Classification of global solutions of a free boundary problem in the plane

CLASSIFICATION OF GLOBAL SOLUTIONS OF A FREE BOUNDARY PROBLEM IN THE PLANE

SERENA DIPIERRO, ARAM KARAKHANYAN, AND ENRICO VALDINOCI

Abstract. We classify nontrivial, nonnegative, positively homogeneous solutions of the equation

$$\Delta u = \gamma u^{\gamma - 1}$$

in the plane.

The problem is motivated by the analysis of the classical Alt-Phillips free boundary problem, but considered here with negative exponents γ.

The proof relies on several bespoke results for ordinary differential equations.

Contents

1. Introduction

Several problems of interest in the calculus of variations can be reduced to the study of critical points of an energy functional of the type

$$\int \frac{|\nabla u|^2}{2} + F(u)$$

where, up to a normalization, $F(r) \geqslant 0$ for all $r \in \mathbb{R}$ and $F(r) = 0$ for all $r \in (-\infty, 0]$.

An archetypal example of the potential F is given by power-like functions such as

$$F(r) := r^\gamma \, \chi_{(0,+\infty)}(r), \tag{1.1}$$

for a given $\gamma \in \mathbb{R}$. In this case, nonnegative critical points of the energy functional formally correspond to solutions of the equation

$$\Delta u = \gamma u^{\gamma - 1}. \tag{1.2}$$

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When $\gamma \geqslant 2$, we have that $F \in C^{1,1}(\mathbb{R})$ and the right hand side of (1.2) is Lipschitz continuous in u. In particular, in this case one can define $c := -\gamma u^{\gamma-2}$ and deduce that c is continuous if so is u: in this setting, the Strong Maximum Principle (see e.g. Theorem 1.7 in [HL11]) yields that nonnegative solutions of (1.2) are actually strictly positive inside the domain in which the equation takes place.

When $\gamma \in (0, 2)$, the situation changes significantly: for instance, it is readily checked that $u(x) := \left(\frac{(2-\gamma)^2\,(x_n)_+^2}{2}\right)^{\frac{1}{2-\gamma}}$ is in this case Lipschitz continuous and, for every $\phi \in C_0^\infty(B_1)$, the partial integration yields the identity

$$\int_{B_1} \nabla u(x)\cdot\nabla\phi(x) + \gamma u^{\gamma-1}(x)\phi(x)\,dx = \int_{B_1\cap\{x_n>0\}} \nabla u(x)\cdot\nabla\phi(x) + \gamma u^{\gamma-1}(x)\phi(x)\,dx$$
$$= \int_{B_1\cap\{x_n>0\}} \frac{(2-\gamma)^{\frac{\gamma}{2-\gamma}}}{2^{\frac{\gamma-1}{2-\gamma}}} x_n^{\frac{\gamma}{2-\gamma}}\,\partial_n\phi(x) + \gamma\left(\frac{(2-\gamma)^2\,x_n^2}{2}\right)^{\frac{\gamma-1}{2-\gamma}}\phi(x)\,dx$$
$$= \int_{B_1\cap\{x_n>0\}} \partial_n\left(\frac{(2-\gamma)^{\frac{\gamma}{2-\gamma}}}{2^{\frac{\gamma-1}{2-\gamma}}} x_n^{\frac{\gamma}{2-\gamma}}\,\phi(x)\right)dx = 0,$$

providing an example of weak solution of (1.2) with a vanishing point (actually, a vanishing region) in the interior of the domain.

For this reason, equation (1.2) when $\gamma \in (0, 2)$ has been widely investigated in the context of free boundary problems and it is indeed the main topic of a classical article by H. W. Alt and D. Phillips, see [AP86].

From the point of view of applications, equation (1.2) also models a reaction-diffusion problem of a gas distribution in a porous catalyst pellet (see e.g. [Rut75]). To understand the regularity of the minimizers of the associated energy functional and the way in which the free boundary separates the zero set of the solution from the positive region, one of the main tools relies on the blow-up analysis of the problem, as well as on the understanding of the corresponding homogeneous solutions, see e.g. Sections 1.15 and 1.16 in [AP86] (see also Theorem 5.1 in [Spr83] for the range $\gamma \in (1, 2)$).

The case $\gamma = 1$ in (1.1) corresponds to an obstacle problem and is covered by the classical work in [Caf77]. Similarly, the case $\gamma = 0$ in (1.1) produces the seminal case studied in [AC81]. The case $\gamma \in (0, 1)$ has also been considered in [ST19].

To the best[1] of our knowledge, however, the case of negative exponents γ has never been studied in the literature, and one can naturally wonder whether there is any structural reason for it.

The main goal of this paper is to answer this question by considering all possible ranges of γ, focusing on the two-dimensional case. A natural assumption for us, in view of the degree a of homogeneity of the solution, is to consider the case in which $u^{\frac{1}{a}}$ meets the zero set in a C^1 fashion. In this situation, as expected, one obtains positive and rotationally invariant solutions, as well as "one dimensional" one phase solutions whose positivity set is a halfplane. But, perhaps more surprisingly, when $a = 1/2$ one also detects a "resonance"

[1]Once we have completed this paper, the preprint [DSS22] has become available online, where the case $\gamma \in (-2, 0)$ has been taken into account. Our perspective here is however quite different from that in [DSS22], since we do not focus our attention on the regularity of the local minimizers of the energy functional but rather on classification results for global solutions, without energy constraints.

which produces new solutions whose positivity set is a nontrivial cone (and even the union of different cones whose opening is an acute angle).

