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A Logical Theory of Localization

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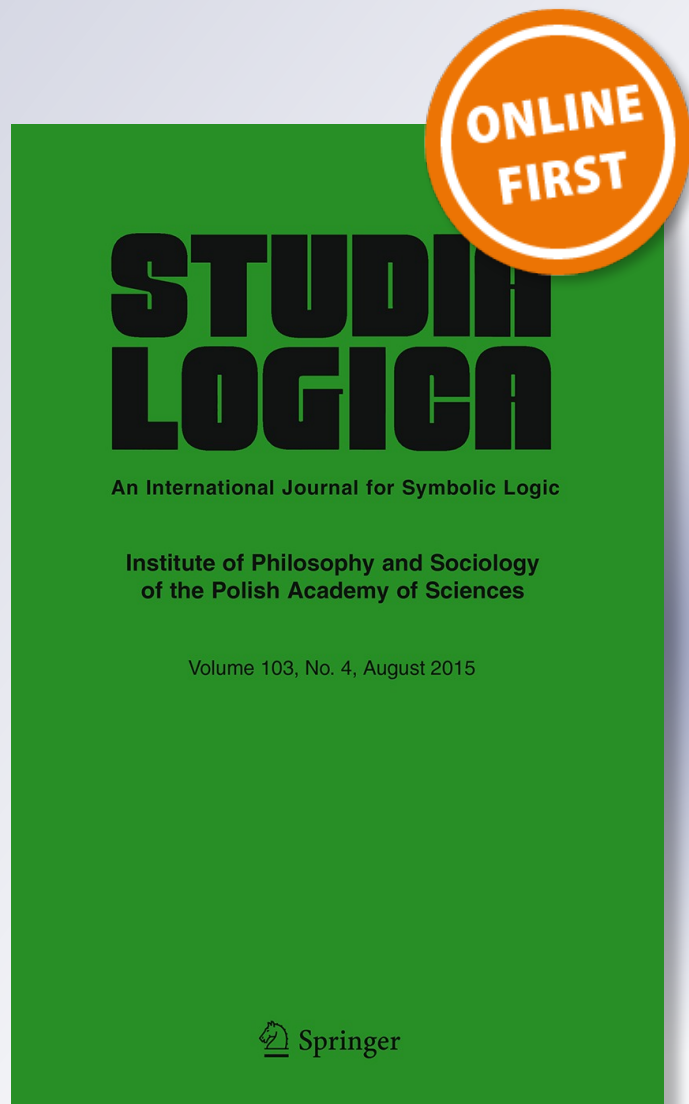
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A Logical Theory of Localization

Abstract. A central problem in applying logical knowledge representation formalisms to traditional robotics is that the treatment of belief change is categorical in the former, while probabilistic in the latter. A typical example is the fundamental capability of localization where a robot uses its noisy sensors to situate itself in a dynamic world. Domain designers are then left with the rather unfortunate task of abstracting probabilistic sensors in terms of categorical ones, or more drastically, completely abandoning the inner workings of sensors to black-box probabilistic tools and then interpreting their outputs in an abstract way. Building on a first-principles approach by Bacchus, Halpern and Levesque, and a recent continuous extension to it by Belle and Levesque, we provide an axiomatization that shows how localization can be realized wrt a basic action theory, thereby demonstrating how such capabilities can be enabled in a single logical framework. We then show how the framework can also enable localization for multiple agents, where an agent can appeal to the sensing already performed by another agent and the knowledge of their relative positions to localize itself.

Keywords: Knowledge representation; Reasoning about action; Reasoning about knowledge and belief; Multi-agent logics.

1. Introduction

Cognitive robotics, as envisioned in [33, 35], is a *high-level control* paradigm that attempts to apply knowledge representation (KR) technologies to the reasoning problems faced by an autonomous agent, such as a robot, in an incompletely known dynamic world. It is a challenging endeavor: in the least, reasonable features of action and change, such as the frame and ramification problems, need addressing, but if the robot has limited information then acting, sensing, knowledge and belief change also need to be taken into account. To this end, in the case of a popular action formalism such as the situation calculus [46], one usually provides a set of logical sentences called a *basic action theory* which explicates in a precise way the properties of the world and their relation to the agent's sensors and effectors. The

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benefit of appealing to such logical languages, of course, is that they admit non-trivial action specifications and arbitrary kinds of strict uncertainty, via disjunctions and quantification. When that is further supported using complex actions and procedures in the sense of a programming language [36, 54], one obtains a powerful and general methodology for designing intelligent agents, seen for example in [12, 24, 33].

Although a tight pairing of sensor data and high-level control is indeed what is desired, typical sensor data is best treated *probabilistically* [55] while many knowledge change accounts are *categorical* [17], that is, they do not represent and reason about changing degrees of belief. This has led to a major criticism that these logical formalisms are not realistic for applications involving actual physical robots. Indeed, a domain designer is now left with the rather unfortunate task of modeling probabilistic sensors in terms of non-probabilistic ones, an extreme being noise-free sensors [49], which would lead to an inaccurate model. More drastically, designers may completely abandon the inner workings of sensors to black-box probabilistic tools, in which case their outputs would need to be interpreted in some qualitative fashion and so is not straightforward. Regardless of application domains where such a move might be appropriate, for computational reasons or otherwise, both of these limitations are very serious since they challenge the underlying theory as a genuine characterization of the agent. Other major concerns include: (a) the loss of granularity, as it is not clear at the outset which aspect of the sensor data is being approximated and by how much, and (b) the domain designer is at the mercy of her intuition to imagine the various ways sensors might get used.

A first-principles proposal by Bacchus et al. [2], BHL henceforth, is perhaps the most general account to rectify this problem. Embedded in the usual machinery of a basic action theory, the BHL scheme enriches the situation calculus with an account of probabilistic nondeterminism. The enrichment allows us to talk about belief change in the formalism, which is compatible with earlier accounts on knowledge [49] while also subsuming Bayesian conditioning [44]. In contrast to many probabilistic formalisms (see the penultimate section for more on this), it allows for partial specifications, that is, distributions where only some of the fluents in the domain may be provided, as well as strict uncertainty. Recently, we [5] have further extended the BHL framework to reason about noise that is *continuous*. Building on these results, we now consider the most basic capability needed for an autonomous agent to situate itself: the *localization* problem. Roughly speaking, given a spatial characterization of the robot's environment, the

robot is to identify its pose (location and orientation) to a reasonable certainty using the sensors at its disposal.¹ We then show how the framework can also enable localization for multiple agents, where an agent can appeal to the sensing already performed by another agent and the knowledge of their relative positions to localize itself.

Localization has been addressed using a number of algorithmic techniques for more than two decades in the robotics literature [14, 55], including in the multi-agent context [21]. Our objective will not be to compete with these techniques; in fact, this paper will not concern itself with algorithms at all. Rather, we want to show how localization can be understood as part of a larger effort in a single logical framework [46]. To the best of our knowledge, this has not been attempted before. Nevertheless, we remark that owing to the first-order nature of the formalism, our account of localization, among other capabilities, is significantly more general than most, if not all, probabilistic formalisms.

The agenda for this paper will be as follows. We first introduce the preliminaries for reasoning about degrees of belief in the logical language of the situation calculus. We then iteratively develop the steps needed to localize a robot in an uncertain world, and finally consider a multi-agent setting. As one would expect, given the domain axiomatization, we show that localization is realized entirely within the logic in terms of belief change. Perhaps most significantly, we demonstrate how the framework subsumes probabilistic formalisms by using the full range of situation calculus successor state axioms and sensing axioms. We then discuss related efforts and conclude.

2. The Situation Calculus

The language \mathcal{L} of the situation calculus [40] is a many-sorted dialect of predicate calculus, with sorts for *actions*, *situations* and *objects*. It is a special-purpose knowledge representation for reasoning about action and change. We assume familiarity with the language and discuss it informally below; for a comprehensive treatment, interested readers are referred to [46].

A situation represents a world history as a sequence of actions. A set of initial situations correspond to the ways the world might be initially. Successor situations are the result of doing actions, where the term $do(a, s)$ denotes the unique situation obtained on doing a in situation s . The term $do(\alpha, s)$, where α is the sequence $[a_1, \dots, a_n]$ abbreviates $do(a_n, do(\dots, do(a_1, s) \dots))$.

¹In the absence of an intricate spatial setup, a simpler account of a robot situating itself is possible, as in [7], for example.

For example, $do([drop(o), repair(o)], s)$ denotes the situation obtained on dropping o in s and then repairing it. Initial situations are defined as those without a predecessor:

$$Init(s) \doteq \neg \exists a, s'. s = do(a, s').$$

We let the constant S_0 denote the actual initial situation (that is, the state of the world before any actions have occurred), and we use the variable ι to range over initial situations only (that is, the states of all worlds before any actions have occurred). (When describing the domain, the modeler would provide sentences about S_0 , thereby indicating what the domain is actually like initially.) In each model of \mathcal{L} , the situations can be structured into a set of trees, where the root of each tree is an initial situation and the edges are actions.

