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Fourier-Mukai Transforms and Bridgeland Stability Conditions on Abelian Threefolds

Antony Maciocia and Dulip Piyaratne

ABSTRACT

We use the ideas of Bayer, Bertram, Macrì and Toda to construct a Bridgeland stability condition on a principally polarized abelian threefold (X, L) with $\text{NS}(X) = \mathbb{Z}[\ell]$ by establishing their Bogomolov-Gieseker type inequality for certain tilt stable objects associated to the pair $(\mathcal{A}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}, Z_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell})$ on X . This is done by proving the stronger result that $\mathcal{A}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}$ is preserved by a suitable Fourier-Mukai transform.

Introduction

In [5] Bridgeland introduced the notion of stability conditions on triangulated categories and these now have many applications to the study of the geometry of the underlying spaces and highlight the role played by the derived categories of the suitable categories of sheaves on the spaces. The space of stability conditions is known precisely for curves and for abelian surfaces and Bridgeland's geometric stability conditions provide examples for all projective surfaces (see, for example, [6], [11], [1], [15]). A conjectural construction of Bridgeland stability conditions for projective threefolds was introduced in [4] and the problem is reduced to proving an inequality, which the authors call a Bogomolov-Gieseker (B-G for short) type inequality, holds for certain tilt stable objects. This inequality has been shown to hold for three dimensional projective space (see [4] and [12]) and smooth quadric threefold (see [17]), and some progress has been made for more general threefolds (see [19] and [9]). However, there is no known example of a stability condition on a projective Calabi-Yau threefold and this case is especially significant because of the interest from Mathematical Physics and also in connection with Donaldson-Thomas invariants. In this paper, we establish the existence of a particular stability condition on a particular Calabi-Yau threefold (namely, a principally polarized abelian threefold with Picard rank one). However, it is likely that the method will generalize to other Calabi-Yau threefolds while the extension to other stability conditions for the abelian threefold case will be the subject of a forthcoming article.

We reduce the requirement of the B-G type inequality to a smaller class of tilt stable objects as defined in the Definition 2.2. Moreover, they are essentially minimal objects (also called simple objects in the literature) of the heart of the stability condition. In this paper we use Fourier-Mukai theory to prove the B-G type inequality for these minimal objects by showing that the heart is

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preserved by a suitable Fourier-Mukai transform (or FMT for short). For the surface case, the fact that a countable family of (Bridgeland's) geometric stability conditions satisfy the numerical conditions for being a stability condition is actually equivalent to the existence of a Fourier-Mukai transform preserving the heart. The forward implication was proved by Huybrechts ([8]) and the reverse implication is a fairly straightforward exercise (partly done in [20]). For the threefold case, we build on these ideas to establish the reverse implication for our case.

Throughout this paper our abelian varieties will be principally polarized abelian varieties with Picard rank one over \mathbb{C} . Let (X, L) be an abelian variety of dimension three and let ℓ be $c_1(L)$. We use L to canonically identify X with $\mathbf{Pic}^0(X)$. Let $\Phi : D^b(X) \rightarrow D^b(X)$ be the (classical) FMT with the Poincaré line bundle on $X \times X$ as the kernel. Then the image of the category $\mathrm{Coh}(X)$ under the FMT Φ is a subcategory of $D^b(X)$ with non-zero $\mathrm{Coh}(X)$ -cohomologies in 0, 1, 2 and 3 positions. In section 4, we study the slope stability of $\mathrm{Coh}(X)$ -cohomologies under the FM transform Φ in great detail. In particular, we investigate the images under Φ of torsion sheaves supported in dimensions 1 and 2, and of torsion free sheaves whose Harder-Narasimhan (or H-N for short) semistable factors satisfy certain slope bounds.

In [4] and [2], the authors construct their conjectural stability condition hearts as a tilt of a tilt. The first tilt of $\mathrm{Coh}(X)$ associated to the H-N filtration with respect to the twisted slope $\mu_{\omega, B}$ stability is denoted $\mathcal{B}_{\omega, B}$ and the second $\mathcal{A}_{\omega, B}$ associated to the H-N filtration with respect to the tilt slope $\nu_{\omega, B}$ stability. We shall consider the particular case where $\omega = \sqrt{3}\ell/2$ and $B = \ell/2$. Let $\Psi := L\Phi$ and $\hat{\Psi} := \Phi L^{-1}[1]$. At the end of section 4 we prove the images of the abelian category $\mathcal{B}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}$ under the Fourier-Mukai transforms Ψ and $\hat{\Psi}$ have non-zero $\mathcal{B}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}$ -cohomologies only in positions 0, 1 and 2 (see Theorem 4.20). On the other hand, we have the isomorphisms ([14])

$$\Psi \circ \hat{\Psi} \cong (-1)^* \mathrm{id}_{D^b(X)}[-2], \text{ and } \hat{\Psi} \circ \Psi \cong (-1)^* \mathrm{id}_{D^b(X)}[-2].$$

Therefore the abelian category $\mathcal{B}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}$ behaves somewhat similarly to the category of coherent sheaves on an abelian surface under the Fourier-Mukai transform (see [3], [10], [20] for further details). Now Theorem 4.20 becomes the key technical tool to show that the second tilt $\mathcal{A}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}$ is preserved by Ψ .

Under this auto-equivalence, minimal objects are mapped to minimal objects and this provides us with an inequality which bounds the top component of the Chern character of the object. This is the main idea to show that the B-G type inequality is satisfied by our restricted class of minimal objects in $\mathcal{A}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}$. In section 5, we have to show that the B-G type inequality is satisfied by a very special class of minimal objects by showing that they actually do not exist. This result is of interest in its own right as it shows that if a bundle E of such a threefold satisfies $c_1(E) = 0 = c_2(E)$ then it cannot carry a non-flat Hermitian-Einstein connection.

Notation

(i) For $0 \leq i \leq \dim X$

$$\mathrm{Coh}^{\leq i}(X) := \{E \in \mathrm{Coh}(X) : \dim \mathrm{Supp}(E) \leq i\},$$

$$\mathrm{Coh}^{\geq i}(X) := \{E \in \mathrm{Coh}(X) : \text{for } 0 \neq F \subset E, \dim \mathrm{Supp}(F) \geq i\}, \text{ and}$$

$$\mathrm{Coh}^i(X) := \mathrm{Coh}^{\leq i}(X) \cap \mathrm{Coh}^{\geq i}(X).$$

(ii) For an interval $I \subset \mathbb{R} \cup \{+\infty\}$, $\mathrm{HN}_{\omega, B}^{\mu}(I) := \{E \in \mathrm{Coh}(X) : [\mu_{\omega, B}^-(E), \mu_{\omega, B}^+(E)] \subset I\}$. Similarly the subcategory $\mathrm{HN}_{\omega, B}^{\nu}(I) \subset \mathcal{B}_{\omega, B}$ is defined.

- (iii) For a FMT Υ and a heart \mathfrak{A} of a t-structure for which $D^b(X) \cong D^b(\mathfrak{A})$, $\Upsilon_{\mathfrak{A}}^k(E) := H_{\mathfrak{A}}^k(\Upsilon(E))$.
- (iv) For a sequence of integers i_1, \dots, i_s ,

$$V_{\mathfrak{A}}^{\Upsilon}(i_1, \dots, i_s) := \{E \in D^b(X) : \Upsilon_{\mathfrak{A}}^j(E) = 0 \text{ for } j \notin \{i_1, \dots, i_s\}\}.$$

Then $E \in \text{Coh}(X)$ being Υ -WIT $_i$ is equivalent to $E \in V_{\text{Coh}(X)}^{\Upsilon}(i)$.

- (v) Let (X, L) be a principally polarized abelian variety. Then we write Φ for the FMT from X to X with the Poincaré line bundle $\mathcal{P} := m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$ on $X \times X$ as the kernel.
- (vi) For $E \in \text{Coh}(X)$, $E^k := \Phi_{\text{Coh}(X)}^k(E)$.
- (vii) $\Psi := L\Phi$ and $\hat{\Psi} := \Phi L^{-1}[1]$. Here and elsewhere we abuse notation to write L for the functor $L \otimes -$.
- (viii) For a polarized projective threefold (X, L) with Picard rank 1 over \mathbb{C} , the Chern character of E is $\text{ch}(E) = (a_0, a_1\ell, a_2\frac{\ell^2}{2}, a_3\frac{\ell^3}{6})$ for some $a_i \in \mathbb{Q}$. For simplicity we write $\text{ch}(E) = (a_0, a_1, a_2, a_3)$. Here $a_i \in \mathbb{Z}$ for the principally polarized abelian threefold case.

1. Preliminaries

1.1 Construction of stability conditions

We recall the conjectural construction of stability conditions as introduced in [4].

Let X be a smooth projective threefold over \mathbb{C} . Let ω, B be in $\text{NS}_{\mathbb{R}}(X)$ with ω an ample class. The twisted Chern character ch^B with respect to B is defined by $\text{ch}^B(-) = e^{-B} \text{ch}(-)$. So we have

$$\begin{aligned} \text{ch}_0^B &= \text{ch}_0, & \text{ch}_1^B &= \text{ch}_1 - B \text{ch}_0, \\ \text{ch}_2^B &= \text{ch}_2 - B \text{ch}_1 + \frac{B^2}{2} \text{ch}_0, & \text{ch}_3^B &= \text{ch}_3 - B \text{ch}_2 + \frac{B^2}{2} \text{ch}_1 - \frac{B^3}{6} \text{ch}_0. \end{aligned}$$

The twisted slope $\mu_{\omega, B}$ on $\text{Coh}(X)$ is defined by

$$\mu_{\omega, B}(E) = \begin{cases} +\infty & \text{if } E \text{ is a torsion sheaf} \\ \frac{\omega^2 \text{ch}_1^B(E)}{\text{ch}_0^B(E)} & \text{otherwise} \end{cases}$$

for $E \in \text{Coh}(X)$. Then E is said to be $\mu_{\omega, B}$ -(semi)stable, if for any $0 \neq F \subsetneq E$, we have $\mu_{\omega, B}(F) < (\leq) \mu_{\omega, B}(E/F)$. The H-N filtration of E with respect to $\mu_{\omega, B}$ -stability enables us to define the following slopes:

$$\mu_{\omega, B}^+(E) = \max_{0 \neq G \subsetneq E} \mu_{\omega, B}(G), \quad \mu_{\omega, B}^-(E) = \min_{G \subsetneq E} \mu_{\omega, B}(E/G).$$

For an interval $I \subset \mathbb{R} \cup \{+\infty\}$, the subcategory $\text{HN}_{\omega, B}^{\mu}(I) \subset \text{Coh}(X)$ is defined by

$$\text{HN}_{\omega, B}^{\mu}(I) = \{E \in \text{Coh}(X) : [\mu_{\omega, B}^-(E), \mu_{\omega, B}^+(E)] \subset I\}.$$

Define the subcategories $\mathcal{T}_{\omega, B}$ and $\mathcal{F}_{\omega, B}$ of $\text{Coh}(X)$ by setting

$$\mathcal{T}_{\omega, B} = \text{HN}_{\omega, B}^{\mu}(0, +\infty], \quad \mathcal{F}_{\omega, B} = \text{HN}_{\omega, B}^{\mu}(-\infty, 0].$$

Then $(\mathcal{T}_{\omega, B}, \mathcal{F}_{\omega, B})$ forms a torsion pair on $\text{Coh}(X)$. Let the abelian category $\mathcal{B}_{\omega, B} = \langle \mathcal{F}_{\omega, B}[1], \mathcal{T}_{\omega, B} \rangle \subset D^b(X)$ be the corresponding tilt of $\text{Coh}(X)$.

The central charge function $Z_{\omega, B} : K(X) \rightarrow \mathbb{C}$ is defined by

$$Z_{\omega, B}(E) = - \int_X e^{-B-i\omega} \text{ch}(E).$$

So $Z_{\omega,B}(E) = \left(-\text{ch}_3^B(E) + \frac{\omega^2}{2} \text{ch}_1^B(E)\right) + i \left(\omega \text{ch}_2^B(E) - \frac{\omega^3}{6} \text{ch}_0^B(E)\right)$. The following result is very useful:

LEMMA 1.1. [4, Lemma 3.2.1] *For any $0 \neq E \in \mathcal{B}_{\omega,B}$, one of the following conditions holds:*

- (i) $\omega^2 \text{ch}_1^B(E) > 0$,
- (ii) $\omega^2 \text{ch}_1^B(E) = 0$ and $\Im Z_{\omega,B}(E) > 0$,
- (iii) $\omega^2 \text{ch}_1^B(E) = \Im Z_{\omega,B}(E) = 0$, $-\Re Z_{\omega,B}(E) > 0$ and $E \cong T$ for some $0 \neq T \in \text{Coh}^0(X)$.

As a result of this Lemma, they go on to remark that the vector $(\omega^2 \text{ch}_1^B, \Im Z_{\omega,B}, -\Re Z_{\omega,B})$ for objects in $\mathcal{B}_{\omega,B}$ behaves like the Chern character vector $\text{ch} = (\text{ch}_0, \text{ch}_1, \text{ch}_2)$ for coherent sheaves on a surface.

Following [4], the tilt-slope $\nu_{\omega,B}$ on $\mathcal{B}_{\omega,B}$ is defined by

$$\nu_{\omega,B}(E) = \begin{cases} +\infty & \text{if } \omega^2 \text{ch}_1^B(E) = 0 \\ \frac{\Im Z_{\omega,B}(E)}{\omega^2 \text{ch}_1^B(E)} & \text{otherwise} \end{cases}$$

for $E \in \mathcal{B}_{\omega,B}$. Then E is said to be $\nu_{\omega,B}$ -(semi)stable, if for any $0 \neq F \subsetneq E$ in $\mathcal{B}_{\omega,B}$, we have $\nu_{\omega,B}(F) < (\leq) \nu_{\omega,B}(E/F)$. In [4] it is proved that the abelian category $\mathcal{B}_{\omega,B}$ satisfies the H-N property with respect to the tilt-slope stability. So the following slopes can be defined for $E \in \mathcal{B}_{\omega,B}$:

$$\nu_{\omega,B}^+(E) = \max_{0 \neq G \subsetneq E} \nu_{\omega,B}(G), \quad \nu_{\omega,B}^-(E) = \min_{G \subsetneq E} \nu_{\omega,B}(E/G).$$

For an interval $I \subset \mathbb{R} \cup \{+\infty\}$, the subcategory $\text{HN}_{\omega,B}^\nu(I) \subset \mathcal{B}_{\omega,B}$ is defined by

$$\text{HN}_{\omega,B}^\nu(I) = \{E \in \mathcal{B}_{\omega,B} : [\nu_{\omega,B}^-(E), \nu_{\omega,B}^+(E)] \subset I\}.$$

Define the subcategories $\mathcal{T}'_{\omega,B}$ and $\mathcal{F}'_{\omega,B}$ of $\mathcal{B}_{\omega,B}$ by setting

$$\mathcal{T}'_{\omega,B} = \text{HN}_{\omega,B}^\nu(0, +\infty], \quad \mathcal{F}'_{\omega,B} = \text{HN}_{\omega,B}^\nu(-\infty, 0].$$

Then $(\mathcal{T}'_{\omega,B}, \mathcal{F}'_{\omega,B})$ forms a torsion pair on $\mathcal{B}_{\omega,B}$. Let the abelian category $\mathcal{A}_{\omega,B} = \langle \mathcal{F}'_{\omega,B}[1], \mathcal{T}'_{\omega,B} \rangle \subset D^b(X)$ be the corresponding tilt of $\mathcal{B}_{\omega,B}$.