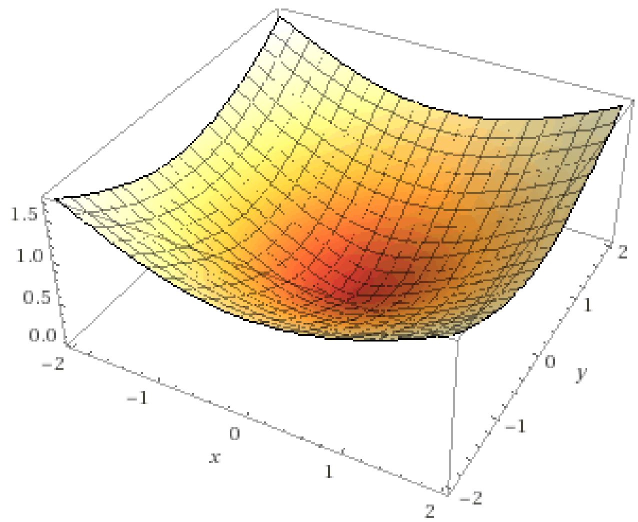

FIGURE 1. *The solution in (1.7) with $a := 4/3$.*

The precise result that we have is the following:

Theorem 1.1. *Let $a > 0$ and $\gamma \neq 0$. Assume that $u \in C(\mathbb{R}^2)$, that*

$$u^{\frac{1}{a}} \in C^1\left(\overline{(B_2 \setminus B_{1/2}) \cap \{u > 0\}}\right), \tag{1.3}$$

and that u is a nontrivial, nonnegative, positively homogeneous solution of degree a of the equation

$$\Delta u = \gamma u^{\gamma-1} \qquad in\ \ \mathbb{R}^2 \cap \{u > 0\}. \tag{1.4}$$

Then,

$$\gamma < 2 \qquad and \qquad a = \frac{2}{2-\gamma}. \tag{1.5}$$

Moreover, we have the following, non-exclusive, possible scenarios:

[i] *either*

$$\gamma \in (0, 2) \tag{1.6}$$

and

$$u(x) = C_a\,|x|^a, \qquad with \quad C_a := \frac{(2(a-1))^{a/2}}{a^{3a/2}}, \tag{1.7}$$

[ii] *or, up to a rotation and a reflection,*

$$u(x) = \frac{2^{\frac{a}{2}}}{a^a}\,(x_2)_+^a, \tag{1.8}$$

[iii] *or the following situation occurs:*

- $a = 1/2$,

- *given $c \in \mathbb{R} \setminus \{0\}$, up to a rotation and a reflection, the positivity set of u contains the cone*

$$\mathcal{C}_c := \Big\{ (r\cos\theta, r\sin\theta), \quad r > 0 \quad \text{and} \quad \theta \in (0, T_c) \Big\}, \tag{1.9}$$

with

$$T_c := \begin{cases} 2\pi - 2\arctan(1/c) & \text{when } c > 0, \\ -2\arctan(1/c) & \text{when } c < 0, \end{cases} \tag{1.10}$$

- *$u = 0$ on $\partial\mathcal{C}_c$,*
- *for every $x \in \mathcal{C}_c$,*

$$u(x) = 2^{\frac{3}{4}} \sqrt{x_2 - cx_1 + c|x|}. \tag{1.11}$$

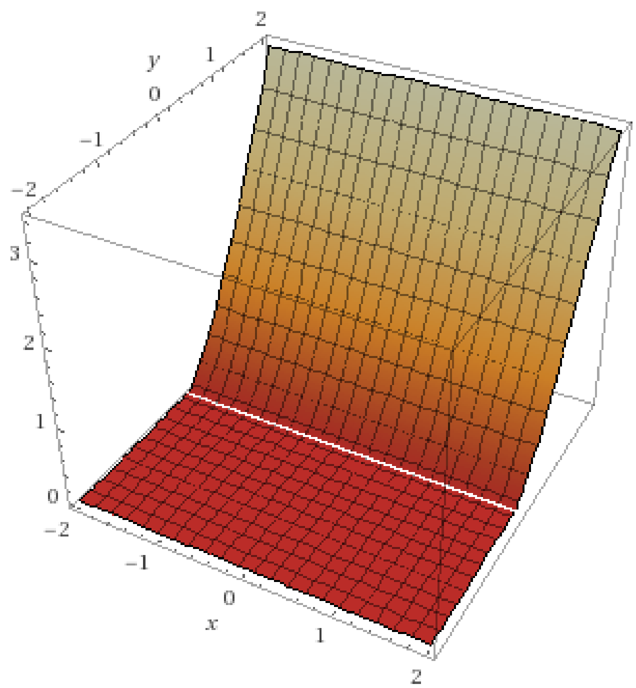

FIGURE 2. *The solution in (1.8) with $a := 4/3$.*

We stress that the scenarios described in [i], [ii] and [iii] of Theorem 1.1 are non-exclusive: namely, when $a = 1/2$, the solution u can take any of the forms in (1.8) and (1.11) (but not the form in (1.7), since this requires $\gamma > 0$, that is $a > 1$).

Similarly, when $\gamma \in (0, 2)$, the solution can take both the expressions in (1.7) and (1.8).

Another interesting feature of Theorem 1.1 is that the "degenerate" case in which the free boundary reduces to a single point, as described by (1.7), can only occur when $\gamma \in (0, 2)$, as detailed in (1.6). Instead, the case $\gamma < 0$ only produces a "flat free boundary", as given in (1.8), with the only possible exception of $\gamma = -2$, in which a resonance can produce the situation described in (1.11).

Some of the solutions described in Theorem 1.1 are depicted in Figures 1, 2 and 3.

In relation to (1.10), we also remark that $T_c \in (\pi, 2\pi)$ when $c > 0$, and $T_c \in (0, \pi)$ when $c < 0$. In particular, the case $c < 0$ produces acute cones in (1.9): in this scenario, the