In dynamical domains, we want the values of predicate and functions to vary from situation to situation. For this purpose, \mathcal{L} includes *fluents* whose last argument is always a situation. Here we assume without loss of generality that all fluents are functional.

We follow some notational conventions. Free variables are assumed to be implicitly quantified from the outside. We often suppress the situation argument in a formula ϕ , or use a distinguished variable *now*. Either way, $\phi[t]$ is used to denote the formula with that variable replaced by t . We often introduce formula and term abbreviations that are meant to expand as \mathcal{L} -formulas. For example, we might introduce a new formula A (not in the language) by $A \doteq \phi$ where $\phi \in \mathcal{L}$. Then any expression $E(A)$ containing A is assumed to mean $E(\phi)$. Analogously, if we introduce a new term t (not in the language) by $t = u \doteq \phi(u)$ then any expression $E(t)$ is assumed to mean $\exists u(E(u) \wedge \phi(u))$. These conventions are used for convenience only, and do not require any extension to the version of the formalism introduced in [46].

2.1. Basic Action Theory

Following [46], we model dynamic domains in \mathcal{L} by means of a *basic action theory* \mathcal{D} , which are a set of sentences consisting of:²

1. Sentences \mathcal{D}_0 that describe what is true in the initial states, including S_0 ;
2. Precondition axioms of the form $Poss(a, s) \equiv \varphi$ describing the conditions under which actions are executable;

²As usual, free variables in any of these axioms should be understood as universally quantified from the outside.

3. Successor state axioms of the form $f(do(a, s)) = u \equiv \gamma_f(a, u, s)$ determining the fluent values on executing actions, following the format of Reiter's monotonic solution to the frame problem [46];
4. Domain-independent *foundational* axioms, including, for example, a second-order induction axiom enabling a tree of situations as described above. The details of this need not concern us here; see [46].

Given such a basic action theory \mathcal{D} , a fundamental reasoning task is that of *projection* [46] where given a property ϕ and a sequence of actions a_1, \dots, a_k , we are interested in whether:

$$\mathcal{D} \models \phi(do([a_1, \dots, a_k], S_0))?$$

Projection underlies a number of other intricate tasks, including plan synthesis where we are to verify whether some action sequence satisfies a given goal condition. Entailment is understood wrt standard first-order (Tarskian) models. We assume henceforth that models *also* assign the usual interpretations to $=, <, >, 0, 1, +, \times, /, -, e, \pi$ and x^y (exponentials).

Following [5], in the sequel, we will be assuming that f_1, \dots, f_k are all the fluents in \mathcal{L} , and that they only take a single situation term as an argument.³ Note that we still allow these fluents to range over any set, including the reals \mathbb{R} .

2.2. Belief, Likelihood and Continuous Noise

A first-principles approach to enrich the standard situation calculus to reason about noisy sensors and belief change was developed by BHL [2]. It builds on an earlier treatment of knowledge [42, 49] that mirrors modal logic [28, 31] in putting forth a possible-worlds interpretation, that is, situations are treated as possible worlds. (As a consequence, unlike classical modal logic, worlds are thus reified in the language.) In the main, it is based on two distinguished binary fluents l and p .

The term $l(a, s)$ is intended to denote the *likelihood* of action a in situation s . The axioms for l vary from domain to domain, but they have the general form of $l(A(\vec{x}), s) = u \equiv \phi_A(\vec{x}, u, s)$ which characterizes the conditions under which action type A has likelihood u in s . For example, suppose $sonar(z)$ is the action of reading the value z from a sensor that measures

³Basically, as observed in [5], if fluents are allowed to take arguments from infinite sets, then that can be seen to result in joint probability distributions over uncountably many random variables. We have as yet no good ideas about how to deal with it. (It remains to be seen whether probability theory on high dimensions [10, 15] can be adapted for the purposes of a model of belief.)

the distance to the wall, h . We might assume that this action is characterized by a Gaussian error model:

$$l(\text{sonar}(z), s) = u \equiv (z \geq 0 \wedge u = \mathcal{N}(z - h(s); \mu, \sigma^2)) \vee (z < 0 \wedge u = 0)$$

which stipulates that the difference between a nonnegative reading of z and the true value h is normally distributed with a variance of σ^2 and mean of μ .⁴

Next, the p fluent determines a probability distribution on situations. The term $p(s', s)$ denotes the relative *weight* accorded to situation s' when the agent happens to be in situation s . The properties of p in initial states, which vary from domain to domain, are specified by axioms as part of \mathcal{D}_0 , as one would for any other functional fluent (examples are discussed shortly). Now, to give p the required properties, so that it behaves like a probability distributions, two axioms are needed. First, a nonnegative constraint on p is assumed to be included in \mathcal{D}_0 :

$$\forall \iota, s. p(s, \iota) \geq 0 \wedge (p(s, \iota) > 0 \supset \text{Init}(s)).$$

(While this is indeed a stipulation about initial states ι only, by means of the next axiom, the nonnegative constraint continues to hold everywhere.) Second, a successor state axiom of the following form is provided for the p fluent:

$$\begin{aligned} p(s', do(a, s)) = u \equiv \\ \exists s'' [s' = do(a, s'') \wedge Poss(a, s'') \wedge u = p(s'', s) \times l(a, s'')] \\ \vee \neg \exists s'' [s' = do(a, s'') \wedge Poss(a, s'') \wedge u = 0]. \end{aligned}$$

This successor state axiom states that, given an appropriate action likelihood axiom, the weight of situations s' relative to $do(a, s)$ is the weight of their predecessors s'' times the likelihood of a contingent on the successful execution of a at s'' . One consequence of these two axioms is that $(p(s', s) > 0)$ will be true only when s' and s share the same history of actions.

Putting it all together, if ϕ is a formula with a single free variable of sort situation, then the *degree of belief* in ϕ is simply defined as the following abbreviation:

$$Bel(\phi, s) \doteq \frac{1}{\gamma} \sum_{\{s': \phi[s']\}} p(s', s) \tag{B}$$

⁴Note that \mathcal{N} is a continuous distribution involving π , e , exponentiation, and so on. Therefore, BHL always consider discrete probability distributions that *approximate* the continuous ones.

where γ is the normalization factor and is understood throughout as the same expression as the numerator but with ϕ replaced by *true*. Here, then, γ is $\sum_{s'} p(s', s)$. So, similar to proposals such as [20], belief in ϕ is simply the total weight of worlds satisfying ϕ . (We remark that the summation symbol is not an extension to the logical language \mathcal{L} but stands for a formula using second-order quantification; see BHL for details.)

While the BHL definition of belief is simple and intuitive, a major limitation of their work is the restriction to discrete probability distributions, since the *Bel*-expression is well-defined only when the sum over situations is well-defined. This is in contrast to the continuous noise usually encountered in robotics [55]. This limitation has been recently lifted in [5]. The main idea, which we only summarize here, is to first insist on a precise space of initial situations. So, in addition to the non-negative stipulation and the successor state axiom for p , a new axiom is included in \mathcal{D}_0 to impress exactly one initial situation for any vector of fluent values, following [37]. We list these three axioms in Table 1. With this addition, letting ϕ be any formula with a single free variable of sort situation, the *degree of belief* in ϕ is simply defined as a logical term by the following abbreviation:

$$Bel(\phi, s) \doteq \frac{1}{\gamma} \int_{\vec{x}} Density(\vec{x}, \phi, s)$$

where $Density(\vec{x}, \phi, s)$ is an abbreviation that, roughly speaking, returns the unnormalized density associated with ϕ at s :

$$\begin{aligned} Density(\vec{x}, \phi, do(\alpha, S_0)) &= u \doteq \\ \exists \iota [\bigwedge f_i(\iota) = x_i \wedge \phi[do(\alpha, \iota)] \wedge u = p(do(\alpha, \iota), do(\alpha, S_0))] \\ \vee \neg \exists \iota [\bigwedge f_i(\iota) = x_i \wedge \phi[do(\alpha, \iota)] \wedge u = 0]. \end{aligned}$$

The intuition is as follows. Using (iii), we obtain a bijection between initial situations and fluent values. By integrating over \vec{x} in the usual mathematical sense, we simply pick the appropriate initial situation, test whether ϕ holds after doing α and use the corresponding p value. We have assumed for simplicity that all fluents take values over \mathbb{R} , and so for discrete fluents, one would simply replace the integral with a summation (over its possible values) where appropriate. (We remark that, similar to the summation symbol in BHL's definition of belief, \int_x is not an addition to the logical language but only a term formalized using second-order logic that corresponds to mathematical integration; see [5] for details.)

