



THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

Fourier-Mukai Transforms and Bridgeland Stability Conditions on Abelian Threefolds

Citation for published version:

Maciocia, A & Piyaratne, D 2015, 'Fourier-Mukai Transforms and Bridgeland Stability Conditions on Abelian Threefolds', *Algebraic Geometry*, vol. 2, no. 3, pp. 270-297. <https://doi.org/10.14231/AG-2015-012>

Digital Object Identifier (DOI):

[10.14231/AG-2015-012](https://doi.org/10.14231/AG-2015-012)

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Early version, also known as pre-print

Published In:

Algebraic Geometry

General rights

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact openaccess@ed.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.



FOURIER-MUKAI TRANSFORMS AND BRIDGELAND STABILITY CONDITIONS ON ABELIAN THREEFOLDS

ANTONY MACIOCIA & DULIP PIYARATNE

ABSTRACT. We use the ideas of Bayer, Bertram, Macrì and Toda to construct a Bridgeland stability condition on a principally polarized abelian threefold (X, L) with $\text{NS}(X) = \mathbb{Z}[\ell]$ by establishing their Bogomolov-Gieseker type inequality for certain tilt stable objects associated to the pair $(\mathcal{A}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}, \mathcal{Z}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell})$ on X . This is done by proving the stronger result that $\mathcal{A}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}$ is preserved by a suitable Fourier-Mukai transform.

INTRODUCTION

In [5] Bridgeland introduced the notion of stability conditions on triangulated categories and these now have many applications to the study of the geometry of the underlying spaces and highlight the role played by the derived categories of the suitable categories of sheaves on the spaces. The space of stability conditions is known precisely for curves and for abelian surfaces and Bridgeland's geometric stability conditions provide examples for all projective surfaces (see, for example, [6], [11], [1], [15]). A conjectural construction of Bridgeland stability conditions for projective threefolds was introduced in [4] and the problem is reduced to proving an inequality, which the authors call a Bogomolov-Gieseker (B-G for short) type inequality, holds for certain tilt stable objects. This inequality has been shown to hold for three dimensional projective space (see [4] and [12]) and some progress has been made for more general threefolds (see [17] and [9]). However, there is no known example of a stability condition on a projective Calabi-Yau threefold and this case is especially significant because of the interest from Mathematical Physics and also in connection with Donaldson-Thomas invariants. In this paper, we establish the existence of a particular stability condition on a particular Calabi-Yau threefold (namely, a principally polarized abelian threefold with Picard rank one). However, it is likely that the method will generalize to other Calabi-Yau threefolds while the extension to other stability conditions for the abelian threefold case will be the subject of a forthcoming article.

We reduce the requirement of the B-G type inequality to a smaller class of tilt stable objects as defined in the Definition 2.2. Moreover, they are essentially minimal objects (also called simple objects in the literature) of the heart of the stability condition. In this paper we use Fourier-Mukai theory to prove the B-G type inequality for these minimal objects by showing that the heart is preserved by a suitable Fourier-Mukai transform (or FMT for short). For the surface case, the fact that a countable family of (Bridgeland's) geometric stability conditions satisfy the numerical conditions for being a stability condition is actually equivalent to the

Date: September 2, 2013.

2010 Mathematics Subject Classification. Primary 14F05; Secondary 14J30, 14J32, 14J60, 14K99, 18E30, 18E35, 18E40.

Key words and phrases. Bridgeland stability conditions, Fourier-Mukai transforms, Abelian threefolds, Bogomolov-Gieseker inequality.

existence of a Fourier-Mukai transform preserving the heart. The forward implication was proved by Huybrechts ([8]) and the reverse implication is a fairly straightforward exercise (partly done in [18]). For the threefold case, we build on these ideas to establish the reverse implication for our case.

Throughout this paper our abelian varieties will be principally polarized abelian varieties with Picard rank one over \mathbb{C} . Let (X, L) be an abelian variety of dimension three and let ℓ be $c_1(L)$. We use L to canonically identify X with $\mathbf{Pic}^0(X)$. Let $\Phi : D^b(X) \rightarrow D^b(X)$ be the (classical) FMT with the Poincaré line bundle on $X \times X$ as the kernel. Then the image of the category $\mathrm{Coh}(X)$ under the FMT Φ is a subcategory of $D^b(X)$ with non-zero $\mathrm{Coh}(X)$ -cohomologies in 0, 1, 2 and 3 positions.

In [4] and [2], the authors construct their conjectural stability condition hearts as a tilt of a tilt. The first tilt of $\mathrm{Coh}(X)$ associated to the Harder-Narasimhan (or H-N for short) filtration with respect to the twisted slope $\mu_{\omega, B}$ stability is denoted $\mathcal{B}_{\omega, B}$ and the second $\mathcal{A}_{\omega, B}$ associated to the H-N filtration with respect to the tilt slope $\nu_{\omega, B}$ stability. We shall consider the particular case where $\omega = \sqrt{3}\ell/2$ and $B = \ell/2$. Let $\Psi := L\Phi$ and $\widehat{\Psi} := \Phi L^{-1}[1]$. Then we prove the images of the abelian category $\mathcal{B}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}$ under the Fourier-Mukai transforms Ψ and $\widehat{\Psi}$ have non-zero $\mathcal{B}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}$ -cohomologies only in positions 0, 1 and 2 (see Theorem 4.19). On the other hand, we have the isomorphisms ([14])

$$\Psi \circ \widehat{\Psi} \cong (-1)^* \mathrm{id}_{D^b(X)}[-2], \text{ and } \widehat{\Psi} \circ \Psi \cong (-1)^* \mathrm{id}_{D^b(X)}[-2].$$

Therefore the abelian category $\mathcal{B}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}$ behaves somewhat similarly to the category of coherent sheaves on an abelian surface under the Fourier-Mukai transform (see [3], [10], [18] for further details). For us, the key technical tool is to restrict our sheaves to certain families of curves and this provides us with the link to the geometry of the underlying space (see Lemma 4.12). Now Theorem 4.19 becomes the key technical tool to show that the second tilt $\mathcal{A}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}$ is preserved by Ψ .

Under this auto-equivalence, minimal objects are mapped to minimal objects and this provides us with an inequality which bounds the top component of the Chern character of the object. This is the main idea to show that the B-G type inequality is satisfied by our restricted class of minimal objects in $\mathcal{A}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}$. In section 5, we have to show that the B-G type inequality is satisfied by a very special class of minimal objects by showing that they actually do not exist. This result is of interest in its own right as it shows that if a bundle E of such a threefold satisfies $c_1(E) = 0 = c_2(E)$ then it cannot carry a non-flat Hermitian-Einstein connection.

NOTATION

- (i) For $0 \leq i \leq \dim X$

$$\mathrm{Coh}^{\leq i}(X) := \{E \in \mathrm{Coh}(X) : \dim \mathrm{Supp}(E) \leq i\},$$

$$\mathrm{Coh}^{\geq i}(X) := \{E \in \mathrm{Coh}(X) : \text{for } 0 \neq F \subset E, \dim \mathrm{Supp}(F) \geq i\}, \text{ and}$$

$$\mathrm{Coh}^i(X) := \mathrm{Coh}^{\leq i}(X) \cap \mathrm{Coh}^{\geq i}(X).$$

- (ii) For an interval $I \subset \mathbb{R} \cup \{+\infty\}$, $\mathrm{HN}_{\omega, B}^{\mu}(I) := \{E \in \mathrm{Coh}(X) : [\mu_{\omega, B}^-(E), \mu_{\omega, B}^+(E)] \subset I\}$. Similarly the subcategory $\mathrm{HN}_{\omega, B}^{\nu}(I) \subset \mathcal{B}_{\omega, B}$ is defined.

- (iii) For a FMT Υ and a heart \mathfrak{A} of a t-structure for which $D^b(X) \cong D^b(\mathfrak{A})$, $\Upsilon_{\mathfrak{A}}^k(E) := H_{\mathfrak{A}}^k(\Upsilon(E))$.
- (iv) For a sequence of integers i_1, \dots, i_s ,

$$V_{\mathfrak{A}}^{\Upsilon}(i_1, \dots, i_s) := \{E \in D^b(X) : \Upsilon_{\mathfrak{A}}^j(E) = 0 \text{ for } j \notin \{i_1, \dots, i_s\}\}.$$

Then $E \in \text{Coh}(X)$ being Υ -WIT $_i$ is equivalent to $E \in V_{\text{Coh}(X)}^{\Upsilon}(i)$.

- (v) Let (X, L) be a principally polarized abelian variety. Then we write Φ for the FMT from X to X with the Poincaré line bundle $\mathcal{P} := m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$ on $X \times X$ as the kernel.
- (vi) For $E \in \text{Coh}(X)$, $E^k := \Phi_{\text{Coh}(X)}^k(E)$.
- (vii) $\Psi := L\Phi$ and $\widehat{\Psi} := \Phi L^{-1}[1]$. Here and elsewhere we abuse notation to write L for the functor $L \otimes -$.
- (viii) For a polarized projective threefold (X, L) with Picard rank 1 over \mathbb{C} , the Chern character of E is $\text{ch}(E) = (a_0, a_1\ell, a_2\frac{\ell^2}{2}, a_3\frac{\ell^3}{6})$ for some $a_i \in \mathbb{Q}$. For simplicity we write $\text{ch}(E) = (a_0, a_1, a_2, a_3)$. Here $a_i \in \mathbb{Z}$ for the principally polarized abelian threefold case.

1. PRELIMINARIES

1.1. Construction of stability conditions. We recall the conjectural construction of stability conditions as introduced in [4].

Let X be a smooth projective threefold over \mathbb{C} . Let ω, B be in $\text{NS}_{\mathbb{R}}(X)$ with ω an ample class. The twisted Chern character ch^B with respect to B is defined by $\text{ch}^B(-) = e^{-B} \text{ch}(-)$. So we have

$$\begin{aligned} \text{ch}_0^B &= \text{ch}_0, & \text{ch}_1^B &= \text{ch}_1 - B \text{ch}_0, \\ \text{ch}_2^B &= \text{ch}_2 - B \text{ch}_1 + \frac{B^2}{2} \text{ch}_0, & \text{ch}_3^B &= \text{ch}_3 - B \text{ch}_2 + \frac{B^2}{2} \text{ch}_1 - \frac{B^3}{6} \text{ch}_0. \end{aligned}$$

The twisted slope $\mu_{\omega, B}$ on $\text{Coh}(X)$ is defined by

$$\mu_{\omega, B}(E) = \begin{cases} +\infty & \text{if } E \text{ is a torsion sheaf} \\ \frac{\omega^2 \text{ch}_1^B(E)}{\text{ch}_0^B(E)} & \text{otherwise} \end{cases}$$

for $E \in \text{Coh}(X)$. Then E is said to be $\mu_{\omega, B}$ -(semi)stable, if for any $0 \neq F \subsetneq E$, we have $\mu_{\omega, B}(F) < (\leq) \mu_{\omega, B}(E/F)$. The H-N filtration of E with respect to $\mu_{\omega, B}$ -stability enables us to define the following slopes:

$$\mu_{\omega, B}^+(E) = \max_{0 \neq G \subsetneq E} \mu_{\omega, B}(G), \quad \mu_{\omega, B}^-(E) = \min_{G \subsetneq E} \mu_{\omega, B}(E/G).$$

For an interval $I \subset \mathbb{R} \cup \{+\infty\}$, the subcategory $\text{HN}_{\omega, B}^{\mu}(I) \subset \text{Coh}(X)$ is defined by

$$\text{HN}_{\omega, B}^{\mu}(I) = \{E \in \text{Coh}(X) : [\mu_{\omega, B}^-(E), \mu_{\omega, B}^+(E)] \subset I\}.$$

Define the subcategories $\mathcal{T}_{\omega, B}$ and $\mathcal{F}_{\omega, B}$ of $\text{Coh}(X)$ by setting

$$\mathcal{T}_{\omega, B} = \text{HN}_{\omega, B}^{\mu}(0, +\infty], \quad \mathcal{F}_{\omega, B} = \text{HN}_{\omega, B}^{\mu}(-\infty, 0].$$

Then $(\mathcal{T}_{\omega, B}, \mathcal{F}_{\omega, B})$ forms a torsion pair on $\text{Coh}(X)$. Let the abelian category $\mathcal{B}_{\omega, B} = \langle \mathcal{F}_{\omega, B}[1], \mathcal{T}_{\omega, B} \rangle \subset D^b(X)$ be the corresponding tilt of $\text{Coh}(X)$.

The central charge function $Z_{\omega,B} : K(X) \rightarrow \mathbb{C}$ is defined by

$$Z_{\omega,B}(E) = - \int_X e^{-B-i\omega} \text{ch}(E).$$

So $Z_{\omega,B}(E) = \left(-\text{ch}_3^B(E) + \frac{\omega^2}{2} \text{ch}_1^B(E)\right) + i \left(\omega \text{ch}_2^B(E) - \frac{\omega^3}{6} \text{ch}_0^B(E)\right)$. The following result is very useful:

Lemma 1.1. [4, Lemma 3.2.1] *For any $0 \neq E \in \mathcal{B}_{\omega,B}$, one of the following conditions holds:*

- (i) $\omega^2 \text{ch}_1^B(E) > 0$,
- (ii) $\omega^2 \text{ch}_1^B(E) = 0$ and $\Im Z_{\omega,B}(E) > 0$,
- (iii) $\omega^2 \text{ch}_1^B(E) = \Im Z_{\omega,B}(E) = 0$, $-\Re Z_{\omega,B}(E) > 0$ and $E \cong T$ for some $0 \neq T \in \text{Coh}^0(X)$.

As a result of this Lemma, they go on to remark that the vector $(\omega^2 \text{ch}_1^B, \Im Z_{\omega,B}, -\Re Z_{\omega,B})$ for objects in $\mathcal{B}_{\omega,B}$ behaves like the Chern character vector $\text{ch} = (\text{ch}_0, \text{ch}_1, \text{ch}_2)$ for coherent sheaves on a surface.

Following [4], the tilt-slope $\nu_{\omega,B}$ on $\mathcal{B}_{\omega,B}$ is defined by

$$\nu_{\omega,B}(E) = \begin{cases} +\infty & \text{if } \omega^2 \text{ch}_1^B(E) = 0 \\ \frac{\Im Z_{\omega,B}(E)}{\omega^2 \text{ch}_1^B(E)} & \text{otherwise} \end{cases}$$

for $E \in \mathcal{B}_{\omega,B}$. Then E is said to be $\nu_{\omega,B}$ -(semi)stable, if for any $0 \neq F \subsetneq E$ in $\mathcal{B}_{\omega,B}$, we have $\nu_{\omega,B}(F) < (\leq) \nu_{\omega,B}(E/F)$. In [4] it is proved that the abelian category $\mathcal{B}_{\omega,B}$ satisfies the H-N property with respect to the tilt-slope stability. So the following slopes can be defined for $E \in \mathcal{B}_{\omega,B}$:

$$\nu_{\omega,B}^+(E) = \max_{0 \neq G \subseteq E} \nu_{\omega,B}(G), \quad \nu_{\omega,B}^-(E) = \min_{G \subsetneq E} \nu_{\omega,B}(E/G).$$

For an interval $I \subset \mathbb{R} \cup \{+\infty\}$, the subcategory $\text{HN}_{\omega,B}^\nu(I) \subset \mathcal{B}_{\omega,B}$ is defined by

$$\text{HN}_{\omega,B}^\nu(I) = \{E \in \mathcal{B}_{\omega,B} : [\nu_{\omega,B}^-(E), \nu_{\omega,B}^+(E)] \subset I\}.$$

Define the subcategories $\mathcal{T}'_{\omega,B}$ and $\mathcal{F}'_{\omega,B}$ of $\mathcal{B}_{\omega,B}$ by setting

$$\mathcal{T}'_{\omega,B} = \text{HN}_{\omega,B}^\nu(0, +\infty], \quad \mathcal{F}'_{\omega,B} = \text{HN}_{\omega,B}^\nu(-\infty, 0].$$

Then $(\mathcal{T}'_{\omega,B}, \mathcal{F}'_{\omega,B})$ forms a torsion pair on $\mathcal{B}_{\omega,B}$. Let the abelian category $\mathcal{A}_{\omega,B} = \langle \mathcal{F}'_{\omega,B}[1], \mathcal{T}'_{\omega,B} \rangle \subset D^b(X)$ be the corresponding tilt of $\mathcal{B}_{\omega,B}$.

Conjecture 1.2. [4, Conjecture 3.2.6] *The pair $(Z_{\omega,B}, \mathcal{A}_{\omega,B})$ is a Bridgeland stability condition on $D^b(X)$.*

Definition 1.3. *Let $\mathcal{C}_{\omega,B}$ be the class of $\nu_{\omega,B}$ -stable objects $E \in \mathcal{B}_{\omega,B}$ with $\nu_{\omega,B}(E) = 0$.*

Then $E[1] \in \mathcal{A}_{\omega,B}$ for any $E \in \mathcal{C}_{\omega,B}$.