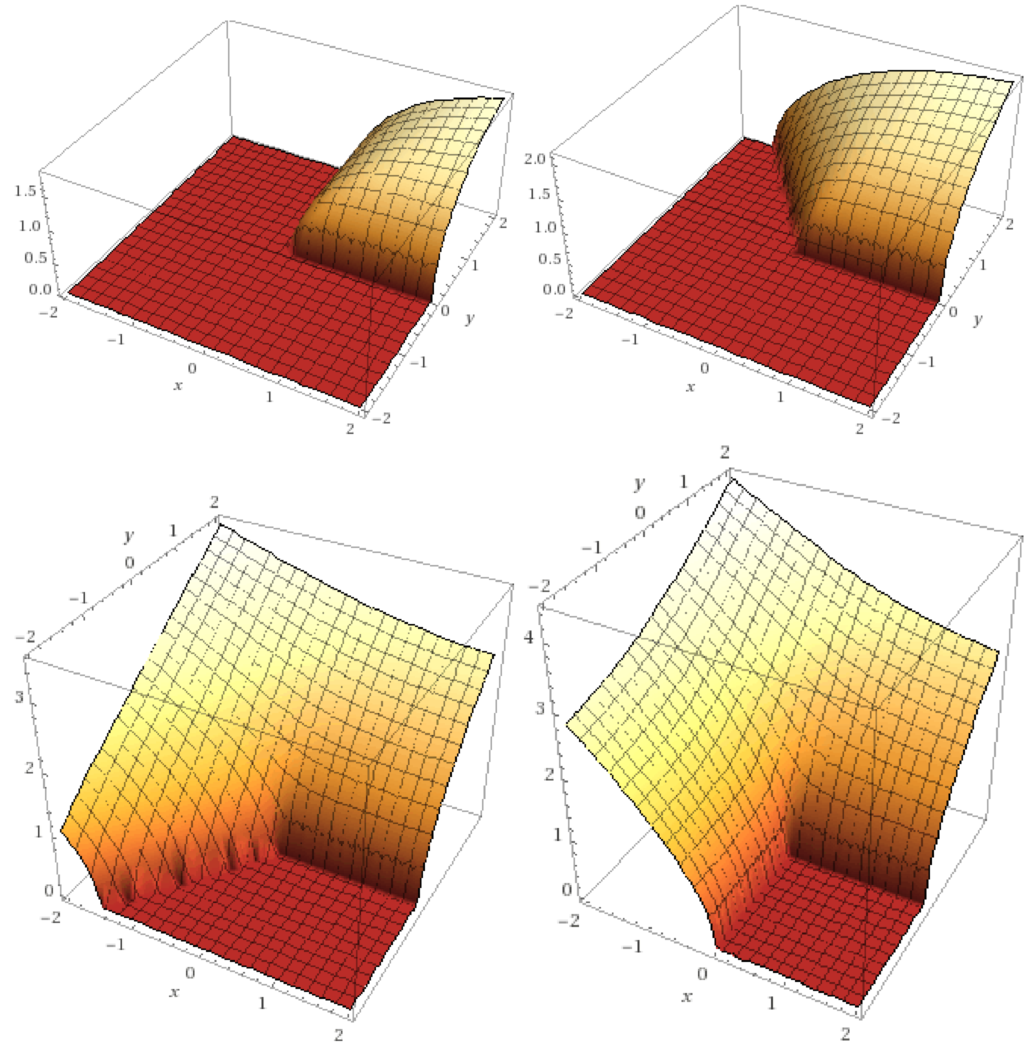

FIGURE 3. *The solution in* (1.11) *with* $c = -1, \ -1/2, \ 1/2, \ 1$.

solutions in (1.11) can be rotated and glued to form solutions with positive sets in multi-flaps cones, see e.g. Figure 4 (and, as a matter of fact, these superpositions can be iterated, thus producing also solutions whose positive sets is a cone with countably many disjoint flaps).

The proof of Theorem 1.1 is given in Section 3 here below. For that, in Section 2 we provide a series of tailored results related to the analysis of ordinary differential equations which will play a crucial role in the proof of the main theorem.

2. ODE METHODS

This section contains some bespoke result on solutions of ordinary differential equations which will be used in Section 3 to establish Theorem 1.1.

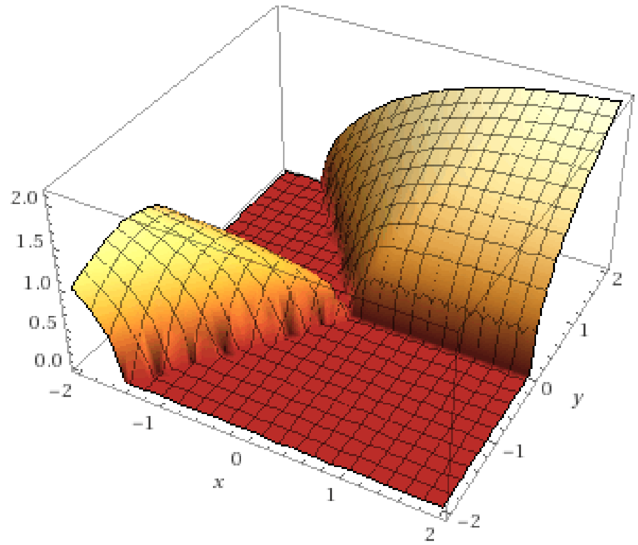

FIGURE 4. *Superposition of the solutions in (1.11) with $c = -1/2$ and $c = -2$.*

Lemma 2.1. *Let $a > 0$ and $b \in \{-1, 1\}$. Let $T_0 > 0$ and $y \in C^1([0, T_0)) \cap C^2((0, T_0))$ be a solution of the Cauchy problem*

$$\begin{cases} y^2 + yy'' + (a-1)(y^2 + (y')^2 - 1) = 0, \\ y(0) = 0, \\ y'(0) = b. \end{cases} \tag{2.1}$$

Then,

$$y(t) = \begin{cases} b \sin t & \text{if } a \neq 1/2, \\ b \sin t + c\,\big(1 - \cos t\big) & \text{if } a = 1/2, \end{cases} \tag{2.2}$$

with $c \in \mathbb{R}$.

Proof. If y has the form claimed in (2.2), then it solves (2.1) by a direct computation.

Hence, it remains to prove that if y solves (2.1), then it is of the form claimed in (2.2). To establish this, we first observe that

(2.3)
if $y(t) = b \sin t + c\,\big(1 - \cos t\big)$, for some $c \in \mathbb{R}$, for all t in an interval $I = [0, T) \subseteq [0, T_0)$, then $y(t) = b \sin t + c\,\big(1 - \cos t\big)$ for all $t \in [0, T_0)$.