The above, put together, is the full proposal. In sum, the following components were needed:

- abbreviations *Bel* and *Density* that expand as \mathcal{L} -expressions;

<p>i. $\forall \iota, s. p(s, \iota) \geq 0 \wedge (p(s, \iota) > 0 \supset \text{Init}(s)).$</p> <p>ii. $p(s', do(a, s)) = u \equiv$ $\exists s'' [s' = do(a, s'') \wedge \text{Poss}(a, s'') \wedge u = p(s'', s) \times l(a, s'')]$ $\vee \neg \exists s'' [s' = do(a, s'') \wedge \text{Poss}(a, s'') \wedge u = 0].$</p> <p>iii. $(\forall \vec{x} \exists \iota \wedge f_i(\iota) = x_i) \wedge (\forall \iota, \iota'. \wedge f_i(\iota) = f_i(\iota') \supset \iota = \iota').$</p>

Table 1. Axioms in \mathcal{D} for p

- an initial theory about S_0 , including (iii) to accommodate multiple initial situations and p 's initial constraint (i);
- action likelihood axioms using l ;
- successor state and precondition axioms, including (ii) for p .

In the sequel, we assume action theories to include (i), (ii) and (iii).

Finally, it is worth noting that the account of belief change using *Bel* subsumes Bayesian conditioning [2, 5], which is requisite for capturing mechanisms such as localization [55].

3. Axiomatizing Localization

One of the significant features about the BHL scheme and its continuous variant is that robot localization, among other capabilities, follows logically from a basic action theory. No new foundational axioms are necessary. In fact, localization is a certain degree of belief regarding position and orientation, and so by reasoning about belief change in terms of *projection* [46], the robot would get localized. On the one hand, this is perhaps expected as many state estimation techniques in robotics are based on Bayesian conditioning, but on the other, we are demonstrating this capability in a very rich first-order framework.

In this section, we develop a simple example, adapted from [55], and a basic action theory corresponding to this example. Localization will then be demonstrated in terms of logical entailments of the action theory. We think many of the features of our example are suggestive of how one would approach more complex domains. In the main, the example involves the following steps:

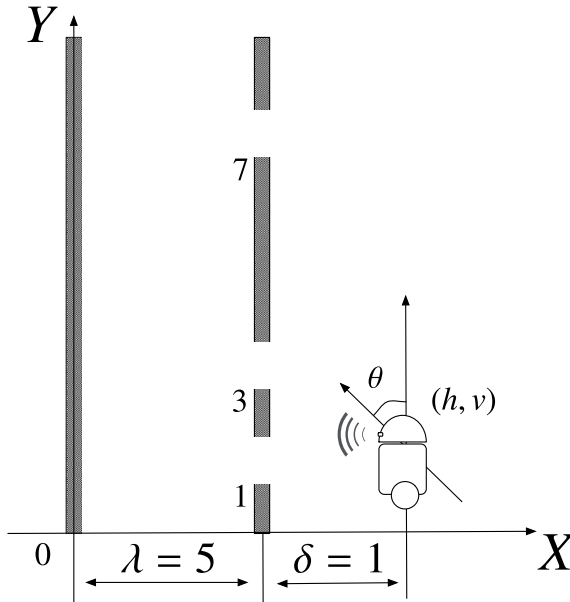


Figure 1. Two walls and a robot

- a characterization of the environment (walls, doors, *etc.*);
- a characterization of the uncertainty of the robot about this environment (its position and orientation); and
- a characterization of the robot's actions and sensors, and how they depend on and affect the environment.

Our example imagines a robot in a two-dimensional grid, equipped with a moving action and distance sensor, and facing two parallel walls as in Fig. 1. The basic action theory \mathcal{D} developed for the above items of this domain will be built using three fluents h (horizontal position), v (vertical position) and θ (orientation) that will determine the pose of the robot, a single rigid predicate *Solid* used to axiomatize the environment, two action types $move(z, w)$ and $rotate(z)$ that determine how the robot moves and how these affect the fluents using successor state axioms, a single sensing action $sonar(z)$, and convenient abbreviations that expand into formulas involving the aforementioned logical symbols. Of course, we assume \mathcal{D} to also mention $Poss, l$ and p , which are distinguished \mathcal{L} -symbols. We reiterate that we will not need any machinery beyond Reiter's version [46].

3.1. Environment

The very first item on the agenda is the notion of a *map*, which for our purpose will simply mean an axiomatic formulation of the physical space. In our example, suppose that the two parallel walls are 10 units long. The one on the *extreme* left of the robot, which we refer to as WALLFAR in the sequel, is without any doors, while the one that is adjacent to the robot, referred to as WALLCLOSE, has 3 open doors. The doors extend for one unit each. As can be seen in Fig. 1, we are imagining a coordinate system that has WALLFAR on the Y-axis, and puts the bottom edge of WALLFAR at the origin.

For our purposes, we develop a simple axiomatization to describe this physical space. (More general formalizations are possible, of course; see, for example, [23, 34].) We think of the walls in terms of continuous solid segments, that is, WALLFAR is considered to be a single chunk, while WALLCLOSE is thought of as 4 components. We will ignore the thickness of walls for simplicity. In precise terms, let $Solid(x, y, d)$ indicate that beginning at the coordinate (x, y) , one finds a solid structure of length d extending from (x, y) to $(x, y + d)$. Of course, we are using a rigid predicate because walls are stationary; for dynamic objects, such as the robot, fluents will

- | |
|--|
| <p>iv. $\{Solid(0, 0, 10), Solid(5, 0, 1), Solid(5, 2, 1), Solid(5, 4, 3), Solid(5, 8, 2)\}$.</p> <p>v. $Poss(a) \equiv true$.</p> <p>vi. $h(do(a, s)) = u \equiv$
 $\neg \exists z, w(a = move(z, w)) \wedge u = h(s) \vee$
 $\exists z, w(a = move(z, w) \wedge u = \max(\delta(s), h(s) - z \cdot \cos(w)))$.</p> <p>vii. $v(do(a, s)) = u \equiv$
 $\neg \exists z, w(a = move(z, w)) \wedge u = v(s) \vee$
 $\exists z, w(a = move(z, w) \wedge u = v(s) + z \cdot \sin(w))$.</p> <p>viii. $\theta(do(a, s)) = u \equiv$
 $\neg \exists z(a = rotate(z) \wedge u = \theta(s)) \vee$
 $\exists z(a = rotate(z) \wedge u = (((\theta(s) + z) \bmod 360) - 180))$.</p> <p>ix. $\{l(move(z, w), s) = 1, l(rotate(z), s) = 1\}$.</p> <p>x. $l(sonar(z), s) = u \equiv$
 $Blocked(s) \wedge u = \mathcal{N}(\delta / \cos(\theta) - z; 0, 1)[s] \vee$
 $\neg Blocked(s) \wedge u = \mathcal{N}((\delta + \lambda) / \cos(\theta) - z; 0, 1)[s]$.</p> |
|--|

Table 2. A basic action theory for the domain

be used. With this idea, we could characterize (say) WALLFAR by including $Solid(0, 0, 10)$ in \mathcal{D}_0 . For both walls, then, \mathcal{D}_0 is assumed to include the formulas (iv) from Table 2.

It should be clear that one may easily extract various directional and spatial relationships between such objects as appropriate. For example, although entirely obvious here, to calculate the distance between the walls, one may define an abbreviation λ as follows:

$$\lambda = u \doteq \exists x, y, d, x', y', d'. Solid(x, y, d) \wedge Solid(x', y', d') \wedge x \neq x' \wedge u = |x - x'|.$$

3.2. Robot: Physical Actions

Here, we characterize the robot's position, and how that is affected using physical actions.

The pose of the robot is given by three fluents: h, v and θ , where h is the horizontal position, v is the vertical position and so (h, v) is the robot's location, and θ is the orientation. We let θ range from -180 to 180 (degrees), with $\theta = 0$ indicating that the robot is perpendicular to WALLFAR and directed towards it, and $\theta = 90$ indicating that the robot is perpendicular to the X -axis and directed towards the positive half of the Y -axis.

We imagine two physical action types at the robot's disposal, $move(z, w)$ and $rotate(z)$. We are thinking that the robot is capable of moving z units along the orientation w (degrees) wrt its angular frame. That is, for $w = 0$, the robot would move z units towards WALLFAR, and for $w = 90$, the robot would move z units along the positive Y -axis, *i.e.* parallel to WALLFAR. The robot can also orient itself in-place, using $rotate(z)$. For these actions, one also needs to specify their preconditions, and their likelihood axioms. In this work, we make two simplifying assumptions. First, we assume these and all other actions in domain (including the sensing action to be discussed shortly) are always executable, given by (v). Second, we assume the physical actions are noise-free. (The sensor will be noisy, however, as we shall see.) Thus, likelihood axioms, which are used to specify probabilistic nondeterminism, will be given by (ix) for these actions. Essentially, (ix) says that the likelihood is 1 for these actions. (Noisy physical actions in the BHL scheme are illustrated, for example, in [9].)