CONJECTURE 1.2. [4, Conjecture 3.2.6] *The pair $(Z_{\omega,B}, \mathcal{A}_{\omega,B})$ is a Bridgeland stability condition on $D^b(X)$.*

DEFINITION 1.3. *Let $\mathcal{C}_{\omega,B}$ be the class of $\nu_{\omega,B}$ -stable objects $E \in \mathcal{B}_{\omega,B}$ with $\nu_{\omega,B}(E) = 0$.*

Then $E[1] \in \mathcal{A}_{\omega,B}$ for any $E \in \mathcal{C}_{\omega,B}$.

CONJECTURE 1.4. [4, Conjecture 3.2.7] *Any $E \in \mathcal{C}_{\omega,B}$ satisfies the so called **Bogomolov-Gieseker Type Inequality**:*

$$\Re Z_{\omega,B}(E[1]) < 0, \text{ i.e. } \text{ch}_3^B(E) < \frac{\omega^2}{2} \text{ch}_1^B(E).$$

Assume $B \in \text{NS}_{\mathbb{Q}}(X)$ and $\omega \in \text{NS}_{\mathbb{R}}(X)$ be an ample class with ω^2 is rational. Then the abelian category $\mathcal{A}_{\omega,B}$ satisfies the following important property. This was proved for rational classes ω in [4]. However a similar proof can be used when we have a weaker condition, namely ω^2 is rational. For example, a different parametrization given by $\omega \mapsto \sqrt{3}\omega$ is considered in [12].

LEMMA 1.5. [4, Proposition 5.2.2] *The abelian category $\mathcal{A}_{\omega,B}$ is Noetherian.*

As a corollary we have the following

COROLLARY 1.6. [4, Corollary 5.2.4] *The Conjectures 1.2 and 1.4 are equivalent.*

1.2 Fourier-Mukai transforms on abelian threefolds

Let us quickly recall the notion of Fourier-Mukai transform on abelian threefolds. See [3], [7] for further details on Fourier-Mukai theory.

Let (X, L) be a principally polarized abelian threefold with Picard rank 1. Let $\ell := c_1(L)$. Then $\chi(L) = \frac{\ell^3}{6} = 1$. Let $\mathcal{P} = m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$ be the Poincaré line bundle on $X \times X$. Then the Fourier-Mukai transform $\Phi : D^b(X) \rightarrow D^b(X)$ with kernel \mathcal{P} is defined by

$$\Phi(-) := \mathbf{R}p_{2*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} p_1^*(-)).$$

Here $X \xleftarrow{p_1} X \times X \xrightarrow{p_2} X$ are the projection maps. In [14] Mukai proved that Φ is an auto-equivalence of the derived category $D^b(X)$ and also

$$\Phi \circ \Phi \cong (-1)^* \text{id}_{D^b(X)}[-3].$$

The Chern character of any $E \in D^b(X)$ is of the form $\text{ch}(E) = (a_0, a_1\ell, a_2\frac{\ell^2}{2}, a_3\frac{\ell^3}{6})$ for some integers a_i . Then we have (see [7, Lemma 9.23]):

$$\text{ch}(\Phi(E)) = (a_3, -a_2\ell, a_1\frac{\ell^2}{2}, -a_0\frac{\ell^3}{6}).$$

2. Minimal objects of $\mathcal{A}_{\omega, B}$ and B-G Type Inequality of Threefolds

2.1 Some minimal objects of $\mathcal{A}_{\omega, B}$

We identify some classes of minimal objects of the abelian category $\mathcal{A}_{\omega, B}$ of a projective threefold X . See [8] for a detailed discussion on minimal objects of some abelian categories associated to Bridgeland stability conditions on a surface.

PROPOSITION 2.1. *For any $x \in X$, the skyscraper sheaf \mathcal{O}_x is a minimal object in $\mathcal{A}_{\omega, B}$.*

Proof. For any $x \in X$, $\mathcal{O}_x \in \mathcal{T}_{\omega, B}$ and also $\mathcal{O}_x \in \mathcal{T}'_{\omega, B}$. Therefore $\mathcal{O}_x \in \mathcal{A}_{\omega, B}$. Let

$$0 \rightarrow a \rightarrow \mathcal{O}_x \rightarrow b \rightarrow 0$$

be a short exact sequence (SES for short) in $\mathcal{A}_{\omega, B}$ such that $a \neq 0$. Then in order to prove $\mathcal{O}_x \in \mathcal{A}_{\omega, B}$ is minimal, we need to show $b = 0$. We obtain the following long exact sequence (LES for short) of $\mathcal{B}_{\omega, B}$ -cohomologies associated to the above $\mathcal{A}_{\omega, B}$ -SES:

$$0 \rightarrow A_{-1} \rightarrow 0 \rightarrow B_{-1} \rightarrow A_0 \rightarrow \mathcal{O}_x \rightarrow B_0 \rightarrow 0.$$

Here $A_k := H_{\mathcal{B}_{\omega, B}}^k(a)$ and $B_k := H_{\mathcal{B}_{\omega, B}}^k(b)$. We have $A_{-1} = 0$ and so $a \cong A_0 \neq 0$. Let $C := A_0/B_{-1}$. Then

$$0 \rightarrow C \rightarrow \mathcal{O}_x \rightarrow B_0 \rightarrow 0$$

is a SES in $\mathcal{B}_{\omega, B}$. We obtain the following LES of $\text{Coh}(X)$ -cohomologies associated to the above $\mathcal{B}_{\omega, B}$ -SES:

$$0 \rightarrow C^{-1} \rightarrow 0 \rightarrow B_0^{-1} \rightarrow C^0 \rightarrow \mathcal{O}_x \rightarrow B_0^0 \rightarrow 0.$$

Here $C^k := H_{\text{Coh}(X)}^k(C)$ and $B_0^k := H_{\text{Coh}(X)}^k(B_0)$. We have $C^{-1} = 0$ and so $C \cong C^0$.

If $B_0^0 \neq 0$ then $\mathcal{O}_x \cong B_0^0$ and $B_0^{-1} \cong C^0 \in \mathcal{T}_{\omega,B} \cap \mathcal{F}_{\omega,B} = \{0\}$. So $C = 0$ and $B_{-1} \cong A_0 \in \mathcal{T}'_{\omega,B} \cap \mathcal{F}'_{\omega,B} = \{0\}$ which implies $A_0 = 0$. This is not possible and so $B_0^0 = 0$. Therefore $B_0 \cong B_0^{-1}[1]$ and

$$0 \rightarrow B_0^{-1} \rightarrow C^0 \rightarrow \mathcal{O}_x \rightarrow 0$$

is a SES in $\text{Coh}(X)$. Here $\text{ch}(\mathcal{O}_x) = (0, 0, 0, 1)$. If $B_0^{-1} \neq 0$ then

$$0 \geq \mu_{\omega,B}(B_0^{-1}) = \mu_{\omega,B}(C^0) > 0.$$

This is not possible and so $B_0^{-1} = 0$ and $C^0 \cong \mathcal{O}_x$. Therefore $b \cong B_{-1}[1]$ and we have the following SES in $\mathcal{B}_{\omega,B}$:

$$0 \rightarrow B_{-1} \rightarrow A_0 \rightarrow \mathcal{O}_x \rightarrow 0.$$

Since $\text{ch}(\mathcal{O}_x) = (0, 0, 0, 1)$, if $B_{-1} \neq 0$ then

$$0 \geq \nu_{\omega,B}(B_{-1}) = \nu_{\omega,B}(A_0) > 0.$$

This is not possible and so $B_{-1} = 0$. Therefore $b = 0$ and so $\mathcal{O}_x \in \mathcal{A}_{\omega,B}$ is a minimal object as required. \square

We now identify further minimal objects.

DEFINITION 2.2. Let $\mathcal{M}_{\omega,B}$ be the class of all objects $E \in \mathcal{B}_{\omega,B}$ such that

- (i) E is $\nu_{\omega,B}$ -stable,
- (ii) $\nu_{\omega,B}(E) = 0$, and
- (iii) $\text{Ext}^1(\mathcal{O}_x, E) = 0$ for any skyscraper sheaf \mathcal{O}_x of $x \in X$.

Then clearly $\mathcal{M}_{\omega,B} \subset \mathcal{C}_{\omega,B}$.

LEMMA 2.3. Let $E \in \mathcal{M}_{\omega,B}$. Then $E[1]$ is a minimal object of $\mathcal{A}_{\omega,B}$.

Proof. By definition $\mathcal{M}_{\omega,B} \subset \mathcal{F}'_{\omega,B}$ and so $E[1] \in \mathcal{A}_{\omega,B}$. Let

$$0 \rightarrow a \rightarrow E[1] \rightarrow b \rightarrow 0$$

be a SES in $\mathcal{A}_{\omega,B}$ such that $b \neq 0$. Now we need to show that $a = 0$ or equivalently $b \cong E[1]$. We have the following LES of $\mathcal{B}_{\omega,B}$ -cohomologies associated to the above $\mathcal{A}_{\omega,B}$ -SES:

$$0 \rightarrow A_{-1} \rightarrow E \rightarrow B_{-1} \rightarrow A_0 \rightarrow 0 \rightarrow B_0 \rightarrow 0.$$

Here $A_k := H_{\mathcal{B}_{\omega,B}}^k(a)$ and $B_k := H_{\mathcal{B}_{\omega,B}}^k(b)$. We have $B_0 = 0$ and so $b \cong B_{-1}[1]$ which implies $B_{-1} \neq 0$.

Case (I) $A_{-1} \neq 0$:

Subcase (i) $E/A_{-1} \neq 0$:

Then $E/A_{-1} \hookrightarrow B_{-1}$ and $\nu_{\omega,B}^+(B_{-1}) \leq 0$ implies $\nu_{\omega,B}(E/A_{-1}) \leq 0$. On the other hand $\nu_{\omega,B}(E/A_{-1}) > 0$ as $A_{-1} \neq 0$ and E is $\nu_{\omega,B}$ -stable with $\nu_{\omega,B}(E) = 0$. But this not possible.

Subcase (ii) $E/A_{-1} = 0$:

Then $A_{-1} \cong E$ and $B_{-1} \cong A_0 \in \mathcal{F}'_{\omega,B} \cap \mathcal{T}'_{\omega,B} = \{0\}$. This is not possible as $B_{-1} \neq 0$.

Case (II) $A_{-1} = 0$:

Then we have the following SES in $\mathcal{B}_{\omega,B}$:

$$0 \rightarrow E \rightarrow B_{-1} \rightarrow A_0 \rightarrow 0. \quad (\blacklozenge)$$

Subcase (i) $A_0 \neq 0$:

Here $\nu_{\omega,B}(E) = 0$ implies $\omega^2 \text{ch}_1^B(E) > 0$ and $\Im Z_{\omega,B}(E) = 0$. Then

$$\nu_{\omega,B}(B_{-1}) = \frac{\Im Z_{\omega,B}(A_0)}{\omega^2 \text{ch}_1^B(E) + \omega^2 \text{ch}_1^B(A_0)} \leq 0$$

implies $\Im Z_{\omega,B}(A_0) \leq 0$. If $\omega^2 \text{ch}_1^B(A_0) \neq 0$ then $\nu_{\omega,B}(A_0) > 0$ implies $\Im Z_{\omega,B}(A_0) > 0$; which is not possible. Hence $\omega^2 \text{ch}_1^B(A_0) = 0$ and by Lemma 1.1, $\Im Z_{\omega,B}(A_0) \geq 0$. So $\Im Z_{\omega,B}(A_0) = 0$ and $A_0 \cong T$ for some $0 \neq T \in \text{Coh}^0(X)$. Then the $\mathcal{B}_{\omega,B}$ -SES (\blacklozenge) corresponds to an element from $\text{Ext}^1(A_0, E) = \text{Ext}^1(T, E)$. But we have $\text{Ext}^1(\mathcal{O}_x, E) = 0$ for any $x \in X$ and so $\text{Ext}^1(T, E) = 0$. So $B_{-1} \cong T \oplus E$. Then T is a subobject of B_{-1} . But this is not possible as $\nu_{\omega,B}(T) = +\infty$ and $E \in \mathcal{M}_{\omega,B}$.

Subcase (ii) $A_0 = 0$:

Then $a = 0$ and $b \cong B_{-1}[1] \cong E[1]$ as required.

This completes the proof of the lemma. \square

Some classes of tilt stable candidates have been identified in [4].

Recall, for $E \in D^b(X)$ the discriminant $\overline{\Delta}_\omega$ in the sense of Drézet is defined by

$$\overline{\Delta}_\omega(E) = (\omega^2 \text{ch}_1^B(E))^2 - 2\omega^3 \text{ch}_0^B(E) \cdot \omega \text{ch}_2^B(E).$$

PROPOSITION 2.4. [4, Proposition 7.4.1] *Let E be a $\mu_{\omega,B}$ -stable locally free sheaf on X with $\overline{\Delta}_\omega(E) = 0$. Then either E or $E[1]$ in $\mathcal{B}_{\omega,B}$ is $\nu_{\omega,B}$ -stable.*

Example 2.5. Let (X, L) be a polarized projective threefold and let $\ell := c_1(L)$. Consider the classes $B = \frac{1}{2}\ell$ and $\omega = \frac{\sqrt{3}}{2}\ell$. Then $\overline{\Delta}_\omega(\mathcal{O}) = \overline{\Delta}_\omega(L) = 0$. So, by Proposition 2.4, $\mathcal{O}[1], L \in \mathcal{B}_{\omega,B}$ are $\nu_{\omega,B}$ -stable. Also $\Im Z_{\omega,B}(\mathcal{O}[1]) = \Im Z_{\omega,B}(L) = 0$. Therefore $\nu_{\omega,B}(\mathcal{O}[1]) = \nu_{\omega,B}(L) = 0$. So by Lemma 2.3, $\mathcal{O}[2], L[1] \in \mathcal{A}_{\omega,B}$ are minimal objects.

NOTE 2.6. *The tilt stable objects associated to minimal objects in Example 2.5 clearly satisfy the corresponding B-G type inequalities.*

2.2 Reduction of B-G type inequality for minimal objects

The following propositions are important.

PROPOSITION 2.7. [9, Proposition 3.1] *Let $E \in \mathcal{B}_{\omega,B}$ be a $\nu_{\omega,B}$ -semistable object with $\nu_{\omega,B} < +\infty$. Then $H_{\text{Coh}(X)}^{-1}(E)$ is a reflexive sheaf.*

PROPOSITION 2.8. [9, Proposition 3.5] *Let $0 \rightarrow E \rightarrow E' \rightarrow Q \rightarrow 0$ be a non splitting SES in $\mathcal{B}_{\omega,B}$ with $Q \in \text{Coh}^0(X)$, $\text{Hom}(\mathcal{O}_x, E') = 0$ for all $x \in X$, and $\omega^2 \text{ch}_1^B(E) \neq 0$. If E is $\nu_{\omega,B}$ -stable then E' is $\nu_{\omega,B}$ -stable.*

Recall that $\mathcal{C}_{\omega,B}$ is the class of $\nu_{\omega,B}$ -stable objects $E \in \mathcal{B}_{\omega,B}$ with $\nu_{\omega,B}(E) = 0$.

PROPOSITION 2.9. *Let $E \in \mathcal{C}_{\omega,B}$. Then there exists $E' \in \mathcal{M}_{\omega,B}$ (i.e. $E'[1]$ is a minimal object of $\mathcal{A}_{\omega,B}$) such that*

$$0 \rightarrow E \rightarrow E' \rightarrow Q \rightarrow 0$$

is a SES in $\mathcal{B}_{\omega,B}$ for some $Q \in \text{Coh}^0(X)$.