To check this, we argue by contradiction, taking I to be the maximal interval for which $y(t) = b \sin t + c\,\big(1 - \cos t\big)$, and suppose that $T < T_0$. Then, we have that

$$y(T) = 0, \tag{2.4}$$

otherwise we could divide by $y(t)$ the equation in (2.1) and obtain a regular Cauchy problem with initial data $y(T)$ and $y'(T)$, thus extending the solution in an interval larger than I.

From now on, we use the notation t_k to denote a sequence in I such that $t_k \to T$ as $k \to +\infty$. Then, by (2.4),

$$0 = y(T) = \lim_{k\to+\infty} y(t_k) = \lim_{k\to+\infty} b\sin t_k + c\big(1-\cos t_k\big) = b\sin T + c\big(1-\cos T\big).$$

From this, we find that

$$\sin T = \frac{c}{b}\big(\cos T - 1\big). \tag{2.5}$$

We claim that

$$y'(T) \in \{-1, 1\}. \tag{2.6}$$

To check this, we first observe that

$$y'(T) = \lim_{k\to+\infty} y'(t_k) = \lim_{k\to+\infty} b\cos t_k + c\sin t_k = b\cos T + c\sin T. \tag{2.7}$$

Also, in view of (2.5),

$$1-\cos^2 T = \sin^2 T = \frac{c^2}{b^2}\big(\cos T - 1\big)^2 = c^2\big(\cos^2 T + 1 - 2\cos T\big).$$

This says that

$$(c^2+1)\cos^2 T - 2c^2\cos T + (c^2-1) = 0,$$

and therefore, solving the quadratic equation in the unknown $\cos T$,

$$\cos T = \frac{c^2 \pm \sqrt{c^4 - (c^2+1)(c^2-1)}}{c^2+1} = \frac{c^2\pm 1}{c^2+1}.$$

In other words,

$$\text{either } \cos T = 1, \tag{2.8}$$

$$\text{or } \cos T = \tfrac{c^2-1}{c^2+1}. \tag{2.9}$$

If (2.8) holds true, then $\sin T = 0$, and therefore (2.7) becomes

$$y'(T) = b,$$

that gives (2.6). If instead (2.9) holds true, we exploit (2.5) to see that

$$\sin T = \frac{c}{b}\left(\frac{c^2-1}{c^2+1} - 1\right) = -\frac{2c}{b\,(c^2+1)}.$$

As a result, in this case (2.7) becomes

$$y'(T) = \frac{b\,(c^2-1)}{c^2+1} - \frac{2c^2}{b\,(c^2+1)} = \frac{b\,(c^2-1)}{c^2+1} - \frac{2bc^2}{c^2+1} = \frac{-bc^2-b}{c^2+1} = -b,$$

thus completing the proof of (2.6).

As a consequence of (2.4) and (2.6), we can write

$$y(t) = c_1\,(t-T) + c_2(t-T)^2 + O((t-T)^3)$$

for all $t \in I$ in the vicinity of T, with $c_1 \in \{-1, 1\}$ and $c_2 \in \mathbb{R}$. In particular, we can find an open interval $J \subseteq I$ with $T \in \partial J$ and such that $y(t) \neq 0$ for all $t \in J$. Hence, in J, we can write (2.1) as

$$y'' = (1-a)\left(y + \frac{(y')^2-1}{y}\right) - y = -ay + (1-a)\frac{(y')^2-1}{y}. \tag{2.10}$$

Also, in J, we have that

$$\begin{aligned} g(t) := \frac{(y'(t))^2-1}{y(t)} &= \frac{\big(c_1+2c_2(t-T)+O((t-T)^2)\big)^2-1}{c_1\,(t-T)+c_2(t-T)^2+O((t-T)^3)} \\ &= \frac{c_1^2+4c_1c_2(t-T)+O((t-T)^2)-1}{c_1\,(t-T)+c_2(t-T)^2+O((t-T)^3)} \\ &= \frac{4c_1c_2(t-T)+O((t-T)^2)}{c_1\,(t-T)+c_2(t-T)^2+O((t-T)^3)} \\ &= \frac{4c_1c_2+O(t-T)}{c_1+c_2(t-T)+O((t-T)^2)}, \end{aligned}$$

which is a regular function in the vicinity of T. Hence, we can define

$$f(y,t) := -ay+(1-a)g(t),$$

and write (2.10) in the form of a regular Cauchy problem $y''=f(y,t)$, with initial data $y(T)$ and $y'(T)$. This allows again to extend the solution beyond the maximal interval I, and this contradiction completes the proof of (2.3).

Now, in light of (2.3), it is sufficient to prove that if y solves (2.1), then it is of the form claimed in (2.2) for all $t\in(0,\eta)$, with $\eta>0$ to be taken conveniently small.

In particular, from (2.1), we can write $y'(t)=b+o(1)$ and thus $y(t)=bt+o(t)$, hence

$$y \text{ does not vanish in } (0,\eta) \text{ and } \inf_{[0,\eta]}|y'|\geqslant\frac12, \tag{2.11}$$

as long as η is conveniently small.

Now, we define

$$w(t) := y^2(t)+(y'(t))^2-1. \tag{2.12}$$

We observe that

$$w' = 2yy'+2y'y'' = 2y'(y+y''), \tag{2.13}$$

hence, in light of (2.11), we can divide by $2y'$ and find that

$$\frac{w'}{2y'} = y+y''.$$

This and (2.1) give that

$$0 = y^2+yy''+(a-1)w = y(y+y'')+(a-1)w = \frac{w'y}{2y'}+(a-1)w,$$

which produces

$$\frac{d}{dt}(\log|w|) = \frac{w'}{w} = -2(a-1)\frac{y'}{y} = 2(1-a)\frac{d}{dt}(\log|y|).$$

As a result, given $\varepsilon\in(0,\eta)$,

$$\log\frac{|w(t)|}{|w(\varepsilon)|} = 2(1-a)\log\frac{|y(t)|}{|y(\varepsilon)|}.$$

