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Citation for published version:

Cordero, FF, Kraus, M, Lin, G, Ruf, MS & Zeng, M 2023, 'Conservative Binary Dynamics with a Spinning Black Hole at $O(G^3)$ from Scattering Amplitudes', *Physical Review Letters*, vol. 130, no. 2, 021601, pp. 1-11. <https://doi.org/10.1103/PhysRevLett.130.021601>

Digital Object Identifier (DOI):

[10.1103/PhysRevLett.130.021601](https://doi.org/10.1103/PhysRevLett.130.021601)

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Publisher's PDF, also known as Version of record

Published In:

Physical Review Letters

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Conservative Binary Dynamics with a Spinning Black Hole at $\mathcal{O}(G^3)$ from Scattering Amplitudes


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 (Received 9 June 2022; revised 15 September 2022; accepted 29 November 2022; published 12 January 2023)

We compute the conservative two-body Hamiltonian of a compact binary system with a spinning black hole through $\mathcal{O}(G^3)$ to all orders in velocity, including linear and quadratic spin terms. To obtain our results we calculate the classical limit of the two-loop amplitude for the scattering of a massive scalar particle with a massive spin-1 particle minimally coupled to gravity. We employ modern scattering amplitude and loop integration techniques, in particular numerical unitarity, integration-by-parts identities, and the method of regions. The conservative potential in terms of rest-frame spin vectors is extracted by matching to a nonrelativistic effective field theory. We also apply the Kosower-Maybee-O’Connell (KMOC) formalism to calculate the impulse in the covariant spin formalism directly from the amplitude. We work systematically in conventional dimensional regularization and explicitly evaluate all divergent integrals that appear in full- and effective-theory amplitudes, as well as in the phase-space integrals that arise in the KMOC formalism.

DOI: [10.1103/PhysRevLett.130.021601](https://doi.org/10.1103/PhysRevLett.130.021601)

Introduction.—After the momentous observation of gravitational waves produced by the merging of a black hole pair [1], the LIGO-Virgo-KAGRA Collaboration continues to explore gravitational waves generated by compact binary systems. The systems so far detected [2] are mainly formed by a pair of black holes with masses in the range of 5 to 100 solar masses, although a handful of events have been associated to systems involving one or two neutron stars. Extracting such details about the binary systems that produced the detected gravitational waves hinges on our ability to precisely predict the structure of such signals based on the parameters of the systems, like for example the masses of the coalescent compact objects, especially in light of the next-generation gravitational wave detectors [3,4]. A critical theoretical input to produce these predictions is the conservative Hamiltonian of the two-body system.

Great efforts have been devoted to improve the predictions of binary dynamics through perturbative means. We work in the post-Minkowskian (PM) framework [5–14]

which expands in Newton’s constant while keeping the full dependence on velocity. Long considered difficult, high-order PM corrections to the conservative dynamics of a spinless binary system enjoyed recent progress using scattering amplitudes and their classical limits [15–17], in combination with effective field theory and advanced integration techniques. The computation of the conservative two-body dynamics at $\mathcal{O}(G^3)$ [18–23] and $\mathcal{O}(G^4)$ [24,25] for spinless binary systems has been completed [26].

Including spin effects is expected to be of high relevance in order to describe some of the detected binary systems (see, e.g., Ref. [2]). This is particularly the case due to known degeneracies between the mass ratio of the binary system and their spin, which leads to strongly biased parameter inference [41–44]. Therefore obtaining predictions for spin-dependent terms in the conservative potential is a necessity. Related results for up to fifth post-Newtonian (PN) order have appeared employing classical [45–64] and effective field theory methods [65–95]. In the PM approach, progress has been made at the first two PM orders [96–109]. At $\mathcal{O}(G^2)$, the scattering amplitudes approach has produced results up to quadratic-in-spin terms [110,111], up to quartic-in-spin terms [112], fifth power in spin [113], and even including all-orders-in-spin contributions [114,115]. Furthermore, recently, a study based on the worldline quantum field theory (QFT) formalism [116–119] has been

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presented on the impulse and spin kick in the scattering of two spinning compact objects including quadratic-in-spin contributions and up to order $\mathcal{O}(G^3)$ [120]. Proposals to express observables for spinning binaries through a generating function known as the eikonal exponent have appeared in Refs. [110,121].