Proof. Let $E \in \mathcal{C}_{\omega,B} \setminus \mathcal{M}_{\omega,B}$. Assume the opposite of the claim in the proposition for E . Then there exists a sequence of non-splitting SESs in $\mathcal{B}_{\omega,B}$, for $i \geq 1$

$$0 \rightarrow E_{i-1} \rightarrow E_i \rightarrow \mathcal{O}_{y_i} \rightarrow 0,$$

where $E_0 = E$, $E_i \in \mathcal{C}_{\omega,B}$ (see Proposition 2.8). So for each $i \geq 1$,

$$0 \rightarrow \mathcal{O}_{y_i} \rightarrow E_{i-1}[1] \rightarrow E_i[1] \rightarrow 0$$

is a SES in $\mathcal{A}_{\omega,B}$. Therefore

$$E[1] = E_0[1] \twoheadrightarrow E_1[1] \twoheadrightarrow E_2[1] \twoheadrightarrow \dots$$

is an infinite chain of quotients in $\mathcal{A}_{\omega,B}$. But this is not possible as $\mathcal{A}_{\omega,B}$ is Noetherian by Lemma 1.5. This is a contradiction. \square

It follows that $E \in \mathcal{C}_{\omega,B}$ satisfies the B-G type inequality if the corresponding $E' \in \mathcal{M}_{\omega,B}$ satisfies the B-G type inequality.

3. Abelian category $\mathcal{A}_{\sqrt{3}B,B}$, FMT and stability conditions

3.1 Some properties of $\mathcal{A}_{\sqrt{3}B,B}$

We discuss some of the properties of the abelian category $\mathcal{A}_{\sqrt{3}B,B}$ for an arbitrary polarized projective threefold (X, L) with Picard rank 1. Let $\ell := c_1(L)$. Let $B = b\ell$ for $b \in \mathbb{Q}_{>0}$. Then for $E \in D^b(X)$

$$\Im Z_{\sqrt{3}B,B}(E) = \sqrt{3}b\ell(\text{ch}_2(E) - b\ell \text{ch}_1(E)).$$

PROPOSITION 3.1. *Let $E \in \mathcal{B}_{\sqrt{3}B,B}$ and let $E_i = H_{\text{Coh}(X)}^i(E)$. Let E_i^\pm be the H-N semistable factors of E_i with highest and lowest $\mu_{\sqrt{3}B,B}$ slopes. Then we have the following:*

- (i) if $E \in \text{HN}_{\sqrt{3}B,B}^\nu(-\infty, 0)$ and $E_{-1} \neq 0$, then $\ell^2 \text{ch}_1(E_{-1}^+) < 0$;
- (ii) if $E \in \text{HN}_{\sqrt{3}B,B}^\nu(0, +\infty)$ and $\text{rk}(E_0) \neq 0$, then $\ell^2 \text{ch}_1(E_0^-) > 2b\ell^3 \text{ch}_0(E_0^-)$; and
- (iii) if E is tilt semistable with $\nu_{\sqrt{3}B,B}(E) = 0$, then
 - (a) for $E_{-1} \neq 0$, $\ell^2 \text{ch}_1(E_{-1}) \leq 0$ with equality if and only if $\text{ch}_2(E_{-1}) = 0$,
 - (b) for $\text{rk}(E_0) \neq 0$, $\ell^2 \text{ch}_1(E_0) \geq 2b\ell^3 \text{ch}_0(E_0)$ with equality if and only if $(\text{ch}_1(E_0))^2 = 2 \text{ch}_0(E_0) \text{ch}_2(E_0)$.

Proof. $E \in \mathcal{B}_{\sqrt{3}B,B}$ fits in to the $\mathcal{B}_{\sqrt{3}B,B}$ -SES

$$0 \rightarrow E_{-1}[1] \rightarrow E \rightarrow E_0 \rightarrow 0.$$

- (i) Since $E \in \text{HN}_{\sqrt{3}B,B}^\nu(-\infty, 0)$, $E_{-1}[1] \in \text{HN}_{\sqrt{3}B,B}^\nu(-\infty, 0)$. We have $0 \neq E_{-1}^+ \subseteq E_{-1}$. Hence $E_{-1}^+[1] \in \text{HN}_{\sqrt{3}B,B}^\nu(-\infty, 0)$.

Let $\text{ch}(E_{-1}^+) = (a_0, a_1, a_2, a_3)$. Assume the opposite for a contradiction; so that $a_1 \geq 0$. We have

$$\begin{aligned} \nu_{\sqrt{3}B,B}(E_{-1}^+[1]) &= \frac{-\Im Z_{\sqrt{3}B,B}(E_{-1}^+)}{-3B^2 \text{ch}_1^B(E_{-1}^+)} \\ &= \frac{\sqrt{3}ba_1(ba_0 - a_1) + \frac{\sqrt{3}}{2}ba_1^2 + \frac{\sqrt{3}}{2}b(a_1^2 - a_0a_2)}{3a_0b^2(ba_0 - a_1)}. \end{aligned}$$

Since E_{-1}^+ is $\mu_{\sqrt{3}B,B}$ -semistable we have, by the usual B-G inequality,

$$a_1^2 - a_0 a_2 \geq 0$$

and since $E_{-1}^+ \in \mathcal{F}_{\sqrt{3}B,B}$, we have $\nu_{\sqrt{3}B,B}(E_{-1}^+[1]) \neq +\infty$ and so $ba_0 - a_1 > 0$. Hence, as $a_0 > 0$, we have $\nu_{\sqrt{3}B,B}(E_{-1}^+[1]) \geq 0$. But this is not possible as $E_{-1}^+[1] \in \text{HN}_{\sqrt{3}B,B}^\nu(-\infty, 0)$. This is the required contradiction to complete the proof.

- (ii) Since $E \in \text{HN}_{\sqrt{3}B,B}^\nu(0, +\infty]$, $E_0 \in \text{HN}_{\sqrt{3}B,B}^\nu(0, +\infty]$. We have $0 \neq E_0^-$ is a torsion free quotient of E_0 . Since $E_0 \in \text{HN}_{\sqrt{3}B,B}^\nu(0, +\infty]$ we have $E_0^- \in \text{HN}_{\sqrt{3}B,B}^\nu(0, +\infty]$.

Let $\text{ch}(E_0^-) = (a_0, a_1, a_2, a_3)$. Assume the opposite for a contradiction; so that $a_1 \leq 2ba_0$. We have

$$\begin{aligned} \nu_{\sqrt{3}B,B}(E_0^-) &= \frac{\Im Z_{\sqrt{3}B,B}(E_0^-)}{3B^2 \text{ch}_1^B(E_0^-)} \\ &= \frac{-\frac{\sqrt{3}}{2}b(a_1^2 - a_0 a_2) + \frac{\sqrt{3}}{2}ba_1(a_1 - 2ba_0)}{3b^2 a_0(a_1 - ba_0)}. \end{aligned}$$

Here $E_0^- \in \mathcal{T}_{\sqrt{3}B,B}$ is torsion free which implies

$$a_1 - ba_0 > 0;$$

E_0^- is $\mu_{\sqrt{3}B,B}$ -semistable which implies (by the usual B-G inequality)

$$a_1^2 - a_0 a_2 \geq 0.$$

Therefore $\nu_{\sqrt{3}B,B}(E_0^-) \leq 0$. But this is not possible as $E_0^- \in \text{HN}_{\sqrt{3}B,B}^\nu(0, +\infty]$. This is the required contradiction to complete the proof.

- (iii) Similar to (i) one can show that if $E \in \text{HN}_{\sqrt{3}B,B}^\nu(-\infty, 0]$ and $E_{-1} \neq 0$, then $\ell^2 \text{ch}_1(E_{-1}^+) \leq 0$. Therefore for $E \in \text{HN}_{\sqrt{3}B,B}^\nu[0]$ we have $\ell^2 \text{ch}_1(E_{-1}) \leq 0$. The equality holds if and only if E_{-1} is slope semistable, and so it satisfies the usual B-G inequality. Since $\nu_{\sqrt{3}B,B}(E_{-1}) \leq 0$ we have $\ell^2 \text{ch}_1(E_{-1}) = 0$ if and only if $\text{ch}_2(E_{-1}) = 0$.

Proof of (b) is similar to that of (a). □

3.2 Relation of FMT to stability conditions

Let (X, L) be a principally polarized abelian threefold with Picard rank 1. Let $\ell := c_1(L)$. Then $\chi(L) = \frac{\ell^3}{6} = 1$ and the Chern character of $E \in D^b(X)$ is of the form $\text{ch}(E) = (a_0, a_1 \ell, a_2 \frac{\ell^2}{2}, a_3 \frac{\ell^3}{6})$ for some integers a_i . Define the classes $B = \frac{1}{2}\ell$ and $\omega = \frac{\sqrt{3}}{2}\ell$.

The following is a key result in this paper.

PROPOSITION 3.2. *If $\Phi(L^{-1}E)[2] \in \mathcal{B}_{\omega,B}$ for any $E \in \mathcal{M}_{\omega,B} \setminus \{L\mathcal{P}_x : x \in X\}$, then the B-G type inequality holds for the objects in $\mathcal{C}_{\omega,B}$.*

Proof. By Proposition 2.9, it is enough to check that the B-G type inequality is satisfied by each object in $\mathcal{M}_{\omega,B}$. Moreover, the objects in $\{L\mathcal{P}_x : x \in X\} \subset \mathcal{M}_{\omega,B}$ satisfy the B-G type inequality (see Note 2.6). Then we only need to check the inequality for objects in $\mathcal{M}_{\omega,B} \setminus \{L\mathcal{P}_x : x \in X\}$.

Let $E \in \mathcal{M}_{\omega,B} \setminus \{L\mathcal{P}_x : x \in X\}$ and assume $\Phi(L^{-1}E)[2] \in \mathcal{B}_{\omega,B}$. Let $\text{ch}(E) = (a_0, a_1 \ell, a_2 \frac{\ell^2}{2}, a_3 \frac{\ell^3}{6})$ and then $\Im Z_{\omega,B}(E) = 0$ implies $a_1 = a_2$. Now the B-G type inequality says

$$\Delta := -a_0 + 3a_1 - a_3 > 0.$$

By Proposition 3.1, we have $\ell^2 \text{ch}_1(E_{-1}) \leq 0$ and $\ell^2 \text{ch}_1(E_0) \geq 0$. Here $E_i = H_{\text{Coh}(X)}^i(E)$. So $a_1 \ell^3 = \ell^2 \text{ch}_1(E) = \ell^2 \text{ch}_1(E_0) - \ell^2 \text{ch}_1(E_{-1}) \geq 0$.

Let $F = \Phi(L^{-1}E)[2]$ and let $\text{ch}(F) = (b_0, b_1 \ell, b_2 \frac{\ell^2}{2}, b_3 \frac{\ell^3}{6})$. Then $b_0 = a_3 - a_0$ and $b_1 = b_2 = a_1 - a_0$. Now $b_1 = b_2$ implies $\Im Z_{\omega, B}(F) = 0$. Also $F \in \mathcal{B}_{\omega, B}$ implies $\omega^2 \text{ch}_1^B(F) \geq 0$, i.e. $2b_1 - b_0 \geq 0$. If $\omega^2 \text{ch}_1^B(F) = 0$ then $\Im Z_{\omega, B}(F) = 0$ implies $F \cong T$ for some $T \in \text{Coh}^0(X)$ (see Lemma 1.1). If $T \neq 0$ then E has a filtration with factors of the form $L\mathcal{P}_x[1] \notin \mathcal{M}_{\omega, B}$. This is not possible and so $\omega^2 \text{ch}_1^B(F) > 0$. That is $2b_1 - b_0 = -a_0 + 2a_1 - a_3 > 0$.

Hence $\Delta > 0$ and so E satisfies the B-G type inequality. This completes the proof as required. \square

Our main goal in the rest of this paper is to prove that $\Phi L^{-1}[2]$ and its quasi-inverse $L\Phi[1]$ are auto-equivalences of the abelian category $\mathcal{A}_{\omega, B}$. Under an equivalence of abelian categories minimal objects are mapped to minimal objects and so the hypothesis of Proposition 3.2 is satisfied. Therefore, by Corollary 1.6, we have the following:

THEOREM 3.3. *The pair $(\mathcal{A}_{\omega, B}, Z_{\omega, B})$ is a Bridgeland stability condition on $D^b(X)$.*

4. Fourier-Mukai transforms on $\text{Coh}(X)$ of Abelian Threefolds

From here onward, we always assume (X, L) is a principally polarized abelian threefold with Picard rank 1. Let $\ell := c_1(L)$. Then $\chi(L) = \frac{\ell^3}{6} = 1$ and the Chern character of any $E \in D^b(X)$ is of the form $\text{ch}(E) = (a_0, a_1 \ell, a_2 \frac{\ell^2}{2}, a_3 \frac{\ell^3}{6})$ for some integers a_i . Define the classes $B = \frac{1}{2}\ell$ and $\omega = \frac{\sqrt{3}}{2}\ell$.

If $E \in \text{Coh}(X)$ then the slope $\mu(E)$ is defined by $\mu(E) := \mu_{\frac{1}{\sqrt{6}}\ell, 0}(E)$. That is $\mu(E) = \frac{a_1}{a_0}$ when $a_0 \neq 0$, and $\mu(E) = +\infty$ when $a_0 = 0$. In the rest of the paper we mostly use μ slope for coherent sheaves and we simply write $\text{HN} = \text{HN}_{\frac{1}{\sqrt{6}}\ell, 0}^\mu$. Then $\mu_{\omega, B}(E) = \frac{9}{2}(\mu(E) - \frac{1}{2})$. Moreover define $\mathcal{T}_0 = \text{HN}(0, +\infty]$ and $\mathcal{F}_0 = \text{HN}(-\infty, 0]$. Also for simplicity we write $\mathcal{T} = \mathcal{T}_{\omega, B}$, $\mathcal{F} = \mathcal{F}_{\omega, B}$, $\mathcal{B} = \mathcal{B}_{\omega, B}$, $\nu = \nu_{\omega, B}$, $\text{HN}_{\omega, B}^\nu = \text{HN}^\nu$, $\mathcal{T}' = \mathcal{T}'_{\omega, B}$, $\mathcal{F}' = \mathcal{F}'_{\omega, B}$, and $\mathcal{A} = \mathcal{A}_{\omega, B}$. Then by the definitions, we have $\mathcal{F} = \text{HN}(-\infty, \frac{1}{2}]$ and $\mathcal{T} = \text{HN}(\frac{1}{2}, +\infty]$.

Let Φ be the Fourier-Mukai transform with kernel the Poincaré line bundle \mathcal{P} . The isomorphism $\Phi \circ \Phi \cong (-1)^* \text{id}_{\text{Coh}(X)}[-3]$ gives us the following convergence of spectral sequence.

MUKAI SPECTRAL SEQUENCE 4.1.

$$E_2^{p, q} = \Phi_{\text{Coh}(X)}^p \Phi_{\text{Coh}(X)}^q(E) \implies H_{\text{Coh}(X)}^{p+q-3}((-1)^* E),$$

for E . Here $\Phi_{\text{Coh}(X)}^i(F) = H_{\text{Coh}(X)}^i(\Phi(F))$.

For $E \in \text{Coh}(X)$, we write

$$E^k = \Phi_{\text{Coh}(X)}^k(E).$$

Then for example $E^{120} = \Phi_{\text{Coh}(X)}^0 \Phi_{\text{Coh}(X)}^2 \Phi_{\text{Coh}(X)}^1(E)$. Using this notation, we can deduce the following immediately from the spectral sequence:

$$E^{00} = E^{01} = E^{32} = E^{33} = 0, \quad E^{10} \cong E^{02} \quad \text{and} \quad E^{31} \cong E^{23}.$$

Let $\mathbf{R}\Delta$ denote the derived dualizing functor $\mathbf{R}\mathcal{H}om(-, \mathcal{O})[3]$. Then due to Mukai,

$$(\Phi \circ \mathbf{R}\Delta)[3] \cong (-1)^* \mathbf{R}\Delta \circ \Phi$$

(see [14, (3.8)]). This gives us the convergence of the following spectral sequences.

“DUALITY” SPECTRAL SEQUENCE 4.2.

$$\Phi_{\text{Coh}(X)}^p(\mathcal{E}xt^{q+3}(E, \mathcal{O})) \implies ? \longleftarrow (-1)^* \mathcal{E}xt^{p+3}(\Phi_{\text{Coh}(X)}^{3-q}(E), \mathcal{O})$$

for $E \in \text{Coh}(X)$.

