Spaces of Extremal Magnitude

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Abstract

Magnitude is a numerical invariant of compact metric spaces. Its theory is most mature for spaces satisfying the classical condition of being of negative type, and the magnitude of such a space lies in the interval $[1, \infty]$. Until now, no example with magnitude ∞ was known. We construct some, thus answering a question open since 2010. We also give a sufficient condition for the magnitude of a space to converge to 1 as it is scaled down to a point, unifying and generalizing previously known conditions.

1 Introduction

Magnitude is an invariant defined in the wide generality of enriched categories and specializing to an invariant of metric spaces. (See [10], or [14] for a survey and [12] for a bibliography.) It carries abundant geometric information. For example, for compact $X \subseteq \mathbb{R}^N$, consider the function assigning to each $t > 0$ the magnitude of the rescaled space tX. The large-scale asymptotics of this function determine the Minkowski dimension of X, its volume, and, under hypotheses, its surface area (Corollary 7.4 of [18], Theorem 1 of [4], and Theorem 2(d) of [6]). Magnitude is also closely related to certain measures of biodiversity, which themselves are essentially entropies ([13], [15] and Chapter 6 of [11]).

The definitions are as follows. For a finite metric space A, write Z_A for the matrix $(e^{-d(a,b)})_{a,b\in A}$. If Z_A is invertible, the **magnitude** $|A|$ of A is the sum of all the entries of Z_A^{-1}. A compact metric space X is **positive definite** if Z_A is positive definite for all finite $A \subseteq X$, and its **magnitude** $|X| \in [0, \infty]$ is then defined as $\sup\{|A| : \text{finite } A \subseteq X\}$. Positive definiteness ensures that this definition is consistent when X is finite and, as shown in [17], allows the theory to be developed satisfactorily.

A stronger condition is that tX is positive definite for all $t > 0$, where tX is shorthand for X equipped with the rescaled metric td_X; this is equivalent to the classical condition that X is of **negative type** ([17], Theorem 3.3). When X is a subset of a Banach space equipped with the subspace metric, we can equivalently instead consider tX to be the usual dilatation of X again equipped with the subspace metric.

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Until now, no example was known of a compact positive definite space with magnitude ∞. The question of whether such a space exists was first raised in the paper [17] (text preceding Lemma 2.1) posted to the arXiv in 2010 and published in 2013. It was raised again in [18] (after Definition 3.3), and once again in [14] (as Open Problem 5(1)).

In section 2 we construct a family of such spaces X. They are moreover of negative type, and we prove not only that $|X| = \infty$, but also that $|tX| = \infty$ for all $t > 0$.

A complementary question involves the behavior of the magnitude when a space shrinks to a point. The magnitude of any nonempty positive definite space lies in the interval $[1, \infty]$, with the lower bound achieved only by the one-point space. We say that a compact metric space X has the **one-point property** if $\lim_{t\to 0^+} |tX| = 1$. Example 2.2.8 of [10], due to Willerton, shows that even a finite space of negative type may fail to have this property. In section 3 we prove that a broad class of compact spaces of negative type do have the one-point property. Our result unifies and generalizes some previously known sufficient conditions, namely that X is isometric to a subset of $\mathbb{R}^N$ equipped with either the Euclidean metric or the metric induced by the 1-norm.

The main tool used to prove both of our main results is Theorem 4.6 in [14], which provides an upper bound, and frequently an exact formula, for the magnitude of a compact, convex subset of ℓ_1^N. Here ℓ_1^N denotes $\mathbb{R}^N$ equipped with the metric induced by the 1-norm. The new ingredient in both proofs is to combine the formula for magnitude in ℓ_1^N with finite-dimensional approximations in order to draw conclusions in the infinite-dimensional spaces ℓ_1 and L_1.