Hence, since w and y do not change sign in $(0,\eta)$,

$$\log\frac{w(t)}{w(\varepsilon)}=2(1-a)\log\frac{y(t)}{y(\varepsilon)},$$

and consequently

$$w(t)=w(\varepsilon)\left(\frac{y(t)}{y(\varepsilon)}\right)^{2(1-a)}. \tag{2.14}$$

Now, if

$$a=\frac{1}{2}, \tag{2.15}$$

we use (2.14) to see that

$$w(t)=\bar c\, y(t), \qquad \text{where } \bar c:=\frac{w(\varepsilon)}{y(\varepsilon)},$$

and thus, recalling (2.13),

$$2y'(y+y'')=w'=\bar c\, y'.$$

For this reason, and recalling (2.11), we can write

$$y+y''=\frac{\bar c}{2} \qquad \text{in } (0,\eta).$$

This standard linear equation has solution

$$y(t)=c+c_1\sin t+c_2\cos t,$$

for some c, c_1 and $c_2\in\mathbb{R}$ (with $c=\bar c/2$). Then, since

$$0=c+c_2 \qquad \text{and} \qquad b=y'(0)=c_1,$$

we conclude that

$$y(t)=b\sin t+c\big(1-\cos t\big).$$

This proves that y has the form claimed in (2.2).

Therefore, in light of (2.15), from now on we can assume

$$a\neq\frac{1}{2}. \tag{2.16}$$

We claim that, in this situation,

$$\lim_{\varepsilon\searrow 0}\frac{w(\varepsilon)}{|y(\varepsilon)|^{2(1-a)}}=0. \tag{2.17}$$

To check this, we first observe that, if $a\geqslant 1$, then

$$\lim_{\varepsilon\searrow 0}|y(\varepsilon)|^{2(a-1)}=\begin{cases}0 & \text{if } a>1,\\ 1 & \text{if } a=1,\end{cases} \tag{2.18}$$

thanks to the initial value in (2.1).

Similarly, recalling (2.12)

$$\lim_{\varepsilon\searrow 0}w(\varepsilon)=\lim_{\varepsilon\searrow 0}y^2(\varepsilon)+(y'(\varepsilon))^2-1=0+b^2-1=0. \tag{2.19}$$

Hence, when $a \geqslant 1$, the claim in (2.17) plainly follows from (2.18) and (2.19), and therefore, to complete the proof of (2.17), we can focus on the case

$$a \in (0,1). \tag{2.20}$$

In this situation, we claim that

$$y \in C^3([0,\eta)). \tag{2.21}$$

To check this, we assume that $b = 1$ (the case $b = -1$ being similar). Hence, by (2.11), we know that y and y' are strictly positive in $(0, \eta)$, as long as η is small enough. Then, we define

$$\begin{aligned} m &:= \frac{w(\varepsilon)}{(y(\varepsilon))^{2(1-a)}} \\ \text{and} \qquad \Psi(r) &:= \int_0^r \frac{d\tau}{\sqrt{1-\tau^2+m\tau^{2(1-a)}}}. \end{aligned}$$

In view of (2.12) and (2.14), we can write

$$y^2(t) + (y'(t))^2 - 1 = w(t) = m(y(t))^{2(1-a)},$$

and accordingly

$$\frac{d}{dt}\Psi(y(t)) = \frac{y'(t)}{\sqrt{1-y^2(t)+m(y(t))^{2(1-a)}}} = 1.$$

Therefore, since $\Psi(y(0)) = \Psi(0) = 0$, we find that

$$\Psi(y(t)) = t, \qquad \text{for all } t \in (0,\eta).$$

Accordingly, taking derivatives,

$$\begin{aligned} & 1 = \Psi'(y(t))\, y'(t), \\ & 0 = \Psi''(y(t))\, (y'(t))^2 + \Psi'(y(t))\, y''(t) \\ \text{and} \qquad & 0 = \Psi'''(y(t))\, (y'(t))^3 + 3\Psi''(y(t))\, y'(t) y''(t) + \Psi'(y(t))\, y'''(t). \end{aligned}$$

From this, since $\Psi'(y(0)) = \Psi'(0) = 1$, we conclude that y'' and y''' are continuous up to $t = 0$, and hence the proof of (2.21) is complete.

As a result, in view of (2.21), we can use a Taylor expansion of the type

$$y(\varepsilon) = b\varepsilon + \frac{y''(0)\varepsilon^2}{2} + \frac{y'''(0)\varepsilon^3}{6} + o(\varepsilon^3),$$

which, in light of (2.12), gives that

$$\begin{aligned} w(\varepsilon) &= \left(b\varepsilon + \frac{y''(0)\varepsilon^2}{2} + \frac{y'''(0)\varepsilon^3}{6} + o(\varepsilon^3)\right)^2 + \left(b + y''(0)\varepsilon + \frac{y'''(0)\varepsilon^2}{2} + o(\varepsilon^2)\right)^2 - 1 \\ &= b^2 - 1 + 2b\, y''(0)\,\varepsilon + \big(b^2 + (y''(0))^2 + by'''(0)\big)\varepsilon^2 + o(\varepsilon^2) \\ &= 2b\, y''(0)\,\varepsilon + \big(1 + (y''(0))^2 + by'''(0)\big)\varepsilon^2 + o(\varepsilon^2). \end{aligned}$$

As a result,

$$\begin{aligned}\frac{w(\varepsilon)}{|y(\varepsilon)|^{2(1-a)}} \\ = \frac{2b\,y''(0)\,\varepsilon^{2a-1} + \big(1 + (y''(0))^2 + by'''(0)\big)\varepsilon^{2a} + o(\varepsilon^{2a})}{\Big|b + \frac{y''(0)\varepsilon}{2} + \frac{y'''(0)\varepsilon^2}{6} + o(\varepsilon^2)\Big|^{2(1-a)}}.\end{aligned} \tag{2.22}$$

Now we distinguish two cases, recalling (2.20) and (2.16), namely $a \in (0, 1/2)$ and $a \in (1/2, 1)$. If $a \in (0, 1/2)$, we know from (2.14) that the ratio $\frac{w(\varepsilon)}{(y(\varepsilon))^{2(1-a)}}$ must remain bounded as $\varepsilon \searrow 0$, and consequently, since in this case $2a-1 < 0$, necessarily (2.22) gives that $y''(0) = 0$. Thus, plugging this information back in (2.22), we obtain that

$$\frac{w(\varepsilon)}{(y(\varepsilon))^{2(1-a)}} = \frac{\big(1 + by'''(0)\big)\varepsilon^{2a} + o(\varepsilon^{2a})}{\Big|b + \frac{y'''(0)\varepsilon^2}{6} + o(\varepsilon^2)\Big|^{2(1-a)}},$$

which converges to 0, since $a > 0$.

