



THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

Dilaton and Massive Hadrons in a Conformal Phase

Citation for published version:

Debbio, LD & Zwicky, R 2022, 'Dilaton and Massive Hadrons in a Conformal Phase', *Journal of High Energy Physics*, vol. 2022, no. 8, 7, pp. 1-20. [https://doi.org/10.1007/JHEP08\(2022\)007](https://doi.org/10.1007/JHEP08(2022)007)

Digital Object Identifier (DOI):

[10.1007/JHEP08\(2022\)007](https://doi.org/10.1007/JHEP08(2022)007)

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Publisher's PDF, also known as Version of record

Published In:

Journal of High Energy Physics

General rights

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact openaccess@ed.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.



Dilaton and massive hadrons in a conformal phase

Luigi Del Debbio and Roman Zwicky

*Higgs Centre for Theoretical Physics, School of Physics and Astronomy,
The University of Edinburgh,
Peter Guthrie Tait Road, Edinburgh EH9 3FD, Scotland, U.K.*

E-mail: luigi.del.debbio@ed.ac.uk, roman.zwicky@ed.ac.uk

ABSTRACT: As the number of fermion fields is increased, gauge theories are expected to undergo a transition from a QCD-like phase, characterised by confinement and chiral symmetry breaking, to a conformal phase, where the theory becomes scale-invariant at large distances. In this paper, we discuss some properties of a third phase, where spontaneously broken conformal symmetry is characterised by its Goldstone boson, the dilaton. In this phase, which we refer to as conformal dilaton phase, the massless pole corresponding to the Goldstone boson guarantees that the conformal Ward identities are satisfied in the infrared despite the other hadrons carrying mass. In particular, using renormalisation group arguments in Euclidean space, we show that for massless quarks the trace of the energy momentum tensor vanishes on all physical states as a result of the fixed point. This implies the vanishing of the gluon condensate and suggests that the scale breaking is driven by the quark condensate which has implications for the cosmological constant. In addition form factors obey an exact constraint for every hadron and are thus suitable probes to identify this phase in the context of lattice Monte Carlo studies. For this purpose we examine how the system behaves under explicit symmetry breaking, via quark-mass and finite-volume deformations. The dilaton mass shows hyperscaling under mass deformation, viz. $m_D = \mathcal{O}(m_q^{1/(1+\gamma^*)})$. This provides another clean search pattern.

KEYWORDS: Nonperturbative Effects, Renormalization and Regularization, Renormalization Group, Spontaneous Symmetry Breaking

ARXIV EPRINT: [2112.11363](https://arxiv.org/abs/2112.11363)

Contents

1	Introduction	1
1.1	Axial and dilatation Ward identities	2
1.2	Overview of the extended conformal window	3
2	Gravitational form factors of spin-0	4
2.1	The gravitational form factors in the conformal phase	5
2.2	Verification of dilatation Ward identity at $q^2 = 0$ via the LSZ formalism	5
3	Perturbations of the conformal dilaton phase	6
3.1	Quark mass-deformation	6
3.2	Scaling in the presence of a dynamical scale Λ	8
3.3	Dilaton Ward identity in the vicinity of the IR fixed point	10
3.4	Finite volume scaling	12
4	Conformal dilaton signatures	12
4.1	Lattice Monte Carlo simulations	12
4.2	The Higgs and the $f_0(500)$ as pseudo-dilatons	13
5	Discussions and conclusions	14
A	Conventions	16
B	Gravitational form factors of spin-1/2	16

1 Introduction

It is well-known, since the seminal work of ref. [1], that gauge theories in $d = 4$ show very different infrared (IR) behaviour depending on the matter representation, the number of flavours N_f and colours N_c . As the matter content is varied, these theories undergo a transition between a QCD-like phase where chiral symmetry is spontaneously broken, and hadron confinement takes place, and a phase where conformal symmetry is exhibited by the scaling of the correlation functions in the IR.¹ The latter phase is referred to as the “conformal window”. Recent results are summarised in ref. [4].

In this work we would like to investigate some properties of a third phase where conformal symmetry is spontaneously broken, leading to the appearance of a Goldstone boson (GB),

¹It is generally believed that confinement and chiral symmetry go hand in hand because the pion match the anomaly of the quark in the IR. In some supersymmetric gauge theories the role of the pions can be taken by massless baryons and this phase is referred to as s-confinement [2, 3].

the dilaton.² The dilaton has been widely studied in the literature as a candidate model for a composite version of the Higgs [6, 7] with various effective Lagrangians [8–12], or as a driving field theory version force of inflation [13]. In this work we focus on the dilaton as the catalyst to the massive hadronic spectrum; indeed the massless pole corresponding to the dilaton allows for the conformal Ward identity (WI) to be satisfied even in the presence of massive states in the spectrum. In particular the trace of the energy momentum tensor (EMT) vanishes on physical states ϕ_i , $\langle \phi_2 | T^\mu_\mu(x) | \phi_1 \rangle \rightarrow 0$ as shown in section 3.3.

1.1 Axial and dilatation Ward identities

It is well-known that the pion decay constant F_π is the order parameter of spontaneous chiral symmetry breaking. The dilaton decay constant F_D plays the analogous role for the spontaneous breaking of dilatation or scale symmetry. It seems beneficial to treat them in parallel here. The decay constants are defined as³

$$\begin{aligned} \Gamma_{5\mu}^{(ab)}(q) &= \langle 0 | J_{5\mu}^a(0) | \pi^b(q) \rangle = iF_\pi q_\mu \delta^{ab}, \\ \Gamma_\mu(q) &= \langle 0 | J_\mu^D(0) | D(q) \rangle = iF_D q_\mu, \end{aligned} \tag{1.1}$$

where the Noether currents associated to the broken symmetries are respectively $J_{5\mu}^a(x) = \bar{q}(x) T^a \gamma_\mu \gamma_5 q(x)$ and $J_\mu^D(x) = x^\nu T_{\mu\nu}(x)$, where $F_\pi \approx 92 \text{ MeV}$ in QCD and T^a is a generator of the broken axial flavour symmetry $SU(N_F)$. The divergences of the currents are given by the explicit and anomalous symmetry breaking; using⁴

$$\begin{aligned} \partial \cdot J_5^a(x) &= 2m_q P^a(x) = 2m_q \bar{q}(x) T^a i\gamma_5 q(x), \\ \partial \cdot J^D(x) &= T^\mu_\mu(x) = \frac{\beta}{2g} G^2(x) + m_q(1+\gamma) \bar{q}(x) q(x), \end{aligned} \tag{1.2}$$

one obtains

$$i^{-1} q^\mu \Gamma_{5\mu}^{(ab)}(0) = 2m_q \langle 0 | P^a(0) | \pi^b(q) \rangle = F_\pi m_\pi^2 \delta^{ab} \xrightarrow{\text{sym}} 0, \tag{1.3}$$

$$i^{-1} q^\mu \Gamma_\mu(0) = \langle 0 | \frac{\beta}{2g} G^2(0) + m_q(1+\gamma) \bar{q}q(0) | D(q) \rangle = F_D m_D^2 \xrightarrow{\text{sym}} 0. \tag{1.4}$$

These equations vanish in the symmetry limit $m_q \rightarrow 0$. For the dilaton WI (1.4) this is not obvious as there is anomalous breaking of scale symmetry in addition. However in section 3.3 we prove, using renormalisation group (RG) arguments in Euclidean space, that the equation holds. When expressed in terms of hadronic quantities, the divergences of the Noether currents are given by products of decay constants times masses, as shown on the right-hand side of eqs. (1.3), (1.4); their vanishing occurs through $m_{\pi,D} \rightarrow 0$ as required

²Throughout we will not distinguish conformal and scale (dilatation) invariance. It is widely believed that scale invariance implies conformal invariance in a wide class of theories in four dimensions, see e.g. ref. [5] for a review.

³The second equation below is consistent with $\langle 0 | T_{\mu\nu} | D(q) \rangle = \frac{F_D m_D^2}{d-1} (\eta_{\mu\nu} - q_\mu q_\nu / m_D^2)$ cf. also eq. (1.4).

⁴All our conventions are specified in appendix A. The trace anomaly [14–16] contains further equation of motion terms which vanish on physical states and are not of interest to our work.

particle	non-Goldstone $\left(\frac{\eta_{F_{D',\pi'}}}{1+\gamma_*} \geq 1\right)$	Goldstone
$J^{PC} = 0^{-+}$	$m_{\pi'} = \mathcal{O}(\Lambda) \quad F_{\pi'} = \mathcal{O}\left(m_q^{\frac{\eta_{F_{\pi'}}}{1+\gamma_*}}\right)$	$m_{\pi} = \mathcal{O}\left(m_q^{\frac{1}{1+\gamma_*}}\right) \quad F_{\pi} = \mathcal{O}(\Lambda)$
$J^{PC} = 0^{++}$	$m_{D'} = \mathcal{O}(\Lambda) \quad F_{D'} = \mathcal{O}\left(m_q^{\frac{\eta_{F_{D'}}}{1+\gamma_*}}\right)$	$m_D = \mathcal{O}\left(m_q^{\frac{1}{1+\gamma_*}}\right) \quad F_D = \mathcal{O}(\Lambda)$

Table 1. Overview of how the important parameters entering the explicitly and anomalously broken Ward identities behave in the conformal dilaton phase. The scale Λ stands for a generic hadronic scale which in QCD is usually referred to as Λ_{QCD} . The behaviour of $m_{D,\pi}$ and $F_{D',\pi'}$ ($\eta_{F_{D',\pi'}}/(1+\gamma_*) \geq 1$) under mass-deformation will be discussed in section 3.1. The quantity γ_* is the mass anomalous dimension at the IR fixed point.

by the Goldstone nature of the pions and the dilaton. The decay constants are the order parameters and do not vanish. Heuristically one has

$$\text{SSB:} \quad Q|0\rangle \neq 0, \quad Q = \int d^{d-1}x J^0, \tag{1.5}$$

the signal of spontaneous symmetry breaking (SSB), is equivalent to (1.1).⁵ For the non-GBs, which we denote by π^{Ia} and D' , it is just the opposite, the WIs (1.3), (1.4) are satisfied by a zero decay constant as the hadronic masses are non-zero. For the D' this is a subtle statement in view of the anomalous breaking of scale symmetry but in the end this is implied by the WI which holds for higher states cf. the remark above. An overview of the parametric behaviour in the conformal dilaton phase is given in table 1 and the precise mass scalings are discussed in section 3.1. Equipped with the broad picture we summarise the characteristics of the three phases before getting to the heart of the paper.

