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SOBOLEV IMPROVING FOR AVERAGES OVER CURVES IN $\mathbb{R}^4$

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Abstract. We study L^p-Sobolev improving for averaging operators A_γ given by convolution with a compactly supported smooth density μ_γ on a non-degenerate curve. In particular, in 4 dimensions we show that A_γ maps $L^p(\mathbb{R}^4)$ to the Sobolev space $L^p_{1/p}(\mathbb{R}^4)$ for all $6 < p < \infty$. This implies the complete optimal range of L^p-Sobolev estimates, except possibly for certain endpoint cases. The proof relies on decoupling inequalities for a family of cones which decompose the wave front set of μ_γ. In higher dimensions, a new non-trivial necessary condition for $L^p(\mathbb{R}^n) \to L^p_{1/p}(\mathbb{R}^n)$ boundedness is obtained, which motivates a conjectural range of estimates.

1. Introduction

For $n \geqslant 2$ let $\gamma\colon I \to \mathbb{R}^n$ be a smooth curve,[1] where $I \subset \mathbb{R}$ is a compact interval, and $\chi \in C^\infty(\mathbb{R})$ be a bump function supported on the interior of I. Consider the averaging operator

$$A_\gamma f(x) := \int_{\mathbb{R}} f(x - \gamma(s))\, \chi(s)\, \mathrm{d}s; \tag{1.1}$$

in particular, $A_\gamma f = \mu_\gamma * f$, where μ_γ is the measure given by the push-forward of $\chi(s)\mathrm{d}s$ under γ.

The goal of this paper is to study sharp L^p-Sobolev improving bounds for the operator A_γ for a wide class of curves in $\mathbb{R}^4$. To state the main theorem, we say a smooth curve $\gamma\colon I \to \mathbb{R}^n$ is *non-degenerate* if there is a constant $c_0 > 0$ such that

$$|\det(\gamma'(s), \cdots, \gamma^{(n)}(s))| \geqslant c_0 \qquad \text{for all } s \in I \tag{1.2}$$

or, equivalently, the $n-1$ curvature functions of γ are all bounded away from 0.

Theorem 1.1. *If $\gamma\colon I \to \mathbb{R}^4$ is non-degenerate and $6 < p < \infty$, then*

$$\|A_\gamma f\|_{L^p_{1/p}(\mathbb{R}^4)} \lesssim_{p,\gamma,\chi} \|f\|_{L^p(\mathbb{R}^4)}.$$

This result is sharp up to $p = 6$ in the sense that the $L^p \to L^p_{1/p}$ bound fails whenever $2 \leqslant p < 6$: see Proposition 1.2 below. Furthermore, interpolation with the elementary $L^2 \to L^2_{1/4}$ inequality and duality give the complete range of $L^p \to L^p_\alpha$ estimates for all $1 \leqslant p \leqslant \infty$, except possibly for endpoint cases.

In higher dimensions no $L^p \to L^p_{1/p}$ estimates are currently known to hold for such averaging operators, although it is natural to conjecture that the following holds.

Conjecture 1. *If $\gamma\colon I \to \mathbb{R}^n$ is non-degenerate and $2n-2 < p < \infty$, then*

$$\|A_\gamma f\|_{L^p_{1/p}(\mathbb{R}^n)} \lesssim_{p,\gamma,\chi} \|f\|_{L^p(\mathbb{R}^n)}. \tag{1.3}$$

If true, then the above conjectured range would be sharp except for some endpoint cases, viz.

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[1]Throughout, any curve is tacitly assumed to be simple (that is, γ is injective) and regular (γ' is non-vanishing).

Proposition 1.2. *Let $2 \leqslant p \leqslant \infty$. If $\gamma\colon I \to \mathbb{R}^n$ is non-degenerate and the inequality*

$$\|A_\gamma f\|_{L^p_\alpha(\mathbb{R}^n)} \lesssim_{p,\gamma,\chi} \|f\|_{L^p(\mathbb{R}^n)}$$

holds, then we must have $\alpha \leqslant \min\left\{\frac{1}{n}\left(\frac{1}{2}+\frac{1}{p}\right), \frac{1}{p}\right\}$.

As in the case of Theorem 1.1, the sharp estimates for $1 \leqslant p \leqslant 2n-2$ would follow from Conjecture 1 by interpolation with the $L^2 \to L^2_{1/n}$ inequality and duality, except the endpoint regularity estimates for $\frac{2n-2}{2n-3} \leqslant p \leqslant 2n-2$, $p \neq 2$.

In the euclidean plane Conjecture 1 is an elementary consequence of the decay of the Fourier transform of the measure μ_γ. In higher dimensions the problem is significantly more difficult, owing to the weaker rate of Fourier decay.[2] The $n = 3$ case was established up to the $p = 4$ endpoint by Pramanik and the fourth author [17], conditional on the sharp Wolff-type 'ℓ^p-decoupling' inequality for the light cone. The sharp decoupling inequality was later proved by Bourgain–Demeter [4], thus establishing the bounds for the averaging operators unconditionally. Theorem 1.1 verifies the $n = 4$ case of Conjecture 1 up to the $p = 6$ endpoint. The proof strategy behind Theorem 1.1 is based on that used to study the $n = 3$ case in [17], although significant new features and additional complications arise in the four-dimensional setting. To overcome these difficulties, advantage is taken of recent new advances in the understanding of decoupling theory. A key tool is the Bourgain–Demeter–Guth decoupling theorem for curves [5].

The first stage of the argument relies on a careful decomposition of the operator in the frequency domain. This part of the proof is inspired by the analysis of the helical maximal function appearing in [1] (see also [17]). Indeed, the maximal problem treated in [1] shares a number of essential features with Theorem 1.1. In particular, for both problems it is natural to microlocalise the operator with respect to a pair of nested cones in the frequency domain (see the introductory discussion in [1] for more details). However, a quick comparison between this paper and [1] shows that the methods and overall proof scheme differ on a number of key points. For instance, the frequency decomposition used here is significantly more involved than that used in [1], owing to additional complications which arise when working in $\mathbb{R}^4$ rather than $\mathbb{R}^3$. Furthermore, whilst decoupling plays an important rôle in the current paper, the analysis in [1] relies on square function estimates. One useful feature of decoupling (as opposed to the use of square functions) is that decoupling inequalities are readily iterated. We make use of this fact in a fundamental way when decomposing the operator with respect to the different frequency cones.

1.1. **Corollaries.**

Theorem 1.1 has a number of consequences which follow immediately from known arguments.

Extension to finite type curves. Using arguments from [17], one can show that Theorem 1.1 implies bounds for a more general class of curves. We say a smooth curve $\gamma\colon I \to \mathbb{R}^n$ is of *finite maximal type* if there exists $d \in \mathbb{N}$ and a constant $c_0 > 0$ such that

$$\sum_{j=1}^{d} |\langle \gamma^{(j)}(s), \xi\rangle| \geqslant c_0|\xi| \qquad \text{for all } s \in I,\ \xi \in \mathbb{R}^n. \tag{1.4}$$

For fixed s, the smallest d for which (1.4) holds for some $c_0 > 0$ is called the *type of γ at s*. The type is an upper semicontinuous function, and the supremum of the types over all $s \in I$ is referred to as the *maximal type* of γ.

Corollary 1.3. *If $\gamma\colon I \to \mathbb{R}^4$ is of maximal type $d \in \mathbb{N}$ and $\max\{6, d\} < p < \infty$, then*

$$\|A_\gamma f\|_{L^p_{1/p}(\mathbb{R}^4)} \lesssim_{p,\gamma,\chi} \|f\|_{L^p(\mathbb{R}^4)}.$$

[2] In particular, the curve is no longer a Salem set.

This result is sharp up to endpoints (for further discussion of endpoint cases, see §3.6) regarding the range of p for which the regularity of order $1/p$ holds. In the range $2 \leqslant p \leqslant \max\{6, d\}$, the inequalities resulting from interpolation with the $L^2(\mathbb{R}^4) \to L^2_{1/d}(\mathbb{R}^4)$ estimates are also sharp, up to the regularity endpoint, for $d \geqslant 6$ and the non-degenerate $d = 4$ case; for $d = 5$ one expects, however, better bounds to hold in this range (see Figure 3.4). There are also natural extensions of Conjecture 1 and Proposition 1.2 which deal with finite maximal type curves in higher dimensions: see §3 below.

Endpoint lacunary maximal estimates. For the measure μ_γ introduced above, define the family of dyadic dilates μ_γ^k for $k \in \mathbb{Z}$ by

$$\langle \mu_\gamma^k, f \rangle = \langle \mu_\gamma, f(2^k \cdot) \rangle$$

and consider the associated convolution operators $A_\gamma^k f := \mu_\gamma^k * f$. If γ is of finite maximal type, then a well-known and classical result (see, for instance, [9]) states that the associated lacunary maximal function

$$\mathcal{M}_\gamma f := \sup_{k \in \mathbb{Z}} |A_\gamma^k f|$$

is bounded on L^p for all $1 < p \leqslant \infty$. A difficult problem is to understand the endpoint behaviour of these operators near L^1. By an off-the-shelf application of the main theorem from [18], Corollary 1.3 implies an endpoint bound for $\mathcal{M}_\gamma$ in the $n = 4$ case.

Corollary 1.4. *If $\gamma \colon I \to \mathbb{R}^4$ is of finite maximal type, then the lacunary maximal function $\mathcal{M}_\gamma$ maps the (standard isotropic) Hardy space $H^1(\mathbb{R}^4)$ to $L^{1,\infty}(\mathbb{R}^4)$.*

In particular, by [18, Theorem 1.1], Corollary 1.4 follows from *any* $L^p \to L^p_{1/p}$ bound for the associated averaging operator A_γ for $2 \leqslant p < \infty$ (that is, one does not require $L^p \to L^p_{1/p}$ for the sharp range of p for this application). Note that, prior to this paper, no such bounds L^p-Sobolev bounds were known for $n \geqslant 4$; thus the question of the $H^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ boundedness of lacunary maximal associated to finite maximal type (or even non-degenerate) curves remains open for $n \geqslant 5$.

Outline of the paper. This paper is structured as follows:

- In §2 we discuss a simple reductions to a class of model curves.
- In §3 we derive necessary conditions for L^p-Sobolev improving inequalities for our averaging operators. In particular, we establish Proposition 1.2.
- In §§4–6 we present the proof of Theorem 1.1.
- In §7 we discuss certain decoupling inequalities used in the proof of Theorem 1.1.
- There are three appendices which deal with various auxiliary results and technical lemmas used in the main argument.

Notational conventions. Given a (possibly empty) list of objects L, for real numbers $A_p, B_p \geqslant 0$ depending on some Lebesgue exponent p or dimension parameter n the notation $A_p \lesssim_L B_p$, $A_p = O_L(B_p)$ or $B_p \gtrsim_L A_p$ signifies that $A_p \leqslant C B_p$ for some constant $C = C_{L,p,n} \geqslant 0$ depending on the objects in the list, p and n. In addition, $A_p \sim_L B_p$ is used to signify that both $A_p \lesssim_L B_p$ and $A_p \gtrsim_L B_p$ hold. Given $a, b \in \mathbb{R}$ we write $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. The length of a multiindex $\alpha \in \mathbb{N}_0^n$ is given by $|\alpha| = \sum_{i=1}^n \alpha_i$.

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2. Reduction to perturbations of the moment curve

A prototypical example of a smooth curve satisfying the non-degeneracy condition (1.2) is the *moment curve* $\gamma_\circ\colon \mathbb{R} \to \mathbb{R}^n$, given by

$$\gamma_\circ(s) := \Big(s, \frac{s^2}{2}, \dots, \frac{s^n}{n!}\Big).$$

Indeed, in this case the determinant appearing in (1.2) is everywhere equal to 1. Moreover, at small scales, any non-degenerate curve can be thought of as a perturbation of an affine image of $\gamma_\circ$. To see why this is so, fix a non-degenerate curve $\gamma\colon I \to \mathbb{R}^n$ and $\sigma \in I$, $\lambda > 0$ such that $[\sigma - \lambda, \sigma + \lambda] \subseteq I$. Denote by $[\gamma]_\sigma$ the $n \times n$ matrix

$$[\gamma]_\sigma := \begin{bmatrix} \gamma^{(1)}(\sigma) & \cdots & \gamma^{(n)}(\sigma) \end{bmatrix},$$

where the vectors $\gamma^{(j)}(\sigma)$ are understood to be *column* vectors. Note that this is precisely the matrix appearing in the definition of the non-degeneracy condition (1.2) and is therefore invertible by our hypothesis. It is also convenient to let $[\gamma]_{\sigma,\lambda}$ denote the $n \times n$ matrix

$$[\gamma]_{\sigma,\lambda} := [\gamma]_\sigma \cdot D_\lambda, \tag{2.1}$$

where $D_\lambda := \mathrm{diag}(\lambda, \dots, \lambda^n)$, the diagonal matrix with eigenvalues λ, $\lambda^2, \dots, \lambda^n$. Consider the portion of the curve γ lying over the subinterval $[\sigma - \lambda, \sigma + \lambda]$. This is parametrised by the map $s \mapsto \gamma(\sigma + \lambda s)$ for $s \in [-1, 1]$. The degree n Taylor polynomial of $s \mapsto \gamma(\sigma + \lambda s)$ around σ is given by

$$s \mapsto \gamma(\sigma) + [\gamma]_{\sigma,\lambda} \cdot \gamma_\circ(s), \tag{2.2}$$

which is indeed an affine image of $\gamma_\circ$. Furthermore, by Taylor's theorem, the original curve γ agrees with the polynomial curve (2.2) to high order at σ.

Inverting the affine transformation $x \mapsto \gamma(\sigma) + [\gamma]_{\sigma,\lambda} \cdot x$ from (2.2), we can map the portion of γ over $[\sigma - \lambda, \sigma + \lambda]$ to a small perturbation of the moment curve.

Definition 2.1. *Let $\gamma \in C^{n+1}(I; \mathbb{R}^n)$ be a non-degenerate curve and $\sigma \in I, \lambda > 0$ be such that $[\sigma - \lambda, \sigma + \lambda] \subseteq I$. The (σ, λ)-rescaling of γ is the curve $\gamma_{\sigma,\lambda} \in C^{n+1}([-1, 1]; \mathbb{R}^n)$ given by*

$$\gamma_{\sigma,\lambda}(s) := [\gamma]_{\sigma,\lambda}^{-1}\big(\gamma(\sigma + \lambda s) - \gamma(\sigma)\big).$$

It follows from the preceding discussion that

$$\gamma_{\sigma,\lambda}(s) = \gamma_\circ(s) + [\gamma]_{\sigma,\lambda}^{-1}\mathcal{E}_{\gamma,\sigma,\lambda}(s)$$

where $\mathcal{E}_{\gamma,\sigma,\lambda}$ is the remainder term for the Taylor expansion (2.2). In particular, if γ satisfies the non-degeneracy condition (1.2) with constant c_0, then

$$\|\gamma_{\sigma,\lambda} - \gamma_\circ\|_{C^{n+1}([-1,1];\mathbb{R}^n)} \lesssim c_0^{-1}\lambda \,\|\gamma\|^n_{C^{n+1}(I)}.$$

Thus, if $\lambda > 0$ is chosen to be small enough, then the rescaled curve $\gamma_{\sigma,\lambda}$ is a minor perturbation of the moment curve. In particular, given any $0 < \delta < 1$, we can choose λ so as to ensure that $\gamma_{\sigma,\lambda}$ belongs to the following class of *model curves*.

Definition 2.2. *Given $n \geqslant 2$ and $0 < \delta < 1$, let $\mathfrak{G}_n(\delta)$ denote the class of all smooth curves $\gamma\colon [-1, 1] \to \mathbb{R}^n$ that satisfy the following conditions:*

i) $\gamma(0) = 0$ *and* $\gamma^{(j)}(0) = \vec{e}_j$ *for* $1 \leqslant j \leqslant n$;

ii) $\|\gamma - \gamma_\circ\|_{C^{n+1}([-1,1])} \leqslant \delta$.

Here $\vec{e}_j$ denotes the jth standard Euclidean basis vector and

$$\|\gamma\|_{C^{n+1}(I)} := \max_{1\leqslant j\leqslant n+1} \sup_{s\in I} |\gamma^{(j)}(s)| \qquad \textit{for all } \gamma \in C^{n+1}(I;\mathbb{R}^n).$$

Given any $\gamma \in \mathfrak{G}_n(\delta)$, condition ii) and the multilinearity of the determinant ensures that $\det[\gamma]_s = \det[\gamma_\circ]_s + O(\delta) = 1 + O(\delta)$. Thus, there exists a dimensional constant $c_n > 0$ such that if $0 < \delta < c_n$, then any curve $\gamma \in \mathfrak{G}_n(\delta)$ is non-degenerate and, moreover, satisfies $\det[\gamma]_s \geqslant 1/2$. Henceforth, it is always assumed that $\delta > 0$ satisfies this condition, which we express succinctly as $0 < \delta \ll 1$.

Turning back to the Sobolev improving problem for the averages A_γ, the above observations facilitate a reduction to the class of model curves. To precisely describe this reduction, it is useful to make the choice of cutoff function explicit in the notation by writing $A[\gamma,\chi]$ for the operator A_γ as defined in (1.1).

Proposition 2.3. *Let $\gamma\colon I \to \mathbb{R}^n$ be a non-degenerate curve, $\chi \in C^\infty_c(\mathbb{R})$ be supported on the interior of I and $0 < \delta \ll 1$. There exists some $\gamma^* \in \mathfrak{G}_n(\delta)$ and $\chi^* \in C^\infty_c(\mathbb{R})$ such that*

$$\|A[\gamma,\chi]\|_{L^p(\mathbb{R}^n)\to L^p_\alpha(\mathbb{R}^n)} \sim_{\gamma,\chi,\delta,p,\alpha} \|A[\gamma^*,\chi^*]\|_{L^p(\mathbb{R}^n)\to L^p_\alpha(\mathbb{R}^n)}$$

for all $1 \leqslant p < \infty$ and $0 \leqslant \alpha \leqslant 1$. Furthermore, χ^* may be chosen to satisfy $\operatorname{supp}\chi^* \subseteq [-\delta,\delta]$.

Proof. The proof follows by decomposing the domain of γ into small intervals and applying the rescaling described in Definition 2.1 on each interval. This decomposition in s induces a decomposition of the derived operator $(1-\Delta)^{\alpha/2}A[\gamma,\chi]$. The upper bound then follows from the triangle inequality and the stability of the estimates under affine transformation (together with a simple pigeonholing argument).

The proof of the lower bound is more subtle since one must take into account possible cancellation between the different pieces of the decomposition. To get around this, we observe that

$$\|A[\gamma,\chi_0]\|_{L^p(\mathbb{R}^n)\to L^p_\alpha(\mathbb{R}^n)} \lesssim_{\gamma,\chi_0} \|A[\gamma,\chi_1]\|_{L^p(\mathbb{R}^n)\to L^p_\alpha(\mathbb{R}^n)} \tag{2.3}$$

holds whenever χ_0, $\chi_1 \in C^\infty_c(\mathbb{R})$ are supported in I and $\chi_1(s) = 1$ for all $s \in \operatorname{supp}\chi_0$. Once this is established, it is possible to localise in s and rescale to deduce the desired bound.

To prove (2.3) note, after possibly applying a translation and a dilation, one may write

$$A[\gamma,\chi_0]f(x) = \int_{\mathbb{R}} f(x-\gamma(s))\tilde{\chi}_0\circ\gamma(s)\chi_1(s)\,\mathrm{d}s$$

where the function $\tilde{\chi}_0 \in C^\infty_c(\mathbb{R}^n)$ is supported in $[-\pi,\pi]^n$. Consequently, by performing a Fourier series decomposition,

$$\tilde{\chi}_0\circ\gamma(s) = \frac{1}{(2\pi)^n}\sum_{k\in\mathbb{Z}^n} a_k e^{i\langle x,k\rangle}e^{-i\langle x-\gamma(s),k\rangle}$$

where the sequence $(a_k)_{k\in\mathbb{Z}^n}$ of Fourier coefficients is rapidly decaying. Thus, if Mod_k denotes the modulation operator $\mathrm{Mod}_k\, g(x) := e^{i\langle x,k\rangle}g(x)$, then

$$A[\gamma,\chi_0]f(x) = \frac{1}{(2\pi)^n}\sum_{k\in\mathbb{Z}^n} a_k\cdot\mathrm{Mod}_k\circ A[\gamma,\chi_1]\circ\mathrm{Mod}_{-k} f(x).$$

By analytic interpolation, it follows that

$$\|\mathrm{Mod}_k\|_{L^p_\alpha(\mathbb{R}^n)\to L^p_\alpha(\mathbb{R}^n)} \lesssim (1+|k|)^\alpha \qquad \text{for all } 0\leqslant\alpha\leqslant 1$$

and therefore

$$\begin{aligned}\|A[\gamma,\chi_0]f\|_{L^p_\alpha(\mathbb{R}^n)} &\lesssim \sum_{k\in\mathbb{Z}^n}|a_k|(1+|k|)^\alpha\cdot\|A[\gamma,\chi_1]\circ\mathrm{Mod}_{-k} f\|_{L^p_\alpha(\mathbb{R}^n)}\\ &\lesssim_{\gamma,\chi_0} \|A[\gamma,\chi_1]\|_{L^p(\mathbb{R}^n)\to L^p_\alpha(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

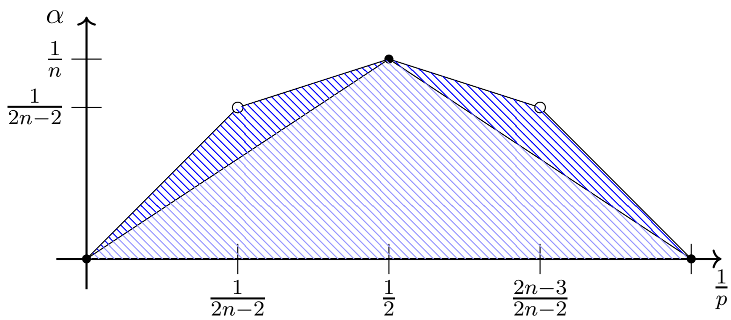

FIGURE 1. Conjectured range of $A_\gamma \colon L^p(\mathbb{R}^n) \to L^p_\alpha(\mathbb{R}^n)$ boundedness for γ non-degenerate. The inner triangle follows from the elementary L^2 estimate. The goal is to establish the $L^p(\mathbb{R}^n) \to L^p_{1/p}(\mathbb{R}^n)$ bound at the 'kink' point $p_{\mathrm{cr}} = 2n-2$ (or, equivalently, $p'_{\mathrm{cr}} = \frac{2n-3}{2n-2}$).

using the rapid decay of the Fourier coefficients. □

As a consequence of Proposition 2.3, it suffices to fix $\delta_0 > 0$ and prove Theorem 1.1 and Proposition 1.2 in the special case where $\gamma \in \mathfrak{G}_4(\delta_0)$ and $\operatorname{supp} \chi \subseteq I_0 := [-\delta_0, \delta_0]$. Thus, henceforth, we work with some fixed δ_0, chosen to satisfy the forthcoming requirements of the proofs. For the sake of concreteness, the choice of $\delta_0 := 10^{-10^5}$ is more than enough for our purposes.

3. Necessary conditions

3.1. **General $L^p \to L^p_\alpha$ estimates.** If $\gamma \colon I \to \mathbb{R}^n$ is of maximal type d, then the van der Corput lemma shows that the Fourier transform of any smooth density μ_γ on γ satisfies

$$|\hat{\mu}_\gamma(\xi)| \lesssim_\gamma (1 + |\xi|)^{-1/d}. \tag{3.1}$$

This readily implies that

$$\|A_\gamma f\|_{L^2_{1/d}(\mathbb{R}^n)} \lesssim_\gamma \|f\|_{L^2(\mathbb{R}^n)}. \tag{3.2}$$

Consider the case where γ is non-degenerate, so that $d = n$. By interpolating against (3.2), Conjecture 1 formally implies that A_γ maps L^p to L^p_α for all $p \geqslant 2$ and

$$\alpha < \alpha_{\mathrm{cr}}(p) := \min\Big\{\frac{1}{n}\Big(\frac{1}{2} + \frac{1}{p}\Big), \frac{1}{p}\Big\}, \tag{3.3}$$

with the equality case also holding in the restricted range $p > 2n-2$. It is an interesting question what happens at the endpoint in the range $2 < p \leqslant 2n-2$.

The range of conjectured bounds is represented in Figure 1. The two constraints appearing in the definition of the critical regularity exponent $\alpha_{\mathrm{cr}}(p)$ agree precisely when p corresponds to the critical Lebesgue exponent

$$p_{\mathrm{cr}} := 2n-2,$$

which manifests as a 'kink' in the L^p-Sobolev diagram.

By a simple scaling argument (see, for instance, [17, pp.81-82]), Conjecture 1 further implies bounds for A_γ under a finite type hypothesis. In view of Corollary 1.3, it is reasonable to conjecture the following.

Conjecture 2. *If $\gamma \colon I \to \mathbb{R}^n$ is of maximal type d, then the operator A_γ maps L^p to L^p_α for all $p \geqslant 2$ and*

$$\alpha \leqslant \alpha_{\mathrm{cr}}(d;p) := \min\Big\{\alpha_{\mathrm{cr}}(p), \frac{1}{d}\Big\} \tag{3.4}$$

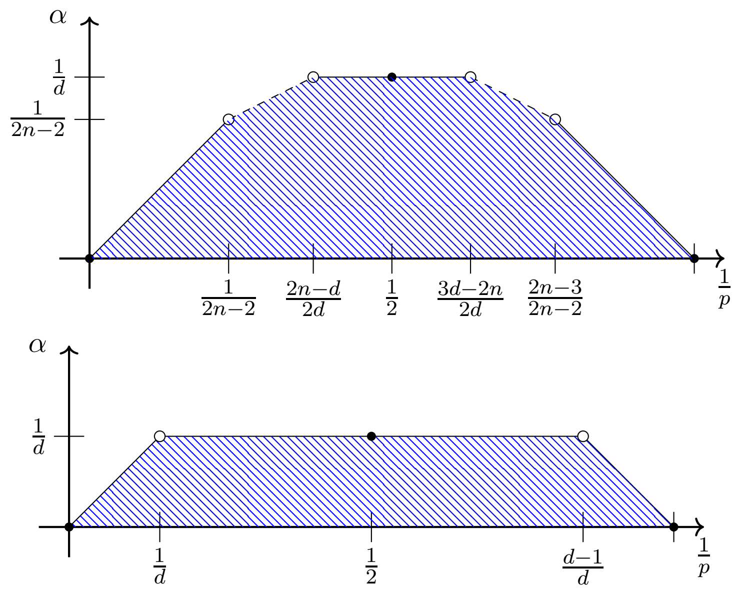

FIGURE 2. Conjectured range of $A_\gamma \colon L^p(\mathbb{R}^n) \to L^p_\alpha(\mathbb{R}^n)$ boundedness for γ of maximal type d. The upper diagram corresponds to $d < 2n-2$ whilst the lower diagram corresponds to $d \geqslant 2n-2$.

with strict inequality if $\min\{2n-2, d\} \leqslant p \leqslant \max\{2n-2, d\}$.

The range of conjectured bounds is represented in Figure 2.

Remark. Using the fact that $(I-\Delta)^{\alpha/2} \colon L^p_{\alpha+\beta}(\mathbb{R}^n) \to L^p_\beta(\mathbb{R}^n)$ is an isomorphism together with a duality argument, any $L^p \to L^p_\alpha$ estimate for A_γ immediately implies a corresponding $L^{p'} \to L^{p'}_\alpha$ estimate.

The condition $\alpha \leqslant 1/d$ is clearly necessary. Indeed, by duality and interpolation, any $L^p \to L^p_\alpha$ estimate implies an $L^2 \to L^2_\alpha$ estimate for the same value of α. However, a slight refinement of (3.1) shows that the L^2 estimate (3.2) is sharp in the sense that the regularity exponent on the left-hand side cannot be taken larger than $1/d$.

3.2. **Band-limited examples.** The remainder of this section discusses the necessity of the conditions (3.3) and (3.4). To begin, given $\lambda > 0$, consider the family of band-limited Schwartz functions

$$\mathscr{Z}_\lambda := \big\{ f \in \mathcal{S}(\mathbb{R}^n) : \operatorname{supp} \hat{f} \subset \{\xi \in \hat{\mathbb{R}}^n : \lambda/2 \leqslant |\xi| \leqslant 2\lambda\} \big\}.$$

By elementary Sobolev space theory, the desired necessary conditions are a consequence of the following proposition.

Proposition 3.1. *If* $\gamma \colon I \to \mathbb{R}^n$ *is a smooth curve satisfying the non-degeneracy hypothesis* (1.2) *and* $p \geqslant 2$*, then*

$$\sup\big\{ \|A_\gamma f\|_{L^p(\mathbb{R}^n)} : f \in L^p \cap \mathscr{Z}_\lambda, \quad \|f\|_{L^p(\mathbb{R}^n)} = 1 \big\} \gtrsim_{p,\gamma} \lambda^{-\alpha_{\mathrm{cr}}(p)}.$$

This directly implies Proposition 1.2 and, moreover, shows that the (p,α)-ranges in (3.3) and Conjecture 2 are optimal up to endpoints.[3]

[3] If γ is finite type curve, then the points s for which the type of γ at s is strictly larger than d are isolated. Consequently, any necessary condition for the non-degenerate problem is automatically a necessary condition for the finite type problem. The necessity of the additional constraint $\alpha \leqslant 1/d$ is discussed in the previous subsection.

Proposition 3.1 is based on testing the estimate against two examples, corresponding to the two constraints inherent in the minimum appearing in the definition of $\alpha_{\mathrm{crit}}(p)$.

3.3. **Dimensional constraint: $\alpha \leqslant 1/p$.** The condition $\alpha \leqslant 1/p$ is well-known and appears to be folkloric; in lieu of a precise reference, the details are given presently.

Lemma 3.2. *If $\gamma \colon I \to \mathbb{R}^n$ is a smooth curve and $p \geqslant 2$, then*

$$\sup\big\{\|A_\gamma f\|_{L^p(\mathbb{R}^n)} : f \in L^p(\mathbb{R}^n) \cap \mathcal{Z}_\lambda, \quad \|f\|_{L^p(\mathbb{R}^n)} = 1\big\} \gtrsim_{p,\gamma} \lambda^{-1/p}.$$

Proof. Since the operator A_γ is self-adjoint and commutes with frequency projections, given $1 \leqslant p \leqslant 2$ it suffices to show

$$\sup\big\{\|A_\gamma f\|_{L^p(\mathbb{R}^n)} : f \in L^p(\mathbb{R}^n) \cap \mathcal{Z}_\lambda, \quad \|f\|_{L^p(\mathbb{R}^n)} = 1\big\} \gtrsim_{p,\gamma} \lambda^{-1/p'}.$$

Fix $\beta \in C^\infty_c(\hat{\mathbb{R}}^n)$ a real-valued even function with (inverse) Fourier transform $\check{\beta}$ satisfying $\check{\beta}(0) = 1$ and

$$\operatorname{supp}\beta \subseteq \{\xi \in \hat{\mathbb{R}}^n : 1/2 \leqslant |\xi| \leqslant 2\}. \tag{3.5}$$

In addition, let $\psi \in C^\infty_c(\mathbb{R}^n)$ be non-zero, non-negative and supported in a ball centred at the origin of radius c, where $0 < c < 1$ is a sufficiently small constant (independent of λ) chosen to satisfy the requirements of the forthcoming argument. With these bump functions define

$$f := (\beta_{\lambda^{-1}})^{\vee} * \psi_\lambda$$

where $\beta_{\lambda^{-1}}(\xi) := \beta(\lambda^{-1}\xi)$ and $\psi_\lambda(x) := \psi(\lambda x)$. The condition (3.5) implies $f \in L^p(\mathbb{R}^n) \cap \mathcal{Z}_\lambda$, whilst direct calculation shows that

$$\|f\|_{L^p(\mathbb{R}^n)} \sim \lambda^{-n/p}. \tag{3.6}$$

By a simple computation, $A_\gamma f = K^\lambda * \psi_\lambda$ where

$$K^\lambda(x) := \lambda^n \int_{\mathbb{R}} \check{\beta}\big(\lambda(x - \gamma(s))\big)\, \chi(s)\mathrm{d}s.$$

The key claim is that, provided $0 < c < 1$ is chosen sufficiently small (independently of λ), the pointwise inequality

$$K^\lambda * \psi_\lambda(x) \gtrsim \lambda^{-1} \qquad \text{for all } x \in \mathcal{N}_{c\lambda^{-1}}(\gamma) \tag{3.7}$$

holds, where $\mathcal{N}_{c\lambda^{-1}}(\gamma)$ denotes the $c\lambda^{-1}$-neighbourhood of the curve

$$\{\gamma(s) : s \in \operatorname{supp}\chi\}.$$

To see this, choose c sufficiently small so that $\check{\beta}$ is bounded away from zero on a ball of radius $10c$ centred at the origin. If $x \in \mathcal{N}_{c\lambda^{-1}}(\gamma)$, then there exists some $s_0 \in \operatorname{supp}\chi$ such that

$$|x - y - \gamma(s)| < 10c\lambda^{-1} \qquad \text{whenever } |s - s_0| \lesssim_\gamma \lambda^{-1} \text{ and } |y| \leqslant c\lambda^{-1},$$

from which (3.7) follows.

Combining (3.6) and (3.7), one concludes that

$$\sup_{f \in L^p(\mathbb{R}^n)\cap\mathcal{Z}_\lambda} \frac{\|A_\gamma f\|_{L^p(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}} \gtrsim \frac{\lambda^{-1}\lambda^{-(n-1)/p}}{\lambda^{-n/p}} = \lambda^{-1/p'},$$

as desired. ☐

Remark. More generally, suppose A_Σ is an averaging operator defined as in (1.1) but now with respect to Σ a (regular parametrisation of a) surface in $\mathbb{R}^n$ of arbitrary dimension. Then

$$\sup\big\{\|A_\Sigma f\|_{L^p(\mathbb{R}^n)} : f \in L^p(\mathbb{R}^n) \cap \mathcal{Z}_\lambda, \quad \|f\|_{L^p(\mathbb{R}^n)} = 1\big\} \gtrsim_{p,\Sigma} \lambda^{-\dim\Sigma/p}.$$

This general necessary condition follows from the proof of Lemma 3.2 *mutatis mutandis*. Further generalisations hold for appropriate classes of variable coefficient averaging operators: see, for instance, [3].

3.4. **Fourier decay constraint:** $\alpha \leqslant \frac{1}{n}\big(\frac{1}{2}+\frac{1}{p}\big)$. Establishing the second condition is more involved. Here, in contrast with Lemma 3.2, the non-degeneracy hypothesis (1.2) plays a rôle via certain refinements of the Fourier decay estimate (3.1).

Recall the desired bound.

Proposition 3.3. *If $\gamma\colon I \to \mathbb{R}^n$ is a smooth curve satisfying the non-degeneracy hypothesis (1.2) and $p \geqslant 2$, then*

$$\sup\big\{\|A_\gamma f\|_{L^p(\mathbb{R}^n)} : f \in L^p(\mathbb{R}^n) \cap \mathscr{Z}_\lambda, \quad \|f\|_{L^p(\mathbb{R}^n)} = 1\big\} \gtrsim_{p,\gamma} \lambda^{-\frac{1}{n}(\frac{1}{2}+\frac{1}{p})}.$$

This conclusion was shown in three dimensions by Oberlin and Smith [14] for the model example of the helix in $\mathbb{R}^3$, $t \mapsto (\cos t, \sin t, t)$, by using DeLeeuw's restriction theorem and an analysis of a Bessel multiplier in $\mathbb{R}^2$. Here the more general statement in Proposition 3.3 is shown by combining a sharp example of Wolff [22] for ℓ^p-decoupling inequalities with a stationary phase analysis of the Fourier multiplier $\hat{\mu}_\gamma$.

The proof of Proposition 3.3 is broken into stages.

The worst decay cone. At any given large scale, the decay estimate (3.1) is only sharp for ξ belonging to a narrow region around a low-dimensional cone in the frequency space. To prove Proposition 3.3, it is natural to test the L^p-Sobolev estimate against functions which are Fourier supported in a neighbourhood of this 'worst decay cone'.