For non-conservative dynamics it is useful to separate conservative and dissipative contributions motivated e.g., by a separation of timescales or as a means to organize contributions to physical observables. This split is not unique and multiple proposals are found in the literature. For the conservative sector we follow [14,25,122] and adopt a prescription based on a time-symmetric propagator for the graviton, thereby excluding contributions from radiation-reaction and in particular contributions from zero-frequency graviton exchanges. These effects conserve the total energy of the system but in general not the orbital angular momentum. In this context, to determine the dissipative contributions it is efficient to employ complementary approaches based on linear response relations or soft theorems (see e.g., Refs. [120] and [123] for recent applications involving spin.).

In this Letter we compute for the first time the conservative binary Hamiltonian with a spinning black hole up to $\mathcal{O}(G^3)$ including terms linear and quadratic in spin.

Scattering amplitudes.—We make use of the fact that $2 \rightarrow 2$ scattering amplitudes for processes involving spin- s massive particles can be used to fully characterize the conservative spinning dynamics through $2s$ powers of spin [124]. Therefore, we consider a theory of a massive scalar ϕ and a massive spin-1 (vector) field A_μ minimally coupled to gravity, which is described by the following Lagrangian:

$$\mathcal{L} = \sqrt{-g} \left[-\frac{2R}{\kappa^2} + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m_\phi^2 \phi^2 - \frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} F_{\mu\nu} + \frac{1}{2} m_A^2 g^{\mu\nu} A_\mu A_\nu \right], \quad (1)$$

where $\kappa = \sqrt{32\pi G}$, $g_{\mu\nu}$ is the metric tensor, $g = \det(g_{\mu\nu})$, R is the Ricci scalar associated to $g_{\mu\nu}$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Furthermore we work in the weak field approximation, i.e., $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, where $h_{\mu\nu}$ is the graviton field. We consider the elastic scattering process $A(p_1, \epsilon_1) + \phi(p_2) \rightarrow \phi(p_3) + A(p_4, \epsilon_4)$, where $p_1^2 = p_4^2 = m_A^2$, $p_2^2 = p_3^2 = m_\phi^2$, and $\epsilon_{1,4}$ are the polarization vectors of the incoming and outgoing vector particles. We work in the center-of-mass (c.m.) frame in which the momentum transfer takes the form $q^\mu = p_2^\mu - p_3^\mu = (0, \mathbf{q})$. Momenta are decomposed [125,126] as $p_1 = \bar{p}_1 - q/2$, $p_2 = \bar{p}_2 + q/2$, $p_3 = \bar{p}_2 - q/2$, $p_4 = \bar{p}_1 + q/2$, with $\bar{p}_i \cdot q = 0$. In the c.m. frame the spatial momenta read $\mathbf{p}_1 = -\mathbf{p}_2 = \bar{\mathbf{p}} - \mathbf{q}/2 \equiv \mathbf{p}$ and $E_{A,\phi} = \sqrt{m_{A,\phi}^2 + \mathbf{p}^2}$. We express the scattering amplitude in terms of the variables $\{m_A, m_\phi, \sigma, q^2\}$ with $\sigma = [(p_1 \cdot p_2)/m_A m_\phi]$.

We numerically compute the tree-level, one- and two-loop scattering amplitudes employing the multiloop numerical unitarity method [127–131], as implemented in the CARAVEL framework [132], using finite field arithmetic to allow for the reconstruction of corresponding analytic expressions [133,134]. This framework has already been used in the calculation of two-loop amplitudes in gravity [135] and we have extended it here to allow the handling of massive particles. We have also included the interaction vertices needed in our calculation from the Lagrangian in Eq. (1) with the help of the XACT [136,137] package.

We decompose the scattering amplitudes in terms of five form factors $\mathcal{M} = \sum_{n=1}^5 M_n T_n$, $T_n = \epsilon_{1,\mu} T_n^{\mu\nu} \epsilon_{4,\nu}^*$, with

$$\{T_1^{\mu\nu}, \dots, T_5^{\mu\nu}\} = \{\eta^{\mu\nu}, q^\mu q^\nu, q^2 \bar{p}_2^\mu \bar{p}_2^\nu, \bar{p}_2^{[\mu} q^{\nu]}, \bar{p}_2^{(\mu} q^{\nu)}\}, \quad (2)$$

where $a^{(\mu} b^{\nu)} = a^\mu b^\nu + a^\nu b^\mu$ and $a^{[\mu} b^{\nu]} = a^\mu b^\nu - a^\nu b^\mu$. The form factors M_n are obtained by computing five different helicity amplitudes and solving numerically for them. Parity invariance and crossing symmetry imply that only $M_{1,2,3,4}$ are present in the amplitude, though M_5 is present in the integrand before loop integration.

