_\tau$  and  $\mathcal{T}_h$  are proportional to the responses of varying the matter action with respect to  $\tau_\mu$  and  $h_{\mu\nu}$ , respectively, as reviewed in more detail in Section 7.1.

In the case at hand, the sources that follow from the LO particle action are

$$\mathcal{T}_\tau^\mu = -m\int d\lambda\frac{\delta(x-x(\lambda))}{e}\dot{x}^\mu, \tag{6.33}$$

and  $\mathcal{T}_h^{\mu\nu} = 0$ . The equations of motion of the NRG action (6.27) are coupled to the point particle action (6.30) and then reproduce the covariant form of the Newtonian Poisson equation:

$$\bar{R}_{\mu\nu} = 8\pi G\frac{d-2}{d-1}\rho\tau_\mu\tau_\nu, \tag{6.34}$$

with  $\rho = m\int d\lambda\delta(x-x(\lambda))/e$  (in the gauge  $\tau_\mu\dot{x}^\mu = 1$ ). In addition to these NC equations of motion, there are additional decoupled equations of motion for the field  $\Phi_{\mu\nu}$  and the Lagrange multiplier. It is also important to stress that  $\rho$  in (6.34) is not a Bargmann mass density but rather the leading contribution to the energy density. Overall, we see that the NRG action (6.27) for type II TNC geometry provides a complete off-shell description of NC gravity with a point particle source.

## 7 Other aspects of NRG

This section discusses further properties of the large speed of light expansion of general relativity. This discussion is non-exhaustive, and we will comment on other aspects and directions in Section 8.

### 7.1 Expansion of matter sources

We already discussed the non-relativistic particle coupling to NRG in Section 6.5. Here, we present some highlights of the large speed of light expansion of more general matter couplings. These expansions can be obtained by applying the same methods used for the EH Lagrangian to a generic matter Lagrangian. The matter actions that we obtain in this way act as sources for gravity in the  $1/c^2$  expansion, particularly for the NRG Lagrangian (6.27).

The expansion of a generic matter Lagrangian takes the following form:

$$\begin{aligned} \mathcal{L}_{\text{mat}}(c^2, \phi, \partial_\mu \phi) &= c^N \tilde{\mathcal{L}}_{\text{mat}}(\sigma) \\ &= c^N \mathcal{L}_{\text{mat, LO}}^{(-N)} + c^{N-2} \mathcal{L}_{\text{mat, NLO}}^{(2-N)} + c^{N-4} \mathcal{L}_{\text{mat, NNLO}}^{(4-N)} \\ &\quad + \mathcal{O}(c^{N-6}). \end{aligned} \tag{7.1}$$

At each order  $n \in \mathbb{Z}_{\geq 0}$ , we define matter currents as responses to variations of the geometric fields. For example,

$$T_h^{\alpha\beta} \equiv 2e^{-1} \frac{\delta \mathcal{L}_{\text{mat, N}^n \text{LO}}^{(2n-N)}}{\delta h_{\alpha\beta}}, \quad T_\tau^\alpha \equiv e^{-1} \frac{\delta \mathcal{L}_{\text{mat, N}^n \text{LO}}^{(2n-N)}}{\delta \tau_\alpha}. \tag{7.2}$$

In this way, one can get the EOMs of the sourced gravity coupled to matter at any order in the large  $c$  expansion [10]. In particular, if we define the EOMs from the variations of the  $N^n$ LO gravity Lagrangian with respect to  $h_{\alpha\beta}$  and  $\tau_\alpha$  for  $n \in \mathbb{Z}_{\geq 0}$  as

$$\begin{aligned} \frac{1}{16\pi G} G_h^{(2n-6)\alpha\beta} &\equiv -e^{-1} \frac{\delta \mathcal{L}_{\text{N}^n \text{LO}}^{(2n-6)}}{\delta h_{\alpha\beta}}, \\ \frac{1}{8\pi G} G_\tau^{(2n-6)\alpha} &\equiv -e^{-1} \frac{\delta \mathcal{L}_{\text{N}^n \text{LO}}^{(2n-6)}}{\delta \tau_\alpha}, \end{aligned} \tag{7.3}$$

we get the following sourced EOMs at any order  $2m \geq -6$ :

$$G_h^{(2m)\alpha\beta} = 8\pi G T_h^{(2m)\alpha\beta}, \quad G_\tau^{(2m)\alpha} = 8\pi G T_\tau^{(2m)\alpha}. \tag{7.4}$$

The equations of motion and currents for the higher-order fields are defined analogously. Relatedly, one can derive the Ward identities resulting from various gauge invariances, see [10] for details. Amongst other things, these Ward identities provide the non-relativistic analog of the conservation of the relativistic energy-momentum tensor.