Conjecture 1.4. [4, Conjecture 3.2.7] *Any $E \in \mathcal{C}_{\omega,B}$ satisfies the so called **Bogomolov-Gieseker Type Inequality**:*

$$\Re Z_{\omega,B}(E[1]) < 0, \text{ i.e. } \text{ch}_3^B(E) < \frac{\omega^2}{2} \text{ch}_1^B(E).$$

Assume $B \in \text{NS}_{\mathbb{Q}}(X)$ and $\omega \in \text{NS}_{\mathbb{R}}(X)$ be an ample class with ω^2 is rational. Then the abelian category $\mathcal{A}_{\omega,B}$ satisfies the following important property. This was proved for rational classes ω in [4]. However a similar proof can be used when we have a weaker condition, namely ω^2 is rational. For example, a different parametrisation given by $\omega \mapsto \sqrt{3}\omega$ is considered in [12].

Lemma 1.5. [4, Proposition 5.2.2] *The abelian category $\mathcal{A}_{\omega,B}$ is Noetherian.*

As a corollary we have the following

Corollary 1.6. [4, Corollary 5.2.4] *The Conjectures 1.2 and 1.4 are equivalent.*

1.2. Fourier-Mukai transforms on abelian threefolds. Let us quickly recall the notion of Fourier-Mukai transform on abelian threefolds. See [3], [7] for further details on Fourier-Mukai theory.

Let (X, L) be a principally polarized abelian threefold with Picard rank 1. Let $\ell := c_1(L)$. Then $\chi(L) = \frac{\ell^3}{6} = 1$. Let $\mathcal{P} = m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$ be the Poincaré line bundle on $X \times X$. Then the Fourier-Mukai transform $\Phi : D^b(X) \rightarrow D^b(X)$ with kernel \mathcal{P} is defined by

$$\Phi(-) := \mathbf{R}p_{2*}(\mathcal{P} \otimes^{\mathbf{L}} p_1^*(-)).$$

Here $X \xleftarrow{p_1} X \times X \xrightarrow{p_2} X$ are the projection maps. In [14] Mukai proved that Φ is an auto-equivalence of the derived category $D^b(X)$ and also

$$\Phi \circ \Phi \cong (-1)^* \text{id}_{D^b(X)}[-3].$$

The Chern character of any $E \in D^b(X)$ is of the form $\text{ch}(E) = (a_0, a_1\ell, a_2\frac{\ell^2}{2}, a_3\frac{\ell^3}{6})$ for some integers a_i . Then we have (see [7, Lemma 9.23]):

$$\text{ch}(\Phi(E)) = (a_3, -a_2\ell, a_1\frac{\ell^2}{2}, -a_0\frac{\ell^3}{6}).$$

2. MINIMAL OBJECTS OF $\mathcal{A}_{\omega,B}$ AND B-G TYPE INEQUALITY OF THREEFOLDS

2.1. Some minimal objects of $\mathcal{A}_{\omega,B}$. We identify some classes of minimal objects of the abelian category $\mathcal{A}_{\omega,B}$ of a projective threefold X . See [8] for a detailed discussion on minimal objects of some abelian categories associated to Bridgeland stability conditions on a surface.

Proposition 2.1. *For any $x \in X$, the skyscraper sheaf \mathcal{O}_x is a minimal object in $\mathcal{A}_{\omega,B}$.*

Proof. For any $x \in X$, $\mathcal{O}_x \in \mathcal{T}_{\omega,B}$ and also $\mathcal{O}_x \in \mathcal{T}'_{\omega,B}$. Therefore $\mathcal{O}_x \in \mathcal{A}_{\omega,B}$. Let

$$0 \rightarrow a \rightarrow \mathcal{O}_x \rightarrow b \rightarrow 0$$

be a short exact sequence (SES for short) in $\mathcal{A}_{\omega,B}$ such that $a \neq 0$. Then in order to prove $\mathcal{O}_x \in \mathcal{A}_{\omega,B}$ is minimal, we need to show $b = 0$. We obtain the following long exact sequence (LES for short) of $\mathcal{B}_{\omega,B}$ -cohomologies associated to the above $\mathcal{A}_{\omega,B}$ -SES:

$$0 \rightarrow A_{-1} \rightarrow 0 \rightarrow B_{-1} \rightarrow A_0 \rightarrow \mathcal{O}_x \rightarrow B_0 \rightarrow 0.$$

Here $A_k := H_{\mathcal{B}_{\omega,B}}^k(a)$ and $B_k := H_{\mathcal{B}_{\omega,B}}^k(b)$. We have $A_{-1} = 0$ and so $a \cong A_0 \neq 0$. Let $C := A_0/B_{-1}$. Then

$$0 \rightarrow C \rightarrow \mathcal{O}_x \rightarrow B_0 \rightarrow 0$$

is a SES in $\mathcal{B}_{\omega,B}$. We obtain the following LES of $\text{Coh}(X)$ -cohomologies associated to the above $\mathcal{B}_{\omega,B}$ -SES:

$$0 \rightarrow C^{-1} \rightarrow 0 \rightarrow B_0^{-1} \rightarrow C^0 \rightarrow \mathcal{O}_x \rightarrow B_0^0 \rightarrow 0.$$

Here $C^k := H_{\text{Coh}(X)}^k(C)$ and $B_0^k := H_{\text{Coh}(X)}^k(B_0)$. We have $C^{-1} = 0$ and so $C \cong C^0$.

If $B_0^0 \neq 0$ then $\mathcal{O}_x \cong B_0^0$ and $B_0^{-1} \cong C^0 \in \mathcal{T}_{\omega,B} \cap \mathcal{F}_{\omega,B} = \{0\}$. So $C = 0$ and $B_{-1} \cong A_0 \in \mathcal{T}'_{\omega,B} \cap \mathcal{F}'_{\omega,B} = \{0\}$ which implies $A_0 = 0$. This is not possible and so $B_0^0 = 0$. Therefore $B_0 \cong B_0^{-1}[1]$ and

$$0 \rightarrow B_0^{-1} \rightarrow C^0 \rightarrow \mathcal{O}_x \rightarrow 0$$

is a SES in $\text{Coh}(X)$. Here $\text{ch}(\mathcal{O}_x) = (0, 0, 0, 1)$. If $B_0^{-1} \neq 0$ then

$$0 \geq \mu_{\omega,B}(B_0^{-1}) = \mu_{\omega,B}(C^0) > 0.$$

This is not possible and so $B_0^{-1} = 0$ and $C^0 \cong \mathcal{O}_x$. Therefore $b \cong B_{-1}[1]$ and we have the following SES in $\mathcal{B}_{\omega,B}$:

$$0 \rightarrow B_{-1} \rightarrow A_0 \rightarrow \mathcal{O}_x \rightarrow 0.$$

Since $\text{ch}(\mathcal{O}_x) = (0, 0, 0, 1)$, if $B_{-1} \neq 0$ then

$$0 \geq \nu_{\omega,B}(B_{-1}) = \nu_{\omega,B}(A_0) > 0.$$

This is not possible and so $B_{-1} = 0$. Therefore $b = 0$ and so $\mathcal{O}_x \in \mathcal{A}_{\omega,B}$ is a minimal object as required. \square

We now identify further minimal objects.

Definition 2.2. Let $\mathcal{M}_{\omega,B}$ be the class of all objects $E \in \mathcal{B}_{\omega,B}$ such that

- (i) E is $\nu_{\omega,B}$ -stable,
- (ii) $\nu_{\omega,B}(E) = 0$, and
- (iii) $\text{Ext}^1(\mathcal{O}_x, E) = 0$ for any skyscraper sheaf \mathcal{O}_x of $x \in X$.

Then clearly $\mathcal{M}_{\omega,B} \subset \mathcal{C}_{\omega,B}$.

Lemma 2.3. Let $E \in \mathcal{M}_{\omega,B}$. Then $E[1]$ is a minimal object of $\mathcal{A}_{\omega,B}$.

Proof. By definition $\mathcal{M}_{\omega,B} \subset \mathcal{F}'_{\omega,B}$ and so $E[1] \in \mathcal{A}_{\omega,B}$. Let

$$0 \rightarrow a \rightarrow E[1] \rightarrow b \rightarrow 0$$

be a SES in $\mathcal{A}_{\omega,B}$ such that $b \neq 0$. Now we need to show that $a = 0$ or equivalently $b \cong E[1]$. We have the following LES of $\mathcal{B}_{\omega,B}$ -cohomologies associated to the above $\mathcal{A}_{\omega,B}$ -SES:

$$0 \rightarrow A_{-1} \rightarrow E \rightarrow B_{-1} \rightarrow A_0 \rightarrow 0 \rightarrow B_0 \rightarrow 0.$$

Here $A_k := H_{\mathcal{B}_{\omega,B}}^k(a)$ and $B_k := H_{\mathcal{B}_{\omega,B}}^k(b)$. We have $B_0 = 0$ and so $b \cong B_{-1}[1]$ which implies $B_{-1} \neq 0$.

Case (I) $A_{-1} \neq 0$:

Subcase (i) $E/A_{-1} \neq 0$:

Then $E/A_{-1} \hookrightarrow B_{-1}$ and $\nu_{\omega,B}^+(B_{-1}) \leq 0$ implies $\nu_{\omega,B}(E/A_{-1}) \leq 0$. On the other hand $\nu_{\omega,B}(E/A_{-1}) > 0$ as $A_{-1} \neq 0$ and E is $\nu_{\omega,B}$ -stable with $\nu_{\omega,B}(E) = 0$. But this not possible.

Subcase (ii) $E/A_{-1} = 0$:

Then $A_{-1} \cong E$ and $B_{-1} \cong A_0 \in \mathcal{F}'_{\omega,B} \cap \mathcal{T}'_{\omega,B} = \{0\}$. This is not possible as $B_{-1} \neq 0$.

Case (II) $A_{-1} = 0$:

Then we have the following SES in $\mathcal{B}_{\omega,B}$:

$$(\blacklozenge) \quad 0 \rightarrow E \rightarrow B_{-1} \rightarrow A_0 \rightarrow 0.$$

Subcase (i) $A_0 \neq 0$:

Here $\nu_{\omega,B}(E) = 0$ implies $\omega^2 \text{ch}_1^B(E) > 0$ and $\Im Z_{\omega,B}(E) = 0$. Then

$$\nu_{\omega,B}(B_{-1}) = \frac{\Im Z_{\omega,B}(A_0)}{\omega^2 \text{ch}_1^B(E) + \omega^2 \text{ch}_1^B(A_0)} \leq 0$$

implies $\Im Z_{\omega,B}(A_0) \leq 0$. If $\omega^2 \text{ch}_1^B(A_0) \neq 0$ then $\nu_{\omega,B}(A_0) > 0$ implies $\Im Z_{\omega,B}(A_0) > 0$; which is not possible. Hence $\omega^2 \text{ch}_1^B(A_0) = 0$ and by Lemma 1.1, $\Im Z_{\omega,B}(A_0) \geq 0$. So $\Im Z_{\omega,B}(A_0) = 0$ and $A_0 \cong T$ for some $0 \neq T \in \text{Coh}^0(X)$. Then the $\mathcal{B}_{\omega,B}$ -SES (\blacklozenge) corresponds to an element from $\text{Ext}^1(A_0, E) = \text{Ext}^1(T, E)$. But we have $\text{Ext}^1(\mathcal{O}_x, E) = 0$ for any $x \in X$ and so $\text{Ext}^1(T, E) = 0$. So $B_{-1} \cong T \oplus E$. Then T is a subobject of B_{-1} . But this is not possible as $\nu_{\omega,B}(T) = +\infty$ and $E \in \mathcal{M}_{\omega,B}$.

Subcase (i) $A_0 = 0$:

Then $a = 0$ and $b \cong B_{-1}[1] \cong E[1]$ as required.

This completes the proof of the lemma. \square

Some classes of tilt stable candidates have been identified in [4].

Recall, for $E \in D^b(X)$ the discriminant $\overline{\Delta}_\omega$ in the sense of Dr ezet is defined by

$$\overline{\Delta}_\omega(E) = (\omega^2 \text{ch}_1^B(E))^2 - 2\omega^3 \text{ch}_0^B(E) \cdot \omega \text{ch}_2^B(E).$$

Proposition 2.4. [4, Proposition 7.4.1] *Let E be a $\mu_{\omega,B}$ -stable locally free sheaf on X with $\overline{\Delta}_\omega(E) = 0$. Then either E or $E[1]$ in $\mathcal{B}_{\omega,B}$ is $\nu_{\omega,B}$ -stable.*

Example 2.5. Let (X, L) be a polarized projective threefold and let $\ell := c_1(L)$. Consider the classes $B = \frac{1}{2}\ell$ and $\omega = \frac{\sqrt{3}}{2}\ell$. Then $\overline{\Delta}_\omega(\mathcal{O}) = \overline{\Delta}_\omega(L) = 0$. So, by Proposition 2.4, $\mathcal{O}[1], L \in \mathcal{B}_{\omega,B}$ are $\nu_{\omega,B}$ -stable. Also $\Im Z_{\omega,B}(\mathcal{O}[1]) = \Im Z_{\omega,B}(L) = 0$. Therefore $\nu_{\omega,B}(\mathcal{O}[1]) = \nu_{\omega,B}(L) = 0$. So by Lemma 2.3, $\mathcal{O}[2], L[1] \in \mathcal{A}_{\omega,B}$ are minimal objects.

Note 2.6. *The tilt stable objects associated to minimal objects in Example 2.5 clearly satisfy the corresponding B-G type inequalities.*

2.2. Reduction of B-G type inequality for minimal objects. The following propositions are important.

Proposition 2.7. [9, Proposition 3.1] *Let $E \in \mathcal{B}_{\omega,B}$ be a $\nu_{\omega,B}$ -semistable object with $\nu_{\omega,B} < +\infty$. Then $H_{\text{Coh}(X)}^{-1}(E)$ is a reflexive sheaf.*

Proposition 2.8. [9, Proposition 3.5] *Let $0 \rightarrow E \rightarrow E' \rightarrow Q \rightarrow 0$ be a non splitting SES in $\mathcal{B}_{\omega,B}$ with $Q \in \text{Coh}^0(X)$, $\text{Hom}(\mathcal{O}_x, E') = 0$ for all $x \in X$, and $\omega^2 \text{ch}_1^B(E) \neq 0$. If E is $\nu_{\omega,B}$ -stable then E' is $\nu_{\omega,B}$ -stable.*

Recall that $\mathcal{C}_{\omega,B}$ is the class of $\nu_{\omega,B}$ -stable objects $E \in \mathcal{B}_{\omega,B}$ with $\nu_{\omega,B}(E) = 0$.

Proposition 2.9. *Let $E \in \mathcal{C}_{\omega,B}$. Then there exists $E' \in \mathcal{M}_{\omega,B}$ (i.e. $E'[1]$ is a minimal object of $\mathcal{A}_{\omega,B}$) such that*

$$0 \rightarrow E \rightarrow E' \rightarrow Q \rightarrow 0$$

is a SES in $\mathcal{B}_{\omega,B}$ for some $Q \in \text{Coh}^0(X)$.

Proof. Let $E \in \mathcal{C}_{\omega,B} \setminus \mathcal{M}_{\omega,B}$. Assume the opposite of the claim in the proposition for E . Then there exists a sequence of non-splitting SESs in $\mathcal{B}_{\omega,B}$, for $i \geq 1$

$$0 \rightarrow E_{i-1} \rightarrow E_i \rightarrow \mathcal{O}_{y_i} \rightarrow 0,$$

where $E_0 = E$, $E_i \in \mathcal{C}_{\omega,B}$ (see Proposition 2.8). So for each $i \geq 1$,

$$0 \rightarrow \mathcal{O}_{y_i} \rightarrow E_{i-1}[1] \rightarrow E_i[1] \rightarrow 0$$

is a SES in $\mathcal{A}_{\omega,B}$. Therefore

$$E[1] = E_0[1] \twoheadrightarrow E_1[1] \twoheadrightarrow E_2[1] \twoheadrightarrow \dots$$

is an infinite chain of quotients in $\mathcal{A}_{\omega,B}$. But this is not possible as $\mathcal{A}_{\omega,B}$ is Noetherian by Lemma 1.5. This is a contradiction. \square

It follows that $E \in \mathcal{C}_{\omega,B}$ satisfies the B-G type inequality if the corresponding $E' \in \mathcal{M}_{\omega,B}$ satisfies the B-G type inequality.