To perform the loop integration we employ integration-by-parts (IBP) identities [138], obtained with the FIRE6 [139] program. This step is highly simplified by employing finite field values for the kinematic invariants and truncating the IBP relations by taking into account the power counting of the contributions relevant to the classical result. After doing an univariate rational reconstruction of the master-integral coefficients in q^2 , we expand in a Laurent series all master integrals and their coefficients for small momentum transfer q^2 and in ϵ , using the results of Ref. [126] and keeping terms up to $\mathcal{O}((q^2)^0)$. The coefficients in this double expansion are yet-unknown rational functions in the variables $\{m_A, m_\phi, \sigma\}$. We then employ multiple numerical evaluations to perform a multivariate rational reconstruction [134]. We are able to reconstruct the analytic expressions for the relevant classical terms of all required amplitudes by employing modest computational resources. The resulting amplitudes are given in the Supplemental Material, which includes computer-readable files [140].

Effective field theory.—We employ effective field theory (EFT) techniques [16,110,141] to extract the two-body Hamiltonian for our system. We construct a nonrelativistic EFT with nonlocal contact interactions described by the Lagrangian

$$\begin{aligned} L_{\text{EFT}} = & \int_{\mathbf{k}} \hat{\phi}^\dagger(-\mathbf{k})(i\partial_t - \sqrt{\mathbf{k}^2 + m_\phi^2})\hat{\phi}(\mathbf{k}) \\ & + \int_{\mathbf{k}} \hat{A}^{\dagger,i}(-\mathbf{k})(i\partial_t - \sqrt{\mathbf{k}^2 + m_A^2})\hat{A}^i(\mathbf{k}) \\ & - \int_{\mathbf{k},\mathbf{k}'} \tilde{V}_{ij}(\mathbf{k},\mathbf{k}')\hat{A}^{\dagger,i}(\mathbf{k}')\hat{A}^j(\mathbf{k})\hat{\phi}^\dagger(-\mathbf{k}')\hat{\phi}(-\mathbf{k}), \quad (3) \end{aligned}$$

with $\int_{\mathbf{k}} \equiv \int [d^3\mathbf{k}/(2\pi)^3]$. The fields $\hat{\phi}$ and \hat{A} are defined in their respective rest frames. The potential \tilde{V}_{ij} can be decomposed into different spin operators according to

$$\tilde{V}_{ij}(\mathbf{k}, \mathbf{k}') = \sum_{n=1}^4 \tilde{V}^{(n)}(\mathbf{k}, \mathbf{k}') O_n^{ij} \left(\frac{\mathbf{k} + \mathbf{k}'}{2}, \mathbf{k}' - \mathbf{k} \right), \quad (4)$$

where the nonrelativistic tensor structures $O_n^{ij}(\bar{\mathbf{p}}, \mathbf{q})$ are

$$\{O_1^{ij}, \dots, O_4^{ij}\} = \{\delta^{ij}, \bar{\mathbf{p}}^i \mathbf{q}^j, \mathbf{q}^i \mathbf{q}^j, \mathbf{q}^2 \bar{\mathbf{p}}^i \bar{\mathbf{p}}^j\}. \quad (5)$$

To connect to the classical spin we relate the nonrelativistic tensor structures to spin operators using the representation $(\mathbf{S}_i)_{jk} = -i\epsilon_{ijk}$. We find

$$\begin{aligned} O_1 &= \mathbb{1}, & O_2 &= -i(\mathbf{q} \times \bar{\mathbf{p}}) \cdot \mathbf{S}, \\ O_3 &= \frac{1}{2} \mathbf{q}^2 \mathbf{S}^2 - (\mathbf{q} \cdot \mathbf{S})^2, \\ O_4 &= \mathbf{q}^2 \left(\frac{1}{2} \bar{\mathbf{p}}^2 \mathbf{S}^2 - (\bar{\mathbf{p}} \cdot \mathbf{S})^2 \right), \end{aligned} \quad (6)$$