The aim of this section is to use mainly the Mukai and “Duality” spectral sequences to study the slope stability of sheaves under the FM transform Φ . More precisely, we consider the $\text{Coh}(X)$ cohomology sheaves of the images under Φ of torsion sheaves supported in dimensions 1 and 2. We also study the transforms of torsion free sheaves whose H-N semistable factors satisfy certain slope bounds.

NOTATION 4.3. Any $E \in \text{Coh}(X)$ fits into $\text{Coh}(X)$ -SES

$$0 \rightarrow T \rightarrow E \rightarrow F \rightarrow 0$$

for some $T \in \mathcal{T}_0$ and $F \in \mathcal{F}_0$. Denote $T(E) = T$ and $F(E) = F$.

Any torsion free sheaf E fits into a non-splitting $\text{Coh}(X)$ -SES

$$0 \rightarrow E \rightarrow E^{**} \rightarrow T \rightarrow 0$$

for some $T \in \text{Coh}^{\leq 1}(X)$. Here E^{**} is a reflexive sheaf. If E is rank 1 then E^{**} is a line bundle and so $E^{**} \cong L^k \mathcal{P}_x$ for some $k \in \mathbb{Z}$ and $x \in X$.

NOTATION 4.4. If E is a rank 1 torsion free sheaf with $c_1(E) = k\ell$ then we can write $E = L^k \mathcal{P}_x \mathcal{I}_C$. Here \mathcal{I}_C is the ideal sheaf of the structure sheaf $\mathcal{O}_C := L^{-k} \mathcal{P}_{-x} \otimes (E^{**}/E) \in \text{Coh}^{\leq 1}(X)$ of a subscheme $C \subset X$ of dimension ≤ 1 .

PROPOSITION 4.5. Let $E \in \text{Coh}(X)$. If $E^0 \neq 0$ then E^0 is a reflexive sheaf.

Proof. Let $x \in X$. Then for $0 \leq i \leq 2$, we have

$$\text{Hom}(\mathcal{O}_x, E^0[i]) \cong \text{Hom}(\Phi(\mathcal{O}_x), \Phi(E^0)[i]) \cong \text{Hom}(\mathcal{P}_x, E^{02}[-2+i])$$

from the convergence of the Mukai Spectral Sequence 4.1 for E . So $\text{Hom}(\mathcal{O}_x, E^0) = \text{Ext}^1(\mathcal{O}_x, E^0) = 0$, and

$$\begin{aligned} \text{Ext}^2(\mathcal{O}_x, E^0) &\cong \text{Hom}(\mathcal{P}_x, E^{02}) \\ &\cong \text{Hom}(\mathcal{P}_x, E^{10}), \text{ by the Mukai Spectral Sequence for } E \\ &\cong \text{Hom}(\Phi(\mathcal{O}_x), \Phi(E^1)) \\ &\cong \text{Hom}(\mathcal{O}_x, E^1). \end{aligned}$$

Hence, as any map $\mathcal{O}_x \rightarrow E^1$ must factor through the torsion subsheaf of E^1 and E^1 is coherent, only finitely many of these can be non-zero. So $\dim\{x \in X : \text{Ext}^2(\mathcal{O}_x, E^0) \neq 0\} \leq 0$. Therefore E^0 is a reflexive sheaf. \square

PROPOSITION 4.6. Let $E \in \text{Coh}(X)$. Then we have the following:

- (i) if $E \in \mathcal{T}_0$ then $E^3 = 0$, and
- (ii) if $E \in \mathcal{F}_0$ then $E^0 = 0$.

Proof. (i) Let $E \in \mathcal{T}_0$. Then for any $x \in X$, we have

$$\begin{aligned} \mathrm{Hom}(E^3, \mathcal{O}_x) &\cong \mathrm{Hom}(\Phi(E)[3], \Phi(\mathcal{P}_{-x})[3]) \\ &\cong \mathrm{Hom}(E, \mathcal{P}_{-x}) = 0, \end{aligned}$$

as $\mathcal{P}_{-x} \in \mathcal{F}_0$. Therefore $E^3 = 0$ as required.

(ii) Let $E \in \mathcal{F}_0$. We can assume E is μ -stable using H-N and Jordan-Hölder filtrations. For generic $x \in X$ and $i = 1, 2$ we have

$$\mathrm{Hom}(E^1, \mathcal{O}_x[i]) = \mathrm{Hom}(E^2, \mathcal{O}_x[i+1]) = \mathrm{Hom}(E^3, \mathcal{O}_x[i+2]) = 0.$$

Hence for generic $x \in X$,

$$\begin{aligned} \mathrm{Hom}(E^0, \mathcal{O}_x) &\cong \mathrm{Hom}(\Phi(E), \mathcal{O}_x) \\ &\cong \mathrm{Hom}(\Phi(E), \Phi(\mathcal{P}_{-x})[3]) \\ &\cong \mathrm{Hom}(E, \mathcal{P}_{-x}[3]) \\ &\cong \mathrm{Hom}(\mathcal{P}_{-x}, E)^*. \end{aligned}$$

(a) Case $\mu(E) < 0$:

Then $\mathrm{Hom}(\mathcal{P}_{-x}, E) = 0$.

(b) Case $\mu(E) = 0$:

Since E is assumed to be μ -stable, any map in $\mathrm{Hom}(\mathcal{P}_{-x}, E)$ must be an isomorphism and so $E^0 = 0$.

Therefore for generic $x \in X$, $\mathrm{Hom}(E^0, \mathcal{O}_x) = 0$. By Proposition 4.5 if $E^0 \neq 0$ then it is reflexive. So $E^0 = 0$. □

PROPOSITION 4.7. *Let $E \in \mathrm{Coh}(X)$. Then*

(i) $E^3 \in \mathcal{T}_0$, and

(ii) $E^0 \in \mathcal{F}_0$.

Proof. (i) Let $T = T(E^3) \in \mathcal{T}_0$ and $F = F(E^3) \in \mathcal{F}_0$, so that

$$0 \rightarrow T \rightarrow E^3 \rightarrow F \rightarrow 0$$

is a SES in $\mathrm{Coh}(X)$. Now we need to show that $F = 0$. Apply Φ to the above SES and consider the LES of $\mathrm{Coh}(X)$ -cohomologies. Then we have $F \in V_{\mathrm{Coh}(X)}^\Phi(1)$, $T \in V_{\mathrm{Coh}(X)}^\Phi(0, 1, 2)$ (for the definition of V see the notation section of the introduction) and

$$0 \rightarrow T^1 \rightarrow E^{31} \rightarrow F^1 \rightarrow T^2 \rightarrow 0$$

is a LES in $\mathrm{Coh}(X)$. Here $E^{31} \cong E^{23}$ (from the Mukai Spectral Sequence 4.1 for E) and so

$$\begin{aligned} \mathrm{Hom}(E^{31}, F^1) &\cong \mathrm{Hom}(E^{23}, F^1) \\ &\cong \mathrm{Hom}(\Phi(E^2)[3], \Phi(F)[1]) \\ &\cong \mathrm{Hom}(E^2, F[-2]) = 0. \end{aligned}$$

Hence $F \cong (-1)^*F^{12} \cong (-1)^*T^{22} = 0$ (from the Mukai Spectral Sequence 4.1 for T) as required.

(ii) Similar to the proof of (i). □

PROPOSITION 4.8. *Let $E \in \mathcal{F}_0$. If $E^1 \neq 0$ then E^1 is a reflexive sheaf.*

Proof. By Proposition 4.6, $E^0 = 0$. Let $x \in X$. Then from the convergence of the Mukai Spectral Sequence 4.1 for E and $0 \leq i \leq 2$, we have

$$\begin{aligned} \mathrm{Hom}(\mathcal{O}_x, E^1[i]) &\cong \mathrm{Hom}(\Phi(\mathcal{O}_x), \Phi(E^1)[i]) \\ &\cong \mathrm{Hom}(\mathcal{P}_x, E^{12}[i-2]) \end{aligned}$$

as $\mathrm{Hom}(\mathcal{P}_x, \tau_{>2}\Phi(E^1)[i]) \cong \mathrm{Hom}(\mathcal{P}_x, E^{13}[i-3]) = 0$. Therefore $\mathrm{Hom}(\mathcal{O}_x, E^1) = \mathrm{Ext}^1(\mathcal{O}_x, E^1) = 0$ and $\mathrm{Ext}^2(\mathcal{O}_x, E^1) \cong \mathrm{Hom}(\mathcal{P}_x, E^{12})$.

From the convergence of the Mukai Spectral Sequence 4.1 for E

$$0 \rightarrow E^{20} \rightarrow E^{12} \rightarrow F \rightarrow 0$$

is a SES in $\mathrm{Coh}(X)$. Here F is a subobject of $(-1)^*E$. By applying the functor $\mathrm{Hom}(\mathcal{P}_x, -)$ we obtain the exact sequence

$$0 \rightarrow \mathrm{Hom}(\mathcal{P}_x, E^{20}) \rightarrow \mathrm{Hom}(\mathcal{P}_x, E^{12}) \rightarrow \mathrm{Hom}(\mathcal{P}_x, F) \rightarrow \cdots$$

Now $F \in \mathcal{F}_0$ and by Proposition 4.7 E^{20} is also in \mathcal{F}_0 . Therefore we have $\mathrm{Hom}(\mathcal{P}_x, F) \neq 0$ or $\mathrm{Hom}(\mathcal{P}_x, E^{20}) \neq 0$ for at most a finite number of points $x \in X$. That is $\dim\{x \in X : \mathrm{Ext}^2(\mathcal{O}_x, E^1) \neq 0\} \leq 0$. Therefore E^1 is a reflexive sheaf. \square

PROPOSITION 4.9. *If E is a torsion sheaf then $E^2 \in \mathcal{T}_0$.*

Proof. Let $T = T(E^2)$ and $F = F(E^2)$. Then $0 \rightarrow T \rightarrow E^2 \rightarrow F \rightarrow 0$ is a SES in $\mathrm{Coh}(X)$. By applying Φ we obtain the LES

$$0 \rightarrow T^1 \rightarrow E^{21} \rightarrow F^1 \rightarrow T^2 \rightarrow 0$$

in $\mathrm{Coh}(X)$. Here $F \in V_{\mathrm{Coh}(X)}^\Phi(1)$. From the convergence of the Mukai Spectral Sequence 4.1 for E , E^{21} fits into the $\mathrm{Coh}(X)$ -SES

$$0 \rightarrow Q \rightarrow E^{21} \rightarrow E^{13} \rightarrow 0,$$

where Q is a quotient of $(-1)^*E$. So Q is a torsion sheaf and $\mathrm{Hom}(Q, F^1) = 0$ as F^1 is a reflexive sheaf (see Proposition 4.8). Therefore

$$\begin{aligned} \mathrm{Hom}(E^{21}, F^1) &\cong \mathrm{Hom}(E^{13}, F^1) \\ &\cong \mathrm{Hom}(\Phi(E^1)[3], \Phi(F)[1]) \\ &\cong \mathrm{Hom}(E^1, F[-2]) = 0. \end{aligned}$$

Hence $F^1 \cong T^2$ and so $F \cong (-1)^*F^{12} \cong (-1)^*T^{22} = 0$ (from the Mukai Spectral Sequence 4.1 for T) as required. \square

For $x \in X$, let L_x denote $L\mathcal{P}_x$. Since $h^0(X, L_x) = \chi(L_x) = 1$, let the divisor D_x be the zero locus of the unique (up to scale) section $0 \neq s_x \in H^0(X, L_x)$. Moreover, as $t_x^*L \otimes L^{-1} = \mathcal{P}_x$, we have $D_x = t_x^*D_e$, where $e \in X$ is the identity element. For positive integer m , let mD_x be the non-reduced divisor in the linear system $|m\ell|$ topologically supported on D_x . So mD_x is the zero locus of the section $s_x^{\otimes m}$ of L_x^m , and we have the SES

$$0 \rightarrow L_x^{-m} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{mD_x} \rightarrow 0$$

in $\mathrm{Coh}(X)$. For $E \in \mathrm{Coh}(X)$, apply the functor $E \otimes^{\mathbf{L}} (-)$ to the above SES and consider the LES of $\mathrm{Coh}(X)$ -cohomologies. Since L_x^{-m} and \mathcal{O}_X are locally free, we have $\mathrm{Tor}_i(E, \mathcal{O}_{mD_x}) = 0$

for $i \geq 2$. Now assume $E \in \text{Coh}^k(X)$ for some $k \in \{0, 1, 2, 3\}$. For generic $x \in X$, we have $\dim(\text{Supp}(E) \cap D_x) \leq (k-1)$ and so $\text{Tor}_1(E, \mathcal{O}_{mD_x}) \in \text{Coh}^{\leq k-1}(X)$. However, $L_x^{-m}E \in \text{Coh}^k(X)$, and so $\text{Tor}_1(E, \mathcal{O}_{mD_x}) = 0$. Therefore, we have the SES

$$0 \rightarrow L_x^{-m}E \rightarrow E \rightarrow E|_{mD_x} \rightarrow 0 \quad (\dagger)$$

in $\text{Coh}(X)$. Since any $E \in \text{Coh}(X)$ is an extension of sheaves from $\text{Coh}^k(X)$, for generic $x \in X$, $\text{Tor}_i(E, \mathcal{O}_{mD_x}) = 0$ for $i \geq 1$ and so we have the SES (\dagger) . Moreover, when $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$ is a SES in $\text{Coh}(X)$, for generic $x \in X$ we have $\text{Tor}_i(E_j, \mathcal{O}_{mD_x}) = 0$, $i \geq 1$ for each j , and so

$$0 \rightarrow E_1|_{mD_x} \rightarrow E_2|_{mD_x} \rightarrow E_3|_{mD_x} \rightarrow 0$$

is a SES in $\text{Coh}(X)$.

PROPOSITION 4.10. *Let $E \in \text{Coh}^{\leq 1}(X)$. Then $E^1 \in \mathcal{T}_0$.*

Proof. $E \in \text{Coh}^{\leq 1}(X)$ fits into the torsion sequence $0 \rightarrow E_0 \rightarrow E \rightarrow E_1 \rightarrow 0$, where $E_0 \in \text{Coh}^0(X)$ and $E_1 \in \text{Coh}^1(X)$. Here $E_0 \in V_{\text{Coh}(X)}^\Phi(0)$ and so $E^1 = E_1^1$. Therefore we only need to prove the claim for a pure dimension 1 torsion sheaf E . Then for sufficiently large $m > 0$ and suitable $x \in X$, $L_x^{-m}E \in V_{\text{Coh}(X)}^\Phi(1)$, and

$$0 \rightarrow L_x^{-m}E \rightarrow E \rightarrow E|_{mD_x} \rightarrow 0$$

is a SES in $\text{Coh}(X)$ for $E|_{mD_x} \in \text{Coh}^0(X)$. By applying the FMT Φ , we have $(L_x^{-m}E)^1 \rightarrow E^1$. Therefore, we only need to show $(L_x^{-m}E)^1 \in \mathcal{T}_0$. Let us show this by proving the claim for a pure dimension one torsion sheaf $E \in V_{\text{Coh}(X)}^\Phi(1)$. Then $\text{ch}(E) = (0, 0, \alpha, \beta)$, where $\alpha > 0$ and $\beta \leq 0$ since $\beta = -\text{rk}(E^1)$.

Let $T = T(E^1)$ and $F = F(E^1)$. Then $0 \rightarrow T \rightarrow E^1 \rightarrow F \rightarrow 0$ is a SES in $\text{Coh}(X)$. Now we need to show $F = 0$. So suppose $F \neq 0$ for a contradiction. Apply the FMT Φ and consider the LES of $\text{Coh}(X)$ -cohomologies. Then we have $T \in V_{\text{Coh}(X)}^\Phi(2)$, $F \in V_{\text{Coh}(X)}^\Phi(1, 2)$ and

$$0 \rightarrow F^1 \rightarrow T^2 \rightarrow E \rightarrow F^2 \rightarrow 0$$

is a LES in $\text{Coh}(X)$.