By Proposition 2.3 we may assume without loss of generality that $\gamma \in \mathfrak{G}_n(\delta_0)$ for some small $0 < \delta_0 \ll 1$ and that the cutoff χ in the definition of A_γ is supported in $I_0 = [-\delta_0, \delta_0]$. In view of the van der Corput lemma, the worst decay cone should correspond to the ξ for which the derivatives $\langle\gamma^{(j)}(s), \xi\rangle$, $1 \leqslant j \leqslant n-1$, all simultaneously vanish for some $s \in I_0$. In order to describe this region, first note that

$$\langle\gamma^{(n-1)}(s_0), \xi_0\rangle = 0 \quad \text{and} \quad \frac{\partial}{\partial s}\langle\gamma^{(n-1)}(s), \xi\rangle\Big|_{\substack{s=s_0\\ \xi=\xi_0}} = 1$$

for $(s_0, \xi_0) = (0, \vec{e}_n)$, by the reduction $\gamma^{(j)}(0) = \vec{e}_j$ for $1 \leqslant j \leqslant n$. Consequently, provided the support of χ is chosen sufficiently small, by the implicit function theorem and homogeneity there exists a constant $c > 0$ and a smooth mapping

$$\theta\colon \Xi \to I_0, \qquad \text{where} \quad \Xi := \big\{\xi = (\xi', \xi_n) \in \hat{\mathbb{R}}^n \backslash \{0\} : |\xi'| \leqslant c|\xi_n|\big\},$$

such that $s = \theta(\xi)$ is the unique solution in I to the equation $\langle\gamma^{(n-1)}(s), \xi\rangle = 0$ whenever $\xi \in \Xi$. Note that θ is homogeneous of degree one.

Further consider the system of n equations in $n+1$ variables given by

$$\begin{cases} \langle\gamma^{(j)}(s), \xi\rangle = 0 & \text{for } 1 \leqslant j \leqslant n-1, \\ \xi_n = 1. \end{cases} \tag{3.8}$$

Again, by the reduction $\gamma^{(j)}(0) = \vec{e}_j$ for $1 \leqslant j \leqslant n$, this can be solved for suitably localised ξ using the implicit function theorem, expressing s, ξ_1, ... ξ_{n-2} as functions of ξ_{n-1}. Thus (3.8) holds if and only if

$$\begin{aligned} \xi_i &= \Gamma_i(\xi_{n-1}), \qquad 1 \leqslant i \leqslant n-2, \\ s &= \theta(\Gamma_1(\xi_{n-1}), \dots, \Gamma_{n-2}(\xi_{n-1}), \xi_{n-1}, 1), \end{aligned} \tag{3.9a}$$

for some smooth functions Γ_i, $i = 1, \dots, n-2$ satisfying $\Gamma_i(0) = 0$. On I we form the $\mathbb{R}^n$-valued function $\tau \mapsto \Gamma(\tau)$ with the first $n-2$ components as defined in (3.9a) and

$$\Gamma_{n-1}(\tau) := \tau, \qquad \Gamma_n(\tau) := 1. \tag{3.9b}$$

With this definition, the formulæ in (3.9a) can be succinctly expressed as

$$\xi = \Gamma(\xi_{n-1}), \qquad s = \theta \circ \Gamma(\xi_{n-1}).$$

Moreover, the 'worst decay cone' can then be defined as the cone generated by the curve Γ, given by

$$\mathcal{C} := \big\{\lambda\Gamma(\tau) : \lambda > 0 \text{ and } \tau \in I\big\}.$$

Remark. For the model case $\gamma(s) = \sum_{i=1}^{n} \frac{s^i}{i!}\vec{e}_i$ one may explicitly compute that

$$\Gamma(\tau) = \sum_{i=1}^{n} \frac{(-\tau)^{n-i}}{(n-i)!}\vec{e}_i.$$

The Wolff example revisited. In analogy with the example in [22], here we consider functions with Fourier support on a union of balls with centres lying on the worst decay cone $\mathcal{C}$. To this end, let $\varepsilon > 0$ be a small dimensional constant, chosen to satisfy the forthcoming requirements of the argument, and

$$\mathfrak{N}_\varepsilon(\lambda) := \mathbb{Z} \cap \{s \in \mathbb{R} : |s| \leqslant \varepsilon\lambda^{1/n}\}.$$

The centres of the aforementioned balls are then given by

$$\xi^\nu := \lambda\Gamma(\nu\lambda^{-1/n}) \qquad \text{for all } \nu \in \mathfrak{N}_\varepsilon(\lambda). \tag{3.10}$$

Fix $\eta \in C_c^\infty(\hat{\mathbb{R}}^n)$ satisfying $\eta(\xi) = 1$ if $|\xi| \leqslant 1/2$ and $\eta(\xi) = 0$ if $|\xi| \geqslant 1$. Let $0 < \rho < 1$ be another dimensional constant, again chosen small enough to satisfy the forthcoming requirements of the argument, and define Schwartz functions g_ν for $\nu \in \mathfrak{N}_\varepsilon(\lambda)$ via the Fourier transform by

$$\hat{g}_\nu(\xi) := \eta\big(\lambda^{-1/n}\rho^{-1}(\xi - \xi^\nu)\big). \tag{3.11}$$

We consider randomised sums of the functions (3.11). In particular, set

$$g^\omega(x) := \sum_{\nu \in \mathfrak{N}_\varepsilon(\lambda)} r_\nu(\omega) g_\nu(x) \qquad \text{for } \omega \in [0,1], \tag{3.12}$$

where $\{r_\nu\}_{\nu=1}^\infty$ is the sequence of Rademacher functions. We claim

$$\Big(\int_0^1 \|g^\omega\|_{L^p(\mathbb{R}^n)}^p \,\mathrm{d}\omega\Big)^{1/p} \sim \lambda^{1-\frac{1}{p}+\frac{1}{2n}}. \tag{3.13}$$

To prove this we apply Fubini's theorem and Khinchine's inequality (see, for instance, [19, Appendix D]) to see that the left hand side is (3.13) is equal to

$$\Big\|\Big(\int_0^1 |g^\omega|^p \,\mathrm{d}\omega\Big)^{1/p}\Big\|_{L^p(\mathbb{R}^n)} \sim \Big\|\Big(\sum_{\nu \in \mathfrak{N}_\varepsilon(\lambda)} |g_\nu|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}.$$

The right-hand side of the last display is equal to

$$\Big\|\Big(\sum_{\nu \in \mathfrak{N}_\varepsilon(\lambda)} |\lambda\rho^n\check{\eta}(\lambda^{1/n}\rho\,\cdot\,)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} = \big[\#\mathfrak{N}_\varepsilon(\lambda)\big]^{1/2}\|\lambda\rho^n\check{\eta}(\lambda^{1/n}\rho\,\cdot\,)\|_{L^p(\mathbb{R}^n)} \sim \lambda^{\frac{1}{2n}+1-\frac{1}{p}},$$

which yields (3.13). The above estimates depend on ρ , but since this parameter is chosen to be a dimensional constant (independently of λ) this dependence is suppressed. Also note that so far the argument is independent of the choice of the ξ_ν.

Asymptotics. The next step is to study the behaviour of the multiplier $\hat{\mu}_\gamma$ near the support of the $\hat{g}_\nu$. The key result is Lemma 3.4 below, which relies on the asymptotics of $\hat{\mu}_\gamma$ near the worst decay cone and the observation that the functions $\hat{g}_\nu$ with the choice of ξ^ν as in (3.10) are supported near that cone.

Set $\hat{g}_{+,\nu}(\xi) := \eta_+(\lambda^{-1/n}\rho^{-1}(\xi - \xi^\nu))$ where $\eta_+ \in C_c^\infty(\widehat{\mathbb{R}}^n)$ is such that $\eta_+(\xi) = 1$ for $|\xi| \leqslant 1$ and $\eta_+(\xi) = 0$ for $|\xi| > 3/2$, so that $\hat{g}_\nu = \hat{g}_{+,\nu} \cdot \hat{g}_\nu$. Let

$$\phi(\xi) := \langle \gamma \circ \theta(\xi), \xi \rangle. \tag{3.14}$$

Lemma 3.4. *If ε, $\rho > 0$ are chosen sufficiently small, then for all $\lambda \geqslant 1$ and $\nu \in \mathfrak{N}_\varepsilon(\lambda)$ the identity*

$$\hat{\mu}_\gamma(\xi) = e^{-i\phi(\xi)} m(\xi)$$

holds on $\operatorname{supp} \hat{g}_{+,\nu}$ *where*

i) $|m(\xi)| \gtrsim \lambda^{-1/n}$ *for* $\xi \in \operatorname{supp} \hat{g}_{+,\nu}$*;*

ii) The function $a_\nu := m^{-1} \cdot \hat{g}_{+,\nu}$ *satisfies*

$$|\partial_\xi^\alpha a_\nu(\xi)| \leqslant C_\alpha \lambda^{(1-|\alpha|)/n} \qquad \text{for all } \alpha \in \mathbb{N}_0^n.$$

The proof, which is based on the stationary phase method and, in particular, oscillatory integral estimates from [6], is postponed until §3.5 below.

Lower bounds for the operator norm. For each $\nu \in \mathfrak{N}_\varepsilon(\lambda)$ define f_ν by

$$\hat{f}_\nu(\xi) := \frac{\hat{g}_\nu(\xi)}{\hat{\mu}_\gamma(\xi)}$$

and consider the randomised sums

$$f^\omega(x) := \sum_{\nu \in \mathfrak{N}_\varepsilon(\lambda)} r_\nu(\omega) f_\nu(x) \qquad \text{for } \omega \in [0,1].$$

Note that, by Lemma 3.4, the f^ω are well-defined smooth function with compact support (and with bounds depending on λ). Furthermore, if g^ω is the function defined in (3.12), then

$$g^\omega = A_\gamma f^\omega. \tag{3.15}$$

We proceed to estimate the L^p norm of f^ω, uniformly in ω.

We have $f_\nu = K_\nu * g_\nu$ where the kernel K_ν is given by

$$K_\nu(x) := \frac{1}{(2\pi)^n} \int_{\hat{\mathbb{R}}^n} e^{i\langle x, \xi\rangle} \hat{\mu}_\gamma(\xi)^{-1} \hat{g}_{+,\nu}(\xi)\, \mathrm{d}\xi.$$

By Lemma 3.4 one may write $\hat{\mu}_\gamma(\xi)^{-1}\hat{g}_{+,\nu}(\xi) = e^{i\phi(\xi)} a_\nu(\xi)$ where a_ν is supported where $|\xi - \xi^\nu| \leqslant 2\rho\lambda^{1/n}$ and satisfies $\partial_\xi^\alpha a_\nu(\xi) = O(\lambda^{(1-|\alpha|)/n})$. Setting

$$\begin{aligned} \mathcal{E}_\nu(\xi) &:= \phi(\xi) - \phi(\xi^\nu) - \langle \partial_\xi \phi(\xi^\nu), \xi - \xi^\nu \rangle, \\ x^\nu &:= -\partial_\xi \phi(\xi^\nu), \end{aligned} \tag{3.16}$$

it follows that

$$\hat{\mu}_\gamma(\xi)^{-1}\hat{g}_{+,\nu}(\xi) = e^{i\phi(\xi^\nu)} e^{-i\langle x^\nu, \xi - \xi^\nu\rangle} e^{i\mathcal{E}_\nu(\xi)} a_\nu(\xi).$$

Applying a change of variable,

$$K_\nu(x) = e^{i(\langle x, \xi^\nu\rangle + \phi(\xi^\nu))} \frac{\lambda}{(2\pi)^n} \int_{\hat{\mathbb{R}}^n} e^{i\langle \lambda^{1/n}(x - x^\nu), \xi\rangle} e^{i\mathcal{E}_\nu(\xi^\nu + \lambda^{1/n}\xi)} a_\nu(\xi^\nu + \lambda^{1/n}\xi)\, \mathrm{d}\xi.$$

By the homogeneity of ϕ, it follows that $|\partial_\xi^\alpha e^{i\mathcal{E}_\nu(\xi^\nu+\lambda^{1/n}\xi)}| \lesssim_\alpha 1$ for all multiindices $\alpha \in \mathbb{N}_0^n$. On the other hand, Lemma 3.4 implies that

$$\int_{\hat{\mathbb{R}}^n} |\partial_\xi^\alpha a_\nu(\xi^\nu + \lambda^{1/n}\xi)|\,\mathrm{d}\xi \lesssim_\alpha \lambda^{1/n} \qquad \text{for all } \alpha \in \mathbb{N}_0^n.$$

Repeated integration-by-parts therefore yields

$$|K_\nu(x)| \lesssim_N \lambda^{(n+1)/n}(1 + \lambda^{1/n}|x - x^\nu|)^{-N} \qquad \text{for all } N \in \mathbb{N}_0^n.$$

Consequently, the pointwise inequality

$$|f_\nu(x)| \lesssim \lambda^{(n+1)/n}(1 + \lambda^{1/n}|x - x^\nu|)^{-N} \lesssim \lambda^{(n+1)/n} \sum_{\ell \geqslant 0} 2^{-\ell N} \mathbb{1}_{B^\nu_{\ell,\lambda}}(x)$$

holds with $B^\nu_{\ell,\lambda} := \{x \in \mathbb{R}^n : |x - x^\nu| \leqslant 2^\ell \lambda^{-1/n}\}$. Hence,

$$\|f^\omega\|_{L^p(\mathbb{R}^n)} \lesssim \lambda^{(n+1)/n} \sum_{\ell \geqslant 0} 2^{-\ell N} \Big\| \sum_{\nu \in \mathfrak{N}_\varepsilon(\lambda)} \mathbb{1}_{B^\nu_{\ell,\lambda}} \Big\|_{L^p(\mathbb{R}^n)} \qquad \text{for all } \omega \in [0,1]. \tag{3.17}$$

To estimate the terms of (3.17) for $2^\ell > \varepsilon\lambda^{1/n}$ use the immediate bound

$$\Big\| \sum_{\nu \in \mathfrak{N}_\varepsilon(\lambda)} \mathbb{1}_{B^\nu_{\ell,\lambda}} \Big\|_{L^p(\mathbb{R}^n)} \lesssim \#\mathfrak{N}_\varepsilon(\lambda) \cdot 2^{\ell n/p}\lambda^{-1/p} \lesssim 2^{\ell n/p}\lambda^{1/n - 1/p}.$$

For $2^\ell \leqslant \varepsilon\lambda^{1/n}$ this may be improved upon using a separation property of the x^ν: namely,

$$|x^\nu - x^{\nu'}| \gtrsim \lambda^{-1/n}|\nu - \nu'|, \tag{3.18}$$

provided the parameter $\varepsilon > 0$ is chosen sufficiently small (independently of λ). The property (3.18) implies that the balls $B^\nu_{\ell,\lambda}$, $B^{\nu'}_{\ell,\lambda}$ are disjoint for $|\nu - \nu'| \gtrsim 2^\ell$. Assuming (3.18) for a moment and taking $2^\ell \leqslant \varepsilon\lambda^{1/n}$, we obtain

$$\begin{aligned}
\Big\| \sum_{\nu \in \mathfrak{N}_\varepsilon(\lambda)} \mathbb{1}_{B^\nu_{\ell,\lambda}} \Big\|_{L^p(\mathbb{R}^n)} &\leqslant \sum_{i=0}^{2^\ell - 1} \Big\| \sum_{\substack{m \in \mathbb{Z} \\ |m| \leqslant \varepsilon 2^{-\ell}\lambda^{1/n}}} \mathbb{1}_{B^{2^\ell m + i}_{\ell,\lambda}} \Big\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \sum_{i=0}^{2^\ell - 1} \Big(\sum_{\substack{m \in \mathbb{Z} \\ |m| \leqslant \varepsilon 2^{-\ell}\lambda^{1/n}}} \|\mathbb{1}_{B^{2^\ell m + i}_{\ell,\lambda}}\|^p_{L^p(\mathbb{R}^n)} \Big)^{1/p} \\
&\lesssim 2^\ell (\lambda^{1/n}2^{-\ell})^{1/p}(2^\ell\lambda^{-1/n})^{n/p}.
\end{aligned}$$

Applying the preceding bounds to estimate the terms in (3.17) and choosing $N > n - n/p$, this leads to the uniform estimate

$$\sup_{\omega \in [0,1]} \|f^\omega\|_{L^p(\mathbb{R}^n)} \lesssim \lambda^{(n+1)/n - (n-1)/np}. \tag{3.19}$$

Thus, one concludes from (3.15), (3.13) and (3.19), together with the fact that the f^ω are Fourier supported where $|\xi| \sim \lambda$, that

$$\sup_{f \in L^p \cap \mathcal{Z}_\lambda} \frac{\|A_\gamma f\|_{L^p(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}} \geqslant \frac{\big(\int_0^1 \|A_\gamma f^\omega\|^p_{L^p(\mathbb{R}^n)}\,\mathrm{d}\omega\big)^{1/p}}{\sup_{\omega \in [0,1]} \|f^\omega\|_{L^p(\mathbb{R}^n)}} = \frac{\big(\int_0^1 \|g^\omega\|^p_{L^p(\mathbb{R}^n)}\,\mathrm{d}\omega\big)^{1/p}}{\sup_{\omega \in [0,1]} \|f^\omega\|_{L^p(\mathbb{R}^n)}} \gtrsim \lambda^{-\frac{1}{n}(\frac{1}{2} + \frac{1}{p})},$$

which is the desired bound stated in Proposition 3.3.

It remains to verify the crucial separation property (3.18). Recall from (3.16) and (3.10) that $x^\nu = -\partial_\xi\phi\big(\lambda\Gamma(\nu\lambda^{-1/n})\big)$. Thus, by homogeneity, one wishes to bound

$$x^\nu - x^{\nu'} = -\big[(\partial_\xi\phi) \circ \Gamma(\nu\lambda^{-1/n}) - (\partial_\xi\phi) \circ \Gamma(\nu'\lambda^{-1/n})\big]. \tag{3.20}$$

In particular, it suffices to show that

$$\frac{d}{d\tau}(\partial_\xi \phi) \circ \Gamma(\tau) = -\vec{e}_1 + O(\tau). \tag{3.21}$$

Indeed, applying Taylor's theorem to (3.20) and using (3.21) to bound the linear term yields

$$x^\nu - x^{\nu'} = \frac{\nu - \nu'}{\lambda^{1/n}} \cdot \vec{e}_1 + O\Big(\tfrac{(|\nu|+|\nu'|)|\nu-\nu'|}{\lambda^{2/n}}\Big) = \frac{\nu - \nu'}{\lambda^{1/n}} \cdot \vec{e}_1 + O\Big(\varepsilon \tfrac{|\nu-\nu'|}{\lambda^{1/n}}\Big)$$

for all $\nu, \nu' \in \mathfrak{N}_\varepsilon(\lambda)$. Choosing $\varepsilon > 0$ sufficiently small so as to control the error term establishes (3.18).

Turning to the proof of (3.21), we have $\partial_\xi \phi(\xi) = \gamma \circ \theta(\xi) + \langle \gamma' \circ \theta(\xi), \xi \rangle \partial_\xi \theta(\xi)$ by the definition of ϕ from (3.14). Since $\langle \gamma' \circ \theta(\xi), \xi \rangle = 0$ when $\xi = \Gamma(\tau)$, this yields

$$(\partial_\xi \phi) \circ \Gamma(\tau) = \gamma \circ \theta(\Gamma(\tau))$$

and, consequently,

$$\frac{d}{d\tau}(\partial_\xi \phi) \circ \Gamma(\tau) = \gamma' \circ \theta(\Gamma(\tau)) \cdot \langle \partial_\xi \theta(\Gamma(\tau)), \Gamma'(\tau) \rangle.$$

By the initial reductions, $\gamma^{(j)}(0) = \vec{e}_j$ for $1 \leqslant j \leqslant n$, and so

$$\frac{d}{d\tau}(\partial_\xi \phi) \circ \Gamma(\tau) = \langle \partial_\xi \theta(\Gamma(0)), \Gamma'(0) \rangle \cdot \vec{e}_1 + O(\tau). \tag{3.22}$$

Thus, to prove (3.21) it suffices to show that the inner product in the above display is equal to -1. Differentiating the defining equation $\langle \gamma^{(n-1)} \circ \theta(\xi), \xi \rangle = 0$, one deduces that

$$\partial_\xi \theta(\xi) = -\frac{1}{\langle \gamma^{(n)} \circ \theta(\xi), \xi \rangle} \gamma^{(n-1)} \circ \theta(\xi).$$

Since, by uniqueness in (3.9a) and (3.9b) together with the initial reductions, $\Gamma(0) = \vec{e}_n$ and $\theta(\vec{e}_n) = 0$, it follows that $(\partial_\xi \theta) \circ \Gamma(0) = -\vec{e}_{n-1}$. On the other hand, from (3.9b) it is clear that $\langle \vec{e}_{n-1}, \Gamma'(0) \rangle = 1$. Applying these observations to the formula in (3.22) concludes the proof.

3.5. **Proof of Lemma 3.4.** It remains to prove Lemma 3.4. To this end, we recall an asymptotic expansion from [6], based on the following formula:

$$\int_{-\infty}^{\infty} e^{i\lambda s^n} \, ds = \alpha_n \lambda^{-1/n} \qquad \text{for } n = 2, 3, \dots \text{ and } \lambda > 0, \tag{3.23}$$

where α_n is given by

$$\alpha_n := \begin{cases} \frac{2}{n}\Gamma(\frac{1}{n}) \sin(\frac{(n-1)\pi}{2n}) & \text{if } n \text{ is odd,} \\ \frac{2}{n}\Gamma(\frac{1}{n}) \exp(i\frac{\pi}{2n}) & \text{if } n \text{ is even.} \end{cases} \tag{3.24}$$

The derivation of (3.23) relies on contour integration arguments, whilst the formula itself yields asymptotic expansions for integrals $\int_{\mathbb{R}} e^{i\lambda s^n} \chi(s) \, ds$ with $\chi \in C_0^\infty$: see, for instance, [20, VIII.1.3] or [13, §7.7]. Similar asymptotic expansions remain valid under slight perturbation of the phase function $s \mapsto s^n$, as demonstrated by the following lemma proved in [6]. We use the notation $\|g\|_{C^m(I)} := \max_{0 \leqslant j \leqslant m} \sup_{x \in I} |g^{(j)}(x)|$.

Lemma 3.5 ([6], Lemma 5.1)**.** *Let $0 < r \leqslant 1$, $I = [-r, r]$, $I^* = [-2r, 2r]$ and let $g \in C^2(I^*)$. Suppose that*

$$r \leqslant \frac{1}{10(1 + \|g\|_{C^2(I^*)})}$$

and let $\eta \in C_c^1(\mathbb{R})$ be supported in I and satisfy the bounds

$$\|\eta\|_\infty + \|\eta'\|_1 \leqslant A_0, \;\; \text{and} \;\; \|\eta'\|_\infty \leqslant A_1.$$

Let $n \geqslant 2$, define

$$I_\lambda(\eta, w) := \int_{\mathbb{R}} \eta(s) \exp\big(i\lambda(\sum_{j=1}^{n-2} w_j s^j + s^n + g(s)s^{n+1})\big)\mathrm{d}s$$

and let α_n be as in (3.24). Suppose $|w_j| \leqslant \delta\lambda^{(j-n)/n}$, $j = 1, \dots, n-2$. Then there is an absolute constant C such that, for $\lambda > 2$,

$$|I_\lambda(\eta, w) - \eta(0)\alpha_n\lambda^{-1/n}| \leqslant C[A_0\delta\lambda^{-1/n} + A_1\lambda^{-2/n}(1 + \beta_n \log\lambda)];$$

here $\beta_2 := 1$ and $\beta_n := 0$ for $n > 2$.

Lemma 3.4 is obtained via a fairly direct application of the above result.

Proof (of Lemma 3.4). Taylor expand the phase $\langle\gamma(s), \xi\rangle$ with $s = \theta(\xi) + h$ to obtain

$$\langle\gamma(\theta(\xi) + h), \xi\rangle = \phi(\xi) + \sum_{j=1}^{n} u_j(\xi)\frac{h^j}{j!} + u_{n+1}(\xi, h)\frac{h^{n+1}}{(n+1)!}$$

where

$$u_j(\xi) := \langle\gamma^{(j)} \circ \theta(\xi), \xi\rangle, \text{ for } 1 \leqslant j \leqslant n,$$
$$u_{n+1}(\xi, h) := \int_0^1 (n+1)(1-t)^n\langle\gamma^{(n+1)}(\theta(\xi) + th), \xi\rangle\,\mathrm{d}t.$$

Recall that $u_{n-1}(\xi) \equiv 0$ by the definition of $\theta(\xi)$, whilst $u_n(\xi) \sim |\xi| \sim \lambda$ for $|\xi'| \leqslant c\xi_n$. Thus, writing

$$\hat{\mu}_\gamma(\xi) = e^{-i\phi(\xi)}m(\xi)$$

as in the statement of the lemma, it follows that the function m is given by

$$m(\xi) := \int_{\mathbb{R}} e^{-i(\sum_{j=1}^n u_j(\xi)\frac{h^j}{j!} + u_{n+1}(\xi,h)\frac{h^{n+1}}{(n+1)!})}\chi(\theta(\xi) + h)\mathrm{d}h.$$

Thus, defining

$$\Psi(\xi, h) := \sum_{j=1}^{n-2} w_j(\xi)h^j + h^n + g(\xi, h)h^{n+1} \tag{3.25}$$

where

$$w_j(\xi) := \frac{1}{j!}\cdot\frac{u_j(\xi)}{u_n(\xi)} \qquad \text{and} \qquad g(\xi, h) := \frac{1}{(n+1)!}\cdot\frac{u_{n+1}(\xi, h)}{u_n(\xi)},$$

one may succinctly express m as

$$m(\xi) = \int_{\mathbb{R}} e^{-iu_n(\xi)\Psi(\xi,h)}\chi(\theta(\xi) + h)\mathrm{d}h.$$

We now turn to proving the bounds on m stated in Lemma 3.4.

i) The desired pointwise lower bound on m follows from a direct application of Lemma 3.5. In particular, by the definition of the ξ^ν we have $w_j(\xi^\nu) = 0$ for $1 \leqslant j \leqslant n-1$ and therefore

$$|w_j(\xi)| \lesssim \rho\lambda^{(1-n)/n} \qquad \text{and} \qquad |g(\xi, h)| \lesssim 1 \qquad \text{for } \xi \in \operatorname{supp}\hat{g}_{\nu,+}. \tag{3.26}$$

Thus, provided ρ and ε are chosen small enough, Lemma 3.5 can be applied to show that

$$|m(\xi)| \gtrsim \lambda^{-1/n} \qquad \text{for all } \xi \in \operatorname{supp}\hat{g}_{+,\nu}, \tag{3.27}$$

as desired.

ii) It remains to show that

$$|\partial_\xi^\alpha[m^{-1}\cdot\hat{g}_{+,\nu}](\xi)| \lesssim_\alpha \lambda^{(1-|\alpha|)/n} \qquad \text{for all } \alpha \in \mathbb{N}_0^n. \tag{3.28}$$

The derivative $\partial_\xi^\alpha m(\xi)$ can be expressed as a sum of functions of the form

$$m^\alpha_{\kappa,d}(\xi) := \int_{\mathbb{R}} e^{-i(\sum_{j=1}^n u_j(\xi)\frac{h^j}{j!} + u_{n+1}(\xi,h)\frac{h^{n+1}}{(n+1)!})} h^\kappa b^\alpha_d(\xi,h)\,\mathrm{d}h, \quad d+\kappa \geqslant |\alpha|,$$

where $b^\alpha_d \in C^\infty(\hat{\mathbb{R}}^n\backslash\{0\}\times\mathbb{R})$ and homogeneous of degree $-d$ in the ξ-variable. The key claim is

$$|m^\alpha_{\kappa,d}(\xi)| \lesssim_\alpha \lambda^{-(1+\kappa)/n-d} \qquad \text{for } \xi \in \operatorname{supp}\hat{g}_{+,\nu}. \tag{3.29}$$

Indeed, once (3.29) is established it can be combined with (3.27) and the Leibniz rule the deduce the desired bound (3.28).

The asserted bound (3.29) follows from

$$\Big|\int_{\mathbb{R}} e^{-iu_n(\xi)\Psi(\xi,h)}\chi_1(\xi,h)h^\kappa \mathrm{d}h\Big| \lesssim \lambda^{-(1+\kappa)/n}, \tag{3.30}$$

where $\chi_1 \in C^\infty(\hat{\mathbb{R}}^n\backslash\{0\}\times\mathbb{R})$ is homogeneous of degree zero with respect to ξ and vanishes unless $|h| \lesssim \varepsilon$. To prove (3.30) we form a dyadic decomposition of the integral. Fix $\zeta_0 \in C_0^\infty(\mathbb{R})$ such that $\zeta_0(h) = 1$ for $|h| < 1/2$ with $\operatorname{supp}\zeta \subseteq [-1,1]$. For $\ell \in \mathbb{N}$ set $\zeta_\ell(h) = \zeta_0(2^{-\ell}h) - \zeta_0(2^{-\ell-1}h)$ and define

$$J_{\ell,\lambda}(\xi) := \int_{\mathbb{R}} e^{-iu_n(\xi)\Psi(\xi,h)}\zeta_\ell(\lambda^{1/n}h)\chi_1(\xi,h))h^\kappa\,\mathrm{d}h. \tag{3.31}$$

By just a size estimate we have $|J_{\ell,\lambda}(\xi)| \lesssim (2^\ell\lambda^{-1/n})^{1+\kappa}$, which we use for $\ell \leqslant C$. For larger ℓ we use integration-by-parts.

Recall that the assumption $\xi \in \operatorname{supp}\hat{g}_{+,\nu}$ implies the bounds (3.26). Consequently, on the support of the integrand in (3.31), the dominant term in the formula for Ψ as given in (3.25) is h^n. Moreover,

$$\Big|\frac{\partial}{\partial h}\Psi(\xi,h)\Big| \sim (2^\ell\lambda^{-1/n})^{n-1}$$

and, similarly,

$$\Big|\frac{\partial^i}{\partial h^i}\Psi(\xi,h)\Big| \lesssim \min\{(2^\ell\lambda^{-1/n})^{n-i}, 1\}.$$

Also, it is not difficult to show that

$$\Big|\frac{\partial^i}{\partial h^i}\big[\zeta_\ell(\lambda^{1/n}h)\chi_1(\xi,h))h^\kappa\big]\Big| \lesssim (2^\ell\lambda^{-1/n})^{\kappa-i}.$$

Using these bounds we derive, by N-fold integration-by-parts,

$$|J_{\ell,\lambda}(\xi)| \lesssim_N (2^\ell\lambda^{-1/n})^{1+\kappa}2^{-\ell nN}$$

and, by summing in ℓ, obtain (3.30). □

3.6. **The Christ example.** We close this section by making an observation regarding an endpoint case. We may rule out $L^d(\mathbb{R}^n) \to L^d_{1/d}(\mathbb{R}^n)$ boundedness under the maximal type d hypothesis for $d \geqslant 3$. Note that this corresponds to the critical vertices in the lower diagram in Figure 2. To show the failure of the estimate, suppose that for some t_0 with $\chi(t_0) \neq 0$, there is a unit vector u with $\langle u, \gamma^{(k)}(t_0)\rangle = 0$ for $k = 1,\dots,d-1$ and $\langle u, \gamma^{(d)}(t_0)\rangle \neq 0$. By a rotation we can assume that $\gamma'(t_0) = \vec{e}_1$ and $u = \vec{e}_2$, the standard coordinate vectors. The $L^d(\mathbb{R}^n) \to L^d_{1/d}(\mathbb{R}^n)$ boundedness is equivalent with the statement that the multiplier

$$|\xi|^{1/d}\upsilon(\xi)\int_{\mathbb{R}} e^{i\langle\gamma(t),\xi\rangle}\chi(t)\,\mathrm{d}t$$

belongs to the multiplier class $M^d(\mathbb{R}^n)$; here $\upsilon \in C^\infty$, equal to 1 for large ξ and vanishing in a neighborhood of the origin. Since $(\xi_1^2+\xi_2^2)^{1/2d}|\xi|^{-1/d}$ belongs to $M^p(\mathbb{R}^n)$ for $1 < p < \infty$ we may replace in the display $|\xi|^{1/d}$ with $(\xi_1^2 + \xi_2^2)^{1/2d}$. Now apply the theorem by de Leeuw [8] on the restriction of multipliers to subspaces to see that

$$(\xi_1^2 + \xi_2^2)^{1/2d} \upsilon(\xi_1, \xi_2, 0, \dots, 0) \int \chi(t) e^{i(\gamma_1(t)\xi_1 + \gamma_2(t)\xi_2)} \, \mathrm{d}t$$

is a multiplier in $M^d(\mathbb{R}^2)$ which implies the $L^d(\mathbb{R}^2) \to L^d_{1/d}(\mathbb{R}^2)$ boundedness of the averaging operator associated to the plane curve $(\gamma_1(t), \gamma_2(t))$. However the latter statement can be disproved by using the argument of Christ [7], who considered the curve (t, t^d).

4. Initial reductions and auxiliary results

The remainder of the paper deals with the proof of Theorem 1.1. This section contains some preliminary results, the most significant of which is the decoupling result in Theorem 4.4 which lies at the heart of the proof.

4.1. **Multiplier notation.** From the reduction described in Proposition 2.3 it suffices to consider $\gamma \in \mathfrak{G}_4(\delta_0)$ where δ_0 is a small parameter, as described at the end of §2. If f belongs to a suitable *a priori* class, then the Fourier transform of $A_\gamma f$ is the product of $\hat{f}$ and the multiplier

$$\hat{\mu}_\gamma(\xi) = \int_{\mathbb{R}} e^{-i\langle \gamma(s), \xi \rangle} \chi(s) \, \mathrm{d}s. \tag{4.1}$$

Recall, again from the reduction described in Proposition 2.3, that we may assume $\chi \in C_c^\infty(\mathbb{R})$ satisfies $\operatorname{supp} \chi \subseteq I_0 = [-\delta_0, \delta_0]$.

Given $m \in L^\infty(\hat{\mathbb{R}}^4)$, define the associated multiplier operator $m(D)$ by

$$m(D)f(x) := \frac{1}{(2\pi)^4} \int_{\hat{\mathbb{R}}^4} e^{i\langle x, \xi \rangle} m(\xi) \hat{f}(\xi) \, \mathrm{d}\xi$$

so that, in this notation, $A_\gamma = \hat{\mu}_\gamma(D)$. We also define the associated L^p multiplier norms

$$\|m\|_{M^p(\mathbb{R}^4)} := \|m(D)\|_{L^p(\mathbb{R}^4) \to L^p(\mathbb{R}^4)} \qquad \text{for } 1 \leqslant p \leqslant \infty.$$

To prove Theorem 1.1, we analyse various multipliers obtained by decomposing (4.1). To this end, given $a \in C^\infty(\hat{\mathbb{R}}^4 \backslash \{0\} \times \mathbb{R})$, define

$$m[a](\xi) := \int_{\mathbb{R}} e^{-i\langle \gamma(s), \xi \rangle} a(\xi; s) \chi(s) \, \mathrm{d}s. \tag{4.2}$$

Any decomposition of the symbol a results in a corresponding decomposition of the multiplier. We will also use the notation $\operatorname{supp}_\xi a$ to denote the projection of $\operatorname{supp} a \subseteq \hat{\mathbb{R}}^4 \backslash \{0\} \times \mathbb{R}$ into $\hat{\mathbb{R}}^4 \backslash \{0\}$.