A few remarks are in order. Recall that the LO EOMs  $G_h^{(-6)\alpha\beta}$  and  $G_\tau^{(-6)\alpha}$  are equivalent to  $\tau \wedge d\tau = 0$ . To avoid spacetimes, which violate the twistless torsion condition  $\tau \wedge d\tau = 0$ , we must have matter actions such that the overall scaling  $N$  in (7.1) is at most equal to four, so that  $T_h^{(-6)\alpha\beta} = T_\tau^{(-6)\alpha} = 0$  and no violation of the twistless torsion condition is sourced. Hansen et al. [10] showed that  $N \leq 4$  for all examples considered, including the large  $c$  expansion of a real or complex scalar field, the Maxwell field, and fluids. The non-relativistic point particle case discussed in Section 6.5 corresponds to  $N = 2$ .

Thus, the matter sector determines whether the geometry has torsion or not. This is seen, for example, in the expansion of perfect fluids, for which different regimes depend on how we expand the energy and pressure as a function of  $1/c^2$ . Furthermore, Hansen et al. [10] discussed the resulting actions for various field theory examples, a complex and real scalar field, and electrodynamics. In particular, the expansion of a complex scalar field coupled to GR leads to the Lagrangian for the Schrödinger–Newton equation. This off-shell description of the Schrödinger–Newton system includes fields whose equations of motion inform us that the clock one-form must be closed. Furthermore, in the case of Maxwell’s theory, there are two limits (a magnetic and an electric limit, see [64–67]) depending on how we expand the gauge connection. One can obtain the Lagrangian descriptions for both using this procedure.

## 7.2 Strong gravity expansion of the Schwarzschild metric

One way to generate solutions of NRG is by considering the  $1/c^2$  expansion of solutions of GR. To illustrate this, we now discuss the  $1/c^2$  expansion of the Schwarzschild solution. Interestingly, this can be performed in two interesting ways: through a weak field expansion related to the post-Newtonian expansion or through a strong field expansion that leads to an exact torsionful solution of NRG. Although the precise physical interpretation of this latter expansion remains under construction, the fact that NRG includes solutions with torsion (so that time is no longer absolute) shows that it is richer than just Newtonian gravity.

Consider the Schwarzschild metric, including factors of  $c$ :

$$ds^2 = -c^2 \left(1 - \frac{2Gm}{c^2 r}\right) dt^2 + \left(1 - \frac{2Gm}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega_2^2. \tag{7.5}$$

As first noticed in [8], we can perform two different expansions, both physically relevant, depending on how we treat the mass parameter  $m$  as a function of  $c^2$ . The first option is to consider  $m$  constant in  $c^2$  as we expand. Using (6.1) and (6.11), we see that the resulting types II TNC fields are as follows:

$$\tau_\mu dx^\mu = dt, \tag{7.6a}$$

$$m_\mu dx^\mu = -\frac{Gm}{r} dt = \Phi dt, \tag{7.6b}$$

$$h_{\mu\nu} dx^\mu dx^\nu = dr^2 + r^2 d\Omega_2^2, \tag{7.6c}$$

$$\Phi_{\mu\nu} dx^\mu dx^\nu = \frac{2Gm}{r} dr^2 = -2\Phi dr^2. \tag{7.6d}$$

This is a flat torsionless NC spacetime with non-zero subleading fields  $m_\mu$  and  $\Phi_{\mu\nu}$ . It can be verified that this is a vacuum solution of the NRG equations of motion. The solution has zero torsion and is expressed in terms of the Newtonian potential  $\Phi = -\nu^\mu m_\mu = -Gm/r$ . This geometry receives non-trivial corrections in subleading order.

In the second approach, we take the mass to be of order  $c^2$ , so that  $M = m/c^2$  is constant in  $c^2$ , as in [8]. This provides an approximation of GR that is distinct from the post-Newtonian expansion. In this case, the expansion terminates at NLO, and the resulting geometry is described by the following type II TNC fields:

$$\tau_\mu dx^\mu = \sqrt{1 - \frac{2GM}{r}} dt, \tag{7.7a}$$

$$m_\mu dx^\mu = 0, \tag{7.7b}$$

$$h_{\mu\nu} dx^\mu dx^\nu = \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{7.7c}$$

$$\Phi_{\mu\nu} dx^\mu dx^\nu = 0. \tag{7.7d}$$

This is a torsionful NC spacetime. It is a vacuum solution of the equations of motion of the NLO EH Lagrangian (6.25), which describes Galilean gravity, as it does not involve the subleading fields. This strong gravity expansion of the Schwarzschild metric cannot be captured by Newtonian gravity because it has non-zero torsion. However, it can be

described as a torsionful NC geometry. Studying the geodesics in this novel geometry, one can show that the three classical tests of GR—perihelion precession, deflection of light, and gravitational red-shift—are passed perfectly [61]. Thus, it is possible to have a non-relativistic causal structure and still correctly include the effects of gravitational time dilation, albeit in a conceptually different way from that in GR.