1.2 Overview of the extended conformal window

Let us summarise the different phases of gauge theories. First, we know from the Banks-Zaks analysis [1] that there is a conformal phase for $N_f \approx 16$ and $N_c = 3$ and probably well below. The range in N_f before conformal symmetry is (dynamically) broken is known as the conformal window and its determination is the topic of ongoing efforts of continuum [18–21] and lattice Monte Carlo studies [22–32] (cf. [33] for a recent review). In $\mathcal{N} = 1$ supersymmetric gauge theories this boundary is known exactly. Below the conformal window chiral symmetry is spontaneously broken and quark confinement takes place. In particular this happens in QCD where $N_c = 3$, $N_f = 3$ (three light flavours) and quarks are in the fundamental representation of $SU(N_c)$. What we are advertising here is that there might be a third phase embedded in the conformal window where conformal symmetry is spontaneously broken. It would seem reasonable to assume that this phase lives on the boundary of the conformal window as sketched in figure 1.

The paper is organised as follows. In section 2 we define the gravitational form factors and show how the dilaton restores the dilatational WI. In section 3 matter mass and finite

⁵There is a subtlety with this argument in that the norm of the state created in eq. (1.5) is proportional to square root of the spatial volume. This can be seen by considering the 2-point function of the currents and integrating over the spatial parts. A careful treatment for chiral symmetry can be found in ref. [17].

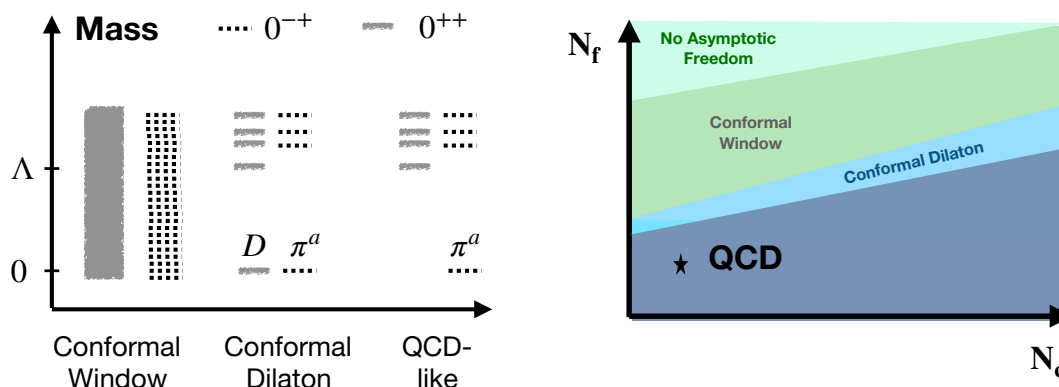


Figure 1. (left) Schematic spectra of conformal window, conformal dilaton and a QCD like theory in the $m_q \rightarrow 0$ limit. (right) Qualitative phase diagram of a given matter representation as a function of the number of colours and flavours. The boundary of asymptotic freedom is well-established and known as the Banks-Zaks region. The boundary with the QCD region is a matter of debate. The light-blue conformal dilaton phase is the one discussed in this paper. We wish to emphasise that this is just schematic and that the region of this phase could be rather different (should it exist at all). In this paper we discuss its logical possibility and speculate in section 4.2 that QCD itself could be of this type.

volume effects are discussed. Specific search strategies for the conformal dilaton phase with lattice Monte Carlo simulations are assembled in section 4 along with a discussion on whether the dilaton could be the $f_0(500)$ or the Higgs in QCD or the electroweak sector. The paper ends with discussion and conclusions in section 5. Appendices A, B deal with conventions and the spin-1/2 form factors.

2 Gravitational form factors of spin-0

The gravitational form factors parametrise the matrix elements of the energy momentum tensor (EMT) between physical states; they can serve as quantum corrections to external gravitational fields [34], or as probes of the nucleon structure [35, 36]. The spin-1/2 case is discussed in appendix B and the spin-1, parameterised in [34], amounts to an interplay between F_1 and F_2 at zero momentum transfer. Here we focus on the spin-0 case since it illustrates all the important points without unnecessary complications. The dimensionless gravitational form factors for a generic scalar hadron, denoted by ϕ , are defined as follows

$$T_{\mu\nu}^{(\phi)}(p, p') = \langle \phi(p') | T_{\mu\nu}(0) | \phi(p) \rangle = 2\mathcal{P}_\mu \mathcal{P}_\nu G_1(q^2) + \left(\frac{q_\mu q_\nu}{q^2} - \eta_{\mu\nu} \right) m_\phi^2 G_2(q^2), \quad (2.1)$$

where $q \equiv p - p'$ is the momentum transfer, $\mathcal{P} \equiv \frac{1}{2}(p + p')$ and $q^\mu T_{\mu\nu}^{(\phi)} = 0$, as required by translational invariance. Note that the limit $q^2 \rightarrow 0$ of (2.1) is still well defined, despite the pole in G_2 , because for diagonal form factors the limit implies $q_\mu \rightarrow 0$ at the same time. Since the EMT is related to the momentum, $P_\mu = \int d^{d-1}x T_\mu^0(x)$, by the usual conserved current procedure, the form factor G_1 must satisfy

$$G_1(0) = 1, \quad (2.2)$$

where we use the conventional state normalisation $\langle \phi(p') | \phi(p) \rangle = 2E_p (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$. The second structure is related to the improved energy momentum tensor which renders the free scalar field conformal in dimension other than two [37]. Everything in this section, up to now, was completely general. In the next section we discuss the conformal IR phases with particular emphasis on the dilaton case.

2.1 The gravitational form factors in the conformal phase

In section 3.3 we show that $T_\mu^{(\phi)\mu}(p, p')$, as defined in (2.1), vanishes when there is an IR fixed point. This yields one constraint on the form factors for any spin and in particular for spin-0 this results in

$$2m_\phi^2 G_1(q^2) - (d-1)m_\phi^2 G_2(q^2) = 0. \tag{2.3}$$

The most straightforward solution is the one of unbroken conformal symmetry for which $m_\phi^2 = 0$ implies $G_2(q^2) = 0$. This is the classic conformal window scenario. However, there is another possibility: the second term cancels the first one. In particular this implies $G_2(0) = 2/(d-1)$, taking into account eqs. (2.3) and (2.2). And this is where the dilaton pole and spontaneous breaking of scale symmetry come into play. In summary one has the three phases depicted in figure 1^{6,7}

Conformal Window:	$G_2(q^2) \neq 0,$	$\langle \phi T_\mu^\mu \phi \rangle = 2m_\phi^2 = 0,$
Conformal Dilaton:	$G_2(q^2) = \frac{2}{d-1} G_1(q^2),$	$\langle \phi T_\mu^\mu \phi \rangle = 0, m_\phi^2 \neq 0,$
QCD-like:	$G_2(q^2) \neq 0,$	$\langle \phi T_\mu^\mu \phi \rangle = 2m_\phi^2 \neq 0.$

(2.4)

In order to avoid confusion it seems crucial to state that in this scenario the usual relation $2m_\phi^2 = \langle \phi | T_\mu^\mu | \phi \rangle$ does *not* hold, cf. above, as it would either not allow for hadron masses or the dilation WI to be obeyed.⁸ The possibility of such a scenario was mentioned prior to the discovery of the trace anomaly [41] but not worked out, for example in terms of hadronic parameters. The doing thereof is the topic of the next section.

2.2 Verification of dilatation Ward identity at $q^2 = 0$ via the LSZ formalism

It is advantageous to represent the form factor G_2 in terms of a subtracted dispersion relation

$$G_2(q^2) = G_2(0) + \frac{q^2}{\pi} \int_{0^+}^\infty \frac{ds \operatorname{Im}[G_2(s)]}{s(s - q^2 - i0)}, \tag{2.5}$$

where 0^+ indicates that the single dilaton has been removed from the integral. From (2.3) we infer the low energy theorem $G_2(0) = 2/(d-1)$ which we are able to verify explicitly,

⁶The second relation can be seen as a cousin of the Goldberger-Treiman relation for the nucleons. The analogy is not strictly close as there the partially conserved axial current (PCAC) gives a non-vanishing term on the r.h.s. (which though vanishes in the limit $m_q \rightarrow 0$). This results in $g_A = 1 + \mathcal{O}(m_\pi^2) \approx 1.23$ (e.g. [38]) and not an exact relation like $G_1(0) = 1$.

⁷This is the only, straightforward, logical possibility as the $J^{\text{PC}} = 0^{++}$ state does not contribute to the G_1 form factor and a composite massless $J^{\text{PC}} = 2^{++}$ is forbidden by the Weinberg-Witten theorem [39].