where we defined $O_n \equiv O_n^{ij}(\bar{\mathbf{p}}, \mathbf{q}) \hat{A}^{\dagger, i}(\bar{\mathbf{p}} - \mathbf{q}/2) \hat{A}^j(\bar{\mathbf{p}} + \mathbf{q}/2)$. We ignore the Casimir operator $O_5 = \mathbf{q}^2 \mathbf{S}^2$ in the amplitude calculation for simplicity in this initial study, though we determine its coefficient in the potential by requiring consistency with the results of Ref. [120] in the aligned-spin limit [142]. Note that the polarization tensor representation of O_n^{ij} in Eq. (5) can be used within conventional dimensional regularization for computing divergent EFT amplitudes, while the spin operators in Eq. (6) are intrinsically three dimensional and are used to represent the finite potential obtained by EFT matching. We normalize the coefficients of the operators in Eq. (4) including dimensional regularization factors as

$$\begin{aligned} \tilde{V}^{(n)}(\mathbf{k}, \mathbf{k}') &= \frac{(4\pi)^{3/2-\epsilon}}{q^2} \sum_{L, Q=0}^{\infty} \left(\frac{\mu^2}{q^2} \right)^{L\epsilon} \frac{G^{L+1} |\mathbf{q}|^{L+Q}}{2^{(L+1)(1-2\epsilon)}} \\ &\times \frac{\Gamma[1 - \frac{1}{2}(1-2\epsilon)L]}{\Gamma[\frac{1}{2}(1-2\epsilon)(L+1)]} \tilde{c}_{L+1, Q}^{(n)}(\mathbf{k}^2), \end{aligned} \quad (7)$$

where $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ and μ is the dimensional regularization scale, while L denotes the loop order and Q the additional powers of \hbar beyond the classical limit. The unknown coefficients $\tilde{c}_{L, Q}^{(n)}(\mathbf{k}^2)$ are fixed by matching the EFT and full-theory amplitudes.

The EFT amplitude is given by iterated bubble diagrams [16], see Fig. 1. We start by decomposing the numerator of each diagram in terms of the operators O_n by explicitly constructing projectors. Then, we follow the procedure outlined in Ref. [16], i.e., we integrate out energy components of the loop momenta using the residue theorem. Then the propagators are expanded in the soft region $\ell \sim |\mathbf{q}|$, yielding the amplitude in terms of the *linearized*

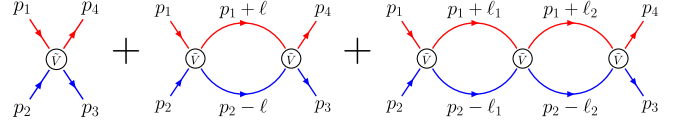


FIG. 1. The EFT amplitude is given by iterated bubble diagrams. Blue (red) lines refer to scalar (vector) particles. Each potential \tilde{V} has a perturbative expansion in terms of κ .

$(3 - 2\epsilon)$ -dimensional master integrals of Ref. [126]. We then obtain the L -loop EFT amplitude as an expansion in $|\mathbf{q}|$,

$$\mathcal{M}_{\text{EFT}}^{(L)} = \left(\frac{\kappa}{2} \right)^{2+2L} \left(\frac{\mu^2}{q^2} \right)^{L\epsilon} \sum_{n=1}^4 \sum_{k=0}^{\infty} \frac{d_{L, k}^{(n)}}{|\mathbf{q}|^{2-k}} O_n. \quad (8)$$

As explained in Refs. [102, 110] to perform the matching to the relativistic loop amplitude the relativistic tensor structures T_n have to be converted into nonrelativistic counterparts O_n . The necessary relations are given in the Supplemental Material [140].

Let us discuss the differences between the EFT matching procedure that we have employed and the one introduced in Ref. [16] and later used in Refs. [18, 19] at the two-loop order and in Ref. [110] for the case of spin. In these references, the integration of the relativistic amplitude is performed by an expansion and resummation in the velocity. In this process IR-divergent integrals are kept unevaluated and explicitly matched to the corresponding EFT integrals, therefore strictly speaking no dimensional regularization is required. However, this strategy relies on subtle arguments for the cancellation of evanescent effects, i.e., terms subleading in ϵ or $|\mathbf{q}|$ at lower loops being promoted to physically significant terms after multiplying divergent iteration integrals. Here we explicitly evaluate all such subleading terms in dimensional regularization in a transparent manner. This makes it natural to use dimensional regularization in a relativistic approach to conservative classical dynamics as introduced in Ref. [126] in conjunction with treating the EFT in dimensional regularization as described in this Letter.

