Multivariate permutation entropy via the Cartesian graph product to analyse two-phase flow

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1 Introduction

Entropy metrics are nonlinear measures to quantify the complexity of time series. Among them, permutation entropy is a common metric due to its robustness and fast computation. Multivariate entropy metrics techniques are needed to analyse data consisting of more than one time series. To this end, we present a multivariate permutation entropy, $\mathrm{MPE_G}$, using a graph-based approach.

Most biomedical and physical systems are multivariate. Therefore, univariate entropy metrics have been generalised to a multivariate setting, including multivariate sample entropy, multivariate dispersion entropy [1], among others. A multivariate multiscale permutation entropy (MMPE) to analyse physiological signals is proposed in [2]. However, such algorithm extracts the permutation patterns from each channel separately regardless of their cross-channel dependencies. MMPE treated multichannel signals as a unique block and without interactions between channels. Thus, it works appropriately only when the components of a multivariate signal are statistically independent.

Given a multivariate signal, the proposed algorithm $\mathrm{MPE_G}$ involves two main steps: 1) we construct an underlying graph G as the Cartesian product of two graphs G_1 and G_2, where G_1 preserves temporal information of each time series together with G_2 that models the relations between different channels, and 2) we consider the multivariate signal as samples defined on the regular graph G and apply the recently introduced permutation entropy for graphs [3].

Our graph-based approach gives the flexibility to consider diverse types of cross channel relationships and signals, and it overcomes with the limitations of current multivariate permutation entropy.

2 Multivariate permutation entropy $\mathrm{MPE_G}$

Let $\mathbf{U} = \{u_{t,s}\}_{t=1,2,\dots,n}^{s=1,2,\dots,p}$ be a multivariate signal; we will construct a 2D graph (using the Cartesian product). One dimension will preserve the temporal information, and another will preserve the cross-channel information.

Dimension 1. Temporal information. We associate the directed path with a time series, where a vertex represents each sample time. A *directed path* on n vertices is a directed graph that joins a sequence of different vertices with all the edges in the same direction and is denoted by $\overrightarrow{P_n}$. An example is depicted in Fig. 1(a).

Dimension 2. Relationships between channels. Let I_p be the graph with p vertices representing the interaction between different channels (or time series).

If we do not have any a priori information about the interactions between channels, by default, we will consider equal interactions between all channels. Complete graphs represent such relations, i.e., we will set $I_p = K_p$ as the complete graph with p vertices, see an example in Fig. 1(b). Any other interaction between directed (temporal) and undirected (spatial) edges, can be included in the weighted adjacency matrix.

Let $G_1 = (\mathscr{V}_1, \mathscr{E}_1)$ and $G = (\mathscr{V}_2, \mathscr{E}_2)$ be two graphs, we denote the *Cartesian product* (or box product) of two graph by $G_1 \Box G_2$. The square symbol $\Box$ shows visually the Cartesian product of two edges.

Let G as the graph associated with $\mathbf{U}$ and given by $G := \overrightarrow{P_n} \Box I_p$, observe G is not a periodic graph, but it can be considered as a geometrical perturbation of a periodic graph [4] and some centrality measures to time-dependant networks are studied in [5].

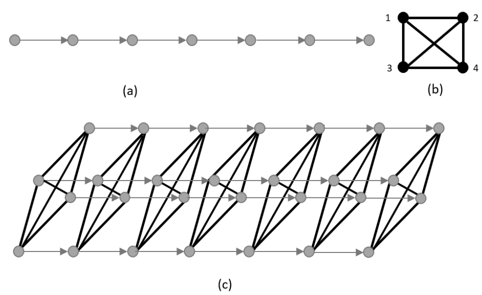

Fig. 1. (a) Directed path with seven vertices, denoted by $\overrightarrow{P_7}$. (b) Interactions between the four channels are encoded with the complete graph on four vertices, denoted by K_4. (c) The Cartesian product $\overrightarrow{P_7} \Box K_4$.

Once the graph is constructed, we use the permutation entropy for graph signals $\mathrm{PE_G}$ introduced in [3]. $\mathrm{PE_G}$ is an entropy metric to analyse signals measured over irregular graphs by generalising permutation entropy, a well-established nonlinear metric based on the comparison of neighbouring values within patterns in a time series. $\mathrm{PE_G}$ is based on comparing signal values on neighbouring nodes, using the adjacency matrix. This generalisation preserves the properties of classical permutation for time series and the recent permutation entropy for images [6], and it can be applied to any graph structure.

Definition 1. ***Multivariate permutation entropy.*** *Let* $\mathbf{U}$ *be a multivariate time series with interaction graph* I_p *between channels.*

1. ***Graph construction.*** *Construct the graph* G *using the Cartesian graph product, i.e.,* $G := \overrightarrow{P_n} \Box I_p$.
2. ***Graph signal.*** $\mathbf{U}$ *can be considered as a signal defined on the graph* G.
3. *The* ***multivariate permutation entropy*** ($\mathrm{MPE_G}$) *is defined as the permutation entropy for the graph signal* $\mathrm{PE_G}$ *(see [3, 7]) for the signal* $\mathbf{U}$ *and the graph* G*, i.e.,* $\mathrm{MPE_G} := \mathrm{PE_G}(\mathbf{U})$.

3 Results

We apply the algorithm to a set of multivariate synthetic signals used in the study of dynamical systems, included the Hénon map and the Lorenz System. On both settings, MPE_G is able to detect chaotic behaviour and windows of stability (see Fig. 2). We show that MPE_G is a general frame where classical Permutation entropy, MMPE and PE_G are particular cases [7].

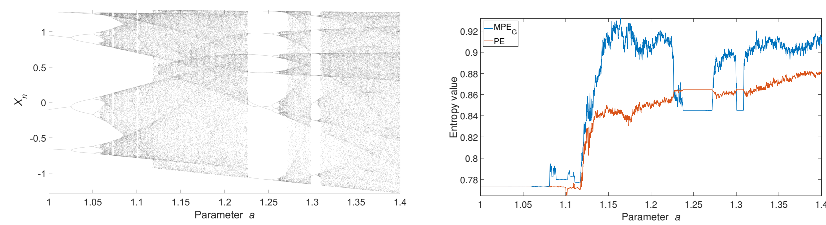

Fig. 2. Orbit diagram for the Hénon map and its entropy values.

As application in real-world data, we apply MPE_G to two-phase flow data. MPE_G is sensitive to complexity change of flow patterns. It also reveals the transition between flow patterns when flow velocity changes. We show that MPE_G is effective to analyse two-phase flow pattern transition.

Summary. We introduced a multivariate permutation entropy to quantify the complexity of multivariate time series. The algorithm proposed use the Cartesian product of graphs and the recently introduced permutation entropy for graph signals [3]. Our graph-based approach considers diverse type of cross channel relationships and overcomes with the limitations of current multivariate permutation entropy.

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