⁸This implies that the gluon condensate definition [40], which departs from this relation, does hold in QCD-like but not in the conformal dilaton phase.

using the LSZ formalism (e.g. [38, 42, 43]), as this point corresponds to the on-shell process $\phi \rightarrow \phi D$. The effective Lagrangian for the $\phi \rightarrow \phi D$ process is $\mathcal{L}^{\text{eff}} = g_{\phi\phi D} \frac{1}{2} \phi^2 D$, which yields the amplitude

$$\langle D\phi|\phi\rangle = i(2\pi)^d \delta\left(\sum p_i\right) g_{\phi\phi D}. \tag{2.6}$$

To achieve our goal two steps are needed. First we need to determine $g_{\phi\phi D}$ in terms of other parameters and then we apply the LSZ reduction to extract $\langle D\phi|\phi\rangle$ and match to (2.6). The $g_{\phi\phi D}$ coupling can be determined by writing an effective Lagrangian for the dilaton, where the field $e^{D(x)/F_D}$ plays the role of a conformal compensator, see e.g. [44]. Namely, terms in the Lagrangian which scale like $\sqrt{-g}\mathcal{L} \rightarrow e^{-n\alpha}\sqrt{-g}\mathcal{L}$ under dilatations $g_{\mu\nu} \rightarrow e^{2\alpha}g_{\mu\nu}$ can be made invariant by adding a prefactor e^{nD/F_D} , where $D \rightarrow D + \alpha F_D$ under scale transformations.⁹ Applied to the mass terms this gives the following effective Lagrangian

$$\mathcal{L}^{\text{eff}} = e^{2D/F_D} \frac{1}{2} m_\phi^2 \phi^2 \quad \Rightarrow \quad g_{\phi\phi D} = \frac{2m_\phi^2}{F_D}. \tag{2.7}$$

Second the matrix element in (2.6) can be obtained in another way, directly from the form factor (2.1), by using the EMT as an interpolating operator of the dilaton. Assuming external momenta with no entries in the third direction, then $T_{33}^{(\phi)}$ projects on $(m_\phi^2/q^2)G_2$.

$$\begin{aligned} \langle D\phi|\phi\rangle &= \lim_{q^2 \rightarrow 0} i \frac{q^2}{Z_D} \int d^d x e^{iq \cdot x} T_{33}^{(\phi)}(p, p', x) \\ &= \lim_{q^2 \rightarrow 0} i \frac{q^2}{Z_D} \frac{m_\phi^2}{q^2} G_2(q^2) (2\pi)^d \delta\left(\sum p_i\right), \end{aligned} \tag{2.8}$$

where $Z_D = F_D/(d-1)$ by footnote 3 and identifying the two equations one gets, using (2.7)

$$\lim_{q^2 \rightarrow 0} G_2(q^2) = \frac{g_{\phi\phi D} Z_D}{m_\phi^2} = \frac{2}{d-1}, \tag{2.9}$$

which satisfies (2.4) when $G_1(0) = 1$ is taken into account. This matches (2.3) in the $q^2 \rightarrow 0$ limit and thus shows that a dilaton phase seems a logical possibility indeed. The interplay of the dilaton residue and the vanishing of the trace of the EMT is an encouraging result.

3 Perturbations of the conformal dilaton phase

3.1 Quark mass-deformation

We turn now to the question of how the hadronic quantities change when the quark mass is turned on. At a scale $\sqrt{q^2} \ll \Lambda$, introduced in table 1, all states except the dilaton and the pion decouple from the spectrum and we essentially have a conformal theory with a dilaton and pions. This situation is similar to the mass-deformed conformal window scenario extensively discussed in our previous papers [45–47], provided that $m_q \ll \Lambda$ (as otherwise

⁹Here $g_{\mu\nu} = \eta_{\mu\nu}$ and g denotes the determinant. In the case where the transformation parameter is chosen to be a local function one often refers to these transformations as Weyl scaling. The term below is Weyl invariant.

the quarks would decouple). The result that is sufficient for this section is that a matrix element of an operator \mathcal{O} , of scaling dimension $\Delta_{\mathcal{O}} = d_{\mathcal{O}} + \gamma_{\mathcal{O}}$, between physical states $\phi_{1,2}$ in the vicinity of the fixed point behaves like¹⁰

$$\langle \phi_2 | \mathcal{O} | \phi_1 \rangle = m_q^{\frac{\eta}{1+\gamma_*}}, \quad \eta = \Delta_{\mathcal{O}} + d_{\phi_1} + d_{\phi_2}, \quad (3.1)$$

where we have assumed zero momentum transfer ($p = p'$) for the time being. Above $d_{\mathcal{O}}$ and $\gamma_{\mathcal{O}} = -\frac{d}{d \ln \mu} \ln \mathcal{O}$ stand for the engineering and anomalous dimensions respectively. The relation (3.1) has limited applicability in our case because of the presence of the additional scale Λ , a point we will return to in the next section. We can only apply it to the dilaton and the pion mass. Starting from (3.1) one can obtain a differential equation, using the trace anomaly, which leads to [46]

$$m_D^2 \propto m_q^{\frac{2}{1+\gamma_*}}, \quad m_{\pi}^2 \propto m_q^{\frac{2}{1+\gamma_*}}. \quad (3.2)$$

Alternatively this result can be obtained following other techniques [45] which correspond to setting $\Lambda = 0$ in section 3.2.

It is also of interest to investigate the scaling of $F_{D',\pi'}$ which can be done by using the dilaton WI (1.4) applied to D', π'

$$\begin{aligned} F_{D'} m_{D'}^2 &= \langle 0 | m_q (1 + \gamma_*) \bar{q} q | D'(q) \rangle \propto m_q^{\frac{\eta_{F_{D'} m_{D'}^2}}{1+\gamma_*}}, \quad \frac{\eta_{F_{D'} m_{D'}^2}}{1 + \gamma_*} \geq 1, \\ F_{\pi'} m_{\pi'}^2 &= \langle 0 | m_q (1 + \gamma_*) \bar{q} q | \pi'(q) \rangle \propto m_q^{\frac{\eta_{F_{\pi'} m_{\pi'}^2}}{1+\gamma_*}}, \quad \frac{\eta_{F_{\pi'} m_{\pi'}^2}}{1 + \gamma_*} \geq 1, \end{aligned} \quad (3.3)$$

where the matrix element proportional to the β -function has been neglected, as it is subleading for $m_q \approx 0$. The statement $\eta_{F_{D',\pi'} m_{D',\pi'}^2} / (1 + \gamma_*) \geq 1$ then follows from the assumption that the matrix element $\langle 0 | \bar{q} q | D', \pi'(q) \rangle$ is finite for $m_q \rightarrow 0$. Since $m_{D'} = \mathcal{O}(\Lambda)$ it then follows that

$$F_{D',\pi'} \propto m_q^{\frac{\eta_{F_{D',\pi'}}}{1+\gamma_*}}, \quad \frac{\eta_{F_{D',\pi'}}}{1 + \gamma_*} \geq 1. \quad (3.4)$$

These observations are interesting per se and complete table 1 but we would like to understand how (2.3) is altered. We may use the same WI as above but applied to a diagonal matrix element and conclude

$$T_{\mu}^{(\phi)\mu}(0) = m_q (1 + \gamma_*) \langle \phi | \bar{q} q | \phi \rangle \propto m_q^{\frac{\eta_{T_{\phi}}}{1+\gamma_*}}, \quad \frac{\eta_{T_{\phi}}}{1 + \gamma_*} \geq 1. \quad (3.5)$$

The first scaling follows from the hyperscaling relation (3.1) and the second one, once more, from the assumption that the matrix element $\langle \phi | \bar{q} q | \phi \rangle$ is finite as $m_q \rightarrow 0$. The

¹⁰In our previous work this was shown to hold on the lowest state in each channel, except for the masses where it was shown in generality [46]. However, our arguments at the end of section 3.3 shows that it holds for all states.

correction to the form factor constraint (2.3) then follows from the correction to the on-shell coupling (2.7) and leads to

$$G_2(0) = \frac{2}{d-1} \left[1 + \mathcal{O} \left(m_q^{\frac{\eta_{T_\phi}}{1+\gamma_*}} \right) \right], \quad (3.6)$$

since $G_1(0) = 1$ in general. Hence the scaling correction will come from the second term in (3.6) so that the r.h.s. can match (3.5).

An interesting question is how this changes when the momentum transfer is non-zero. We may assess this question by expanding in q^2

$$T_\mu^{(\phi)\mu}(q^2) = T_\mu^{(\phi)\mu}(0) + q^2 \frac{d}{dq^2} T_\mu^{(\phi)\mu}(0) + \frac{q^4}{2} \left(\frac{d}{dq^2} \right)^2 T_\mu^{(\phi)\mu}(0) + \mathcal{O}(q^6), \quad (3.7)$$

and demanding that the expansion converges which amounts to determine the scaling of the derivatives. First we note, cf. section 3.3 for more details, that $T_\mu^{(\phi)\mu}(q^2) = 0$ for $m_q \rightarrow 0$ and thus we may apply the RG analysis in section 3 of our previous work [47] as applied to the pion form factor. We infer that

$$\left(\frac{d}{dq^2} \right)^n T_\mu^{(\phi)\mu}(0) \propto \left(\frac{m_q}{\Lambda} \right)^{\frac{\eta_{T_\phi}}{1+\gamma_*}} \left(\frac{1}{\Lambda_m^2} \right)^n, \quad \Lambda_m \equiv m_q^{\frac{1}{1+\gamma_*}} \Lambda^{\frac{\gamma_*}{1+\gamma_*}}, \quad (3.8)$$

where the first factor is just the previous result in (3.5) and Λ_m sets the new scale. Note that the relative coefficients, unlike η_{T_ϕ} itself, of the form factor derivatives follows the straightforward hyperscaling law as they are not affected by the dynamical scale Λ to be assessed in the next section. Hence $T_\mu^{(\phi)\mu}(q^2)$ ceases to be close to $T_\mu^{(\phi)\mu}(0) \propto (m_q/\Lambda)^{\eta_{T_\phi}/(1+\gamma_*)}$ for momentum transfers $q^2 \gg \Lambda_m^2$. In some sense the scale Λ_m defines the deep IR for which the TEMT reveals its IR fixed-point in the presence of an explicit quark mass m_q .