### 7.2.1 Other solutions

We also mention here some other types of solutions that have been studied in the literature. One can consider the Tolman–Oppenheimer–Volkoff fluid, where it turns out that the TOV equation can be derived entirely from the NR framework. Another interesting case is that of cosmological solutions, where one can show that the FLRW spacetime is also an exact solution of NRG.

One important remark is that the results of the  $1/c^2$  expansion of a given Lorentzian spacetime can depend on the coordinate chart used to construct the expansion. Given two different charts on a Lorentzian spacetime related by a diffeomorphism that is not analytic in  $c$ , the expansion of the spacetime in these charts will yield distinct non-relativistic spacetimes, which are not related by gauge transformations. An example is the expansion of flat spacetime in the usual Minkowski coordinates *versus* Rindler coordinates, leading to non-gauge-equivalent NC geometries.

As an additional example of the dependence of the expansion on the scaling we choose for the parameters in a solution, we consider two inequivalent  $1/c^2$  expansions of  $\text{AdS}_{d+1}$ . In global coordinates and with explicit factors of  $c$ , the  $\text{AdS}_{d+1}$  metric is

$$ds^2 = -c^2 \cosh^2 \rho dt^2 + l^2 (d\rho^2 + \sinh^2 \rho d\Omega_{d-1}^2), \tag{7.8}$$

where  $l$  is the AdS radius,  $\rho > 0$  is dimensionless, and  $t$  has dimensions of time. The corresponding type II TNC geometry can be read off as follows:

$$\tau_\mu dx^\mu = \cosh \rho dt, \tag{7.9a}$$

$$h_{\mu\nu} dx^\mu dx^\nu = l^2 (d\rho^2 + \sinh^2 \rho d\Omega_{d-1}^2), \tag{7.9b}$$

$$m_\mu = 0, \tag{7.9c}$$

$$\Phi_{\mu\nu} = 0. \tag{7.9d}$$

Obviously, the  $1/c^2$  expansion terminates immediately. This is a torsionful NC spacetime. Likewise, the  $1/c^2$  expansion of AdS in Poincaré coordinates also leads to a torsionful type II TNC geometry.

Conversely, if we do the coordinate transformation  $r = l \sinh \rho$  in (7.8), we obtain the metric

$$ds^2 = -c^2 \left( 1 + \frac{r^2}{l^2} \right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{l^2}} + r^2 d\Omega_{d-1}^2. \tag{7.10}$$

This metric describes the static patch of de Sitter if we replace  $l^2$  with  $-l^2$ . Then, if we define  $l = \frac{c}{H}$ , where  $H$  is independent of  $c$ , we find

$$ds^2 = -c^2 dt^2 \mp H^2 r^2 dt^2 + \frac{dr^2}{1 \pm \frac{H^2 r^2}{c^2}} + r^2 d\Omega_{d-1}^2, \tag{7.11}$$

where the upper sign is for AdS and the lower sign is for dS. Expanding this to NLO, the resulting type II TNC geometry is

$$\tau = dt \quad h_{\mu\nu} dx^\mu dx^\nu = d\vec{x} \cdot d\vec{x}, \quad m_\mu dx^\mu = \pm \frac{1}{2} H^2 \vec{x}^2 dt, \tag{7.12}$$

where  $\Phi_{\mu\nu}$  is omitted, and we transformed to Cartesian coordinates. This TNC geometry is known as the Newton–Hooke spacetime. Grosvenor et al. [68] showed that such a spacetime could be written in the form of a non-relativistic FLRW geometry with flat spatial slices by a sequence of NC gauge transformations. Furthermore, they were related to a null reduction of a pp-wave geometry in [69].