Case (i) The map $T^2 \rightarrow E$ is zero:

Then $T \cong (-1)^*T^{21} \cong (-1)^*F^{11} = 0$ from the Mukai Spectral Sequence 4.1 as $F \in V_{\text{Coh}(X)}^\Phi(1, 2)$. So $E = F^2$ and hence $F \in V_{\text{Coh}(X)}^\Phi(2)$. Therefore $F \cong (-1)^*E^1$ and so $\text{ch}(F) = (-\beta, \alpha, 0, 0)$. Here $\alpha > 0$ and which is not possible as $\mu(F) \leq 0$.

Case (ii) The map $T^2 \rightarrow E$ is non-zero:

Let $K = \text{im}(T^2 \rightarrow E)$. Then $K \in \text{Coh}^1(X)$ and the $\text{Coh}(X)$ -SES $0 \rightarrow F^1 \rightarrow T^2 \rightarrow K \rightarrow 0$ corresponds to an element from $\text{Ext}^1(K, F^1)$. Here F^1 is a reflexive sheaf and so there exists a locally free sheaf U and a torsion free sheaf V such that $0 \rightarrow F^1 \rightarrow U \rightarrow V \rightarrow 0$ is a non-splitting SES in $\text{Coh}(X)$. By applying the functor $\text{Hom}(K, -)$, we obtain the following exact sequence:

$$\cdots \rightarrow \text{Hom}(K, V) \rightarrow \text{Ext}^1(K, F^1) \rightarrow \text{Ext}^1(K, U) \rightarrow \cdots$$

Here $\text{Hom}(K, V) = 0$ and $\text{Ext}^1(K, U) \cong \text{Ext}^2(U, K)^* \cong H^2(X, U^* \otimes K)^* = 0$ as $K \in \text{Coh}^{\leq 1}(X)$. So $\text{Ext}^1(K, F^1) = 0$ implies $T^2 \cong F^1 \oplus K$. Here $T^2 \in V_{\text{Coh}(X)}^\Phi(1)$ implies $F^1 = 0$ and so $K \cong T^2$. Then $F^2 \cong E/T^2$ and also $F \in V_{\text{Coh}(X)}^\Phi(2)$. Since $F^2 \in V_{\text{Coh}(X)}^\Phi(1)$, it is a pure dimension 1 torsion sheaf. So $\text{ch}(F^2) = (0, 0, \alpha', \beta')$, where $\alpha' > 0$ and $\beta' \leq 0$. Therefore $\text{ch}(F) = (-\beta', \alpha', 0, 0)$ and which is not possible as $\mu(F) \leq 0$ implies $\alpha' \leq 0$.

Therefore $F = 0$ as required to complete the proof. \square

Recall from [13, Prop 6.16] for any positive integer s , the semi-homogeneous bundle $(L^s)^0$ is slope stable. In the rest of this section we abuse notation to write $(L^s)^0$ for the functor $(L^s)^0 \otimes -$.

PROPOSITION 4.11. *Let $E_n \in \text{HN}[0, +\infty)$, $n \in \mathbb{N}$ be a sequence of coherent sheaves on X . Assume that for any $s > 0$, there is $N(s) > 0$ such that for any $n > N(s)$, $(L^s)^0 E_n \in V_{\text{Coh}(X)}^\Phi(3)$. Then $\mu^+(E_n) \rightarrow 0$ as $n \rightarrow +\infty$.*

Proof. Assume $\mu^+(E_n) \not\rightarrow 0$ as $n \rightarrow +\infty$ for a contradiction. Then there exists $\varepsilon > 0$ such that for any $N' > 0$ there is $n > N'$ satisfying $\mu^+(E_n) > \varepsilon$.

Let T_n be the slope semistable H-N factor of E_n with the highest slope, i.e. $\mu(T_n) = \mu^+(E_n)$. There is $s \in \mathbb{N}$ such that $\mu((L^s)^0) > -\varepsilon$. Then for any $N' > 0$ there is $n > N'$ such that $(L^s)^0 T_n \in \mathcal{T}_0$. Therefore, for some $n > N(s)$ we have

$$\begin{aligned} \text{Hom}((L^s)^0 T_n, (L^s)^0 E_n) &\cong \text{Hom}(\Phi((L^s)^0 T_n), \Phi((L^s)^0 E_n)) \\ &\cong 0, \end{aligned}$$

as $((L^s)^0 T_n)^3 = 0$ (from Proposition 4.6) and $(L^s)^0 E_n \in V_{\text{Coh}(X)}^\Phi(3)$. This is the required contradiction to complete the proof. \square

Let s be a positive integer. Consider the Fourier-Mukai functor defined by

$$\Pi = \Phi \circ (L^s)^0 \circ \Phi[3].$$

Then $\Pi_{\text{Coh}(X)}^i(\mathcal{O}_x) = 0$ for $i \neq 0$ and $\Pi_{\text{Coh}(X)}^0(\mathcal{O}_x) = L^s \mathcal{P}_y$ for some $y \in X$. Define the Fourier-Mukai functor

$$\widehat{\Pi} = \Phi \circ (L^{-s})^3 \circ \Phi.$$

One can show that $\widehat{\Pi}[3]$ is right and left adjoint to Π (and vice versa). We have $\widehat{\Pi}_{\text{Coh}(X)}^i(\mathcal{O}_x) = 0$ for $i \neq 0$, and $\widehat{\Pi}_{\text{Coh}(X)}^0(\mathcal{O}_x) = L^{-s} \mathcal{P}_z$ for some $z \in X$. Therefore Π is a Fourier-Mukai functor with kernel a locally free sheaf \mathcal{U} on $X \times X$.

We have the spectral sequence

$$\Phi^p((L^s)^0 \Phi^q(E)) \implies \Pi^{p+q-3}(E) \quad (\heartsuit)$$

for E .

PROPOSITION 4.12. *Let E be a coherent sheaf such that $L^{-n}E \in V_{\text{Coh}(X)}^\Phi(k)$ for sufficiently large n , where $k \in \{0, \dots, 3\}$. Then $\mu^+((L^{-n}E)^k) \rightarrow 0$ as $n \rightarrow +\infty$.*

Proof. Since $L^{-n}E \in V_{\text{Coh}(X)}^\Phi(k)$ for sufficiently large n , $E \in \text{Coh}^k(X)$. If $k = 0$ then $E \in \text{Coh}^0(X)$ and so we have $\mu^+((L^{-n}E)^0) = 0$. Otherwise, by Propositions 4.10, 4.9 and 4.7, for $E \in \text{Coh}^k(X)$ we have $(L^{-n}E)^k \in \mathcal{T}_0$. Let s be a positive integer. Consider the convergence of the Spectral Sequence (\heartsuit) . For large enough n , we also have $L^{-n}E \in V_{\text{Coh}(X)}^\Pi(k)$. Therefore $(L^s)^0(L^{-n}E)^k \in V_{\text{Coh}(X)}^\Phi(3)$. By Proposition 4.11 we have $\mu^+((L^{-n}E)^k) \rightarrow 0$ as $n \rightarrow +\infty$. \square

PROPOSITION 4.13. *Let E be a reflexive sheaf. Then for sufficiently large $n > 0$,*

- (i) $L^{-n}E \in V_{\text{Coh}(X)}^\Phi(2, 3)$, and
- (ii) $(L^{-n}E)^2 \cong (T_0)^0$ for some $T_0 \in \text{Coh}^0(X)$.

Proof. (i) Consider a minimal locally free resolution of E ,

$$0 \rightarrow F_2 \rightarrow F_1 \rightarrow E \rightarrow 0.$$

By applying the FMT ΦL^{-n} for sufficiently large $n > 0$, we obtain $L^{-n}E \in V_{\text{Coh}(X)}^\Phi(2, 3)$.

(ii) Since E is a reflexive sheaf, there is a locally free sheaf P and a torsion free sheaf Q such that

$$0 \rightarrow E \rightarrow P \rightarrow Q \rightarrow 0$$

is a SES in $\text{Coh}(X)$. By applying the FMT ΦL^{-n} for sufficiently large n we have $(L^{-n}E)^2 \cong (L^{-n}Q)^1$.

The torsion free sheaf Q fits into the SES $0 \rightarrow Q \rightarrow Q^{**} \rightarrow T \rightarrow 0$ for some $T \in \text{Coh}^{\leq 1}(X)$. Apply the FMT ΦL^{-n} for sufficiently large n and consider the LES of $\text{Coh}(X)$ -cohomologies. Since $L^{-n}Q^{**} \in V_{\text{Coh}(X)}^\Phi(2, 3)$, we have $(L^{-n}Q)^1 \cong (L^{-n}T)^0$. The torsion sheaf $T \in \text{Coh}^{\leq 1}(X)$ fits into a SES $0 \rightarrow T_0 \rightarrow T \rightarrow T_1 \rightarrow 0$ in $\text{Coh}(X)$ for $T_i \in \text{Coh}^i(X)$, $i = 0, 1$. Therefore $(L^{-n}T)^0 \cong (T_0)^0$, and so $(L^{-n}E)^2 \cong (T_0)^0$ as required. \square

PROPOSITION 4.14. *Let $E \in \text{Coh}^1(X)$ with $E \in V_{\text{Coh}(X)}^\Phi(1)$. If $0 \neq T \in \text{HN}[0, +\infty]$ is a subsheaf of E^1 , then $\ell \text{ch}_2(T) \leq 0$.*

Proof. For $n > 0$ and generic $z \in X$, we have the $\text{Coh}(X)$ -SES

$$0 \rightarrow L_z^{-n}E \rightarrow E \rightarrow T_0 \rightarrow 0$$

for $T_0 := E|_{nD_z} \in \text{Coh}^0(X)$. By applying the FMT Φ , we get the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_0^0 & \longrightarrow & (L_z^{-n}E)^1 & \longrightarrow & E^1 \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & T_0^0 & \longrightarrow & A & \longrightarrow & T \longrightarrow 0, \end{array}$$

for some $A \in \text{HN}[0, +\infty]$. So we have $\text{ch}_k(A) = \text{ch}_k(T)$ for $k = 1, 2, 3$.

Let G be a slope semistable H-N factor of A . From the usual B-G inequality, $\ell(\text{ch}_1(G)^2 - 2\text{ch}_0(G)\text{ch}_2(G)) \geq 0$. So we have

$$\begin{aligned} 2\ell \text{ch}_2(G) &\leq \frac{\ell \text{ch}_1(G)^2}{\text{ch}_0(G)} \\ &= \ell^2 \text{ch}_1(G) \mu(G) \\ &\leq \ell^2 \text{ch}_1(A) \mu(G) \\ &\leq \ell^2 \text{ch}_1(T) \mu^+((L_z^{-n}E)^1). \end{aligned}$$

By Proposition 4.12, $\mu^+((L_z^{-n}E)^1) \rightarrow 0$ as $n \rightarrow +\infty$. So choose large enough $n > 0$ such that $\ell^2 \text{ch}_1(T) \mu^+((L_z^{-n}E)^1) < \ell^3$. Since $2\ell \text{ch}_2(G) \in \ell^3 \mathbb{Z}$ we have $\ell \text{ch}_2(G) \leq 0$. So $\ell \text{ch}_2(T) = \ell \text{ch}_2(A) \leq 0$. \square

PROPOSITION 4.15. *We have the following:*

- (i) *Let $E \in \mathcal{F}_0$ be a reflexive sheaf. If $0 \neq T \in \mathcal{T}_0$ is a subsheaf of E^1 then $\ell \text{ch}_2(T) \leq 0$.*
- (ii) *Let $E \in \mathcal{T}_0$ be a torsion free. If $0 \neq F \in \mathcal{F}_0$ is a quotient of E^2 then $\ell \text{ch}_2(F) \leq 0$.*

Proof. (i) Recall that, for any positive integer m , non-reduced divisors mD_x of L_x^m are topologically supported on D_x .

Since E is a reflexive sheaf, one can choose $x, y \in X$ such that

- $\dim(D_x \cap D_y) = 1$,
- $E|_{D_x}$ is locally free on D_x , and
- $E|_{D_y}$ is locally free on D_y .

By Proposition 4.13, for sufficiently large $m > 0$, $L_x^{-m}E \in V_{\text{Coh}(X)}^\Phi(2, 3)$. By applying the FMT Φ to the $\text{Coh}(X)$ -SES

$$0 \rightarrow L_x^{-m}E \rightarrow E \rightarrow E|_{mD_x} \rightarrow 0$$

$E|_{mD_x} \in V_{\text{Coh}(X)}^\Phi(1, 2)$ and $E^1 \hookrightarrow (E|_{mD_x})^1$. Since $E|_{D_x}$ is locally free on D_x , for large enough $n > 0$, $L_y^{-n}E|_{mD_x} \in V_{\text{Coh}(X)}^\Phi(2)$. By applying the FMT Φ to the $\text{Coh}(X)$ -SES

$$0 \rightarrow L_y^{-n}E|_{mD_x} \rightarrow E|_{mD_x} \rightarrow E|_{mD_x \cap nD_y} \rightarrow 0,$$

$E|_{mD_x \cap nD_y} \in V_{\text{Coh}(X)}^\Phi(1)$ and $(E|_{mD_x})^1 \hookrightarrow (E|_{mD_x \cap nD_y})^1$. Therefore we have

$$T \hookrightarrow E^1 \hookrightarrow (E|_{mD_x \cap nD_y})^1.$$

The result follows from Proposition 4.14.

- (ii) Since $F \neq 0$ is a quotient of E^2 , we have $F^* \hookrightarrow (E^2)^*$. Here $F^* \in \text{HN}[0, +\infty)$ fits into $\text{Coh}(X)$ -SES $0 \rightarrow T \rightarrow F^* \rightarrow F_0 \rightarrow 0$ for some $T \in \mathcal{T}_0$ and $F_0 \in \text{HN}[0]$. By the usual B-G inequality $\ell \text{ch}_2(F_0) \leq 0$.

By Proposition 4.6, $E^3 = 0 = (E^*)^0$. Therefore from the convergence of the ‘‘Duality’’ Spectral Sequence 4.2 for E , we have the $\text{Coh}(X)$ -SES

$$0 \rightarrow (-1)^*(E^*)^1 \rightarrow (E^2)^* \rightarrow P \rightarrow 0,$$

for some subsheaf P of $(\mathcal{E}xt^1(E, \mathcal{O}))^0$. By Proposition 4.7, $(\mathcal{E}xt^1(E, \mathcal{O}))^0 \in \mathcal{F}_0$ and so $P \in \mathcal{F}_0$. Therefore $\text{Hom}(T, P) = 0$ and so $P \hookrightarrow (-1)^*(E^*)^1$. Here $E^* \in \mathcal{F}_0$ and so by part (I), $\ell \text{ch}_2(T) \leq 0$. Therefore $\ell \text{ch}_2(F) \leq \ell \text{ch}_2(F^{**}) = \ell \text{ch}_2(F^*) = \ell \text{ch}_2(F_0) + \ell \text{ch}_2(T) \leq 0$. □

PROPOSITION 4.16. *For $E \in \text{Coh}(X)$*

- (i) *if $E \in \mathcal{F}_0$ then $E^1 \in \mathcal{F}_0$, and*
- (ii) *if $E \in \text{HN}[0, +\infty)$ with $E^3 = 0$ then $E^2 \in \text{HN}[0, +\infty]$.*

Proof. (i) Assume the opposite for a contradiction. Let $T = T(E^1)$ and $F = F(E^1)$. Then $0 \rightarrow T \rightarrow E^1 \rightarrow F \rightarrow 0$ is a SES in $\text{Coh}(X)$. By Proposition 4.8 E^1 is reflexive and so non-trivial T is reflexive. So $\ell^2 \text{ch}_1(T) > 0$. By applying the FMT Φ to this SES we obtain that $T \in V_{\text{Coh}(X)}^\Phi(2)$ and $F \in V_{\text{Coh}(X)}^\Phi(1, 2)$. Moreover, we have the $\text{Coh}(X)$ -SES

$$0 \rightarrow F^1 \rightarrow T^2 \rightarrow E_1 \rightarrow 0$$

for some subsheaf E_1 of E^{12} . From the Mukai Spectral Sequence 4.1 for E we have the $\text{Coh}(X)$ -SES

$$0 \rightarrow E^{20} \rightarrow E^{12} \rightarrow E_2 \rightarrow 0,$$

for some subsheaf E_2 of $(-1)^*E$. Therefore $E_2 \in \mathcal{F}_0$ and by Proposition 4.7 $E^{20} \in \mathcal{F}_0$. So we have $E^{12} \in \mathcal{F}_0$. Hence $E_1 \in \mathcal{F}_0$.