3.2 Scaling in the presence of a dynamical scale Λ

Let us now revisit the RG scaling for field correlators in the case where scale invariance is spontaneously broken. We closely follow the derivations in our previous studies [46], allowing for the dependence on an extra scale Λ that is dynamically generated as a result of the spontaneous breaking. If it was not for the scale Λ one would directly conclude that $\eta_{T_\phi} = 2$ and $\eta_{E_{D'} m_{D'}^2} = 3$. In this section we shall see why this conclusion does not hold in the presence of the dynamical scale Λ .

We consider both 2-point and 3-point functions in Euclidean space, which are defined respectively as

$$C_{\mathcal{O}\Phi}(t, \mathbf{p}, \Lambda; \delta g, \hat{m}_q, L^{-1}, \mu) = \frac{1}{V} \int d^{d-1}x e^{-i\mathbf{p}\cdot\mathbf{x}} \langle 0 | \mathcal{O}(t, \mathbf{x}) \Phi(0)^\dagger | 0 \rangle, \quad (3.9)$$

where $p^0 = iE_p = i\sqrt{\mathbf{p}^2 + m_\phi^2}$, and

$$\begin{aligned} C_{\mathcal{O}\Phi\Phi}(T, t, \mathbf{p}, \mathbf{p}', \Lambda; \delta g, \hat{m}_q, L^{-1}, \mu) &= \\ &= \frac{1}{V^2} \int d^{d-1}x d^{d-1}y e^{-i(\mathbf{p}\cdot\mathbf{x} + \mathbf{p}'\cdot\mathbf{y})} \langle 0 | \mathcal{O}(t, \mathbf{x}) \Phi(T, \mathbf{y}) \Phi(0)^\dagger | 0 \rangle, \end{aligned} \quad (3.10)$$

with Φ an interpolating field for the particle ϕ . The theory is assumed to be defined in a finite volume of linear size L and at a scale μ , in the neighbourhood of a RG fixed point, located at $\delta g = \hat{m}_q = 0$. The couplings δg and \hat{m}_q are both dimensionless. If necessary, dimensionful couplings are rescaled by the appropriate powers of the scale μ . The spatial volume is $V = L^{d-1}$.

Using unitarity and usual RG scaling arguments — see e.g. our previous publications [45–47] for details — we obtain

$$C_{\mathcal{O}\Phi}(t, \mathbf{p}, \Lambda; \delta g, \hat{m}_q, L^{-1}, \mu) = \frac{e^{-E_p t}}{2E_p V} \langle 0 | \mathcal{O}(0) | \phi(\mathbf{p}) \rangle \langle \phi(\mathbf{p}) | \Phi(0)^\dagger | 0 \rangle + \dots$$

$$= b^{-\Delta_{\mathcal{O}} - \Delta_{\Phi}} C_{\mathcal{O}\Phi}(b^{-1}t, b\mathbf{p}, b\Lambda; b^{y_g} \delta g, b^{y_m} \hat{m}_q, bL^{-1}, \mu), \quad (3.11)$$

where we have used the identification $(2\pi)^{(d-1)}\delta^{(d-1)}(0) \leftrightarrow V$ and $\Delta_{\mathcal{O}}$ and Δ_{Φ} are the scaling dimensions of the operators \mathcal{O} and Φ . The quantities y_g and $y_m \equiv 1 + \gamma_*$ are the critical exponents that characterise the running of the couplings determined by the linearised RG equations in the vicinity of the fixed point. $\phi(\mathbf{p})$ is the lightest state in the spectrum with the same quantum numbers as $\Phi(x)$ and energy $E_p = \sqrt{\mathbf{p}^2 + m_\phi^2}$. The ellipses represent the contributions from excited states in the spectrum, which are exponentially suppressed.

The scaling formula for the 2-point function can be used as usual to derive the scaling of the masses of the hadronic states. Setting $\mathbf{p} = 0$, and $b^{y_m} \hat{m}_q = 1$ yields

$$C_{\mathcal{O}\Phi}(t, 0, \Lambda; \delta g, \hat{m}_q, L^{-1}, \mu) = \hat{m}_q^{\frac{\Delta_{\mathcal{O}} + \Delta_{\Phi}}{y_m}} C_{\mathcal{O}\Phi}(\hat{m}_q^{1/y_m} t, 0, \hat{m}_q^{-1/y_m} \Lambda; 0, 1, \mu)$$

$$+ \mathcal{O}(m_q^{-y_g/y_m} \delta g).$$
(3.12)

We may parameterise the large- t behaviour as

$$C_{\mathcal{O}\Phi}(t, 0, \Lambda; \delta g, \hat{m}_q, L^{-1}, \mu) \rightarrow \hat{m}_q^{\frac{\Delta_{\mathcal{O}} + \Delta_{\Phi}}{y_m}} e^{-\left(\hat{m}_q^{1/y_m} \mathcal{F}(\hat{m}_q^{-1/y_m} \Lambda)\right) \Lambda t} f(\hat{m}_q^{-1/y_m} \Lambda, \mu), \quad (3.13)$$

where both functions, \mathcal{F} and f , can and will overturn the hyperscaling behaviour found in (3.1) for masses and matrix elements. Specifically we may read off the behaviour of the ϕ -mass

$$m_\phi \propto \hat{m}_q^{1/y_m} \mathcal{F}(\hat{m}_q^{-1/y_m} \Lambda). \quad (3.14)$$

We are interested in the scaling of the masses as the fermion mass $m_q \rightarrow 0$, which corresponds to $\hat{m}_q^{-1/y_m} \Lambda \rightarrow \infty$. We can then distinguish two different regimes

$$\lim_{x \rightarrow \infty} \mathcal{F}(x) = \begin{cases} \kappa & \Rightarrow m_\phi = \mathcal{O}\left(m_q^{\frac{1}{1+\gamma_*}}\right) \\ \kappa x & \Rightarrow m_\phi = \mathcal{O}(\Lambda) \end{cases}, \quad (3.15)$$

where κ is a constant and the first case is an alternative derivation of the mass scaling quoted earlier. The first regime corresponds to the conformal scaling already discussed in our previous study [46]. Interestingly, the second regime yields the scaling with Λ that is expected in the theory with spontaneously broken symmetry and a dilaton. We further note

that, using arguments about the finiteness of matrix elements in the $m_q \rightarrow 0$ limit (as done in section 3.1), it may be possible to make further statements about the function $f(x, \mu)$ as $x \rightarrow \infty$. We refrain from doing so as it does not add anything to the key messages of this paper.

3.3 Dilaton Ward identity in the vicinity of the IR fixed point

A similar analysis for the 3-point function allows us to derive a crucial result for the WI in the neighbourhood of a fixed point. Once again we start from the RG equation,

$$\begin{aligned}
 C_{\mathcal{O}\Phi\Phi} \left(T, t, \mathbf{p}, \mathbf{p}', \Lambda; \delta g, \hat{m}_q, L^{-1}, \mu \right) &= \\
 &= \frac{e^{-E_p(T-t)} e^{-E_{p'}t}}{2E_p 2E_{p'} V^2} \langle 0 | \Phi(0) | \phi(\mathbf{p}) \rangle \langle \phi(\mathbf{p}) | \mathcal{O}(0) | \phi(\mathbf{p}') \rangle \langle \phi(\mathbf{p}') | \Phi(0)^\dagger | 0 \rangle + \dots \\
 &= b^{-\Delta_{\mathcal{O}} - 2\Delta_{\Phi}} C_{\mathcal{O}\Phi\Phi} \left(b^{-1}T, b^{-1}t, b\mathbf{p}, b\mathbf{p}', b\Lambda; b^{y_g} \delta g, b^{y_m} \hat{m}_q, bL^{-1}, \mu \right). \quad (3.16)
 \end{aligned}$$