Classical Hamiltonian.—The result of the EFT calculation described before are the matching coefficients $\tilde{c}_{L+1}^{(n)} = \tilde{c}_{L+1, 0}^{(n)}$. For the spin-orbit coupling we find

$$\tilde{c}_{L+1}^{(2)} = \tilde{c}_{L+1, \text{red}}^{(2)} + \tilde{c}_{L+1, \text{iter}}^{(2)} + \frac{\tilde{c}_{L+1, \text{red}}^{(1)}}{m_A^2 (\gamma_1 + 1)}, \quad (9)$$

with $\gamma_1 = E_A/m_A$. The coefficients $\tilde{c}_{L+1, \text{red}}^{(1)}$ are related to the scalar coupling [18, 19], while the $\tilde{c}_{L+1, \text{iter}}^{(2)}$ are given in terms of lower-order coefficients. We find

$$\begin{aligned} \tilde{c}_{1, \text{red}}^{(1)}(\mathbf{k}^2) &= \frac{m_A^2 m_\phi^2}{E^2 \xi} (1 - 2\sigma^2), \\ \tilde{c}_{2, \text{red}}^{(1)}(\mathbf{k}^2) &= \frac{3(m_\phi + m_A) m_\phi^2 m_A^2}{4E^2 \xi} (1 - 5\sigma^2), \end{aligned} \quad (10)$$

$$\begin{aligned} \tilde{c}_{3,\text{red}}^{(1)}(\mathbf{k}^2) &= \frac{m_A^2 m_\phi^2}{E^2 \xi} \left[-\frac{2}{3} m_A m_\phi \left(\frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}} (-12\sigma^4 + 36\sigma^2 + 9) + 22\sigma^3 - 19\sigma \right) - 2(m_\phi^2 + m_A^2)(6\sigma^2 + 1) \right] \\ &\quad + \frac{3Em_A^2 m_\phi^2}{4E^2 \xi} (m_A + m_\phi) \frac{(1 - 2\sigma^2)(1 - 5\sigma^2)}{(\sigma^2 - 1)} - \frac{3m_A^4 m_\phi^4}{E^2 \xi k^2}, \end{aligned} \quad (11)$$

$$\tilde{c}_{1,\text{red}}^{(2)}(\mathbf{k}^2) = -\frac{2\sigma m_\phi}{E\xi}, \quad \tilde{c}_{2,\text{red}}^{(2)}(\mathbf{k}^2) = \frac{m_\phi(4m_A + 3m_\phi)\sigma(5\sigma^2 - 3)}{4E\xi(\sigma^2 - 1)}, \quad (12)$$

$$\begin{aligned} \tilde{c}_{3,\text{red}}^{(2)}(\mathbf{k}^2) &= \frac{m_\phi}{E\xi(\sigma^2 - 1)^2} \left[-2m_A^2 \sigma(3 - 12\sigma^2 + 10\sigma^4) - \left(\frac{83}{6} + 27\sigma^2 - 52\sigma^4 + \frac{44}{3}\sigma^6 \right) m_A m_\phi - m_\phi^2 \sigma \left(\frac{7}{2} - 14\sigma^2 + 12\sigma^4 \right) \right. \\ &\quad \left. + \frac{(4m_A + 3m_\phi)E}{4} \sigma(2\sigma^2 - 1)(5\sigma^2 - 3) + 4m_A m_\phi \sigma(\sigma^2 - 6)(2\sigma^2 + 1) \sqrt{\sigma^2 - 1} \text{arccosh}(\sigma) \right], \end{aligned} \quad (13)$$

$$\tilde{c}_{1,\text{iter}}^{(2)}(\mathbf{k}^2) = 0, \quad \tilde{c}_{2,\text{iter}}^{(2)}(\mathbf{k}^2) = E\xi \tilde{c}_1^{(2)} \frac{\partial \tilde{c}_1^{(1)}}{\partial \mathbf{k}^2} + \tilde{c}_1^{(1)} \left(E\xi \frac{\partial \tilde{c}_1^{(2)}}{\partial \mathbf{k}^2} + \frac{\tilde{c}_1^{(2)} \left(\frac{2E^2 \xi}{k^2} + \frac{1}{\xi} - 3 \right)}{2E} \right), \quad (14)$$