3. ABELIAN CATEGORY $\mathcal{A}_{\sqrt{3}B,B}$, FMT AND STABILITY CONDITIONS

3.1. Some properties of $\mathcal{A}_{\sqrt{3}B,B}$. We discuss some of the properties of the abelian category $\mathcal{A}_{\sqrt{3}B,B}$ for an arbitrary polarized projective threefold (X, L) with Picard rank 1. Let $\ell := c_1(L)$. Let $B = b\ell$ for $b \in \mathbb{Q}_{>0}$. Then for $E \in D^b(X)$

$$\Im Z_{\sqrt{3}B,B}(E) = \sqrt{3}b\ell(\text{ch}_2(E) - b\ell \text{ch}_1(E)).$$

Proposition 3.1. *Let $E \in \mathcal{B}_{\sqrt{3}B,B}$ and let $E_i = H_{\text{Coh}(X)}^i(E)$. Let E_i^\pm be the H-N semistable factors of E_i with highest and lowest $\mu_{\sqrt{3}B,B}$ slopes. Then we have the following:*

- (i) if $E \in \text{HN}_{\sqrt{3}B,B}^\nu(-\infty, 0)$ and $E_{-1} \neq 0$, then $\ell^2 \text{ch}_1(E_{-1}^+) < 0$;
- (ii) if $E \in \text{HN}_{\sqrt{3}B,B}^\nu(0, +\infty]$ and $\text{rk}(E_0) \neq 0$, then $\ell^2 \text{ch}_1(E_0^-) > 2b\ell^3 \text{ch}_0(E_0^-)$; and
- (iii) if E is tilt-stable with $\nu_{\sqrt{3}B,B}(E) = 0$, then
 - (a) for $E_{-1} \neq 0$, $\ell^2 \text{ch}_1(E_{-1}) \leq 0$ with equality if and only if $\text{ch}_2(E_{-1}) = 0$,
 - (b) for $\text{rk}(E_0) \neq 0$, $\ell^2 \text{ch}_1(E_0) \geq 2b\ell^3 \text{ch}_0(E_0)$ with equality if and only if $(\text{ch}_1(E_0))^2 = 2 \text{ch}_0(E_0) \text{ch}_2(E_0)$.

Proof. (i) $E \in \text{HN}_{\sqrt{3}B,B}^\nu(-\infty, 0)$ fits in to the $\mathcal{B}_{\sqrt{3}B,B}$ -SES

$$0 \rightarrow E_{-1}[1] \rightarrow E \rightarrow E_0 \rightarrow 0.$$

Since $E \in \text{HN}_{\sqrt{3}B,B}^\nu(-\infty, 0)$, $E_{-1}[1] \in \text{HN}_{\sqrt{3}B,B}^\nu(-\infty, 0)$. We have $0 \neq E_{-1}^+ \subseteq E_{-1}$. Hence $E_{-1}^+[1] \in \text{HN}_{\sqrt{3}B,B}^\nu(-\infty, 0)$.

Let $\text{ch}(E_{-1}^+) = (a_0, a_1, a_2, a_3)$. Assume the opposite for a contradiction; so that $a_1 \geq 0$. We have

$$\begin{aligned} \nu_{\sqrt{3}B,B}(E_{-1}^+[1]) &= \frac{-\Im Z_{\sqrt{3}B,B}(E_{-1}^+)}{-3B^2 \text{ch}_1^B(E_{-1}^+)} \\ &= \frac{\sqrt{3}ba_1(ba_0 - a_1) + \frac{\sqrt{3}}{2}ba_1^2 + \frac{\sqrt{3}}{2}b(a_1^2 - a_0a_2)}{3a_0b^2(ba_0 - a_1)}. \end{aligned}$$

Since E_{-1}^+ is $\mu_{\sqrt{3}B,B}$ -semistable we have, by the usual B-G inequality,

$$a_1^2 - a_0a_2 \geq 0$$

and since $E_{-1}^+ \in \mathcal{F}_{\sqrt{3}B,B}$, $ba_0 - a_1 \geq 0$. Hence, as $a_0 > 0$, we have $\nu_{\sqrt{3}B,B}(E_{-1}^+[1]) \geq 0$. But this is not possible as $E_{-1}^+[1] \in \text{HN}_{\sqrt{3}B,B}^\nu(-\infty, 0)$. This is the required contradiction to complete the proof.

(ii) $E \in \text{HN}_{\sqrt{3}B,B}^\nu(0, +\infty]$ fits in to the $\mathcal{B}_{\sqrt{3}B,B}$ -SES

$$0 \rightarrow E_{-1}[1] \rightarrow E \rightarrow E_0 \rightarrow 0.$$

Then $E \in \text{HN}_{\sqrt{3}B,B}^\nu(0, +\infty]$ implies $E_0 \in \text{HN}_{\sqrt{3}B,B}^\nu(0, +\infty]$. We have $0 \neq E_0^-$ is a torsion free quotient of E_0 . Since $E_0 \in \text{HN}_{\sqrt{3}B,B}^\nu(0, +\infty]$ we have $E_0^- \in \text{HN}_{\sqrt{3}B,B}^\nu(0, +\infty]$.

Let $\text{ch}(E_0^-) = (a_0, a_1, a_2, a_3)$. Assume the opposite for a contradiction; so that $a_1 \leq 2ba_0$. We have

$$\begin{aligned} \nu_{\sqrt{3}B,B}(E_0^-) &= \frac{\Im Z_{\sqrt{3}B,B}(E_0^-)}{3B^2 \text{ch}_1^B(E_0^-)} \\ &= \frac{-\frac{\sqrt{3}}{2}b(a_1^2 - a_0a_2) + \frac{\sqrt{3}}{2}ba_1(a_1 - 2ba_0)}{3b^2a_0(a_1 - ba_0)}. \end{aligned}$$

Here $E_0^- \in \mathcal{T}_{\sqrt{3}B,B}$ is torsion free which implies

$$a_1 - ba_0 > 0;$$

E_0^- is $\mu_{\sqrt{3}B,B}$ -semistable which implies (by the usual B-G inequality)

$$a_1^2 - a_0a_2 \geq 0.$$

Therefore $\nu_{\sqrt{3}B,B}(E_0^-) \leq 0$. But this is not possible as $E_0^- \in \text{HN}_{\sqrt{3}B,B}^\nu(0, +\infty]$. This is the required contradiction to complete the proof.

(iii) Similar to the proofs of (i) and (ii). □

3.2. Relation of FMT to stability conditions. Let (X, L) be a principally polarized abelian threefold with Picard rank 1. Let $\ell := c_1(L)$. Then $\chi(L) = \frac{\ell^3}{6} = 1$ and the Chern character of $E \in D^b(X)$ is of the form $\text{ch}(E) = (a_0, a_1\ell, a_2\frac{\ell^2}{2}, a_3\frac{\ell^3}{6})$ for some integers a_i . Define the classes $B = \frac{1}{2}\ell$ and $\omega = \frac{\sqrt{3}}{2}\ell$.

The following is a key result in this paper.

Proposition 3.2. *If $\Phi(L^{-1}E)[2] \in \mathcal{B}_{\omega,B}$ for any $E \in \mathcal{M}_{\omega,B} \setminus \{L\mathcal{P}_x : x \in X\}$, then the B-G type inequality holds for the objects in $\mathcal{C}_{\omega,B}$.*

Proof. By Proposition 2.9, it is enough to check that the B-G type inequality is satisfied by each object in $\mathcal{M}_{\omega,B}$. Moreover, the objects in $\{L\mathcal{P}_x : x \in X\} \subset \mathcal{M}_{\omega,B}$ satisfy the B-G type inequality (see Note 2.6). Then we only need to check the inequality for objects in $\mathcal{M}_{\omega,B} \setminus \{L\mathcal{P}_x : x \in X\}$.

Let $E \in \mathcal{M}_{\omega,B} \setminus \{L\mathcal{P}_x : x \in X\}$ and assume $\Phi(L^{-1}E)[2] \in \mathcal{B}_{\omega,B}$. Let $\text{ch}(E) = (a_0, a_1\ell, a_2\frac{\ell^2}{2}, a_3\frac{\ell^3}{6})$ and then $\Im Z_{\omega,B}(E) = 0$ implies $a_1 = a_2$. Now the B-G type inequality says

$$\Delta := -a_0 + 3a_1 - a_3 > 0.$$

By Proposition 3.1, we have $\ell^2 \text{ch}_1(E_{-1}) \leq 0$ and $\ell^2 \text{ch}_1(E_0) \geq 0$. Here $E_i = H_{\text{Coh}(X)}^i(E)$. So $a_1\ell^3 = \ell^2 \text{ch}_1(E) = \ell^2 \text{ch}_1(E_0) - \ell^2 \text{ch}_1(E_{-1}) \geq 0$.

Let $F = \Phi(L^{-1}E)[2]$ and let $\text{ch}(F) = (b_0, b_1\ell, b_2\frac{\ell^2}{2}, b_3\frac{\ell^3}{6})$. Then $b_0 = a_3 - a_0$ and $b_1 = b_2 = a_1 - a_0$. Now $b_1 = b_2$ implies $\Im Z_{\omega,B}(F) = 0$. Also $F \in \mathcal{B}_{\omega,B}$ implies $\omega^2 \text{ch}_1^B(F) \geq 0$, i.e. $2b_1 - b_0 \geq 0$. If $\omega^2 \text{ch}_1^B(F) = 0$ then $\Im Z_{\omega,B}(F) = 0$ implies $F \cong T$ for some $T \in \text{Coh}^0(X)$ (see Lemma 1.1). If $T \neq 0$ then E has a filtration with factors of the form $L\mathcal{P}_x[1] \notin \mathcal{M}_{\omega,B}$. This is not possible and so $\omega^2 \text{ch}_1^B(F) > 0$. That is $2b_1 - b_0 = -a_0 + 2a_1 - a_3 > 0$.

Hence $\Delta > 0$ and so E satisfies the B-G type inequality. This completes the proof as required. \square

Our main goal in the rest of this paper is to prove that $\Phi L^{-1}[2]$ and its quasi-inverse $L\Phi[1]$ are auto-equivalences of the abelian category $\mathcal{A}_{\omega,B}$. Under an equivalence of abelian categories minimal objects are mapped to minimal objects and so the hypothesis of Proposition 3.2 is satisfied. Therefore, by Corollary 1.6, we have the following:

Theorem 3.3. *The pair $(\mathcal{A}_{\omega,B}, Z_{\omega,B})$ is a Bridgeland stability condition on $D^b(X)$.*

4. FOURIER-MUKAI TRANSFORMS ON $\text{Coh}(X)$ OF ABELIAN THREEFOLDS

From here onward, we always assume (X, L) is a principally polarized abelian threefold with Picard rank 1. Let $\ell := c_1(L)$. Then $\chi(L) = \frac{\ell^3}{6} = 1$ and the Chern character of any $E \in D^b(X)$ is of the form $\text{ch}(E) = (a_0, a_1\ell, a_2\frac{\ell^2}{2}, a_3\frac{\ell^3}{6})$ for some integers a_i . Define the classes $B = \frac{1}{2}\ell$ and $\omega = \frac{\sqrt{3}}{2}\ell$.

If $E \in \text{Coh}(X)$ then the slope $\mu(E)$ is defined by $\mu(E) := \mu_{\frac{1}{\sqrt{6}}\ell, 0}(E)$. That is $\mu(E) = \frac{a_1}{a_0}$ when $a_0 \neq 0$, and $\mu(E) = +\infty$ when $a_0 = 0$. In the rest of the paper we mostly use μ slope for coherent sheaves and we simply write $\text{HN} = \text{HN}_{\frac{1}{\sqrt{6}}\ell, 0}^\mu$. Then $\mu_{\omega,B}(E) = \frac{9}{2}(\mu(E) - \frac{1}{2})$. Moreover define $\mathcal{T}_0 = \text{HN}(0, +\infty]$ and $\mathcal{F}_0 = \text{HN}(-\infty, 0]$. Also for simplicity we write $\mathcal{T} = \mathcal{T}_{\omega,B}$, $\mathcal{F} = \mathcal{F}_{\omega,B}$, $\mathcal{B} = \mathcal{B}_{\omega,B}$, $\nu = \nu_{\omega,B}$, $\text{HN}_{\omega,B}^\nu = \text{HN}^\nu$, $\mathcal{T}' = \mathcal{T}'_{\omega,B}$, $\mathcal{F}' = \mathcal{F}'_{\omega,B}$, and $\mathcal{A} = \mathcal{A}_{\omega,B}$. Then by the definitions, we have $\mathcal{F} = \text{HN}(-\infty, \frac{1}{2}]$ and $\mathcal{T} = \text{HN}(\frac{1}{2}, +\infty]$.

Let Φ be the Fourier-Mukai transform with kernel the Poincaré line bundle \mathcal{P} . The isomorphism $\Phi \circ \Phi \cong (-1)^* \text{id}_{\text{Coh}(X)}[-3]$ gives us the following convergence of spectral sequence.

Mukai Spectral Sequence 4.1.

$$E_2^{p,q} = \Phi_{\text{Coh}(X)}^p \Phi_{\text{Coh}(X)}^q(E) \implies H_{\text{Coh}(X)}^{p+q-3}((-1)^*E),$$

for E . Here $\Phi_{\text{Coh}(X)}^i(F) = H_{\text{Coh}(X)}^i(\Phi(F))$.

For $E \in \text{Coh}(X)$, we write

$$E^k = \Phi_{\text{Coh}(X)}^k(E).$$

Then for example $E^{120} = \Phi_{\text{Coh}(X)}^0 \Phi_{\text{Coh}(X)}^2 \Phi_{\text{Coh}(X)}^1(E)$. Using this notation, we can deduce the following immediately from the spectral sequence:

$$E^{00} = E^{01} = E^{32} = E^{33} = 0, E^{10} \cong E^{02} \text{ and } E^{31} \cong E^{23}.$$

Let $\mathbf{R}\Delta$ denote the derived dualizing functor $\mathbf{R}\mathcal{H}om(-, \mathcal{O})[3]$. Then due to Mukai,

$$(\Phi \circ \mathbf{R}\Delta)[3] \cong (-1)^* \mathbf{R}\Delta \circ \Phi \text{ (see [14, (3.8)])}.$$

This gives us the convergence of the following spectral sequences.

“Duality” Spectral Sequence 4.2.

$$\Phi_{\text{Coh}(X)}^p(\mathcal{E}xt^{q+3}(E, \mathcal{O})) \implies ? \iff (-1)^* \mathcal{E}xt^{p+3}(\Phi_{\text{Coh}(X)}^{3-q}(E), \mathcal{O})$$

for $E \in \text{Coh}(X)$.

Notation 4.3. Any $E \in \text{Coh}(X)$ fits into a non-splitting $\text{Coh}(X)$ -SES

$$0 \rightarrow T \rightarrow E \rightarrow F \rightarrow 0$$

for some $T \in \mathcal{T}_0$ and $F \in \mathcal{F}_0$. Denote $T(E) = T$ and $F(E) = F$.

Any torsion free sheaf E fits into a non-splitting $\text{Coh}(X)$ -SES

$$0 \rightarrow E \rightarrow E^{**} \rightarrow T \rightarrow 0$$

for some $T \in \text{Coh}^{\leq 1}(X)$. Here E^{**} is a reflexive sheaf. If E is rank 1 then E^{**} is a line bundle and so $E^{**} \cong L^k \mathcal{P}_x$ for some $k \in \mathbb{Z}$ and $x \in X$.

Notation 4.4. If E is a rank 1 torsion free sheaf with $c_1(E) = kl$ then we can write $E = L^k \mathcal{P}_x \mathcal{I}_C$. Here \mathcal{I}_C is the ideal sheaf of the structure sheaf $\mathcal{O}_C := L^{-k} \mathcal{P}_{-x} \otimes (E^{**}/E) \in \text{Coh}^{\leq 1}(X)$ of a subscheme $C \subset X$ of dimension ≤ 1 .

Proposition 4.5. Let $E \in \text{Coh}(X)$. If $E^0 \neq 0$ then E^0 is a reflexive sheaf.

Proof. Let $x \in X$. Then for $0 \leq i \leq 2$, we have

$$\text{Hom}(\mathcal{O}_x, E^0[i]) \cong \text{Hom}(\Phi(\mathcal{O}_x), \Phi(E^0)[i]) \cong \text{Hom}(\mathcal{P}_x, E^{02}[-2+i])$$

from the convergence of the Mukai Spectral Sequence 4.1 for E . So $\text{Hom}(\mathcal{O}_x, E^0) = \text{Ext}^1(\mathcal{O}_x, E^0) = 0$, and

$$\begin{aligned} \text{Ext}^2(\mathcal{O}_x, E^0) &\cong \text{Hom}(\mathcal{P}_x, E^{02}) \\ &\cong \text{Hom}(\mathcal{P}_x, E^{10}), \text{ by the Mukai Spectral Sequence for } E \\ &\cong \text{Hom}(\Phi(\mathcal{O}_x), \Phi(E^1)) \\ &\cong \text{Hom}(\mathcal{O}_x, E^1). \end{aligned}$$

Hence $\dim\{x \in X : \text{Ext}^2(\mathcal{O}_x, E^0) \neq 0\} \leq 0$. Therefore E^0 is a reflexive sheaf. \square

Proposition 4.6. Let $E \in \text{Coh}(X)$. Then we have the following:

- (i) if $E \in \mathcal{T}_0$ then $E^3 = 0$, and
- (ii) if $E \in \mathcal{F}_0$ then $E^0 = 0$.

Proof. (i) Let $E \in \mathcal{T}_0$. Then for any $x \in X$, we have

$$\begin{aligned} \text{Hom}(E^3, \mathcal{O}_x) &\cong \text{Hom}(\Phi(E)[3], \Phi(\mathcal{P}_{-x})[3]) \\ &\cong \text{Hom}(E, \mathcal{P}_{-x}) = 0, \end{aligned}$$

as $\mathcal{P}_{-x} \in \mathcal{F}_0$. Therefore $E^3 = 0$ as required.