### 7.3 Odd powers in $1/c$

So far, we have focused on even powers of  $1/c$  in our large  $c$  expansion. This is a consistent subsector in the purely gravitational sector. In earlier work [26], using a weak field assumption along with physical constraints on the energy-momentum tensor, odd terms only appear at subleading orders beyond the 1PN order in post-Newtonian expansions. A complete analysis of odd powers, including ones that can appear at pre-Newtonian orders and going beyond the weak field assumption, was performed in [11]. The motivation for this is that an energy-momentum tensor that sources torsion can also source the leading-order odd term in the metric at order  $c^1$  when there is dynamics. Another motivation is that a solution such as the Kerr metric admits several  $1/c$  expansions depending on how one scales the mass and angular momentum, and some of these lead to odd powers in  $1/c$  [11]. Finally, odd powers in  $1/c$  can capture retardation effects.

The starting point of [11] follows from writing the line element as follows:

$$ds^2 = -e^{-\Psi} (cdt + C_i dx^i)^2 + e^\Psi k_{ij} dx^i dx^j + \mathcal{O}(c) dt^2 + \mathcal{O}_i(c^0) dt dx^i + \mathcal{O}_{ij}(c^{-1}) dx^i dx^j. \tag{7.13}$$

The authors of [11] then obtain the LO equations of motion of the LO physical fields, which consist of a scalar potential  $\Psi$ , a vector potential  $C_i$  and a spatial metric  $k_{ij}$ . When these fields are time-independent, they can be shown to be solutions to the Einstein equations for stationary metrics. Interestingly, when we allow time dependence, the LO fields satisfy the same equations as when no time dependence is involved. This means that the time dependence sits in the integration “constants” when solving the Einstein equations for stationary metrics. These time-dependent integration constants source the following subleading equations. Thus, we can view the  $1/c$  expansion as an expansion around a stationary GR solution, which has been illustrated for the Kerr metric in [11]. Similarly, the  $1/c^2$  expansion can be viewed as an expansion around a static sector of GR [8].

## 8 Discussion

In this review, we have emphasized introducing the reader to the various notions of NC geometry and how they enter the non-relativistic expansions of general relativity. We only mentioned a few applications, but there are numerous further developments and extensions related to the topic of this review, which we briefly mention here.

## 8.1 Connection to the post-Newtonian expansion

The post-Newtonian expansion is performed in harmonic gauge. The Einstein equations can be formally integrated using a retarded Green's function (to obey a no-incoming-radiation boundary condition in the past). Then, one solves this integral equation by performing a  $1/c$  expansion (in a large but finite region containing the source) and a  $G$  expansion (outside the source) and matching the two in their overlap region. However, a calculational scheme that allows one to do this in an arbitrary gauge does not currently exist.

The covariant  $1/c$  expansion seems the ideal starting point to generalize the existing approaches, such as the ones by Blanchet–Damour and Will–Wiseman. However, this would be very reminiscent of what is sometimes called the “classic” approach. The latter has been abandoned because the  $1/c$  expansion beyond the 1PN order has a finite regime of validity, so one cannot impose asymptotic boundary conditions on the  $1/c$  solution. Nevertheless, we want to advocate a hybrid approach that combines the Blanchet–Damour or the Will–Wiseman approach with the classic approach. We refer to [36] for more details.

## 8.2 Carroll expansion of gravity

The study of the small speed of light limit and expansion of GR goes back to [70] and later [71]. This expansion is also called an ultra-local or Carroll expansion because the Poincaré group contracts to the Carroll group [72–74] in the  $c \rightarrow 0$  limit. The systematic study of the small speed of light expansion of GR, paralleling the approach of [10], was recently obtained in [75], to which we refer for further references on Carroll geometry and Carrollian gravity theories. Interestingly, although the LO action in the large  $c$  expansion is just the TTNC condition, in the Carroll expansion, the LO action already involves non-trivial (though ultra-local) dynamics.

## 8.3 Other formulations

Different approaches to frame and/or first-order formulations of non-relativistic (and ultra-local) limits and expansions of gravity have been considered in [62, 75–77]. Furthermore, the Palatini action for GR was reformulated in [60] in terms of moving frames that exhibit local Galilean covariance in a large speed of light expansion.

There are also interesting connections between the  $1 + 3$  formulation used in GR and the non-relativistic expansion. A  $1 + 3$  formulation of Newton's equations was discussed in [78] and [78] and applied to cosmology in [80]. The relation to the  $1 + 3$  formulation was performed more systematically in [81], extending the computation of the effective Lagrangian to a higher order and making some new all-order observations.

## 8.4 NR gravity models in two and three spacetime dimensions

For GR, special types of models exist when considering NRG in two and three spacetime dimensions. Three-dimensional CS theories of NRG based on extended Galilei algebras were first obtained in [82–84] and further studied and generalized in [85–89]. Likewise, non-relativistic (and Carrollian) versions of JT gravity were first given in [89, 91] and generalized in [92].