$$\begin{aligned} \tilde{c}_{3,\text{iter}}^{(2)}(\mathbf{k}^2) &= (\tilde{c}_1^{(1)})^2 \left\{ -\frac{2}{3} E^2 \xi^2 \frac{\partial^2 \tilde{c}_1^{(2)}}{\partial (\mathbf{k}^2)^2} + \left[\xi \left(3 - \frac{E^2 \xi}{k^2} \right) - 1 \right] \frac{\partial \tilde{c}_1^{(2)}}{\partial \mathbf{k}^2} + \tilde{c}_1^{(2)} \left(\frac{1}{2\xi} - 2 + \frac{3\xi - 1}{k^2} \right) \right\} \\ &\quad + \tilde{c}_1^{(1)} \left\{ \tilde{c}_1^{(2)} \left[\left(-\frac{3E^2 \xi^2}{k^2} + 6\xi - 2 \right) \frac{\partial \tilde{c}_1^{(1)}}{\partial \mathbf{k}^2} - \frac{4}{3} E^2 \xi^2 \frac{\partial^2 \tilde{c}_1^{(1)}}{\partial (\mathbf{k}^2)^2} \right] \right. \\ &\quad \left. + \frac{4}{3} E\xi \left(\frac{\partial \tilde{c}_2^{(2)}}{\partial \mathbf{k}^2} - 2E\xi \frac{\partial \tilde{c}_1^{(1)}}{\partial \mathbf{k}^2} \frac{\partial \tilde{c}_1^{(2)}}{\partial \mathbf{k}^2} \right) + \frac{E^2 \xi^2 (\tilde{c}_1^{(2)})^2}{2k^2} + \tilde{c}_2^{(2)} \left(\frac{2}{3\xi} - 2 + \frac{E\xi}{k^2} \right) \right\} - \frac{1}{6} E^2 \xi^2 (\tilde{c}_1^{(2)})^3 \\ &\quad + \tilde{c}_1^{(2)} \left\{ \frac{2}{3} E\xi \left[\frac{\partial \tilde{c}_2^{(1)}}{\partial \mathbf{k}^2} - 2E\xi \left(\frac{\partial \tilde{c}_1^{(1)}}{\partial \mathbf{k}^2} \right)^2 \right] + \frac{\tilde{c}_2^{(1)} \left(\frac{3E^2 \xi}{k^2} + \frac{1}{\xi} - 3 \right)}{3E} \right\} + \frac{2}{3} E\xi \tilde{c}_2^{(1)} \frac{\partial \tilde{c}_1^{(2)}}{\partial \mathbf{k}^2} + \frac{4}{3} E\xi \tilde{c}_2^{(2)} \frac{\partial \tilde{c}_1^{(1)}}{\partial \mathbf{k}^2}, \end{aligned} \quad (15)$$

where $E = E_A + E_\phi$ and $\xi = E_A E_\phi / E^2$. These expressions represent our results for the spin-orbit term in the momentum-space potential in Eq. (7). The corresponding results for the quadratic-in-spin terms are given in the Supplemental Material [140]. Finally, we convert to position space and obtain the Hamiltonian by computing the Fourier transform of the potential in Eq. (7). The resulting Hamiltonian reads

$$\begin{aligned} H &= \sqrt{\mathbf{p}^2 + m_A^2} + \sqrt{\mathbf{p}^2 + m_\phi^2} + V^{(1)}(\mathbf{r}^2, \mathbf{p}^2) \\ &\quad + V^{(2)}(\mathbf{r}^2, \mathbf{p}^2) \frac{\mathbf{L} \cdot \mathbf{S}}{r^2} + V^{(3)}(\mathbf{r}^2, \mathbf{p}^2) \frac{(\mathbf{r} \cdot \mathbf{S})^2}{r^4} \\ &\quad + V^{(4)}(\mathbf{r}^2, \mathbf{p}^2) \frac{(\mathbf{p} \cdot \mathbf{S})^2}{r^2} + V^{(5)}(\mathbf{r}^2, \mathbf{p}^2) \frac{\mathbf{S}^2}{r^2}, \end{aligned} \quad (16)$$

where each coefficient is expanded as

$$V^{(n)}(\mathbf{r}^2, \mathbf{p}^2) = \sum_{L=1}^3 \left(\frac{G}{r} \right)^L c_L^{(n)}(\mathbf{p}^2) + \mathcal{O}(G^4). \quad (17)$$

The scalar potential $V^{(1)}(\mathbf{r}^2, \mathbf{p}^2)$ coincides with the results of Refs. [18,19]. The $c_L^{(n)}(\mathbf{p}^2)$ coefficients are provided in the Supplemental Material [140] as linear sums of the $\tilde{c}_{L,0}^{(n)}(\mathbf{p}^2)$ coefficients in Eq. (7).