- (ii) Let $E \in \mathcal{F}_0$. We can assume E is μ -stable using H-N and Jordan-Hölder filtrations. For generic $x \in X$ and $i = 1, 2$ we have

$$\text{Hom}(E^1, \mathcal{O}_x[i]) = \text{Hom}(E^2, \mathcal{O}_x[i+1]) = \text{Hom}(E^3, \mathcal{O}_x[i+2]) = 0.$$

Hence for generic $x \in X$,

$$\text{Hom}(E^0, \mathcal{O}_x) \cong \text{Hom}(\Phi(E), \mathcal{O}_x)$$

$$\begin{aligned}
 &\cong \mathrm{Hom}(\Phi(E), \Phi(\mathcal{P}_{-x}[3])) \\
 &\cong \mathrm{Hom}(E, \mathcal{P}_{-x}[3]) \\
 &\cong \mathrm{Hom}(\mathcal{P}_{-x}, E)^*.
 \end{aligned}$$

(a) Case $\mu(E) < 0$:

Then $\mathrm{Hom}(\mathcal{P}_{-x}, E) = 0$.

(b) Case $\mu(E) = 0$:

Since E is assumed to be μ -stable, any map in $\mathrm{Hom}(\mathcal{P}_{-x}, E)$ must be an isomorphism and so $E^0 = 0$.

Therefore for generic $x \in X$, $\mathrm{Hom}(E^0, \mathcal{O}_x) = 0$. By Proposition 4.5 if $E^0 \neq 0$ then it is reflexive. So $E^0 = 0$. □

Proposition 4.7. *Let $E \in \mathrm{Coh}(X)$. Then $E^3 \in \mathcal{T}_0$.*

Proof. Let $T = T(E^3) \in \mathcal{T}_0$ and $F = F(E^3) \in \mathcal{F}_0$, so that

$$0 \rightarrow T \rightarrow E^3 \rightarrow F \rightarrow 0$$

is a non splitting SES in $\mathrm{Coh}(X)$. Now we need to show that $F = 0$. Apply Φ to the above SES and consider the LES of $\mathrm{Coh}(X)$ -cohomologies. Then we have $F \in V_{\mathrm{Coh}(X)}^\Phi(1)$, $T \in V_{\mathrm{Coh}(X)}^\Phi(0, 1, 2)$ (for the definition of V see the notation section of the introduction) and

$$0 \rightarrow T^1 \rightarrow E^{31} \rightarrow F^1 \rightarrow T^2 \rightarrow 0$$

is a LES in $\mathrm{Coh}(X)$. Here $E^{31} \cong E^{23}$ (from the Mukai Spectral Sequence 4.1 for E) and so

$$\begin{aligned}
 \mathrm{Hom}(E^{31}, F^1) &\cong \mathrm{Hom}(E^{23}, F^1) \\
 &\cong \mathrm{Hom}(\Phi(E^2)[3], \Phi(F)[1]) \\
 &\cong \mathrm{Hom}(E^2, F[-2]) = 0.
 \end{aligned}$$

Hence $F \cong (-1)^* F^{12} \cong (-1)^* T^{22} = 0$ (from the Mukai Spectral Sequence 4.1 for T) as required. □

Proposition 4.8. *Let $E \in \mathrm{Coh}(X)$. Then $E^0 \in \mathcal{F}_0$.*

Proof. This can be proved in a same way as Proposition 4.7; but we give a shorter proof to illustrate the use of the ‘‘Duality’’ Spectral Sequence 4.2.

From the ‘‘Duality’’ Spectral Sequence 4.2 for E^* we have

$$\Phi_{\mathrm{Coh}(X)}^0(E^{**}) \cong (-1)^* \left(\Phi_{\mathrm{Coh}(X)}^3(E^*) \right)^*.$$

By Proposition 4.7, $\Phi_{\mathrm{Coh}(X)}^3(E^*) \in \mathcal{T}_0$. So $(E^{**})^0 \in \mathcal{F}_0$. However E fits into the following structure sequence

$$0 \rightarrow E \rightarrow E^{**} \rightarrow T \rightarrow 0$$

for some $T \in \mathrm{Coh}^{\leq 1}(X)$. Then $E^0 \hookrightarrow (E^{**})^0$ which implies $E^0 \in \mathcal{F}_0$ as required. □

Proposition 4.9. *Let $E \in \mathcal{F}_0$. If $E^1 \neq 0$ then E^1 is a reflexive sheaf.*

Proof. By Proposition 4.6, $E^0 = 0$. Let $x \in X$. Then from the convergence of the Mukai Spectral Sequence 4.1 for E and $0 \leq i \leq 2$, we have

$$\mathrm{Hom}(\mathcal{O}_x, E^1[i]) \cong \mathrm{Hom}(\Phi(\mathcal{O}_x), \Phi(E^1)[i])$$

$$\cong \mathrm{Hom}(\mathcal{P}_x, E^{12}[i-2])$$

as $\mathrm{Hom}(\mathcal{P}_x, \tau_{>2}\Phi(E^1)[i]) \cong \mathrm{Hom}(\mathcal{P}_x, E^{13}[i-3]) = 0$. Therefore $\mathrm{Hom}(\mathcal{O}_x, E^1) = \mathrm{Ext}^1(\mathcal{O}_x, E^1) = 0$ and $\mathrm{Ext}^2(\mathcal{O}_x, E^1) \cong \mathrm{Hom}(\mathcal{P}_x, E^{12})$.

From the convergence of the Mukai Spectral Sequence 4.1 for E

$$0 \rightarrow E^{20} \rightarrow E^{12} \rightarrow F \rightarrow 0$$

is a SES in $\mathrm{Coh}(X)$. Here F is a subobject of $(-1)^*E$. By applying the functor $\mathrm{Hom}(\mathcal{P}_x, -)$ we obtain the exact sequence

$$0 \rightarrow \mathrm{Hom}(\mathcal{P}_x, E^{20}) \rightarrow \mathrm{Hom}(\mathcal{P}_x, E^{12}) \rightarrow \mathrm{Hom}(\mathcal{P}_x, F) \rightarrow \cdots$$

Now $F \in \mathcal{F}_0$ and by Proposition 4.8 E^{20} is also in \mathcal{F}_0 . Therefore we have $\mathrm{Hom}(\mathcal{P}_x, F) \neq 0$ or $\mathrm{Hom}(\mathcal{P}_x, E^{20}) \neq 0$ for at most a finite number of points $x \in X$. That is $\dim\{x \in X : \mathrm{Ext}^2(\mathcal{O}_x, E^1) \neq 0\} \leq 0$. Therefore E^1 is a reflexive sheaf. \square

Proposition 4.10. *If E is a torsion sheaf then $E^2 \in \mathcal{T}_0$.*

Proof. Let $T = T(E^2)$ and $F = F(E^2)$. Then $0 \rightarrow T \rightarrow E^2 \rightarrow F \rightarrow 0$ is a non-splitting SES in $\mathrm{Coh}(X)$. By applying Φ we obtain the LES

$$0 \rightarrow T^1 \rightarrow E^{21} \rightarrow F^1 \rightarrow T^2 \rightarrow 0$$

in $\mathrm{Coh}(X)$. Here $F \in V_{\mathrm{Coh}(X)}^\Phi(1)$. From the convergence of the Mukai Spectral Sequence 4.1 for E , E^{21} fits into the $\mathrm{Coh}(X)$ -SES

$$0 \rightarrow Q \rightarrow E^{21} \rightarrow E^{13} \rightarrow 0,$$

where Q is a quotient of $(-1)^*E$. So Q is a torsion sheaf and $\mathrm{Hom}(Q, F^1) = 0$ as F^1 is a reflexive sheaf (see Proposition 4.9). Therefore

$$\begin{aligned} \mathrm{Hom}(E^{21}, F^1) &\cong \mathrm{Hom}(E^{13}, F^1) \\ &\cong \mathrm{Hom}(\Phi(E^1)[3], \Phi(F)[1]) \\ &\cong \mathrm{Hom}(E^1, F[-2]) = 0. \end{aligned}$$

Hence $F^1 \cong T^2$ and so $F \cong (-1)^*F^{12} \cong (-1)^*T^{22} = 0$ (from the Mukai Spectral Sequence 4.1 for T) as required. \square

Proposition 4.11. *Let $E \in \mathrm{Coh}^{\leq 1}(X)$. Then $E^1 \in \mathcal{T}_0$.*

Proof. $E \in \mathrm{Coh}^{\leq 1}(X)$ fits into the torsion sequence $0 \rightarrow E_0 \rightarrow E \rightarrow E_1 \rightarrow 0$, where $E_0 \in \mathrm{Coh}^0(X)$ and $E_1 \in \mathrm{Coh}^1(X)$. Here $E_0 \in V_{\mathrm{Coh}(X)}^\Phi(0)$ and so $E^1 = E_1^1$. Therefore we only need to prove the claim for a pure dimension 1 torsion sheaf E . Then for sufficiently large $n > 0$ and suitable $x \in X$, $L_x^{-n}E \in V_{\mathrm{Coh}(X)}^\Phi(1)$, where $L_x = L\mathcal{P}_x$, and

$$0 \rightarrow L_x^{-n}E \rightarrow E \rightarrow Q \rightarrow 0$$

is a SES in $\mathrm{Coh}(X)$ for some $Q \in \mathrm{Coh}^0(X)$. Then we have $(L_x^{-n}E)^1 \twoheadrightarrow E^1$. Therefore we only need to show $(L_x^{-n}E)^1 \in \mathcal{T}_0$. Let us show this by proving the claim for pure dimension one torsion sheaf $E \in V_{\mathrm{Coh}(X)}^\Phi(1)$. Then $\mathrm{ch}(E) = (0, 0, \alpha, \beta)$, where $\alpha > 0$ and $\beta \leq 0$.

Let $T = T(E^1)$ and $F = F(E^1)$. Then $0 \rightarrow T \rightarrow E^1 \rightarrow F \rightarrow 0$ is a non-splitting SES in $\mathrm{Coh}(X)$. Now we need to show $F = 0$. So suppose $F \neq 0$ for a contradiction. Apply

the FMT Φ and consider the LES of $\text{Coh}(X)$ -cohomologies. Then we have $T \in V_{\text{Coh}(X)}^\Phi(2)$, $F \in V_{\text{Coh}(X)}^\Phi(1, 2)$ and

$$0 \rightarrow F^1 \rightarrow T^2 \rightarrow E \rightarrow F^2 \rightarrow 0$$

is a LES in $\text{Coh}(X)$.

Case (i) The map $T^2 \rightarrow E$ is zero:

Then $T \cong (-1)^*T^{21} \cong (-1)^*F^{11} = 0$ from the Mukai Spectral Sequence 4.1 as $F \in V_{\text{Coh}(X)}^\Phi(1, 2)$. So $E = F^2$ and hence $F \in V_{\text{Coh}(X)}^\Phi(2)$. Therefore $F \cong (-1)^*E^1$ and so $\text{ch}(F) = (-\beta, \alpha, 0, 0)$. Here $\alpha > 0$ and which is not possible as $\mu(F) \leq 0$.

Case (ii) The map $T^2 \rightarrow E$ is non-zero:

Let $K = \text{im}(T^2 \rightarrow E)$. Then $K \in \text{Coh}^1(X)$ and the $\text{Coh}(X)$ -SES $0 \rightarrow F^1 \rightarrow T^2 \rightarrow K \rightarrow 0$ corresponds to an element from $\text{Ext}^1(K, F^1)$. Here F^1 is a reflexive sheaf and so there exists a locally free sheaf U and a torsion free sheaf V such that $0 \rightarrow F^1 \rightarrow U \rightarrow V \rightarrow 0$ is a non-splitting SES in $\text{Coh}(X)$. By applying the functor $\text{Hom}(K, -)$, we obtain the following exact sequence:

$$\cdots \rightarrow \text{Hom}(K, V) \rightarrow \text{Ext}^1(K, F^1) \rightarrow \text{Ext}^1(K, U) \rightarrow \cdots$$

Here $\text{Hom}(K, V) = 0$ and $\text{Ext}^1(K, U) \cong \text{Ext}^2(U, K)^* \cong H^2(X, U^* \otimes K)^* = 0$ as $K \in \text{Coh}^{\leq 1}(X)$. So $\text{Ext}^1(K, F^1) = 0$ implies $T^2 \cong F^1 \oplus K$. Here $T^2 \in V_{\text{Coh}(X)}^\Phi(1)$ implies $F^1 = 0$ and so $K \cong T^2$. Then $F^2 \cong E/T^2$ and also $F \in V_{\text{Coh}(X)}^\Phi(2)$. Since $F^2 \in V_{\text{Coh}(X)}^\Phi(1)$, it is a pure dimension 1 torsion sheaf. So $\text{ch}(F^2) = (0, 0, \alpha', \beta')$, where $\alpha' > 0$ and $\beta' \leq 0$. Therefore $\text{ch}(F) = (-\beta', \alpha', 0, 0)$ and which is not possible as $\mu(F) \leq 0$ implies $\alpha' \leq 0$.

Therefore $F = 0$ as required to complete the proof. \square

Let E be a reflexive sheaf. Then from a minimal projective resolution of E we have a $\text{Coh}(X)$ -SES

$$0 \rightarrow Q \rightarrow F \rightarrow E \rightarrow 0$$

for some locally free sheaves F and Q . Now by applying the FMT ΦL^{-n} to the above SES for $n > 0$, we obtain

$$\text{for sufficiently large } n > 0, L^{-n}E \in V_{\text{Coh}(X)}^\Phi(2, 3).$$

Let $L_z := L\mathcal{P}_z$ for $z \in X$ and let D_z be the divisor of the line bundle L_z .

Then for $n > 0$, any reflexive sheaf E fits into the $\text{Coh}(X)$ -SES

$$(†) \quad 0 \rightarrow L_x^{-n}E \rightarrow E \rightarrow E|_{nD_x} \rightarrow 0.$$

Moreover for $m > 0$, $E|_{nD_x}$ fits into the $\text{Coh}(X)$ -SES

$$(††) \quad 0 \rightarrow L_y^{-m}E|_{nD_x} \rightarrow E|_{nD_x} \rightarrow E|_{nD_x \cap mD_y} \rightarrow 0,$$

for $x \neq y$. Then we have the following key technical lemma.

Lemma 4.12. *Let E be a line bundle on X . The μ^+ slope of $(E|_{nD_x \cap mD_y})^1$ tends to 0 as either n or m tends to $+\infty$.*

Proof. We have $E \cong L^k \mathcal{P}_x$ for some $k \in \mathbb{Z}$ and $x \in X$. Then for $n, m > 0$ we have

$$\left(L^k \mathcal{P}_z|_{nD_x \cap mD_y} \right)^1 \hookrightarrow \left(L_y^{-m} L^k \mathcal{P}_z|_{nD_x} \right)^2 \hookrightarrow \left(L^{k-m-n} \mathcal{P}_{z-my-nx} \right)^3.$$

On the other hand for $p > 0$, it is well known that $(L^{-p})^3$ is stable (see [13, Prop 6.16]) and its μ slope tends to 0 as $p \rightarrow +\infty$. Therefore the result follows by taking the limit either n or m tends to $+\infty$. \square

Lemma 4.13. *Let E be a reflexive sheaf with $E^0 = 0$. Then $E^1 \in \mathcal{F}_0$.*

Proof. Topologically the divisors nD_x of the line bundles L_x^n are same for $n > 0$. Also nD_x is the support of \mathcal{O}/L_x^{-n} . Let us denote $C_{x,y}^{n,m} = nD_x \cap mD_y$ and so it is the support of $(\mathcal{O}/L_x^{-n})|_{mD_y} = (\mathcal{O}/L_y^{-m})|_{nD_x}$.

Since L is ample, $E = E_1$ fits into a filtration of rank 1 torsion free sheaves of form $L^{k_i} \mathcal{P}_{z_i} \mathcal{I}_{C_i}$ as follows:

$$(*) \quad L^{k_r} \mathcal{P}_{z_r} \mathcal{I}_{C_r} = E_r \xrightarrow{L^{k_{r-1}} \mathcal{P}_{z_{r-1}} \mathcal{I}_{C_{r-1}}} E_{r-1} \cdots \cdots \xrightarrow{L^{k_1} \mathcal{P}_{z_1} \mathcal{I}_{C_1}} E_1 .$$

Here $r = \text{rk}(E)$. Now consider one such particular filtration for E .

Then one can choose $x, y \in X$ such that

- (i) $C_{x,y}^{1,1} \cap C_i = \emptyset$ for all $i = 1, \dots, r$;
- (ii) $E|_{D_x}$ is locally free on D_x ; and
- (iii) $E|_{D_y}$ is locally free on D_y .