Combining these expressions, we obtain the matrix elements from taking the large-time limits of correlators. In particular, we have

$$\begin{aligned}
 \langle 0 | \mathcal{O}(0) | \phi(\mathbf{p}) \rangle &= K_2 \lim_{b \rightarrow \infty} e^{E_p b t} C_{\mathcal{O}\Phi} \left(b t, \mathbf{p}, \Lambda; \delta g, \hat{m}_q, L^{-1}, \mu \right) \\
 &= K_2 \lim_{b \rightarrow \infty} e^{E_p b t} b^{-\Delta_{\mathcal{O}} - \Delta_{\Phi}} C_{\mathcal{O}\Phi} \left(t, b\mathbf{p}, b\Lambda; b^{y_g} \delta g, b^{y_m} \hat{m}_q, bL^{-1}, \mu \right), \quad (3.17)
 \end{aligned}$$

and similarly

$$\begin{aligned}
 \langle \phi(\mathbf{p}) | \mathcal{O}(0) | \phi(\mathbf{p}') \rangle &= \\
 &= K_3 \lim_{b \rightarrow \infty} e^{E_p b(T-t)} e^{E_{p'} b t} C_{\mathcal{O}\Phi\Phi} \left(bT, b t, \mathbf{p}, \mathbf{p}', \Lambda; \delta g, \hat{m}_q, L^{-1}, \mu \right) \quad (3.18) \\
 &= K_3 \lim_{b \rightarrow \infty} e^{E_p b(T-t)} e^{E_{p'} b t} b^{-\Delta_{\mathcal{O}} - 2\Delta_{\Phi}} C_{\mathcal{O}\Phi\Phi} \left(T, t, b\mathbf{p}, b\mathbf{p}', b\Lambda; b^{y_g} \delta g, b^{y_m} \hat{m}_q, bL^{-1}, \mu \right),
 \end{aligned}$$

where

$$K_2 = \left(\frac{\langle \phi(\mathbf{p}) | \Phi(0)^\dagger | 0 \rangle}{2E_p V} \right)^{-1}, \quad (3.19)$$

$$K_3 = \left(\frac{\langle 0 | \Phi(0) | \phi(\mathbf{p}) \rangle \langle \phi(\mathbf{p}') | \Phi(0)^\dagger | 0 \rangle}{2E_p 2E_{p'} V^2} \right)^{-1}. \quad (3.20)$$

Eqs. (3.17) and (3.18) are the master formulae needed in order to understand the IR behaviour of the dilatation WI and the scaling of finite-volume effects. From these formulae one infers that evaluating the correlation functions at infinite time separation is the same as evaluating them at finite time with other dimensionful parameters appropriately rescaled. Now, taking the infinite-volume limit first, we are able to show that the on-shell WIs are insensitive to the anomalous breaking in the presence of an IR fixed point, provided that the explicit breaking of scale invariance due to the mass is tuned to zero. In order to prove this statement, we are going to consider the anomalous contribution in eq. (1.2) due to the gauge field, for $\hat{m}_q = 0$, namely

$$\langle 0 | \frac{\beta}{2g} G^2 | D(q) \rangle. \quad (3.21)$$

This matrix element can be obtained from the large- t behaviour of the correlator $C_{T^\mu_\mu \Phi_D}(t)$ where $\mathcal{O} = T^\mu_\mu = \frac{\beta}{2g} G^2$ and Φ_D is a generic interpolating operator that has an overlap with the dilaton field but not the vacuum (e.g. $\Phi_D \rightarrow \Phi_D - \langle \Phi_D \rangle |0\rangle \langle 0|$ is a realisation thereof). Starting from the infinite-volume theory and setting $L^{-1} = 0$, we obtain from eq. (3.17)

$$\langle 0 | \frac{\beta}{2g} G^2 | D(q) \rangle \propto \lim_{b \rightarrow \infty} e^{E_q b t} b^{-\Delta_{\mathcal{O}} - \Delta_{\Phi}} C_{T^\mu_\mu \Phi_D}(t, b\mathbf{q}, b\Lambda; b^{y_g} \delta g, b^{y_m} \hat{n}_q, 0, \mu). \quad (3.22)$$

Note that we need to keep a finite, non-vanishing mass, or a non-vanishing spatial momentum, in order to guarantee the exponential fall-off of the correlator. We see from the expression above that in the neighbourhood of an IR fixed point, the coupling g is irrelevant (that is the critical exponent is negative $y_g < 0$). Assuming that the matrix element of G^2 does not diverge in the IR, the matrix element therefore vanishes

$$\langle 0 | \frac{\beta}{2g} G^2 | D(q) \rangle \propto \frac{\beta(g)}{g} \Big|_{b^{y_g} \delta g} \propto \frac{b^{y_g} \delta g}{g^*} \left[1 - \frac{\delta g}{g^*} \right] \rightarrow 0. \quad (3.23)$$

Above g^* is the value of the coupling at the IR fixed point. Eq. (3.23) shows that the anomalous breaking does not contribute to the WI between the vacuum and the dilaton state. The only assumption needed is that the gluonic matrix element remains finite when $\mu \rightarrow 0$. The order of the limits is relevant here: the mass of the matter fields guarantees that the dilaton is massive and its correlators decay exponentially, then in the large-time limit the contribution from the running of the gauge coupling vanishes. A similar argument applied to the 3-point functions shows that the matrix element of the anomalous breaking term between two one-particle states also vanishes in the presence of an IR fixed point. Note that these statements are true not only for the lowest state as one may choose an interpolating operator which has no overlap with the lowest state. This is particularly clear in the finite volume formulation where the fields can be represented in form of a discrete spectral sum. In summary we thus have that¹¹

$$\langle 0 | T^\mu_\mu | 0 \rangle \xrightarrow{\text{sym}} 0, \quad \langle 0 | T^\mu_\mu | \phi_1(q) \rangle \xrightarrow{\text{sym}} 0, \quad \langle \phi_2(p') | T^\mu_\mu | \phi_1(p) \rangle \xrightarrow{\text{sym}} 0, \quad (3.24)$$

in the $m_q \rightarrow 0$ limit where $\phi_{1,2}$ are any physical states and the equation also holds for the vacuum expectation value if Φ is chosen to have overlap with the vacuum. Colloquially speaking, the physical matrix elements “see” the TEMT at large distances and since there is an IR fixed point this means effectively that $T^\mu_\mu \rightarrow 0$ between physical states. This is an important result of our paper and in agreement with statements found in [7]. In order to delimit this result we stress that correlation functions with T^μ_μ -insertions are generically non-vanishing. For example in the context of the flow theorems they constitute the main observables [48–50]. It seems worthwhile to clarify that the TEMT does not need to vanish on quark and gluon external states since they are not (asymptotic) physical states even in the absence of confinement. This is the case since quarks and gluons can emit soft coloured

¹¹ Since the trace of the EMT is a RG invariant this implies $\langle 0 | G^2(\mu) | 0 \rangle = 0$, $\langle 0 | G^2(\mu) | \phi_1(p) \rangle = 0$ and $\langle \phi_2(p') | G^2(\mu) | \phi_1(p) \rangle = 0$ for any scale $\mu > 0$. By continuity it is then also implied for $\mu = 0$. This implies that the gluon condensate is not the operator that breaks the dilatation symmetry spontaneously.

gluons and thus colour is not a good asymptotic quantum number. Another aspect, that has the same root, is that quark and gluons correlation functions can have unphysical singularities on the first sheet.

3.4 Finite volume scaling

Finally, by keeping the size of the system L finite, the solutions of the RG equations presented above allow us to quantify the scaling of the correlators in finite (but sufficiently large) volumes. Choosing a reference scale L_0 and setting $b = L/L_0$, we obtain for the 2-point function (3.9)

$$\begin{aligned} & \frac{e^{-E_p t}}{2E_p V} \langle 0 | \mathcal{O}(0) | \phi(\mathbf{p}) \rangle \langle \phi(\mathbf{p}) | \Phi(0)^\dagger | 0 \rangle + \dots \\ &= \left(\frac{L}{L_0} \right)^{-\Delta_{\mathcal{O}} - \Delta_{\Phi}} \times \\ & \quad \times C_{\mathcal{O}\Phi} \left(\left(\frac{L}{L_0} \right)^{-1} t, \left(\frac{L}{L_0} \right) \mathbf{p}, \left(\frac{L}{L_0} \right) \Lambda; \left(\frac{L}{L_0} \right)^{y_g} \delta g, \left(\frac{L}{L_0} \right)^{y_m} \hat{m}_q, L_0, \mu \right). \end{aligned} \tag{3.25}$$

This equation allows us to derive the scaling of the energy and of the matrix elements with the size of the lattice; ignoring the contribution of the irrelevant coupling, we obtain

$$E_p L = f(L^{y_m} \hat{m}_q, L \Lambda), \tag{3.26}$$

$$\langle 0 | \mathcal{O}(0) | \phi(\mathbf{p}) \rangle \propto \left(\frac{L}{L_0} \right)^{-\Delta_{\mathcal{O}}}, \tag{3.27}$$

as already discussed in our previous studies [45]. It is interesting to emphasise that in a finite volume the anomalous contribution to the WI from the irrelevant coupling is proportional to $\left(\frac{L}{L_0} \right)^{y_g}$. Hence, the finite volume explicitly breaks the scale symmetry by acting as an IR regulator and this is reflected in the WI for $\mathcal{O} = T^\mu_\mu$. Once again, because $y_g < 0$, the breaking term vanishes when $L \rightarrow \infty$, which is consistent with the fact that SSB cannot occur in a finite volume, as otherwise tunnelling rates prohibit SSB [51].

4 Conformal dilaton signatures

In section 4.1 we discuss concretely how the conformal dilaton phase can be searched for on the lattice and in section 4.2 we comment on the ideas that the $f_0(500)$ in QCD and the Higgs could be dilatons from the perspective of this paper and the newly obtained scaling formula for its mass.

4.1 Lattice Monte Carlo simulations

In order to discriminate a conformal dilaton phase from the QCD or unbroken conformal phase we propose the following two strategies.