This establishes (2.17) in this case, and we now suppose that $a \in (1/2, 1)$. In this case, we have that (2.17) follows directly from (2.22), since $2a - 1 > 0$. In this way, the proof of (2.17) is complete.

Now, by (2.14) and (2.17), we infer that

$$w(t) = 0 \qquad \text{for all } t \in (0, \eta).$$

Hence, in view of (2.12), we find that

$$y^2 + (y')^2 - 1 = 0,$$

and consequently

$$\frac{dy}{\sqrt{1 - y^2}} = b\,dt,$$

yielding that

$$y(t) = \sin(bt) = b \sin t,$$

since $b = \pm 1$, that is (2.2), as desired. □

The counterpart of Lemma 2.1 for the non-singular equations is given by the following result:

Lemma 2.2. *Let $a > 0$ and $b \in \{-1, 1\}$. Assume that there exists a solution $y \in C^2(\mathbb{S}^1)$ of the problem*

$$\begin{cases} y^2 + yy'' + (a-1)(y^2 + (y')^2 - 1) = 0, \\ \min\limits_{\mathbb{S}^1} y > 0. \end{cases} \tag{2.23}$$

Then,

$$a > 1 \tag{2.24}$$

and

$$y \text{ is constantly equal to } \sqrt{\frac{a-1}{a}}. \tag{2.25}$$

Proof. With a slight abuse of notation, we will write $y : \mathbb{R} \to \mathbb{R}$ (assuming y to be 2π-periodic), and we will freely interchange notations involving $\mathbb{S}^1$ and $\mathbb{R}$. Let $t_0 \in \mathbb{R}$ be such that

$$y(t_0) = \min_{\mathbb{S}^1} y > 0.$$

Then, we have that $y'(t_0) = 0$ and $y''(t_0) \geqslant 0$. This and the equation in (2.23) give that

$$\begin{aligned} 0 &= y^2(t_0) + y(t_0)y''(t_0) + (a-1)(y^2(t_0) - 1) \\ &= ay^2(t_0) + y(t_0)y''(t_0) - a + 1 > -a + 1, \end{aligned} \tag{2.26}$$

which yields (2.24), as desired.

It is also useful to remark that, in view of (2.26),

$$0 = ay^2(t_0) + y(t_0)y''(t_0) - a + 1 \geqslant ay^2(t_0) - a + 1,$$

and therefore

$$y(t_0) \leqslant \sqrt{\frac{a-1}{a}}. \tag{2.27}$$

Similarly, if t_1 is such that

$$y(t_1) = \max_{\mathbb{S}^1} y > 0, \tag{2.28}$$

we have that $y'(t_1) = 0$ and $y''(t_1) \leqslant 0$, whence the equation in (2.23) gives that

$$0 = y^2(t_1) + y(t_1)y''(t_1) + (a-1)(y^2(t_1) - 1) \leqslant ay^2(t_1) - a + 1,$$

and accordingly

$$y(t_1) \geqslant \sqrt{\frac{a-1}{a}}.$$

We claim that

$$y(t_1) = \sqrt{\frac{a-1}{a}}. \tag{2.29}$$

For this, we argue by contradiction, supposing that

$$y(t_1) > \sqrt{\frac{a-1}{a}}. \tag{2.30}$$

We define

$$W(t) := 1 - y^2(t) - (y'(t))^2$$

and we observe that, in light of (2.27),

$$W(t_0) = 1 - y^2(t_0) \geqslant 1 - \frac{a-1}{a} = \frac{1}{a} > 0. \tag{2.31}$$

Therefore, W is strictly positive in some interval $I := (t_0 - \delta, t_0 + \delta)$, for a suitable $\delta > 0$. As a consequence, we can consider the logarithm of W in I and exploit the equation in (2.23) to see that

$$\begin{aligned} \frac{d}{dt}\log W &= \frac{W'}{W} = \frac{-2yy' - 2y'y''}{1 - y^2 - (y')^2} = \frac{-2y'(y + y'')}{1 - y^2 - (y')^2} = \frac{-2y'(y^2 + yy'')}{y(1 - y^2 - (y')^2)} \\ &= \frac{2(a-1)y'(y^2 + (y')^2 - 1)}{y(1 - y^2 - (y')^2)} = \frac{-2(a-1)y'}{y} = -2(a-1)\frac{d}{dt}\log y, \end{aligned}$$

and, as a result, for all $t \in I$,

$$\log \frac{W(t)}{W(t_0)} = -2(a-1) \log \frac{y(t)}{y(t_0)} = \log \left(\frac{y(t)}{y(t_0)} \right)^{2(1-a)}.$$

Therefore, setting

$$\kappa := \frac{W(t_0)}{(y(t_0))^{2(1-a)}}, \tag{2.32}$$

we find that, for all $t \in I$,

$$1 - y^2(t) - (y'(t))^2 = W(t) = \kappa (y(t))^{2(1-a)}. \tag{2.33}$$

We also remark that y is an analytic function, since it is a solution of an analytic Cauchy problem (the sign condition in (2.23) ensuring that the source term of the differential equation is non-singular, after a division by y), see e.g. page 124 in [Chi99]. Consequently, the relation in (2.33) is globally valid, namely

$$(y'(t))^2 = 1 - y^2(t) - \kappa (y(t))^{2(1-a)} \qquad \text{for all } t \in \mathbb{R}. \tag{2.34}$$

Moreover, recalling (2.27), (2.31) and (2.32),

$$\kappa \geqslant \frac{1/a}{((a-1)/a)^{1-a}} = \frac{1}{a} \left(\frac{a}{a-1} \right)^{1-a}.$$

For this reason and (2.34), we have that

$$0 \leqslant 1 - y^2(t) - \kappa (y(t))^{2(1-a)} \leqslant 1 - y^2(t) - \frac{1}{a} \left(\frac{a}{a-1} \right)^{1-a} (y(t))^{2(1-a)} \qquad \text{for all } t \in \mathbb{R}.$$

From this and (2.30), we find that

$$0 < 1 - \frac{a-1}{a} - \frac{1}{a} \left(\frac{a}{a-1} \right)^{1-a} \left(\frac{a-1}{a} \right)^{1-a} = 0.$$

This is a contradiction, and thus (2.29) is established.