4.2. **Reduction to band-limited functions.** Given a symbol $a \in C^\infty(\hat{\mathbb{R}}^4 \backslash \{0\} \times \mathbb{R})$ we perform a dyadic decomposition in the frequency variable ξ as follows. Fix $\eta \in C_c^\infty(\mathbb{R})$ non-negative and such that

$$\eta(r) = 1 \quad \text{if } r \in [-1, 1] \quad \text{and} \quad \operatorname{supp} \eta \subseteq [-2, 2]$$

and define $\beta^k \in C_c^\infty(\mathbb{R})$ by

$$\beta^k(r) := \eta(2^{-k} r) - \eta(2^{-k+1} r) \tag{4.3}$$

for each $k \in \mathbb{Z}$. By a slight abuse of notation we also let η, $\beta^k \in C_c^\infty(\hat{\mathbb{R}}^4)$ denote the functions $\eta(\xi) := \eta(|\xi|)$ and $\beta^k(\xi) := \beta^k(|\xi|)$. One may then decompose

$$a = \sum_{k=0}^\infty a_k \qquad \text{where} \qquad a_k(\xi; s) := \begin{cases} a(\xi; s) \cdot \beta^k(\xi) & \text{for } k \geqslant 1 \\ a(\xi; s) \cdot \eta(\xi) & \text{for } k = 0 \end{cases}. \tag{4.4}$$

Theorem 1.1 is a direct consequence of the following result for multipliers localised to some dyadic frequency band. Here we work with additional absolute constants $0 < \delta_j \leqslant \delta_0$ for $1 \leqslant j \leqslant 3$, chosen sufficiently small for the purposes of the forthcoming arguments. In practice, we may simply take $\delta_j := \delta_0$ for $j = 1, 3$ and $\delta_2 := \delta_0^3$. It is also convenient to define $\delta_4 := 9/10$.

Theorem 4.1. *Let $\gamma \in \mathfrak{G}_4(\delta_0)$ and $1 \leqslant J \leqslant 4$. Suppose that $a \in C^\infty(\hat{\mathbb{R}}^4 \backslash \{0\} \times \mathbb{R})$ satisfies*

$$|\partial_\xi^\alpha \partial_s^N a(\xi; s)| \lesssim_{\alpha,N} |\xi|^{-|\alpha|} \qquad \text{for all } \alpha \in \mathbb{N}_0^4 \text{ and } N \in \mathbb{N}_0 \tag{4.5}$$

and

$$\begin{cases} \inf_{s \in I_0} |\langle \gamma^{(J)}(s), \xi \rangle| \geqslant \delta_J |\xi| \\ \inf_{s \in I_0} |\langle \gamma^{(j)}(s), \xi \rangle| \leqslant 4\delta_j |\xi| \quad \text{for } 1 \leqslant j \leqslant J-1 \end{cases} \qquad \text{for all } \xi \in \operatorname{supp}_\xi a. \tag{4.6}$$

If a_k is defined as in (4.4), then

$$\|m[a_k]\|_{M^p(\mathbb{R}^4)} \lesssim_p 2^{-k/p} \tag{4.7}$$

for $k \geqslant 1$ and $p > \max\{2(J-1), 1\}$.

The hypothesis (4.6) implies that

$$|\langle \gamma^{(j)}(s), \xi \rangle| \leqslant 8\delta_0 |\xi| \qquad \text{for all } \xi \in \operatorname{supp}_\xi a,\ s \in I_0 \text{ and } 1 \leqslant j \leqslant J-1. \tag{4.8}$$

Indeed, suppose $s_0 \in I_0$ realises the infimum in (4.6) and let $s \in I_0$. Then the mean value theorem implies

$$|\langle \gamma^{(j)}(s), \xi \rangle| \leqslant |\langle \gamma^{(j)}(s_0), \xi \rangle| + \sup_{t \in I_0} |\gamma^{(j+1)}(t)||s - s_0||\xi| \leqslant 8\delta_0 |\xi|, \tag{4.9}$$

using the fact that $\delta_j \leqslant \delta_0$ and the uniform derivative bounds for $\gamma \in \mathfrak{G}_4(\delta_0)$.

Proof of Theorem 1.1 given Theorem 4.1. By the reduction from §2 it suffices to consider $\gamma \in \mathfrak{G}_4(\delta_0)$ and $\chi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \chi \subseteq I_0 = [-\delta_0, \delta_0]$. For $1 \leqslant j \leqslant 4$ define the sets

$$\mathscr{E}_j := \Big\{ \xi \in S^3 : \inf_{s \in I_0} |\langle \gamma^{(j)}(s), \xi \rangle| < \delta_j \Big\} \quad \text{and} \quad U_j := \begin{cases} N_{\delta_j} \mathscr{E}_j \cap S^3 & \text{if } 1 \leqslant j \leqslant 3 \\ N_{\delta_0} \mathscr{E}_j \cap S^3 & \text{if } j = 4 \end{cases},$$

where $N_{\delta_j} \mathscr{E}_j$ denotes the δ_j-neighbourhood of $\mathscr{E}_j$ and S^3 denotes the unit sphere in $\hat{\mathbb{R}}^4$. Since U_j is an open subset of S^3 containing the compact subset $\operatorname{clos} \mathscr{E}_j$, there exists a smooth function $\rho_j \colon S^3 \to [0, \infty)$ such that

$$\rho_j(\omega) = 1 \quad \text{for } \omega \in \operatorname{clos} \mathscr{E}_j \qquad \text{and} \qquad \operatorname{supp} \rho_j \subseteq U_j.$$

For $1 \leqslant J \leqslant 4$ define $\chi_J \in C^\infty(S^3)$ by

$$\chi_J := \Big(\prod_{j=1}^{J-1} \rho_j \Big) \cdot (1 - \rho_J)$$

These functions satisfy the following properties:

i) If $\xi \in S^3$ and $\xi \in \operatorname{supp} \chi_J$, then (4.6) holds;
ii) $\sum_{J=1}^4 \chi_J \equiv 1$, as functions on S^3.

Indeed, to see property i), note that if $\xi \in \operatorname{supp} \chi_J$, then $\xi \notin \operatorname{clos} \mathscr{E}_J$ which implies the first bound in (4.6). On the other hand, for $1 \leqslant j \leqslant J - 1 \leqslant 3$ it follows that $\xi \in U_j$ and so there exists some $\xi_0 \in \mathscr{E}_j$ with $|\xi - \xi_0| < \delta_j$. Consequently, there exists some $s_0 \in I_0$ such that

$$|\langle \gamma^{(j)}(s_0), \xi \rangle| \leqslant |\langle \gamma^{(j)}(s_0), \xi_0 \rangle| + |\gamma^{(j)}(s_0)||\xi - \xi_0| \leqslant 4\delta_j, \tag{4.10}$$

which is the second bound in (4.6). For property ii), note that (4.9) can be combined with the argument in (4.10) to conclude that

$$\sup_{s \in I_0} |\langle \gamma^{(j)}(s), \xi \rangle| \leqslant 8\delta_0 \quad \text{for } \xi \in U_j,\ 1 \leqslant j \leqslant 3, \quad \text{and} \quad \sup_{s \in I_0} |\langle \gamma^{(4)}(s), \xi \rangle| \leqslant \tfrac{9}{10} + 4\delta_0 \quad \text{for } \xi \in U_4.$$

Provided δ_0 is sufficiently small, the non-degeneracy of $\gamma \in \mathfrak{G}_4(\delta_0)$ implies $\bigcap_{j=1}^4 U_j = \varnothing$. Since $\sum_{J=1}^4 \chi_J = 1 - \prod_{j=1}^4 \rho_j$, property ii) follows from the support conditions of the ρ_j.

In view of the above, we may apply Theorem 4.1 with $a(\xi) := \chi_J(\xi/|\xi|)$ for $J = 1, 2, 3, 4$ and sum in J to conclude that

$$\|\beta^k(D)A_\gamma f\|_{L^p(\mathbb{R}^4)} \lesssim 2^{-k/p}\|f\|_{L^p(\mathbb{R}^4)} \qquad \text{for all } k \geqslant 0 \tag{4.11}$$

and all $p > 6$; note that the case $k = 0$ trivially follows as A_γ is an averaging operator. To pass from the frequency localised estimates (4.11) to genuine L^p-Sobolev bounds, one may apply a Calderón–Zygmund estimate from [16]. This argument is described in the Appendix A: see Proposition A.2. ☐

The hypothesis (4.5) implies that $\|\mathcal{F}_\xi^{-1} a_k(\,\cdot\,; s)\|_{L^1(\mathbb{R}^4)} \lesssim 1$, where $\mathcal{F}_\xi^{-1}$ denotes the inverse Fourier transform in the ξ variable. Consequently, it is not difficult to show that the $p = \infty$ case of (4.7) holds for all $1 \leqslant J \leqslant 4$ (see also Lemma C.2). The problem is therefore to deduce the estimate for p near to $\max\{2(J-1), 1\}$.

Note that the proof of the $J = 1$ case of Theorem 4.1 is trivial. Indeed, here the phase function of (4.2) does not admit a critical point and the desired result follows by repeated integration-by-parts.

The proof of the $J = 2$ case of Theorem 4.1 is also straightforward. Suppose $\gamma \in \mathfrak{G}_4(\delta_0)$ and $a \in C^\infty(\mathbb{R}^4\backslash\{0\} \times \mathbb{R})$ satisfies the hypotheses Theorem 4.1 for $J = 2$, with $\delta_1 := \delta_0$ and $\delta_2 := \delta_0^3$.[4] Note, in particular, that

$$|\langle \gamma''(s), \xi\rangle| \geqslant \delta_0^3|\xi| \qquad \text{for all } (\xi; s) \in \mathrm{supp}_\xi\, a \times I_0.$$

Thus, the van der Corput lemma (see, for instance, [20, Chapter VIII, Proposition 2]) implies

$$\|m[a_k]\|_{M^2(\mathbb{R}^4)} = \|m[a_k]\|_{L^\infty(\mathbb{R}^4)} \lesssim 2^{-k/2}.$$

On the other hand, by the triangle inequality, Fubini's theorem, translation-invariance and integration-by-parts (see Lemma C.2),

$$\|m[a_k]\|_{M^\infty(\mathbb{R}^4)} \leqslant \|\mathcal{F}^{-1}a_k\|_{L^1(\mathbb{R}^4)} \lesssim 1.$$

Interpolation yields

$$\|m[a_k]\|_{M^p(\mathbb{R}^4)} \lesssim 2^{-k/p} \qquad \text{for all } 2 \leqslant p \leqslant \infty,$$

which concludes the proof for the $J = 2$ case.

From now on, we focus on the $J = 3$ and $J = 4$ cases of Theorem 4.1. These are proved in Sections 5 and 6 respectively. Of these, the $J = 4$ is the heart of the matter, and its proof is the main contribution of this paper. Before turning to the proofs, we state some auxiliary results.

4.3. **The Frenet frame.** At this juncture it is convenient to recall some elementary concepts from differential geometry which feature in our proof. Given a smooth non-denegenate curve $\gamma : I \to \mathbb{R}^n$, the Frenet frame is the orthonormal basis resulting from applying the Gram–Schmidt process to the vectors

$$\{\gamma'(s), \dots, \gamma^{(n)}(s)\},$$

which are linearly independent in view of the condition (1.2). Defining the functions[5]

$$\tilde{\kappa}_j(s) := \langle \mathbf{e}_j'(s), \mathbf{e}_{j+1}(s)\rangle \qquad \text{for } j = 1, \dots, n-1,$$

[4]The choice of δ_1, δ_2 is not important for the argument in the $J = 2$ case, but is kept for consistency.

[5]Note that the $\tilde{\kappa}_j$ depend on the choice of parametrisation and only agree with the (geometric) curvature functions

$$\kappa_j(s) := \frac{\langle \mathbf{e}_j'(s), \mathbf{e}_{j+1}(s)\rangle}{|\gamma'(s)|}$$

if γ is unit speed parametrised. Here we do not assume unit speed parametrisation.

one has the classical Frenet formulæ

$$\begin{aligned}\mathbf{e}_1'(s) &= \tilde{\kappa}_1(s)\mathbf{e}_2(s),\\ \mathbf{e}_i'(s) &= -\tilde{\kappa}_{i-1}(s)\mathbf{e}_{i-1}(s) + \tilde{\kappa}_i(s)\mathbf{e}_{i+1}(s),\ i = 2, \dots, n-1,\\ \mathbf{e}_n'(s) &= -\tilde{\kappa}_{n-1}(s)\mathbf{e}_{n-1}(s).\end{aligned}$$

Repeated application of these formulæ shows that

$$\mathbf{e}_i^{(k)}(s) \perp \mathbf{e}_j(s) \qquad \text{whenever} \qquad 0 \leqslant k < |i-j|.$$

Consequently, by Taylor's theorem

$$|\langle \mathbf{e}_i(s_1), \mathbf{e}_j(s_2)\rangle| \lesssim_\gamma |s_1 - s_2|^{|i-j|} \qquad \text{for } 1 \leqslant i, j \leqslant n \text{ and } s_1, s_2 \in I. \tag{4.12}$$

Furthermore, one may deduce from the definition of $\{\mathbf{e}_j(s)\}_{j=1}^n$ that

$$|\langle \gamma^{(i)}(s_1), \mathbf{e}_j(s_2)\rangle| \lesssim_\gamma |s_1 - s_2|^{(j-i)\vee 0} \qquad \text{for } 1 \leqslant i, j \leqslant n \text{ and } s_1, s_2 \in I. \tag{4.13}$$

In this paper, much of the microlocal geometry of the operator A_γ is expressed in terms of the Frenet frame.

4.4. **A decoupling inequality for regions defined by the Frenet frame.** Let $\gamma\colon I \to \mathbb{R}^n$ be a non-degenerate curve.

Definition 4.2. *Given $2 \leqslant d \leqslant n-1$ and $0 < r \leqslant 1$, for each $s \in I$ let $\pi_{d-1}(s;r)$ denote the set of all $\xi \in \hat{\mathbb{R}}^n$ satisfying the following conditions:*

$$|\langle \mathbf{e}_j(s), \xi\rangle| \leqslant r^{d+1-j} \qquad \text{for } 1 \leqslant j \leqslant d, \tag{4.14a}$$

$$1/2 \leqslant |\langle \mathbf{e}_{d+1}(s), \xi\rangle| \leqslant 1 \tag{4.14b}$$

$$|\langle \mathbf{e}_j(s), \xi\rangle| \leqslant 1 \qquad \text{for } d+2 \leqslant j \leqslant n. \tag{4.14c}$$

Such sets $\pi_{d-1}(s;r)$ are referred to as $(d-1,r)$-Frenet boxes.

Definition 4.3. *A collection $\mathcal{P}_{d-1}(r)$ of $(d-1,r)$-Frenet boxes is a Frenet box decomposition for γ if it consists of precisely the $(d-1,r)$-Frenet boxes $\pi_{d-1}(s;r)$ for s varying over a r-net in I.*

In some instances it is useful to highlight the underlying curve and write $\pi_{d-1,\gamma}(s;r)$ for $\pi_{d-1}(s;r)$. The relevance of the $d-1$ index is made apparent in Definition 7.4.

Central to the proof of Theorem 1.1 is the following decoupling inequality.

Theorem 4.4. *Let $2 \leqslant d \leqslant n-1$, $0 \leqslant \delta \ll 1$, $0 < r \leqslant 1$ and $\mathcal{P}_{d-1}(r)$ be a $(d-1,r)$-Frenet box decomposition for $\gamma \in \mathfrak{G}_n(\delta)$. For all $2 \leqslant p \leqslant \infty$ and $\varepsilon > 0$ the inequality*

$$\Big\| \sum_{\pi \in \mathcal{P}_{d-1}(r)} f_\pi \Big\|_{L^p(\mathbb{R}^n)} \lesssim_{n,\gamma,\varepsilon} r^{-\alpha(p)-\varepsilon} \Big(\sum_{\pi \in \mathcal{P}_{d-1}(r)} \|f_\pi\|_{L^p(\mathbb{R}^n)}^p \Big)^{1/p}$$

holds with exponent

$$\alpha(p) := \begin{cases} \frac{1}{2} - \frac{1}{p} & \text{if} \quad 2 \leqslant p \leqslant d(d+1) \\ 1 - \frac{d(d+1)+2}{2p} & \text{if} \quad d(d+1) \leqslant p \leqslant \infty \end{cases}$$

for any tuple of functions $(f_\pi)_{\pi \in \mathcal{P}_{d-1}(r)}$ satisfying $\operatorname{supp} \hat{f}_\pi \subseteq \pi$.

This theorem corresponds to a conic version of the Bourgain–Guth–Demeter decoupling inequality for the moment curve [5]. Theorem 4.4 can be deduced from the moment curve decoupling via rescaling and induction-on-scale arguments, following a scheme originating in [17]. The details of this argument are presented in §7.

5. The proof of Theorem 4.1: The $J = 3$ case

We now turn to the proof of Theorem 1.1 proper. Recall, it remains to prove the $J = 3$ and $J = 4$ cases of Theorem 4.1. Here we present the analysis of the $J = 3$ case, which essentially mirrors that of [17]. The present section can therefore be thought of as a warm up for the significantly more involved argument used to treat $J = 4$ in §6.

5.1. **Preliminaries.** Suppose $\gamma \in \mathfrak{G}_4(\delta_0)$ and $a \in C^\infty(\hat{\mathbb{R}}^4 \backslash \{0\} \times \mathbb{R})$ satisfies the hypotheses Theorem 4.1 for $J = 3$, with $\delta_1 :=: \delta_3 := \delta_0$ and $\delta_2 := \delta_0^3$. Note, in particular, that

$$\begin{cases} |\langle \gamma^{(3)}(s), \xi \rangle| \geqslant \delta_0 |\xi| \\ |\langle \gamma^{(j)}(s), \xi \rangle| \leqslant 8\delta_0 |\xi| \quad \text{for } j = 1, 2 \end{cases} \qquad \text{for all } (\xi; s) \in \operatorname{supp}_\xi a \times I_0, \tag{5.1}$$

as a consequence of (4.8). If $a_k := a \cdot \beta^k$, as introduced in §4.2, this implies, via van der Corput's lemma with third order derivatives, that

$$\|m[a_k](\xi)\| \lesssim 2^{-k/3}. \tag{5.2}$$

Arguing as for $J = 2$, Plancherel's theorem and interpolation with a trivial L^∞ estimate yields

$$\|m[a_k]\|_{M^p(\mathbb{R}^4)} \lesssim 2^{-2k/3p} \qquad \text{for all } 2 \leqslant p \leqslant \infty.$$

In order to obtain the improved bound $\|m[a_k]\|_{M^p(\mathbb{R}^4)} \lesssim 2^{-k/p}$, we decompose the symbol a_k into localised pieces which admit more refined decay rates than (5.2).

5.2. **Geometry of the slow decay cone.** The first step is to isolate regions of the frequency space where the multiplier $m[a]$ decays relatively slowly. Owing to stationary phase considerations, this corresponds to a region around the cone

$$\Gamma := \Big\{ \xi \in \operatorname{supp}_\xi a : \langle \gamma^{(j)}(s), \xi \rangle = 0, \ j = 1, 2, \ \text{for some } s \in I_0 \Big\}.$$

To analyse this region, and the corresponding decay rates for $m[a]$, we make the following simple observation.

Lemma 5.1. *If $\xi \in \operatorname{supp}_\xi a$, then the equation $\langle \gamma''(s), \xi \rangle = 0$ has a unique solution in $\frac{5}{4} \cdot I_0$.*

The above lemma follows from the localisation of the symbol in (5.1) and (4.6) via the mean value theorem. The details are left to the interested reader (see [1, Lemma 6.1] for a proof using similar arguments).

Using Lemma 5.1, we construct a smooth mapping $\theta \colon \operatorname{supp}_\xi a \to [-1, 1]$ such that

$$\langle \gamma'' \circ \theta(\xi), \xi \rangle = 0 \qquad \text{for all } \xi \in \operatorname{supp}_\xi a.$$

It is easy to see that θ is homogeneous of degree 0. This function can be used to construct a natural Whitney decomposition with respect to the cone Γ defined above. In particular, let

$$u(\xi) := \langle \gamma' \circ \theta(\xi), \xi \rangle \qquad \text{for all } \xi \in \operatorname{supp}_\xi a; \tag{5.3}$$

this quantity plays a central rôle in our analysis. If $u(\xi) = 0$, then $\xi \in \Gamma$ and so, roughly speaking, $u(\xi)$ measures the distance of ξ from Γ.

5.3. **Decomposition of the symbols.** Consider the frequency localised symbols $a_k := a \cdot \beta^k$, as introduced in §4.2. We decompose each a_k with respect to the size of $|u(\xi)|$. In particular, write

$$a_k = \sum_{\ell=0}^{\lfloor k/3 \rfloor} a_{k,\ell} \tag{5.4}$$

where $\lfloor k/3 \rfloor$ denotes the greatest integer less than or equal to $k/3$ and[6]

$$a_{k,\ell}(\xi; s) := \begin{cases} a_k(\xi; s)\beta\big(2^{-k+2\ell}u(\xi)\big) & \text{if } 0 \leqslant \ell < \lfloor k/3 \rfloor \\ a_k(\xi; s)\eta\big(2^{-k+2\lfloor k/3 \rfloor}u(\xi)\big) & \text{if } \ell = \lfloor k/3 \rfloor \end{cases}. \tag{5.5}$$

The $J = 3$ case of Theorem 4.1 is a consequence of the following bound for the localised pieces of the multiplier.

Proposition 5.2. *Let $4 \leqslant p \leqslant 6$, $k \in \mathbb{N}$ and $\varepsilon > 0$. For all $0 \leqslant \ell \leqslant \lfloor k/3 \rfloor$,*

$$\|m[a_{k,\ell}]\|_{M^p(\mathbb{R}^4)} \lesssim_{\varepsilon,p} 2^{-k/p - \ell(1/2 - 2/p - \varepsilon)}.$$

Proof of $J = 3$ case of Theorem 4.1, assuming Proposition 5.2. Let $4 < p \leqslant 6$ and define $\varepsilon_p := \frac{1}{2}\big(\frac{1}{2} - \frac{2}{p}\big) > 0$. Apply the decomposition (5.4) and Proposition 5.2 to deduce that

$$\|m[a_k]\|_{M^p(\mathbb{R}^4)} \leqslant \sum_{\ell=0}^{\lfloor k/3 \rfloor} \|m[a_{k,\ell}]\|_{M^p(\mathbb{R}^4)} \lesssim_p 2^{-k/p} \sum_{\ell=0}^{\infty} 2^{-\ell(1/2 - 2/p - \varepsilon_p)} \lesssim_p 2^{-k/p}.$$

This establishes the desired result for $4 < p \leqslant 6$. The remaining range $6 < p \leqslant \infty$ follows by interpolation with a trivial L^∞ estimate. ☐

The rest of §5 is devoted to establishing Proposition 5.2. Before proceeding, it is instructive to reflect on the rationale behind the decomposition (5.4). A lower bound on $|u(\xi)|$ ensures that the functions $s \mapsto \langle \gamma'(s), \xi \rangle$ and $s \mapsto \langle \gamma''(s), \xi \rangle$ do not vanish simultaneously. Quantifying this observation, one obtains, via the van der Corput lemma, the decay estimate

$$|m[a_{k,\ell}](\xi)| \lesssim 2^{-k/2 + \ell/2}; \tag{5.6}$$

see Lemma 5.6 below. This improves upon the trivial decay rate (5.2) since ℓ varies over the range $0 \leqslant \ell \leqslant \lfloor k/3 \rfloor$. Note that $\ell = k/3$ corresponds to the critical value where (5.2) and (5.6) agree.

By Plancherel's theorem, (5.6) implies

$$\|m[a_{k,\ell}]\|_{M^2(\mathbb{R}^4)} \lesssim 2^{-k/2 + \ell/2}.$$

As ℓ increases this estimate becomes weaker. To compensate for this, we attempt to establish stronger estimates for the $M^\infty(\mathbb{R}^4)$ norm. This is not possible, however, for the entire multiplier and a further decomposition is required. The $u(\xi)$ localisation means that $m[a_{k,\ell}]$ is supported in a neighbourhood of the cone Γ. Consequently, one may apply a decoupling theorem for this cone (in particular, an instance of Theorem 4.4) to *radially* decompose the multipliers. It transpires that each radially localised piece is automatically localised along the curve in the physical space, and this leads to favourable $M^\infty(\mathbb{R}^4)$ bounds: see Lemma 5.5 and Lemma 5.7 below.

[6]The β function should be defined slightly differently compared with (4.3) and, in particular, here $\beta(r) := \eta(2^{-2}r) - \eta(r)$. Such minor changes are ignored in the notation.

5.4. **Fourier localisation and decoupling.** The first step towards Proposition 5.2 is to radially decompose the symbols in terms of $\theta(\xi)$. Fix a smooth cutoff $\zeta \in C^\infty(\mathbb{R})$ with $\operatorname{supp} \zeta \subseteq [-1,1]$ such that $\sum_{k\in\mathbb{Z}} \zeta(\,\cdot\, - k) \equiv 1$ and write

$$a_{k,\ell}(\xi;s) = \sum_{\mu\in\mathbb{Z}} a^{\mu}_{k,\ell}(\xi;s) \qquad \text{where} \qquad a^{\mu}_{k,\ell}(\xi;s) := a_{k,\ell}(\xi;s)\zeta(2^\ell\theta(\xi) - \mu). \tag{5.7}$$

Given $0 < r \leqslant 1$ and $s \in I$, recall the definition of the $(1,r)$-*Frenet boxes* $\pi_1(s;r)$ introduced in Definition 4.2:

$$\pi_1(s;r) := \big\{\xi \in \hat{\mathbb{R}}^4 : |\langle \mathbf{e}_j(s), \xi\rangle| \lesssim r^{3-j} \quad \text{for } j = 1,\, 2, \quad |\langle \mathbf{e}_3(s), \xi\rangle| \sim 1, \quad |\langle \mathbf{e}_4(s), \xi\rangle| \lesssim 1\big\}. \tag{5.8}$$

Here $(\mathbf{e}_j)_{j=1}^4$ denotes the Frenet frame, as introduced in §4.3. The multipliers $a^{\mu}_{k,\ell}$ satisfy the following support properties.

Lemma 5.3. *With the above definitions,*

$$\operatorname{supp}_\xi a^{\mu}_{k,\ell} \subseteq 2^k \cdot \pi_1(s_\mu; 2^{-\ell})$$

for all $0 \leqslant \ell \leqslant \lfloor k/3 \rfloor$ *and* $\mu \in \mathbb{Z}$, *where* $s_\mu := 2^{-\ell}\mu$.

Proof. For $\xi \in \operatorname{supp}_\xi a^{\mu}_{k,\ell}$ observe that

$$|\langle \gamma^{(i)} \circ \theta(\xi), \xi\rangle| \lesssim 2^{k-(3-i)\ell\vee 0} \qquad \text{for } 1 \leqslant i \leqslant 4, \qquad |\langle \gamma^{(3)} \circ \theta(\xi), \xi\rangle| \sim 2^k.$$

Since the Frenet vectors $\mathbf{e}_i \circ \theta(\xi)$ are obtained from the $\gamma^{(i)} \circ \theta(\xi)$ via the Gram–Schmidt process, the matrix corresponding to change of basis from $\big(\mathbf{e}_i \circ \theta(\xi)\big)_{i=1}^4$ to $\big(\gamma^{(i)} \circ \theta(\xi)\big)_{i=1}^4$ is lower triangular. Furthermore, the initial localisation implies that this matrix is an $O(\delta)$ perturbation of the identity. Consequently, provided $\delta > 0$ is chosen sufficiently small,

$$|\langle \mathbf{e}_i \circ \theta(\xi), \xi\rangle| \lesssim 2^{k-(3-i)\ell\vee 0} \qquad \text{for } 1 \leqslant i \leqslant 4, \qquad |\langle \mathbf{e}_3 \circ \theta(\xi), \xi\rangle| \sim 2^k.$$

On the other hand, by (5.7) we also have $|\theta(\xi) - s_\mu| \lesssim 2^{-\ell}$ and so (4.12) implies that

$$|\langle \mathbf{e}_i \circ \theta(\xi), \mathbf{e}_j(s_\mu)\rangle| \lesssim |\theta(\xi) - s_\mu|^{|i-j|} \lesssim 2^{-(i-j)\ell}.$$

Writing ξ with respect to the orthonormal basis $\big(\mathbf{e}_j \circ \theta(\xi)\big)_{j=1}^4$, it follows that

$$|\langle \mathbf{e}_j(s_\mu), \xi\rangle| \leqslant \sum_{i=1}^4 |\langle \mathbf{e}_i \circ \theta(\xi), \xi\rangle||\langle \mathbf{e}_i \circ \theta(\xi), \mathbf{e}_j(s_\mu)\rangle| \lesssim 2^{k-(3-j)\ell\vee 0}.$$

Thus, ξ satisfies all the required upper bounds appearing in (5.8). Provided the parameter $\delta > 0$ is sufficiently small, the argument can easily be adapted to prove the remaining lower bound for $\langle \mathbf{e}_3(s_\mu), \xi\rangle$. □

In view of the Fourier localisation described above, we have the following decoupling inequality.

Proposition 5.4. *For all* $2 \leqslant p \leqslant 6$ *and* $\varepsilon > 0$ *one has*

$$\Big\| \sum_{\mu\in\mathbb{Z}} m[a^{\mu}_{k,\ell}](D)f \Big\|_{L^p(\mathbb{R}^4)} \lesssim_\varepsilon 2^{\ell(1/2-1/p)+\varepsilon\ell} \Big(\sum_{\mu\in\mathbb{Z}} \|m[a^{\mu}_{k,\ell}](D)f\|^p_{L^p(\mathbb{R}^4)} \Big)^{1/p}.$$

Proof. In view of the support conditions from Lemma 5.3, after a simple rescaling, the desired result follows from Theorem 4.4 with $d - 1 = 1$, $n = 4$ and $r = 2^{-\ell}$. □

5.5. **Localisation along the curve.** The $\theta(\xi)$ localisation introduced in the previous subsection induces a corresponding localisation along the curve in the physical space. In particular, the main contribution to $m[a^{\mu}_{k,\ell}]$ arises from the portion of the curve defined over the interval $|s-s_\mu| \leqslant 2^{-\ell}$. This is made precise in Lemma 5.5 below.

Here it is convenient to introduce a 'fine tuning' constant $\rho > 0$. This is a small (but absolute) constant which plays a minor technical rôle in the forthcoming arguments: taking $\rho := 10^{-6}$ more than suffices for our purposes.

For $0 \leqslant \ell \leqslant \lfloor k/3 \rfloor$, $\mu \in \mathbb{Z}$ and $\varepsilon > 0$, define

$$a^{\mu,(\varepsilon)}_{k,\ell}(\xi;s) := a_{k,\ell}(\xi;s)\zeta(2^{\ell}\theta(\xi)-\mu)\eta(\rho 2^{\ell(1-\varepsilon)}(s-s_\mu)). \tag{5.9}$$

The key contribution to the multiplier comes from the symbol $a^{\mu,(\varepsilon)}_{k,\ell}$.

Lemma 5.5. *Let $2 \leqslant p < \infty$ and $\varepsilon > 0$. For all $0 \leqslant \ell \leqslant \lfloor k/3 \rfloor$,*

$$\|m[a^{\mu}_{k,\ell} - a^{\mu,(\varepsilon)}_{k,\ell}]\|_{M^p(\mathbb{R}^4)} \lesssim_{N,\varepsilon,p} 2^{-kN} \qquad \text{for all } N \in \mathbb{N}.$$

Proof. It is clear that the multipliers satisfy a trivial L^∞-estimate with operator norm $O(2^{Ck})$ for some absolute constant $C \geqslant 1$. Thus, by interpolation, it suffices to prove the rapid decay estimate for $p = 2$ only. This amounts to showing that, under the hypotheses of the lemma,

$$\|m[a^{\mu}_{k,\ell} - a^{\mu,(\varepsilon)}_{k,\ell}]\|_{L^\infty(\hat{\mathbb{R}}^4)} \lesssim_{N,\varepsilon} 2^{-kN} \qquad \text{for all } N \in \mathbb{N}.$$

Here the localisation of the $a_{k,\ell}$ symbols ensures that

$$|u(\xi)| \lesssim 2^{k-2\ell} \qquad \text{for all } (\xi;s) \in \operatorname{supp}(a^{\mu}_{k,\ell} - a^{\mu,(\varepsilon)}_{k,\ell}), \tag{5.10}$$

where u is the function introduced in (5.3). On the other hand, provided ρ is sufficiently small, the additional localisation in (5.7) and (5.9) implies, via the triangle inequality,

$$|s-\theta(\xi)| \gtrsim \rho^{-1}2^{-\ell(1-\varepsilon)} \quad \text{for all } (\xi;s) \in \operatorname{supp}(a^{\mu}_{k,\ell} - a^{\mu,(\varepsilon)}_{k,\ell}). \tag{5.11}$$

Fix $\xi \in \operatorname{supp}_\xi(a^{\mu}_{k,\ell} - a^{\mu,(\varepsilon)}_{k,\ell})$ and consider the oscillatory integral $m[a^{\mu}_{k,\ell} - a^{\mu,(\varepsilon)}_{k,\ell}](\xi)$, which has phase $s \mapsto \langle \gamma(s), \xi\rangle$. Taylor expansion around $\theta(\xi)$ yields

$$\langle \gamma'(s), \xi\rangle = u(\xi) + \omega_1(\xi;s)\cdot(s-\theta(\xi))^2, \tag{5.12}$$

$$\langle \gamma''(s), \xi\rangle = \omega_2(\xi;s)\cdot(s-\theta(\xi)) \tag{5.13}$$

where the ω_i arise from the remainder terms and satisfy $|\omega_i(\xi;s)| \sim 2^k$. Provided ρ is sufficiently small, (5.10) and (5.11) imply that the $\omega_1(\xi;s)\cdot(s-\theta(\xi))^2$ term dominates the right-hand side of (5.12) and therefore

$$|\langle \gamma'(s), \xi\rangle| \gtrsim 2^k|s-\theta(\xi)|^2 \qquad \text{for all } (\xi;s) \in \operatorname{supp}(a^{\mu}_{k,\ell} - a^{\mu,(\varepsilon)}_{k,\ell}). \tag{5.14}$$

Furthermore, (5.13), (5.14) and the localisation (5.11) immediately imply

$$|\langle \gamma''(s), \xi\rangle| \lesssim 2^{-k+3\ell(1-\varepsilon)}|\langle \gamma'(s), \xi\rangle|^2,$$

$$|\langle \gamma^{(j)}(s), \xi\rangle| \lesssim 2^k \lesssim_j 2^{-(k-3\ell(1-\varepsilon))(j-1)}|\langle \gamma'(s), \xi\rangle|^j \qquad \text{for all } j \geqslant 3$$

for all $(\xi;s) \in \operatorname{supp}(a^{\mu}_{k,\ell} - a^{\mu,(\varepsilon)}_{k,\ell})$.

On the other hand, by the definition of the symbols, (5.14) and the localisation (5.11),

$$|\partial_s^N(a^{\mu}_{k,\ell} - a^{\mu,(\varepsilon)}_{k,\ell})(\xi;s)| \lesssim_N 2^{\ell N} \lesssim 2^{-(k-3\ell)N-2\varepsilon\ell N}|\langle \gamma'(s), \xi\rangle|^N \qquad \text{for all } N \in \mathbb{N}.$$

Thus, by repeated integration-by-parts (via Lemma C.1, with $R = 2^{k-3\ell+2\varepsilon\ell} \geqslant 1$),

$$|m[a^{\mu}_{k,\ell} - a^{\mu,(\varepsilon)}_{k,\ell}](\xi)| \lesssim_N 2^{-(k-3\ell)N-2\varepsilon\ell N} \qquad \text{for all } N \in \mathbb{N}.$$

Since $0 \leqslant \ell \leqslant \lfloor k/3 \rfloor \leqslant k/3$, the desired bound follows. □

5.6. **Estimating the localised pieces.** The multiplier operators $m[a^{\mu,(\varepsilon)}_{k,\ell}](D)$ satisfy favourable L^2 and L^∞ bounds, owing to the $u(\xi)$ and s localisation, respectively.