Let $T_1 := T(F^1)$ and $F_1 := F(T^2)$. They fit into the following commutative diagram for some $F_2 \in \mathcal{F}_0$.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & E_1 \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \parallel \\
 0 & \longrightarrow & F^1 & \longrightarrow & T^2 & \longrightarrow & E_1 \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \\
 & & T_1 & \xlongequal{\quad} & T_1 & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

By Proposition 4.15, $\ell \operatorname{ch}_2(F_1) \leq 0$.

By applying the FMT Φ to the $\operatorname{Coh}(X)$ -SES $0 \rightarrow T_1 \rightarrow F^1 \rightarrow F_2 \rightarrow 0$ we obtain the $\operatorname{Coh}(X)$ -SES

$$0 \rightarrow F_2^1 \rightarrow T_1^2 \rightarrow F_3 \rightarrow 0$$

for some subsheaf F_3 of F^{12} . Also $T_1 \in V_{\operatorname{Coh}(X)}^\Phi(2)$. By considering the Mukai Spectral Sequence 4.1 for F , one can show $F_3 \in \mathcal{F}_0$. By Proposition 4.8 F_2^1 is reflexive. So T_1^2 is torsion free and it fits into $\operatorname{Coh}(X)$ -SES

$$0 \rightarrow T_1^2 \rightarrow (T_1^2)^{**} \rightarrow Q \rightarrow 0,$$

for some $Q \in \operatorname{Coh}^{\leq 1}(X)$. The torsion sheaf Q fits into $\operatorname{Coh}(X)$ -SES

$$0 \rightarrow Q_0 \rightarrow Q \rightarrow Q_1 \rightarrow 0$$

for $Q_0 \in \operatorname{Coh}^0(X)$ and $Q_1 \in \operatorname{Coh}^1(X)$. By Proposition 4.13, for large enough $m > 0$, $(L^{-m}T_1^2)^1 \cong (L^{-m}Q)^0 \cong Q_0^0$. Also $(L^{-m}Q_1)^1 \cong (L^{-m}Q)^1$ and $(L^{-m}(T_1^2)^{**})^2 \cong R_0^0$ for some $R_0 \in \operatorname{Coh}^0(X)$. So we have the $\operatorname{Coh}(X)$ -SES

$$0 \rightarrow (L^{-m}Q_1)^1 \rightarrow (L^{-m}T_1^2)^2 \rightarrow R_0^0 \rightarrow 0.$$

By Proposition 4.10, $(L^{-m}T_1^2)^2 \in \operatorname{HN}[0, +\infty)$, and $\ell \operatorname{ch}_2((L^{-m}T_1^2)^2) = 0$.

The torsion free sheaf F_3 also fits into $\operatorname{Coh}(X)$ -SES $0 \rightarrow F_3 \rightarrow F_3^{**} \rightarrow S \rightarrow 0$ for some $S \in \operatorname{Coh}^{\leq 1}(X)$.

Choose $x, y \in X$ such that

- $\dim(D_x \cap D_y) = 1$,
- $D_x \cap \operatorname{Supp}(Q_0) = \emptyset$,
- $\dim(\operatorname{Supp}(Q_1) \cap D_x) \leq 0$,
- $D_x \cap D_y \cap \operatorname{Supp}(Q) = \emptyset$,
- $D_x \cap D_y \cap \operatorname{Supp}(S) = \emptyset$,
- since F_2^1 is reflexive, $F_2^1|_{D_x}$ is locally free on D_x , and $F_2^1|_{D_y}$ is locally free on D_y ,
- since F_3^{**} is reflexive, $F_3^{**}|_{D_x}$ is locally free on D_x , and $F_3^{**}|_{D_y}$ is locally free on D_y .

From the Mukai Spectral Sequence for F_2 , $F_2^1 \in V_{\operatorname{Coh}(X)}^\Phi(2, 3)$. Since it is a reflexive sheaf, for large enough $m > 0$, $L_x^{-m}F_2^1 \in V_{\operatorname{Coh}(X)}^\Phi(2, 3)$, and since $F_2^1|_{D_x}$ is locally free on D_x , $L_y^{-n}F_2^1|_{mD_x} \in V_{\operatorname{Coh}(X)}^\Phi(2)$. So $F_2^1|_{mD_x \cap nD_y} \in V_{\operatorname{Coh}(X)}^\Phi(1)$. Since $D_x \cap D_y \cap \operatorname{Supp}(S) = \emptyset$, similarly one can show $F_3|_{mD_x \cap nD_y} \cong F_3^{**}|_{mD_x \cap nD_y} \in V_{\operatorname{Coh}(X)}^\Phi(1)$. Therefore we have $T_1^2|_{mD_x \cap nD_y} \cong (T_1^2)^{**}|_{mD_x \cap nD_y} \in V_{\operatorname{Coh}(X)}^\Phi(1)$.

By applying the FMT Φ to the $\text{Coh}(X)$ -SES $0 \rightarrow L_x^{-m}T_1^2 \rightarrow T_1^2 \rightarrow T_1^2|_{mD_x} \rightarrow 0$, for large enough $m > 0$ we have the $\text{Coh}(X)$ -LES

$$0 \rightarrow Q_0^0 \rightarrow (-1)^*T_1 \rightarrow (T_1^2|_{mD_x})^1 \rightarrow (L^{-m}T_1^2)^2 \rightarrow 0.$$

So $(T_1^2|_{mD_x})^1 \in \text{HN}[0, +\infty]$ and $\text{ch}_2((T_1^2|_{mD_x})^1) = \text{ch}_2(T_1)$. Moreover we have the $\text{Coh}(X)$ -SES

$$0 \rightarrow T_1^2|_{mD_x} \rightarrow (T_1^2)^{**}|_{mD_x} \rightarrow Q_1|_{mD_x} \rightarrow 0.$$

Here $Q_1|_{mD_x} \in \text{Coh}^0(X)$. So for large enough $n > 0$, $(L^{-n}T_1^2|_{mD_x})^1 \cong (Q_1|_{mD_x})^0$.

By applying the FMT Φ to the $\text{Coh}(X)$ -SES $0 \rightarrow L_y^{-n}T_1^2|_{mD_x} \rightarrow T_1^2|_{mD_x} \rightarrow T_1^2|_{mD_x \cap nD_y} \rightarrow 0$ we have the $\text{Coh}(X)$ -LES

$$0 \rightarrow (Q_1|_{mD_x})^0 \xrightarrow{\alpha} (T_1^2|_{mD_x})^1 \rightarrow (T_1^2|_{mD_x \cap nD_y})^1 \rightarrow \dots$$

Let $T_2 := \text{coker}(\alpha)$. Then $T_2 \in \text{HN}[0, +\infty]$ and $\text{ch}_2(T_2) = \text{ch}_2(T_1)$. By Proposition 4.14, we have $\ell \text{ch}_2(T_2) \leq 0$. So $\ell \text{ch}_2(T^2) = \ell \text{ch}_2(T_1) + \ell \text{ch}_2(F_1) \leq 0$. Therefore we have $\ell^2 \text{ch}_1(T) \leq 0$. This is the required contradiction to complete the proof.

- (ii) Since $E^* \in \text{HN}(-\infty, 0]$, from (i) $(E^*)^1 \in \text{HN}(-\infty, 0]$. By the convergence of the ‘‘Duality’’ Spectral Sequence 4.2 for E we have $(E^2)^* \in \text{HN}(-\infty, 0]$. So $E^2 \in \text{HN}[0, +\infty]$ as required. \square

COROLLARY 4.17. *Let $E \in \mathcal{T}_0$. Then $E^2 \in \mathcal{T}_0$.*

Proof. Let $T = T(E^2)$ and $F = F(E^2)$. Then $0 \rightarrow T \rightarrow E^2 \rightarrow F \rightarrow 0$ is a SES in $\text{Coh}(X)$. Now we need to show $F = 0$. Apply the FMT Φ and consider the LES of $\text{Coh}(X)$ -cohomologies. So we have $F \in V_{\text{Coh}(X)}^\Phi(1)$ and

$$0 \rightarrow T^1 \rightarrow E^{21} \rightarrow F^1 \rightarrow T^2 \rightarrow 0$$

is a LES in $\text{Coh}(X)$. From the convergence of the Mukai Spectral Sequence 4.1 for E we have the $\text{Coh}(X)$ -SES

$$0 \rightarrow Q \rightarrow E^{21} \rightarrow E^{13} \rightarrow 0,$$

where Q is a quotient of $(-1)^*E$. Then $Q \in \mathcal{T}_0$ and, by Proposition 4.7, $E^{13} \in \mathcal{T}_0$ and so $E^{21} \in \mathcal{T}_0$. On the other hand, by Proposition 4.16, $F^1 \in \mathcal{F}_0$. So the map $E^{21} \rightarrow F^1$ is zero and $F^1 \cong T^2$. Hence $F \cong (-1)^*F^{12} \cong (-1)^*T^{22} = 0$ (from the Mukai Spectral Sequence 4.1 for T) as required. \square

PROPOSITION 4.18. *Let $E \in \text{HN}(0, 1]$. Then $E^0 \in \text{HN}(-\infty, -\frac{1}{2}]$.*

Proof. Due to Mukai, $\Phi L \Phi \cong (-1)^*L^{-1}\Phi L^{-1}$. Therefore we have the following convergence of spectral sequence:

$$E_2^{p,q} = \Phi_{\text{Coh}(X)}^p L \Phi_{\text{Coh}(X)}^q(E) \implies (-1)^*L^{-1}\Phi_{\text{Coh}(X)}^{p+q}(L^{-1}E).$$

Here $L^{-1}E \in \text{HN}(-1, 0]$, and so by Proposition 4.6, $(L^{-1}E)^0 = 0$. So from the convergence of the above spectral sequence for E we have $(LE^0)^0 = 0$. Also $(LE^0)^1 \hookrightarrow L^{-1}(L^{-1}E)^1$. By Proposition 4.16 $(L^{-1}E)^1 \in \mathcal{F}_0$ and so $(LE^0)^1 \in \text{HN}(-\infty, -1] \subset \mathcal{F}_0$.

Let $F \subset E^0$ be the H-N semistable factor of E^0 with the highest slope and let $\mu := \mu(F)$. Then $(LF)^0 \hookrightarrow (LE^0)^0$ and so $(LF)^0 = 0$. Let $\text{ch}(F) = (a_0, \mu a_0, a_2, a_3)$. Now suppose $\mu > -\frac{1}{2}$ for a contradiction. Then $LF \in \mathcal{T}_0$ and F fits into the $\text{Coh}(X)$ -SES

$$0 \rightarrow F \rightarrow E^0 \rightarrow G \rightarrow 0, \quad (\star)$$

That is

$$L\Phi(\mathcal{T}) \subset \langle \mathcal{B}, \mathcal{B}[-1], \mathcal{B}[-2] \rangle.$$

Hence $L\Phi(\mathcal{B}) \subset \langle \mathcal{B}, \mathcal{B}[-1], \mathcal{B}[-2] \rangle$, as $\mathcal{B} = \langle \mathcal{F}[1], \mathcal{T} \rangle$.

- (ii) We can use Propositions 4.6, 4.7 and 4.19, and Proposition 4.16 in a similar way to the proof of (i). □

5. (Semi)stable sheaves with the Chern character $(r, 0, 0, \chi)$

In this section we shall consider sheaves E with $\text{ch}_k(E) = 0$ for $k = 1, 2$ which arise as the $\text{Coh}(X)$ -cohomology of some of the tilt-stable objects. For example, when $F \in \mathcal{B}$ is a tilt stable object with $\nu(F) = 0$ and $F_i := H_{\text{Coh}(X)}^i(F)$. By Proposition 3.1, if $\mu(F_{-1}) = 0$ then $\text{ch}_k(F_{-1}) = 0$, and if $\mu(F_0) = 1$ then $\text{ch}_k(L^{-1}F_0) = 0$ for $k = 1$ and 2 .

We would like to show that such sheaves can only take a very special form:

THEOREM 5.1. *Let E be a slope semistable sheaf with $\text{ch}_k(E) = 0$ for $k = 1, 2$. Then E^{**} is a homogeneous bundle. In other words, E^{**} is filtered with quotients from $\mathbf{Pic}^0(X)$.*

Proof. Assume the opposite for a contradiction. Then there exists a stable reflexive sheaf E with $\text{ch}_k(E) = 0$ for $k = 1, 2$, and $H^k(X, E \otimes \mathcal{P}_x) = 0$ for $k = 0, 3$ and any $x \in X$. By a result of Simpson ([18, Theorem 2]) we have $\text{ch}_3(E) = 0$. Therefore, $\text{ch}(E) = (r, 0, 0, 0)$ for some positive integer r .

Since $H^k(X, E \otimes \mathcal{P}_x) = 0$ for $k = 0, 3$ and any $x \in X$, we have $E^0 = E^3 = 0$. By Proposition 4.16, $E^1 \in \text{HN}(-\infty, 0]$ and $E^2 \in \text{HN}[0, +\infty]$. So we have $\ell^2 \text{ch}_1(E^1) \leq 0$ and $\ell^2 \text{ch}_1(E^2) \geq 0$. Therefore, $\ell^2 \text{ch}_1(\Phi(E)) \geq 0$ which implies $\ell \text{ch}_2(E) \leq 0$. Since $\text{ch}_2(E) = 0$, we obtain $\text{ch}_1(E^1) = \text{ch}_1(E^2) = 0$. Then we have

$$\text{ch}(E^1) = (a, 0, -b, c), \quad \text{ch}(E^2) = (a, 0, -b, -r + c),$$

for some $a > 0$ and $b \geq 0$. Moreover we have $E^1 \in \text{HN}[0]$.