As a consequence of (2.28) and (2.29), we have that

$$y(t_1) = \sqrt{\frac{a-1}{a}} \qquad \text{and} \qquad y'(t_1) = 0.$$

Since, by inspection, the function $y_\star$ constantly equal to $\sqrt{\frac{a-1}{a}}$ is also a solution of (2.23), by the uniqueness result of the standard Cauchy problem we infer that $y(t) = y_\star(t)$ for every $t \in \mathbb{R}$, and this proves the desired claim in (2.25). □

3. Proof of Theorem 1.1

We write u in polar coordinates $(r, \theta) \in (0, +\infty) \times \mathbb{S}^1$, namely

$$u = r^a \, g(\theta) \tag{3.1}$$

for some function $g \in C^2(\mathbb{S}^1)$, and we deduce from (1.4) that, in the positivity set of g,

$$a(a-1) r^{a-2} g + a r^{a-2} g + r^{a-2} g'' = \gamma (r^a g)^{\gamma - 1},$$

and thus

$$a^2 g + g'' = \gamma r^{a(\gamma-1)-a+2} g^{\gamma-1}. \tag{3.2}$$

Now, we can suppose that g does not vanish identically (otherwise u would be the trivial solution), and we take $\theta_0 \in \mathbb{S}^1$ such that $g(\theta_0) \neq 0$. Then we deduce from (3.2) that

$$\mathbb{R} \ni \frac{g^{1-\gamma}(\theta_0)}{\gamma} \Big(a^2 g(\theta_0) + g''(\theta_0) \Big) = \lim_{r\to 0} r^{a(\gamma-1)-a+2} = \lim_{r\to+\infty} r^{a(\gamma-1)-a+2}.$$

From this, we obtain that

$$\gamma < 2 \qquad \text{and} \qquad a = \frac{2}{2-\gamma}, \tag{3.3}$$

that establishes (1.5), as desired.

Now, we plug (3.3) into (3.2), finding that, in the positivity set of g,

$$a^2 g + g'' = \frac{2(a-1)}{a} g^{\frac{a-2}{a}}. \tag{3.4}$$

Hence, we can define

$$y(\theta) := K g^{\frac{1}{a}}(\theta), \tag{3.5}$$

for some $K > 0$ to be chosen in what follows, and obtain from (3.4) that, in the positivity set of y,

$$\frac{a^2 y^a}{K^a} + \frac{a(a-1)y^{a-2}(y')^2 + a y^{a-1} y''}{K^a} = \frac{2(a-1)}{aK^{a-2}} y^{a-2},$$

that is

$$a y^2 + (a-1)(y')^2 + y y'' = \frac{2K^2(a-1)}{a^2}.$$

In particular, choosing

$$K := a/\sqrt{2}, \tag{3.6}$$

we find

$$a y^2 + (a-1)(y')^2 + y y'' = a - 1,$$

which says that y solves the ordinary differential equation in (2.1) and (2.23).

Now, to distinguish between the settings in (2.1) and (2.23), we recall that y is nonnegative, hence two cases may hold:

$$\text{either } \inf_{\mathbb{S}^1} y > 0, \tag{3.7}$$

$$\text{or } y \text{ vanishes somewhere.} \tag{3.8}$$

Assume first that (3.7) holds true. Then, y is as in (2.23), whence we can apply Lemma 2.2 and infer that

$$a > 1 \tag{3.9}$$

and, for all $\theta \in \mathbb{S}^1$,

$$\sqrt{\frac{a-1}{a}} = y(\theta) = K g^{\frac{1}{a}}(\theta) = \frac{a}{\sqrt{2}} g^{\frac{1}{a}}(\theta).$$

This and (3.3) give that

$$u = \frac{(2(a-1))^{a/2}}{a^{3a/2}} r^a,$$

hence (1.7) is established.

We also remark that the function in (1.7) is indeed a solution of (1.4) since

$$\begin{aligned} &\frac{(2(a-1))^{a/2}a(a-1)}{a^{3a/2}}\,r^{a-2}+\frac{(2(a-1))^{a/2}a}{a^{3a/2}}\,r^{a-2}-\gamma\left(\frac{(2(a-1))^{a/2}}{a^{3a/2}}\,r^{a}\right)^{\gamma-1}\\ =\;&\frac{(2(a-1))^{a/2}a^2}{a^{3a/2}}\,r^{a-2}-\frac{2(a-1)}{a}\left(\frac{(2(a-1))^{a/2}}{a^{3a/2}}\,r^{a}\right)^{(a-2)/a}\\ =\;&\left(\frac{(2(a-1))^{a/2}}{a^{(3a-4)/2}}-\frac{2(a-1)}{a}\,\frac{(2(a-1))^{(a-2)/2}}{a^{3(a-2)/2}}\right)r^{a-2}\\ =\;&0. \end{aligned}$$

Finally, (1.6) follows from (3.3) and (3.9).

Then, we can now focus on the case in which (3.8) is satisfied. Hence, up to a reflection, we can focus on the case in which $y>0$ in $(\theta_\star,\theta_\star+T)$, with $y(\theta_\star)=y(\theta_\star+T)=0$ for some $\theta_\star\in\mathbb{R}$ and $T\in(0,2\pi]$. Up to a rotation, we can assume that $\theta_\star=0$. We remark that, by (1.3),

$$y=\frac{a}{\sqrt{2}}\,u^{\frac{1}{a}}\in C^{1,1}([0,T]),$$

and accordingly we can take limits in (2.1) as $\theta\searrow 0$ and $\theta\nearrow T$, finding that

$$(a-1)((y'(0))^2-1)=0=(a-1)((y'(T))^2-1),$$

from which it follows that $y'(0)=1$ and $y'(T)=-1$.