The ℓ_1 **intrinsic volumes** of a compact, convex set $A \subseteq \ell_1^N$ are defined by

$$V'_k(A) = \sum_{1\le i_1 < \cdots < i_k \le N} \mathrm{Vol}_k\big(\pi_{i_1,\ldots,i_k} A\big)$$

where $\pi_{i_1,\ldots,i_k}$ is orthogonal projection onto the subspace spanned by the standard basis vectors $e_{i_1}, \ldots, e_{i_k}$. These quantities were introduced in [9], where it was shown that there exists a version of integral geometry adapted to the 1-norm, with the ℓ_1 intrinsic volumes playing the role of the classical intrinsic volumes V_k in Euclidean integral geometry (see e.g. [8]). In fact, this ℓ_1 integral geometry is valid for the wider class of ℓ_1-convex sets (defined in [9]), as are some of the results in section 3 below; but for simplicity, we state our results for ordinary convex sets only.

The aforementioned Theorem 4.6 in [14] is the following.

Theorem 1.1 *If $A \subseteq \ell_1^N$ is compact and convex, then*

$$|A| \le \sum_{i=0}^{N} \frac{1}{2^i} V'_i(A) = \sum_{i=0}^{N} V'_i\big(\tfrac{1}{2}A\big), \tag{1}$$

with equality if A has nonempty interior.

We note that $\sum_{i=0}^N V'_i$ can also be considered an ℓ_1 analogue of the Wills functional $W = \sum_{i=0}^N V_i$ (see e.g. [2]).

2 Spaces with infinite magnitude

As usual, ℓ_1 denotes the space of real sequences (x_i) whose 1-norm $\sum |x_i|$ is finite, with the metric induced by the 1-norm. Write e_i for the ith standard basis vector $(0, \ldots, 0, 1, 0, \ldots)$ of ℓ_1 or ℓ_1^N.

Let (a_i) be a sequence of positive reals converging to 0, with $\sum a_i = \infty$. Denote by X the closed convex hull in ℓ_1 of $\{a_1e_1, a_2e_2, \ldots\}$, with the subspace metric. Equivalently,

$$X = \Big\{(x_1, x_2, \ldots) : x_i \geq 0, \ \sum x_i/a_i \leq 1\Big\}.$$

Theorem 2.1 *The metric space X is compact and of negative type, and $|tX| = \infty$ for all $t > 0$.*

Spaces similar to X, but with the ℓ_2 metric, have been studied in the geometry of Banach spaces (e.g. by Ball and Pajor [3]).

Proof First note that $\lim_{i\to\infty} a_ie_i = 0$, which implies that $\{a_ie_i : i \geq 1\} \cup \{0\}$ is compact and that its closed convex hull is X. But in a Banach space, the closed convex hull of a compact set is compact (Theorem 5.35 of [1]), so X is compact.

That X is of negative type is immediate, since ℓ_1 is of negative type (Theorem 3.6(2) of [17]).

It remains to prove that $|tX| = \infty$ for all $t > 0$. Since tX is of the same form as X, we may assume that $t = 1$.

For $N \geq 1$, write X_N for the convex hull of $\{a_1e_1, \ldots, a_Ne_N, 0\}$ in ℓ_1^N, with the subspace metric. Theorem 1.1 implies that

$$|X_N| \geq \tfrac{1}{2}V_1'(X_N) = \tfrac{1}{2}\sum_{i=1}^{N} a_i.$$

Now $\sum_{i=1}^{\infty} a_i = \infty$, so $|X_N| \to \infty$ as $N \to \infty$.

The standard isometry $\ell_1^N \hookrightarrow \ell_1$ restricts to an isometry $X_N \to X$ for every N. For compact positive definite spaces, magnitude is monotone with respect to inclusion, so $|X_N| \leq |X|$ for all N. Hence $|X| = \infty$. $\square$

Remark 2.2 If (a_i) is a sequence of positive reals such that $a_i \to 0$ but $\sum a_i < \infty$, then $|X| < \infty$. Indeed, X is a subspace of the infinite-dimensional box $Y = \prod_{i=1}^{\infty}[0, a_i]$ in ℓ_1, so

$$|X| \leq |Y| = \prod_{i=1}^{\infty}\big(1 + \tfrac{a_i}{2}\big) \leq e^{\sum a_i/2} < \infty,$$

as observed in Open Problem 5(1) of [14]. Thus, for 0-convergent sequences (a_i), the space X has finite magnitude if and only if the sum $\sum a_i$ is finite.

Spaces X of the class considered above clearly have the property that if X has finite magnitude, then its magnitude function is finite for every $t > 0$. This latter phenomenon holds in greater generality, as the following results show.