If $E^{13} \neq 0$ then E^1 fits into a $\text{Coh}(X)$ -SES of the form $0 \rightarrow K_1 \rightarrow E^1 \rightarrow \mathcal{P}_{z_1} \mathcal{I}_{C_1} \rightarrow 0$. Then $K_1 \in \text{HN}[0]$ and we have the following exact sequence

$$\cdots \rightarrow K_1^3 \rightarrow E^{13} \rightarrow \mathcal{O}_{-z_1} \rightarrow 0$$

in $\text{Coh}(X)$. If $K_1^3 \neq 0$ then K_1 fits into a $\text{Coh}(X)$ -SES $0 \rightarrow K_2 \rightarrow K_1 \rightarrow \mathcal{P}_{z_2} \mathcal{I}_{C_2} \rightarrow 0$. Then $K_2 \in \text{HN}[0]$ and we have the following exact sequence

$$\cdots \rightarrow K_2^3 \rightarrow K_1^3 \rightarrow \mathcal{O}_{-z_2} \rightarrow 0$$

in $\text{Coh}(X)$. We can continue this process for only a finite number of steps since $\text{rk}(E^1) < +\infty$ and hence E^{13} is filtered by skyscraper sheaves. Moreover from the convergence of the Mukai Spectral Sequence 4.1 for E , we have the $\text{Coh}(X)$ -SES

$$0 \rightarrow E^{20} \rightarrow E^{12} \rightarrow Q \rightarrow 0$$

where Q is a subsheaf of $(-1)^*E$ and so $Q \in \text{HN}(-\infty, 0]$. By Proposition 4.7, $E^{20} \in \text{HN}(-\infty, 0]$. This implies $E^{12} \in \text{HN}(-\infty, 0]$. Then $\ell^2 \text{ch}_1(\Phi(E^1)) \leq 0$ and so $-b\ell^3 = 2\ell \text{ch}_2(E^1) \geq 0$. Hence $b = 0$. By Proposition 4.8, E^1 is a reflexive sheaf and since $E^1 \in \text{HN}[0]$ it is slope semistable. So by [18, Theroem 2] we have $c = \text{ch}_3(E^1) = 0$. Therefore $\text{ch}(\Phi(E^1)) = (0, 0, 0, -a)$. Since $E^{13} \in \text{Coh}^0(X)$, we have $\text{ch}_k(E^{12}) = 0$ for $k = 0, 1, 2$. So $E^{12} \in \text{HN}(0, +\infty]$. Therefore $E^{12} = 0$ and we have the Coh(X)-SES

$$0 \rightarrow (-1)^*E \rightarrow E^{21} \rightarrow E^{13} \rightarrow 0.$$

Since $E^{13} \in \text{Coh}^0(X)$ and E is locally free, $\text{Ext}^1(E^{13}, (-1)^*E) = 0$. Therefore $E^{21} \cong (-1)^*E \oplus E^{13}$. Since $E^{21} \in V_{\text{Coh}(X)}^\Phi(2)$ we have $E^{13} = 0$ and so $E \in V_{\text{Coh}(X)}^\Phi(2)$. Therefore $\text{ch}(E^2) = (0, 0, 0, -r)$. But it is not possible to have $-r > 0$ and this is the required contradiction to complete the proof. \square

Remark 5.2. Theorem 5.1 can be interpreted as saying that if a vector bundle E over X satisfies $c_1(E) = 0 = c_2(E)$ then it cannot carry a non-flat Hermitian-Einstein connection. This is analogous to the case where there are no charge 1 $\text{SU}(r)$ instantons on an abelian surface. This is proved in a slick way using the Fourier-Mukai transform and it would be good to avoid the direct proof given for Theorem 5.1 as it would follow more directly from Theorem 6.10.

6. Auto-equivalences of $\mathcal{A}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}$ under the FMTs

Let denote the FMTs $\Psi = L\Phi$ and $\hat{\Psi} = \Phi L^{-1}[1]$. Then by Theorem 4.20, we have that the images of an object from \mathcal{B} under Ψ and $\hat{\Psi}$ are complexes whose \mathcal{B} -cohomologies can only be non-zero in the 0, 1 or 2 positions. We have $\Psi \circ \hat{\Psi} \cong (-1)^* \text{id}_{D^b(X)}[-2]$ and $\hat{\Psi} \circ \Psi \cong (-1)^* \text{id}_{D^b(X)}[-2]$. This gives us the following convergence of spectral sequences.

SPECTRAL SEQUENCE 6.1.

$$\begin{aligned} E_2^{p,q} &= \Psi_{\mathcal{B}}^p \hat{\Psi}_{\mathcal{B}}^q(E) \implies H_{\mathcal{B}}^{p+q-2}((-1)^*E), \\ E_2^{p,q} &= \hat{\Psi}_{\mathcal{B}}^p \Psi_{\mathcal{B}}^q(E) \implies H_{\mathcal{B}}^{p+q-2}((-1)^*E), \end{aligned}$$

for E . Here $\Psi_{\mathcal{B}}^i(F) := H_{\mathcal{B}}^i(\Psi(F))$ and $\hat{\Psi}_{\mathcal{B}}^i(F) := H_{\mathcal{B}}^i(\hat{\Psi}(F))$.

These convergence of the spectral sequences for $E \in \mathcal{B}$ look similar to the convergence of some spectral sequences in an abelian surface for coherent sheaves. See [3], [10], [20] for further details.

Recall that if $B_1, B_2 \in \mathcal{B}$ then $\text{Ext}^i(B_1, B_2) = 0$ for any $i < 0$.

PROPOSITION 6.2. For $E \in D^b(X)$ we have

$$\mathfrak{S} Z(\Psi(E)) = -\mathfrak{S} Z(E), \text{ and } \mathfrak{S} Z(\hat{\Psi}(E)) = -\mathfrak{S} Z(E).$$

Proof. Let $\text{ch}(E) = (a_0, a_1, a_2, a_3)$. Then $\mathfrak{S} Z(E) = \frac{3\sqrt{3}}{4}(a_2 - a_1)$. Also we have $\text{ch}(\Psi(E)) = (*, a_3 - a_2, a_3 - 2a_2 + a_1, *)$ and $\text{ch}(\hat{\Psi}(E)) = (*, a_2 - 2a_1 + a_0, -a_1 + a_0, *)$. Then $\mathfrak{S} Z(\Psi(E)) = \mathfrak{S} Z(\hat{\Psi}(E)) = -\frac{3\sqrt{3}}{4}(a_2 - a_1)$ as required. \square

From Propositions 2.7, 3.1 and Theorem 5.1 we make the following

NOTE 6.3. Let $E \in \mathcal{B}$. Then we have the following:

- (I) if $E \in \text{HN}^\nu(-\infty, 0)$, then $\mu^+(E_{-1}) < 0$;

- (II) if $E \in \text{HN}^\nu(0, +\infty]$, then $\mu^-(E_0) > 1$; and
- (III) for tilt stable E with $\nu(E) = 0$, we have
- (i) $\mu^+(E_{-1}) \leq 0$, and $\mu^-(E_0) \geq 1$,
 - (ii) if $\mu(E_{-1}) = 0$ then $E_{-1} = \mathcal{P}_x$ for some $x \in X$, and
 - (iii) if $\mu(E_0) = 1$ then $E_0^{**} = L\mathcal{P}_x$ for some $x \in X$.

PROPOSITION 6.4. *Let $E \in \mathcal{T}'$. Then we have the following:*

- (i) $H_{\text{Coh}(X)}^0(\hat{\Psi}_{\mathcal{B}}^2(E)) = 0$, and
- (ii) if $\hat{\Psi}_{\mathcal{B}}^2(E) \neq 0$ then $\Im Z(\hat{\Psi}_{\mathcal{B}}^2(E)) > 0$.

Proof. (i) For any $x \in X$,

$$\begin{aligned} \text{Hom}(\hat{\Psi}_{\mathcal{B}}^2(E), \mathcal{O}_x) &\cong \text{Hom}(\hat{\Psi}_{\mathcal{B}}^2(E), \hat{\Psi}_{\mathcal{B}}^2(L\mathcal{P}_{-x})) \\ &\cong \text{Hom}(\hat{\Psi}(E), \hat{\Psi}(L\mathcal{P}_{-x})) \\ &\cong \text{Hom}(E, L\mathcal{P}_{-x}) = 0, \end{aligned}$$

since $E \in \mathcal{T}'$ and $L\mathcal{P}_{-x} \in \mathcal{F}'$. Therefore $H_{\text{Coh}(X)}^0(\hat{\Psi}_{\mathcal{B}}^2(E)) = 0$ as required.

- (ii) From (i), we have $\hat{\Psi}_{\mathcal{B}}^2(E) \cong A[1]$ for some $0 \neq A \in \text{HN}(-\infty, \frac{1}{2}]$.

Consider the convergence of the spectral sequence:

$$E_2^{p,q} = \hat{\Psi}_{\text{Coh}(X)}^p(H_{\text{Coh}(X)}^q(E)) \implies \hat{\Psi}_{\text{Coh}(X)}^{p+q}(E)$$

for E . Let $E_i := H_{\text{Coh}(X)}^i(E)$. Then by Note 6.3, $E_0 \in \text{HN}(1, +\infty]$ and so by Corollary 4.17 and Proposition 4.7 we have

$$(L^{-1}E_0)^2, (L^{-1}E_{-1})^3 \in \text{HN}(0, +\infty].$$

Therefore from the convergence of the above spectral sequence for E , we have

$$A \in \text{HN}(-\infty, \frac{1}{2}] \cap \text{HN}(0, +\infty] = \text{HN}(0, \frac{1}{2}].$$

Let $\text{ch}(A) = (a_0, a_1, a_2, a_3)$. Then from the B-G inequalities for all the H-N semistable factors of A , we have

$$\Im Z(\hat{\Psi}_{\mathcal{B}}^2(E)) = \Im Z(A[1]) = \frac{3\sqrt{3}}{4}(a_1 - a_2) > 0$$

as required. □

PROPOSITION 6.5. *Let $E \in \mathcal{F}'$. Then we have the following:*

- (i) $H_{\text{Coh}(X)}^{-1}(\hat{\Psi}_{\mathcal{B}}^0(E)) = 0$, and
- (ii) if $\hat{\Psi}_{\mathcal{B}}^0(E) \neq 0$ then $\Im Z(\hat{\Psi}_{\mathcal{B}}^0(E)) < 0$.

Proof. (i) Let $x \in X$. Then

$$\begin{aligned} \text{Hom}(\hat{\Psi}_{\mathcal{B}}^0(E), \mathcal{O}_x[1]) &\cong \text{Hom}(\Psi\hat{\Psi}^0(E), \Psi(\mathcal{O}_x[1])) \\ &\cong \text{Hom}(\Psi_{\mathcal{B}}^2\hat{\Psi}_{\mathcal{B}}^0(E)[-2], L\mathcal{P}_x[1]) \\ &\cong \text{Hom}(\Psi_{\mathcal{B}}^2\hat{\Psi}_{\mathcal{B}}^0(E), L\mathcal{P}_x[3]) \\ &\cong \text{Hom}(L\mathcal{P}_x, \Psi_{\mathcal{B}}^2\hat{\Psi}_{\mathcal{B}}^0(E))^*. \end{aligned}$$

From the convergence of the Spectral Sequence 6.1 for E , we have the \mathcal{B} -SES

$$0 \rightarrow \Psi_{\mathcal{B}}^0 \hat{\Psi}_{\mathcal{B}}^1(E) \rightarrow \Psi_{\mathcal{B}}^2 \hat{\Psi}_{\mathcal{B}}^0(E) \rightarrow F \rightarrow 0,$$

where F is a subobject of $(-1)^*E$ and so $F \in \mathcal{F}'$. Moreover by the H-N filtration, F fits into the following \mathcal{B} -SES

$$0 \rightarrow F_0 \rightarrow F \rightarrow F_1 \rightarrow 0,$$

where $F_0 \in \text{HN}^{\nu}[0]$ and $F_1 \in \text{HN}^{\nu}(-\infty, 0)$. Since $L\mathcal{P}_x \in \text{HN}^{\nu}[0]$,

$$\text{Hom}(L\mathcal{P}_x, F_1) = 0.$$

Moreover F_0 has a filtration of ν -stable objects $F_{0,i}$ with $\nu(F_{0,i}) = 0$. By Proposition 2.9, each $F_{0,i}$ fits into a non-splitting \mathcal{B} -SES

$$0 \rightarrow F_{0,i} \rightarrow M_i \rightarrow T_i \rightarrow 0,$$

for some $T_i \in \text{Coh}^0(X)$ such that $M_i[1] \in \mathcal{A}$ is a minimal object. Moreover $L\mathcal{P}_x[1] \in \mathcal{A}$ is a minimal object. So finitely many $x \in X$ we can have $L\mathcal{P}_x \cong M_i$ for some i . So for a generic $x \in X$, $\text{Hom}(L\mathcal{P}_x, M_i) = 0$ and so $\text{Hom}(L\mathcal{P}_x, F_{0,i}) = 0$ which implies $\text{Hom}(L\mathcal{P}_x, F_0) = 0$. Therefore for a generic $x \in X$, $\text{Hom}(L\mathcal{P}_x, F) = 0$.

On the other hand

$$\begin{aligned} \text{Hom}(L\mathcal{P}_x, \Psi_{\mathcal{B}}^0 \hat{\Psi}_{\mathcal{B}}^1(E)) &\cong \text{Hom}(\Psi_{\mathcal{B}}^0(\mathcal{O}_x), \Psi_{\mathcal{B}}^0 \hat{\Psi}_{\mathcal{B}}^1(E)) \\ &\cong \text{Hom}(\Psi(\mathcal{O}_x), \Psi \hat{\Psi}_{\mathcal{B}}^1(E)) \\ &\cong \text{Hom}(\mathcal{O}_x, \hat{\Psi}_{\mathcal{B}}^1(E)). \end{aligned}$$

Here $\hat{\Psi}_{\mathcal{B}}^1(E)$ fits into the \mathcal{B} -SES

$$0 \rightarrow H_{\text{Coh}(X)}^{-1}(\hat{\Psi}_{\mathcal{B}}^1(E))[1] \rightarrow \hat{\Psi}_{\mathcal{B}}^1(E) \rightarrow H_{\text{Coh}(X)}^0(\hat{\Psi}_{\mathcal{B}}^1(E)) \rightarrow 0,$$

where $H_{\text{Coh}(X)}^{-1}(\hat{\Psi}_{\mathcal{B}}^1(E))$ is torsion free and $H_{\text{Coh}(X)}^0(\hat{\Psi}_{\mathcal{B}}^1(E))$ can have torsion supported on a 0-scheme of finite length. Hence for generic $x \in X$, $\text{Hom}(\mathcal{O}_x, \hat{\Psi}_{\mathcal{B}}^1(E)) = 0$. Therefore for generic $x \in X$, $\text{Hom}(L\mathcal{P}_x, \Psi_{\mathcal{B}}^0 \hat{\Psi}_{\mathcal{B}}^1(E)) = \text{Hom}(L\mathcal{P}_x, F) = 0$ implies $\text{Hom}(L\mathcal{P}_x, \Psi_{\mathcal{B}}^2 \hat{\Psi}_{\mathcal{B}}^0(E)) = 0$. Hence $\text{Hom}(\hat{\Psi}_{\mathcal{B}}^0(E), \mathcal{O}_x[1]) = 0$ for generic $x \in X$. But $H_{\text{Coh}(X)}^{-1}(\hat{\Psi}_{\mathcal{B}}^0(E))$ is torsion free and so $H_{\text{Coh}(X)}^{-1}(\hat{\Psi}_{\mathcal{B}}^0(E)) = 0$ as required.

- (ii) From (i), we have $\hat{\Psi}_{\mathcal{B}}^0(E) \cong A$ for some coherent sheaf $0 \neq A \in \text{HN}(\frac{1}{2}, +\infty]$. For any $x \in X$ we have

$$\begin{aligned} \text{Ext}^1(\mathcal{O}_x, A) &\cong \text{Ext}^1(\mathcal{O}_x, \hat{\Psi}_{\mathcal{B}}^0(E)) \cong \text{Hom}(\Psi(\mathcal{O}_x), \Psi \hat{\Psi}_{\mathcal{B}}^0(E)[1]) \\ &\cong \text{Hom}(L\mathcal{P}_x, \Psi^2 \hat{\Psi}_{\mathcal{B}}^0(E)[-1]) = 0. \end{aligned}$$

So $A \in \text{Coh}^{\geq 2}(X)$, and if $\text{ch}(A) = (a_0, a_1, a_2, a_3)$ then we have $a_1 > 0$.

Apply the FMT Ψ to $\hat{\Psi}_{\mathcal{B}}^0(E)$. Since $\hat{\Psi}_{\mathcal{B}}^0(E) \in V_{\mathcal{B}}^{\Psi}(2)$, $\Psi_{\mathcal{B}}^2 \hat{\Psi}_{\mathcal{B}}^0(E) \in \mathcal{B}$ has $\text{Coh}(X)$ -cohomologies:

- LA^1 in position -1 , and
- LA^2 in position 0 .