- To test the subtle cancellation in the trace anomaly between the form factors $G_1(0) = 1$ and $G_2(0) = 2/(d-1) \left(1 + \mathcal{O} \left(m_q^{2/(1+\gamma^*)} \right) \right)$ (or more generally (2.3) with

m_q -corrections), which allows the hadrons to carry mass and the trace anomaly to vanish (in the IR). As stated in the main text $G_1(0) = 1$ holds irrespective of the phase and it is (3.6) that provides the test. Neither in QCD nor in the unbroken conformal phase does G_2 exhibit a pole at $m_D = \mathcal{O}\left(m_q^{1/(1+\gamma^*)}\right)$ as there is no dilaton. The same applies for the spin-1/2 form factors, presented in appendix B, for which $g_1(0) = 1$ and $g_3(0) = 4m_f^2/(d-1)\left(1 + \mathcal{O}\left(m_q^{2/(1+\gamma^*)}\right)\right)$ (or more generally (B.2) with m_q -corrections) provide the test.

- Out of the peculiar properties of the dilaton phase summarised in table 1, $m_D = \mathcal{O}\left(m_q^{1/(1+\gamma^*)}\right)$ seems the most promising to test. The signal of the conformal dilaton phase is then that for $J^{PC} = 0^{++}$

$$m_D = \mathcal{O}\left(m_q^{\frac{1}{1+\gamma^*}}\right), \quad m_{\pi^a} = \mathcal{O}\left(m_q^{\frac{1}{1+\gamma^*}}\right), \quad m_{\text{other}} = \mathcal{O}(\Lambda). \quad (4.1)$$

This contrasts QCD where all masses, except the pions, are $\mathcal{O}(\Lambda)$ and the unbroken conformal window where all masses scale like $m_{\text{all}} = \mathcal{O}\left(m_q^{1/(1+\gamma^*)}\right)$.

We would think that the first test is more spectacular but it might be more costly as the form factor necessitates 3-point functions whereas masses (and decay constants) can be extracted from 2-point functions.

4.2 The Higgs and the $f_0(500)$ as pseudo-dilatons

In this section we briefly discuss whether the Higgs or the $f_0(500)$ are (pseudo)-dilatons in the electroweak and the QCD sector respectively. Our work is distinct from other approaches in the scaling formula for the dilaton (4.1) and we mainly focus on this aspect. We would like to stress that the $\langle 0|T^\mu_\mu|0\rangle = 0$ for $m_q = 0$, in the context of an IR fixed point, is of course of interest to the cosmological constant problem. Moreover, if masses are added for quarks and techniquarks, they would decouple in the deep IR. The question of IR conformality is then shifted to the pure Yang-Mills sector. Whereas lattice studies indicate that pure Yang-Mills is confining, it is, to the best of our knowledge, an open question whether these theories show an IR fixed point not. If this was the case then the Higgs sector and QCD would give a vanishing contribution to the cosmological constant.

- The Higgs boson could in principle be a dilaton e.g. [7] as it couples to mass via the compensator mechanism (2.7). At leading order in the low energy effective theory it is equivalent to the coupling of the Higgs. The basic idea is similar to technicolor (cf. [52, 53] for reviews) in that a new gauge group is added with techniquarks q' which are in addition coupled to the weak force such that the techniquark condensate breaks electroweak symmetry spontaneously, this usually implies $F_\pi = v \approx 246$ GeV. Whereas technicolor would be classed as a Higgsless theory the same is not true in the dilaton case as it takes on the role of the Higgs. Unlike technicolor the generation of fermion mass terms is not aimed to be explained dynamically.

In our scenario the dilaton is a true GB in the $m_{q'} \rightarrow 0$ limit and acquires its mass by explicit symmetry breaking

$$m_D = \mathcal{O}(1) m_{q'}^{\frac{1}{1+\gamma_*}} \Lambda'^{\frac{\gamma_*}{1+\gamma_*}}, \tag{4.2}$$

where Λ' is the hadronic scale of the new gauge sector. Eq. (4.2) suggests that a mass gap between m_D and Λ' can be reached by making $m_{q'}$ small. Whether or not Λ' can be sufficiently large,¹² in order to avoid electroweak and LHC constraints, is another question and beyond the scope of this paper. The crawling technicolor scenario in [7], based on the dilaton, is different in that the techniquarks are assumed to be massless and the dilaton/Higgs acquires its mass by the hypothesis that the IR fixed point is not (quite) reached. According to [7] the dilaton mass is then governed and made small by the derivative of the beta function.

- In this work, cf. figure 1, we have distinguished the conformal dilaton phase from QCD but one might ask the question whether they are one and the same. Could it be that the so called $f_0(500)$ (cf. [54] for a generic review on this particle), with pole on the second sheet at $m_{f_0} = 449(20) - i 275(12)$ MeV [55], is a dilaton with mass $m_{f_0(500)} \propto m_q^{1/(1+\gamma_*)}$? The first thing to note is that if one assumes the Gell-Mann Oakes Renner relation, $F_\pi^2 m_\pi^2 = -2m_q \langle \bar{q}q \rangle$ (e.g. [38]) and $\langle \bar{q}q \rangle = \mathcal{O}(\Lambda^3)$ then the mass scaling relation (3.2) implies $\gamma_* = 1$. This is a logical possibility that is deserving of further studies. In QCD, where $m_s \gg m_{u,d}$, it is not immediate how to apply the mass scaling relation (3.2). The $f_0(500)$ surely has a strange quark component and its mass scale can be considered to be of $\mathcal{O}(m_K)$. This is the case in terms of the actual masses and in scale chiral perturbation theory [56–58]. For more details about this EFT approach we refer the reader to a series of works by Crewther and Tunstall [56–58]. Our approach is though different in that we consider the gluonic part proportional to the β -function as subleading. Setting this aside, there are interesting consequences for $K \rightarrow \pi\pi$ and the famous $\Delta I = \frac{1}{2}$ rule. Such a scenario is also welcomed in dense nuclear interactions, combined with hidden local symmetry [59, 60].

5 Discussions and conclusions

In this paper we have analysed the possibility of a conformal dilaton phase, in addition to the QCD and conformal phase (cf. table 2 for comparison), where hadrons carry mass but the theory is IR conformal. The mechanism whereby this can happen is that conformal symmetry is spontaneously broken and it is the corresponding Goldstone boson, the dilaton, that restores the dilatation Ward identity (2.3). More generally, we have shown, using renormalisation group arguments in Euclidean space that the trace of the EMT vanishes on all physical states (3.24). This implies the vanishing of the gluon condensates and suggests that the scale breaking is driven by the quark condensate (cf. footnote 11). This is an important result of our paper with consequences. For example, it imposes an exact

¹²By large, we mean larger than the naive estimate $\Lambda' \approx 4\pi F_\pi = 4\pi v \approx 3$ TeV.

phase	GB	$\langle \phi T^\mu_\mu \phi' \rangle$	O_{had}	(m_D^2, F_D)	(m_π^2, F_π)
conformal window	$(-, -)$	$= 0$	$m_q^{\frac{\eta}{1+\gamma_*}}$	$\left(m_q^{\frac{2}{1+\gamma_*}}, m_q^{\frac{1}{1+\gamma_*}}\right)$	$\left(m_q^{\frac{2}{1+\gamma_*}}, m_q^{\frac{1}{1+\gamma_*}}\right)$
conformal dilaton	(D, π^a)	$= 0$	$\mathcal{O}(\Lambda)$	$\left(m_q^{\frac{2}{1+\gamma_*}}, \mathcal{O}(\Lambda)\right)$	$\left(m_q^{\frac{2}{1+\gamma_*}}, \mathcal{O}(\Lambda)\right)$
QCD	$(-, \pi^a)$	$\neq 0$	$\mathcal{O}(\Lambda)$	$(\mathcal{O}(\Lambda^2), \mathcal{O}(\Lambda))$	$(m_q, \mathcal{O}(\Lambda))$

Table 2. Comparison table between the three different phases discussed in this paper. GB stands for Goldstone boson. The states in the second column are physical states and zero quark mass is assumed. The columns three to five indicate the scaling when a quark mass is turned on. Above O_{had} is a generic hadronic observable and the conformal window scaling law has been discussed in (3.1). and the masses and the decay constant for the Goldstone boson singled out in the least two columns. The notation is the as in table 1 where further information on excited states in the conformal dilaton phase can be found.

constraint on the gravitational form factors (2.3). At zero momentum transfer we have shown that this constraint is satisfied, (2.9), using the effective Lagrangian (2.7). As far as we are aware this is a new result.

Such phases can be searched for in lattice Monte Carlo simulations for which we have proposed concrete signatures in section 4.1. First, the test of the exact constraint on the gravitational form factor at zero momentum transfer in (2.9) and the scaling of the dilaton mass (4.1), as compared to all other ones. It will be interesting to see whether this new perspective can resolve some of the debates in the lattice conformal window literature.

Moreover we have speculated in section 4.2 whether a pseudo-dilaton is present in QCD and or the electroweak sector in terms of the $f_0(500)$ and the Higgs boson. If the former were true then this would suggest that the conformal dilaton- and the QCD-phase are one and the same.¹³ In our view the study of whether the Higgs is a composite dilaton is deserving of further attention also because it has the potential to ameliorate the cosmological constant problem as emphasised earlier.

Finally we wish to comment on the different (pseudo) Goldstone bosons, the pions, the η' and the dilaton associated with breaking of the $SU_A(N_f)$, the $U_A(1)$ and the dilatation symmetry. In all three cases the quark masses are a form of explicit symmetry breaking. This is manifested by the corresponding WIs (1.2) and $\partial \cdot J_5 = 2m_q P + \frac{g^2}{16\pi^2} G\tilde{G}$ for the $U_A(1)$ -case. The axial non-singlet case stands out in that there is no anomalous piece and it is indeed the case, as well-known, that in the $m_q \rightarrow 0$ limit the pions become true Goldstone bosons. In the axial singlet case the anomalous piece does contribute to the large η' mass which constitutes the resolution to the $U_A(1)$ -problem [62]. It is noted that the η' becomes a Goldstone boson in the $N_c \rightarrow \infty$ limit as the anomalous term is $1/N_c$ suppressed. On the other hand, according to our analysis, the anomalous breaking of the scale symmetry does not affect the dilaton mass in the $m_q \rightarrow 0$ limit and it thus is, remarkably, a genuine Goldstone boson.