The values of fluents may change after actions. The sentence (ii) already specifies how p behaves in successor situations. We now do the same for h, v and θ . Since $move(z)$ and $rotate(z)$ are the only physical actions, the successor state axioms for h, v and θ will only mention these actions. They are given as (vi), (vii) and (viii) respectively. Let us consider them in order.

In the case of h , we would like $move(z, 0)$ to bring the robot z units towards the wall on its left, but that motion should stop if the robot hits the wall. For this, it is perhaps easiest to first infer the distance between the robot and the closest wall on its left. This can be done as follows. For an arbitrary coordinate (x^*, y^*) , we define an abbreviation for the nearest wall on its left:

$$\begin{aligned} NearestLeft(x^*, y^*) = d \doteq & \exists x, y, d. Solid(x, y, d) \wedge y^* \in [y, y + d] \wedge \\ & \neg \exists x', y', d'. Solid(x', y', d') \wedge y^* \in [y', y' + d'] \wedge \\ & (x^* - x') < (x^* - x) \wedge d = (x^* - x). \end{aligned}$$

We use $u \in [v, w]$ to mean $u \geq v \wedge u \leq w$, as usual. To now extract the distance between the *robot* and the nearest wall on its left, simply define an abbreviation δ as follows:

$$\delta(s) = u \doteq u = NearestLeft(h(s), v(s)).$$

This now allows us to dissect (vi). It says that $move(z, w)$ is the only action affecting h , thereby incorporating Reiter's monotonic solution to the frame problem, and it decrements h by $z \cos(w)$ units but stops if the robot hits the nearest wall on its left. Note that, then, the value of h will become δ . For example, if $\theta = 0$, then the new value of h is simply decremented by z , and if $\theta = 180$, which would mean the robot is facing away from WALLCLOSE then h would be incremented by z (since $\cos(180)$ is -1 .)

For the fluent v , the treatment is analogous, as shown in (vii). That is, the action $move(z, w)$ would increment v by $z \cdot \sin(\theta)$. For example, if $z = 90$, then the move action would simply increment v since the motion would be along the Y -axis in an incremental fashion. Naturally, if one were to give a negative argument, say -3 , to $move$, then the robot would move from (h, v) to $(h, v - 3)$.

Finally, θ is manipulated using $rotate(z)$ in an incremental manner while keeping its range in $[-180, 180]$ in (viii).

3.3. Robot: Sensors

The robot is assumed to have a sonar unit on its frontal surface, that is, along θ . We take this sensor to be noisy. What this means is that if the robot is facing WALLCLOSE, then a reading z from the sensor may *differ* from δ , but perhaps in some reasonable way. Most sensors have additive Gaussian noise [55], which is to say the likelihood of z is obtained from a normal curve whose mean is δ .

The complication here is that there are two walls and depending on the robot's pose, the sensor might be measuring either δ or $\lambda + \delta$. For example,

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if $h \in [0, 1]$ and $\theta = 0$, we understand that the sonar's signals would likely be centered around δ . However, if $v < 1$ but the robot's orientation is such that the sonar's signals advance through the gap at $[1, 2]$, then the robot's sonar unit would suggest values closer to $\delta + \lambda$ rather than δ alone. To provide a satisfactory l -axiom for the sensor, let us first introduce an abbreviation for what it means for a sensor's signals to stop at WALLCLOSE:

$$\begin{aligned} \text{Blocked}(s) \doteq \exists x, y, d. \text{Solid}(x, y, d) \wedge h(s) = x + \delta(s) \wedge \\ (v + \delta \cdot \tan(\theta))[s] \in [y, y + d]. \end{aligned}$$

To make sense of this in (converse) terms of when signals would reach WALLFAR, note that if $v < 1$ and yet $v + \tan(\theta) \in [1, 2]$, then the signal advances through the gap. Analogously, if $\theta < 0$ and $v > 2$ and yet $v + \tan(\theta) \in [1, 2]$, then the signal advances through as well. This then allows us to define an l axiom for the sonar in (x). Intuitively, when *Blocked* holds at situation s , we assume the sonar's reading to have additive Gaussian noise (with unit variance) centered around δ , but when the sonar's signals can reach WALLFAR, we assume its reading to have additive Gaussian noise (with unit variance) centered around $\delta + \lambda$. (The \mathcal{N} term is an abbreviation for the mathematical formula defining a Gaussian density.)

3.4. Initial Constraints

The final step is to decide on a p specification for the domain. Recall that the p fluent is used to formalize the (probabilistic) uncertainty that the robot has about the domain. This perhaps accounts for a major difference between the work here and almost all probabilistic formalisms. For us, in a sense, p is just another fluent function, allowing the domain modeler to provide incomplete and partial specifications. But since our objective in this paper will be to show, in the least, that robot localization behaves as it does in standard probabilistic formalisms, we discuss two examples with fully known joint distributions in the next section. There are other possibilities still, a discussion of which we defer to later.

4. Properties

Before looking at the two examples, let us briefly reflect on what is expected. A reasonable belief change mechanism would support the following:

- Suppose the agent believes v to be uniformly distributed on the interval $[0, 10]$. If the robot then uses its sonar and senses a value close to $\lambda + \delta$ say 5.9, it should come to believe that it is located at a door, which would

$$\text{xi. } p(\iota, S_0) = \begin{cases} .1 & \text{if } (h = 6 \wedge v \in [0, 10] \wedge \theta = 0)[\iota] \\ 0 & \text{otherwise} \end{cases}$$

Table 3. Certainty about θ

deflate its beliefs about every point not in $[1, 2] \cup [3, 4] \cup [7, 8]$ (*i.e.* open gaps in WALLCLOSE.).

- Suppose the robot moves 2 units away from the X axis and then uses its sonar obtaining a reading of 5.8. It should then believe, rather confidently, that it must be in $[3, 4]$ since that is the only trajectory that supports a door initially and a second door after 2 units.

We now confirm these intuitions below, which essentially amount to the robot situating itself.

4.1. Knowing the Orientation

The first case we study will be the simpler one among the examples. We imagine (xi) from Table 3 to be the p specification which says that the agent believes v to be uniformly distributed on the interval $[0, 10]$, $h = 6$ and $\theta = 0$.⁵ This is a complete specification, in the sense that a unique joint distribution is provided. Moreover, owing to the exact knowledge that the robot has about its orientation, it is very certain on when the sonar would reach WALLFAR and when it would stop at WALLCLOSE, *viz.* the situations where $v \in [1, 2]$ or $v \in [3, 4]$ or $v \in [7, 8]$ are the only epistemically possible ones where *Blocked* will not hold. Therefore, the agent initially believes v to be uniform, as shown in Fig. 2, but after sensing 5.9, v values in the gaps will be considered with high probability (and equally likely) while the remaining v values will be given low p values.

Here are some properties of the basic action theory stated more formally:

THEOREM 4.1. *Let \mathcal{D} be a basic action theory that includes the sentences in Tables 2 and 3. Then:*

⁵We use the usual “case” notation with curly braces:

$$z = \begin{cases} t_1 & \text{if } \psi \\ t_2 & \text{otherwise} \end{cases} \doteq (\psi \supset z = t_1) \wedge (\neg\psi \supset z = t_2).$$

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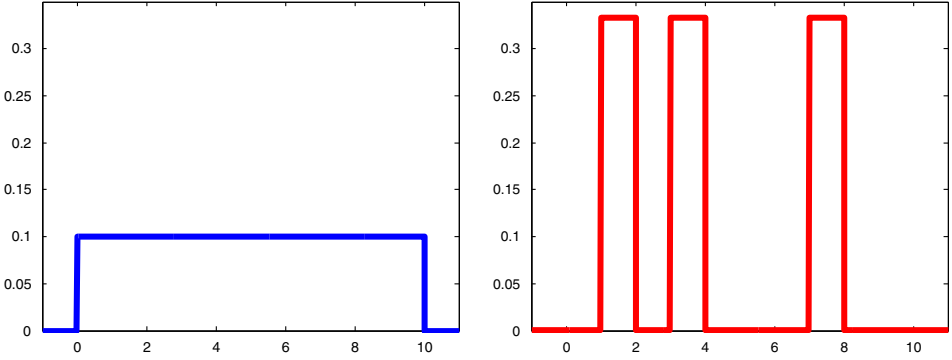


Figure 2. Beliefs initially and after sensing 5.9 when $\theta = 0$

1. $\mathcal{D} \models Bel(v \in [3, 4.57], S_0) = .157$

To see how this number is obtained, let us first expand Bel to obtain:

$$\frac{1}{\gamma} \int_x \int_y \int_z \begin{cases} p(\iota, S_0) & \exists \iota (h = x \wedge v = y \wedge \theta = z \wedge v \in [3, 4.57])[\iota] \\ 0 & \text{otherwise} \end{cases}$$