Lemma 5.6. *For all* $0 \leqslant \ell \leqslant \lfloor k/3 \rfloor$, $\mu \in \mathbb{Z}$ *and* $\varepsilon > 0$, *we have*

$$\|m[a^{\mu,(\varepsilon)}_{k,\ell}]\|_{M^2(\mathbb{R}^4)} \lesssim 2^{-k/2+\ell/2}.$$

Proof. If $\ell = \lfloor k/3 \rfloor$, then the desired estimate follows from Plancherel's theorem and van der Corput lemma with third order derivatives, as the localisation (5.1) implies

$$|\langle \gamma^{(3)}(s), \xi \rangle| \gtrsim 2^k \qquad \text{for all } (\xi, s) \in \operatorname{supp} a_{k,\ell}.$$

For the remaining case, it suffices to show that

$$|\langle \gamma'(s), \xi \rangle| + 2^{-\ell}|\langle \gamma''(s), \xi \rangle| \gtrsim 2^{k-2\ell} \qquad \text{for all } (\xi; s) \in \operatorname{supp} a^{\mu,(\varepsilon)}_{k,\ell}. \tag{5.15}$$

Here the localisation of the symbol ensures the key property

$$|u(\xi)| \sim 2^{k-2\ell} \qquad \text{for all } (\xi; s) \in \operatorname{supp} a^{\mu,(\varepsilon)}_{k,\ell}. \tag{5.16}$$

Indeed, this follows from (5.5) together with the hypothesis $0 \leqslant \ell < \lfloor k/3 \rfloor$.

By Taylor expansion around $\theta(\xi)$, one has

$$\langle \gamma'(s), \xi \rangle = u(\xi) + \omega_1(\xi; s) \cdot (s - \theta(\xi))^2, \tag{5.17}$$

$$\langle \gamma''(s), \xi \rangle = \omega_2(\xi; s) \cdot (s - \theta(\xi)), \tag{5.18}$$

where the functions ω_i arise from the remainder terms and satisfy $|\omega_i(\xi; s)| \sim 2^k$ for $i = 1, 2$.

The analysis now splits into two cases.

Case 1: $|s - \theta(\xi)| < \rho 2^{-\ell}$. Provided ρ is sufficiently small, (5.16) implies that the $u(\xi)$ term dominates in the right-hand side of (5.17) and therefore $|\langle \gamma'(s), \xi \rangle| \gtrsim 2^{k-2\ell}$.

Case 2: $|s - \theta(\xi)| \geqslant \rho 2^{-\ell}$. In this case, (5.18) implies that $|\langle \gamma''(s), \xi \rangle| \gtrsim \rho 2^{k-\ell}$.

In either case, the desired bound (5.15) holds. □

Lemma 5.7. *For all* $0 \leqslant \ell \leqslant \lfloor k/3 \rfloor$, $\mu \in \mathbb{Z}$ *and* $\varepsilon > 0$,

$$\|m[a^{\mu,(\varepsilon)}_{k,\ell}]\|_{M^\infty(\mathbb{R}^4)} \lesssim 2^{-\ell(1-\varepsilon)}.$$

Proof. By Lemma 5.3, we have $\operatorname{supp}_\xi a^{\mu}_{k,\ell} \subseteq 2^k \cdot \pi_1(s_\mu; 2^{-\ell})$. Consequently, an integration-by-parts argument (see Lemma C.2) reduces the problem to showing

$$|\nabla^N_{\boldsymbol{v}_j} a^{\mu}_{k,\ell}(\xi)| \lesssim_N 2^{-(k-(3-j)\ell \vee 0)N} \qquad \text{for all } 1 \leqslant j \leqslant 4 \text{ and all } N \in \mathbb{N}_0, \tag{5.19}$$

where $\nabla_{\boldsymbol{v}_j}$ denotes the directional derivative in the direction of the vector $\boldsymbol{v}_j := \mathbf{e}_j(s_\mu)$.

Given $\xi \in \operatorname{supp}_\xi a^{\mu}_{k,\ell}$, we claim that

$$2^{\ell}|\nabla^N_{\boldsymbol{v}_j} \theta(\xi)| \lesssim_N 2^{-(k-(3-j)\ell \vee 0)N} \qquad \text{and} \qquad 2^{-k+2\ell}|\nabla^N_{\boldsymbol{v}_j} u(\xi)| \lesssim_N 2^{-(k-(3-j)\ell \vee 0)N} \tag{5.20}$$

for all $N \in \mathbb{N}$. Assuming that this is so, the derivative bounds (5.19) follow directly from the chain and Leibniz rule, applying (5.20).

The claimed bounds in (5.20) follow from repeated application of the chain rule, provided

$$|\langle \gamma^{(3)} \circ \theta(\xi), \xi \rangle| \gtrsim 2^k, \tag{5.21a}$$

$$|\langle \gamma^{(K)} \circ \theta(\xi), \xi \rangle| \lesssim_K 2^{k+\ell(K-3)}, \tag{5.21b}$$

$$|\langle \gamma^{(K)} \circ \theta(\xi), \boldsymbol{v}_j \rangle| \lesssim_K 2^{(3-j)\ell \vee 0 + \ell(K-3)} \tag{5.21c}$$

hold for all $K \geqslant 2$ and all $\xi \in \operatorname{supp}_\xi a^\mu_{k,\ell}$. In particular, assuming (5.21a), (5.21b) and (5.21c), the bounds in (5.20) are then a consequence of Lemma B.1 in the appendix: (5.20) corresponds to (B.2) and (B.4) whilst the hypotheses in the above display correspond to (B.1) and (B.3). Here the parameters featured in the appendix are chosen as follows:

g	h	A	B	M_1	M_2	$\mathbf{e}$
γ''	γ'	$2^{k-\ell}$	$2^{k-2\ell}$	$2^{-k+(3-j)\ell\vee 0}$	2^{ℓ}	$\boldsymbol{v}_j$

See Example B.2.

The conditions (5.21a), (5.21b) and (5.21c) are direct consequences of the support properties of the $a^\mu_{k,\ell}$. Indeed, (5.21a) and the $K \geqslant 3$ case of (5.21b) are trivial consequences of the localisation of the symbol a_k. The remaining $K = 2$ case of (5.21b) follows immediately since $\langle \gamma'' \circ \theta(\xi), \xi\rangle = 0$. Finally, (4.13) together with the θ localisation imply

$$|\langle \gamma^{(K)} \circ \theta(\xi), \boldsymbol{v}_j\rangle| \lesssim_K |\theta(\xi) - s_\mu|^{(j-K)\vee 0} \lesssim 2^{-((j-K)\vee 0)\ell}$$

and this is easily seen to imply (5.21c). ☐

Lemma 5.6 and Lemma 5.7 can be combined to obtain the following L^p bounds.

Corollary 5.8. *Let $0 \leqslant \ell \leqslant \lfloor k/3 \rfloor$ and $\varepsilon > 0$. For all $2 \leqslant p \leqslant \infty$,*

$$\Big(\sum_{\mu\in\mathbb{Z}} \|m[a^{\mu,(\varepsilon)}_{k,\ell}](D)f\|^p_{L^p(\mathbb{R}^4)}\Big)^{1/p} \lesssim 2^{-k/p-\ell(1-3/p)+\varepsilon\ell}\|f\|_{L^p(\mathbb{R}^4)}.$$

When $p = \infty$ the left-hand ℓ^p-sum is interpreted as a supremum in the usual manner.

Proof. For $p = 2$ the estimate follows by combining the L^2 bounds from Lemma 5.6 with a simple orthogonality argument. For $p = \infty$ the estimate is a restatement of the L^∞ bound from Lemma 5.7. Interpolating these two endpoint cases, using mixed norm interpolation (see, for instance, [21, §1.18.4]), concludes the proof. ☐

5.7. **Putting everything together.** We are now ready to combine the ingredients to conclude the proof of Proposition 5.2.

Proof of Proposition 5.2. By Proposition 5.4, for all $2 \leqslant p \leqslant 6$ and all $\varepsilon > 0$ one has

$$\|m[a_{k,\ell}](D)f\|_{L^p(\mathbb{R}^4)} = \Big\|\sum_{\mu\in\mathbb{Z}} m[a^\mu_{k,\ell}](D)f\Big\|_{L^p(\mathbb{R}^4)} \lesssim_\varepsilon 2^{\ell(1/2-1/p)+\varepsilon\ell}\Big(\sum_{\mu\in\mathbb{Z}} \|m[a^\mu_{k,\ell}](D)f\|^p_{L^p(\mathbb{R}^4)}\Big)^{1/p}.$$

Moreover, for all $2 \leqslant p < \infty$, $\mu \in \mathbb{Z}$ and all $\varepsilon > 0$, Lemma 5.5 implies that

$$\|m[a^\mu_{k,\ell}]\|_{M^p(\mathbb{R}^4)} \lesssim_{N,\varepsilon,p} \Big\|\sum_{\mu\in\mathbb{Z}} m[a^{\mu,(\varepsilon)}_{k,\ell}]\Big\|_{M^p(\mathbb{R}^4)} + 2^{-kN} \qquad \text{for all } N \in \mathbb{N}.$$

Combining the above, we obtain that for all $2 \leqslant p \leqslant 6$ and all $\varepsilon > 0$,

$$\|m[a_{k,\ell}](D)f\|_{L^p(\mathbb{R}^4)} \lesssim_{\varepsilon,p} 2^{\ell(1/2-1/p)+\varepsilon\ell}\Big(\sum_{\mu\in\mathbb{Z}} \|m[a^\mu_{k,\ell}](D)f\|^p_{L^p(\mathbb{R}^4)}\Big)^{1/p} + 2^{-kN}\|f\|_{L^p(\mathbb{R}^4)}$$

which together with Corollary 5.8 yields

$$\|m[a_{k,\ell}](D)f\|_{L^p(\mathbb{R}^4)} \lesssim_\varepsilon 2^{-k/p-\ell(1/2-2/p-2\varepsilon)}\|f\|_{L^p(\mathbb{R}^4)}.$$

Since $\varepsilon > 0$ was chosen arbitrarily, this is the required bound. ☐

We have established Proposition 5.2 and therefore completed the proof of the $J = 3$ case of Theorem 4.1.

6. The proof of Theorem 4.1: The $J = 4$ case

The analysis used to prove the $J = 4$ case of Theorem 4.1 is much more involved than that for $J = 3$. This case constitutes to the main content of Theorem 4.1.

6.1. Preliminaries.

Suppose $\gamma \in \mathfrak{G}_4(\delta_0)$ and $a \in C^\infty(\hat{\mathbb{R}}^4\backslash\{0\} \times \mathbb{R})$ satisfies the hypotheses of Theorem 4.1 for $J = 4$ with $\delta_1 :=: \delta_3 := \delta_0$, $\delta_2 := \delta_0^3$ and $\delta_4 := 9/10$.[7] Note, in particular, that

$$\begin{cases} |\langle \gamma^{(4)}(s), \xi\rangle| \geqslant \frac{9}{10} \cdot |\xi| \\ |\langle \gamma^{(j)}(s), \xi\rangle| \leqslant 8\delta_0|\xi| \quad \text{for } j = 1, 2, 3 \end{cases} \qquad \text{for all } (\xi; s) \in \operatorname{supp}_\xi a \times I_0, \tag{6.1}$$

as a consequence of (4.8). We note two further consequences of this technical reduction:

- Recall $\gamma^{(j)}(0) = \vec{e}_j$ for $1 \leqslant j \leqslant 4$ and so (6.1) immediately implies that
$$|\xi_4| \geqslant \tfrac{9}{10} \cdot |\xi| \quad \text{and} \quad |\xi_j| \leqslant 8\delta_0|\xi| \quad \text{for } j = 1, 2, 3, \qquad \text{for all } \xi \in \operatorname{supp}_\xi a.$$
- Since $\gamma \in \mathfrak{G}_4(\delta_0)$, we have $\|\gamma^{(5)}\|_\infty \leqslant \delta_0$. Thus, provided δ_0 is sufficiently small,
$$|\langle \gamma^{(4)}(s), \xi\rangle| \geqslant \tfrac{1}{2} \cdot |\xi| \qquad \text{for all } (\xi; s) \in \operatorname{supp}_\xi a \times [-1, 1]. \tag{6.2}$$
Observe that this inequality holds on the large interval $[-1, 1]$, rather than just I_0.

Henceforth, we also assume that $\xi_4 > 0$ for all $\xi \in \operatorname{supp}_\xi a$. In particular,

$$\langle \gamma^{(4)}(s), \xi\rangle > 0 \qquad \text{for all } (\xi; s) \in \operatorname{supp}_\xi a \times [-1, 1] \tag{6.3}$$

and thus, for each $\xi \in \operatorname{supp}_\xi a$, the function $s \mapsto \langle \gamma''(s), \xi\rangle$ is strictly convex on $[-1, 1]$. The analysis for the portion of the symbol supported on the set $\{\xi_4 < 0\}$ follows by symmetry.

If $a_k := a \cdot \beta^k$, as introduced in §4.2, the derivative bound (6.1) implies, via the van der Corput lemma, that

$$|m[a_k](\xi)| \lesssim 2^{-k/4}. \tag{6.4}$$

Thus, Plancherel's theorem and interpolation with a trivial L^∞ estimate, as in the $J = 2$ case, yields

$$\|m[a_k]\|_{M^p(\mathbb{R}^4)} \lesssim 2^{-k/2p} \qquad \text{for all } 2 \leqslant p \leqslant \infty.$$

As in the $J = 3$ case, to obtain the improved bound $\|m[a_k]\|_{M^p(\mathbb{R}^4)} \lesssim 2^{-k/p}$, we decompose the symbol a_k into localised pieces which admit more refined decay rates than (6.4). This decomposition is, however, significantly more involved than that used in the previous section.

6.2. Geometry of the slow decay cones.

The first step is to isolate regions of the frequency space where the multiplier $m[a]$ decays relatively slowly. Owing to stationary phase considerations, this corresponds to the regions around the conic varieties

$$\Gamma_{d-1} := \{\xi \in \operatorname{supp}_\xi a : \langle \gamma^{(j)}(s), \xi\rangle = 0,\ 1 \leqslant j \leqslant d,\ \text{for some } s \in I_0\}, \qquad 2 \leqslant d \leqslant 3.$$

Note that Γ_{d-1} has codimension $d - 1$, which motivates the choice of index. Since $\Gamma_2 \subseteq \Gamma_1$, the decay rate for the multiplier $m[a]$ depends on the relative position with respect to both cones. To analyse this, we begin with the following observation, which helps us to understand the geometry of Γ_2.

Lemma 6.1. *If $\xi \in \operatorname{supp}_\xi a$, then the equation $\langle \gamma^{(3)}(s), \xi\rangle = 0$ has a unique solution in $s \in [-1, 1]$, which corresponds to the unique global minimum of the function $s \mapsto \langle \gamma''(s), \xi\rangle$. Furthermore, the solution has absolute value $O(\delta_0)$.*

[7] The choice $\delta_2 := \delta_0^3$ is not relevant to this part of the argument (we may simply take $\delta_2 := \delta_0$) but is used for consistency with the previous section.

The above lemma quickly follows from (6.3) and the localisation of the symbol via the mean value theorem. A detailed proof (of a very similar result) can be found in [1, Lemma 6.1].

By Lemma 6.1, there exists a unique smooth mapping $\theta_2 : \mathrm{supp}_\xi\, a \to [-1,1]$ such that

$$\langle \gamma^{(3)} \circ \theta_2(\xi), \xi \rangle = 0 \qquad \text{for all } \xi \in \mathrm{supp}_\xi\, a.$$

It is easy to see that θ_2 is homogeneous of degree 0. Define the quantities

$$u_{1,2}(\xi) := \langle \gamma' \circ \theta_2(\xi), \xi \rangle \quad \text{and} \quad u_2(\xi) := \langle \gamma'' \circ \theta_2(\xi), \xi \rangle \qquad \text{for all } \xi \in \mathrm{supp}_\xi\, a.$$

Note that $\xi \in \Gamma_2$ if and only if $u_{1,2}(\xi) = u_2(\xi) = 0$ and thus, roughly speaking, together the quantities $|u_2(\xi)|$ and $|u_{1,2}(\xi)|$ measure the distance of ξ to Γ_2.

The next observation helps us to understand the geometry of the cone Γ_1.

Lemma 6.2. *Let $\xi \in \mathrm{supp}_\xi\, a$ and consider the equation*

$$\langle \gamma''(s), \xi \rangle = 0. \tag{6.5}$$

i) If $u_2(\xi) > 0$, then the equation (6.5) has no solution on $[-1,1]$.
ii) If $u_2(\xi) = 0$, then the equation (6.5) has only the solution $s = \theta_2(\xi)$ on $[-1,1]$.
iii) If $u_2(\xi) < 0$, then the equation (6.5) has precisely two solutions on $[-1,1]$. Both solutions have absolute value $O(\delta_0^{1/2})$.

Again, this lemma quickly follows using the information in Lemma 6.1, the localisation of the symbol and Taylor expansion. The relevant details can be found in [1, Lemma 6.2].

Using Lemma 6.2, we construct a (unique) pair of smooth mappings

$$\theta_1^{\pm} \colon \{\xi \in \mathrm{supp}_\xi\, a : u_2(\xi) < 0\} \to [-1,1]$$

with $\theta_1^-(\xi) \leqslant \theta_1^+(\xi)$ which satisfies

$$\langle \gamma'' \circ \theta_1^{\pm}(\xi), \xi \rangle = 0 \qquad \text{for all } \xi \in \mathrm{supp}_\xi\, a \text{ with } u_2(\xi) < 0.$$

Define the functions

$$u_1^{\pm}(\xi) := \langle \gamma' \circ \theta_1^{\pm}(\xi), \xi \rangle \quad \text{and} \quad u_{3,1}^{\pm}(\xi) := \langle \gamma^{(3)} \circ \theta_1^{\pm}(\xi), \xi \rangle \qquad \text{for all } \xi \in \mathrm{supp}_\xi\, a \text{ with } u_2(\xi) < 0$$

and note that $\xi \in \Gamma_1$ if and only if $u_1^+(\xi) = 0$ or $u_1^-(\xi) = 0$. For this reason, we introduce

$$u_1(\xi) := \begin{cases} u_1^+(\xi) & \text{if } |u_1^+(\xi)| = \min_{\pm} |u_1^{\pm}(\xi)| \\ u_1^-(\xi) & \text{if } |u_1^-(\xi)| = \min_{\pm} |u_1^{\pm}(\xi)| \end{cases} \quad \text{and} \quad \theta_1(\xi) := \begin{cases} \theta_1^+(\xi) & \text{if } u_1(\xi) = u_1^+(\xi) \\ \theta_1^-(\xi) & \text{if } u_1(\xi) = u_1^-(\xi) \end{cases},$$

which clearly satisfy

$$u_1(\xi) = \langle \gamma' \circ \theta_1(\xi), \xi \rangle.$$

Roughly speaking, the quantity $|u_1(\xi)|$ measures the distance of ξ from Γ_1. Furthermore, if $\xi \in \Gamma_1$ satisfies $u_{3,1}(\xi) = 0$ where

$$u_{3,1}(\xi) := \langle \gamma^{(3)} \circ \theta_1(\xi), \xi \rangle,$$

then $\xi \in \Gamma_2$. Thus, again, $|u_{3,1}(\xi)|$ may be interpreted as measuring the distance of $\xi \in \Gamma_1$ to Γ_2.

The following lemma relates important information regarding the functions $\theta_2(\xi)$, $\theta_1^{\pm}(\xi)$ and the associated quantities $u_2(\xi)$, $u_{1,2}(\xi)$, $u_1^{\pm}(\xi)$, $u_{3,1}^{\pm}(\xi)$.

Lemma 6.3. *Let $\xi \in \mathrm{supp}_\xi\, a$ with $u_2(\xi) < 0$. Then the following hold:*

i) $\big|u_{3,1}^{\pm}\big(\frac{\xi}{|\xi|}\big)\big| \sim |\theta_1^{\pm}(\xi) - \theta_2(\xi)| \sim |\theta_1^+(\xi) - \theta_1^-(\xi)| \sim \big|u_2\big(\frac{\xi}{|\xi|}\big)\big|^{1/2}$,

ii) $\big|u_{1,2}\big(\frac{\xi}{|\xi|}\big) - u_1^{\pm}\big(\frac{\xi}{|\xi|}\big)\big| \lesssim \big|u_2\big(\frac{\xi}{|\xi|}\big)\big|^{3/2}$, *iii)* $\big|u_1^+\big(\frac{\xi}{|\xi|}\big) - u_1^-\big(\frac{\xi}{|\xi|}\big)\big| \sim \big|u_2\big(\frac{\xi}{|\xi|}\big)\big|^{3/2}$.

Proof. i) This is almost immediate from Taylor expansion around $\theta_2(\xi)$, and around $\theta_1^-(\xi)$ in the last display. The interested reader is referred to [1, Lemma 6.3] for details of a closely related calculation.

ii) By Taylor expansion around $\theta_1^\pm(\xi)$,

$$u_{1,2}(\xi) = u_1^\pm(\xi) + u_{3,1}^\pm(\xi) \cdot \frac{(\theta_2(\xi) - \theta_1^\pm(\xi))^2}{2} + \omega_4(\xi) \cdot (\theta_2(\xi) - \theta_1^\pm(\xi))^3,$$

where $|\omega_4(\xi)| \sim |\xi|$. The desired estimate follows from the above expansion and part i).

iii) By part i), it suffices to show

$$|u_1^+(\xi) - u_1^-(\xi)| \sim |u_2(\xi)||\theta_1^+(\xi) - \theta_1^-(\xi)|. \tag{6.6}$$

To this end, note that

$$u_1^+(\xi) - u_1^-(\xi) = \int_{\theta_1^-(\xi)}^{\theta_1^+(\xi)} \langle \gamma''(s), \xi\rangle \, \mathrm{d}s.$$

By Lemma 6.1, $u_2(\xi) \leqslant \langle \gamma''(s), \xi\rangle \leqslant 0$ for $\theta_1^-(\xi) \leqslant s \leqslant \theta_1^+(\xi)$. Thus, the upper bound in (6.6) immediately follows from the above identity and the triangle inequality. To see the lower bound in (6.6), recall from (6.3) that the function $s \mapsto \phi(s) := \langle \gamma''(s), \xi\rangle$ is strictly convex in $[-1, 1]$ and that $\phi \circ \theta_1^+(\xi) = \phi \circ \theta_1^-(\xi) = 0$. As $\theta_1^-(\xi) \leqslant \theta_2(\xi) \leqslant \theta_1^+(\xi)$ and $\phi \circ \theta_2(\xi) = u_2(\xi)$, the convexity of ϕ implies

$$\int_{\theta_1^-}^{\theta_1^+} |\langle \gamma''(s), \xi\rangle| \, \mathrm{d}s \geqslant \frac{1}{2}|u_2(\xi)||\theta_1^+(\xi) - \theta_1^-(\xi)|,$$

and thus (6.6) follows from the constant sign of $\phi(s)$ on $[\theta_1^-(\xi), \theta_1^+(\xi)]$. ☐

6.3. **Decomposition of the symbols.** For $k \geqslant 1$ consider the frequency localised symbols $a_k := a \cdot \beta^k$ as defined in §4.2. We begin by decomposing each a_k in relation to the codimension 2 cone Γ_2 corresponding to the directions of slowest decay for $\hat{\mu}$. In order to measure the distance to this cone, we consider the two quantities $u_{1,2}$ and u_2 introduced in the previous subsection and, in particular, form a simultaneous dyadic decomposition according to the relative sizes of each.

Here it is convenient to introduce a 'fine tuning' constant $\rho > 0$. This is a small (but absolute) constant which plays a minor technical rôle in the forthcoming arguments: taking $\rho := 10^{-6}$ more than suffices for our purposes.

Decomposition with respect to Γ_2. Let $\beta, \eta \in C_c^\infty(\hat{\mathbb{R}})$ be the functions used to perform a Littlewood–Paley decomposition in §4.2. Let $\beta_+, \beta_- \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \beta_+ \subset (0, \infty)$ and $\operatorname{supp} \beta_- \subset (-\infty, 0)$ be such that $\beta = \beta_+ + \beta_-$. For each $m \in \mathbb{N}$, write

$$\eta(2^{-1}r_1)\eta(r_2) = \sum_{\ell=0}^{m} \beta(2^{\ell-1}r_1)\eta(2^\ell r_2) + \sum_{\ell=0}^{m-1} \eta(2^\ell r_1)\big(\beta_+(2^\ell r_2) + \beta_-(2^\ell r_2)\big) + \eta(2^m r_1)\eta(2^m r_2).$$

The above formula corresponds to a smooth decomposition of $[-2, 2] \times [-1, 1]$ into axis-parallel dyadic rectangles: see Figure 3. We apply this decomposition[8] with $r_1 = 2^{-k}u_{1,2}(\xi)$ and $r_2 = \rho^{-1}2^{-k}u_2(\xi)$. This is then used to split the symbol a_k as a sum

$$a_k = \sum_{\ell=0}^{\lfloor k/4 \rfloor} a_{k,\ell,1} + a_{k,\ell,2} + \sum_{\ell=0}^{\lfloor k/4 \rfloor - 1} b_{k,\ell}$$

[8]Here the β function should be defined slightly differently compared with (4.3). In particular, when acting on r_1 we have $\beta(r_1) := \eta(2^{-2}r_1) - \eta(r_1)$ and when acting on r_2 we have $\beta(r_2) := \eta(2^{-3}r_2) - \eta(r_2)$. Such minor changes are ignored in the notation.

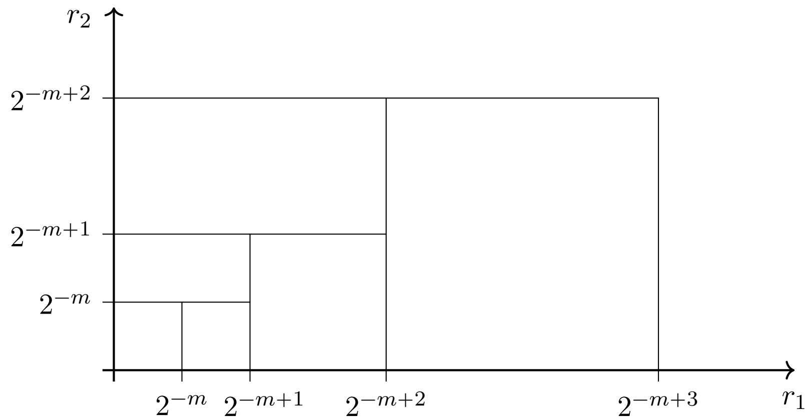

FIGURE 3. Two parameter dyadic decomposition in the upper-left quadrant.

where $\lfloor k/4 \rfloor$ denotes the greatest integer less than or equal to $k/4$ and

$$a_{k,\ell,1}(\xi;s) := a_k(\xi;s)\beta(2^{-k+3\ell}u_{1,2}(\xi))\eta(\rho^{-1}2^{-k+2\ell}u_2(\xi)) \qquad 0 \leqslant \ell \leqslant \lfloor k/4 \rfloor,$$

$$a_{k,\ell,2}(\xi;s) := \begin{cases} a_k(\xi;s)\eta(2^{-k+3\ell}u_{1,2}(\xi))\beta_+(\rho^{-1}2^{-k+2\ell}u_2(\xi)) & \text{if } 0 \leqslant \ell < \lfloor k/4 \rfloor \\ a_k(\xi;s)\eta(2^{-k+3\lfloor k/4 \rfloor}u_{1,2}(\xi))\eta(\rho^{-1}2^{-k+2\lfloor k/4 \rfloor}u_2(\xi)) & \text{if } \ell = \lfloor k/4 \rfloor \end{cases},$$

$$b_{k,\ell}(\xi;s) := a_k(\xi;s)\eta(2^{-k+3\ell}u_{1,2}(\xi))\beta_-(\rho^{-1}2^{-k+2\ell}u_2(\xi)) \qquad 0 \leqslant \ell < \lfloor k/4 \rfloor.$$

The following remarks help to motive the above decomposition:

For $\xi \in \operatorname{supp}_\xi a_{k,\ell,1}$, the functions $s \mapsto \langle \gamma'(s), \xi \rangle$ and $s \mapsto \langle \gamma''(s), \xi \rangle$ do not vanish simultaneously. This is due, in part, to the lower bound on $|u_{2,1}(\xi)|$. On the other hand, for $\xi \in \operatorname{supp}_\xi a_{k,\ell,2}$ we have $u_2(\xi) > 0$ and therefore $s \mapsto \langle \gamma''(s), \xi \rangle$ is non-vanishing by Lemma 6.2. Quantifying these observations, one obtains the decay estimate

$$|m[a_{k,\ell,\iota}](\xi)| \lesssim 2^{-k/2+\ell} \qquad \text{for } \iota = 1, 2 \tag{6.7}$$

via the van der Corput lemma. See Lemma 6.12 a) for details. This improves upon the trivial decay rate (6.4) since ℓ varies over the range $0 \leqslant \ell \leqslant \lfloor k/4 \rfloor$. Note that $\ell = k/4$ corresponds to the critical value where (6.4) and (6.7) agree.

For $\xi \in \operatorname{supp}_\xi b_{k,\ell}$, as $u_2(\xi) < 0$, the function $s \mapsto \langle \gamma''(s), \xi \rangle$ vanishes at $s = \theta_1^{\pm}(\xi)$ by Lemma 6.2. Moreover, the lack of a lower bound for $|u_{1,2}(\xi)|$ allows for simultaneous vanishing of $s \mapsto \langle \gamma'(s), \xi \rangle$ and $s \mapsto \langle \gamma''(s), \xi \rangle$, in contrast with the situation considered above. However, the lower bound on $|u_2(\xi)|$ implies that the functions $s \mapsto \langle \gamma''(s), \xi \rangle$ and $s \mapsto \langle \gamma^{(3)}(s), \xi \rangle$ do not vanish simultaneously. Quantifying these observations, one obtains, via the van der Corput lemma, the decay estimate

$$|m[b_{k,\ell}](\xi)| \lesssim 2^{-k/3+\ell/3}. \tag{6.8}$$

Again, this improves upon the trivial decay rate (6.4) since $0 \leqslant \ell < \lfloor k/4 \rfloor$ and, furthermore, $\ell = k/4$ corresponds to the critical value where (6.4) and (6.8) agree. However, the estimate (6.8) can be further improved by decomposing each $b_{k,\ell}$ with respect to the codimension 1 cone Γ_1. Recall that this cone corresponds to directions of slow (but not necessarily minimal) decay for $\hat{\mu}$. We proceed by performing a secondary dyadic decomposition with respect to the function u_1, which measures the distance to Γ_1.

Decomposition with respect to Γ_1. If $\xi \in \mathrm{supp}_\xi\, b_{k,\ell}$, then $u_2(\xi) < 0$ and therefore the roots $\theta_1^{\pm}(\xi) \in [-1,1]$ are well-defined by Lemma 6.2. Observe that

$$|u_2(\xi)| \sim \rho 2^{k-2\ell} \qquad \text{and} \qquad |u_{1,2}(\xi)| \lesssim 2^{k-3\ell} \quad \text{for all } \xi \in \mathrm{supp}_\xi\, b_{k,\ell},$$

and so it follows from Lemma 6.3 ii) that

$$|u_1(\xi)| \lesssim 2^{k-3\ell} \quad \text{for all } \xi \in \mathrm{supp}_\xi\, b_{k,\ell}.$$

Consequently, provided ρ is chosen sufficiently small,

$$b_{k,\ell_2}(\xi;s) = b_{k,\ell_2}(\xi;s)\eta(\rho 2^{-k+3\ell_2}u_1(\xi)). \tag{6.9}$$

For every $k \in \mathbb{N}$ define the indexing set

$$\Lambda(k) := \Big\{\boldsymbol{\ell} = (\ell_1,\ell_2) \in \mathbb{Z}^2 : 0 \leqslant \ell_2 < \lfloor k/4 \rfloor,\ \ell_2 \leqslant \ell_1 \leqslant \big\lfloor \tfrac{2k+\ell_2}{9} \big\rfloor \Big\}$$

and, for each $0 \leqslant \ell_2 < \lfloor k/4 \rfloor$, consider the fibre associated to its projection in the ℓ_2-variable,

$$\Lambda(k,\ell_2) := \big\{\boldsymbol{\ell} \in \Lambda(k) : \boldsymbol{\ell} = (\ell_1,\ell_2) \text{ for some } \ell_1 \in \mathbb{Z}\big\}.$$

In view of (6.9), we may decompose

$$b_{k,\ell_2}(\xi;s) = b_{k,\ell_2}(\xi;s)\eta(\rho 2^{-k+3\ell_2}u_1(\xi)) = a_{k,\ell_2,3}(\xi;s) + a_{k,\ell_2,4}(\xi;s) + \sum_{\boldsymbol{\ell} \in \Lambda(k,\ell_2)} b_{k,\boldsymbol{\ell}}(\xi;s)$$

where

$$a_{k,\ell_2,3}(\xi;s) := b_{k,\ell_2}(\xi;s)\big(\eta(\rho 2^{-k+3\ell_2}u_1(\xi)) - \eta(\rho^{-4}2^{-k+3\ell_2}u_1(\xi))\big),$$
$$a_{k,\ell_2,4}(\xi;s) := b_{k,\ell_2}(\xi;s)\eta(\rho^{-4}2^{-k+3\ell_2}u_1(\xi))\big(1 - \eta(\rho^{-1}2^{\ell_2}(s-\theta_1(\xi)))\big)$$

and

$$b_{k,\boldsymbol{\ell}}(\xi;s) := \begin{cases} b_{k,\ell_2}(\xi;s)\beta(\rho^{-4}2^{-k+3\ell_1}u_1(\xi))\eta(\rho^{-1}2^{\ell_2}(s-\theta_1(\xi))) & \text{if } \ell_1 < \lfloor (2k+\ell_2)/9 \rfloor \\ b_{k,\ell_2}(\xi;s)\eta(\rho^{-4}2^{-k+3\ell_1}u_1(\xi))\eta(\rho^{-1}2^{\ell_2}(s-\theta_1(\xi))) & \text{if } \ell_1 = \lfloor (2k+\ell_2)/9 \rfloor \end{cases}$$

for $\boldsymbol{\ell} = (\ell_1,\ell_2) \in \Lambda(k)$.

The final decomposition. Combining the preceding definitions, we have

$$a_k = \sum_{\ell=0}^{\lfloor k/4 \rfloor} \sum_{\iota=1}^{4} a_{k,\ell,\iota} + \sum_{\boldsymbol{\ell} \in \Lambda(k)} b_{k,\boldsymbol{\ell}} \tag{6.10}$$

where for $\iota = 3,4$ it is understood that $a_{k,\ell,\iota} \equiv 0$ for $\ell = \lfloor k/4 \rfloor$. This concludes the initial frequency decomposition.

The following remarks help to motive the above decomposition:

For $\xi \in \mathrm{supp}_\xi\, a_{k,\ell,3}$ or $\xi \in \mathrm{supp}_\xi\, a_{k,\ell,4}$ it transpires that the functions $s \mapsto \langle \gamma'(s), \xi \rangle$ and $s \mapsto \langle \gamma''(s), \xi \rangle$ do not vanish simultaneously. Quantifying these observations, one obtains the decay estimate

$$|m[a_{k,\ell,\iota}](\xi)| \lesssim 2^{-k/2+\ell} \qquad \text{for } \iota = 3,\, 4,$$

exactly as in (6.7). See Lemma 6.12 a) for details. Here, however, the attendant stationary phase arguments are a little more delicate than those used to prove (6.7) and, in particular, they rely on a careful analysis involving both Γ_1 and Γ_2. The lower bounds on $|u_1(\xi)|$ and $|s - \theta_1(\xi)|$ are fundamental in each case.

Turning to the $b_{k,\boldsymbol{\ell}}$ symbols, the localisation $|s - \theta_1(\xi)| \lesssim \rho 2^{-\ell_2}$ leads to the following key observation.