Validation: We performed several checks of our computational framework in dimensional regularization. We checked that the EFT amplitudes reduce to their scalar counterparts given in Eq. (10.6) of Ref. [19] when setting $\tilde{c}_{L,Q}^{(n)} = 0$ for $n > 1$. Even more, we reproduced the two-loop matching coefficients of Refs. [18,19]. For the EFT calculation including spin, we cross checked our tree-level and one-loop matching coefficients against Refs. [111,143]. The scalar coefficients up to one loop are also in agreement with Ref. [144] to all orders in ϵ . Beyond one loop we checked explicitly the cancellation of the *superclassical* terms, i.e., unphysical terms behaving as powers of $1/\hbar \sim 1/|\mathbf{q}|$, where we observe a nontrivial interplay between the various matching coefficients. Furthermore the coefficients extracted in this way are finite in the nonrelativistic limit $|\mathbf{p}| \ll 1$ (this can be seen explicitly from the post-Newtonian expanded coefficients provided in the Supplemental Material [140]).

We solved the equation of motion based on the computed conservative potential to obtain the impulse in either the limit of an aligned spin or the limit of a spinless probe moving in a Kerr background. After translating to the covariant spin formalism using the procedure of Refs. [92,98], the impulse agrees with the recent worldline QFT result of Ref. [120] in either of the two above limits, when specializing to the Kerr

black hole case. The result is also consistent with the scattering angle in the probe limit obtained in Ref. [99]. Furthermore, the velocity expansion of our Hamiltonian is consistent with the next-to-next-to-leading order post-Newtonian results for spin-orbit and spin-squared coupling [80,81] calculated by the EFTofPNG package [145], after finding a canonical transformation [146].

Observables from the KMOC formalism.—We employ the full-theory amplitudes we obtained to compute classical gravitational observables using the Kosower-Maybee-O’Connell (KMOC) formalism [17]. In this formalism, a change of classical variable O , associated with any operator \mathbb{O} , between *in* and *out* states is measured as $\Delta O = \langle \text{out} | \mathbb{O} | \text{out} \rangle - \langle \text{in} | \mathbb{O} | \text{in} \rangle$. Relating *in* and *out* states by the S matrix allows us to reformulate the problem in terms of scattering amplitudes. In the classical limit ΔO is given by [17]

$$\Delta O = \int \frac{d^D q}{(2\pi)^{D-2}} \delta(-2p_1 \cdot q) \delta(2p_2 \cdot q) e^{ib \cdot q} (\mathcal{I}_{O,v} + \mathcal{I}_{O,r}) \quad (18)$$

where $\mathcal{I}_{O,v}$ is the *virtual* kernel obtained from virtual amplitudes, while $\mathcal{I}_{O,r}$ is the *real* kernel depending on cut amplitudes with phase-space integration.

We begin with obtaining kernels \mathcal{I}_v and \mathcal{I}_r in order to compute the momentum impulse Δp_i (corresponding to the momentum operator \mathbb{P}_i). The spinless case has been discussed in Ref. [147] and we highlight new features introduced by polarizations and spins here. The *virtual* kernel $\mathcal{I}_{\Delta p,v}^\mu$ is simply $q^\mu \mathcal{M}$, which can be expanded with the tensors in Eq. (2). For the conservative dynamics we only need to consider two-particle cuts and the real kernel takes the form

$$\mathcal{I}_{\Delta p,r}^\mu = \int d\Phi_2 \sum_{\epsilon_c} \ell^\mu \mathcal{M}_L \mathcal{M}_R^*, \quad (19)$$

where $d\Phi_2$ is the two-body phase-space measure and \mathcal{M}_R and \mathcal{M}_L^* are amplitudes for the processes $A(p_1, \epsilon_1) + \phi(p_2) \rightarrow \phi(p_1 + \ell) + A(p_2 - \ell, \epsilon_c)$, and $\phi(p_1 + \ell) + A(p_2 - \ell, \epsilon_c) \rightarrow \phi(p_3) + A(p_4, \epsilon_4)$, respectively. The sum runs over the polarizations of the vector particles in the cut. These one-loop integrals can be evaluated directly through subloop integration, for details see Ref. [21]. To expand the double cut in terms of form factors, we need the additional tensor structures

$$\{T_6^{\mu\nu}, \dots, T_9^{\mu\nu}\} = \{\bar{p}_2^{[\mu} n^{\nu]}, \bar{p}_2^{(\mu} n^{\nu)}, q^{[\mu} n^{\nu]}, q^{(\mu} n^{\nu)}\}, \quad (20)$$

where $n^\mu = \epsilon^{\mu\nu\sigma\rho} q_\nu p_{1,\sigma} p_{2,\rho}$. After combining \mathcal{I}_r and \mathcal{I}_v , we find that all superclassical terms cancel.