¹³IR conformality is important for renormalisation group flow theorems e.g. in the topological Euler term [48, 61], and the $\square R$ term [50].

Acknowledgments

LDD and RZ are supported by an STFC Consolidated Grant, ST/P0000630/1. We are grateful to Tyler Corbett, Oliver Witzel and especially to Lewis Tunstall for very useful discussions on the topic.

A Conventions

We write the gauge theory Lagrangian as

$$\mathcal{L} = -\frac{1}{4}G^2 + N_f m_q \bar{q}(i\not{D} - m_q)q, \quad (\text{A.1})$$

where $G^2 = G_{\mu\nu}^A G^{A\mu\nu}$ is the field strength tensor squared and A is the adjoint index of the gauge group, N_f is the number of flavours, which we assume to have degenerate mass m_q . The beta function and anomalous dimension are defined by $\beta = \frac{d}{d\ln\mu}g$ and $\gamma = -\frac{d}{d\ln\mu}\ln m_q$. Our Minkowski metric is $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

B Gravitational form factors of spin-1/2

In this appendix we present the spin-1/2 case, completing the spin-0 form factor discussion in the main text. The analogous definition of the scalar case (2.1), denoting the spin-1/2 fermion by f , is given by

$$\begin{aligned} T_{\mu\nu}^{(f)}(p, p') &= \langle f(p') | T_{\mu\nu}(0) | f(p) \rangle \\ &= \bar{u}(p') \left(\frac{1}{2} \gamma_{\{\mu} \mathcal{P}_{\nu\}} g_1(q^2) + \frac{i \mathcal{P}_{\{\mu} \sigma_{\nu\}} q}{4m_f} g_2(q^2) + \left(\frac{q_\mu q_\nu}{q^2} - \eta_{\mu\nu} \right) \frac{m_f}{4} g_3(q^2) \right) u(p), \end{aligned} \quad (\text{B.1})$$

where $\sigma_{\mu q} = \sigma_{\mu\nu} q^\nu$. Now, $g_1(0) = 1$ in order to get the mass relation correct and $g_2(0) = 0$ is equivalent to the vanishing of the nucleon's gravitomagnetic moment [36] ($(g_1, g_2, g_3) = (A, B, D \frac{q^2}{m_f^2})$ in their notation). The third form factor, sometimes referred to as the D -term but denoted by g_3 in order to avoid confusion with the dilaton, is unknown although related to the pressure and the shear.

The above remarks are general and we now turn to the conformal dilaton scenario. It is the g_3 term that plays the analogue role with respect to the dilaton pole. For $T_\mu^{(f)\mu}(p, p) = 0$, $m_f^2 \neq 0$ to be true the following constraint on the form factors has to hold

$$m_f g_1(q^2) - \frac{q^2}{8m_f} g_2(q^2) - \frac{(d-1)}{4} m_f g_3(q^2) = 0. \quad (\text{B.2})$$

Since $g_1(0) = 1$ and $g_2(0) = 0$ we must have that

$$g_3(0) = \frac{4}{d-1}, \quad (\text{B.3})$$

holds in analogy with (2.4) or more precisely (2.3). It then remains to check that this is indeed the case from the hadronic viewpoint. It is readily verified that $g_{ffD} = \frac{m_f}{F_D}$ using the same steps as in (2.7) with

$$\langle Df | f \rangle = i(2\pi)^d \delta \left(\sum p_i \right) g_{ffD} \bar{u}(p') u(p). \quad (\text{B.4})$$

In analogy to (2.8) we have

$$\begin{aligned} \langle Df|f \rangle &= \lim_{q^2 \rightarrow 0} i \frac{q^2}{Z'_D} \int d^d x e^{iq \cdot x} T_{33}^{(f)}(p, p', x) \\ &= \lim_{q^2 \rightarrow 0} i \frac{q^2}{Z'_D} \bar{u}(p') \frac{m_f^2}{q^2} g_3(q^2) u(p) (2\pi)^d \delta\left(\sum p_i\right), \end{aligned} \quad (\text{B.5})$$

where here $Z'_D = 4F_D m_f / (d - 1)$ is the appropriate LSZ factor. Equating (B.4) and (B.5) one finally gets

$$g_3(0) = \frac{g_{ffD} Z'_D}{m_f^2} = \frac{4}{d - 1}, \quad (\text{B.6})$$

as required by (B.3).

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited. SCOAP³ supports the goals of the International Year of Basic Sciences for Sustainable Development.

References

- [1] T. Banks and A. Zaks, *On the Phase Structure of Vector-Like Gauge Theories with Massless Fermions*, *Nucl. Phys. B* **196** (1982) 189 [[INSPIRE](#)].
- [2] N. Seiberg, *Exact results on the space of vacua of four-dimensional SUSY gauge theories*, *Phys. Rev. D* **49** (1994) 6857 [[hep-th/9402044](#)] [[INSPIRE](#)].
- [3] J. Terning, *Modern supersymmetry: Dynamics and duality*, Oxford University Press, Oxford, U.K. (2006) [[DOI](#)].
- [4] D. Nogradi and A. Patella, *Strong dynamics, composite Higgs and the conformal window*, *Int. J. Mod. Phys. A* **31** (2016) 1643003 [[arXiv:1607.07638](#)] [[INSPIRE](#)].
- [5] Y. Nakayama, *Scale invariance vs conformal invariance*, *Phys. Rept.* **569** (2015) 1 [[arXiv:1302.0884](#)] [[INSPIRE](#)].
- [6] K. Yamawaki, M. Bando and K.-i. Matumoto, *Scale Invariant Technicolor Model and a Technidilaton*, *Phys. Rev. Lett.* **56** (1986) 1335 [[INSPIRE](#)].
- [7] O. Catà, R.J. Crewther and L.C. Tunstall, *Crawling technicolor*, *Phys. Rev. D* **100** (2019) 095007 [[arXiv:1803.08513](#)] [[INSPIRE](#)].
- [8] T. Appelquist, J. Ingoldby and M. Piai, *Nearly Conformal Composite Higgs Model*, *Phys. Rev. Lett.* **126** (2021) 191804 [[arXiv:2012.09698](#)] [[INSPIRE](#)].
- [9] T. Appelquist, J. Ingoldby and M. Piai, *Dilaton EFT Framework For Lattice Data*, *JHEP* **07** (2017) 035 [[arXiv:1702.04410](#)] [[INSPIRE](#)].
- [10] T. Appelquist, J. Ingoldby and M. Piai, *Analysis of a Dilaton EFT for Lattice Data*, *JHEP* **03** (2018) 039 [[arXiv:1711.00067](#)] [[INSPIRE](#)].
- [11] M. Golterman and Y. Shamir, *Low-energy effective action for pions and a dilatonic meson*, *Phys. Rev. D* **94** (2016) 054502 [[arXiv:1603.04575](#)] [[INSPIRE](#)].