Lemma 6.4. *Let $k \in \mathbb{N}$ and $\boldsymbol{\ell} = (\ell_1, \ell_2) \in \Lambda(k)$. Then*

$$|\langle \gamma^{(3)}(s), \xi\rangle| \sim \rho^{1/2} 2^{k-\ell_2} \qquad \text{for all } (\xi; s) \in \operatorname{supp} b_{k,\boldsymbol{\ell}}. \tag{6.11}$$

Proof. The localisation of the symbol ensures the key properties

$$|u_2(\xi)| \sim \rho 2^{k-2\ell_2}, \quad |s - \theta_1(\xi)| \lesssim \rho 2^{-\ell_2} \qquad \text{for all } (\xi; s) \in \operatorname{supp} b_{k,\boldsymbol{\ell}}. \tag{6.12}$$

By the mean value theorem we obtain

$$\langle \gamma^{(3)}(s), \xi\rangle = u_{3,1}(\xi) + \omega(\xi; s) \cdot (s - \theta_1(\xi)), \tag{6.13}$$

where ω satisfies $|\omega(\xi; s)| \sim 2^k$. Observe that (6.12) and Lemma 6.3 i) imply $|u_{3,1}(\xi)| \sim \rho^{1/2} 2^{k-\ell_2}$. Consequently, provided ρ is sufficiently small, the second inequality in (6.12) implies that the $u_{3,1}$ term dominates the right-hand side of (6.13) and therefore the desired bound (6.11) holds. ☐

The condition (6.11) reveals that the symbol $b_{k,\boldsymbol{\ell}}$ essentially corresponds to a scaled version of the multiplier $a_{k,\ell}$ from the $J = 3$ case, for a suitable choice of ℓ and k. Of course, the condition (6.11) immediately implies

$$|m[b_{k,\boldsymbol{\ell}}(\xi)]| \lesssim 2^{-k/3 + \ell_2/3}, \tag{6.14}$$

as in (6.8). However, arguing as in Lemma 5.6, one may improve the decay rate to

$$|m[b_{k,\boldsymbol{\ell}}(\xi)]| \lesssim 2^{-k/2 + (3\ell_1 + \ell_2)/4}; \tag{6.15}$$

see Lemma 6.12 b). Indeed, for each $0 \leqslant \ell_2 < \lfloor k/4 \rfloor$, the decomposition of the $a_{k,\ell}$ for the $J = 3$ case in §5.3 matches that of the $b_{k,\boldsymbol{\ell}}$ above, with the identification

$$k \longleftrightarrow k - \ell_2 \qquad \text{and} \qquad \ell \longleftrightarrow \tfrac{3\ell_1 - \ell_2}{2}.$$

The bound (6.15) corresponds to the conclusion of Lemma 5.6 once we substitute in these indices. Observe that (6.15) is indeed an improvement over the trivial decay rate (6.14) since, for ℓ_2 fixed, ℓ_1 varies over the range $0 \leqslant \ell_1 \leqslant \lfloor \frac{2k+\ell_2}{9} \rfloor$. Note that $\ell_1 = \frac{2k+\ell_2}{9}$ corresponds to the critical index where (6.14) and (6.15) agree.

Remark. The symbols in the above decomposition are in fact smooth. This is not entirely obvious, since the function u_1 is defined pointwise as the minimum of $|u_1^-|$ and $|u_1^+|$. Thus, u_1 fails to be smooth whenever $u_1^-(\xi) = \pm u_1^+(\xi)$. However, the decomposition ensures that $|u_2(\xi)| \sim \rho 2^{k-2\ell_2}$ and $|u_1(\xi)| \lesssim \rho^4 2^{k-3\ell_2}$ for all $\xi \in \operatorname{supp}_\xi a_{k,\ell_2,4}$ or $\xi \in \operatorname{supp}_\xi b_{k,\boldsymbol{\ell}}$. Combining these facts with Lemma 6.3, one easily deduces that

$$|u_1^-(\xi) \pm u_1^+(\xi)| \gtrsim \rho^{3/2} 2^{k-3\ell_2}$$

and so u_1 is smooth on the ξ-support of either $a_{k,\ell_2,4}$ or b_{k,ℓ_2}. Furthermore, these observations also imply that the function $\theta_1(\xi)$ is smooth on the supports. The symbol $a_{k,\ell_2,3}$ can be treated in a similar manner, by writing it as a difference of the symbols

$$b_{k,\ell_2}(\xi; s)\eta(\rho 2^{-k+3\ell_2} u_1(\xi)) = b_{k,\ell_2}(\xi; s) \qquad \text{and} \qquad b_{k,\ell_2}(\xi; s)\eta(\rho^{-4} 2^{-k+3\ell_2} u_1(\xi))$$

and showing that both are smooth.

Given the above decomposition, in order to prove the $J = 4$ case of Theorem 4.1, it suffices to establish the following.

Proposition 6.5. *Let $6 \leqslant p \leqslant 12$, $k \in \mathbb{N}$ and $\varepsilon > 0$.*

a) For all $0 \leqslant \ell \leqslant \lfloor k/4 \rfloor$ and $1 \leqslant \iota \leqslant 4$,

$$\|m[a_{k,\ell,\iota}]\|_{M^p(\mathbb{R}^4)} \lesssim_{p,\varepsilon} 2^{-k/p - \ell(1/2 - 3/p - \varepsilon)}.$$

b) For all $\boldsymbol{\ell} = (\ell_1, \ell_2) \in \Lambda(k)$,

$$\|m[b_{k,\boldsymbol{\ell}}]\|_{M^p(\mathbb{R}^4)} \lesssim_{p,\varepsilon} 2^{-3(\ell_1 - \ell_2)(1/2p - \varepsilon) - \ell_2(1/2 - 3/p - \varepsilon)}.$$

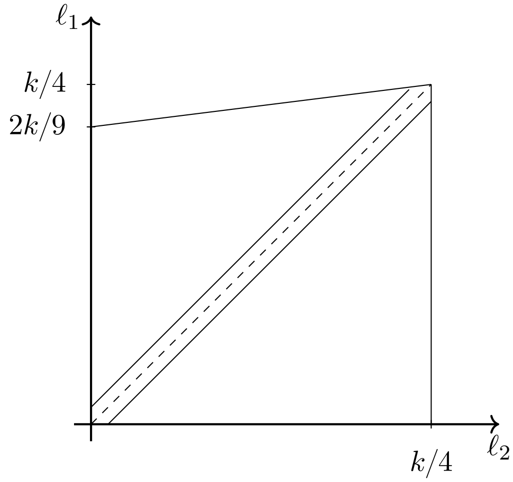

FIGURE 4. Setting $\ell = \ell_1$ for $a_{k,\ell,1}$ and $\ell = \ell_2$ for $a_{k,\ell,\iota}$, $2 \leqslant \iota \leqslant 4$, one can interpret the decomposition (6.10) in the (ℓ_2, ℓ_1)-plane as follows. The symbols $a_{k,\ell,1}$ correspond to horizontal lines in the lower triangle, whilst the symbols $a_{k,\ell,2}$ correspond to vertical lines in the diagonal and upper triangle whenever $u_2(\xi) > 0$. If $u_2(\xi) < 0$, the symbols $a_{k,\ell,3}$ correspond to vertical lines in the fattened diagonal, the symbols $a_{k,\ell,4}$ correspond to vertical lines in the upper triangle (under the additional condition that $|s - \theta_1(\xi)| \gtrsim 2^{-\ell_2}$) and the symbols $b_{k,\boldsymbol{\ell}}$ correspond to integer points in the upper triangle (under the additional condition that $|s-\theta_1(\xi)| \lesssim 2^{-\ell_2}$).

Proof of $J = 4$ case of Theorem 4.1, assuming Proposition 6.5. Let $6 < p \leqslant 12$ and define

$$\varepsilon_p := \frac{1}{2} \cdot \min\Big\{\frac{1}{2} - \frac{3}{p}, \frac{1}{2p}\Big\} > 0.$$

Apply the decomposition (6.10) to deduce that

$$\|m[a_k]\|_{M^p(\mathbb{R}^4)} \leqslant \sum_{\iota=1}^{4} \sum_{\ell=0}^{\lfloor k/4 \rfloor} \|m[a_{k,\ell,\iota}]\|_{M^p(\mathbb{R}^4)} + \sum_{\boldsymbol{\ell} \in \Lambda(k)} \|m[b_{k,\boldsymbol{\ell}}]\|_{M^p(\mathbb{R}^4)}.$$

By Proposition 6.5 a), we have

$$\sum_{\iota=1}^{4} \sum_{\ell=0}^{\lfloor k/4 \rfloor} \|m[a_{k,\ell,\iota}]\|_{M^p(\mathbb{R}^4)} \lesssim_p 2^{-k/p} \sum_{\ell=0}^{\infty} 2^{-\ell(1/2-3/p-\varepsilon_p)} \lesssim_p 2^{-k/p},$$

Similarly, by Proposition 6.5 b), we have

$$\sum_{\boldsymbol{\ell} \in \Lambda(k)} \|m[b_{k,\boldsymbol{\ell}}]\|_{M^p(\mathbb{R}^4)} \lesssim_p 2^{-k/p} \sum_{\ell_2=0}^{\infty} 2^{-\ell_2(1/2-3/p-\varepsilon_p)} \sum_{\ell_1=\ell_2}^{\infty} 2^{-3(\ell_1-\ell_2)(1/2p-\varepsilon_p)} \lesssim_p 2^{-k/p}.$$

Combining these observations establishes the desired result for $6 < p \leqslant 12$. The remaining range $12 < p \leqslant \infty$ follows by interpolation with a trivial L^∞ estimate. □

The rest of §6 is devoted to establishing Proposition 6.5. Before proceeding, it is instructive to describe the general strategy.

By Plancherel's theorem, (6.7) and (6.15) imply

$$\|m[a_{k,\ell,\iota}]\|_{M^2(\mathbb{R}^4)} \lesssim 2^{-k/2+\ell/2} \qquad \text{and} \qquad \|m[b_{k,\boldsymbol{\ell}}]\|_{M^2(\mathbb{R}^4)} \lesssim 2^{-k/2+(3\ell_1+\ell_2)/4}.$$

As ℓ, ℓ_1 and ℓ_2 increase, these estimates become weaker. To compensate for this, we attempt to establish stronger estimates for the $M^\infty(\mathbb{R}^4)$ norm. This is not possible, however, for the entire multipliers and a further decomposition is required. The $u_2(\xi)$ localisation means that $m[a_{k,\ell,\iota}]$ and $m[b_{k,\boldsymbol{\ell}}]$ are supported in a neighbourhood of the cone Γ_2. Furthermore, the $u_1(\xi)$ localisation means that $m[b_{k,\boldsymbol{\ell}}]$ is localised in a neighbourhood of the cone Γ_1. Consequently, one may apply a decoupling theorem for such cones (in particular, instances of Theorem 4.4) to *radially* decompose the multipliers. In the case of the $m[b_{k,\boldsymbol{\ell}}]$, we first decouple with respect to the cone Γ_2. After rescaling, the localised pieces can be treated in a similar manner to the multipliers from the $J = 3$ case. In particular, we apply a second decoupling to each rescaled piece with respect to the cone Γ_1 to further decompose into smaller pieces. For both the $a_{k,\ell,\iota}$ and $b_{k,\boldsymbol{\ell}}$, it transpires that each radially localised piece is automatically localised along the curve in the physical space, and this leads to favourable $M^\infty(\mathbb{R}^4)$ bounds: see Lemma 6.11 and Lemma 6.13 below.

6.4. **Fourier localisation and decoupling.** The first step towards proving Proposition 6.5 is to further decompose the symbols $a_{k,\ell,\iota}$ and $b_{k,\boldsymbol{\ell}}$ in terms of $\theta_2(\xi)$ and $\theta_1(\xi)$ respectively. Fix $\zeta \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\zeta \subseteq [-1,1]$ such that $\sum_{l\in\mathbb{Z}} \zeta(\,\cdot\, - l) \equiv 1$. For $0 \leqslant \ell \leqslant \lfloor k/4 \rfloor$, $1 \leqslant \iota \leqslant 4$ and $\boldsymbol{\ell} = (\ell_1, \ell_2) \in \Lambda(k)$, write

$$a_{k,\ell,\iota} = \sum_{\mu\in\mathbb{Z}} a^\mu_{k,\ell,\iota} \qquad \text{and} \qquad b_{k,\boldsymbol{\ell}} = \sum_{\nu\in\mathbb{Z}} b^\nu_{k,\boldsymbol{\ell}}$$

where

$$a^\mu_{k,\ell,\iota}(\xi; s) := a_{k,\ell,\iota}(\xi; s)\zeta(2^\ell\theta_2(\xi) - \mu), \tag{6.16}$$

$$b^\nu_{k,\boldsymbol{\ell}}(\xi; s) := b_{k,\boldsymbol{\ell}}(\xi; s)\zeta(2^{(3\ell_1-\ell_2)/2}\theta_1(\xi) - \nu). \tag{6.17}$$

In the case of the $b_{k,\boldsymbol{\ell}}$, we also consider symbols formed by grouping the $b^\nu_{k,\boldsymbol{\ell}}$ into pieces at the larger scale $2^{-\ell_2}$. Given $\boldsymbol{\ell} = (\ell_1, \ell_2) \in \Lambda(k)$ we write $\mathbb{Z} = \bigcup_{\mu\in\mathbb{Z}} \mathfrak{N}_{\boldsymbol{\ell}}(\mu)$, where the sets $\mathfrak{N}_{\boldsymbol{\ell}}(\mu)$ are disjoint and satisfy

$$\mathfrak{N}_{\boldsymbol{\ell}}(\mu) \subseteq \{\nu \in \mathbb{Z} : |\nu - 2^{3(\ell_1-\ell_2)/2}\mu| \leqslant 2^{3(\ell_1-\ell_2)/2}\}.$$

For each $\mu \in \mathbb{Z}$, we then define

$$b^{*,\mu}_{k,\boldsymbol{\ell}} := \sum_{\nu\in\mathfrak{N}_{\boldsymbol{\ell}}(\mu)} b^\nu_{k,\boldsymbol{\ell}}$$

and note that $|\theta_1(\xi) - s_\mu| \lesssim 2^{-\ell_2}$ on $\operatorname{supp} b^{*,\mu}_{k,\boldsymbol{\ell}}$, where $s_\mu := 2^{-\ell_2}\mu$. Of course, by the definition of the sets $\mathfrak{N}_{\boldsymbol{\ell}}(\mu)$,

$$b_{k,\boldsymbol{\ell}} = \sum_{\mu\in\mathbb{Z}} b^{*,\mu}_{k,\boldsymbol{\ell}} = \sum_{\mu\in\mathbb{Z}} \sum_{\nu\in\mathfrak{N}_{\boldsymbol{\ell}}(\mu)} b^\nu_{k,\boldsymbol{\ell}}.$$

Given $0 < r \leqslant 1$ and $s \in I$, recall the definition of the $(2, r)$-*Frenet boxes* $\pi_2(s; r)$ introduced in Definition 4.2:

$$\pi_2(s; r) := \big\{\xi \in \hat{\mathbb{R}}^4 : |\langle \mathbf{e}_j(s), \xi\rangle| \lesssim r^{4-j} \text{ for } 1 \leqslant j \leqslant 3, \quad |\langle \mathbf{e}_4(s), \xi\rangle| \sim 1\big\}. \tag{6.18}$$

The symbols $a^\mu_{k,\ell,\iota}$ and $b^{*,\mu}_{k,\boldsymbol{\ell}}$ satisfy the following support properties.

Lemma 6.6. *With the above definitions,*

a) $\operatorname{supp}_\xi a^\mu_{k,\ell,\iota} \subseteq 2^k \cdot \pi_2(s_\mu; 2^{-\ell})$ *for all* $0 \leqslant \ell \leqslant \lfloor k/4 \rfloor$, $1 \leqslant \iota \leqslant 4$ *and* $\mu \in \mathbb{Z}$, *where* $s_\mu := 2^{-\ell}\mu$;

b) $\operatorname{supp}_\xi b^{*,\mu}_{k,\boldsymbol{\ell}} \subseteq 2^k \cdot \pi_2(s_\mu; 2^{-\ell_2})$ *for all* $\boldsymbol{\ell} = (\ell_1, \ell_2) \in \Lambda(k)$ *and* $\mu \in \mathbb{Z}$, *where* $s_\mu := 2^{-\ell_2}\mu$.

It is convenient to set up a unified framework in order to treat parts a) and b) of Lemma 6.6 simultaneously. Given n, $s \in \mathbb{R}$, let $\Xi_2(k, n; s)$ denote the set of all $\xi \in \operatorname{supp}_\xi a_k$ which lie in the domain of θ_2 and satisfy

$$|\theta_2(\xi) - s| \lesssim 2^{-n}, \qquad |u_{1,2}(\xi)| \lesssim 2^{k-3n}, \qquad |u_2(\xi)| \lesssim 2^{k-2n}. \tag{6.19}$$

Note in particular that:

a) $\mathrm{supp}_\xi\, a^{\mu}_{k,\ell,\iota} \subseteq \Xi_2(k,\ell;s_\mu)$ for $1 \leqslant \iota \leqslant 4$.
b) $\mathrm{supp}_\xi\, b^{\nu}_{k,\boldsymbol{\ell}} \subseteq \Xi_2(k,\ell_2;s_\nu)$ and $\mathrm{supp}_\xi\, b^{\nu}_{k,\boldsymbol{\ell}} \subseteq \Xi_2(k,\ell_2;s_\mu)$ for all $\nu \in \mathfrak{N}_{\boldsymbol{\ell}}(\mu)$.

Indeed, for the respective parameter values, all the desired properties stated in (6.19) hold as an immediate consequence of the definition of the symbols, with the exception of the bounds $|\theta_2(\xi) - s_\nu| \lesssim 2^{-\ell_2}$ for $\xi \in \mathrm{supp}_\xi\, b^{\nu}_{k,\ell}$ and $|\theta_2(\xi) - s_\mu| \lesssim 2^{-\ell_2}$ for $\xi \in \mathrm{supp}_\xi\, b^{\nu}_{k,\ell}$ and $\nu \in \mathfrak{N}_{\boldsymbol{\ell}}(\mu)$. However, by Lemma 6.3, it follows from the localisation of the symbol that

$$|\theta_2(\xi) - s_\nu| \lesssim \big|u_2\big(\tfrac{\xi}{|\xi|}\big)\big|^{1/2} + |\theta_1(\xi) - s_\nu| \lesssim 2^{-\ell_2} \quad \text{for all } \xi \in \mathrm{supp}_\xi\, b^{\nu}_{k,\ell},$$

which further implies

$$|\theta_2(\xi) - s_\mu| \lesssim |\theta_2(\xi) - s_\nu| + |s_\nu - s_\mu| \lesssim 2^{-\ell_2} \quad \text{for all } \xi \in \mathrm{supp}_\xi\, b^{\nu}_{k,\ell}, \;\; \nu \in \mathfrak{N}_{\boldsymbol{\ell}}(\mu),$$

by the condition $|s_\mu - s_\nu| \lesssim 2^{-\ell_2}$ for $s_\nu := 2^{-(3\ell_1-\ell_2)/2}\nu$. Thus, all the required bounds hold.

Note that the support property in b) immediately implies that $\mathrm{supp}_\xi\, b^{*,\mu}_{k,\boldsymbol{\ell}} \subseteq \Xi_2(k,\ell_2;s_\mu)$.

Proof of Lemma 6.6. Let n, $s \in \mathbb{R}$. As a consequence of the preceding discussion, it suffices to show that

$$\Xi_2(k,n;s) \subseteq 2^k \cdot \pi_2(s;2^{-n}).$$

Let $\xi \in \Xi_2(k,n;s)$ and observe that the localisation of a_k, the implicit definition of θ_2 and the latter two conditions in (6.19) imply

$$|\langle \gamma^{(i)} \circ \theta_2(\xi), \xi\rangle| \lesssim 2^{k-(4-i)n} \qquad \text{for } 1 \leqslant i \leqslant 4.$$

Since the Frenet vectors $\mathbf{e}_i \circ \theta_2(\xi)$ are obtained from the $\gamma^{(i)} \circ \theta_2(\xi)$ via the Gram–Schmidt process,

$$|\langle \mathbf{e}_i \circ \theta_2(\xi), \xi\rangle| \lesssim 2^{k-(4-i)n} \qquad \text{for } 1 \leqslant i \leqslant 4.$$

On the other hand, the first condition in (6.19), together with (4.12), implies

$$|\langle \mathbf{e}_i \circ \theta_2(\xi), \mathbf{e}_j(s)\rangle| \lesssim |\theta_2(\xi) - s|^{|i-j|} \lesssim 2^{-(i-j)n}.$$

Writing ξ with respect to the orthonormal basis $\big(\mathbf{e}_j \circ \theta_2(\xi)\big)_{j=1}^4$, it follows that

$$|\langle \mathbf{e}_j(s), \xi\rangle| \leqslant \sum_{i=1}^{4} |\langle \mathbf{e}_i \circ \theta_2(\xi), \xi\rangle||\langle \mathbf{e}_i \circ \theta_2(\xi), \mathbf{e}_j(s_\mu)\rangle| \lesssim 2^{k-(4-j)n}.$$

Thus, ξ satisfies all the required upper bounds arising from (6.18). The remaining condition $|\langle \mathbf{e}_4(s), \xi\rangle| \gtrsim 2^k$ holds as an immediate consequence of the initial localisation of a_k. ☐

The argument used in the proof of Lemma 6.6 can be applied to analyse the support properties of the individual $b^{\nu}_{k,\boldsymbol{\ell}}$, although in this case the geometric significance of the supporting set is only apparent after rescaling (see Lemma 6.8 below). Given $0 < r_1, r_2 \leqslant 1$ and $s \in I$, define the set

$$\pi_1(s;r_1,r_2) := \big\{\xi \in \hat{\mathbb{R}}^4 : |\langle \mathbf{e}_j(s), \xi\rangle| \lesssim r_1^{3-j} \text{ for } j = 1,\, 2,\;\; |\langle \mathbf{e}_3(s), \xi\rangle| \sim 1,\;\; |\langle \mathbf{e}_4(s), \xi\rangle| \lesssim r_2\big\}. \quad (6.20)$$

The multipliers $b^{\nu}_{k,\boldsymbol{\ell}}$ satisfy the following support property.

Lemma 6.7. *With the above definitions,*

$$\mathrm{supp}_\xi\, b^{\nu}_{k,\boldsymbol{\ell}} \subseteq 2^{k-\ell_2} \cdot \pi_1(s_\nu; 2^{-(3\ell_1-\ell_2)/2}, 2^{\ell_2})$$

for all $\boldsymbol{\ell} = (\ell_1,\ell_2) \in \Lambda(k)$ *and* $\nu \in \mathbb{Z}$*, where* $s_\nu := 2^{-(3\ell_1-\ell_2)/2}\nu$*.*

As with the proof of Lemma 6.6, here and in Lemma 6.13, we will work with a more general setup. This abstraction is not particularly useful at this stage, but it will help to unify some of the later arguments. Given $\boldsymbol{n} = (n_1, n_2) \in (0, \infty)^2$ and $s \in \mathbb{R}$, let $\Xi_1(k, \boldsymbol{n}; s)$ denote the set of all $\xi \in \operatorname{supp}_\xi a_k$ which lie in the domain of θ_1 and satisfy

$$|\theta_1(\xi) - s| \lesssim 2^{-n_1}, \qquad |u_1(\xi)| \lesssim 2^{k-2n_1-n_2}, \qquad |u_{3,1}(\xi)| \sim 2^{k-n_2}. \tag{6.21}$$

Note in particular that:

a) $\operatorname{supp}_\xi a^{\mu}_{k,\ell,\iota} \subseteq \Xi_1(k, \ell, \ell; s_\mu)$ for $\iota = 3$ or $\iota = 4$.
b) $\operatorname{supp}_\xi b^{\nu}_{k,\boldsymbol{\ell}} \subseteq \Xi_1\big(k, \frac{3\ell_1-\ell_2}{2}, \ell_2; s_\nu\big)$.

Indeed, the definition of the symbols implies $|u_2(\xi)| \sim 2^{k-2\ell}$, $|u_1(\xi)| \lesssim 2^{k-3\ell}$ and $|\theta_2(\xi) - s_\mu| \lesssim 2^{-\ell}$ for all $\xi \in \operatorname{supp}_\xi a^{\mu}_{k,\ell,\iota}$ when $\iota \in \{3, 4\}$. Consequently, by Lemma 6.3, it follows that

$$|\theta_1(\xi) - s_\mu| \lesssim \big|u_2\big(\tfrac{\xi}{|\xi|}\big)\big|^{1/2} + |\theta_2(\xi) - s_\mu| \lesssim 2^{-\ell}, \qquad \big|u_{3,1}\big(\tfrac{\xi}{|\xi|}\big)\big| \sim \big|u_2\big(\tfrac{\xi}{|\xi|}\big)\big|^{1/2} \sim 2^{\ell}$$

for all $\xi \in \operatorname{supp}_\xi a^{\mu}_{k,\ell,\iota}$, which covers the required bounds for a). Turning to b), all the desired properties hold as an immediate consequence of the definition of the symbols, with the exception of the bound $|u_{3,1}(\xi)| \sim 2^{k-\ell_2}$. However, as in a), the function $u_{3,1}$ can be estimated via Lemma 6.3 using the u_2 localisation.

Proof of Lemma 6.7. Let $\boldsymbol{n} = (n_1, n_2) \in (0, \infty)^2$ and $s \in I_0$. As a consequence of the preceding discussion, it suffices to show that

$$\Xi_1(k, \boldsymbol{n}; s) \subseteq 2^{k-n_2} \cdot \pi_1(s; 2^{-n_1}, 2^{-n_2}).$$

The argument in fact depends on the implicit constants in (6.21) satisfying certain size relations, but we shall ignore this minor technicality. In the case in question (namely, on the support of $b^{\nu}_{k,\ell}$), the required size relations follow provided ρ is chosen sufficiently small.

Let $\xi \in \Xi_1(k, \boldsymbol{n}; s)$ and observe that the localisation of a_k, the implicit definition of θ_1 and the latter two conditions in (6.21) imply

$$|\langle \gamma^{(i)} \circ \theta_1(\xi), \xi\rangle| \lesssim 2^{k-(3-i)n_1-n_2} \quad \text{for } i = 1,\, 2, \quad |\langle \gamma^{(3)} \circ \theta_1(\xi), \xi\rangle| \sim 2^{k-n_2}, \quad |\langle \gamma^{(4)} \circ \theta_1(\xi), \xi\rangle| \sim 2^{k}.$$

Since the Frenet vectors $\mathbf{e}_i \circ \theta_2(\xi)$ are obtained from the $\gamma^{(i)} \circ \theta_2(\xi)$ via the Gram–Schmidt process, the matrix corresponding to change of basis from $\big(\mathbf{e}_i \circ \theta_1(\xi)\big)_{i=1}^4$ to $\big(\gamma^{(i)} \circ \theta_1(\xi)\big)_{i=1}^4$ is lower triangular. Furthermore, the initial localisations imply that this matrix is an $O(\delta)$ perturbation of the identity. Consequently, provided $\delta > 0$ is chosen sufficiently small,

$$|\langle \mathbf{e}_i \circ \theta_1(\xi), \xi\rangle| \lesssim 2^{k-(3-i)n_1-n_2} \quad \text{for } i = 1,\, 2, \quad |\langle \mathbf{e}_3 \circ \theta_1(\xi), \xi\rangle| \sim 2^{k-n_2}, \quad |\langle \mathbf{e}_4 \circ \theta_1(\xi), \xi\rangle| \sim 2^{k}.$$

On the other hand, the first condition in (6.21) together with (4.12) imply

$$|\langle \mathbf{e}_i \circ \theta_1(\xi), \mathbf{e}_j(s)\rangle| \lesssim |s - \theta_1(\xi)|^{|i-j|} \lesssim 2^{-(i-j)n_1}.$$

Writing ξ with respect to the orthonormal basis $\big(\mathbf{e}_j \circ \theta_1(\xi)\big)_{j=1}^4$, it follows that

$$|\langle \xi, \mathbf{e}_j(s)\rangle| \leqslant \sum_{i=1}^{4} |\langle \mathbf{e}_i \circ \theta_1(\xi), \xi\rangle||\langle \mathbf{e}_i \circ \theta_1(\xi), \mathbf{e}_j(s)\rangle| \lesssim 2^{k-((3-j)n_1+n_2)\vee 0}.$$

Thus, ξ satisfies all the required upper bounds arising from (6.20). The above argument can easily be adapted to give the required lower bounds, provided the implied constant in the the hypothesis $|u_{3,1}(\xi)| \sim 2^{k-n_2}$ is large compared to that in the hypothesis $|\theta_1(\xi) - s| \lesssim 2^{-n_1}$. $\square$

Fix some $\boldsymbol{\ell} = (\ell_1, \ell_2) \in \Lambda(k)$ and $\mu \in \mathbb{Z}$ with $s_\mu := 2^{-\ell_2}\mu \in [-1, 1]$. To simplify notation, let $\sigma := s_\mu$, $\lambda := 2^{-\ell_2}$ and let $\tilde{\gamma} := \gamma_{\sigma,\lambda}$ denote the rescaled curve, as defined in Definition 2.1, so that

$$\tilde{\gamma}(s) := \big([\gamma]_{\sigma,\lambda}\big)^{-1}\big(\gamma(\sigma + \lambda s) - \gamma(\sigma)\big). \tag{6.22}$$

Given a symbol $b \in C_c^\infty(\hat{\mathbb{R}}^4 \times I_0)$, let $\tilde{b}$ be the rescaled symbol defined by the relation

$$\tilde{b}(\tilde{\xi}; \tilde{s}) = b(\xi; s) \qquad \text{for} \quad \tilde{\xi} := \big([\gamma]_{\sigma,\lambda}\big)^\top \xi \quad \text{and} \quad \tilde{s} := \lambda^{-1}(s - \sigma). \tag{6.23}$$

Given $f \in \mathscr{S}(\mathbb{R}^4)$, it follows by a simple changes of the variables that

$$m[b](D)f(x) = \lambda \cdot \tilde{m}[\tilde{b}](D)\tilde{f}(\tilde{x}) \tag{6.24}$$

where:

- The multiplier $\tilde{m}[\tilde{b}]$ is defined in the same manner as $m[\tilde{b}]$ but with the curve γ replaced with $\tilde{\gamma}$ and the cut-off $\chi_\circ$ replaced with $\chi_\circ(\sigma + \lambda\,\cdot)$;
- $\tilde{f} := f \circ [\gamma]_{\sigma,\lambda}$;
- $\tilde{x} := \big([\gamma]_{\sigma,\lambda}\big)^{-1}\big(x - \gamma(\sigma)\big)$.

Let $(\tilde{\mathbf{e}}_j)_{j=1}^4$ denote the Frenet frame defined with respect to $\tilde{\gamma}$. Given $0 < r \leqslant 1$ and $s \in I$, recall the definition of the $(1, r)$-*Frenet boxes* (with respect to $(\tilde{\mathbf{e}}_j)_{j=1}^4$) introduced in Definition 4.2:

$$\tilde{\pi}_1(s; r) := \big\{\xi \in \hat{\mathbb{R}}^4 : |\langle \tilde{\mathbf{e}}_j(s), \xi\rangle| \lesssim r^{3-j} \quad \text{for } j = 1,\, 2, \quad |\langle \tilde{\mathbf{e}}_3(s), \xi\rangle| \sim 1, \quad |\langle \tilde{\mathbf{e}}_4(s), \xi\rangle| \lesssim 1\big\}.$$

Note that all these definitions depend of the choice of μ and $\boldsymbol{\ell}$, but it is typographically convenient to suppress this dependence.

The rescaled symbols $\tilde{b}^\nu_{k,\boldsymbol{\ell}}$ satisfy the following support properties.

Lemma 6.8. *With the above definitions,*

$$\operatorname{supp}_\xi \tilde{b}^\nu_{k,\boldsymbol{\ell}} \subseteq 2^{k-4\ell_2} \cdot \tilde{\pi}_1(\tilde{s}_\nu; 2^{-3(\ell_1-\ell_2)/2})$$

for all $\boldsymbol{\ell} = (\ell_1, \ell_2) \in \Lambda(k)$ *and* $\nu \in \mathfrak{N}_{\boldsymbol{\ell}}(\mu)$, *where* $\tilde{s}_\nu := 2^{\ell_2}(s_\nu - s_\mu)$ *for* $s_\nu := 2^{-(3\ell_1-\ell_2)/2}\nu$.

Proof. For $\tilde{\xi} \in \operatorname{supp}_\xi \tilde{b}^\nu_{k,\boldsymbol{\ell}}$, it follows from Lemma 6.7 and the definition of the rescaling in (6.23) that $\xi := \big([\gamma]_{\sigma,\lambda}\big)^{-\top}\tilde{\xi}$ satisfies

$$|\langle \mathbf{e}_j(s_\nu), \xi\rangle| \lesssim 2^{k-(3-j)(3\ell_1-\ell_2)/2-\ell_2} \quad \text{for } j = 1,\, 2, \quad |\langle \mathbf{e}_3(s_\nu), \xi\rangle| \sim 2^{k-\ell_2}, \quad |\langle \mathbf{e}_4(s_\nu), \xi\rangle| \sim 2^k.$$

Since the matrix corresponding to the change of basis from $\big(\mathbf{e}_j(s_\nu)\big)_{j=1}^4$ to $\big(\gamma^{(j)}(s_\nu)\big)_{j=1}^4$ is lower triangular and an $O(\delta_0)$ perturbation of the identity, provided δ_0 is sufficiently small,

$$|\langle \gamma^{(j)}(s_\nu), \xi\rangle| \lesssim 2^{k-(3-j)(3\ell_1-\ell_2)/2-\ell_2} \quad \text{for } j = 1,\, 2, \quad |\langle \gamma^{(3)}(s_\nu), \xi\rangle| \sim 2^{k-\ell_2}, \quad |\langle \gamma^{(4)}(s_\nu), \xi\rangle| \sim 2^k.$$

On the other hand, recalling that $\lambda := 2^{-\ell_2}$, it follows from the definition of $\tilde{\gamma}$ from (6.22) that

$$\langle \tilde{\gamma}^{(j)}(\tilde{s}_\nu), \tilde{\xi}\rangle = 2^{-j\ell_2}\langle \gamma^{(j)}(s_\nu), \xi\rangle \qquad \text{for } j \geqslant 1.$$

Combining the above observations,

$$|\langle \tilde{\gamma}^{(j)}(\tilde{s}_\nu), \tilde{\xi}\rangle| \lesssim 2^{k-(3-j)(3\ell_1-\ell_2)/2-(j+1)\ell_2} \quad \text{for } j = 1,\, 2,$$
$$|\langle \tilde{\gamma}^{(3)}(\tilde{s}_\nu), \tilde{\xi}\rangle| \sim 2^{k-4\ell_2}, \quad |\langle \tilde{\gamma}^{(4)}(\tilde{s}_\nu), \tilde{\xi}\rangle| \sim 2^{k-4\ell_2}.$$

Provided δ_0 is sufficiently small, the desired result now follows since the matrix corresponding to the change of basis from $\big(\tilde{\mathbf{e}}_i(\tilde{s}_\nu)\big)_{i=1}^4$ to $\big(\tilde{\gamma}^{(i)}(\tilde{s}_\nu)\big)_{i=1}^4$ is also lower triangular and an $O(\delta_0)$ perturbation of the identity. ☐

In view of the support conditions from Lemma 6.6 and Lemma 6.8, the multipliers can be effectively decoupled using Theorem 4.4.