## 8.5 Non-relativistic string theory

The topic of non-relativistic string theory is an entire subject on its own, although closely related to that of the present review. Therefore, we refer the reader to a recent review article [93] that, besides the present review article, is part of a larger set of reviews related to modern developments in non-relativistic physics. Non-relativistic string theory on flat spacetime was first formulated in [94]. A natural question, which has only been recently addressed, motivated largely by the developments reviewed here, is the question of what type of curved target spacetime geometry non-relativistic strings move in. This study was initiated in [95–97], and we refer to the review [93] for a complete set of references. See also [98] for a derivation of type I torsional string Newton–Cartan (TSNC) geometry along similar lines as those presented in Section 5, using both the perspective of the limit and null reduction of relativistic string actions, as well as the gauging of the so-called fundamental string Galilei algebra. Moreover, the stringy analog of type II TNC, called type II TSNC, was recently discussed in [99, 100]. We also note that when quantizing non-relativistic string theory, one finds low-energy effective actions [101–104] similar to the type I TNC actions described in this review, though they involve different geometric fields.

## 8.6 Non-relativistic holography

Non-relativistic, or more generally non-Lorentzian, geometry and gravity play a role in non-AdS holography. In its original form, the AdS/CFT correspondence relates a relativistic bulk gravity to a corresponding dual (conformal) relativistic field theory living on the boundary. Beyond this, roughly three other classes of dualities have been found involving some types of non-Lorentzian geometry. The first is the appearance of non-Lorentzian geometry on the boundary. This was uncovered in the context of Lifshitz holography, which led to the discovery of a torsionful generalization of NC geometry [4–6]. The reason that NC geometry appears is that light cones open up as one approaches the boundary. This observation spurred many of the subsequent developments in non-relativistic geometry.

Additionally, it has been suggested that non-relativistic field theories have perhaps a more natural holographic realization with NRG theories in the bulk [84, 105–107]. This also seems to be the case for the holographic bulk duals of Spin Matrix Theory [108], which are quantum-mechanical theories obtained from near-BPS limits of AdS<sub>5</sub>/CFT<sub>4</sub>. On the string theory side, these

are described by novel non-relativistic worldsheet models [96, 109–112], which in the low-energy limit are expected to be described by dynamical non-Lorentzian gravity theories. Finally, we mention that there also exists an example [113] of a non-relativistic bulk gravity theory with a scale-invariant relativistic field theory on the boundary.

## 8.7 Supersymmetry

A natural generalization to consider is supersymmetric extensions of NRG. This is especially relevant in view of the connection of NC gravity with string theory and holography and using supersymmetric localization techniques for non-relativistic field theories. We refer to the recent review article in [114], which is also connected to the present review. Bergshoeff and Rosseel provided an overview of the different non-Lorentzian supergravity theories in diverse dimensions constructed in recent years.

## 8.8 Generalizations

Furthermore, we mention here several further generalizations involving some types of NRG. Hartong and Obers [49] showed that TNC geometry is a natural geometrical framework underlying Hořava–Lifshitz gravity with manifest diffeomorphism invariance. This connection was further studied in [52, 84]. A teleparallel version of NC gravity was considered in [115, 116], whereas a non-relativistic MacDowell–Mansouri type approach was considered in [117]. A generalization of NRG for arbitrary co-dimension foliation was presented in [118]. Two further generalizations include a non-relativistic version of spin-3 CS gravity [119, 120] and multi-metric gravity [121].

## 8.9 Field theory applications

Another point worth mentioning is that TNC geometry plays an important role as the natural background geometry [7, 46, 122, 123] in non-relativistic field theories, which are ubiquitous in condensed matter and biological systems. Further applications, involving energy-momentum tensors, Ward identities, hydrodynamics, and anomalies in the context of non-relativistic field theories, can be found in [58, 124–128]. Last but not least, the use of type II TNC is expected to be important to further examine the physics of non-relativistic quantum matter in non-trivial, non-relativistic spacetime geometry. Developing this framework could allow addressing new signals at low energies for quantum matter in gravitational backgrounds and studying composite systems with potentially measurable decoherence effects.

Overall, we conclude that the field of non-relativistic and generally non-Lorentzian gravity and its relation to field theory,

gravity, and string theory remains growing in scope. Each of the exciting research lines mentioned previously is expected to develop further in the coming years.

## Author contributions

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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## Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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## Supplementary material

The Supplementary Material for this article can be found online at: <https://www.frontiersin.org/articles/10.3389/fphy.2023.1116888/full#supplementary-material>

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