Since E is a reflexive sheaf, for sufficiently large $n > 0$, $L_x^{-n} E \in V_{\text{Coh}(X)}^\Phi(2, 3)$. By SES (\dagger) , $E|_{nD_x} \in V_{\text{Coh}(X)}^\Phi(1, 2)$ and $E^1 \hookrightarrow (E|_{nD_x})^1$. Since $E|_{nD_x}$ is pure, we have $L_y^{-m} E|_{nD_x} \in V_{\text{Coh}(X)}^\Phi(2)$ for sufficiently large $m > 0$. By SES $(\dagger\dagger)$, $E|_{C_{x,y}^{n,m}} \in V_{\text{Coh}(X)}^\Phi(1)$ and $(E|_{nD_x})^1 \hookrightarrow (E|_{C_{x,y}^{n,m}})^1$. Therefore we have

$$E^1 \hookrightarrow (E|_{C_{x,y}^{n,m}})^1 .$$

The claim in the lemma follows from the following. \square

Claim *For any $T \in \mathcal{T}_0$ which is independent of sufficiently large m, n ,*

$$\text{Hom} \left(T, (E|_{C_{x,y}^{n,m}})^1 \right) = 0 .$$

Proof. Let us prove by induction on $\text{rk}(E)$. If $\text{rk}(E) = 1$ then the result follows from Lemma 4.12.

Assume the claim is true for $\text{rk} < \text{rk}(E)$ cases. From the filtration $(*)$, we have the Coh(X)-SES

$$0 \rightarrow F \rightarrow E \rightarrow L^{k_1} \mathcal{P}_{z_1} \mathcal{I}_{C_1} \rightarrow 0$$

for some F . Then F is reflexive and, since $E^0 = 0$, $F^0 = 0$. Also we have the following SES in Coh(X):

$$0 \rightarrow F|_{C_{x,y}^{n,m}} \rightarrow E|_{C_{x,y}^{n,m}} \rightarrow L^{k_1} \mathcal{P}_{z_1} \mathcal{I}_{C_1}|_{C_{x,y}^{n,m}} \rightarrow 0 .$$

Apply the FMT Φ and then consider the LES of Coh(X)-cohomologies. Then we obtain

$$0 \rightarrow (L^{k_1} \mathcal{P}_{z_1} \mathcal{I}_{C_1}|_{C_{x,y}^{n,m}})^0 \rightarrow (F|_{C_{x,y}^{n,m}})^1 \rightarrow (E|_{C_{x,y}^{n,m}})^1 \rightarrow (L^{k_1} \mathcal{P}_{z_1} \mathcal{I}_{C_1}|_{C_{x,y}^{n,m}})^1 \rightarrow 0$$

is a LES in $\text{Coh}(X)$. Due to the choice of x and y , $L^{k_1} \mathcal{P}_{z_1} \mathcal{I}_{C_1}|_{C_{x,y}^{n,m}} \cong L^{k_1} \mathcal{P}_{z_1}|_{C_{x,y}^{n,m}}$. Then $\left(L^{k_1} \mathcal{P}_{z_1} \mathcal{I}_{C_1}|_{C_{x,y}^{n,m}}\right)^0 \cong \left(L^{k_1} \mathcal{P}_{z_1}|_{C_{x,y}^{n,m}}\right)^0 \cong (L^{k_1} \mathcal{P}_{z_1})^0$. The latter isomorphism comes from the SESs (\dagger) and $(\dagger\dagger)$.

Assume the claim is false. Then there exists a non-zero map $T \rightarrow \left(E|_{C_{x,y}^{n,m}}\right)^1$ for some $T \in \mathcal{T}_0$ which is independent of n, m . However

$$\text{Hom}\left(T, \left(E|_{C_{x,y}^{n,m}}\right)^1\right) \cong \text{Hom}\left((-1)^* T^2, E|_{C_{x,y}^{n,m}}\right).$$

Now by the induction hypothesis

$$\text{Hom}\left((-1)^* T^2, F|_{C_{x,y}^{n,m}}\right) \cong \text{Hom}\left((-1)^* T^{21}, \left(F|_{C_{x,y}^{n,m}}\right)^1\right) = 0$$

as $T^{21} \in \mathcal{T}_0$ from the Mukai Spectral Sequence 4.1 for T . Therefore the composition

$$f : (-1)^* T^2 \rightarrow E|_{C_{x,y}^{n,m}} \rightarrow L^{k_1} \mathcal{P}_{z_1} \mathcal{I}_{C_1}|_{C_{x,y}^{n,m}}$$

is non-zero. On the other hand

$$\text{Hom}\left((-1)^* T^2, L^{k_1} \mathcal{P}_{z_1} \mathcal{I}_{C_1}|_{C_{x,y}^{n,m}}\right) \cong \text{Hom}\left(\Phi((-1)^* T^2), \Phi(L^{k_1} \mathcal{P}_{z_1}|_{C_{x,y}^{n,m}})\right);$$

and by the base case

$$\text{Hom}\left(\Phi((-1)^* T^2), \left(L^{k_1} \mathcal{P}_{z_1}|_{C_{x,y}^{n,m}}\right)^1[-1]\right) \cong \text{Hom}\left((-1)^* T^{21}, \left(L^{k_1} \mathcal{P}_{z_1}|_{C_{x,y}^{n,m}}\right)^1\right) = 0.$$

Therefore

$$\text{Hom}\left(\Phi((-1)^* T^2), \left(L^{k_1} \mathcal{P}_{z_1} \mathcal{I}_{C_1}|_{C_{x,y}^{n,m}}\right)^0\right) \rightarrow \text{Hom}\left(\Phi((-1)^* T^2), \Phi(L^{k_1} \mathcal{P}_{z_1}|_{C_{x,y}^{n,m}})\right),$$

and where

$$\text{Hom}\left(\Phi((-1)^* T^2), \left(L^{k_1} \mathcal{P}_{z_1} \mathcal{I}_{C_1}|_{C_{x,y}^{n,m}}\right)^0\right) = \text{Hom}\left(\Phi((-1)^* T^2), \left(L^{k_1} \mathcal{P}_{z_1}\right)^0\right).$$

Case (I) $k_1 \leq 0$:

Then $(L^{k_1} \mathcal{P}_{z_1})^0 = 0$ and so $f = 0$; which is the required contradiction.

Case (II) $k_1 > 0$:

Then $\text{Hom}\left(\Phi((-1)^* T^2), \left(L^{k_1} \mathcal{P}_{z_1}\right)^0\right) \cong \text{Hom}\left((-1)^* T^2, L^{k_1} \mathcal{P}_{z_1}\right)$ and so the composition f factors as

$$f : (-1)^* T^2 \rightarrow E|_{C_{x,y}^{n,m}} \rightarrow L^{k_1} \mathcal{P}_{z_1} \rightarrow L^{k_1} \mathcal{P}_{z_1} \mathcal{I}_{C_1}|_{C_{x,y}^{n,m}}.$$

However $\text{Hom}(E|_{C_{x,y}^{n,m}}, L^{k_1} \mathcal{P}_{z_1}) = 0$ as $E|_{C_{x,y}^{n,m}}$ is a torsion sheaf. Therefore the map f is zero. This is the required contradiction.

This completes the proof of the claim. \square

Corollary 4.14. *Let $E \in \mathcal{F}_0$. Then $E^1 \in \mathcal{F}_0$.*

Proof. Apply the FMT Φ to the structure sequence

$$0 \rightarrow E \rightarrow E^{**} \rightarrow T \rightarrow 0$$

and consider the LES of $\text{Coh}(X)$ -cohomologies. Since $T \in \text{Coh}^{\leq 1}(X)$, $E^{**} \in \mathcal{F}_0$. Then $(E^{**})^0 = E^0 = 0$ by Proposition 4.6. Consequently,

$$0 \rightarrow T^0 \rightarrow E^1 \rightarrow (E^{**})^1 \rightarrow \dots$$

is an exact sequence in $\text{Coh}(X)$. By Lemma 4.13, $(E^{**})^1 \in \mathcal{F}_0$. We also have $T^0 \in \mathcal{F}_0$ by Proposition 4.8. Hence $E^1 \in \mathcal{F}_0$ as required. \square

Corollary 4.15. *Let $E \in \mathcal{T}_0$. Then $E^2 \in \mathcal{T}_0$.*

Proof. Let $T = T(E^2)$ and $F = F(E^2)$. Then $0 \rightarrow T \rightarrow E^2 \rightarrow F \rightarrow 0$ is a non-splitting SES in $\text{Coh}(X)$. Now we need to show $F = 0$. Apply the FMT Φ and consider the LES of $\text{Coh}(X)$ -cohomologies. So we have $F \in V_{\text{Coh}(X)}^\Phi(1)$ and

$$0 \rightarrow T^1 \rightarrow E^{21} \rightarrow F^1 \rightarrow T^2 \rightarrow 0$$

is a LES in $\text{Coh}(X)$. From the convergence of the Mukai Spectral Sequence 4.1 for E we have the $\text{Coh}(X)$ -SES

$$0 \rightarrow Q \rightarrow E^{21} \rightarrow E^{13} \rightarrow 0,$$

where Q is a quotient of $(-1)^*E$. Then $Q \in \mathcal{T}_0$ and, by Proposition 4.7, $E^{13} \in \mathcal{T}_0$ and so $E^{21} \in \mathcal{T}_0$. On the other hand, by Corollary 4.14, $F^1 \in \mathcal{F}_0$. So the map $E^{21} \rightarrow F^1$ is zero and $F^1 \cong T^2$. Hence $F \cong (-1)^*F^{12} \cong (-1)^*T^{22} = 0$ (from the Mukai Spectral Sequence 4.1 for T) as required. \square

Proposition 4.16. *Let E be a coherent sheaf with $E^3 = 0$. Then $E^2 \in \text{HN}[0, +\infty]$.*

Proof. Apply the FMT Φ to the structure sequence

$$0 \rightarrow E \rightarrow E^{**} \rightarrow T \rightarrow 0$$

and consider the LES of $\text{Coh}(X)$ -cohomologies. Then we have

$$\cdots \rightarrow T^1 \rightarrow E^2 \rightarrow (E^{**})^2 \rightarrow 0$$

is an exact sequence in $\text{Coh}(X)$ and $(E^{**})^3 = E^3 = 0$. Here $T \in \text{Coh}^{\leq 1}(X)$ and so, by Proposition 4.11, $T^1 \in \mathcal{T}_0$. So we only need to prove the Lemma for reflexive E with $E^3 = 0$.

Let E be a reflexive sheaf with $E^3 = 0$. By the convergence of the ‘‘Duality’’ Spectral Sequence 4.2 for E , we have $(E^*)^0 \cong (-1)^*(E^3)^* = 0$ and $(E^2)^* \cong (-1)^*(E^*)^1$. Since E^* is reflexive, by Lemma 4.13 we have $(E^*)^1 \in \text{HN}(-\infty, 0]$. Hence $(E^2)^* \in \text{HN}(-\infty, 0]$ which implies $E^2 \in \text{HN}[0, +\infty]$ as required. \square

Proposition 4.17. *Let $E \in \text{HN}(0, 1]$. Then $E^0 \in \text{HN}(-\infty, -\frac{1}{2}]$.*

Proof. Due to Mukai, $\Phi L \Phi \cong (-1)^*L^{-1}\Phi L^{-1}$. Therefore we have the following convergence of spectral sequence:

$$E_2^{p,q} = \Phi_{\text{Coh}(X)}^p L \Phi_{\text{Coh}(X)}^q(E) \implies (-1)^*L^{-1}\Phi_{\text{Coh}(X)}^{p+q}(L^{-1}E).$$

Here $L^{-1}E \in \text{HN}(-1, 0]$ and so $(L^{-1}E)^0 = 0$ by Proposition 4.6. So from the convergence of the above spectral sequence for E we have $(LE^0)^0 = 0$.

Let $F \subset E^0$ be the H-N semistable factor of E^0 with the highest slope and let $\mu := \mu(F)$. Then $(LF)^0 \hookrightarrow (LE^0)^0$ and so $(LF)^0 = 0$. Let $\text{ch}(F) = (a_0, \mu a_0, a_2, a_3)$. Now suppose $\mu > -\frac{1}{2}$ for a contradiction. Then $LF \in \mathcal{T}_0$ and F fits into the $\text{Coh}(X)$ -SES

$$(\star) \quad 0 \rightarrow F \rightarrow E^0 \rightarrow G \rightarrow 0,$$

for some $G \in \text{HN}(-\infty, 0]$. By Proposition 4.5, E^0 is reflexive. Since G is torsion-free, it follows that F is also reflexive. Apply the FMT Φ and consider the LES of $\text{Coh}(X)$ -cohomologies. Then we have $F \in V_{\text{Coh}(X)}^\Phi(2, 3)$ and

$$0 \rightarrow G^1 \rightarrow F^2 \rightarrow E^{02} \rightarrow \cdots$$

is an exact sequence in $\text{Coh}(X)$. From the convergence of the Mukai Spectral Sequence 4.1 for E , $E^{02} \cong E^{10}$ and $E^{10} \in \text{HN}(-\infty, 0]$ by Proposition 4.8. Also by Proposition 4.14, $G^1 \in \text{HN}(-\infty, 0]$. So $F^2 \in \text{HN}(-\infty, 0]$ and we have $\ell^2 \text{ch}_1(F^2) \leq 0$. Moreover, $F^3 \in \text{HN}(0, +\infty]$ by Proposition 4.7 and so $\ell^2 \text{ch}_1(F^3) \geq 0$. Therefore $\ell^2 \text{ch}_1(\Phi(F)) \leq 0$ and so $\text{ch}(\Phi(F)) = (a_3, -a_2, \mu a_0, -a_0)$ implies

$$a_2 \ell^3 = 2\ell \text{ch}_2(F) \geq 0.$$

Apply the FMT ΦL to the SES (\star) and consider the LES of $\text{Coh}(X)$ -cohomologies. Then $LF \in V_{\text{Coh}(X)}^\Phi(1, 2)$ and by Lemma 4.13 and Corollary 4.15 we have $(LF)^1 \in \text{HN}(-\infty, 0]$ and $(LF)^2 \in \text{HN}(0, +\infty]$. So $\ell^2 \text{ch}_1(LF^1) \leq 0$ and $\ell^2 \text{ch}_1(LF^2) \geq 0$ which imply $\ell^2 \text{ch}_1(\Phi(LF)) \geq 0$. Hence

$$(a_0 + 2\mu a_0 + a_2)\ell^3 = 2\ell \text{ch}_2(LF) \leq 0.$$

Here by the assumption $2\mu + 1 > 0$ and we already obtained that $a_2 \geq 0$. Hence $(2\mu + 1)a_0 + a_2 > 0$ and which is not possible. This is the required contradiction to complete the proof. \square

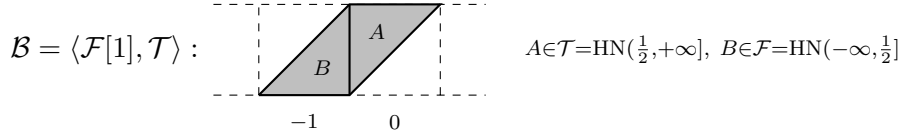
Proposition 4.18. *Let $E \in \text{HN}[-1, 0]$. Then $E^3 \in \text{HN}[\frac{1}{2}, +\infty]$.*

Proof. From the ‘‘Duality’’ Spectral Sequence 4.2 for E we have $(E^*)^0 \cong (-1)^*(E^3)^*$. Here $E^* \in \text{HN}[0, 1]$ and so by Propositions 4.6 and 4.17, $(E^*)^0 \in \text{HN}(-\infty, -\frac{1}{2}]$. Hence $(E^3)^* \in \text{HN}(-\infty, -\frac{1}{2}]$ and so $E^3 \in \text{HN}[\frac{1}{2}, +\infty]$ as required. \square

Theorem 4.19. *We have the following:*

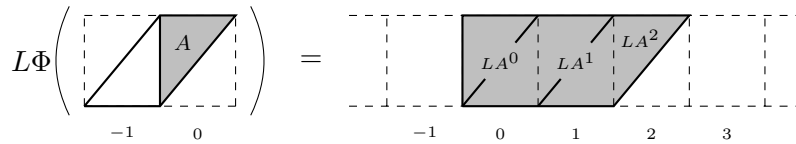
- (i) $L\Phi(\mathcal{B}) \subset \langle \mathcal{B}, \mathcal{B}[-1], \mathcal{B}[-2] \rangle$, and
- (ii) $\Phi L^{-1}[1](\mathcal{B}) \subset \langle \mathcal{B}, \mathcal{B}[-1], \mathcal{B}[-2] \rangle$.

Proof. (i) We can visualize \mathcal{B} as follows:



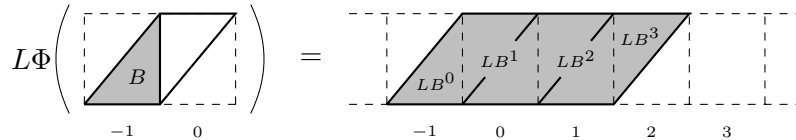
If $E \in \mathcal{F} = \text{HN}(-\infty, \frac{1}{2}]$ then by Propositions 4.6 and 4.17, $LE^0 \in \mathcal{F}$. Also by Proposition 4.7, $LE^3 \in \text{HN}(1, +\infty] \subset \mathcal{F}$. Therefore $L\Phi(E)$ has \mathcal{B} -cohomologies in 1,2,3 positions. That is

$$L\Phi(\mathcal{F})[1] \subset \langle \mathcal{B}, \mathcal{B}[-1], \mathcal{B}[-2] \rangle.$$



On the other hand if $E \in \mathcal{T} = \text{HN}(\frac{1}{2}, +\infty]$ then by Proposition 4.6 $LE^3 = 0$ and by Corollary 4.15 $LE^2 \in \text{HN}(1, +\infty] \subset \mathcal{T}$. So $L\Phi(E)$ has \mathcal{B} -cohomologies in 0,1,2 positions. That is

$$L\Phi(\mathcal{T}) \subset \langle \mathcal{B}, \mathcal{B}[-1], \mathcal{B}[-2] \rangle.$$



Hence $L\Phi(\mathcal{B}) \subset \langle \mathcal{B}, \mathcal{B}[-1], \mathcal{B}[-2] \rangle$, as $\mathcal{B} = \langle \mathcal{F}[1], \mathcal{T} \rangle$.