As a consequence, we can exploit Lemma 2.1 and find that, for every $\theta\in(0,T)$,

$$y(\theta)=\sin\theta+c(1-\cos\theta),$$

with c an arbitrary real constant when $a=1/2$ and $c=0$ when $a\neq 1/2$.

In particular, if $a\neq 1/2$, then $y(\theta)=\sin\theta$ and $T=\pi$. This gives that, for every $x=(x_1,x_2)$ with $x_2>0$,

$$u=r^a g=\frac{2^{\frac{a}{2}}}{a^a}r^a y^a=\frac{2^{\frac{a}{2}}}{a^a}(r\sin\theta)^a=\frac{2^{\frac{a}{2}}}{a^a}x_2^a.$$

This gives two possibilities:

$$\begin{aligned} &\text{either } u(x)=\frac{2^{\frac{a}{2}}}{a^a}(x_2)_+^a\\ &\text{or } u(x)=\frac{2^{\frac{a}{2}}}{a^a}|x_2|^a, \end{aligned}$$

for all $x\in\mathbb{R}^2$. The latter possibility does not satisfy (1.3), therefore (1.8) is established in this case.

If instead $a=1/2$, we have that, for every $\theta\in(0,T)$,

$$y(\theta)=\sin\theta+c(1-\cos\theta),$$

with $c\in\mathbb{R}$, and the case $c=0$ reduces to the previous situation. Hence, we can suppose that $c\neq 0$ and we use the formulae

$$\cos\theta=\frac{1-\tau^2}{1+\tau^2}\qquad\text{and}\qquad\sin\theta=\frac{2\tau}{1+\tau^2},\qquad\text{where}\qquad\tau:=\tan\frac{\theta}{2}.$$

In this way, we have that

$$y = \frac{2\tau(1+c\tau)}{1+\tau^2},$$

which is positive when $\tau \in (-\infty, -1/c) \cup (0, +\infty)$ if $c > 0$, and when $\tau \in (0, -1/c)$ when $c < 0$.

That is, $y(\theta)$ is positive when $\theta \in (0, 2\pi - 2\arctan(1/c))$ if $c > 0$, and when $\theta \in (0, -2\arctan(1/c))$ when $c < 0$. This gives that $T = 2\pi - 2\arctan(1/c) \in (\pi, 2\pi)$ when $c > 0$, and that $T = -2\arctan(1/c) \in (0, \pi)$ when $c < 0$.

Hence, in the cone $\mathcal{C}_c$ introduced in (1.9) we have that

$$u = r^a g = \frac{2^{\frac{a}{2}}}{a^a} r^a y^a = \frac{2^{\frac{a}{2}}}{a^a} r^a \big(\sin\theta + c(1-\cos\theta)\big)^a = \frac{2^{\frac{a}{2}}}{a^a}\big(x_2 - cx_1 + c|x|\big)^a,$$

and this is the setting described in (1.11).

We stress that the function in (1.8) satisfies (1.3) and also is a solution of (1.4), since, in this setting,

$$\begin{aligned}\Delta u - \gamma u^{\gamma-1} &= \frac{2^{\frac{a}{2}} a(a-1)}{a^a} x_2^{a-2} - \gamma\left(\frac{2^{\frac{a}{2}}}{a^a} x_2^a\right)^{\gamma-1} \\ &= \frac{2^{\frac{a}{2}} a(a-1)}{a^a} x_2^{a-2} - \frac{2a-2}{a}\,\frac{2^{\frac{a-2}{2}}}{a^{a-2}} x_2^{a-2} = 0\end{aligned}$$

when $x_2 > 0$, thanks to (1.5).

We also observe that the function in (1.11) satisfies (1.3) and is a solution of (1.4), since

$$\begin{aligned}&\Delta u - \gamma u^{\gamma-1} \\ =\;& \frac{2^{\frac{a}{2}}}{a^{a-1}}\left(x_2 - cx_1 + c|x|\right)^{a-2}\left[(a-1)\left(\left(\frac{cx_1}{|x|} - c\right)^2 + \left(\frac{cx_2}{|x|} + 1\right)^2\right) + \frac{c}{|x|}(x_2 - cx_1 + c|x|)\right] \\ &\quad -\gamma\left(\frac{2^{\frac{a}{2}}}{a^a}\big(x_2 - cx_1 + c|x|\big)^a\right)^{\gamma-1} \\ =\;& \frac{2^{\frac{a}{2}}}{a^{a-1}}\left(x_2 - cx_1 + c|x|\right)^{a-2}\left[(a-1)\left(2c^2 + 1 + \frac{2c}{|x|}(x_2 - cx_1)\right) + \frac{c}{|x|}(x_2 - cx_1 + c|x|)\right] \\ &\quad -\frac{2a-2}{a}\,\frac{2^{\frac{a-2}{2}}}{a^{a-2}}\big(x_2 - cx_1 + c|x|\big)^{a-2} \\ =\;& \frac{1}{2^{1/4}}\left(x_2 - cx_1 + c|x|\right)^{-3/2}\left[-\frac{1}{2}\left(2c^2 + 1 + \frac{2c}{|x|}(x_2 - cx_1)\right) + \frac{c}{|x|}(x_2 - cx_1 + c|x|)\right] \\ &\quad +\frac{1}{2^{5/4}}\big(x_2 - cx_1 + c|x|\big)^{-3/2} \\ =\;& \frac{1}{2^{1/4}}\left(x_2 - cx_1 + c|x|\right)^{-3/2}\left[-c^2 - \frac{1}{2} + c^2\right] + \frac{1}{2^{5/4}}\big(x_2 - cx_1 + c|x|\big)^{-3/2} \\ =\;& 0.\end{aligned}$$

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