- [12] S. Matsuzaki and K. Yamawaki, *Dilaton Chiral Perturbation Theory: Determining the Mass and Decay Constant of the Technidilaton on the Lattice*, *Phys. Rev. Lett.* **113** (2014) 082002 [[arXiv:1311.3784](#)] [[INSPIRE](#)].
- [13] C. Csáki, N. Kaloper, J. Serra and J. Terning, *Inflation from Broken Scale Invariance*, *Phys. Rev. Lett.* **113** (2014) 161302 [[arXiv:1406.5192](#)] [[INSPIRE](#)].
- [14] P. Minkowski, *On the Anomalous Divergence of the Dilatation Current in Gauge Theories*, PRINT-76-0813 (1976).
- [15] J.C. Collins, A. Duncan and S.D. Joglekar, *Trace and Dilatation Anomalies in Gauge Theories*, *Phys. Rev. D* **16** (1977) 438 [[INSPIRE](#)].
- [16] N.K. Nielsen, *The Energy Momentum Tensor in a Nonabelian Quark Gluon Theory*, *Nucl. Phys. B* **120** (1977) 212 [[INSPIRE](#)].
- [17] J. Goldstone, A. Salam and S. Weinberg, *Broken Symmetries*, *Phys. Rev.* **127** (1962) 965 [[INSPIRE](#)].
- [18] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Five-Loop Running of the QCD coupling constant*, *Phys. Rev. Lett.* **118** (2017) 082002 [[arXiv:1606.08659](#)] [[INSPIRE](#)].
- [19] T.A. Ryttov and R. Shrock, *Higher-Loop Corrections to the Infrared Evolution of a Gauge Theory with Fermions*, *Phys. Rev. D* **83** (2011) 056011 [[arXiv:1011.4542](#)] [[INSPIRE](#)].
- [20] T.A. Ryttov and R. Shrock, *Infrared Zero of β and Value of γ_m for an SU(3) Gauge Theory at the Five-Loop Level*, *Phys. Rev. D* **94** (2016) 105015 [[arXiv:1607.06866](#)] [[INSPIRE](#)].
- [21] T.A. Ryttov and R. Shrock, *Scheme-Independent Series Expansions at an Infrared Zero of the β -function in Asymptotically Free Gauge Theories*, *Phys. Rev. D* **94** (2016) 125005 [[arXiv:1610.00387](#)] [[INSPIRE](#)].
- [22] A. Hasenfratz, C. Rebbi and O. Witzel, *Gradient flow step-scaling function for SU(3) with ten fundamental flavors*, *Phys. Rev. D* **101** (2020) 114508 [[arXiv:2004.00754](#)] [[INSPIRE](#)].
- [23] Z. Fodor, K. Holland, J. Kuti, D. Negradi and C.H. Wong, *Case studies of near-conformal β -functions*, *PoS LATTICE2019* (2019) 121 [[arXiv:1912.07653](#)] [[INSPIRE](#)].
- [24] Z. Fodor, K. Holland, J. Kuti, D. Negradi and C.H. Wong, *Extended investigation of the twelve-flavor β -function*, *Phys. Lett. B* **779** (2018) 230 [[arXiv:1710.09262](#)] [[INSPIRE](#)].
- [25] Z. Fodor, K. Holland, J. Kuti, D. Negradi and C.H. Wong, *Is SU(3) gauge theory with 13 massless flavors conformal?*, *PoS LATTICE2018* (2018) 198 [[arXiv:1811.05024](#)] [[INSPIRE](#)].
- [26] T.-W. Chiu, *Improved study of the β -function of SU(3) gauge theory with $N_f = 10$ massless domain-wall fermions*, *Phys. Rev. D* **99** (2019) 014507 [[arXiv:1811.01729](#)] [[INSPIRE](#)].
- [27] LATTICE STRONG DYNAMICS collaboration, *Nonperturbative investigations of SU(3) gauge theory with eight dynamical flavors*, *Phys. Rev. D* **99** (2019) 014509 [[arXiv:1807.08411](#)] [[INSPIRE](#)].
- [28] L. Del Debbio, B. Lucini, A. Patella, C. Pica and A. Rago, *Large volumes and spectroscopy of walking theories*, *Phys. Rev. D* **93** (2016) 054505 [[arXiv:1512.08242](#)] [[INSPIRE](#)].
- [29] R.C. Brower, A. Hasenfratz, C. Rebbi, E. Weinberg and O. Witzel, *Composite Higgs model at a conformal fixed point*, *Phys. Rev. D* **93** (2016) 075028 [[arXiv:1512.02576](#)] [[INSPIRE](#)].
- [30] LSD collaboration, *Lattice simulations with eight flavors of domain wall fermions in SU(3) gauge theory*, *Phys. Rev. D* **90** (2014) 114502 [[arXiv:1405.4752](#)] [[INSPIRE](#)].

- [31] LATKMI collaboration, *Light composite scalar in eight-flavor QCD on the lattice*, *Phys. Rev. D* **89** (2014) 111502 [[arXiv:1403.5000](#)] [[INSPIRE](#)].
- [32] LATKMI collaboration, *Light flavor-singlet scalars and walking signals in $N_f = 8$ QCD on the lattice*, *Phys. Rev. D* **96** (2017) 014508 [[arXiv:1610.07011](#)] [[INSPIRE](#)].
- [33] O. Witzel, *Review on Composite Higgs Models*, *PoS LATTICE2018* (2019) 006 [[arXiv:1901.08216](#)] [[INSPIRE](#)].
- [34] B.R. Holstein, *Metric modifications for a massive spin 1 particle*, *Phys. Rev. D* **74** (2006) 084030 [[gr-qc/0607051](#)] [[INSPIRE](#)].
- [35] E. Leader and C. Lorcé, *The angular momentum controversy: What's it all about and does it matter?*, *Phys. Rept.* **541** (2014) 163 [[arXiv:1309.4235](#)] [[INSPIRE](#)].
- [36] J. Hudson, I.A. Perevalova, M.V. Polyakov and P. Schweitzer, *Structure of the Energy-Momentum Tensor and Applications*, *PoS QCDEV2016* (2017) 007 [[arXiv:1612.06721](#)] [[INSPIRE](#)].
- [37] C.G. Callan, Jr., S.R. Coleman and R. Jackiw, *A New improved energy-momentum tensor*, *Annals Phys.* **59** (1970) 42 [[INSPIRE](#)].
- [38] J.F. Donoghue, E. Golowich and B.R. Holstein, *Dynamics of the standard model*, *Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.* **2** (1992) 1 [[INSPIRE](#)].
- [39] S. Weinberg and E. Witten, *Limits on Massless Particles*, *Phys. Lett. B* **96** (1980) 59 [[INSPIRE](#)].
- [40] L. Del Debbio and R. Zwicky, *Renormalisation group, trace anomaly and Feynman-Hellmann theorem*, *Phys. Lett. B* **734** (2014) 107 [[arXiv:1306.4274](#)] [[INSPIRE](#)].
- [41] M. Gell-Mann, *Symmetry violation in hadron physics*, [DOI](#) (1969) [[INSPIRE](#)].
- [42] S. Weinberg, *The Quantum theory of fields. Vol. 1: Foundations*, Cambridge University Press, Cambridge, U.K. (1995) [[DOI](#)] [[INSPIRE](#)].
- [43] A. Duncan, *The Conceptual Framework of Quantum Field Theory*, Oxford University Press, Oxford, U.K. (2012) [[DOI](#)].
- [44] S. Coleman, *Aspects of Symmetry: Selected Erice Lectures*, Cambridge University Press, Cambridge, U.K. (1985) [[DOI](#)] [[INSPIRE](#)].
- [45] L. Del Debbio and R. Zwicky, *Hyperscaling relations in mass-deformed conformal gauge theories*, *Phys. Rev. D* **82** (2010) 014502 [[arXiv:1005.2371](#)] [[INSPIRE](#)].
- [46] L. Del Debbio and R. Zwicky, *Scaling relations for the entire spectrum in mass-deformed conformal gauge theories*, *Phys. Lett. B* **700** (2011) 217 [[arXiv:1009.2894](#)] [[INSPIRE](#)].
- [47] L. Del Debbio and R. Zwicky, *Conformal scaling and the size of m -hadrons*, *Phys. Rev. D* **89** (2014) 014503 [[arXiv:1306.4038](#)] [[INSPIRE](#)].
- [48] Z. Komargodski and A. Schwimmer, *On Renormalization Group Flows in Four Dimensions*, *JHEP* **12** (2011) 099 [[arXiv:1107.3987](#)] [[INSPIRE](#)].
- [49] G.M. Shore, *The c and a -theorems and the Local Renormalisation Group*, SpringerBriefs in Physics, Springer, Cham, Germany (2017) [[DOI](#)] [[arXiv:1601.06662](#)] [[INSPIRE](#)].
- [50] V. Prochazka and R. Zwicky, *On the Flow of $\square R$ Weyl-Anomaly*, *Phys. Rev. D* **96** (2017) 045011 [[arXiv:1703.01239](#)] [[INSPIRE](#)].

- [51] S. Weinberg, *The quantum theory of fields. Vol. 2: Modern applications*, Cambridge University Press, Cambridge, U.K. (1996) [[DOI](#)] [[INSPIRE](#)].
- [52] C.T. Hill and E.H. Simmons, *Strong Dynamics and Electroweak Symmetry Breaking*, *Phys. Rept.* **381** (2003) 235 [*Erratum ibid.* **390** (2004) 553] [[hep-ph/0203079](#)] [[INSPIRE](#)].
- [53] F. Sannino, *Conformal Dynamics for TeV Physics and Cosmology*, *Acta Phys. Polon. B* **40** (2009) 3533 [[arXiv:0911.0931](#)] [[INSPIRE](#)].
- [54] J.R. Pelaez, *From controversy to precision on the sigma meson: a review on the status of the non-ordinary $f_0(500)$ resonance*, *Phys. Rept.* **658** (2016) 1 [[arXiv:1510.00653](#)] [[INSPIRE](#)].
- [55] PARTICLE DATA GROUP collaboration, *Review of Particle Physics*, *PTEP* **2020** (2020) 083C01 [[INSPIRE](#)].
- [56] R.J. Crewther and L.C. Tunstall, *Origin of $\Delta I = 1/2$ Rule for Kaon Decays: QCD Infrared Fixed Point*, [arXiv:1203.1321](#) [[INSPIRE](#)].
- [57] R.J. Crewther and L.C. Tunstall, *$\Delta I = 1/2$ rule for kaon decays derived from QCD infrared fixed point*, *Phys. Rev. D* **91** (2015) 034016 [[arXiv:1312.3319](#)] [[INSPIRE](#)].
- [58] R.J. Crewther and L.C. Tunstall, *Status of Chiral-Scale Perturbation Theory*, *PoS* **CD15** (2015) 132 [[arXiv:1510.01322](#)] [[INSPIRE](#)].
- [59] M. Rho and Y.-L. Ma, *Manifestation of Hidden Symmetries in Baryonic Matter: From Finite Nuclei to Neutron Stars*, *Mod. Phys. Lett. A* **36** (2021) 2130012 [[arXiv:2101.07121](#)] [[INSPIRE](#)].
- [60] G.E. Brown and M. Rho, *Scaling effective Lagrangians in a dense medium*, *Phys. Rev. Lett.* **66** (1991) 2720 [[INSPIRE](#)].
- [61] M.A. Luty, J. Polchinski and R. Rattazzi, *The a-theorem and the Asymptotics of 4D Quantum Field Theory*, *JHEP* **01** (2013) 152 [[arXiv:1204.5221](#)] [[INSPIRE](#)].
- [62] G. 't Hooft, *Symmetry Breaking Through Bell-Jackiw Anomalies*, *Phys. Rev. Lett.* **37** (1976) 8 [[INSPIRE](#)].