The next step is to translate the polarization vectors into spin vectors. Following Ref. [148], we first boost the

polarization vector ϵ_4^* into the frame of p_1 . With the Pauli-Lubanski operator $\mathbb{W}^\mu = \epsilon^{\mu\nu\sigma\rho} \mathbb{P}_\nu \mathbb{J}_{\rho\sigma} / 2$, we define the spin vector s_1^μ as the expectation value of \mathbb{W}_1^μ / m_1 . It is straightforward to convert from ϵ_1 to s_1 given $s_1^\mu = i \epsilon^{\mu\nu\rho\sigma} p_{1,\nu} \epsilon_\rho(p_1) \epsilon_\sigma^*(p_1) / m_1$. Finally, we perform the Fourier transform from the momentum space to the impact parameter space as in Eq. (18). We compare our momentum impulse results up to $\mathcal{O}(G^3)$ with Refs. [92,120] and find agreement in the Kerr black hole case. This serves as yet another check on the correctness of our amplitude and confirms the practical value of the KMOC formalism at nontrivial PM orders beyond the spinless case [21,147]. There is no obstruction to computing other observables such as the spin kick using the KMOC formalism, which we leave to future work.

Conclusions.—In this Letter we have presented the conservative two-body Hamiltonian for a compact binary system with a spinning black hole at $\mathcal{O}(G^3)$ with exact velocity dependence and including terms linear and quadratic in spin. Though obtained from scattering, the Hamiltonian is directly applicable to bound orbits. This is especially valuable since analytic continuation for observables in these two kinds of orbits, through the boundary-to-bound map [149–151], has not been worked out for generic spin configurations. The computation employs powerful modern scattering amplitude techniques, including numerical unitarity, integration-by-parts identities, expansion-by-regions and functional reconstruction algorithms.

We have shown that the effective field theory approach to study the binary dynamics with spin [110,124] can be applied in complex high-order calculations. Furthermore, we have performed matching calculations for the effective field theory of Ref. [16] at the two-loop level, for both the spinless case and spinning case, fully within dimensional regularization for the first time. We have also demonstrated the power of the KMOC formalism beyond the lowest order in G for spinning binary dynamics.

There are many directions to extend the applicability of our framework. For instance, one can study Hamiltonian terms with higher powers in spin by considering minimally coupled massive higher-spin particles [103,124,152]. Also including finite-size [153–160] and radiation [123,161–180] effects is an interesting possibility of relevance for phenomenological studies in gravitational wave detectors. Finally, our framework can be extended beyond the third-post-Minkowskian order following progress in the spinless case [24,25,28,29].

We thank Harald Ita for collaboration during the initial phase of the project. We thank Jan Steinhoff and Justin Vines for their expertise and timely help in checking the consistency of our Hamiltonian with known post-Newtonian results. We thank Andres Luna, Chia-Hsien Shen, Dimitrios Kosmopoulos, Fei Teng, Justin Vines, Radu Roiban, and Zvi Bern for insightful discussions

and/or comments on the manuscript. The work of F. F. C. and M. K. is supported in part by the U.S. Department of Energy under Grant No. DE-SC0010102. G.L. is supported in part by the National Natural Science Foundation of China Grant No. 11935013. G.L. thanks the Higgs Centre for Theoretical Physics at the University of Edinburgh for the Visiting Researcher Scheme. M. Z.'s work is supported in part by the U.K. Royal Society through Grant URF/R1\20109. M. S. R. thanks the Higgs Centre for Theoretical Physics at the University of Edinburgh for hospitality and the Mani L. Bhaumik Institute for generous support. The computing for this project was performed on the HPC cluster at the Research Computing Center at the Florida State University (FSU). For the purpose of open access, the author have applied a Creative Commons Attribution (CC BY) license to any Author Accepted Manuscript version arising from this submission.

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