- (ii) We can use Propositions 4.6, 4.8 and 4.18, and Corollary 4.14 in a similar way to the proof of (i). □

5. (SEMI)STABLE SHEAVES WITH THE CHERN CHARACTER $(r, 0, 0, \chi)$

The aim of this section is to show the following

Theorem 5.1. *Let E be a slope stable sheaf with $\text{ch}_k(E) = 0$ for $k = 1, 2$. Then $E^{**} = \mathcal{P}_x$ for some $x \in X$.*

We have sheaves of this form as the $\text{Coh}(X)$ -cohomology of some of the tilt-stable objects. Let $F \in \mathcal{B}$ be a tilt stable object with $\nu(F) = 0$ and $F_i := H_{\text{Coh}(X)}^i(F)$. From Proposition 3.1 we have, if $\mu(F_{-1}) = 0$ then $\text{ch}_k(F_{-1}) = 0$, and if $\mu(F_0) = 1$ then $\text{ch}_k(L^{-1}F_0) = 0$ for $k = 1, 2$.

Proposition 5.2. *Let E be a semistable reflexive sheaf with $\text{ch}(E) = (r, 0, 0, \chi)$, and $H^k(X, E \otimes \mathcal{P}_x) = 0$ for $k = 0, 3$ and any $x \in X$. Then we have the following:*

- (i) $\text{ch}_k(E^i) = 0$ for $i, k = 1, 2$,
- (ii) $E^{13} \in \text{Coh}^0(X)$,
- (iii) $E^2 \in \text{HN}[0]$, and $E^2 \in V_{\text{Coh}(X)}^\Phi(1)$.

Proof. Since $H^k(X, E \otimes \mathcal{P}_x) = 0$ for $k = 0, 3$ and any $x \in X$, we have $E^0 = E^3 = 0$. So $E \in V_{\text{Coh}(X)}^\Phi(1, 2)$. By Corollary 4.14 $E^1 \in \text{HN}(-\infty, 0]$, and by Proposition 4.16 $E^2 \in \text{HN}[0, +\infty]$. So we have $\ell^2 \text{ch}_1(E^1) \leq 0$ and $\ell^2 \text{ch}_1(E^2) \geq 0$. Therefore $\ell^2 \text{ch}_1(\Phi(E)) \geq 0$ which implies $\ell \text{ch}_2(E) \leq 0$. Since $\text{ch}_2(E) = 0$, we obtain $\text{ch}_1(E^1) = \text{ch}_1(E^2) = 0$. Then we have

$$\text{ch}(E^1) = (a, 0, -b, c), \quad \text{ch}(E^2) = (\chi + a, 0, -b, -r + c),$$

for some $b \geq 0$. Moreover we have $E^1 \in \text{HN}[0]$.

If $E^{13} \neq 0$ then E^1 fits into a $\text{Coh}(X)$ -SES $0 \rightarrow K_1 \rightarrow E^1 \rightarrow \mathcal{P}_{z_1} \mathcal{I}_{C_1} \rightarrow 0$. Then $K_1 \in \text{HN}[0]$ and we have the following exact sequence

$$\cdots \rightarrow K_1^3 \rightarrow E^{13} \rightarrow \mathcal{O}_{-z_1} \rightarrow 0$$

in $\text{Coh}(X)$. If $K_1^3 \neq 0$ then K_1 fits into a $\text{Coh}(X)$ -SES $0 \rightarrow K_2 \rightarrow K_1 \rightarrow \mathcal{P}_{z_2} \mathcal{I}_{C_2} \rightarrow 0$. Then $K_2 \in \text{HN}[0]$ and we have the following exact sequence

$$\cdots \rightarrow K_2^3 \rightarrow K_1^3 \rightarrow \mathcal{O}_{-z_2} \rightarrow 0$$

in $\text{Coh}(X)$. In this way one can consider a chain of SESs. However $\text{rk}(E^1) < +\infty$ implies we have $E^{13} \in \text{Coh}^0(X)$. So $E^{13} = \mathcal{O}_U$ for some 0-subscheme $U \subset X$. Moreover from the convergence of the Mukai Spectral Sequence 4.1 for E , we have the $\text{Coh}(X)$ -SES

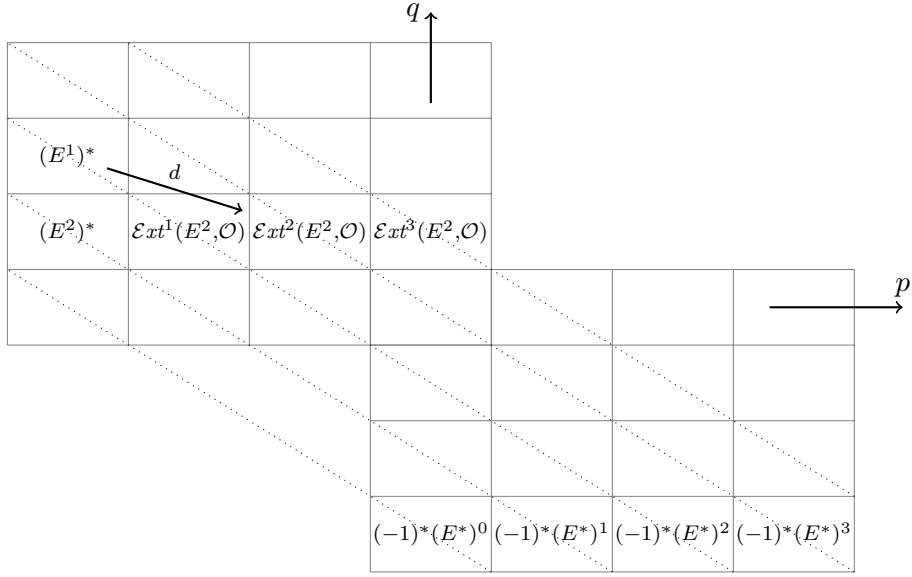
$$0 \rightarrow E^{20} \rightarrow E^{12} \rightarrow Q \rightarrow 0$$

where Q is a subsheaf of $(-1)^*E$ and so $Q \in \text{HN}(-\infty, 0]$. By Proposition 4.8 $E^{20} \in \text{HN}(-\infty, 0]$. This implies $E^{12} \in \text{HN}(-\infty, 0]$. Then $\ell^2 \text{ch}_1(\Phi(E^1)) \leq 0$ and so $-b\ell^3 = 2\ell \text{ch}_2(E^1) \geq 0$. Hence $b = 0$. We also have $\text{ch}_k(E^{20}) = \text{ch}_k(Q) = 0$ for $k = 1, 2$.

On the other hand $\text{ch}_1(E^2) = 0$ implies $E^2 \in \text{HN}[0, +\infty]$ fits into the $\text{Coh}(X)$ -SES

$$0 \rightarrow T \rightarrow E^2 \rightarrow F \rightarrow 0,$$

where $T = T(E^2) \in \text{Coh}^{\leq 1}(X)$ and $F = F(E^2) \in \text{HN}[0]$. Now consider the convergence of the ‘‘Duality’’ Spectral Sequence 4.2 for E :



By Proposition 4.9, E^1 is a reflexive sheaf. So we have $\mathcal{E}xt^3(E^2, \mathcal{O}) = 0$. Therefore T is a pure dimension 1 torsion sheaf, i.e. $T \in \text{Coh}^1(X)$. Apply the FMT Φ to the above SES and consider the LES of $\text{Coh}(X)$ -cohomologies. Then $F \in V_{\text{Coh}(X)}^\Phi(1)$ and $T^0 \cong E^{20}$. So $\text{ch}_k(T^0) = 0$ for $k = 1, 2$. If $T \neq 0$ then $\text{ch}_k(T^0) = 0$ for $k = 1, 2$ implies $\ell^2 \text{ch}_1(T^1) > 0$ and so $\ell^2 \text{ch}_1(F^1) < 0$. Here we have $\text{Coh}(X)$ -SES

$$0 \rightarrow T^1 \rightarrow E^{21} \rightarrow F^1 \rightarrow 0.$$

On the other hand from the convergence of the Mukai Spectral Sequence 4.1 for E , we have the $\text{Coh}(X)$ -SES

$$0 \rightarrow (-1)^*E/Q \rightarrow E^{21} \rightarrow \mathcal{O}_U \rightarrow 0.$$

So there exists an induced surjection map from $(-1)^*E/Q \twoheadrightarrow G$, where G is the kernel of $F^1 \twoheadrightarrow \mathcal{O}_{U'}$ for some $U' \subset U$. Hence, there is a non-zero composition of maps $(-1)^*E \twoheadrightarrow G$. This is not possible as E is slope semistable with $\mu(E) = 0$ and $\mu(G) < 0$. Therefore $T = 0$ as required. \square

Proposition 5.3. *Let E be a slope stable reflexive sheaf with $\text{ch}(E) = (r, 0, 0, \chi)$. Then*

$$E \begin{cases} \in V_{\text{Coh}(X)}^\Phi(1) & \text{if } \chi < 0, \\ \in V_{\text{Coh}(X)}^\Phi(2) & \text{if } \chi > 0, \\ \cong \mathcal{P}_x \text{ for some } x \in X & \text{if } \chi = 0. \end{cases}$$

Proof. First let us consider the case $\chi < 0$. We have $H^k(X, E \otimes \mathcal{P}_x) = 0$ for $k = 0, 3$ and for any $x \in X$. Otherwise $E \cong \mathcal{P}_y$ for some $y \in X$. From Proposition 5.2, we have $E^{13} \cong \mathcal{O}_U$ for a 0-subscheme $U \subset X$, $E^2 \in \text{HN}[0]$ and $E^2 \in V_{\text{Coh}(X)}^\Phi(1)$. Since E is slope stable and $E^{21} \in \text{HN}[0]$, from the convergence of the Mukai Spectral Sequence 4.1 for E we have either $E^{12} = 0$ or $E^{21} \cong E^{13}$.

Case (i) $E^{12} = 0$:

Then E fits into the $\text{Coh}(X)$ -SES

$$0 \rightarrow (-1)^*E \rightarrow E^{21} \rightarrow \mathcal{O}_U \rightarrow 0.$$

However since E is reflexive, if $U \neq \emptyset$ then $\text{Ext}^1(\mathcal{O}_U, E) = 0$; so $E^{21} \cong (-1)^*E \oplus \mathcal{O}_U$. This is not possible as $E^{21} \in V_{\text{Coh}(X)}^\Phi(2)$. Hence $U = \emptyset$ and so $E \in V_{\text{Coh}(X)}^\Phi(2)$. But this is not possible as $\chi < 0$.

Case (ii) $E^{21} \cong E^{13}$:

Since $E^2 \in V_{\text{Coh}(X)}^\Phi(1)$, $E^2 \cong (-1)^*(\mathcal{O}_U)^2 = 0$. Hence $E \in V_{\text{Coh}(X)}^\Phi(1)$ as required.

The proof of the case $\chi > 0$ is similar.

Consider the case $\chi = 0$. If $H^k(X, E \otimes \mathcal{P}_x) = 0$ for $k = 0, 3$ and for any $x \in X$, then E belongs to either $V_{\text{Coh}(X)}^\Phi(1)$ or $V_{\text{Coh}(X)}^\Phi(2)$. But this is not possible as $\text{ch}(\Phi(E)) = (0, 0, 0, -\text{rk}(E))$. Since E is reflexive and slope stable $E \cong \mathcal{P}_x$ for some $x \in X$ as required. \square

Recall that a sheaf E is semi-homogeneous if for all $x \in X$, there is some flat line bundle \mathcal{P}_y such that $t_x^*E \cong E \otimes \mathcal{P}_y$, where $t_x : X \rightarrow X : z \mapsto z + x$. This notion was introduced by Mukai in [13].

Lemma 5.4. *Let E be a slope stable reflexive sheaf with $\text{ch}(E) = (r, 0, 0, \chi)$ and $E \not\cong \mathcal{P}_x$ for any $x \in X$. Then E is a semi-homogeneous bundle on X .*

Proof. If $\chi < 0$ then $\text{ch}_3(E^*) > 0$. Therefore with out loss of generality assume $\chi > 0$.

Now assume the opposite for a contradiction. Then there exists $x \in X$ such that

$$H^0(X, t_x^*E \otimes E^* \otimes \mathcal{P}_y) = H^3(X, t_x^*E \otimes E^* \otimes \mathcal{P}_y) = 0$$

for any $y \in X$. Equivalently we have $(t_x^*E \otimes E^*)^3 = 0$.

Case (i) r is odd:

Let $G := t_x^*E \otimes E^*$. Then $\text{ch}(G) = (r^2, 0, 0, 0)$ and G is a slope semistable reflexive sheaf with $H^0(X, G \otimes \mathcal{P}_y) = H^3(X, G \otimes \mathcal{P}_y) = 0$, for any $y \in X$. Also we have

$$G^* \cong t_{-x}^*(t_x^*E) \otimes (t_x^*E)^*$$

and $H^0(X, G^* \otimes \mathcal{P}_y) = H^3(X, G^* \otimes \mathcal{P}_y) = 0$ for any $y \in X$.

By Proposition 5.2, $G^2, (G^*)^2 \in \text{HN}[0]$ belong to $V_{\text{Coh}(X)}^\Phi(1)$. Now consider the convergence of the ‘‘Duality’’ Spectral Sequence 4.2 for G . Then we have $(G^*)^1 \cong (-1)^*(G^2)^*$. Since $G^2 \in V_{\text{Coh}(X)}^\Phi(1)$, $(G^2)^* \in V_{\text{Coh}(X)}^\Phi(2)$. Therefore $(G^*)^1 \in V_{\text{Coh}(X)}^\Phi(2)$ and so $(G^*)^{13} = 0$. Similarly we have $G^1 \in V_{\text{Coh}(X)}^\Phi(2)$.

From the convergence of the Mukai Spectral Sequence 4.1 for G , we have the Coh(X)-SES

$$0 \rightarrow G^{12} \rightarrow (-1)^*G \rightarrow G^{21} \rightarrow 0.$$

Here $G^{12} \in V_{\text{Coh}(X)}^\Phi(1)$ and $G^{21} \in V_{\text{Coh}(X)}^\Phi(2)$. By Proposition 4.9 G^{21} is reflexive and so G^{12} is reflexive. So we get the Coh(X)-SES

$$0 \rightarrow (G^{21})^* \rightarrow (-1)^*G^* \rightarrow (G^{12})^* \rightarrow 0.$$

Here $(G^{21})^* \in V_{\text{Coh}(X)}^\Phi(1)$ and $(G^{12})^* \in V_{\text{Coh}(X)}^\Phi(2)$. If $\text{rk}(G^{12}) = \alpha$ then $\text{rk}(G^{21}) = r^2 - \alpha$. We have

$$((-1)^*G^*)^1 \cong ((G^{21})^*)^1, \text{ and } ((-1)^*G^*)^2 \cong ((G^{12})^*)^2.$$

Therefore we have $\alpha = \text{rk}(G^{12}) = \text{rk}((G^*)^{12}) = \text{rk}((G^{21})^*) = r^2 - \alpha$. Hence $r^2 = 2\alpha$ and so r is even. This is the required contradiction.

Case (ii) r is even:

By Proposition 5.3, $E \in V_{\text{Coh}(X)}^\Phi(2)$ and E^2 fits into a non-splitting $\text{Coh}(X)$ -SES

$$0 \rightarrow K \rightarrow (-1)^* E^2 \rightarrow \mathcal{O}_z \rightarrow 0,$$

for some $z \in X$. Then $K \in V_{\text{Coh}(X)}^\Phi(1)$ and we have non-splitting $\text{Coh}(X)$ -SES

$$0 \rightarrow \mathcal{P}_z \rightarrow F \rightarrow E \rightarrow 0,$$

where $F = K^1$. Then $\text{rk}(F) = r + 1$. Since \mathcal{P}_z and E are reflexive, we have a non-splitting $\text{Coh}(X)$ -SES

$$0 \rightarrow E^* \rightarrow F^* \rightarrow \mathcal{P}_{-z} \rightarrow 0.$$

Now F fits into the following non-splitting $\text{Coh}(X)$ -SESs:

$$\begin{aligned} 0 \rightarrow t_x^* F \otimes E^* &\rightarrow t_x^* F \otimes F^* \rightarrow t_x^* F \otimes \mathcal{P}_{-z} \rightarrow 0, \\ 0 \rightarrow t_x^* \mathcal{P}_z \otimes E^* &\rightarrow t_x^* F \otimes E^* \rightarrow t_x^* E \otimes E^* \rightarrow 0. \end{aligned}$$

Therefore $\hat{G} := t_x^* F \otimes F^* \in V_{\text{Coh}(X)}^\Phi(1, 2)$ and $\text{rk}(\hat{G}) = (r + 1)^2$ is odd. Now similar to the above case, one can get a contradiction with \hat{G} .