By means of the p -specification in \mathcal{D}_0 , this simplifies to:

$$\frac{1}{\gamma} \int_x \int_y \int_z \begin{cases} .1 & \exists \iota (h = x \wedge v = y \wedge \theta = z \wedge \dots)[\iota] \\ 0 & \text{otherwise} \end{cases}$$

where the ellipsis stands for:

$$(h = 6 \wedge v \in [0, 10] \wedge \theta = 0) \wedge (v \in [3, 4.57])$$

Intuitively, for the numerator of Bel , we are to integrate a function $q(x, y, z)$ (where x corresponds to the fluent h , y corresponds to the fluent v and z corresponds to the fluent θ) that is .1 when $y \in [3, 4.57]$ and 0 otherwise. This equals $.157/\gamma$. (The simplified mathematical expressions in this example and the ones below can be calculated using any software with numerical integration capabilities. We often write $t \approx u$ when the calculation for an expression t gives u after truncating the resulting value to two significant digits.) The normalization factor γ , analogously, is shown to be:

$$\int_x \int_y \int_z \begin{cases} .1 & \exists \iota (h = x \wedge v = y \wedge \theta = z \wedge h = 6 \wedge v \in [0, 10] \wedge \theta = 0)[\iota] \\ 0 & \text{otherwise} \end{cases}$$

Here, $v \in [3, 4.57]$ from the numerator is dropped (*i.e.* replaced by *true*). It is easy to see that $\gamma = 1$ since y is integrated for all values.

2. $Bel(v \in [3, 4], do(sonar(5.9), S_0)) \approx .33$

We do the expansion of Bel in detail for this one too. We have:

$$\frac{1}{\gamma} \int_x \int_y \int_z \begin{cases} p(do(sonar(5.9), \iota), do(sonar(5.9), S_0)) & \text{if } \exists \iota(\dots)[\iota] \\ 0 & \text{otherwise} \end{cases}$$

where the ellipsis stands for

$$h = x \wedge v = y \wedge \theta = z \wedge v(do(sonar(5.9), now)) \in [3, 4].$$

By means of the specification in \mathcal{D}_0 and the l -axiom for the distance sensor, we obtain:

$$\frac{1}{\gamma} \int_x \int_y \int_z \begin{cases} .1 \cdot \mathcal{N}(\delta + \lambda - 5.9; 0, 1) & \text{if } \exists \iota(\dots \wedge \psi)[\iota] \\ .1 \cdot \mathcal{N}(\delta - 5.9; 0, 1) & \text{if } \exists \iota(\dots \wedge \neg\psi)[\iota] \\ 0 & \text{otherwise} \end{cases}$$

where, the ellipsis stands for

$$\begin{aligned} h &= x \wedge v = y \wedge \theta = z \wedge \\ h &= 6 \wedge v \in [0, 10] \wedge \theta = 0 \wedge \\ v &(do(sonar(5.9), now)) \in [3, 4]; \end{aligned}$$

and ψ denotes

$$(v + \tan \theta \in [1, 2]) \vee (v + \tan \theta \in [3, 4]) \vee (v + \tan \theta \in [7, 8]).$$

The idea is simple. First, from (xi), those initial situations where $h = 6, \theta = 0$ and $v \in [0, 10]$ are the only ones with non-zero p values. By means of (iii), for various real values of y , which is the variable corresponding to v , we will be ranging over all the initial situations with non-zero p values. Next, since we are interested in the belief in $v \in [3, 4]$, as in the previous item, we give all other successor situations a density of 0 when calculating the numerator. Third, we note that when $Blocked$ holds (tested using ψ), from (x) and (ii), p values get multiplied by $\mathcal{N}(\delta - 5.9; 0, 1)$, and when not, p values get multiplied by $\mathcal{N}(\delta + \lambda - 5.9; 0, 1)$. Analogously, for γ , we derive a similar formula by replacing $v(do(sonar(5.9), now)) \in [3, 4]$ in the conditional expressions by $true$. As a final simplification in the numerator, because $\tan \theta = 0$, the second condition in the case statement (with $\neg\psi$) is not satisfiable, and so we get

$$\frac{1}{\gamma} \int_3^4 .1 \cdot \mathcal{N}(.1; 0, 1) \approx .33.$$

$\text{xi. } p(\iota, S_0) = \begin{cases} .1 \times \mathcal{N}(\theta(\iota); 0, 9) & \text{if } (h = 6 \wedge v \in [0, 10])[\iota] \\ 0 & \text{otherwise} \end{cases}$

Table 4. Uncertainty about θ

4.2. Uncertainty About the Orientation

We now consider a more interesting p specification determined by the orientation. The p we are thinking of is the one specified in Table 4. Here, $h = 6$, v is uniformly distributed as before, and independently θ is normally distributed around 0 with a variance of 9. This too is a complete specification, in the sense that there is a unique joint distribution corresponding to the p axiom.

Consider for the moment what would happen after sensing once. Unlike in Fig. 2, there is uncertainty regarding θ , which means that sensing (say) 5.9 will not imply full confidence in v being in $[1, 2] \cup [3, 4] \cup [7, 8]$. Indeed, as discussed earlier, even for v values less than 1, the orientation may cause the sonar to sense WALLFAR. Moreover, a larger range of θ values may cause the sonar to sense WALLFAR in the $[3, 4]$ interval rather than the $[1, 2]$ interval due to its lack of wall obstructions, causing a belief density change as shown in Fig. 3. After moving (say) 2 units and sensing values closer to $\lambda + \delta$ will lead to a more definite localization, as also shown in Fig. 3.

Here are some properties of this second basic action theory:

THEOREM 4.2. *Let \mathcal{D} be a basic action theory that includes the sentences in Tables 2 and 4. Then:*

1. $\mathcal{D} \models \text{Bel}(v \in [3, 4.57], S_0) = .157$

We are integrating under the same conditions initially as in the previous example, except that we now have:

$$\frac{1}{\gamma} \int_x \int_y \int_z \begin{cases} .1 \cdot \mathcal{N}(z; 0, 1) & \text{if } \exists \iota. (h = x \wedge v = y \wedge \theta = z \wedge \dots)[\iota] \\ 0 & \text{otherwise} \end{cases}$$

where the ellipsis stands for

$$h = 6 \wedge v \in [0, 10] \wedge v \in [3, 4.57]$$

and where the distribution for θ is also considered in the density expression for initial situations. Here, because we are integrating θ for all values, we obtain $\gamma = 1$ as usual, and so the above expression leads to .157.

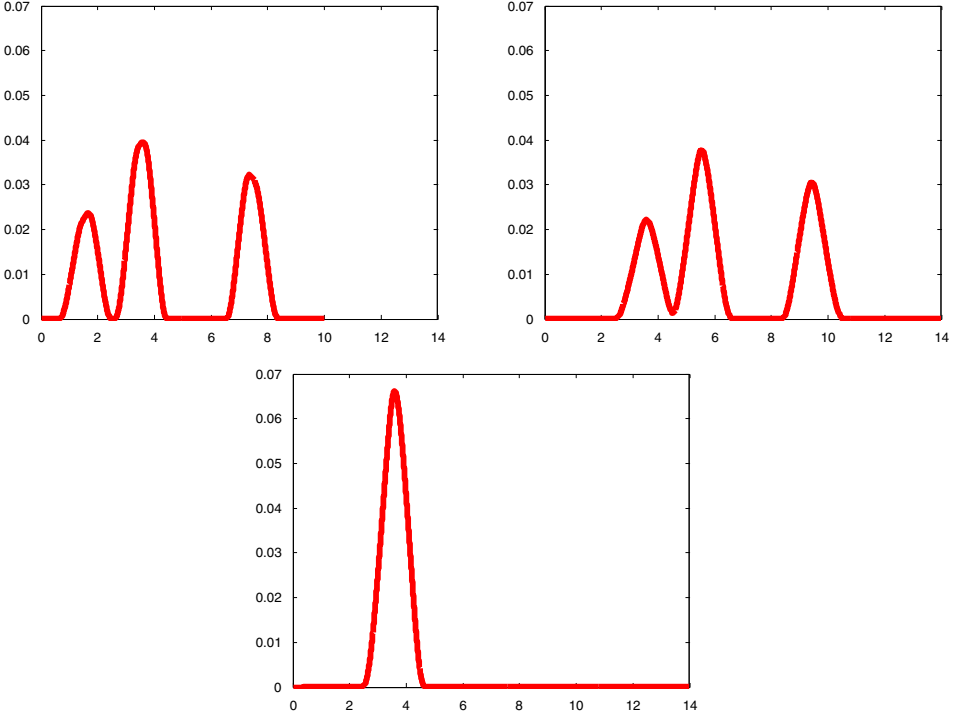


Figure 3. Belief change with normally distributed θ : after sensing 5.9, moving 2 units after that, and sensing 5.83 finally