Proposition 6.9. *For all $2 \leqslant p \leqslant 12$ and all $\varepsilon > 0$, the following inequalities hold:*

a) For all $0 \leqslant \ell \leqslant \lfloor k/4 \rfloor$, $1 \leqslant \iota \leqslant 4$,

$$\Big\| \sum_{\mu \in \mathbb{Z}} m[a^{\mu}_{k,\ell,\iota}](D)f \Big\|_{L^p(\mathbb{R}^4)} \lesssim_\varepsilon 2^{\ell(1/2-1/p)+\varepsilon\ell} \Big(\sum_{\mu \in \mathbb{Z}} \|m[a^{\mu}_{k,\ell,\iota}](D)f\|^p_{L^p(\mathbb{R}^4)} \Big)^{1/p}.$$

b) For all $\boldsymbol{\ell} = (\ell_1, \ell_2) \in \Lambda(k)$,

$$\Big\| \sum_{\mu \in \mathbb{Z}} m[b^{*,\mu}_{k,\boldsymbol{\ell}}](D)f \Big\|_{L^p(\mathbb{R}^4)} \lesssim_\varepsilon 2^{\ell_2(1/2-1/p)+\varepsilon\ell_2} \Big(\sum_{\mu \in \mathbb{Z}} \|m[b^{*,\mu}_{k,\boldsymbol{\ell}}](D)f\|^p_{L^p(\mathbb{R}^4)} \Big)^{1/p}.$$

Proof. In view of the support conditions from Lemma 6.6, after a simple rescaling, the desired result follows from Theorem 4.4 with $d - 1 = 2$, $n = 4$ and $r = 2^{-\ell}$, $2^{-\ell_2}$ for parts a) and b), respectively. □

Proposition 6.10. *For all $\boldsymbol{\ell} = (\ell_1, \ell_2) \in \Lambda(k)$, $6 \leqslant p \leqslant 12$ and $\varepsilon > 0$,*

$$\Big\| \sum_{\nu \in \mathbb{Z}} m[b^{\nu}_{k,\boldsymbol{\ell}}](D)f \Big\|_{L^p(\mathbb{R}^4)} \lesssim_\varepsilon 2^{\ell_2(1/2-1/p+\varepsilon)} 2^{3(\ell_1-\ell_2)(1-4/p+\varepsilon)/2} \Big(\sum_{\nu \in \mathbb{Z}} \|m[b^{\nu}_{k,\boldsymbol{\ell}}](D)f\|^p_{L^p(\mathbb{R}^4)} \Big)^{1/p}.$$

Proof. It suffices to show that, under the hypotheses of the proposition, for all $\mu \in \mathbb{Z}$ one has

$$\Big\| \sum_{\nu \in \mathfrak{N}_{\boldsymbol{\ell}}(\mu)} m[b^{\nu}_{k,\boldsymbol{\ell}}](D)f \Big\|_{L^p(\mathbb{R}^4)} \lesssim_\varepsilon 2^{3(\ell_1-\ell_2)(1-4/p+\varepsilon)/2} \Big(\sum_{\nu \in \mathfrak{N}_{\boldsymbol{\ell}}(\mu)} \|m[b^{\nu}_{k,\boldsymbol{\ell}}](D)f\|^p_{L^p(\mathbb{R}^4)} \Big)^{1/p}. \tag{6.25}$$

Indeed, one may then combine the above inequality with Proposition 6.9 b) to obtain the desired decoupling result. However, by applying a linear change of variables, (6.25) is equivalent to the same inequality but with each $m[b^{\nu}_{k,\boldsymbol{\ell}}]$ replaced with the rescaled multiplier $\tilde{m}[\tilde{b}^{\nu}_{k,\boldsymbol{\ell}}]$ as defined in (6.24). In view of the support conditions from Lemma 6.8, after a simple rescaling, the desired result follows from Theorem 4.4 with $d - 1 = 1$, $n = 4$ and $r = 2^{-3(\ell_1-\ell_2)/2}$. □

6.5. **Localisation along the curve.** The localisation in $\theta_2(\xi)$ and $\theta_1(\xi)$ introduced in the previous subsection induces a corresponding localisation along the curve in the physical space. In particular, the main contribution to $m[a^{\mu}_{k,\ell,\iota}]$ and $m[b^{\nu}_{k,\boldsymbol{\ell}}]$ arises from the portion of the curve defined over the interval $|s - s_\mu| \leqslant 2^{-\ell}$ and $|s - s_\nu| \leqslant 2^{-(3\ell_1-\ell_2)/2}$, respectively. This is made precise by Lemma 6.11 below.

For each $\mu, \nu \in \mathbb{Z}$, let $s_\mu := 2^{-\ell}\mu$ and $s_\nu := 2^{-(3\ell_1-\ell_2)/2}\nu$. Given $\varepsilon > 0$ and for the fine tuning parameter ρ as introduced in §6.3, define

$$a^{\mu,(\varepsilon)}_{k,\ell,\iota}(\xi; s) := a^{\mu}_{k,\ell,\iota}(\xi)\eta(\rho 2^{\ell(1-\varepsilon)}(s - s_\mu)), \tag{6.26}$$

$$b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}}(\xi; s) := b^{\nu}_{k,\boldsymbol{\ell}}(\xi)\eta(\rho 2^{(1-\varepsilon)(3\ell_1-\ell_2)/2}(s - s_\nu)). \tag{6.27}$$

The key contribution to the multipliers comes from the symbols $a^{\mu,(\varepsilon)}_{k,\ell,\iota}$ and $b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}}$ respectively.

Lemma 6.11. *Let $2 \leqslant p < \infty$ and $\varepsilon > 0$.*

a) For all $0 \leqslant \ell \leqslant \lfloor k/4 \rfloor$, $\mu \in \mathbb{Z}$ and $1 \leqslant \iota \leqslant 4$,

$$\|m[a^{\mu}_{k,\ell,\iota} - a^{\mu,(\varepsilon)}_{k,\ell,\iota}]\|_{M^p(\mathbb{R}^4)} \lesssim_{N,\varepsilon,p} 2^{-kN} \qquad \textit{for all } N \in \mathbb{N}.$$

b) For all $\boldsymbol{\ell} = (\ell_1, \ell_2) \in \Lambda(k)$ and $\nu \in \mathbb{Z}$,

$$\|m[b^{\nu}_{k,\boldsymbol{\ell}} - b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}}]\|_{M^p(\mathbb{R}^4)} \lesssim_{N,\varepsilon,p} 2^{-kN} \qquad \textit{for all } N \in \mathbb{N}.$$

Proof. In both part a) and b) it is clear that the multipliers satisfy a trivial L^∞-estimate with operator norm $O(2^{Ck})$ for some absolute constant $C \geqslant 1$. Thus, by interpolation, it suffices to prove the rapid decay for $p = 2$ only. This amounts to showing that, under the hypotheses of the lemma,

$$\|m[a^{\mu}_{k,\ell,\iota} - a^{\mu,(\varepsilon)}_{k,\ell,\iota}]\|_{L^\infty(\hat{\mathbb{R}}^4)} \lesssim_{N,\varepsilon} 2^{-kN} \quad \text{and} \quad \|m[b^{\nu}_{k,\boldsymbol{\ell}} - b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}}]\|_{L^\infty(\hat{\mathbb{R}}^4)} \lesssim_{N,\varepsilon} 2^{-kN} \quad \text{for all } N \in \mathbb{N}. \tag{6.28}$$

This is achieved via a simple (non)-stationary phase analysis.

a) Here the localisation of the $a_{k,\ell,\iota}$ symbols ensures that

$$|u_{1,2}(\xi)| \lesssim 2^{k-3\ell}, \quad |u_2(\xi)| \lesssim \rho 2^{k-2\ell} \qquad \text{for all } (\xi; s) \in \operatorname{supp}(a^{\mu}_{k,\ell,\iota} - a^{\mu,(\varepsilon)}_{k,\ell,\iota}). \tag{6.29}$$

On the other hand, provided ρ is sufficiently small, the additional localisation in (6.16) and (6.26) implies

$$|s - \theta_2(\xi)| \gtrsim \rho^{-1} 2^{-\ell(1-\varepsilon)} \quad \text{for all } (\xi; s) \in \operatorname{supp}(a^{\mu}_{k,\ell,\iota} - a^{\mu,(\varepsilon)}_{k,\ell,\iota}). \tag{6.30}$$

Fix $\xi \in \operatorname{supp}_\xi (a^{\mu}_{k,\ell,\iota} - a^{\mu,(\varepsilon)}_{k,\ell,\iota})$ and consider the oscillatory integral $m[a^{\mu}_{k,\ell,\iota} - a^{\mu,(\varepsilon)}_{k,\ell,\iota}](\xi)$, which has phase $s \mapsto \langle \gamma(s), \xi \rangle$. Taylor expansion around $\theta_2(\xi)$ yields

$$\langle \gamma'(s), \xi \rangle = u_{1,2}(\xi) + \big(u_2(\xi) + \omega_1(\xi; s) \cdot (s - \theta_2(\xi))^2\big) \cdot (s - \theta_2(\xi)), \tag{6.31}$$

$$\langle \gamma''(s), \xi \rangle = u_2(\xi) + \omega_2(\xi; s) \cdot (s - \theta_2(\xi))^2, \tag{6.32}$$

$$\langle \gamma^{(3)}(s), \xi \rangle = \omega_3(\xi; s) \cdot (s - \theta_2(\xi)), \tag{6.33}$$

where the ω_i arise from the remainder terms and satisfy $|\omega_i(\xi; s)| \sim 2^k$. Provided ρ is sufficiently small, (6.29) and (6.30) imply that the $\omega_1(\xi; s) \cdot (s - \theta_2(\xi))^3$ term dominates the right-hand side of (6.31) and therefore

$$|\langle \gamma'(s), \xi \rangle| \gtrsim 2^k |s - \theta_2(\xi)|^3 \quad \text{for all } (\xi; s) \in \operatorname{supp}(a^{\mu}_{k,\ell,\iota} - a^{\mu,(\varepsilon)}_{k,\ell,\iota}). \tag{6.34}$$

Furthermore, by (6.29) and (6.30), the term $\omega_2(\xi; s) \cdot (s - \theta_2(\xi))^2$ dominates in (6.32). This, (6.33), (6.34) and the localisation (6.30) immediately imply

$$\begin{aligned} |\langle \gamma''(s), \xi \rangle| &\lesssim 2^{-k+4\ell(1-\varepsilon)} |\langle \gamma'(s), \xi \rangle|^2, \\ |\langle \gamma^{(3)}(s), \xi \rangle| &\lesssim 2^{-(k-4\ell(1-\varepsilon))2} |\langle \gamma'(s), \xi \rangle|^3, \\ |\langle \gamma^{(j)}(s), \xi \rangle| &\lesssim 2^k \lesssim_j 2^{-(k-4\ell(1-\varepsilon))(j-1)} |\langle \gamma'(s), \xi \rangle|^j \qquad \text{for all } j \geqslant 4 \end{aligned}$$

for all $(\xi; s) \in \operatorname{supp}(a^{\mu}_{k,\ell,\iota} - a^{\mu,(\varepsilon)}_{k,\ell,\iota})$.

On the other hand, by the definition of the symbols, (6.34) and the localisation (6.30),

$$|\partial_s^N (a^{\mu}_{k,\ell,\iota} - a^{\mu,(\varepsilon)}_{k,\ell,\iota})(\xi; s)| \lesssim_N 2^{\ell N} \lesssim 2^{-(k-4\ell)N - 3\varepsilon\ell N} |\langle \gamma'(s), \xi \rangle|^N \qquad \text{for all } N \in \mathbb{N}.$$

Thus, by repeated integration-by-parts (via Lemma C.1, with $R = 2^{k-4\ell+3\varepsilon\ell} \geqslant 1$),

$$|m[a^{\mu}_{k,\ell,\iota} - a^{\mu,(\varepsilon)}_{k,\ell,\iota}](\xi)| \lesssim_N 2^{-(k-4\ell)N} 2^{-3\varepsilon\ell N} \qquad \text{for all } N \in \mathbb{N}.$$

Since $0 \leqslant \ell \leqslant \lfloor k/4 \rfloor \leqslant k/4$, the first bound in (6.28) follows.

b) Here the localisation of the $b_{k,\boldsymbol{\ell}}$ symbols ensures that

$$|u_1(\xi)| \lesssim \rho^4 2^{k-3\ell_1}, \qquad |u_2(\xi)| \sim \rho 2^{k-2\ell_2}, \qquad |s - \theta_1(\xi)| \lesssim \rho 2^{-\ell_2} \tag{6.35}$$

hold for all $(\xi; s) \in \operatorname{supp}(b^{\nu}_{k,\boldsymbol{\ell}} - b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}})$. Furthermore, by Lemma 6.4,

$$|\langle \gamma^{(3)}(s), \xi \rangle| \sim \rho^{1/2} 2^{k-\ell_2} \qquad \text{for all } (\xi; s) \in \operatorname{supp} b_{k,\boldsymbol{\ell}}, \tag{6.36}$$

whilst, provided ρ is sufficiently small, the additional localisation in (6.17) and (6.27) implies

$$|s - \theta_1(\xi)| \gtrsim \rho^{-1} 2^{-(1-\varepsilon)(3\ell_1 - \ell_2)/2} \quad \text{for all } (\xi; s) \in \operatorname{supp}(b^{\nu}_{k,\boldsymbol{\ell}} - b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}}). \tag{6.37}$$

Fix $\xi \in \operatorname{supp}_\xi(b^{\nu}_{k,\boldsymbol{\ell}} - b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}})$ and consider the oscillatory integral $m[b^{\nu}_{k,\boldsymbol{\ell}} - b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}}](\xi)$, which has phase $s \mapsto \langle \gamma(s), \xi\rangle$. Taylor expansion around $\theta_1(\xi)$ yields

$$\langle \gamma'(s), \xi\rangle = u_1(\xi) + \omega_1(\xi; s) \cdot (s - \theta_1(\xi))^2, \tag{6.38}$$

$$\langle \gamma''(s), \xi\rangle = \omega_2(\xi; s) \cdot (s - \theta_1(\xi)), \tag{6.39}$$

where the ω_i arise from the remainder terms and satisfy $|\omega_i(\xi; s)| \sim \rho^{1/2} 2^{k-\ell_2}$ by (6.36). Provided $\rho > 0$ is sufficiently small, (6.35) and (6.37) imply that the second term dominates the right-hand side of (6.38) and therefore

$$|\langle \gamma'(s), \xi\rangle| \gtrsim \rho^{1/2} 2^{k-\ell_2} |s - \theta_1(\xi)|^2 \quad \text{for all } (\xi; s) \in \operatorname{supp}(b^{\nu}_{k,\boldsymbol{\ell}} - b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}}). \tag{6.40}$$

Furthermore, (6.39), (6.36), (6.40) and the localisation (6.37) imply

$$\begin{aligned}
|\langle \gamma''(s), \xi\rangle| &\lesssim 2^{-k+\ell_2+3(1-\varepsilon)(3\ell_1-\ell_2)/2} |\langle \gamma'(s), \xi\rangle|^2, \\
|\langle \gamma^{(3)}(s), \xi\rangle| &\lesssim 2^{k-\ell_2} \lesssim 2^{-2(k-\ell_2-3(1-\varepsilon)(3\ell_1-\ell_2)/2)} |\langle \gamma'(s), \xi\rangle|^3, \\
|\langle \gamma^{(j)}(s), \xi\rangle| &\lesssim 2^{k} \lesssim_j 2^{-(k-\ell_2-3(1-\varepsilon)(3\ell_1-\ell_2)/2)(j-1)} |\langle \gamma'(s), \xi\rangle|^j \qquad \text{for all } j \geqslant 4
\end{aligned}$$

for all $(\xi; s) \in \operatorname{supp}(b^{\nu}_{k,\boldsymbol{\ell}} - b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}})$.

On the other hand, by the definition of the symbols, (6.40) and the localisation (6.37),

$$\begin{aligned}
|\partial_s^N (b^{\nu}_{k,\boldsymbol{\ell}} - b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}})(\xi; s)| &\lesssim_N \max\{\rho^{-N} 2^{\ell_2 N}, 2^{(1-\varepsilon)(3\ell_1-\ell_2)N/2}\} \\
&\lesssim_{N,\rho} 2^{-(k-\ell_2-3(1-\varepsilon)(3\ell_1-\ell_2)/2)N} |\langle \gamma'(s), \xi\rangle|^N \qquad \text{for all } N \in \mathbb{N}
\end{aligned}$$

and all $(\xi; s) \in \operatorname{supp}(b^{\nu}_{k,\boldsymbol{\ell}} - b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}})$, using that $0 \leqslant \ell_2 \leqslant \ell_1$ for $\boldsymbol{\ell} \in \Lambda(k)$. Thus, by repeated integration-by-parts (via Lemma C.1 with $R = 2^{k-\ell_2-3(1-\varepsilon)(3\ell_1-\ell_2)/2} \geqslant 1$),

$$|m[b^{\nu}_{k,\boldsymbol{\ell}} - b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}}](\xi; s)| \lesssim_{N,\rho} 2^{-(k-\ell_2-3(3\ell_1-\ell_2)/2)N - 3\varepsilon(3\ell_1-\ell_2)N/2} \qquad \text{for all } N \in \mathbb{N}.$$

Since $\ell_2 \leqslant \ell_1 \leqslant (2k + \ell_2)/9$ and $0 \leqslant \ell_2 < k/4$ for $\boldsymbol{\ell} \in \Lambda(k)$, the second bound in (6.28) follows. ☐

6.6. **Estimating the localised pieces.** Each piece of the multipliers $m[a^{\mu,(\varepsilon)}_{k,\ell,\iota}]$ and $m[b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}}]$ arising from the preceding decomposition satisfies favourable L^2 and L^∞ bounds.

Lemma 6.12. *a) For $0 \leqslant \ell \leqslant \lfloor k/4 \rfloor$, $\mu \in \mathbb{Z}$, $1 \leqslant \iota \leqslant 4$ and $\varepsilon > 0$, we have*

$$\|m[a^{\mu,(\varepsilon)}_{k,\ell,\iota}]\|_{M^2(\mathbb{R}^4)} \lesssim 2^{-k/2+\ell}.$$

b) For $\boldsymbol{\ell} = (\ell_1, \ell_2) \in \Lambda(k)$, $\nu \in \mathbb{Z}$ and $\varepsilon > 0$, we have

$$\|m[b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}}]\|_{M^2(\mathbb{R}^4)} \lesssim 2^{-k/2+(3\ell_1+\ell_2)/4}.$$

Proof. a) If $\ell = \lfloor k/4 \rfloor$, then the desired bounds follow from Plancherel's theorem and the van der Corput lemma with fourth order derivatives. For the remaining cases, it suffices to show that

$$|\langle \gamma'(s), \xi\rangle| + 2^{-\ell} |\langle \gamma''(s), \xi\rangle| \gtrsim 2^{k-3\ell} \qquad \text{for all } (\xi; s) \in \operatorname{supp} a^{\mu,(\varepsilon)}_{k,\ell,\iota}. \tag{6.41}$$

We treat each class of symbol, as index by the parameter ι, individually.

$\underline{\iota = 1}$. Here the localisation of the symbol ensures the key properties

$$|u_{1,2}(\xi)| \sim 2^{k-3\ell}, \qquad |u_2(\xi)| \lesssim \rho 2^{k-2\ell} \qquad \text{for all } (\xi; s) \in \operatorname{supp} a^{\mu,(\varepsilon)}_{k,\ell,1}. \tag{6.42}$$

By Taylor expansion around $\theta_2(\xi)$, one has

$$\langle\gamma'(s),\xi\rangle = u_{1,2}(\xi) + u_2(\xi)\cdot(s-\theta_2(\xi)) + \omega_1(\xi;s)\cdot(s-\theta_2(\xi))^3, \tag{6.43}$$

$$\langle\gamma''(s),\xi\rangle = u_2(\xi) + \omega_2(\xi;s)\cdot(s-\theta_2(\xi))^2 \tag{6.44}$$

where the functions ω_i arise from the remainder terms and satisfy $|\omega_i(\xi;s)| \sim 2^k$ for $i=1,2$. The argument splits into two cases:

Case 1: $|s-\theta_2(\xi)| \leqslant \rho^{1/4}2^{-\ell}$. Provided $\rho>0$ is chosen sufficiently small, (6.42) implies that the $u_{1,2}(\xi)$ term dominates the right-hand side of (6.43) and therefore $|\langle\gamma'(s),\xi\rangle| \gtrsim 2^{k-3\ell}$.

Case 2: $|s-\theta_2(\xi)| \geqslant \rho^{1/4}2^{-\ell}$. Again provided $\rho>0$ is sufficiently small, (6.42) implies that the second term dominates the right-hand side of (6.44) and therefore $|\langle\gamma''(s),\xi\rangle| \gtrsim \rho^{1/2}2^{k-2\ell}$.

Thus, in either case the desired bound (6.41) holds.

$\underline{\iota=2}$. Suppose $0\leqslant\ell<\lfloor k/4\rfloor$ and $\xi\in\operatorname{supp} a^{\mu,(\varepsilon)}_{k,\ell,2}$. Recall, by Lemma 6.1, that $\theta_2(\xi)$ is the unique global minimum of the function $s\mapsto\langle\gamma''(s),\xi\rangle$ on $[-1,1]$. Thus, $\langle\gamma''(s),\xi\rangle \geqslant u_2(\xi)\sim\rho 2^{k-2\ell}$, as required.

$\underline{\iota=3}$. Here the localisation of the symbol ensures the key properties

$$|u_1(\xi)|\sim\rho^4 2^{k-3\ell}, \qquad |u_2(\xi)|\sim\rho 2^{k-2\ell} \qquad \text{for all } (\xi;s)\in\operatorname{supp} a^{\mu,(\varepsilon)}_{k,\ell,3}. \tag{6.45}$$

The argument splits into two cases:

Case 1: $\min_\pm|s-\theta_1^\pm(\xi)|\leqslant\rho^2 2^{-\ell}$. By Taylor expansion around $\theta_1^\pm(\xi)$, one has

$$\langle\gamma'(s),\xi\rangle = u_1^\pm(\xi) + u_{3,1}^\pm(\xi)\cdot\frac{(s-\theta_1^\pm(\xi))^2}{2} + \omega^\pm(\xi;s)\cdot(s-\theta_1^\pm(\xi))^3, \tag{6.46}$$

where the functions $\omega^\pm$ arise from the third order remainder term and satisfy $|\omega^\pm(\xi;s)|\sim 2^k$. Moreover, (6.45) and Lemma 6.3 i) imply $|u_{3,1}^\pm(\xi)|\sim\rho^{1/2}2^{k-\ell}$. Provided ρ is sufficiently small, (6.45) implies that the $u_1^\pm(\xi)$ term dominates the right-hand side of (6.46) and therefore $|\langle\gamma'(s),\xi\rangle|\gtrsim\rho^4 2^{k-3\ell}$.

Case 2: $\min_\pm|s-\theta_1^\pm(\xi)|\geqslant\rho^2 2^{-\ell}$. In this case, rather than analysing Taylor expansions, we use a convexity argument. Fix $\xi\in\operatorname{supp} a^{\mu,(\varepsilon)}_{k,\ell,3}$ and let

$$\phi\colon[-1,1]\to\mathbb{R}, \quad \phi\colon s\mapsto\langle\gamma''(s),\xi\rangle;$$

by (6.3), this function is strictly convex. Thus, given $t\in[-1,1]$, the auxiliary function

$$q_t\colon[-1,1]\to\mathbb{R}, \quad q_t\colon s\mapsto\frac{\phi(s)-\phi(t)}{s-t} \quad \text{for } s\neq t \quad \text{and} \quad q_t\colon t\mapsto\phi'(t)$$

is increasing. Setting $t:=\theta_1^-(\xi)$ and noting that $\phi\circ\theta_1^-(\xi)=0$, it follows that

$$\frac{\phi(s)}{s-\theta_1^-(\xi)} \leqslant \frac{\phi\circ\theta_2(\xi)}{\theta_2(\xi)-\theta_1^-(\xi)} = \frac{u_2(\xi)}{\theta_2(\xi)-\theta_1^-(\xi)} < 0 \qquad \text{for all } -1\leqslant s\leqslant\theta_2(\xi),$$

where we have used the fact that $u_2(\xi)<0$ on the support of $a_{k,\ell,3}$. If $s\in[\theta_2(\xi),1]$, then we can carry out the same argument with respect to $t=\theta_1^+(\xi)$ to obtain a similar inequality. From this, we deduce the bound

$$|\langle\gamma''(s),\xi\rangle| \geqslant \min_\pm\frac{|u_2(\xi)||s-\theta_1^\pm(\xi)|}{|\theta_2(\xi)-\theta_1^\pm(\xi)|} \qquad \text{for all } -1\leqslant s\leqslant 1. \tag{6.47}$$

Recall from (6.45) that $|u_2(\xi)|\sim\rho 2^{k-2\ell}$ and therefore $|\theta_2(\xi)-\theta_1^\pm(\xi)|\sim\rho^{1/2}2^{-\ell}$ by Lemma 6.3 i). Substituting these bounds and the hypothesis $\min_\pm|s-\theta_1^\pm(\xi)|\geqslant\rho^2 2^{-\ell}$ into (6.47), we conclude that $|\langle\gamma''(s),\xi\rangle|\gtrsim\rho^{5/2}2^{k-2\ell}$.

Thus, in either case the desired bound (6.41) holds.

$\underline{\iota = 4}$. Here the localisation of the symbol ensures the key properties

$$|u_1(\xi)| \lesssim \rho^4 2^{k-3\ell}, \qquad |s - \theta_1(\xi)| \gtrsim \rho 2^{-\ell} \qquad \text{for all } (\xi; s) \in \operatorname{supp} a_{k,\ell,4}^{\mu,(\varepsilon)}. \tag{6.48}$$

By Taylor expansion around $\theta_1(\xi)$, we obtain

$$\langle \gamma'(s), \xi\rangle = u_1(\xi) + u_{3,1}(\xi) \cdot \frac{(s-\theta_1(\xi))^2}{2} + \omega_1(\xi; s) \cdot (s - \theta_1(\xi))^3, \tag{6.49}$$

$$\langle \gamma''(s), \xi\rangle = u_{3,1}(\xi) \cdot (s - \theta_1(\xi)) + \omega_2(\xi; s) \cdot (s - \theta_1(\xi))^2 \tag{6.50}$$

where the functions ω_1 and ω_2 arise from the remainder terms and satisfy $|\omega_i(\xi; s)| \sim 2^k$ for $i = 1$, 2. It is convenient to define the functions

$$\alpha(\xi; s) := u_{3,1}(\xi) + \omega_2(\xi; s) \cdot (s - \theta_1(\xi)),$$
$$\beta(\xi; s) := 2\omega_1(\xi; s) - \omega_2(\xi; s),$$

so that (6.49) and (6.50) can be rewritten as

$$\langle \gamma'(s), \xi\rangle = u_1(\xi) + \big(\alpha(\xi; s) + \beta(\xi; s) \cdot (s - \theta_1(\xi))\big) \cdot \frac{(s-\theta_1(\xi))^2}{2}, \tag{6.51}$$

$$\langle \gamma''(s), \xi\rangle = \alpha(\xi; s) \cdot (s - \theta_1(\xi)). \tag{6.52}$$

The argument splits into two cases:

Case 1: $|\alpha(\xi; s)| \leqslant \rho^2 2^{k-\ell}$. By the integral form of the remainder,

$$\beta(\xi; s) \cdot (s - \theta_1(\xi))^3 = -\int_{\theta_1(\xi)}^{s} \langle \gamma^{(4)}(t), \xi\rangle \cdot (s - t) \cdot (t - \theta_1(\xi))\, \mathrm{d}t.$$

Recall from (6.3) that $\langle \gamma^{(4)}(t), \xi\rangle > 0$ for all $t \in [-1, 1]$. Thus, the integrand in the above display has constant sign. Furthermore, (6.2) also guarantees that $|\langle \gamma^{(4)}(t), \xi\rangle| \sim 2^k$. Combining these observations,

$$|\beta(\xi; s)| \sim 2^k \qquad \text{for all } (\xi; s) \in \operatorname{supp} a_{k,\ell,4}.$$

Thus, provided ρ is chosen sufficiently small, the hypothesis $|\alpha(\xi; s)| \leqslant \rho^2 2^{k-\ell}$ and together with the bound $|s - \theta_1(\xi)| \geqslant \rho 2^{-\ell}$ from (6.42) imply

$$|\beta(\xi; s)||s - \theta_1(\xi)| - |\alpha(\xi; s)| \gtrsim \rho 2^{k-\ell}.$$

Consequently, (6.48) implies that the second term dominates the right-hand side of (6.51) and therefore $|\langle \gamma'(s), \xi\rangle| \gtrsim \rho^3 2^{k-3\ell}$.

Case 2: $|\alpha(\xi; s)| \geqslant \rho^2 2^{k-\ell}$. Here (6.48) and (6.52) immediately imply $|\langle \gamma''(s), \xi\rangle| \gtrsim \rho^3 2^{k-2\ell}$.

Thus, in either case the desired bound (6.41) holds.

b) If $\ell_1 = \lfloor (2k + \ell_2)/9 \rfloor$, then the desired bound follows from Plancherel's theorem and the van der Corput lemma with third order derivatives. Indeed, by Lemma 6.4,

$$|\langle \gamma^{(3)}(s), \xi\rangle| \sim \rho^{1/2} 2^{k-\ell_2} \qquad \text{for all } (\xi; s) \in \operatorname{supp} b_{k,\boldsymbol{\ell}}^{\nu,(\varepsilon)}. \tag{6.53}$$

For the remaining cases, it suffices to show that

$$|\langle \gamma'(s), \xi\rangle| + 2^{-(3\ell_1 - \ell_2)/2} |\langle \gamma''(s), \xi\rangle| \gtrsim 2^{k-3\ell_1} \qquad \text{for all } (\xi; s) \in \operatorname{supp} b_{k,\boldsymbol{\ell}}^{\nu,(\varepsilon)}. \tag{6.54}$$

Here the localisation of the symbol ensures the key properties

$$|u_1(\xi)| \sim \rho^4 2^{k-3\ell_1}, \quad |u_2(\xi)| \sim \rho 2^{k-2\ell_2}, \quad |s - \theta_1(\xi)| \lesssim \rho 2^{-\ell_2} \qquad \text{for all } (\xi; s) \in \operatorname{supp} b_{k,\boldsymbol{\ell}}^{\nu,(\varepsilon)}. \tag{6.55}$$

By Taylor expansion around $\theta_1(\xi)$, we obtain

$$\langle\gamma'(s),\xi\rangle = u_1(\xi) + \omega_1(\xi;s)\cdot(s-\theta_1(\xi))^2, \tag{6.56}$$

$$\langle\gamma''(s),\xi\rangle = \omega_2(\xi;s)\cdot(s-\theta_1(\xi)), \tag{6.57}$$

where the functions ω_1 and ω_2 arise from the remainder terms and satisfy $|\omega_i(\xi;s)| \sim \rho^{1/2}2^{k-\ell_2}$ for $i=1,\,2$ by (6.53). The argument splits into two cases:

Case 1: $|\theta_1(\xi)-s| \leqslant \rho^2 2^{-(3\ell_1-\ell_2)/2}$. Provided $\rho>0$ is chosen sufficiently small, (6.55) and the bound $|\omega_1(\xi;s)| \sim \rho^{1/2}2^{k-\ell_2}$ imply that the $u_1(\xi)$ term dominates the right-hand side of (6.56) and therefore $|\langle\gamma'(s),\xi\rangle| \gtrsim \rho^4 2^{k-3\ell_1}$.

Case 2: $|\theta_1(\xi)-s| \geqslant \rho^2 2^{-(3\ell_1-\ell_2)/2}$. In this case, the bound $|\omega_2(\xi)| \sim \rho^{1/2}2^{k-\ell_2}$ and (6.57) immediately imply $|\langle\gamma''(s),\xi\rangle| \gtrsim \rho^{5/2}2^{k-\ell_2-(3\ell_1-\ell_2)/2}$.

Thus, in either case the desired bound (6.54) holds. □

Lemma 6.13. *a) For all $0\leqslant\ell\leqslant\lfloor k/4\rfloor$, $\mu\in\mathbb{Z}$, $1\leqslant\iota\leqslant 4$ and $\varepsilon>0$, we have*

$$\|m[a^{\mu,(\varepsilon)}_{k,\ell,\iota}]\|_{M^\infty(\mathbb{R}^4)} \lesssim 2^{-(1-\varepsilon)\ell}.$$

b) For $\boldsymbol{\ell}=(\ell_1,\ell_2)\in\Lambda(k)$, $\nu\in\mathbb{Z}$ and $\varepsilon>0$, we have

$$\|m[b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}}]\|_{M^\infty(\mathbb{R}^4)} \lesssim 2^{-(1-\varepsilon)(3\ell_1-\ell_2)/2}.$$

Proof. In view of the support properties of the symbols (see Lemma 6.6 and Lemma 6.7), by an integration-by-parts argument (see Lemma C.2), the problem is reduced to showing

$$|\nabla^N_{\mathbf{e}_j(s_\mu)} a^\mu_{k,\ell,\iota}(\xi;s)| \lesssim_N 2^{-(k-(4-j)\ell)N}, \tag{6.58a}$$

$$|\nabla^N_{\mathbf{e}_j(s_\nu)} b^\nu_{k,\boldsymbol{\ell}}(\xi;s)| \lesssim_N 2^{-(k-((3-j)(3\ell_1-\ell_2)/2+\ell_2)\vee 0)N} \tag{6.58b}$$

for all $1\leqslant j\leqslant 4$ and all $N\in\mathbb{N}_0$.

For all $N\in\mathbb{N}$, we claim the following:

- For all $\xi\in\operatorname{supp}_\xi a^\mu_{k,\ell,\iota}$, $1\leqslant\iota\leqslant 4$,

$$2^{\ell}|\nabla^N_{\mathbf{e}_j(s_\mu)}\theta_2(\xi)|,\quad 2^{-k+2\ell}|\nabla^N_{\mathbf{e}_j(s_\mu)}u_2(\xi)|,\quad 2^{-k+3\ell}|\nabla^N_{\mathbf{e}_j(s_\mu)}u_{1,2}(\xi)| \lesssim_N 2^{-(k-(4-j)\ell)N}; \tag{6.59}$$

- For all $\xi\in\operatorname{supp}_\xi a^\mu_{k,\ell,\iota}$, $3\leqslant\iota\leqslant 4$,

$$2^{\ell}|\nabla^N_{\mathbf{e}_j(s_\mu)}\theta_1(\xi)|,\quad 2^{-k+3\ell}|\nabla^N_{\mathbf{e}_j(s_\mu)}u_1(\xi)| \lesssim_N 2^{-(k-(4-j)\ell)N}; \tag{6.60}$$

- For all $\xi\in\operatorname{supp}_\xi b^\nu_{k,\boldsymbol{\ell}}$,

$$2^{\ell_2}|\nabla^N_{\mathbf{e}_j(s_\nu)}\theta_2(\xi)|,\quad 2^{-k+2\ell_2}|\nabla^N_{\mathbf{e}_j(s_\nu)}u_2(\xi)|,\quad 2^{-k+3\ell_2}|\nabla^N_{\mathbf{e}_j(s_\nu)}u_{1,2}(\xi)| \lesssim_N 2^{-(k-(4-j)\ell_2)N}; \tag{6.61}$$

- For all $\xi\in\operatorname{supp}_\xi b^\nu_{k,\boldsymbol{\ell}}$,

$$2^{(3\ell_1-\ell_2)/2}|\nabla^N_{\mathbf{e}_j(s_\nu)}\theta_1(\xi)|,\quad 2^{-k+3\ell_1}|\nabla^N_{\mathbf{e}_j(s_\nu)}u_1(\xi)| \lesssim_N 2^{-(k-((3-j)(3\ell_1-\ell_2)/2+\ell_2)\vee 0)N}. \tag{6.62}$$

Once the above claims are established, the derivative bounds (6.58a) and (6.58b) follow directly from the chain and Leibniz rule.

In order to prove (6.59)-(6.62) we work with the unified framework introduced in §6.4.