So we have $A \in V_{\text{Coh}(X)}^{\Phi}(1, 2)$, $LA^1 \in \text{HN}(-\infty, \frac{1}{2}]$ and by Corollary 4.17 $A^2 \in \text{HN}(0, +\infty]$.

Therefore $\ell^2 \text{ch}_1(A^1) \leq 0$ and $\ell^2 \text{ch}_1(A^2) \geq 0$. Hence

$$a_2 \ell^3 = 2\ell \text{ch}_2(A) = -\ell^2 \text{ch}_1(\Phi(A)) = -\ell^2 \text{ch}_1(A^2) + \ell^2 \text{ch}_1(A^1) \leq 0.$$

So

$$\Im Z(\hat{\Psi}_{\mathcal{B}}^0(E)) = \Im Z(A) = \frac{3\sqrt{3}}{4}(a_2 - a_1) < 0$$

as required. □

PROPOSITION 6.6. (I) Let $E \in \mathcal{T}'$. Then we have the following:

(i) $H_{\text{Coh}(X)}^0(\Psi_{\mathcal{B}}^2(E)) = 0$, and (ii) if $\Psi_{\mathcal{B}}^2(E) \neq 0$ then $\Im Z(\Psi_{\mathcal{B}}^2(E)) > 0$.

(II) Let $E \in \mathcal{F}'$. Then we have the following:

(i) $H_{\text{Coh}(X)}^{-1}(\Psi_{\mathcal{B}}^0(E)) = 0$, and (ii) if $\Psi_{\mathcal{B}}^0(E) \neq 0$ then $\Im Z(\Psi_{\mathcal{B}}^0(E)) < 0$.

Proof. (I) Let $E \in \mathcal{T}'$.

(i) Similar to the proof of (i) in Proposition 6.4.

(ii) From (i), we have $\Psi_{\mathcal{B}}^2(E) \cong A[1]$ for some coherent sheaf $0 \neq A \in \text{HN}(-\infty, \frac{1}{2}]$. Let $\text{ch}(A) = (a_0, a_1, a_2, a_3)$. Then $\text{ch}(L^{-1}A) = (a_0, a_1 - a_0, a_2 - 2a_1 + a_0, *)$ and so $a_1 - a_0 < 0$. Apply the FMT $\hat{\Psi}$ to $\Psi_{\mathcal{B}}^2(E)$. Since $\Psi_{\mathcal{B}}^2(E) \in V_{\mathcal{B}}^{\hat{\Psi}}(0)$, $\hat{\Psi}_{\mathcal{B}}^0 \Psi_{\mathcal{B}}^2(E) \in \mathcal{B}$ has $\text{Coh}(X)$ -cohomologies:

- $(L^{-1}A)^1$ in position -1 , and
- $(L^{-1}A)^2$ in position 0 .

So we have $(L^{-1}A)^2 \in \text{HN}(\frac{1}{2}, +\infty]$, and by Proposition 4.16, $(L^{-1}A)^1 \in \text{HN}(-\infty, 0]$. Therefore $\ell^2 \text{ch}_1((L^{-1}A)^1) \leq 0$ and $\ell^2 \text{ch}_1((L^{-1}A)^2) \geq 0$. Hence

$$\begin{aligned} (a_2 - 2a_1 + a_0)\ell^3 &= 2\ell \text{ch}_2(L^{-1}A) = -\ell^2 \text{ch}_1(\Phi(L^{-1}A)) \\ &= -\ell^2 \text{ch}_1((L^{-1}A)^2) + \ell^2 \text{ch}_1((L^{-1}A)^1) \leq 0. \end{aligned}$$

So we have

$$\begin{aligned} \Im Z(\Psi_{\mathcal{B}}^2(E)) &= \Im Z(A[1]) = \frac{3\sqrt{3}}{4}(a_1 - a_2) \\ &= -\frac{3\sqrt{3}}{4}((a_1 - a_0) + (a_2 - 2a_1 + a_0)) > 0 \end{aligned}$$

as required.

(II) Let $E \in \mathcal{F}'$.

(i) Similar to the proof of (i) in Proposition 6.5.

(ii) From (i), we have $\Psi_{\mathcal{B}}^0(E) \cong A$ for some $0 \neq A \in \text{HN}(\frac{1}{2}, +\infty]$.

Consider the convergence of the spectral sequence:

$$E_2^{p,q} = \Psi_{\text{Coh}(X)}^p(H_{\text{Coh}(X)}^q(E)) \implies \Psi_{\text{Coh}(X)}^{p+q}(E)$$

for E . Let $E_i := H_{\text{Coh}(X)}^i(E)$. Then by Note 6.3, $E_{-1} \in \text{HN}(-\infty, 0]$ and so by Proposition 4.16 and Proposition 4.7 we have

$$LE_{-1}^1 \in \text{HN}(-\infty, 1], \text{ and } LE_0^0 \in \text{HN}(-\infty, 1].$$

Therefore from the convergence of the above spectral sequence for E , we have

$$A \in \text{HN}(\frac{1}{2}, +\infty] \cap \text{HN}(-\infty, 1] = \text{HN}(\frac{1}{2}, 1].$$

Also A is reflexive, as LE_0^0 and LE_{-1}^1 are reflexive sheaves by Propositions 4.5 and 4.8. Let $\text{ch}(A) = (a_0, a_1, a_2, a_3)$. Then from the B-G inequalities for all the H-N semistable

factors of A , we have

$$\Im Z(\Psi_{\mathcal{B}}^0(E)) = \Im Z(A) = \frac{3\sqrt{3}}{4}(a_2 - a_1) \leq 0.$$

Equality holds when $A \in \text{HN}[1]$ with $\text{ch}(A) = (a_0, a_0, a_0, *)$. Then, by considering a Jordan-Hölder filtration for A together with Theorem 5.1, $L^{-1}A$ has a filtration of ideal sheaves $\mathcal{P}_{x_i} \mathcal{I}_{Z_i}$ of some 0-subschemas. Here $\Psi_{\mathcal{B}}^0(E) \cong A \in V_{\mathcal{B}}^{\hat{\Psi}}(2)$ implies $L^{-1}A \in V_{\text{Coh}(X)}^{\Phi}(2, 3)$. An easy induction on the rank of A also shows that $L^{-1}A \in V_{\text{Coh}(X)}^{\Phi}(1, 3)$ and so $L^{-1}A \in V_{\text{Coh}(X)}^{\Phi}(3)$. But then we have $Z_i = \emptyset$ for all i . Therefore $\hat{\Psi}_{\mathcal{B}}^2 \Psi_{\mathcal{B}}^0(E) \in \text{Coh}^0(X)$. Now consider the convergence of the Spectral Sequence 6.1 for E . Then we have \mathcal{B} -SES

$$0 \rightarrow \hat{\Psi}_{\mathcal{B}}^0 \Psi_{\mathcal{B}}^1(E) \rightarrow \hat{\Psi}_{\mathcal{B}}^2 \Psi_{\mathcal{B}}^0(E) \rightarrow F \rightarrow 0,$$

where F is a subobject of $(-1)^*E$ and so $F \in \mathcal{F}'$. Then $\hat{\Psi}_{\mathcal{B}}^2 \Psi_{\mathcal{B}}^0(E) \in \text{Coh}^0(X) \subset \mathcal{T}'$ which implies $F = 0$ and $\hat{\Psi}_{\mathcal{B}}^0 \Psi_{\mathcal{B}}^1(E) \cong \hat{\Psi}_{\mathcal{B}}^2 \Psi_{\mathcal{B}}^0(E)$. But then we have $\Psi_{\mathcal{B}}^0(E) \cong (-1)^* \Psi_{\mathcal{B}}^0 \hat{\Psi}_{\mathcal{B}}^0 \Psi_{\mathcal{B}}^1(E) = 0$. This is not possible as $\Psi_{\mathcal{B}}^0(E) \neq 0$. Therefore we have the strict inequality $\Im Z(\Psi_{\mathcal{B}}^0(E)) < 0$ as required to complete the proof. \square

LEMMA 6.7. (I) Let $E \in \mathcal{T}'$. Then (i) $\hat{\Psi}_{\mathcal{B}}^2(E) = 0$, and (ii) $\Psi_{\mathcal{B}}^2(E) = 0$.

(II) Let $E \in \mathcal{F}'$. Then (i) $\hat{\Psi}_{\mathcal{B}}^0(E) = 0$, and (ii) $\Psi_{\mathcal{B}}^0(E) = 0$.

Proof. (I) Let $E \in \mathcal{T}'$.

(i) From the convergence of the Spectral Sequence 6.1 for E , we have the \mathcal{B} -SES

$$0 \rightarrow Q \rightarrow \Psi_{\mathcal{B}}^0 \hat{\Psi}_{\mathcal{B}}^2(E) \rightarrow \Psi_{\mathcal{B}}^2 \hat{\Psi}_{\mathcal{B}}^1(E) \rightarrow 0.$$

Here Q is a quotient of $(-1)^*E \in \mathcal{T}'$ and so $Q \in \mathcal{T}'$. Then $\Psi_{\mathcal{B}}^0 \hat{\Psi}_{\mathcal{B}}^2(E)$ fits into the \mathcal{B} -SES

$$0 \rightarrow T \rightarrow \Psi_{\mathcal{B}}^0 \hat{\Psi}_{\mathcal{B}}^2(E) \rightarrow F \rightarrow 0$$

for some $T \in \mathcal{T}'$ and $F \in \mathcal{F}'$. Now apply the FMT $\hat{\Psi}$ and consider the LES of \mathcal{B} -cohomologies. Then we have $\hat{\Psi}_{\mathcal{B}}^0(T) = 0$, $\hat{\Psi}_{\mathcal{B}}^1(T) \cong \hat{\Psi}_{\mathcal{B}}^0(F)$. By Proposition 6.5 $\Im Z(\hat{\Psi}_{\mathcal{B}}^0(F)) \leq 0$ and by Proposition 6.4 $\Im Z(\hat{\Psi}_{\mathcal{B}}^2(T)) \geq 0$. So $\Im Z(\hat{\Psi}(T)) \geq 0$ and by Proposition 6.2 $\Im Z(T) \leq 0$. Since $T \in \mathcal{T}'$, we have $\Im Z(T) = 0$ and $\omega^2 \text{ch}_1^B(T) = 0$. Then by Lemma 1.1, $T \cong T_0$ for some $T_0 \in \text{Coh}^0(X)$. But $\text{Coh}^0(X) \subset V_{\mathcal{B}}^{\hat{\Psi}}(0)$. Hence $T = 0$ and so $Q = 0$. Then $\Psi_{\mathcal{B}}^0 \hat{\Psi}_{\mathcal{B}}^2(E) \cong \Psi_{\mathcal{B}}^2 \hat{\Psi}_{\mathcal{B}}^1(E)$ and so we have $\hat{\Psi}_{\mathcal{B}}^2(E) \cong (-1)^* \hat{\Psi}_{\mathcal{B}}^2 \Psi_{\mathcal{B}}^2 \hat{\Psi}_{\mathcal{B}}^1(E) = 0$ as required.

(ii) Similar to the proof of (i).

(II) Similar to the proofs in (I). \square

COROLLARY 6.8. Let $E \in \mathcal{B}$. Then

(i) $\Psi_{\mathcal{B}}^2(E), \hat{\Psi}_{\mathcal{B}}^2(E) \in \mathcal{T}'$, and

(ii) $\Psi_{\mathcal{B}}^0(E), \hat{\Psi}_{\mathcal{B}}^0(E) \in \mathcal{F}'$.

Proof. (i) By the definition of \mathcal{T}' and \mathcal{F}' , $\Psi_{\mathcal{B}}^2(E)$ fits into \mathcal{B} -SES

$$0 \rightarrow T \rightarrow \Psi_{\mathcal{B}}^2(E) \rightarrow F \rightarrow 0,$$

for some $T \in \mathcal{T}'$ and $F \in \mathcal{F}'$. Now apply the FMT $\hat{\Psi}$ and consider the LES of \mathcal{B} -cohomologies. Then by Lemma 6.7, $F = 0$ as required.

Similarly one can prove $\hat{\Psi}_{\mathcal{B}}^2(E) \in \mathcal{T}'$.

(ii) Similar to the proofs in (i). □

PROPOSITION 6.9. (I) Let $E \in \mathcal{F}'$. Then (i) $\hat{\Psi}_{\mathcal{B}}^1(E) \in \mathcal{F}'$, and (ii) $\Psi_{\mathcal{B}}^1(E) \in \mathcal{F}'$.

(II) Let $E \in \mathcal{T}'$. Then (i) $\hat{\Psi}_{\mathcal{B}}^1(E) \in \mathcal{T}'$, and (ii) $\Psi_{\mathcal{B}}^1(E) \in \mathcal{T}'$.

Proof. (I) (i) By the torsion theory $\hat{\Psi}_{\mathcal{B}}^1(E)$ fits into \mathcal{B} -SES

$$0 \rightarrow T \rightarrow \hat{\Psi}_{\mathcal{B}}^1(E) \rightarrow F \rightarrow 0$$

for some $T \in \mathcal{T}'$ and $F \in \mathcal{F}'$. Now we need to show $T = 0$. Apply the FMT Ψ and consider the LES of \mathcal{B} -cohomologies. We get $\Psi_{\mathcal{B}}^1(T) \hookrightarrow \Psi_{\mathcal{B}}^1 \hat{\Psi}_{\mathcal{B}}^1(E)$ and $T \in V_{\mathcal{B}}^{\Psi}(1)$. Also by the convergence of the Spectral Sequence 6.1 for E , $\Psi_{\mathcal{B}}^1 \hat{\Psi}_{\mathcal{B}}^1(E)$ is a subobject of $(-1)^*E$. Hence $\Psi_{\mathcal{B}}^1(T) \in \mathcal{F}'$ implies $\Im Z(\Psi_{\mathcal{B}}^1(T)) \leq 0$. On the other hand by Proposition 6.2, $\Im Z(\Psi_{\mathcal{B}}^1(T)) = \Im Z(T) \geq 0$ as $T \in \mathcal{T}'$. Hence $\Im Z(T) = 0$ and $T \in \mathcal{T}'$ implies $\omega^2 \text{ch}_1^B(T) = 0$. So by Lemma 1.1, $T \cong T_0$ for some $T_0 \in \text{Coh}^0(X)$. Since any object from $\text{Coh}^0(X)$ belongs to $V_{\mathcal{B}}^{\Psi}(0)$, $\Psi_{\mathcal{B}}^1(T) = 0$. So $T = 0$ as required.

(ii) Similar to the proof of (i).

(II) Similar to the proofs in (I). □

By Lemma 6.7, Corollary 6.8 and Proposition 6.9 we have

$$\Psi[1](\mathcal{F}'[1]) \subset \mathcal{A}, \quad \text{and} \quad \Psi[1](\mathcal{T}') \subset \mathcal{A}.$$

Since $\mathcal{A} = \langle \mathcal{F}'[1], \mathcal{T}' \rangle$, $\Psi[1](\mathcal{A}) \subset \mathcal{A}$.

Similarly we have $\hat{\Psi}[1](\mathcal{A}) \subset \mathcal{A}$. The isomorphisms $\hat{\Psi}[1] \circ \Psi[1] \cong (-1)^* \text{id}_{D^b(X)}$ and $\Psi[1] \circ \hat{\Psi}[1] \cong (-1)^* \text{id}_{D^b(X)}$ give us the following

THEOREM 6.10. *The FMTs $\Psi[1]$ and $\hat{\Psi}[1]$ give the auto-equivalences*

$$\Psi[1](\mathcal{A}) \cong \mathcal{A}, \quad \text{and} \quad \hat{\Psi}[1](\mathcal{A}) \cong \mathcal{A}$$

of the abelian category \mathcal{A} .

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