- 2. $\mathcal{D} \models Bel(v \in [3, 4], do(sonar(5.9), S_0)) \approx .31$
 $\mathcal{D} \models Bel(v \in [2.8, 4.2], do(sonar(5.9), S_0)) \approx .33$

It is worth developing this in detail and contrasting it with what we had in the previous example. Picking the first entailment, *Bel* can be seen to expand as:

$$\frac{1}{\gamma} \int_x \int_y \int_z .1 \cdot \mathcal{N}(z; 0, 9) \cdot \begin{cases} \mathcal{N}(.1; 0, 1) & \text{if } \exists t(\dots \wedge \psi)[t] \\ \mathcal{N}(4.9; 0, 1) & \text{if } \exists t(\dots \wedge \neg\psi)[t] \\ 0 & \text{otherwise} \end{cases}$$

where (analogously) the ellipsis stands for:

$$\begin{aligned} h &= x \wedge \theta = z \wedge v = y \wedge \\ h &= 6 \wedge v \in [0, 10] \wedge \\ v &(do(sonar(5.9), now)) \in [3, 4] \end{aligned}$$

and ψ is (exactly as before):

$$(v + \tan \theta \in [1, 2]) \vee (v + \tan \theta \in [3, 4]) \vee (v + \tan \theta \in [7, 8]).$$

Note the simplification of the l -values for the sensing action as follows:

$$\mathcal{N}(\delta + \lambda - 5.9; 0, 1) = \mathcal{N}(.1; 0, 1), \mathcal{N}(\delta - 5.9; 0, 1) = \mathcal{N}(4.9; 0, 1).$$

What is interesting about *Bel*'s expansion here is that since $\theta \neq 0$, the sensor may read $\delta + \lambda$ even if the robot is not located in $[1, 2]$, $[3, 4]$ and $[7, 8]$. This accounts for belief in (say) exactly $[3, 4]$ being less than $1/3$, which is different from the previous example. Indeed, the degree of belief in a slightly larger interval, such as $[2.8, 4.2]$, approaches $1/3$.

3. $\mathcal{D} \models \text{Bel}(v \in [3, 4], \text{do}([\text{sonar}(5.9), \text{move}(2, 90)], S_0)) \approx .31$

Intuitively, the belief in $[1, 2]$ after sensing 5.9, which is also slightly less than $1/3$ owing to the open door at $[1, 2]$ and the uncertainty about θ , is transferred to $[3, 4]$ after moving laterally by 2 units. (That is, the new values of the points in the interval $[1, 2]$ correspond to the interval $[3, 4]$ and their densities do not change over a noise-free action.)

4. $\mathcal{D} \models \text{Bel}(v \in [3, 4], \text{do}([\text{sonar}(5.9), \text{move}(2, 90), \text{sonar}(5.83)], S_0)) \approx .96$

After sensing a value close to $\lambda + \delta$, moving and sensing $\lambda + \delta$ again, the robot is very confident about the $[3, 4]$ interval. We will not expand *Bel* completely but just point out that the density function is

$$.1 \cdot \mathcal{N}(z; 0, 9) \cdot \mathcal{N}(\delta + \lambda - 5.9; 0, 1) \cdot \mathcal{N}(\delta + \lambda - 5.83; 0, 1)$$

at initial situations where:

$$\begin{aligned} h = & 6 \wedge v \in [0, 10] \wedge \\ & (v + \tan \theta \in [1, 2] \vee v + \tan \theta \in [3, 4] \vee v + \tan \theta \in [7, 8]) \wedge \\ & (v + \tan \theta + 2 \in [1, 2] \vee v + \tan \theta + 2 \in [3, 4] \vee \\ & v + \tan \theta + 2 \in [7, 8]). \end{aligned}$$

Roughly speaking, these situations are those that support the observations of 5.9 and 5.83 in the best possible way. Note that, for the second sensing action, we need to test whether the incremented value of v after *move*(2, 90) is within a gap. It is not hard to see that when v is in the vicinity of $[3, 4]$, we would easily satisfy these constraints, which then has the intended effect.

4.3. Discussions

As seen in much of the work in cognitive robotics [33, 46], a logical language like the situation calculus allows for non-trivial action specifications, including, for example, context-dependent prerequisites and effects. In earlier

work [5], we have demonstrated how such actions can affect probability distributions in interesting ways, such as transforming continuous distributions to mixed ones, and how the language leads itself for reasoning about past and future events, among others.

Most significantly, in comparison to standard (non-logical) probabilistic formalisms, the advantages of our proposal are perhaps most evident in terms of what is allowed in the initial specification of the p fluent. The two examples used in the paper were comparable to unique joint probability distributions, which are standard. But that is not the case for one of the form:

$$\forall \iota(p(\iota, S_0) = \mathcal{U}(v; 0, 10)[\iota]) \vee \forall \iota(p(\iota, S_0) = \mathcal{U}(v; 3, 13)[\iota])$$

This says that the agent believes v to be uniformly distributed on $[0, 10]$ or on $[3, 13]$, without being able to say which. (That is, the \mathcal{U} term is an abbreviation for the mathematical formula defining a uniform density.) As one would expect (in logic), appropriate beliefs will still be entailed. For example:

- initially, it will follow that the robot is certain that $v \notin [30, 40]$, and will believe that $v \in [3, 10]$ with a probability of .7;
- if the robot has sensors to indicate that it is well within (say) the range of $[7, 8]$, after a few sensor readings, the disjunctive uncertainty about v will no longer be significant.

Much weaker specifications are possible still, where the modeler may leave the nature of the distribution of some fluents completely open, which would correspond more closely to incomplete information in the usual categorical sense, among others. As an example, the sentence above in a language with even one other fluent, say h , would mean that the agent has no information about the initial distribution of h . All of these are admitted in the framework.

5. Localization with Multiple Agents

In this section, we sketch how our logical account is extended to achieve localization in a many agent setting, where agents can appeal to the sensing actions they have individually performed to jointly situate themselves. In particular, the situation we consider is one where a robot performs a sensing action, and a second robot can inspect the outcome of this sensing action and its position relative to the first robot to localize itself. (A more drastic reworking is to update the beliefs of the second robot entirely using the first robot's beliefs, as in [21].)

5.1. Multiple Agents in \mathcal{L}

In terms of the logical language \mathcal{L} , a multi-agent version of the underlying model of belief is needed. In particular, we will need to reason about the beliefs of the individual agents, which may differ arbitrarily, as well as the sensing information perceived by the agent who performs the action.

Let us begin by extending \mathcal{L} to reason about multi-agent beliefs. Following the work of [51] on categorical knowledge, we let the binary fluent $p(s', s)$ be replaced by a ternary one $p(x, s', s)$ where x denotes the agent. Intuitively, $p(x, s', s)$ is the weight accorded to s' by x when it is at s . The successor state axiom for p , then, is retrofitted for the ternary version in an obvious way:

$$\begin{aligned} p(x, s', do(a, s)) = u \equiv & \\ & \exists s'' [s' = do(a, s'') \wedge Poss(a, s'') \wedge \\ & \quad u = p(x, s'', s) \times l(a, s'')] \\ \vee \neg \exists s'' [s' = do(a, s'') \wedge Poss(a, s'') \wedge u = 0]. \end{aligned}$$

This says that according to the agent x , the p -value of $do(a, s)$ is obtained from the p -value of s and the likelihood of the action a . Other p axioms in the action theory are modified analogously.

When considering sensing actions in a multi-agent context, we make the assumption that all actions are publicly observable, although the information obtained on the sensors is (naturally) private. (See [30], for example, for an account where the executability of actions is also private.) This can be captured in \mathcal{L} by means of an account of *action nondeterminism*. Following the work of [18], we introduce a new distinguished predicate $Alt(x, a, a')$ to mean that when a is executed, x imagines that a' has occurred. We assume the action theory to include axioms of the form:

$$Alt(x, a, a') \equiv \phi(a, a').$$

In the simplest case, a and a' are the same: $Alt(x, a, a') \equiv a = a'$. We will see examples of non-trivial Alt -axioms shortly on considering sensing actions.⁶

Putting it together, we let x 's degree of belief in ϕ be given by the following abbreviation:

$$Bel(x, \phi, s) \doteq \frac{1}{\gamma} \int_{\vec{y}} Density(x, \vec{y}, \phi, s).$$

⁶This account of alternate actions can also be used to reason about noisy effectors; see [9]. It can also be made context dependent [18], in the sense that the action a' would differ across situations. For simplicity, we omit this feature.

where $Density(x, \vec{y}, \phi, s)$ is an abbreviation that returns the *density* that x associates with ϕ at s while accommodating *Alt*-related actions:

$$Density(x, \vec{y}, \phi, do(a, S_0)) = u \doteq \\ \exists \iota, b [\wedge f_i(\iota) = y_i \wedge Alt(x, a, b) \wedge \phi[do(b, \iota)] \wedge u = p(x, do(b, \iota), do(a, S_0))] \\ \vee \neg \exists \iota, b [\wedge f_i(\iota) = y_i \wedge Alt(x, a, b) \wedge \phi[do(b, \iota)] \wedge u = 0].$$

The normalization factor γ is obtained, as usual, by letting ϕ in the numerator be replaced by *true*.