We start with (6.59) and (6.61). Given $n,\,s\in\mathbb{R}$, recall the set $\Xi_2(k,n;s)$ introduced in (6.19). In particular, if $\xi\in\Xi_2(k,n;s)$, then $\xi\in\operatorname{supp}_\xi a_k$ and ξ lies in the domain of θ_2 and satisfies

$$|\theta_2(\xi)-s| \lesssim 2^{-n} \qquad\text{and}\qquad |u_2(\xi)| \lesssim 2^{k-2n}. \tag{6.63}$$

From the discussion following (6.19), we know that

$$\operatorname{supp}_\xi a^{\mu}_{k,\ell,\iota} \subseteq \Xi_2(k,\ell;s_\mu) \quad \text{for } 1 \leqslant \iota \leqslant 4 \quad \text{and} \quad \operatorname{supp}_\xi b^{\nu}_{k,\boldsymbol{\ell}} \subseteq \Xi_2\big(k,\ell_2;s_\nu\big).$$

Let $\xi \in \Xi_2(k,n;s)$ and for $1 \leqslant j \leqslant 4$ define $\boldsymbol{v}_j := \mathbf{e}_j(s)$. The bounds (6.59) and (6.61) amount to proving that

$$2^n|\nabla^N_{\boldsymbol{v}_j}\theta_2(\xi)|, \quad 2^{-k+2n}|\nabla^N_{\boldsymbol{v}_j}u_{1,2}(\xi)|, \quad 2^{-k+3n}|\nabla^N_{\boldsymbol{v}_j}u_2(\xi)| \lesssim_N 2^{-(k-(4-j)n)N} \tag{6.64}$$

hold for all $N \in \mathbb{N}$. These bounds follow from repeated application of the chain rule, provided

$$|\langle\gamma^{(4)} \circ \theta_2(\xi), \xi\rangle| \gtrsim 2^k, \tag{6.65a}$$

$$|\langle\gamma^{(K)} \circ \theta_2(\xi), \xi\rangle| \lesssim_K 2^{k+n(K-4)}, \tag{6.65b}$$

$$|\langle\gamma^{(K)} \circ \theta_2(\xi), \boldsymbol{v}_j\rangle| \lesssim_K 2^{(K-j)n} \tag{6.65c}$$

hold for all $K \geqslant 2$. In particular, assuming (6.65a), (6.65b) and (6.65c), the bounds in (6.64) are then a consequence of Lemma B.1 in the appendix. More precisely, the desired estimates in (6.64) correspond to (B.2) and two separate instances of (B.4) whilst the hypotheses in the above display correspond to (B.1) and (B.3). Here the parameters featured in the appendix are chosen as follows:

g	h	A	B	M_1	M_2	$\mathbf{e}$
$\gamma^{(3)}$	γ''	2^{k-n}	2^{k-2n}	$2^{-k+(4-j)n}$	2^n	$\boldsymbol{v}_j$
$\gamma^{(3)}$	γ'	2^{k-n}	2^{k-3n}	$2^{-k+(4-j)n}$	2^n	$\boldsymbol{v}_j$

The conditions (6.65a), (6.65b) and (6.65c) follow directly from the definition of $\Xi_2(k,n;s)$. Indeed, (6.65a) and the $K \geqslant 4$ case of (6.65b) are trivial consequences of the localisation of the symbol a_k. The $K = 3$ case of (6.65b) follows immediately since $\langle\gamma^{(3)} \circ \theta_2(\xi), \xi\rangle = 0$ and the $K = 2$ case of (6.65b) is just a restatement of the condition $|u_2(\xi)| \lesssim 2^{k-2n}$ from (6.63). Finally, (4.13) together with the θ_2 localisation hypothesis from (6.63) imply that

$$|\langle\gamma^{(K)} \circ \theta_2(\xi), \boldsymbol{v}_j\rangle| \lesssim_K |\theta_2(\xi) - s|^{(j-K)\vee 0} \lesssim 2^{-((j-K)\vee 0)n}$$

which yields (6.65c).

We next turn to (6.60) and (6.62). Given $\boldsymbol{n} = (n_1, n_2) \in \mathbb{R}^2$ and $s \in \mathbb{R}$, recall the set $\Xi_1(k,\boldsymbol{n};s)$ introduced in (6.21). In particular, if $\xi \in \Xi_1(k,\boldsymbol{n};s)$, then $\xi \in \operatorname{supp}_\xi a_k$ and ξ lies in the domain of θ_1 and satisfies

$$|\theta_1(\xi) - s| \lesssim 2^{-n_1} \quad \text{and} \quad |u_{3,1}(\xi)| \sim 2^{k-n_2}. \tag{6.66}$$

From the discussion following (6.21), we know that

$$\operatorname{supp}_\xi a^{\mu}_{k,\ell,\iota} \subseteq \Xi_1(k,\ell,\ell;s_\mu) \quad \text{for } \iota = 3,\, 4 \quad \text{and} \quad \operatorname{supp}_\xi b^{\nu}_{k,\boldsymbol{\ell}} \subseteq \Xi_1\big(k,\tfrac{3\ell_1-\ell_2}{2},\ell_2;s_\nu\big).$$

Let $\xi \in \Xi_1(k,\boldsymbol{n};s)$ where $\boldsymbol{n} = (n_1,n_2)$ for some $0 < n_2 \leqslant n_1$ and for $1 \leqslant j \leqslant 4$ define $\boldsymbol{v}_j := \mathbf{e}_j(s)$. The bounds (6.60) and (6.62) amount to proving that

$$2^{n_1}|\nabla^N_{\boldsymbol{v}_j}\theta_1(\xi)|, \quad 2^{-k+2n_1+n_2}|\nabla^N_{\boldsymbol{v}_j}u_1(\xi)| \lesssim_N 2^{-(k-((3-j)n_1+n_2)\vee 0)N} \tag{6.67}$$

hold for all $N \in \mathbb{N}$. These bounds follow from repeated application of the chain rule, provided

$$|\langle\gamma^{(3)} \circ \theta_1(\xi), \xi\rangle| \gtrsim 2^{k-n_2}, \tag{6.68a}$$

$$|\langle\gamma^{(K)} \circ \theta_1(\xi), \xi\rangle| \lesssim_K 2^{k+n_1(K-3)-n_2}, \tag{6.68b}$$

$$|\langle\gamma^{(K)} \circ \theta_1(\xi), \boldsymbol{v}_j\rangle| \lesssim_K 2^{n_1(K-3)-n_2+((3-j)n_1+n_2)\vee 0} \tag{6.68c}$$

hold for all $K \geqslant 2$. In particular, assuming (6.68a), (6.68b) and (6.68c), the bounds in (6.67) are then a consequence of Lemma B.1 in the appendix. More precisely, the desired estimates in (6.67) correspond to (B.2) and (B.4) whilst the hypotheses in the above display correspond to (B.1) and (B.3). Here the parameters featured in the appendix are chosen as follows:

g	h	A	B	M_1	M_2	$\mathbf{e}$
γ''	γ'	$2^{k-n_1-n_2}$	$2^{k-2n_1-n_2}$	$2^{-k+((3-j)n_1+n_2)\vee 0}$	2^{n_1}	$\boldsymbol{v}_j$

The conditions (6.68a), (6.68b) and (6.68c) follow directly from the definition of $\Xi_1(k,\boldsymbol{n};s)$. Indeed, (6.68a) and the $K=3$ case of (6.68b) are just a restatement of the condition $|u_{3,1}(\xi)| \sim 2^{k-n_2}$ from (6.66). The $K \geqslant 4$ case of (6.68b) is a trivial consequence of the localisation of the symbol a_k whist the remaining $K=2$ case of (6.68b) follows immediately since $\langle \gamma'' \circ \theta_1(\xi), \xi \rangle = 0$. Finally, (4.13) together with the θ_1 localisation hypothesis from (6.66) imply that

$$|\langle \gamma^{(K)} \circ \theta_1(\xi), \boldsymbol{v}_j \rangle| \lesssim_N |\theta_1(\xi) - s|^{(j-K)\vee 0} \lesssim 2^{((j-K)\vee 0)n_1}$$

which, by directly comparing exponents, yields (6.68c). □

Lemma 6.12 and Lemma 6.13 can be combined to obtain the following L^p bounds.

Corollary 6.14. *For all $2 \leqslant p \leqslant \infty$ and all $\varepsilon > 0$, the following inequalities hold:*
a) For all $0 \leqslant \ell \leqslant \lfloor k/4 \rfloor$ and $1 \leqslant \iota \leqslant 4$,

$$\Big(\sum_{\mu \in \mathbb{Z}} \|m[a^{\mu,(\varepsilon)}_{k,\ell,\iota}](D) f\|^p_{L^p(\mathbb{R}^4)} \Big)^{1/p} \lesssim 2^{-k/p + \ell(4/p-1)+\varepsilon\ell} \|f\|_{L^p(\mathbb{R}^4)}.$$

b) For all $\boldsymbol{\ell} = (\ell_1, \ell_2) \in \Lambda(k)$,

$$\Big(\sum_{\nu \in \mathbb{Z}} \|m[b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}}](D) f\|^p_{L^p(\mathbb{R}^4)} \Big)^{1/p} \lesssim 2^{-k/p + (3\ell_1+\ell_2)/2p - (3\ell_1-\ell_2)(1/2-1/p-\varepsilon)} \|f\|_{L^p(\mathbb{R}^4)}.$$

When $p = \infty$ the left-hand ℓ^p-sums are interpreted as suprema in the usual manner.

Proof. For $p=2$ the estimate a) and b) follow by the combining L^2 bounds from Lemma 6.12 with a simple orthogonality argument, as the supports of $m[a^{\mu,(\varepsilon)}_{k,\ell,\iota}]$ and $m[b^{\nu,(\varepsilon)}_{k,\boldsymbol{\ell}}]$ are essentially disjoint for different μ and ν respectively. For $p = \infty$ the estimate is a restatement of the L^∞ bounds from Lemma 6.13. Interpolating these two endpoint cases, using mixed norm interpolation (see, for instance, [21, §1.18.4]), concludes the proof. □

6.7. **Putting everything together.** We are now ready to combine the ingredients to conclude the proof of Proposition 6.5.

Proof of Proposition 6.5. a) Let $1 \leqslant \iota \leqslant 4$. By Proposition 6.9 a), for all $2 \leqslant p \leqslant 12$ and all $\varepsilon > 0$ one has

$$\|m[a_{k,\ell,\iota}](D) f\|_{L^p(\mathbb{R}^4)} = \Big\| \sum_{\mu \in \mathbb{Z}} m[a^{\mu}_{k,\ell,\iota}](D) f \Big\|_{L^p(\mathbb{R}^4)} \lesssim_\varepsilon 2^{\ell(1/2-1/p)+\varepsilon\ell} \Big(\sum_{\mu \in \mathbb{Z}} \|m[a^{\mu}_{k,\ell,\iota}](D) f\|^p_{L^p(\mathbb{R}^4)} \Big)^{1/p}.$$

Moreover, for all $\mu \in \mathbb{Z}$, Lemma 6.11 a) implies that

$$\|m[a^{\mu}_{k,\ell,\iota}]\|_{M^p(\mathbb{R}^4)} \lesssim_{N,\varepsilon,p} \|m[a^{\mu,(\varepsilon)}_{k,\ell,\iota}]\|_{M^p(\mathbb{R}^4)} + 2^{-k} \quad \text{for all } N \in \mathbb{N}.$$

Combining the above, we obtain

$$\|m[a_{k,\ell,\iota}](D) f\|_{L^p(\mathbb{R}^4)} \lesssim_{\varepsilon,p} 2^{\ell(1/2-1/p)+\varepsilon\ell} \Big(\sum_{\mu \in \mathbb{Z}} \|m[a^{\mu,(\varepsilon)}_{k,\ell,\iota}](D) f\|^p_{L^p(\mathbb{R}^4)} \Big)^{1/p} + 2^{-k} \|f\|_{L^p(\mathbb{R}^4)},$$

which, together with Corollary 6.14 a), yields

$$\|m[a_{k,\ell,\iota}](D)f\|_{L^p(\mathbb{R}^4)} \lesssim_{\varepsilon,p} 2^{-k/p-\ell(1/2-3/p-2\varepsilon)}\|f\|_{L^p(\mathbb{R}^4)}.$$

Since $\varepsilon > 0$ was chosen arbitrarily, this is the required bound.

b) By Proposition 6.10, for all $6 \leqslant p \leqslant 12$ and all $\varepsilon > 0$ one has

$$\|m[b_{k,\boldsymbol{\ell}}](D)f\|_{L^p(\mathbb{R}^4)} \lesssim_{\varepsilon} 2^{\ell_2(1/2-1/p+\varepsilon)}2^{3(\ell_1-\ell_2)(1-4/p+\varepsilon)/2}\Big(\sum_{\nu\in\mathbb{Z}}\|m[b_{k,\boldsymbol{\ell}}^{\nu}](D)f\|_{L^p(\mathbb{R}^4)}^p\Big)^{1/p}.$$

Moreover, for all $\nu \in \mathbb{Z}$, Proposition 6.11 b) implies that

$$\|m[b_{k,\boldsymbol{\ell}}^{\nu}]\|_{M^p(\mathbb{R}^4)} \lesssim_{N,\varepsilon,p} \|m[b_{k,\boldsymbol{\ell}}^{\nu,(\varepsilon)}]\|_{M^p(\mathbb{R}^4)} + 2^{-kN} \qquad \text{for all } N \in \mathbb{N}.$$

Combining the above, we obtain

$$\begin{aligned}\|m[b_{k,\boldsymbol{\ell}}](D)f\|_{L^p(\mathbb{R}^4)} \lesssim_{\varepsilon,p} 2^{\ell_2(1/2-1/p+\varepsilon)}2^{3(\ell_1-\ell_2)(1-4/p+\varepsilon)/2}&\Big(\sum_{\nu\in\mathbb{Z}}\|m[b_{k,\boldsymbol{\ell}}^{\nu,(\varepsilon)}](D)f\|_{L^p(\mathbb{R}^4)}^p\Big)^{1/p}\\ &+ 2^{-k}\|f\|_{L^p(\mathbb{R}^4)},\end{aligned}$$

which, together with Corollary 6.14 b), yields

$$\|m[b_{k,\boldsymbol{\ell}}](D)f\|_{L^p(\mathbb{R}^4)} \lesssim_{\varepsilon,p} 2^{-3(\ell_1-\ell_2)(1/2p-2\varepsilon)-\ell_2(1/2-3/p-2\varepsilon)}\|f\|_{L^p(\mathbb{R}^4)}.$$

Since $\varepsilon > 0$ was chosen arbitrarily, this is the required bound. □

We have established Proposition 6.5 and therefore completed the proof of the $J = 4$ case of Theorem 4.1.

7. Proof of the decoupling inequalities

This section is devoted to the proof of Theorem 4.4.

7.1. **Decoupling inequalities for non-degenerate curves.** The central ingredient in the proof of Theorem 4.4 is the decoupling theorem of Bourgain–Demeter–Guth [5]. We begin by recalling the statement of (one formulation of) this result. Given a non-degenerate curve $g \in C^{d+1}(I;\hat{\mathbb{R}}^d)$ and $0 < r \leqslant 1$, an 'anisotropic r-neighbourhood' of the curve is constructed as follows.

Definition 7.1. *For each $s \in I$ define the parallelepiped*

$$\alpha(s;r) := \big\{\xi \in \hat{\mathbb{R}}^d : \xi = g(s) + \sum_{j=1}^{d}\lambda_j r^j g^{(j)}(s) \quad \text{for some } \lambda_j \in [-2,2],\ 1 \leqslant j \leqslant d\big\};$$

such sets are referred to as r-slabs.

In some cases it is useful to highlight the choice of function g by writing $\alpha(g;s;r)$ for a r-slab $\alpha(s;r)$. Note that the formula for the parallelepiped $\alpha(s;r)$ can be expressed succinctly in terms of the matrix $[g]_{s,r}$ introduced in (2.1). In particular,

$$\alpha(s;r) = g(s) + [g]_{s,r}\big([-2,2]^d\big). \tag{7.1}$$

An anisotropic r-neighbourhood of the curve g is formed by taking the union of all the r-slabs as s varies over I.

Definition 7.2. *A collection $\mathcal{A}(r)$ of r-slabs is a slab decomposition for g if it consists of precisely the r-slabs $\alpha(g;s;r)$ for s varying over a r-net in I.*

With the above definitions, the decoupling theorem may be stated as follows.

Theorem 7.3 (Bourgain–Demeter–Guth [5]). *Let $g \in \mathfrak{G}_d(\delta)$ for some $0 < \delta \ll 1$, $0 < r \leqslant 1$ and $\mathcal{A}(r)$ be a r-slab decomposition for g. For all $2 \leqslant p \leqslant d(d+1)$ and $\varepsilon > 0$ the inequality*

$$\Big\| \sum_{\alpha \in \mathcal{A}(r)} f_\alpha \Big\|_{L^p(\mathbb{R}^d)} \lesssim_\varepsilon r^{-\varepsilon} \Big(\sum_{\alpha \in \mathcal{A}(r)} \|f_\alpha\|^2_{L^p(\mathbb{R}^d)} \Big)^{1/2} \tag{7.2}$$

holds for any tuple of functions $(f_\alpha)_{\alpha \in \mathcal{A}(r)}$ satisfying $\mathrm{supp}\, \hat{f}_\alpha \subseteq \alpha$.

Remark. This is a slight variant of the decoupling inequality of Bourgain–Demeter–Guth [5] which can be found, for instance, in [10].[9] It is also remarked that the result holds for general non-degenerate curves, although not in the uniform fashion described here. Note, in particular, that by restricting to the model curves $g \in \mathfrak{G}_d(\delta)$ for $0 < \delta \ll 1$, the decoupling inequality (7.2) holds with a constant independent of both the choice of g and δ.

7.2. **Geometric observations.** In order to relate Theorem 4.4 to the Bourgain–Demeter–Guth result from Theorem 7.3, we first relate the Frenet boxes $\pi_{d-1,\gamma}(s;r)$ to certain regions which are more similar in form to the slabs $\alpha(g;s,r)$ introduced above. The Frenet boxes $\pi_{d-1,\gamma}(s;r)$ do not correspond precisely to slabs but to related regions referred to as *plates*. These plate regions are formed by extending d-dimensional slabs into n-dimensions by adjoining additional long directions. Moreover, the plates are naturally defined in relation to a cone generated over a family of non-degenerate curves $g_j \colon I \to \mathbb{R}^d$.

A family of cones. Let $\gamma \in \mathfrak{G}_n(\delta)$ for $0 < \delta \ll 1$ and $\mathbf{e}_j \colon [-1,1] \to S^{n-1}$ for $1 \leqslant j \leqslant n$ be the associated Frenet frame. Without loss of generality, in proving Theorem 4.4 we may always localise so that we only consider the portion of the curve lying over the interval $I = [-\delta, \delta]$. In this case

$$\mathbf{e}_j(s) = \vec{e}_j + O(\delta) \qquad \text{for } 1 \leqslant j \leqslant n \tag{7.3}$$

where, as in Definition 2.1, the $\vec{e}_j$ denote the standard basis vectors.

Here we introduce certain conic surfaces which are 'generated' over the curves $s \mapsto \mathbf{e}_j(s)$. The following observations extend the analysis of [17], where a cone in $\mathbb{R}^3$ generated by the binormal vector $\mathbf{e}_3$ features prominently in the proof of the 3-dimensional analogue of Theorem 1.1.

Let $2 \leqslant d \leqslant n-1$ and consider the map $\tilde{\Gamma} \colon \mathbb{R}^{n-d} \times I \to \mathbb{R}^n$ defined by

$$\tilde{\Gamma}(\vec{\lambda}, s) := \sum_{j=d+1}^{n} \lambda_j \mathbf{e}_j(s), \qquad \vec{\lambda} = (\lambda_{d+1}, \dots, \lambda_n).$$

Restricting to λ_{d+1} bounded away from zero, this is a regular parametrisation of a $(n-d+1)$-dimensional surface in $\mathbb{R}^n$, which is denoted $\Gamma_{n,d}$. Indeed, by the Frenet formulæ,

$$\frac{\partial \tilde{\Gamma}}{\partial s}(\vec{\lambda}, s) = -\lambda_{d+1} \tilde{\kappa}_d(s) \mathbf{e}_d(s) + E_d(\vec{\lambda}, s),$$
$$\frac{\partial \tilde{\Gamma}}{\partial \lambda_j}(\vec{\lambda}, s) = \mathbf{e}_j(s), \qquad d+1 \leqslant j \leqslant n,$$

where $E_d(\vec{\lambda}, s)$ lies in the subspace $\langle \mathbf{e}_{d+1}(s), \dots, \mathbf{e}_n(s) \rangle$. Thus, provided λ_{d+1} is bounded away from zero, the non-vanishing of $\tilde{\kappa}_d$ ensures that these tangent vectors are linearly independent.

Reparametrisation. It is convenient to reparametrise $\Gamma_{n,d}$ so that it is realised as a surface 'generated' over an alternative family of curves which is formed by graphs. To this end, let

[9] More precisely, the general version of the decoupling theorem here follows by combining Theorem 1.2 and Lemma 3.6 from [10].

$A\colon I \to \mathrm{GL}(n-d, \mathbb{R})$ be given by

$$A(s) := \begin{bmatrix} \mathbf{e}_{d+1,d+1}(s) & \cdots & \mathbf{e}_{n,d+1}(s) \\ \vdots & & \vdots \\ \mathbf{e}_{d+1,n}(s) & \cdots & \mathbf{e}_{n,n}(s) \end{bmatrix}^{-1},$$

where $\mathbf{e}_{i,j}(s)$ denotes the jth component of $\mathbf{e}_i(s)$. Provided δ is chosen sufficiently small, (7.3) ensures that the above matrix inverse is well-defined and, moreover, is a small perturbation of the identity matrix. Define the reparametrisation

$$\Gamma(\vec{\lambda}, s) := \tilde{\Gamma}(A(s)\vec{\lambda}, s) \qquad \text{for all } (\vec{\lambda}, s) \in \mathbb{R}^{n-d} \times I. \tag{7.4}$$

Consider the restriction of this mapping to the set $\mathcal{R}'_{n,d} \subset \mathbb{R}^{n-d}$ consisting of all vectors $\vec{\lambda} = (\lambda_{d+1}, \dots, \lambda_n)$ satisfying

$$1/4 \leqslant \lambda_{d+1} \leqslant 2 \quad \text{and} \quad |\lambda_j| \leqslant 2 \quad \text{for } d+2 \leqslant j \leqslant n; \tag{7.5}$$

under this restriction, Γ is a regular parametrisation by the preceding observations.

The mapping (7.4) can be expressed in matrix form as

$$\begin{aligned} \Gamma(\vec{\lambda}, s) &= \begin{bmatrix} \mathbf{e}_{d+1}(s) & \cdots & \mathbf{e}_n(s) \end{bmatrix} \cdot A(s)\vec{\lambda}, \\ &= \begin{bmatrix} G_{d+1}(s) & \cdots & G_n(s) \end{bmatrix} \vec{\lambda}, \end{aligned} \tag{7.6}$$

where the $G_j\colon I \to \mathbb{R}^n$ (which form the column vectors of the above matrix) are of the form

$$G_j(s) = \begin{bmatrix} g_j(s) \\ 0 \end{bmatrix} + \vec{e}_j$$

for some smooth function $g_j\colon I \to \mathbb{R}^d$.

Non-degeneracy conditions. Given $\mathbf{a} = (a_{d+1}, \dots, a_n) \in \mathcal{R}'_{n,d}$, define

$$G_{\mathbf{a}} := \sum_{j=d+1}^{n} a_j \cdot G_j \qquad \text{and} \qquad g_{\mathbf{a}} := \sum_{j=d+1}^{n} a_j \cdot g_j, \tag{7.7}$$

noting $G_{\mathbf{a}}(s) = \Gamma(\mathbf{a}, s)$. The curve $g_{\mathbf{a}}\colon I \to \mathbb{R}^d$ is non-degenerate. To see this, first note that $\frac{\partial^i \Gamma}{\partial s^i}(\vec{\lambda}, s)$ can be expressed as a linear combination of vectors of the form

$$\begin{bmatrix} \mathbf{e}_{d+1}^{(\ell)}(s) & \cdots & \mathbf{e}_n^{(\ell)}(s) \end{bmatrix} \cdot A^{(i-\ell)}(s)\, \vec{\lambda}, \qquad 0 \leqslant \ell \leqslant i, \tag{7.8}$$

where $A^{(k)}$ denotes the component-wise kth-derivative of A. Indeed, this follows simply by applying the Leibniz rule to (7.6). Consequently, $\frac{\partial^i \Gamma}{\partial s^i}(\vec{\lambda}, s)$ must lie in the subspace generated by the columns of the left-hand matrix in (7.8), where i is allowed to vary over the stated range. In particular, one concludes from the Frenet formulæ that

$$\frac{\partial^i \Gamma}{\partial s^i}(\vec{\lambda}, s) \in \langle \mathbf{e}_{d+1-i}(s), \dots, \mathbf{e}_n(s) \rangle \qquad \text{for } 0 \leqslant i \leqslant d. \tag{7.9}$$

On the other hand, the Frenet formulæ also show that the $\mathbf{e}_{d+1-i}(s)$ component of $\frac{\partial^i \Gamma}{\partial s^i}(\vec{\lambda}, s)$ arises only from the term in (7.8) corresponding to $\ell = i$ and

$$\langle \frac{\partial^i \Gamma}{\partial s^i}(\vec{\lambda}, s), \mathbf{e}_{d+1-i}(s) \rangle = (-1)^i \Big(\prod_{\ell=d+1-i}^{d} \tilde{\kappa}_\ell(s) \Big) \langle \vec{A}_1(s), \vec{\lambda} \rangle, \tag{7.10}$$

where $\vec{A}_1(s)$ denotes the first row of $A(s)$. Recall A is a small perturbation of the identity matrix. Thus, under the constraint $\vec{\lambda} \in \mathcal{R}'_{n,d}$ from (7.5), if δ is chosen sufficiently small, then (7.10) implies that

$$|\langle \frac{\partial^i \Gamma}{\partial s^i}(\vec{\lambda}, s), \mathbf{e}_{d+1-i}(s)\rangle| \sim 1 \qquad \text{for all } 1 \leqslant i \leqslant d. \tag{7.11}$$

Thus, combining (7.9) and (7.11), it follows that the vectors $\frac{\partial^i \Gamma}{\partial s^i}(\vec{\lambda}, s)$, $1 \leqslant i \leqslant d$, are linearly independent. Moreover, fixing $\vec{\lambda} = \mathbf{a}$ and noting that $G^{(i)}_{\mathbf{a}}(s) = \frac{\partial^i \Gamma}{\partial s^i}(\mathbf{a}, s) \in \mathbb{R}^d \times \{0\}^{n-d}$ for $i \geqslant 1$, one concludes that

$$|\det[g_{\mathbf{a}}]_s| \gtrsim 1 \tag{7.12}$$

for all $s \in I$, which is the claimed non-degeneracy condition. Note this holds uniformly over the choice of original curve $\gamma \in \mathfrak{G}_n(\delta)$ and over $\mathbf{a} \in \mathcal{R}'_{n,d}$.

Frenet boxes revisited. From the preceding observations, the vectors $G^{(i)}_{\mathbf{a}}(s)$ for $1 \leqslant i \leqslant d$ form a basis of $\mathbb{R}^d \times \{0\}^{n-d}$. Fixing $\xi \in \hat{\mathbb{R}}^n$ and $r > 0$, one may write

$$\xi - \sum_{j=d+1}^{n} \xi_j G_j(s) = \sum_{i=1}^{d} r^i \eta_i G^{(i)}_{\mathbf{a}}(s) \tag{7.13}$$

for some vector of coefficients $(\eta_1, \dots, \eta_d) \in \mathbb{R}^d$. The powers of r appearing in the above expression play a normalising rôle below. For each $1 \leqslant k \leqslant d$ form the inner product of both sides of the above identity with the Frenet vector $\mathbf{e}_k(s)$. Combining the resulting expressions with the linear independence relations inherent in (7.9), the coefficients η_k can be related to the numbers $\langle \xi, \mathbf{e}_k(s)\rangle$ via a lower anti-triangular transformation, viz.

$$\begin{bmatrix} \langle \xi, \mathbf{e}_1(s)\rangle \\ \vdots \\ \langle \xi, \mathbf{e}_d(s)\rangle \end{bmatrix} = \begin{bmatrix} 0 & \cdots & \langle G^{(d)}_{\mathbf{a}}(s), \mathbf{e}_1(s)\rangle \\ \vdots & \ddots & \vdots \\ \langle G^{(1)}_{\mathbf{a}}(s), \mathbf{e}_d(s)\rangle & \cdots & \langle G^{(d)}_{\mathbf{a}}(s), \mathbf{e}_d(s)\rangle \end{bmatrix} \begin{bmatrix} r\eta_1 \\ \vdots \\ r^d \eta_d \end{bmatrix}. \tag{7.14}$$

Thus, if $\xi \in \pi_{d-1,\gamma}(s; r)$, then it follows from combining (4.14a) and (7.11) with (7.14) that $|\eta_i| \lesssim_\gamma 1$ for $1 \leqslant i \leqslant d$, provided $\delta > 0$ is sufficiently small. Similarly, the conditions (4.14b), (4.14c) and the localisation (7.3) imply that

$$\pi_{d-1,\gamma}(s; r) \subseteq \mathcal{R}_{n,d} := [-2, 2]^d \times \mathcal{R}'_{n,d}$$

The identity (7.13) can be succinctly expressed using matrices. In particular, collect the functions g_j together as an $(n-d)$-tuple $\mathbf{g} := (g_{d+1}, \dots, g_n)$ and, for $s \in I$ and $r > 0$, define the $n \times n$ matrix

$$[\mathbf{g}]_{\mathbf{a},s,r} := \begin{pmatrix} [g_{\mathbf{a}}]_{s,r} & \mathbf{g}(s) \\ 0 & \mathrm{I}_{n-d} \end{pmatrix}. \tag{7.15}$$

Here the block $[g_{\mathbf{a}}]_{s,r}$ is the $d \times d$ matrix (2.1) with γ here taken to be $g_{\mathbf{a}}$ as defined in (7.7), whilst $\mathbf{g}(s)$ is understood to be the $(n-d) \times d$ matrix with jth column equal to $g_j(s)$ and I_{n-d} is the $(n-d) \times (n-d)$ identity matrix. With this notation, the identity (7.13) may be written as

$$\xi = [\mathbf{g}]_{\mathbf{a},s,r} \cdot \eta \qquad \text{where } \eta = (\eta_1, \dots, \eta_d, \xi_{d+1}, \dots, \xi_n).$$

Moreover, if $\xi \in \pi_{d-1,\gamma}(s; r)$, then the preceding observations show that η in the above equation may be taken to lie in a bounded region and so

$$\pi_{d-1,\gamma}(s; r) \subseteq \bigcap_{\mathbf{a} \in \mathcal{R}'_{n,d}} [\mathbf{g}]_{\mathbf{a},s,Cr}\big([-2, 2]^n\big) \cap \mathcal{R}_{n,d}, \tag{7.16}$$

where $C \geqslant 1$ is a suitably large dimensional constant. The right-hand side of (7.16) should be compared with the matrix definition of the slabs used in the Bourgain–Demeter–Guth theorem from (7.1).

7.3. **Decoupling inequalities for cones generated by non-degenerate curves.** Here the geometric setup described in §7.2 is abstracted. We first generalise the definition (7.4) to arbitrary cones generated over a tuple of curves $(g_{d+1}, \dots, g_n)$.

Definition 7.4. *Let $2 \leqslant d \leqslant n-1$, $\mathbf{g} = (g_{d+1}, \dots, g_n)$ be an $(n-d)$-tuple of functions in $C^{d+1}(I;\mathbb{R}^d)$ and $\Gamma_{\mathbf{g}}$ denote the codimension $d-1$ cone in $\mathbb{R}^n$ parametrised by*

$$(\vec{\lambda}, s) \mapsto \sum_{j=d+1}^{n} \lambda_j \cdot \Big(\begin{bmatrix} g_j(s) \\ 0 \end{bmatrix} + \vec{e}_j \Big) \quad \textit{for } \vec{\lambda} = (\lambda_{d+1}, \dots, \lambda_n) \in \mathcal{R}'_{n,d} \textit{ and } s \in I.$$

In this case, $\Gamma_{\mathbf{g}}$ is referred to as the cone generated by $\mathbf{g}$.

We now take into account the non-degeneracy condition established in (7.12). Given $\mathbf{a} = (a_{d+1}, \dots, a_n) \in \mathcal{R}'_{n,d}$ and $0 < \delta \ll 1$ consider the collection $\mathfrak{G}^{\mathbf{a}}_{n,d}(\delta)$ of all $(n-d)$-tuples of functions

$$\mathbf{g} = (g_{d+1}, \dots, g_n) \in [C^{d+1}(I;\mathbb{R}^d)]^{n-d}$$

with the property

$$g_{\mathbf{a}} := \sum_{j=d+1}^{n} a_j \cdot g_j \in \mathfrak{G}_d(\delta), \tag{7.17}$$

where $\mathfrak{G}_d(\delta)$ is the class of model curves introduced in §2.

In (7.12) we showed that the curves $g_{\mathbf{a}}$ relevant to our study are *non-degenerate*, which is a weaker condition than $g_{\mathbf{a}} \in \mathfrak{G}_d(\delta)$ (provided $0 < \delta \ll 1$). However, by a localisation and scaling argument similar to that used in §2, we will always be able to assume the condition (7.17) holds in what follows (see the proof of Lemma 7.9 for details of the rescaling).

Given $\mathbf{g} \in \mathfrak{G}^{\mathbf{a}}_{n,d}(\delta)$, $s \in [-1,1]$ and $0 < r \leqslant 1$, define the $n \times n$ matrix $[\mathbf{g}]_{\mathbf{a},s,r}$ as in (7.15); that is,

$$[\mathbf{g}]_{\mathbf{a},s,r} := \begin{pmatrix} [g_{\mathbf{a}}]_{s,r} & \mathbf{g}(s) \\ 0 & \mathrm{I}_{n-d} \end{pmatrix}. \tag{7.18}$$

In view of (7.16), one wishes to study decoupling with respect to the plates

$$\theta(s;r) := [\mathbf{g}]_{\mathbf{a},s,r}\big([-2,2]^n\big) \cap \mathcal{R}_{n,d}.$$

In some cases it will be useful to highlight the choice of function $\mathbf{g}$ by writing $\theta(\mathbf{g};s;r)$ for $\theta(s;r)$. Note that each of these plates lies in an r-neighbourhood of the cone $\Gamma_{\mathbf{g}}$. We think of the union of all plates $\theta(s;r)$ as s varies over the domain $[-1,1]$ as forming an anisotropic r-neighbourhood of $\Gamma_{\mathbf{g}}$, similar to the situation for curves described in §7.1.

Rather than work with the $\theta(s;r)$ directly, certain truncated versions are considered.

Definition 7.5. *For $0 < r \leqslant 1$, $\mathbf{a} = (a_{d+1}, \dots, a_n) \in \mathcal{R}'_{n,d}$ and $K \geqslant 1$ an $(\mathbf{a}, K)$-truncated r-plate for $\Gamma_{\mathbf{g}}$ is a set of the form*

$$\theta^{\mathbf{a},K}(s;r) := [\mathbf{g}]_{\mathbf{a},s,r}\big([-2,2]^n\big) \cap Q(\mathbf{a}, K^{-1})$$

for some $s \in I$ and

$$Q(\mathbf{a}, K^{-1}) := \big\{\xi \in \hat{\mathbb{R}}^n : |\xi_j - a_j| \leqslant K^{-1} \textit{ for } d+1 \leqslant j \leqslant n\big\}.$$

Definition 7.6. *A collection $\Theta^{\mathbf{a},K}(r)$ of $(\mathbf{a},K)$-truncated r-plates is an $(\mathbf{a},K)$-truncated plate decomposition for $\mathbf{g}$ if it consists of $\theta^{\mathbf{a},K}(\mathbf{g};s;r)$ for s varying over a r-net in I.*

Theorem 4.4 is a consequence of the following decoupling inequality for cones $\Gamma_{\mathbf{g}}$.

Proposition 7.7. *Let $2 \leqslant d \leqslant n-1$ and $\varepsilon > 0$. There exists some integer $K \geqslant 1$ such that for all $0 < r \leqslant 1$, $\mathbf{a} \in \mathcal{R}'_{n,d}$ and $\mathbf{g} \in \mathfrak{G}^{\mathbf{a}}_{n,d}(\delta)$ for $0 \leqslant \delta \ll 1$ the following holds. If $\Theta^{\mathbf{a},K}(r)$ is an $(\mathbf{a}, K)$-truncated r-plate decomposition for $\Gamma_{\mathbf{g}}$ and $2 \leqslant p \leqslant d(d+1)$, then*

$$\Big\| \sum_{\theta \in \Theta^{\mathbf{a},K}(r)} f_\theta \Big\|_{L^p(\mathbb{R}^n)} \lesssim_\varepsilon r^{-\varepsilon} \Big(\sum_{\theta \in \Theta^{\mathbf{a},K}(r)} \|f_\theta\|^2_{L^p(\mathbb{R}^n)} \Big)^{1/2}$$

holds for any tuple of functions $(f_\theta)_{\theta \in \Theta^{\mathbf{a},K}(r)}$ satisfying $\operatorname{supp} \hat{f}_\theta \subseteq \theta$.