This completes the proof of the Lemma. \square

The universal sheaf associated to the fine moduli space of semi-homogeneous slope stable bundles of a fixed Chern character over an abelian variety gives rise to a (invertible) Fourier-Mukai transform. These were also classified by Orlov ([16]) and when X is a principally polarized abelian threefold with Picard rank 1, any restriction of a universal bundle associated to a Fourier-Mukai transform has Chern character of the form (a^3, a^2b, ab^2, b^3) for some integers $a > 0, b$ with $\text{gcd}(a, b) = 1$. This observation completes the proof of Theorem 5.1.

Remark 5.5. Theorem 5.1 can be interpreted as saying that if a vector bundle E over X satisfies $c_1(E) = 0 = c_2(E)$ then it cannot carry a non-flat Hermitian-Einstein connection. This is analogous to the case where there are no charge 1 $\text{SU}(r)$ instantons on an abelian surface. This is proved in a slick way using the Fourier-Mukai transform and it would be good to avoid the direct proof given for Theorem 5.1 as it would follow more directly from Theorem 6.10.

6. AUTO-EQUIVALENCES OF $\mathcal{A}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}$ UNDER THE FMTs

Let denote the FMTs $\Psi = L\Phi$ and $\hat{\Psi} = \Phi L^{-1}[1]$. Then by Theorem 4.19, we have that the images of an object from \mathcal{B} under Ψ and $\hat{\Psi}$ are complexes whose \mathcal{B} -cohomologies can only be non-zero in the 0, 1 or 2 positions. We have $\Psi \circ \hat{\Psi} \cong (-1)^* \text{id}_{D^b(X)}[-2]$ and $\hat{\Psi} \circ \Psi \cong (-1)^* \text{id}_{D^b(X)}[-2]$. This gives us the following convergence of spectral sequences.

Spectral Sequence 6.1.

$$\begin{aligned} E_2^{p,q} = \Psi_{\mathcal{B}}^p \hat{\Psi}_{\mathcal{B}}^q(E) &\implies H_{\mathcal{B}}^{p+q-2}((-1)^* E), \\ E_2^{p,q} = \hat{\Psi}_{\mathcal{B}}^p \Psi_{\mathcal{B}}^q(E) &\implies H_{\mathcal{B}}^{p+q-2}((-1)^* E), \end{aligned}$$

for E . Here $\Psi_{\mathcal{B}}^i(F) := H_{\mathcal{B}}^i(\Psi(F))$ and $\hat{\Psi}_{\mathcal{B}}^i(F) := H_{\mathcal{B}}^i(\hat{\Psi}(F))$.

These convergence of the spectral sequences for $E \in \mathcal{B}$ look similar to the convergence of some spectral sequences in an abelian surface for coherent sheaves. See [3], [10], [18] for further details.

Recall that if $B_1, B_2 \in \mathcal{B}$ then $\text{Ext}^i(B_1, B_2) = 0$ for any $i < 0$.

Proposition 6.2. *For $E \in D^b(X)$ we have*

$$\mathfrak{S} Z(\Psi(E)) = -\mathfrak{S} Z(E), \text{ and } \mathfrak{S} Z(\widehat{\Psi}(E)) = -\mathfrak{S} Z(E).$$

Proof. Let $\text{ch}(E) = (a_0, a_1, a_2, a_3)$. Then $\mathfrak{S} Z(E) = \frac{3\sqrt{3}}{4}(a_2 - a_1)$. Also we have $\text{ch}(\Psi(E)) = (*, a_3 - a_2, a_3 - 2a_2 + a_1, *)$ and $\text{ch}(\widehat{\Psi}(E)) = (*, a_2 - 2a_1 + a_0, -a_1 + a_0, *)$. Then $\mathfrak{S} Z(\Psi(E)) = \mathfrak{S} Z(\widehat{\Psi}(E)) = -\frac{3\sqrt{3}}{4}(a_2 - a_1)$ as required. \square

From Propositions 2.7, 3.1 and Theorem 5.1 we make the following

Note 6.3. *Let $E \in \mathcal{B}$. Then we have the following:*

- (I) *if $E \in \text{HN}^\nu(-\infty, 0)$, then $\mu^+(E_{-1}) < 0$;*
- (II) *if $E \in \text{HN}^\nu(0, +\infty]$, then $\mu^-(E_0) > 1$; and*
- (III) *for tilt stable E with $\nu(E) = 0$, we have*
 - (i) *$\mu^+(E_{-1}) \leq 0$, and $\mu^-(E_0) \geq 1$,*
 - (ii) *if $\mu(E_{-1}) = 0$ then $E_{-1} = \mathcal{P}_x$ for some $x \in X$, and*
 - (iii) *if $\mu(E_0) = 1$ then $E_0^{**} = L\mathcal{P}_x$ for some $x \in X$.*

Proposition 6.4. *Let $E \in \mathcal{T}'$. Then we have the following:*

- (i) *$H_{\text{Coh}(X)}^0(\widehat{\Psi}_{\mathcal{B}}^2(E)) = 0$, and*
- (ii) *if $\widehat{\Psi}_{\mathcal{B}}^2(E) \neq 0$ then $\mathfrak{S} Z(\widehat{\Psi}_{\mathcal{B}}^2(E)) > 0$.*

Proof. (i) For any $x \in X$,

$$\begin{aligned} \text{Hom}(\widehat{\Psi}_{\mathcal{B}}^2(E), \mathcal{O}_x) &\cong \text{Hom}(\widehat{\Psi}_{\mathcal{B}}^2(E), \widehat{\Psi}_{\mathcal{B}}^2(L\mathcal{P}_{-x})) \\ &\cong \text{Hom}(\widehat{\Psi}(E), \widehat{\Psi}(L\mathcal{P}_{-x})) \\ &\cong \text{Hom}(E, L\mathcal{P}_{-x}) = 0, \end{aligned}$$

since $E \in \mathcal{T}'$ and $L\mathcal{P}_{-x} \in \mathcal{F}'$. Therefore $H_{\text{Coh}(X)}^0(\widehat{\Psi}_{\mathcal{B}}^2(E)) = 0$ as required.

- (ii) From (i), we have $\widehat{\Psi}_{\mathcal{B}}^2(E) \cong A[1]$ for some $0 \neq A \in \text{HN}(-\infty, \frac{1}{2}]$.

Consider the convergence of the spectral sequence:

$$E_2^{p,q} = \widehat{\Psi}_{\text{Coh}(X)}^p(H_{\text{Coh}(X)}^q(E)) \implies \widehat{\Psi}_{\text{Coh}(X)}^{p+q}(E)$$

for E . Let $E_i := H_{\text{Coh}(X)}^i(E)$. Then by Note 6.3, $E_0 \in \text{HN}(1, +\infty]$ and so by Corollary 4.15 and Proposition 4.7 we have

$$(L^{-1}E_0)^2, (L^{-1}E_{-1})^3 \in \text{HN}(0, +\infty].$$

Therefore from the convergence of the above spectral sequence for E , we have

$$A \in \text{HN}(-\infty, \frac{1}{2}] \cap \text{HN}(0, +\infty] = \text{HN}(0, \frac{1}{2}].$$

Let $\text{ch}(A) = (a_0, a_1, a_2, a_3)$. Then from the B-G inequalities for all the H-N semistable factors of A , we have

$$\mathfrak{S} Z(\widehat{\Psi}_{\mathcal{B}}^2(E)) = \mathfrak{S} Z(A[1]) = \frac{3\sqrt{3}}{4}(a_1 - a_2) > 0$$

as required. □

Proposition 6.5. *Let $E \in \mathcal{F}'$. Then we have the following:*

- (i) $H_{\text{Coh}(X)}^{-1}(\widehat{\Psi}_{\mathcal{B}}^0(E)) = 0$, and
- (ii) if $\widehat{\Psi}_{\mathcal{B}}^0(E) \neq 0$ then $\Im Z(\widehat{\Psi}_{\mathcal{B}}^0(E)) < 0$.

Proof. (i) Let $x \in X$. Then

$$\begin{aligned} \text{Hom}(\widehat{\Psi}_{\mathcal{B}}^0(E), \mathcal{O}_x[1]) &\cong \text{Hom}(\Psi\widehat{\Psi}^0(E), \Psi(\mathcal{O}_x[1])) \\ &\cong \text{Hom}(\Psi_{\mathcal{B}}^2\widehat{\Psi}_{\mathcal{B}}^0(E)[-2], L\mathcal{P}_x[1]) \\ &\cong \text{Hom}(\Psi_{\mathcal{B}}^2\widehat{\Psi}_{\mathcal{B}}^0(E), L\mathcal{P}_x[3]) \\ &\cong \text{Hom}(L\mathcal{P}_x, \Psi_{\mathcal{B}}^2\widehat{\Psi}_{\mathcal{B}}^0(E))^*. \end{aligned}$$

From the convergence of the Spectral Sequence 6.1 for E , we have the \mathcal{B} -SES

$$0 \rightarrow \Psi_{\mathcal{B}}^0\widehat{\Psi}_{\mathcal{B}}^1(E) \rightarrow \Psi_{\mathcal{B}}^2\widehat{\Psi}_{\mathcal{B}}^0(E) \rightarrow F \rightarrow 0,$$

where F is a subobject of $(-1)^*E$ and so $F \in \mathcal{F}'$. Moreover by the H-N filtration, F fits into the following \mathcal{B} -SES

$$0 \rightarrow F_0 \rightarrow F \rightarrow F_1 \rightarrow 0,$$

where $F_0 \in \text{HN}^{\nu}[0]$ and $F_1 \in \text{HN}^{\nu}(-\infty, 0)$. Since $L\mathcal{P}_x \in \text{HN}^{\nu}[0]$,

$$\text{Hom}(L\mathcal{P}_x, F_1) = 0.$$

Moreover F_0 has a filtration of ν -stable objects $F_{0,i}$ with $\nu(F_{0,i}) = 0$. By Proposition 2.9, each $F_{0,i}$ fits into a non-splitting \mathcal{B} -SES

$$0 \rightarrow F_{0,i} \rightarrow M_i \rightarrow T_i \rightarrow 0,$$

for some $T_i \in \text{Coh}^0(X)$ such that $M_i[1] \in \mathcal{A}$ is a minimal object. Moreover $L\mathcal{P}_x[1] \in \mathcal{A}$ is a minimal object. So finitely many $x \in X$ we can have $L\mathcal{P}_x \cong M_i$ for some i . So for a generic $x \in X$, $\text{Hom}(L\mathcal{P}_x, M_i) = 0$ and so $\text{Hom}(L\mathcal{P}_x, F_{0,i}) = 0$ which implies $\text{Hom}(L\mathcal{P}_x, F_0) = 0$. Therefore for a generic $x \in X$, $\text{Hom}(L\mathcal{P}_x, F) = 0$.

On the other hand

$$\begin{aligned} \text{Hom}(L\mathcal{P}_x, \Psi_{\mathcal{B}}^0\widehat{\Psi}_{\mathcal{B}}^1(E)) &\cong \text{Hom}(\Psi_{\mathcal{B}}^0(\mathcal{O}_x), \Psi_{\mathcal{B}}^0\widehat{\Psi}_{\mathcal{B}}^1(E)) \\ &\cong \text{Hom}(\Psi(\mathcal{O}_x), \Psi\widehat{\Psi}_{\mathcal{B}}^1(E)) \\ &\cong \text{Hom}(\mathcal{O}_x, \widehat{\Psi}_{\mathcal{B}}^1(E)). \end{aligned}$$

Here $\widehat{\Psi}_{\mathcal{B}}^1(E)$ fits into the \mathcal{B} -SES

$$0 \rightarrow H_{\text{Coh}(X)}^{-1}(\widehat{\Psi}_{\mathcal{B}}^1(E))[1] \rightarrow \widehat{\Psi}_{\mathcal{B}}^1(E) \rightarrow H_{\text{Coh}(X)}^0(\widehat{\Psi}_{\mathcal{B}}^1(E)) \rightarrow 0,$$

where $H_{\text{Coh}(X)}^{-1}(\widehat{\Psi}_{\mathcal{B}}^1(E))$ is torsion free and $H_{\text{Coh}(X)}^0(\widehat{\Psi}_{\mathcal{B}}^1(E))$ can have torsion supported on a 0-scheme of finite length. Hence for generic $x \in X$, $\text{Hom}(\mathcal{O}_x, \widehat{\Psi}_{\mathcal{B}}^1(E)) = 0$. Therefore for generic $x \in X$, $\text{Hom}(L\mathcal{P}_x, \Psi_{\mathcal{B}}^0\widehat{\Psi}_{\mathcal{B}}^1(E)) = \text{Hom}(L\mathcal{P}_x, F) = 0$ implies $\text{Hom}(L\mathcal{P}_x, \Psi_{\mathcal{B}}^2\widehat{\Psi}_{\mathcal{B}}^0(E)) = 0$. Hence $\text{Hom}(\widehat{\Psi}_{\mathcal{B}}^0(E), \mathcal{O}_x[1]) = 0$ for generic $x \in X$. But $H_{\text{Coh}(X)}^{-1}(\widehat{\Psi}_{\mathcal{B}}^0(E))$ is torsion free and so $H_{\text{Coh}(X)}^{-1}(\widehat{\Psi}_{\mathcal{B}}^0(E)) = 0$ as required.

(ii) From (i), we have $\widehat{\Psi}_{\mathcal{B}}^0(E) \cong A$ for some coherent sheaf $0 \neq A \in \text{HN}(\frac{1}{2}, +\infty]$. For any $x \in X$ we have

$$\begin{aligned} \text{Ext}^1(\mathcal{O}_x, A) &\cong \text{Ext}^1(\mathcal{O}_x, \widehat{\Psi}_{\mathcal{B}}^0(E)) \cong \text{Hom}(\Psi(\mathcal{O}_x), \Psi \widehat{\Psi}_{\mathcal{B}}^0(E)[1]) \\ &\cong \text{Hom}(L\mathcal{P}_x, \Psi^2 \widehat{\Psi}_{\mathcal{B}}^0(E)[-1]) = 0. \end{aligned}$$

So $A \in \text{Coh}^{\geq 2}(X)$, and if $\text{ch}(A) = (a_0, a_1, a_2, a_3)$ then we have $a_1 > 0$.

Apply the FMT Ψ to $\widehat{\Psi}_{\mathcal{B}}^0(E)$. Since $\widehat{\Psi}_{\mathcal{B}}^0(E) \in V_{\mathcal{B}}^{\Psi}(2)$, $\Psi_{\mathcal{B}}^2 \widehat{\Psi}_{\mathcal{B}}^0(E) \in \mathcal{B}$ has $\text{Coh}(X)$ -cohomologies:

- LA^1 in position -1 , and
- LA^2 in position 0 .

So we have $A \in V_{\text{Coh}(X)}^{\Phi}(1, 2)$, $LA^1 \in \text{HN}(-\infty, \frac{1}{2}]$ and by Corollary 4.15 $A^2 \in \text{HN}(0, +\infty]$. Therefore $\ell^2 \text{ch}_1(A^1) \leq 0$ and $\ell^2 \text{ch}_1(A^2) \geq 0$. Hence

$$a_2 \ell^3 = 2\ell \text{ch}_2(A) = -\ell^2 \text{ch}_1(\Phi(A)) = -\ell^2 \text{ch}_1(A^2) + \ell^2 \text{ch}_1(A^1) \leq 0.$$

So

$$\Im Z(\widehat{\Psi}_{\mathcal{B}}^0(E)) = \Im Z(A) = \frac{3\sqrt{3}}{4}(a_2 - a_1) < 0$$

as required. □

Proposition 6.6. (I) Let $E \in \mathcal{T}'$. Then we have the following:

(i) $H_{\text{Coh}(X)}^0(\Psi_{\mathcal{B}}^2(E)) = 0$, and (ii) if $\Psi_{\mathcal{B}}^2(E) \neq 0$ then $\Im Z(\Psi_{\mathcal{B}}^2(E)) > 0$.

(II) Let $E \in \mathcal{F}'$. Then we have the following:

(i) $H_{\text{Coh}(X)}^{-1}(\Psi_{\mathcal{B}}^0(E)) = 0$, and (ii) if $\Psi_{\mathcal{B}}^0(E) \neq 0$ then $\Im Z(\Psi_{\mathcal{B}}^0(E)) < 0$.

Proof. (I) Let $E \in \mathcal{T}'$.

(i) Similar to the proof of (i) in Proposition 6.4.

(ii) From (i), we have $\Psi_{\mathcal{B}}^2(E) \cong A[1]$ for some coherent sheaf $0 \neq A \in \text{HN}(-\infty, \frac{1}{2}]$. Let $\text{ch}(A) = (a_0, a_1, a_2, a_3)$. Then $\text{ch}(L^{-1}A) = (a_0, a_1 - a_0, a_2 - 2a_1 + a_0, *)$ and so $a_1 - a_0 < 0$.