The *Density* term can be understood as follows. We allow for x believing that b occurred although a was executed. But precisely because x believes that b occurred, the property ϕ is tested at b -related successor situations, and the likelihood of b is applied to these situations. (The abbreviation is extended for action sequences in a straightforward way.)

5.2. Example Reconsidered

Let us imagine two robots r and r' , both along the vertical axis, as shown in Fig. 4. Of course, then, while the description of the environment does not change between the examples, we will need a natural extension to characterize the robots' uncertainty, effectors and sensors. So, in addition to fluents h, v and θ for the position of robot r , we introduce three companion fluents h', v' and θ' for the position of robot r' .

To provide successor state axioms, let us suppose actions include a parameter for the agent, that is, suppose $move(x, z, w)$, $rotate(x, z)$ and $sonar(x, z)$ are actions that affect the agent x 's horizontal and vertical positions, x 's orientation and measures x 's distance to the walls, respectively. Then, we retrofit the successor state axioms in an obvious way:

$$v(do(a, s)) = u \equiv \\ \neg \exists z, w (a = move(r, z, w)) \wedge u = v(s) \vee \\ \exists z, w (a = move(r, z, w) \wedge u = v(s) + z \cdot \sin(w)).$$

This says that $move(r, z, w)$ is the only action affecting v , and similarly:

$$v'(do(a, s)) = u \equiv \\ \neg \exists z, w (a = move(r', z, w)) \wedge u = v'(s) \vee \\ \exists z, w (a = move(r', z, w) \wedge u = v'(s) + z \cdot \sin(w))$$

says that $move(r', z, w)$ is the only action affecting v' . Analogous definitions are provided for h, h', θ and θ' .

To summarize, the following changes were needed:

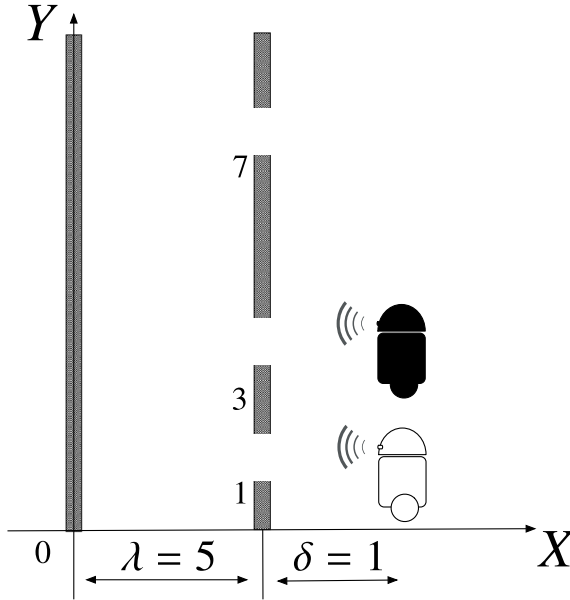


Figure 4. The walls and two robots: r (in white) and r' (in black)

- formulas (i) and (ii) from Table 1 are provided using a ternary version of p ;
- successor state axioms for h, v, θ, h', v' and θ' are defined wrt $move(x, z, w)$, $rotate(x, z)$ and $sonar(x, z)$.

However, formula (iii) from Table 1 remains unchanged as do the formulas on the preconditions and the rigid predicate $Solid(x, y, z)$ in Table 2. We now turn to the likelihood axioms, *Alt*-axioms and the p -specifications for the example.

5.3. Individual Localization

To begin with, let us consider the case where the two robots operate without any communication. Suppose that both r and r' are unsure of their position along the vertical axis, and believe their vertical positions to be uniformly distributed on the interval $[0,10]$. However, they do know that they are a unit away from WALLCLOSE, as well as their relative pose information in the following sense:

$$\forall \iota (v'(\iota) = v(\iota) + 2).$$

So, initially, r' is 2 units from r along the vertical axis. More precisely, the initial theory is as given in Table 5, which says that x has knowledge of h, h', θ and θ' , and believes v and v' are independently uniformly distributed on $[0,10]$. Moreover, because x is universally quantified from the outside, both robots have knowledge of this specification.

$\text{xiii. } p(x, \iota, S_0) = \begin{cases} .01 & \text{if } (h = 6 \wedge h' = 6 \wedge \theta = 0 \wedge \theta' = 0 \wedge \\ & v \in [0, 10] \wedge v' \in [0, 10] \wedge v' = v + 2)[\iota] \\ 0 & \text{otherwise} \end{cases}$

Table 5. Two robots uncertain about their vertical position

To capture that actions are publicly observable whereas the information obtained on the sensors is private, we provide the following *Alt*-axiom:

$$\text{Alt}(x, \text{sonar}(y, z), a') \equiv (x = y \wedge a' = \text{sonar}(y, z)) \vee (x \neq y \wedge a' = \text{null}).$$

This is to be read as follows. Suppose r executes the sonar action, and that works as usual. From the perspective of r' , however, *null* has occurred, which is a special action that has no effects. Basically, the effect of this *Alt*-axiom is that the beliefs of r' remain unchanged after $\text{sonar}(r, 5)$ whereas the beliefs of r are sharpened by means of the sonar's likelihood axioms.

The sonar's likelihood axioms are defined as before; for example:

$$\begin{aligned} l(\text{sonar}(r, z), s) = u &\equiv \\ \text{Blocked}(s) \wedge u &= \mathcal{N}(\delta / \cos(\theta) - z; 0, 1)[s] \vee \\ \neg \text{Blocked}(s) \wedge u &= \mathcal{N}((\delta + \lambda) / \cos(\theta) - z; 0, 1)[s]. \end{aligned}$$

The likelihood for $\text{sonar}(r', z)$ is defined in a symmetric manner using h', v' , and θ' . Of course, we also assume that for all actions outside of sensing actions, including *null*, $l(a, s) = 1$ and $\text{Alt}(x, a, a') \equiv a = a'$. Let \mathcal{D} be a basic action theory that is the union of these sentences.

THEOREM 5.1. *Suppose \mathcal{D} is a basic action theory using the ternary p fluent and the axioms as described above. Then, the following are entailments of \mathcal{D} :*

1. $\mathcal{D} \models (\text{Bel}(r, v \in [1, 2], S_0) = .1) \wedge (\text{Bel}(r', v' \in [3, 4], S_0) = .1)$
2. $\mathcal{D} \models \text{Bel}(r, v \in [1, 2], \text{do}(\text{sonar}(r, 6), S_0)) \approx .33$

We expand $\text{Bel}(r, \phi, s)$ as follows:

$$\frac{1}{\gamma} \int_x \dots \int_{z'} \begin{cases} p(r, \text{do}(b, \iota), \text{do}(\text{sonar}(r, 6), S_0)) & \text{if } \exists \iota, b. \psi[\iota] \\ 0 & \text{otherwise} \end{cases}$$

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where the integration variables range over $\{x, x', y, y', z, z'\}$ and ψ stands for:

$$\begin{aligned} h &= x \wedge h' = x' \wedge v = y \wedge v' = y' \wedge \theta = z \wedge \theta' = z' \wedge v' = v + 2 \wedge \\ h &= 6 \wedge h' = 6 \wedge \theta = 0 \wedge \theta' = 0 \wedge v \in [1, 10] \wedge v' \in [1, 10] \wedge \\ &Alt(r, sonar(r, 6), b) \wedge v(do(sonar(r, 6), now)) \in [1, 2] \end{aligned}$$

So, by means of \mathcal{D} and the RHS of l and Alt axioms, we obtain:

$$\frac{1}{\gamma} \int_x \cdots \int_{z'} \begin{cases} .01 \cdot \mathcal{N}(\delta + \lambda - 6; 0, 1) & \text{if } \exists \iota(\psi \wedge \xi)[\iota] \\ .01 \cdot \mathcal{N}(\delta - 6; 0, 1) & \text{if } \exists \iota(\psi \wedge \neg\xi)[\iota] \\ 0 & \text{otherwise} \end{cases}$$

where ξ denotes:

$$(v + \tan \theta \in [1, 2]) \vee (v + \tan \theta \in [3, 4]) \vee (v + \tan \theta \in [7, 8])$$

that corresponds to the sonar's signals reaching WALLFAR. The beliefs of r about $v \in [1, 2]$, then, becomes $\approx .33$.