Proposition 2.3 *If A is a positive definite compact metric space and $n \in \mathbb{N}$, then nA is positive definite and $|nA| \le |A|^n$.*

Proof The map $x \mapsto (x, \ldots, x)$ is an isometric embedding $nA \hookrightarrow A^n$, where A^n is given the ℓ_1-sum metric. Therefore nA is positive definite and $|nA| \le |A^n| = |A|^n$, by Lemma 3.1.3 and Proposition 3.1.4 of [10]. □

Corollary 2.4 *Suppose that A is a compact and convex subset of a Banach space and is positive definite. Then A is of negative type, and $|A| < \infty$ if and only if $|tA| < \infty$ for every $t > 0$.*

Proof By translation we may assume that $0 \in A$, so by convexity $t_1 A \subseteq t_2 A$ whenever $0 \le t_1 \le t_2$, and in particular $tA \subseteq \lceil t \rceil A$ for every $t > 0$. By Proposition 2.3, $\lceil t \rceil A$ is positive definite and therefore tA is as well, and furthermore $|tA| \le |\lceil t \rceil A| \le |A|^{\lceil t \rceil} < \infty$. □

3 The one-point property

Write $L_1 = L_1[0, 1]$ for the Banach space of measurable functions $f : [0, 1] \to \mathbb{R}$ whose integral 1-norm $\int |f|$ is finite, with the metric induced by the 1-norm. We note that a separable Banach space is a positive definite metric space (equivalently, of negative type), with the metric induced by its norm, if and only if it is isometrically isomorphic to a subspace of L_1 (Corollary 3.5 in [17]). Examples include both ℓ_1^N and $\mathbb{R}^N$ with the Euclidean metric.

Our second main theorem is the following.

Theorem 3.1 *Suppose A is a nonempty compact subset of a finite-dimensional subspace of L_1. Then $|A| < \infty$ and A has the one-point property.*

The rest of this section is devoted to the proof.

The finiteness statement in Theorem 3.1 was previously proved (in a less elementary way) in Proposition 4.13 of [14], following special cases proved earlier in [10] and [17]. The one-point property for compact subsets of ℓ_1^N was first explicitly noted in Proposition 4.4 of [14] (but follows easily from results in [10]). Independent proofs of the one-point property for subsets of $\mathbb{R}^N$ were given in [4, 21, 19]. Theorem 3.1 simultaneously generalizes these facts. (In [19], it was further proved that so-called GB-bodies in a Hilbert space have finite magnitude and the one-point property, with a proof closely related to the proof of Theorem 3.1.)

The proof of Theorem 3.1 has three main ingredients: a classical approximation procedure that allows us to reduce consideration to subspaces of ℓ_1^N, a bound on magnitude in terms of V_1' which follows from Theorem 1.1, and a dimension-independent bound on V_1' for polytopes. Here, a **polytope** is the convex hull of a finite set.

Lemma 3.2 *If $A \subseteq \ell_1^N$ is convex, then*

$$(j+k)!V'_{j+k}(A) \leq \big(j!V'_j(A)\big)\big(k!V'_k(A)\big)$$

for each $j, k \geq 0$.

Proof By the definition of the ℓ_1 intrinsic volumes,

$$(j+k)!V'_{j+k}(A) = \sum_{i_1,\ldots,i_{j+k}=1}^{N} \mathrm{Vol}_{j+k}(\pi_{i_1,\ldots,i_{j+k}}(A)),$$

noting that if $i_1, \ldots, i_{j+k}$ are not all distinct then the corresponding summand vanishes. Hence

$$\begin{aligned}(j+k)!V'_{j+k}(A) &\leq \sum_{i_1,\ldots,i_{j+k}=1}^{N} \mathrm{Vol}_{j+k}\big(\pi_{i_1,\ldots,i_j}(A) \times \pi_{i_{j+1},\ldots,i_{j+k}}(A)\big) \\ &= \big(j!V'_j(A)\big)\big(k!V'_k(A)\big). \qquad \square\end{aligned}$$

The $j = 1$ case of Lemma 3.2 implies the following result by induction.

Proposition 3.3 *If $A \subseteq \ell_1^N$ is compact and convex, then*

$$V'_k(A) \leq \frac{1}{k!} V'_1(A)^k$$

for each $0 \leq k \leq N$.