Apply the FMT $\widehat{\Psi}$ to $\Psi_{\mathcal{B}}^2(E)$. Since $\Psi_{\mathcal{B}}^2(E) \in V_{\mathcal{B}}^{\widehat{\Psi}}(0)$, $\widehat{\Psi}_{\mathcal{B}}^0 \Psi_{\mathcal{B}}^2(E) \in \mathcal{B}$ has $\text{Coh}(X)$ -cohomologies:

- $(L^{-1}A)^1$ in position -1 , and
- $(L^{-1}A)^2$ in position 0 .

So we have $L^{-1}A \in V_{\text{Coh}(X)}^{\Phi}(1, 2)$, by Corollary 4.14 $(L^{-1}A)^1 \in \text{HN}(-\infty, 0]$, and by Proposition 4.16 $(L^{-1}A)^2 \in \text{HN}[0, +\infty]$. Therefore $\ell^2 \text{ch}_1((L^{-1}A)^1) \leq 0$ and $\ell^2 \text{ch}_1((L^{-1}A)^2) \geq 0$. Hence

$$\begin{aligned} (a_2 - 2a_1 + a_0) \ell^3 &= 2\ell \text{ch}_2(L^{-1}A) = -\ell^2 \text{ch}_1(\Phi(L^{-1}A)) \\ &= -\ell^2 \text{ch}_1((L^{-1}A)^2) + \ell^2 \text{ch}_1((L^{-1}A)^1) \leq 0. \end{aligned}$$

So we have

$$\begin{aligned} \Im Z(\Psi_{\mathcal{B}}^2(E)) &= \Im Z(A[1]) = \frac{3\sqrt{3}}{4}(a_1 - a_2) \\ &= -\frac{3\sqrt{3}}{4}((a_1 - a_0) + (a_2 - 2a_1 + a_0)) > 0 \end{aligned}$$

as required.

(II) Let $E \in \mathcal{F}'$.

- (i) Similar to the proof of (i) in Proposition 6.5.
- (ii) From (i), we have $\Psi_{\mathcal{B}}^0(E) \cong A$ for some $0 \neq A \in \text{HN}(\frac{1}{2}, +\infty]$.
Consider the convergence of the spectral sequence:

$$E_2^{p,q} = \Psi_{\text{Coh}(X)}^p(H_{\text{Coh}(X)}^q(E)) \implies \Psi_{\text{Coh}(X)}^{p+q}(E)$$

for E . Let $E_i := H_{\text{Coh}(X)}^i(E)$. Then by Note 6.3, $E_{-1} \in \text{HN}(-\infty, 0]$ and so by Corollary 4.14 and Proposition 4.8 we have

$$LE_{-1}^1 \in \text{HN}(-\infty, 1], \text{ and } LE_0^0 \in \text{HN}(-\infty, 1].$$

Therefore from the convergence of the above spectral sequence for E , we have

$$A \in \text{HN}(\frac{1}{2}, +\infty] \cap \text{HN}(-\infty, 1] = \text{HN}(\frac{1}{2}, 1].$$

Also A is reflexive, as LE_0^0 and LE_{-1}^1 are reflexive sheaves by Propositions 4.5 and 4.9. Let $\text{ch}(A) = (a_0, a_1, a_2, a_3)$. Then from the B-G inequalities for all the H-N semistable factors of A , we have

$$\Im Z(\Psi_{\mathcal{B}}^0(E)) = \Im Z(A) = \frac{3\sqrt{3}}{4}(a_2 - a_1) \leq 0.$$

Equality holds when $A \in \text{HN}[1]$ with $\text{ch}(A) = (a_0, a_0, a_0, *)$. Then, by considering a Jordan-Hölder filtration for A together with Theorem 5.1, $L^{-1}A$ has a filtration of ideal sheaves $\mathcal{P}_{x_i} \mathcal{I}_{Z_i}$ of some 0-subschemes. Here $\Psi_{\mathcal{B}}^0(E) \cong A \in V_{\mathcal{B}}^{\widehat{\Psi}}(2)$ implies $L^{-1}A \in V_{\text{Coh}(X)}^{\Phi}(2, 3)$. An easy induction on the rank of A also shows that $L^{-1}A \in V_{\text{Coh}(X)}^{\Phi}(1, 3)$ and so $L^{-1}A \in V_{\text{Coh}(X)}^{\Phi}(3)$. But then we have $Z_i = \emptyset$ for all i . Therefore $\widehat{\Psi}_{\mathcal{B}}^2 \Psi_{\mathcal{B}}^0(E) \in \text{Coh}^0(X)$. Now consider the convergence of the Spectral Sequence 6.1 for E . Then we have \mathcal{B} -SES

$$0 \rightarrow \widehat{\Psi}_{\mathcal{B}}^0 \Psi_{\mathcal{B}}^1(E) \rightarrow \widehat{\Psi}_{\mathcal{B}}^2 \Psi_{\mathcal{B}}^0(E) \rightarrow F \rightarrow 0,$$

where F is a subobject of $(-1)^*E$ and so $F \in \mathcal{F}'$. Then $\widehat{\Psi}_{\mathcal{B}}^2 \Psi_{\mathcal{B}}^0(E) \in \text{Coh}^0(X) \subset \mathcal{T}'$ which implies $F = 0$ and $\widehat{\Psi}_{\mathcal{B}}^0 \Psi_{\mathcal{B}}^1(E) \cong \widehat{\Psi}_{\mathcal{B}}^2 \Psi_{\mathcal{B}}^0(E)$. But then we have $\Psi_{\mathcal{B}}^0(E) \cong (-1)^* \Psi_{\mathcal{B}}^0 \widehat{\Psi}_{\mathcal{B}}^0 \Psi_{\mathcal{B}}^1(E) = 0$. This is not possible as $\Psi_{\mathcal{B}}^0(E) \neq 0$. Therefore we have the strict inequality $\Im Z(\Psi_{\mathcal{B}}^0(E)) < 0$ as required to complete the proof. \square

Lemma 6.7. (I) Let $E \in \mathcal{T}'$. Then (i) $\widehat{\Psi}_{\mathcal{B}}^2(E) = 0$, and (ii) $\Psi_{\mathcal{B}}^2(E) = 0$.

(II) Let $E \in \mathcal{F}'$. Then (i) $\widehat{\Psi}_{\mathcal{B}}^0(E) = 0$, and (ii) $\Psi_{\mathcal{B}}^0(E) = 0$.

Proof. (I) Let $E \in \mathcal{T}'$.

- (i) From the convergence of the Spectral Sequence 6.1 for E , we have the \mathcal{B} -SES

$$0 \rightarrow Q \rightarrow \Psi_{\mathcal{B}}^0 \widehat{\Psi}_{\mathcal{B}}^2(E) \rightarrow \Psi_{\mathcal{B}}^2 \widehat{\Psi}_{\mathcal{B}}^1(E) \rightarrow 0.$$

Here Q is a quotient of $(-1)^*E \in \mathcal{T}'$ and so $Q \in \mathcal{T}'$.

Then $\Psi_{\mathcal{B}}^0 \widehat{\Psi}_{\mathcal{B}}^2(E)$ fits into the \mathcal{B} -SES

$$0 \rightarrow T \rightarrow \Psi_{\mathcal{B}}^0 \widehat{\Psi}_{\mathcal{B}}^2(E) \rightarrow F \rightarrow 0$$

for some $T \in \mathcal{T}'$ and $F \in \mathcal{F}'$. Now apply the FMT $\widehat{\Psi}$ and consider the LES of \mathcal{B} -cohomologies. Then we have $\widehat{\Psi}_{\mathcal{B}}^0(T) = 0$, $\widehat{\Psi}_{\mathcal{B}}^1(T) \cong \widehat{\Psi}_{\mathcal{B}}^0(F)$. By Proposition 6.5

$\Im Z(\widehat{\Psi}_{\mathcal{B}}^0(F)) \leq 0$ and by Proposition 6.4 $\Im Z(\widehat{\Psi}_{\mathcal{B}}^2(T)) \geq 0$. So $\Im Z(\widehat{\Psi}(T)) \geq 0$ and by Proposition 6.2 $\Im Z(T) \leq 0$. Since $T \in \mathcal{T}'$, we have $\Im Z(T) = 0$ and $\omega^2 \text{ch}_1^B(T) = 0$. Then by Lemma 1.1, $T \cong T_0$ for some $T_0 \in \text{Coh}^0(X)$. But $\text{Coh}^0(X) \subset V_{\mathcal{B}}^{\widehat{\Psi}}(0)$. Hence $T = 0$ and so $Q = 0$. Then $\Psi_{\mathcal{B}}^0 \widehat{\Psi}_{\mathcal{B}}^2(E) \cong \Psi_{\mathcal{B}}^2 \widehat{\Psi}_{\mathcal{B}}^1(E)$ and so we have $\widehat{\Psi}_{\mathcal{B}}^2(E) \cong (-1)^* \widehat{\Psi}_{\mathcal{B}}^2 \Psi_{\mathcal{B}}^2 \widehat{\Psi}_{\mathcal{B}}^1(E) = 0$ as required.

(ii) Similar to the proof of (i).

(II) Similar to the proofs in (I). □

Corollary 6.8. *Let $E \in \mathcal{B}$. Then*

(i) $\Psi_{\mathcal{B}}^2(E), \widehat{\Psi}_{\mathcal{B}}^2(E) \in \mathcal{T}'$, and

(ii) $\Psi_{\mathcal{B}}^0(E), \widehat{\Psi}_{\mathcal{B}}^0(E) \in \mathcal{F}'$.

Proof. (i) By the torsion theory $\Psi_{\mathcal{B}}^2(E)$ fits into \mathcal{B} -SES

$$0 \rightarrow T \rightarrow \Psi_{\mathcal{B}}^2(E) \rightarrow F \rightarrow 0,$$

for some $T \in \mathcal{T}'$ and $F \in \mathcal{F}'$. Now apply the FMT $\widehat{\Psi}$ and consider the LES of \mathcal{B} -cohomologies. Then by Lemma 6.7, $F = 0$ as required.

Similarly one can prove $\widehat{\Psi}_{\mathcal{B}}^2(E) \in \mathcal{T}'$.

(ii) Similar to the proofs in (i). □

Proposition 6.9. (I) *Let $E \in \mathcal{F}'$. Then (i) $\widehat{\Psi}_{\mathcal{B}}^1(E) \in \mathcal{F}'$, and (ii) $\Psi_{\mathcal{B}}^1(E) \in \mathcal{F}'$.*

(II) *Let $E \in \mathcal{T}'$. Then (i) $\widehat{\Psi}_{\mathcal{B}}^1(E) \in \mathcal{T}'$, and (ii) $\Psi_{\mathcal{B}}^1(E) \in \mathcal{T}'$.*

Proof. (I) (i) By the torsion theory $\widehat{\Psi}_{\mathcal{B}}^1(E)$ fits into \mathcal{B} -SES

$$0 \rightarrow T \rightarrow \widehat{\Psi}_{\mathcal{B}}^1(E) \rightarrow F \rightarrow 0$$

for some $T \in \mathcal{T}'$ and $F \in \mathcal{F}'$. Now we need to show $T = 0$. Apply the FMT Ψ and consider the LES of \mathcal{B} -cohomologies. We get $\Psi_{\mathcal{B}}^1(T) \hookrightarrow \Psi_{\mathcal{B}}^1 \widehat{\Psi}_{\mathcal{B}}^1(E)$ and $T \in V_{\mathcal{B}}^{\Psi}(1)$. Also by the convergence of the Spectral Sequence 6.1 for E , $\Psi_{\mathcal{B}}^1 \widehat{\Psi}_{\mathcal{B}}^1(E)$ is a subobject of $(-1)^*E$. Hence $\Psi_{\mathcal{B}}^1(T) \in \mathcal{F}'$ implies $\Im Z(\Psi_{\mathcal{B}}^1(T)) \leq 0$. On the other hand by Proposition 6.2, $\Im Z(\Psi_{\mathcal{B}}^1(T)) = \Im Z(T) \geq 0$ as $T \in \mathcal{T}'$. Hence $\Im Z(T) = 0$ and $T \in \mathcal{T}'$ implies $\omega^2 \text{ch}_1^B(T) = 0$. So by Lemma 1.1, $T \cong T_0$ for some $T_0 \in \text{Coh}^0(X)$. Since any object from $\text{Coh}^0(X)$ belongs to $V_{\mathcal{B}}^{\Psi}(0)$, $\Psi_{\mathcal{B}}^1(T) = 0$. So $T = 0$ as required.

(ii) Similar to the proof of (i).

(II) Similar to the proofs in (I). □

By Lemma 6.7, Corollary 6.8 and Proposition 6.9 we have

$$\Psi[1](\mathcal{F}'[1]) \subset \mathcal{A}, \quad \text{and} \quad \Psi[1](\mathcal{T}') \subset \mathcal{A}.$$

Since $\mathcal{A} = \langle \mathcal{F}'[1], \mathcal{T}' \rangle$, $\Psi[1](\mathcal{A}) \subset \mathcal{A}$.

Similarly we have $\widehat{\Psi}[1](\mathcal{A}) \subset \mathcal{A}$. The isomorphisms $\widehat{\Psi}[1] \circ \Psi[1] \cong (-1)^* \text{id}_{D^b(X)}$ and $\Psi[1] \circ \widehat{\Psi}[1] \cong (-1)^* \text{id}_{D^b(X)}$ give us the following

Theorem 6.10. *The FMTs $\Psi[1]$ and $\widehat{\Psi}[1]$ give the auto-equivalences*

$$\Psi[1](\mathcal{A}) \cong \mathcal{A}, \text{ and } \widehat{\Psi}[1](\mathcal{A}) \cong \mathcal{A}$$

of the abelian category \mathcal{A} .

ACKNOWLEDGEMENTS

The authors would like to thank Arend Bayer and Tom Bridgeland for very useful discussions. We are also grateful to Jason Lo for pointing out several errors. The second author is funded by Principal's Career Development Scholarship programme and Scottish Overseas Research Student Awards Scheme of the University of Edinburgh, and this work forms a part of his PhD thesis.

REFERENCES

- [1] Daniele Arcara, Aaron Bertram, *Bridgeland-stable moduli spaces for K-trivial surfaces*, With an appendix by Max Lieblich, J. Eur. Math. Soc. 15 (2013), no. 1, 1–38.
- [2] Arend Bayer, Aaron Bertram, Emanuele Macrì, Yukinobu Toda, *Bridgeland Stability conditions on threefolds II: An application to Fujita's conjecture*, arXiv:1106.3430, 2011.
- [3] Claudio Bartocci, Ugo Bruzzo, Daniel Hernández Ruipérez, *Fourier-Mukai and Nahm transforms in geometry and mathematical physics*, Progress in Mathematics, 276. Birkhäuser Boston, Boston, MA, 2009.
- [4] Arend Bayer, Emanuele Macrì, Yukinobu Toda, *Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker type inequalities*, arXiv:1103.5010v2, 2012.
- [5] Tom Bridgeland, *Stability conditions on triangulated categories*, Ann. of Math. (2) 166 (2007), no. 2, 317–345.
- [6] Tom Bridgeland, *Stability conditions on K3 surfaces*, Duke Math. J. 141 (2008), no. 2, 241–291.
- [7] Daniel Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2006.
- [8] Daniel Huybrechts, *Derived and abelian equivalence of K3 surfaces*, J. Algebraic Geom. 17 (2008), no. 2, 375–400.
- [9] Jason Lo, Yogesh More, *Some examples of tilt-stable objects on threefolds*, arXiv:1209.2749v1, 2012.
- [10] Antony Maciocia, *Gieseker stability and the Fourier-Mukai transform for abelian surfaces*, Quart. J. Math. Oxford Ser. (2) 47 (1996), no. 185, 87–100.
- [11] Emanuele Macrì, *Stability conditions on curves*, Math. Res. Lett. 14 (2007) 657–672.
- [12] Emanuele Macrì, *A generalized Bogomolov-Gieseker inequality for the three-dimensional projective space*, arXiv:1207.4980, 2012.
- [13] Shigeru Mukai, *Semi-homogeneous vector bundles on an Abelian variety*, J. Math. Kyoto Univ. 18 (1978), no. 2, 239–272.
- [14] Shigeru Mukai, *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves*, Nagoya Math. J. 81 (1981), 153–175.
- [15] So Okada, *Stability manifold of \mathbb{P}^1* , J. Algebraic Geom., 15 (2006), no. 3, 487–505.
- [16] Dmitri Orlov, *Derived categories of coherent sheaves on abelian varieties and equivalences between them*, (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 66 (2002), no. 3, 131–158; translation in Izv. Math. 66 (2002), no. 3, 569–594.
- [17] Yukinobu Toda, *A note on Bogomolov-Gieseker type inequality for Calabi-Yau 3-folds*, arXiv:1201.4911, 2012.
- [18] Kōta Yoshioka, *Stability and the Fourier-Mukai transform II*, Compos. Math. 145 (2009), no. 1, 112–142.

SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, THE KING'S BUILDINGS, MAYFIELD ROAD, EDINBURGH EH9 3JZ, UK.

E-mail address: A.Maciocia@ed.ac.uk, H.D.B.Piyaratne@sms.ed.ac.uk