3. $\mathcal{D} \models Bel(r', v' \in [3, 4], do(sonar(r, 6), S_0)) = .1$

The expansion proceeds analogously, except that $Alt(r', sonar(r, 6), b)$ is true for $b = null$. Recall that its likelihood is 1, and so, the beliefs of r' do not change. Thus, as in item 1, we get .1.

4. $Bel(r', v' \in [3, 4], do([sonar(r, 6), sonar(r', 6)], S_0)) \approx .33$

While the beliefs of r' in $v' \in [3, 4]$ are sharpened after $sonar(r', 6)$ in a manner symmetric to item 2, no such improvement is obtained after $sonar(r, 6)$ owing to the likelihood and Alt -axioms (as in item 3).

5.4. Mutual Localization

We now consider the case where a robot might have performed a sensing action, and a second robot can benefit from the observed outcome and its own position relative to the first for localization. Inspired by [3, 52], we consider the notion of an *announcement*. For simplicity, we lump the announcement and sensing features in a single action *announce* defined by the following likelihood axiom:

$$\begin{aligned} l(announce(x, z), s) &= u \equiv \\ &Blocked(s) \wedge u = \mathcal{N}(\delta / \cos(\theta) - z; 0, 1)[s] \vee \\ &\neg Blocked(s) \wedge u = \mathcal{N}((\delta + \lambda) / \cos(\theta) - z; 0, 1)[s]. \end{aligned}$$

together with the following Alt -axiom:

$$Alt(x, announce(y, z), a') \equiv a' = announce(y, z).$$

Basically, $announce(r, z)$ amounts to sensing and (implicitly) telling every robot how the p -values of situations should change based on what r observed and how that observation relates to the position of r . We can show:

THEOREM 5.2. *Let \mathcal{D} be as in Theorem 5.1 with the addition of the announcement action. Then:*

$$\mathcal{D} \models Bel(r', v' \in [3, 4], do([announce(r, 6), sonar(r', 5.5)], S_0)) \approx .99$$

On expanding Bel , we obtain:

$$\frac{1}{\gamma} \int_x \cdots \int_{z'} Q$$

where Q denotes

$$\begin{cases} .01 \cdot \mathcal{N}(\delta + \lambda - 6; 0, 1) \cdot \mathcal{N}(\delta + \lambda - 5.5; 0, 1) & \text{if } \exists \iota, b, b' (\psi \wedge \xi \wedge \xi')[\iota] \\ .01 \cdot \mathcal{N}(\delta + \lambda - 6; 0, 1) \cdot \mathcal{N}(\delta - 5.5; 0, 1) & \text{if } \exists \iota, b, b' (\psi \wedge \xi \wedge \neg \xi')[\iota] \\ .01 \cdot \mathcal{N}(\delta - 6; 0, 1) \cdot \mathcal{N}(\delta + \lambda - 5.5; 0, 1) & \text{if } \exists \iota, b, b' (\psi \wedge \neg \xi \wedge \xi')[\iota] \\ .01 \cdot \mathcal{N}(\delta - 6; 0, 1) \cdot \mathcal{N}(\delta - 5.5; 0, 1) & \text{if } \exists \iota, b, b' (\psi \wedge \neg \xi \wedge \neg \xi')[\iota] \\ 0 & \text{otherwise} \end{cases}$$

and ψ denotes

$$\begin{aligned} h &= x \wedge h' = x \wedge v = y \wedge v' = y' \wedge \theta = z \wedge \theta' = z \wedge v' = v + 2 \wedge \\ h &= 6 \wedge h' = 6 \wedge v \in [0, 10] \wedge v' \in [0, 10] \wedge \theta = 0 \wedge \theta' = 0 \wedge \\ &Alt(r', announce(r, 6), b) \wedge Alt(r', sonar(r', 5.5), b') \wedge \\ &v'(do([announce(r, 6), sonar(r', 5.5)], now)) \in [3, 4] \end{aligned}$$

and ξ denotes

$$(v + \tan \theta \in [1, 2]) \vee (v + \tan \theta \in [3, 4]) \vee (v + \tan \theta \in [7, 8])$$

and ξ' denotes

$$(v' + \tan \theta' \in [1, 2]) \vee (v' + \tan \theta' \in [3, 4]) \vee (v' + \tan \theta' \in [7, 8]).$$

On simplification, and on substituting $v' = v + 2$, one would observe that conditions other than $\psi \wedge \xi \wedge \xi'$ become false. Indeed, if $\xi \wedge \xi'$ holds, then $v \in [1, 2]$ and $v' \in [3, 4]$, and so $\psi \wedge \xi \wedge \xi'$ admits $v' \in [3, 4]$. In contrast, for example, $\neg \xi'$ does not admit $v' \in [3, 4]$, and so $\psi \wedge \xi \wedge \neg \xi'$ is false because ψ requires $v' \in [3, 4]$. Finally, the normalization factor is obtained by replacing

$$v'(do([announce(r, 6), sonar(r', 5.5)], now)) \in [3, 4]$$

in ψ by *true*. All together, we obtain a value of .99.

6. Related Work

There are three main strands of related work from the representational aspect. They are probabilistic formalisms, relational probabilistic languages and finally action languages. We discuss them in turn, but focus our attention on robot localization where possible.

There are numerous probabilistic formalisms, see [55] for a comprehensive overview, some of which are at the heart of most traditional robotic systems. Much of the results are algorithmic in nature, in the sense of investigating sampling-based techniques, approximating domains with Gaussian distributions, and so on. At the outset, we mentioned already that this paper is about a specification. So, wrt the underlying formal characterization, almost all of these are based on Bayesian conditioning [44]; see [19] for early work, and [21] for a multi-agent version. They also assume a full specification of a joint distribution, specified compactly in the form of (say) conjugate distributions such as Gaussians or dependency structures such as Bayesian networks. Thus, in terms of methodology, none of these are geared to handle strict uncertainty, logical connectives, and partial specifications. Similar limitations also apply to early work on diagnosis in hybrid systems [41]. Moreover, apart from a few cases such as [16] and [25] that are propositional, they do not reason about rich actions explicitly.

Logical formalisms for probabilistic reasoning, such as [1,27], are equipped to handle features such as disjunctions and quantifiers, but they do not explicitly address actions. Relational probabilistic languages and Markov logics [43,47] also do not model actions. Recent temporal extensions, such as [13], treat special cases such as Kalman filtering, but not complex actions. Similar limitations apply to certain fuzzy logic approaches for Bayesian filtering [29].

In this regard, action logics such as dynamic and process logics are closely related. These, and other proposals for action and change [53], in fact, are precisely the kind of logical languages we expect to be used for high-level control. But most of the work in the area, to the best of our knowledge, is limited in terms of one or more of the following: (a) they are propositional, (b) they have not been extended to handle noise that is continuous, and (c) they have not formalized and studied how localization can be realized. For example, in the area of dynamic logic, [56] treat probabilistic nondeterminism, but (a), (b) and (c) hold here. Frameworks such as [26], and probabilistic planning languages [32,48,57], are propositional. Finally, proposals based on the situation and fluent calculi are first-order [2,4,11,22,39,45,50,53],

but none of them deal with continuous noise. Also, (c) holds for these. We remark that, among these first-order formalisms, multi-agent accounts have been previously considered [51, 52], but collaborative localization has not been addressed.

7. Conclusions and Outlook

This paper addresses a fundamental limitation when applying logical knowledge representation formalisms to robotics. One is forced to abstract the sensing results in a categorical fashion, or much worse, abandon its inner workings. In that regard, this paper's essential contribution was to explain and suggest how the modeler may represent her domain in a basic action theory, and how that gets further used to localize mobile robots. We think this clarification and logical study is original, and not only is it fully compatible with existing probabilistic formalisms, but goes well beyond them in allowing non-trivial actions and partial specifications. These expressive capabilities are significant, because they are the very reason why (first-order) logical languages are chosen for modeling and reasoning in the first place. Giving them up would not be preferable for many domain modelers.

An account of localization is, however, only a first step for realizing general mechanisms that situate autonomous agents. For example, a more flexible account would involve a robot discovering the environment on its own [55], and how that can be realized in a basic action theory is an important open question. Of course, for such logical theories to be useful on actual physical robots, much work remains to be done on providing computing machinery for implementing reasoning systems. In very recent work [6, 8], we have investigated ways to address the problem of *projection* in our framework, where one is interested in the properties that hold after actions. Projection is fundamental in planning and high-level programming [46], and is a major concern in reasoning about action [38]. In ongoing work [9], we are also identifying fragments of our representation language that admit real-time behavior.

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