Combining Proposition 3.3 and Theorem 1.1 we obtain the following.

Corollary 3.4 *If $A \subseteq \ell_1^N$ is compact and convex, then*

$$|A| \leq \exp(V'_1(\tfrac{1}{2}A)).$$

Remark 3.5 For the classical intrinsic volumes V_k, the estimate $V_k \leq \frac{1}{k!}V_1^k$ analogous to Proposition 3.3 was independently derived by Chevet (Lemme 4.2 in [5]) and McMullen (Theorem 2 in [16]) from the Alexandrov–Fenchel inequalities. As noted by McMullen, this implies the bound $W \leq \exp(V_1)$ on the Wills functional $W = \sum V_k$, analogous to Corollary 3.4.

Lemma 3.6 *Suppose that $P \subseteq \ell_1^N$ is a polytope with m vertices. Then*

$$V'_1(P) \leq 2(m-1)\,\mathrm{diam}(P),$$

where $\mathrm{diam}(P)$ *is the diameter of P in the ℓ_1 metric.*

Proof By translation, we may assume that one of the vertices of P is at the origin. We write $P = \mathrm{conv}\{v_1, \ldots, v_{m-1}, 0\}$, set $v_m = 0$, and denote $v_k = (v_k(1), \ldots, v_k(N))$. Then

$$\begin{aligned}V'_1(P) &= \sum_{i=1}^{N}\big(\max_k v_k(i) - \min_k v_k(i)\big) \leq \sum_{i=1}^{N} 2\max_k |v_k(i)| \\ &\leq 2\sum_{i=1}^{N}\sum_{k=1}^{m-1} |v_k(i)| = 2\sum_{k=1}^{m-1} \|v_k\|_1 \leq 2(m-1)\,\mathrm{diam}(P). \qquad \square\end{aligned}$$

Corollary 3.4 and Lemma 3.6 immediately imply the following.

Corollary 3.7 *If $P \subseteq \ell_1^N$ is a polytope with m vertices, then*

$$|P| \leq \exp((m-1)\,\mathrm{diam}(P)). \tag{2}$$

Remark 3.8 Theorem 6.2 of [9] and Proposition 3.3 together imply that

$$\mathrm{Vol}_N\big(P + [0,1]^N\big) = \sum_{k=0}^{N} V_k'(P) \leq \exp(V_1'(P))$$

for every polytope $P \subseteq \ell_1^N$. It follows from Lemma 3.6 that

$$\mathrm{Vol}_N\big(P + [0,1]^N\big) \leq \exp(2(m-1)\,\mathrm{diam}(P)),$$

where m is the number of vertices and the diameter is in the ℓ_1 metric. Despite the classical flavor of this estimate we have not seen it stated elsewhere.

Corollary 3.9 *If $P \subseteq L_1$ is the convex hull of m points, then*

$$|P| \leq \exp((m-1)\,\mathrm{diam}(P)).$$

Proof Let $E \subseteq L_1$ be the linear span of P. It is well known (e.g. [20], section 1) that E can be approximated in the Banach–Mazur distance by a sequence of subspaces $E_n \subseteq \ell_1^{N_n}$. It follows (as in section 3.A of [7]) that P is the limit, in the Gromov–Hausdorff distance, of a sequence of polytopes $P_n \subseteq \ell_1^{N_n}$, each with at most m vertices.

The magnitude of compact positive definite metric spaces is lower semicontinuous with respect to the Gromov–Hausdorff distance [17, Theorem 2.6], and diameter is continuous. By Corollary 3.7 we therefore have

$$|P| \leq \liminf_{n\to\infty} |P_n| \leq \liminf_{n\to\infty} e^{(m-1)\,\mathrm{diam}(P_n)} = e^{(m-1)\,\mathrm{diam}(P)}. \qquad \square$$

Proof of Theorem 3.1 Let P be a polytope lying in the linear span of A and containing A. Then for each $t > 0$,

$$1 \leq |tA| \leq |tP| \leq \exp\big((m-1)t\,\mathrm{diam}(P)\big),$$

where m is the number of vertices of P. The theorem follows. $\square$

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