Proposition 7.7 follows from the Bourgain–Demeter–Guth result (namely, Theorem 7.3) via an argument from [4], where decoupling estimates for the light cone in $\mathbb{R}^n$ were obtained as a consequence of decoupling estimates for the paraboloid in $\mathbb{R}^{n-1}$. The key observation is that, at suitably small scales, the cone $\Gamma_{\mathbf{g}}$ can be approximated by a cylinder over the curve g. This approximation is only directly useful for relatively large r values, but rescaling and induction-on-scale arguments allow one to leverage this observation in the small r setting. Arguments of this kind originate in [17] and have been used repeatedly in the context of decoupling theory: see, for instance, [2, 11, 12, 15].

The details of the proof of Proposition 7.7 are postponed until §7.5 below. In the following subsection, we show that Proposition 7.7 implies Theorem 4.4.

7.4. **Relating the decoupling regions.** Theorem 4.4 may now be deduced as a consequence of Proposition 7.7 using the geometric observations from §7.2.

Proof of Theorem 4.4, assuming Proposition 7.7. First note that it suffices to show the desired decoupling inequality in the restricted range $2 \leqslant p \leqslant d(d+1)$; the estimate for the remaining range $d(d+1) \leqslant p \leqslant \infty$ then follows by an interpolation argument and a trivial estimate for $p = \infty$.

Let $\gamma \in \mathfrak{G}_d(\delta)$ for $0 \leqslant \delta \ll 1$. As previously noted, we may restrict attention to the portion of γ over $I = [-\delta, \delta]$ so that the Frenet vectors satisfy (7.3). Fix $2 \leqslant d \leqslant n-1$, $0 < r \leqslant 1$ and $\mathcal{P}_{d-1}(r)$ a Frenet box decomposition of γ.

Define $\mathbf{g} = (g_{d+1}, \dots, g_n)$ as in §7.2 so that the $g_{\mathbf{a}}$ are non-degenerate. Let $\varepsilon > 0$ be given and take $K \geqslant 1$ an integer satisfying the properties described in Proposition 7.7.

Let $(f_\pi)_{\pi \in \mathcal{P}_{d-1}(r)}$ be a tuple of functions satisfying the Fourier support hypothesis from the statement of Theorem 4.4. If $\pi = \pi_{d-1,\gamma}(s; r) \in \mathcal{P}_{d-1}(r)$, then, recalling (7.16), we have

$$\operatorname{supp} \hat{f}_\pi \subseteq \pi_{d-1,\gamma}(s; r) \subseteq \bigcap_{\mathbf{a} \in \mathcal{R}'_{n,d}} [\mathbf{g}]_{\mathbf{a},s,Cr}\big([-2,2]^n\big) \cap \mathcal{R}_{n,d}. \tag{7.19}$$

The frequency domain is decomposed according to the $Q(\mathbf{a}, K^{-1})$ from Definition 7.5. In particular, let

$$\mathcal{R}'_{n,d}(K) := K^{-1}\mathbb{Z}^{n-d} \cap \mathcal{R}'_{n,d}$$

so that the sets $Q(\mathbf{a}, K^{-1})$ for $\mathbf{a} \in \mathcal{R}'_{n,d}(K)$ are finitely-overlapping and cover of $\mathcal{R}_{n,d}$. Form a smooth partition of unity $(\psi_{\mathbf{a},K^{-1}})_{\mathbf{a} \in \mathcal{R}'_{n,d}(K)}$ adapted to the sets $Q(\mathbf{a}, K^{-1})$ and define the frequency projection operators $P_{\mathbf{a}}$ via the Fourier transform by

$$\big(P_{\mathbf{a}} f\big)\hat{} := \psi_{\mathbf{a},K^{-1}} \cdot \hat{f}.$$

These operators are bounded on L^p for $1 \leqslant p \leqslant \infty$ uniformly in $\mathbf{a}$ and K and, furthermore,

$$f_\pi = \sum_{\mathbf{a} \in \mathcal{R}'_{n,d}(K)} P_{\mathbf{a}} f_\pi \qquad \text{for all } \pi \in \mathcal{P}_{d-1}(r).$$

Since $\#\mathcal{R}'_{n,d}(K) \lesssim_{n,\delta,\varepsilon} 1$, by the triangle inequality and the L^p boundedness of the $P_{\mathbf{a}}$, it suffices to show that

$$\Big\| \sum_{\pi \in \mathcal{P}_{d-1}(r)} P_{\mathbf{a}} f_\pi \Big\|_{L^p(\mathbb{R}^n)} \lesssim_{n,\delta,\varepsilon} r^{-(1/2-1/p)-\varepsilon} \Big(\sum_{\pi \in \mathcal{P}_{d-1}(r)} \|P_{\mathbf{a}} f_\pi\|_{L^p(\mathbb{R}^n)}^p \Big)^{1/p} \tag{7.20}$$

uniformly in $\mathbf{a} \in \mathcal{R}'_{n,d}(K)$. However, recalling (7.19), each function $P_{\mathbf{a}} f_\pi$ has frequency support in the set

$$\theta^{\mathbf{a},K}(s, Cr) = [\mathbf{g}]_{\mathbf{a},s,Cr}\big([-2,2]^n\big) \cap Q(\mathbf{a}, K^{-1})$$

and so an ℓ^2 version of (7.20) follows as a consequence of Proposition 7.7.[10] The desired ℓ^p-decoupling (7.20) follows by applying Hölder's inequality to the ℓ^2-sum. $\square$

7.5. **Proof of Proposition 7.7.** It remains to prove the decoupling Proposition 7.7. This is achieved using the argument outlined at the end of §7.3.

Definition 7.8 (Decoupling constant)**.** *For $2 \leqslant d \leqslant n-1$, $0 < r \leqslant 1$, $p \geqslant 2$, $0 < \delta \ll 1$, $\mathbf{a} \in \mathcal{R}'_{n,d}$ and $K \geqslant 1$ let $\mathfrak{D}^{\mathbf{a}}_{n,d}(K;r)$ denote the infimum over all $C \geqslant 1$ for which*

$$\Big\| \sum_{\theta \in \Theta^{\mathbf{a},K}(r)} f_\theta \Big\|_{L^p(\mathbb{R}^n)} \leqslant C \Big(\sum_{\theta \in \Theta^{\mathbf{a},K}(r)} \|f_\theta\|_{L^p(\mathbb{R}^n)}^2 \Big)^{1/2}$$

holds whenever:

i) $\Theta^{\mathbf{a},K}(r)$ is an $(\mathbf{a},K)$-truncated r-plate decomposition for $\Gamma_{\mathbf{g}}$ for some $\mathbf{g} \in \mathfrak{G}^{\mathbf{a}}_{n,d}(\delta)$,

ii) $(f_\theta)_{\theta \in \Theta^{\mathbf{a},K}(r)}$ is a tuple of functions satisfying $\operatorname{supp} \hat{f}_\theta \subseteq \theta$.

Thus, in this notation, Proposition 7.7 states that for all $\varepsilon > 0$ there exists some $K \geqslant 1$, depending only on n and ε, such that

$$\mathfrak{D}^{\mathbf{a}}_{n,d}(K;r) \lesssim_\varepsilon r^{-\varepsilon} \qquad \text{for all } \mathbf{a} \in \mathcal{R}'_{n,d} \tag{7.21}$$

Remark. The definition of the decoupling constants also depends on p and δ but, for simplicity, these parameters are omitted in the notation.

In conjunction to Theorem 7.3, one needs a simple scaling lemma.

Lemma 7.9 (Generalised Lorentz rescaling)**.** *If $0 < r < \rho < 1$, then*

$$\mathfrak{D}^{\mathbf{a}}_{n,d}(K;r) \lesssim \mathfrak{D}^{\mathbf{a}}_{n,d}(K;\rho)\mathfrak{D}^{\mathbf{a}}_{n,d}(K;r/\rho).$$

Temporarily assuming this result, Proposition 7.7 follows by a simple induction-on-scale argument.

Proof of Proposition 7.7. Let $\varepsilon > 0$, $0 < \delta \ll 1$ and $\mathbf{a} = (a_{d+1}, \dots, a_n) \in \mathcal{R}'_{n,d}$ be given. Henceforth, $K = K(\varepsilon) \geqslant 1$ is thought of as a fixed number, depending only on n and ε, which is chosen sufficiently large to satisfy the forthcoming requirements of the proof. It will be shown, by an induction-on-scale in the r parameter, that (7.21) holds for all $0 < r \leqslant 1$.

If $(100K)^{-d} < r \leqslant 1$, then it follows from the triangle and Cauchy–Schwarz inequalities that

$$\mathfrak{D}^{\mathbf{a}}_{n,d}(K;r) \leqslant \mathbf{C}(\varepsilon)$$

for some constant $\mathbf{C}(\varepsilon) \geqslant 1$ depending only on n and ε. This serves as the base case of an inductive argument.

It remains to establish the inductive step. To this end, fix some $0 < r \leqslant (100K)^{-d}$ and assume the following holds.

[10]Strictly speaking, Proposition 7.7 requires the additional hypothesis $\mathbf{g} \in \mathfrak{G}^{\mathbf{a}}_{n,d}(\delta)$. However, by a rescaling argument (see the proof of Lemma 7.9), the decoupling result generalises to arbitrary $\mathbf{g}$ for which $g_{\mathbf{a}}$ is non-degenerate (albeit no longer with a uniform constant).

Induction hypothesis. If $r_\circ \geqslant 2r$, then $\mathfrak{D}^{\mathbf{a}}_{n,d}(K; r_\circ) \leqslant \mathbf{C}(\varepsilon) r_\circ^{-\varepsilon}$.

Given $0 < r < \rho < 1/2$, one may combine the generalised Lorentz rescaling lemma with the induction hypothesis to conclude that

$$\mathfrak{D}^{\mathbf{a}}_{n,d}(K; r) \lesssim \mathfrak{D}^{\mathbf{a}}_{n,d}(K; \rho)\mathfrak{D}^{\mathbf{a}}_{n,d}(K; r/\rho) \leqslant \mathbf{C}(\varepsilon)\rho^{\varepsilon} r^{-\varepsilon}\mathfrak{D}^{\mathbf{a}}_{n,d}(K; \rho). \tag{7.22}$$

Fix $\rho := K^{-1/d}$. Favourable bounds for $\mathfrak{D}^{\mathbf{a}}_{n,d}(K; \rho)$ can be obtained in this case via an appeal to Theorem 7.3. Let $\mathrm{proj}_d : \hat{\mathbb{R}}^n \to \hat{\mathbb{R}}^d$ denote the orthogonal projection onto the coordinate plane spanned by $\vec{e}_1, \dots, \vec{e}_d$. The key observation is that any $(\mathbf{a}, K)$-truncated ρ-plate $\theta^{\mathbf{a},K}(\mathbf{g}; s; \rho)$ on $\Gamma_{\mathbf{g}}$ essentially projects into a ρ-slab $\alpha(g_{\mathbf{a}}; s; \rho)$ on $g_{\mathbf{a}} = \sum_{j=d+1}^n a_j \cdot g_j$ under this mapping, where $\alpha(g_{\mathbf{a}}; s; \rho)$ is as defined in Definition 7.1. In particular,

$$\mathrm{proj}_d\, \theta^{\mathbf{a},K}(\mathbf{g}; s; \rho) \subseteq \alpha(g_{\mathbf{a}}; s; C\rho) \tag{7.23}$$

for some choice of constant $C \geqslant 1$ depending only on n. To see this, fix $\xi \in \theta^{\mathbf{a},K}(\mathbf{g}; s; \rho)$ and note that $\xi = [\mathbf{g}]_{\mathbf{a},s,\rho} \cdot \eta$ for some $\eta \in [-2, 2]^n$ whilst

$$|\xi_j - a_j| \leqslant 1/K \text{ for } d + 1 \leqslant j \leqslant n, \tag{7.24}$$

by Definition 7.5. By the definition of the matrix $[\mathbf{g}]_{\mathbf{a},s,\rho}$ in (7.18), it follows that

$$\xi' = [g_{\mathbf{a}}]_{s,\rho} \cdot \eta' + \sum_{j=d+1}^n \eta_j g_j(s),$$
$$\xi_j = \eta_j \qquad \text{for } d + 1 \leqslant j \leqslant n$$

where $\xi' := \mathrm{proj}_d\, \xi$ and $\eta' := \mathrm{proj}_d\, \eta \in [-2, 2]^d$. In particular,

$$\xi' - g_{\mathbf{a}}(s) = [g_{\mathbf{a}}]_{s,\rho}\Big(\eta' + [g_{\mathbf{a}}]^{-1}_{s,\rho} \cdot \sum_{j=d+1}^n (\eta_j - a_j) g_j(s)\Big)$$

and, for the choice of $\rho = K^{-1/d}$ specified above,

$$\big|[g_{\mathbf{a}}]^{-1}_{s,\rho} \cdot \sum_{j=d+1}^n (\eta_j - a_j) g_j(s)\big| \leqslant \|[g_{\mathbf{a}}]^{-1}_{s,\rho}\|_{\mathrm{op}} \cdot \sum_{j=d+1}^n |\xi_j - a_j||g_j(s)|$$
$$\lesssim \rho^{-d} K^{-1} \leqslant 1.$$

The second inequality follows from the hypothesis $g_{\mathbf{a}} \in \mathfrak{G}_d(\delta)$ from (7.17), which implies that $\|[g_{\mathbf{a}}]^{-1}_{s,\rho}\|_{\mathrm{op}} \lesssim \rho^{-d}$ (with a uniform constant), and the condition (7.24). Recalling Definition 7.1, it follows that $\xi' \in \alpha(g_{\mathbf{a}}; s; C\rho)$, as claimed.

Let $\Theta^{\mathbf{a},K}(\rho)$ be an $(\mathbf{a}, K)$-truncated ρ-plate decomposition of $\Gamma_{\mathbf{g}}$ and $(f_\theta)_{\theta \in \Theta^{\mathbf{a},K}(\rho)}$ be a tuple of functions satisfying $\mathrm{supp}\, \hat{f}_\theta \subseteq \theta$. For any $2 \leqslant p \leqslant d(d + 1)$ and $\tilde{\varepsilon} > 0$, by (7.23) and Theorem 7.3 it follows that

$$\Big\|\sum_{\theta \in \Theta^{\mathbf{a},K}(\rho)} f_\theta(\,\cdot\,, x'')\Big\|_{L^p(\mathbb{R}^d)} \lesssim_{\tilde{\varepsilon}} \rho^{-\tilde{\varepsilon}}\Big(\sum_{\theta \in \Theta^{\mathbf{a},K}(\rho)} \|f_\theta(\,\cdot\,, x'')\|^2_{L^p(\mathbb{R}^d)}\Big)^{1/2}$$

for all $x'' \in \mathbb{R}^{n-d}$. Taking the L^p-norm of both sides of this inequality with respect to x'' and using Minkowski's inequality to bound the resulting right-hand side, one deduces that

$$\mathfrak{D}^{\mathbf{a}}_{n,d}(K; \rho) \lesssim_{\tilde{\varepsilon}} \rho^{-\tilde{\varepsilon}}. \tag{7.25}$$

Taking $\tilde{\varepsilon} := \varepsilon/2$ in (7.25) and substituting this inequality into (7.22), one deduces that

$$\mathfrak{D}^{\mathbf{a}}_{n,d}(K; r) \leqslant \big(C_\varepsilon \rho^{\varepsilon/2}\big)\mathbf{C}(\varepsilon) r^{\varepsilon},$$

where the C_ε factor arises from the various implied constants in the above argument. Thus, if K is chosen from the outset to be sufficiently large, depending only on n and ε, then

$$C_\varepsilon \rho^{\varepsilon/2} = C_\varepsilon K^{-\varepsilon/2d} \leqslant 1$$

and the induction closes. □

It remains to prove the Lorentz rescaling lemma. Before presenting the argument, it is useful to introduce an extension of Definition 2.1 to tuples of curves $\mathbf{g}$.

Definition 7.10. *Let $\mathbf{g} = (g_{d+1}, \dots, g_n) \in \mathfrak{G}^{\mathbf{a}}_{n,d}(\delta)$ and $g_{\mathbf{a}} := \sum_{j=d+1}^{n} a_j \cdot g_j$, as in (7.17) and (7.18). Define the $(\mathbf{a}; b, \rho)$-rescaling of $\mathbf{g}$ to be the $(n-d)$-tuple*

$$\mathbf{g}_{\mathbf{a},b,\rho} = (g_{\mathbf{a},b,\rho,d+1}, \dots, g_{\mathbf{a},b,\rho,n}) \in [C^{d+1}(I, \mathbb{R}^d)]^{n-d}$$

given by

$$\mathbf{g}_{\mathbf{a},b,\rho}(t) = [g_{\mathbf{a}}]^{-1}_{b,\rho}\big(\mathbf{g}(b+\rho t) - \mathbf{g}(b)\big). \tag{7.26}$$

Here $\mathbf{g}_{\mathbf{a},b,\rho}(t)$ and $\mathbf{g}(t)$ are understood to be the $d \times (n-d)$ matrices whose columns are the component functions of $\mathbf{g}_{\mathbf{a},b,\rho}$ and $\mathbf{g}$, respectively, evaluated at $t \in I$.

As a consequence of this definition, the function

$$g_{\mathbf{a},b,\rho} := \sum_{j=d+1}^{n} a_j \cdot g_{\mathbf{a},b,\rho,j}$$

is precisely the (b,ρ)-rescaling of $g_{\mathbf{a}} := \sum_{j=d+1}^{n} a_j \cdot g_j$. Thus, the notation $g_{\mathbf{a},b,\rho}$ used here is consistent in the sense that $g_{\mathbf{a},b,\rho} = (g_{\mathbf{a}})_{b,\rho}$ and, furthermore, since $g_{\mathbf{a},b,\rho} := (\mathbf{g}_{\mathbf{a},b,\rho})_{\mathbf{a}}$, one has

$$[\mathbf{g}_{\mathbf{a},b,\rho}]_{\mathbf{a},u,h} := \begin{pmatrix} [g_{\mathbf{a},b,\rho}]_{u,h} & \mathbf{g}_{\mathbf{a},b,\rho}(s) \\ 0 & \mathrm{I}_{n-d} \end{pmatrix}. \tag{7.27}$$

Proof of Lemma 7.9. Fix $\mathbf{g} \in \mathfrak{G}^{\mathbf{a}}_{n,d}(\delta)$, an $(\mathbf{a}, K)$-truncated r-plate decomposition $\Theta^{\mathbf{a},K}(r)$ for $\Gamma_{\mathbf{g}}$ and let $(f_\theta)_{\theta \in \Theta^{\mathbf{a},K}(r)}$ be a tuple of functions satisfying $\operatorname{supp} \hat{f}_\theta \subseteq \theta$. By a simple pigeonholing argument, there exists an $(\mathbf{a}, K)$-truncated ρ-plate decomposition $\Theta^{\mathbf{a},K}(\rho)$ such that

$$\Big\| \sum_{\theta \in \Theta^{\mathbf{a},K}(r)} f_\theta \Big\|_{L^p(\mathbb{R}^n)} \lesssim \Big\| \sum_{\theta' \in \Theta^{\mathbf{a},K}(\rho)} f_{\theta'} \Big\|_{L^p(\mathbb{R}^n)}$$

where

$$f_{\theta'} := \sum_{\substack{\theta \in \Theta^{\mathbf{a},K}(r) \\ \theta \subset \theta'}} f_\theta \quad \text{for all } \theta' \in \Theta^{\mathbf{a},K}(\rho).$$

Since $\operatorname{supp} \hat{f}_{\theta'} \subseteq \theta'$, by definition

$$\Big\| \sum_{\theta \in \Theta^{\mathbf{a},K}(r)} f_\theta \Big\|_{L^p(\mathbb{R}^n)} \lesssim \mathfrak{D}^{\mathbf{a}}_{n,d}(K;\rho) \Big(\sum_{\theta' \in \Theta^{\mathbf{a},K}(\rho)} \|f_{\theta'}\|^2_{L^p(\mathbb{R}^n)} \Big)^{1/2}. \tag{7.28}$$

The goal here is to show that

$$\|f_{\theta'}\|_{L^p(\mathbb{R}^n)} \lesssim \mathfrak{D}^{\mathbf{a}}_{n,d}(K; r/\rho) \Big(\sum_{\substack{\theta \in \Theta^{\mathbf{a},K}(\rho) \\ \theta \subset \theta'}} \|f_\theta\|^2_{L^p(\mathbb{R}^n)} \Big)^{1/2} \tag{7.29}$$

for each $\theta' \in \Theta^{\mathbf{a},K}(\rho)$. Indeed, once this is established, by combining (7.28) and (7.29) with the definition of $\mathfrak{D}^{\mathbf{a}}_{n,d}(K;r)$, one deduces the desired result.

Fix an $(\mathbf{a}, K)$-truncated ρ-plate $\theta(b;\rho) \in \Theta^{\mathbf{a},K}(\rho)$ and recall that

$$\theta(b;\rho) = \big\{ \xi \in \hat{\mathbb{R}}^n : ([\mathbf{g}]_{\mathbf{a},b,\rho})^{-1}\xi \in [-2,2]^n \big\} \cap Q(\mathbf{a}, K^{-1})$$

for $Q(\mathbf{a}, K^{-1})$ as defined in Definition 7.5. Note that the preimage of $\theta(b;\rho)$ under the $[\mathbf{g}]_{\mathbf{a},b,\rho}$ mapping is the set

$$\big([\mathbf{g}]_{\mathbf{a},b,\rho}\big)^{-1}\theta(b;\rho) = [-2,2]^n \cap Q(\mathbf{a}, K^{-1}).$$

On the other hand, the $(\mathbf{a}, K)$-truncated r-plate $\theta(s;r) \equiv \theta^{\,\mathbf{a},K}(s;r)$ is transformed under $\big([\mathbf{g}]_{\mathbf{a},b,\rho}\big)^{-1}$ into

$$\big\{\xi \in \hat{\mathbb{R}}^n : \big([\mathbf{g}]_{\mathbf{a},s,r}\big)^{-1}\cdot[\mathbf{g}]_{\mathbf{a},b,\rho}\,\xi \in [-2,2]^n\big\} \cap Q(\mathbf{a}, K^{-1}). \tag{7.30}$$

The key observation is that

$$\big([\mathbf{g}]_{\mathbf{a},s,r}\big)^{-1}\cdot[\mathbf{g}]_{\mathbf{a},b,\rho} = \big([\mathbf{g}_{\mathbf{a},b,\rho}]_{\mathbf{a},\frac{s-b}{\rho},\frac{r}{\rho}}\big)^{-1} \tag{7.31}$$

so that (7.30) corresponds to an $(\mathbf{a}, K)$-truncated r/ρ-plate for the cone generated over the rescaled curve tuple $\mathbf{g}_{\mathbf{a},b,\rho}$. Once this established, (7.29) follows easily by a change of variable. Indeed, taking $\theta' = \theta(b,\rho)$ and defining the functions $\tilde{f}_{\theta'}$ and $\tilde{f}_\theta$ for $\theta \in \Theta^{\,\mathbf{a},K}(r)$ via the Fourier transform by

$$\big(\tilde{f}_{\theta'}\big)\hat{} := \hat{f}_{\theta'} \circ [\mathbf{g}]_{\mathbf{a},b,\rho} \qquad \text{and} \qquad \big(\tilde{f}_{\theta}\big)\hat{} := \hat{f}_{\theta} \circ [\mathbf{g}]_{\mathbf{a},b,\rho},$$

by a linear change of variable the desired inequality (7.29) is equivalent to

$$\|\tilde{f}_{\theta'}\|_{L^p(\mathbb{R}^n)} \lesssim \mathfrak{D}^{\mathbf{a}}_{n,d}(K; r/\rho)\Big(\sum_{\substack{\theta\in\Theta^{\,\mathbf{a},K}(\rho)\\ \theta\subset\theta'}} \|\tilde{f}_\theta\|^2_{L^p(\mathbb{R}^n)}\Big)^{1/2}.$$

However, since the preceding observations show that the $\tilde{f}_\theta$ are Fourier supported on $(\mathbf{a}, K)$-truncated r/ρ-plates for the cone generated over the rescaled curve tuple $\mathbf{g}_{\mathbf{a},b,\rho}$, this bound follows directly from the definition of the decoupling constant.

To prove (7.31), first note that it suffices to show

$$[\mathbf{g}]_{\mathbf{a},b,\rho}\cdot[\mathbf{g}_{\mathbf{a},b,\rho}]_{\mathbf{a},\frac{s-b}{\rho},\frac{r}{\rho}} = [\mathbf{g}]_{\mathbf{a},s,r}.$$

Recalling (7.27) (taking $u := (s-b)/\rho$ and $h := r/\rho$) and carrying out the block matrix multiplication, this is equivalent to the pair of identities

$$[g_{\mathbf{a}}]_{b,\rho}\cdot[g_{\mathbf{a},b,\rho}]_{\frac{s-b}{\rho},\frac{r}{\rho}} = [g_{\mathbf{a}}]_{s,r}, \tag{7.32a}$$

$$[g_{\mathbf{a}}]_{b,\rho}\cdot\mathbf{g}_{\mathbf{a},b,\rho}\big(\tfrac{s-b}{\rho}\big) + \mathbf{g}(b) = \mathbf{g}(s). \tag{7.32b}$$

Note that (7.32a) is an identification of $d\times d$ matrices, whilst (7.32b) is an identification of $d\times(n-d)$ matrices.

Recall the definition of the matrix

$$[g_{\mathbf{a},b,\rho}]_x = \begin{bmatrix} g^{(1)}_{\mathbf{a},b,\rho}(x) & \cdots & g^{(d)}_{\mathbf{a},b,\rho}(x)\end{bmatrix}.$$

From the discussion following Definition 7.10, the curve $g_{\mathbf{a},b,\rho}$ is as defined in Definition 2.1 and, in particular, is given by

$$g_{\mathbf{a},b,\rho}(t) = [g_{\mathbf{a}}]^{-1}_{b,\rho}\big(g_{\mathbf{a}}(b+\rho t) - g_{\mathbf{a}}(b)\big).$$

Combining these definitions with the chain rule,

$$[g_{\mathbf{a},b,\rho}]_x = [g_{\mathbf{a}}]^{-1}_{b,\rho}\cdot[g_{\mathbf{a}}]_{b+\rho x,\rho} \qquad \text{for } x\in\mathbb{R} \text{ with } b+\rho x\in[-1,1].$$

Taking $x = \frac{s-b}{\rho}$ and right multiplying the above display by $D_{r/\rho}$ immediately implies (7.32a). On the other hand, (7.32b) follows directly from the definition (7.26). $\square$

Appendix A. Reduction to a frequency localised estimate

Here we discuss the passage from frequency localised used in §4.2. In particular, we fill in the details of the argument reducing Theorem 1.1 to Theorem 4.1. This follows very quickly by using a special case of a result proved in [16].

A.1. A Calderón–Zygmund estimate.

For each $k \in \mathbb{N}$ we are given operators T_k defined on Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$T_k f(x) := \int_{\mathbb{R}^n} K_k(x,y) f(y)\, \mathrm{d}y$$

where each K_k is a continuous and bounded kernel (with no other quantitative assumptions). Let $\zeta \in \mathcal{S}(\mathbb{R}^n)$, define $\zeta_k := 2^{kn}\zeta(2^k\,\cdot\,)$ and set $P_k f := \zeta_k * f$.

Theorem A.1 ([16]). *Let $\varepsilon > 0$ and $1 < p_0 < p < \infty$. Assume for some $A \geqslant 1$ the operators T_k satisfy*

$$\sup_{k>0} 2^{k/p_0} \|T_k\|_{L^{p_0}(\mathbb{R}^n) \to L^{p_0}(\mathbb{R}^n)} \leqslant A,$$

$$\sup_{k>0} 2^{k/p} \|T_k\|_{L^{p}(\mathbb{R}^n) \to L^{p}(\mathbb{R}^n)} \leqslant A\,.$$

Furthermore, assume that there exist $B \geqslant 1$ and for each cube Q a measurable set $\mathcal{E}_Q$ such that

$$|\mathcal{E}_Q| \leqslant B \max\{\mathrm{diam}(Q)^{n-1}, \mathrm{diam}(Q)^n\}$$

and such that, for every $k \in \mathbb{N}$ and every cube Q with $2^k\mathrm{diam}(Q) \geqslant 1$,

$$\sup_{x \in Q} \int_{\mathbb{R}^n \setminus \mathcal{E}_Q} |K_k(x,y)|\, \mathrm{d}y \leqslant A \max\{(2^k \mathrm{diam}(Q))^{-\varepsilon}, 2^{-k\varepsilon}\}. \tag{A.1}$$

Then for every $r > 0$ we have

$$\Big\| \Big(\sum_{k=1}^{\infty} 2^{kr/p} |P_k T_k f_k|^r \Big)^{1/r} \Big\|_{L^p(\mathbb{R}^n)} \lesssim \Big(\sum_{k=1}^{\infty} \|f_k\|_{L^p(\mathbb{R}^n)}^p \Big)^{1/p}$$

for any sequence of functions $(f_k)_{k=1}^{\infty} \in \ell^p(L^p)$, where the implicit constant depends only on A, B, r, ε, p, p_0, n and ζ.

A.2. Application.

We consider a regular curve given by $t \mapsto \gamma(t)$, $t \in [0,1]$, and let A_γ be as in (1.1); that is, $A_\gamma f = \mu_\gamma * f$.

Let $\beta_\circ \in C_c^\infty(\hat{\mathbb{R}}^n)$ be, say, radial, supported in $\{1/2 < |\xi| < 1\}$ and $\beta_\circ(\xi) > 0$ for $2^{-1/2} \leqslant |\xi| \leqslant 2^{1/2}$ and define $L_j f := \beta_\circ(2^{-j}D)f$. We make the assumption that for some $p_0 \geqslant 2$ we have

$$\|L_j A_\gamma f\|_{L^{p_0}(\mathbb{R}^n)} \leqslant C 2^{-j/p_0} \|f\|_{L^{p_0}(\mathbb{R}^n)}, \quad j \geqslant 0. \tag{A.2}$$

For a non-degenerate curve in $\mathbb{R}^4$ such inequalities were proved in the previous sections.

Theorem A.1 facilitates the following reduction.

Proposition A.2. *Assumption (A.2) implies that A_γ maps L^p boundedly to $L^p_{1/p}$ for $p_0 < p < \infty$.*

This result can be used to complete the argument described in §4.2 and thereby reduce Theorem 1.1 to Theorem 4.1.

Proof of Proposition A.2. Let $\Gamma := \{\gamma(t) : t \in I\}$ where I is a compact interval containing the support of χ. Let Q be a cube with center x_Q and define

$$\mathcal{E}_Q := \{y \in \mathbb{R}^n : \mathrm{dist}(y - x_Q, \Gamma) \leqslant 10\,\mathrm{diam}(Q)\}.$$

Thus if Q is small then $\mathcal{E}_Q$ is a tubular neighborhood of $x_Q + \Gamma$ of width $O(\mathrm{diam}(Q))$. It is not hard to see that there is a constant C (depending on B) such that

$$\mathrm{meas}(\mathcal{E}_Q) \leqslant C \begin{cases} \mathrm{diam}(Q)^{n-1} & \text{if } \mathrm{diam}(Q) \leqslant 1, \\ \mathrm{diam}(Q)^n & \text{if } \mathrm{diam}(Q) \geqslant 1. \end{cases}$$

Let $\upsilon \in C_c^\infty(\mathbb{R}^n)$ be supported in $\{x : |x| < 1/4\}$, with the property that $\widehat{\upsilon}(\xi) > 0$ on the support of $\beta_\circ$, $\widehat{\upsilon}(0) = 0$ and $\nabla\widehat{\upsilon}(0) = 0$.[11] Let $\upsilon_k := 2^{kn}\upsilon(2^k\,\cdot\,)$ and define

$$T_k f(x) := \upsilon_k * \mu_\gamma * f \qquad \text{and} \qquad K_k(x,y) := \upsilon_k * \mu_\gamma(x-y).$$

By the support properties of υ_k we have

$$K_k(x,y) = 0 \qquad \text{if } x \in Q,\ y \in \mathbb{R}^n \backslash \mathcal{E}_Q \text{ and } 2^k\mathrm{diam}(Q) > 1$$

and thus (A.1) holds trivially.

We claim that the assumption (A.2) also implies

$$\sup_{k\in\mathbb{N}} 2^{k/p_0}\|T_k\|_{L^{p_0}(\mathbb{R}^n)\to L^{p_0}(\mathbb{R}^n)} < \infty.$$

To see this, choose $\widetilde{\beta}_\circ \in C_c^\infty(\hat{\mathbb{R}}^n)$ supported in $\{\xi : 1/2 < |\xi| < 2\}$ such that

$$\sum_{j\in\mathbb{Z}} \beta_\circ(2^{-j}\xi)\widetilde{\beta}_\circ(2^{-j}\xi) = 1$$

for all $\xi \neq 0$ and define $\widetilde{L}_j := \widetilde{\beta}_\circ(2^{-j}D)$. Thus,

$$\|T_k\|_{L^{p_0}\to L^{p_0}} \leqslant \sum_{j\in\mathbb{Z}} \|L_j\widetilde{L}_jT_k\|_{L^{p_0}\to L^{p_0}} \leqslant \sum_{j\in\mathbb{Z}} \|L_jA_\gamma\|_{L^{p_0}\to L^{p_0}}\|\widetilde{L}_j\upsilon_k\|_1.$$

By straightforward calculations, using scaling and the cancellation and Schwartz properties of υ and $\mathcal{F}^{-1}[\beta_\circ]$, one has $\|\widetilde{L}_j\upsilon_k\|_1 = O(2^{-|j-k|})$ for all $j \in \mathbb{Z}$. Using this together with the hypothesis (A.2), we get

$$\|T_k\|_{L^{p_0}\to L^{p_0}} \leqslant \sum_{j\in\mathbb{Z}} \min\{2^{-j/p}, 1\}2^{-|j-k|} \lesssim 2^{-k/p};$$

note for $j < 0$ we use the trivial bounds $\|L_jA_\gamma f\|_p \lesssim \|f\|_p$. Since $\|T_k\|_{L^\infty\to L^\infty} = O(1)$, interpolation therefore yields

$$\sup_{k\in\mathbb{N}} 2^{k/p}\|T_k\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} < \infty \qquad \text{for all } p_0 \leqslant p \leqslant \infty.$$

Let $\beta \in C_c^\infty(\hat{\mathbb{R}}^n)$ be supported in $\{\xi : 1/4 < |\xi| < 4\}$ such that $\beta_\circ\beta = \beta_\circ$ and $p_0 < p < \infty$. Defining $f_k := \beta(2^{-k}D)f$, by Theorem A.1 we obtain

$$\begin{aligned} \Big\|\Big(\sum_{k=1}^\infty [2^{k/p}|\beta_\circ(2^{-k}D)A_\gamma f|]^r\Big)^{1/r}\Big\|_{L^p(\mathbb{R}^n)} &= \Big\|\Big(\sum_{k=1}^\infty 2^{kr/p}|P_kT_kf_k|^r\Big)^{1/r}\Big\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \Big(\sum_{k=1}^\infty \|f_k\|_{L^p(\mathbb{R}^n)}^p\Big)^{\frac{1}{p}} \\ &\lesssim \Big(\sum_{k=1}^\infty \|\beta(2^{-k}D)f\|_{L^p(\mathbb{R}^n)}^p\Big)^{\frac{1}{p}}. \end{aligned} \tag{A.3}$$

[11] To construct such a function take any compactly supported real valued $u \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} u = 1$, form $u_C = C^d u(C\,\cdot\,)$ for sufficiently large C to ensure $\widehat{u}_C > 0$ on $\mathrm{supp}\,\beta_\circ$ and then take the Laplacian, $\upsilon = \Delta u_C$, to also get the condition $\widehat{\upsilon}(0) = 0$.

Since $\ell^r \hookrightarrow \ell^2$ for $r \leqslant 2$ and $\ell^2 \hookrightarrow \ell^p$ for $2 \leqslant p$, we deduce that

$$\Big\| \Big(\sum_{k=1}^{\infty} [2^{k/p} |\beta_\circ(2^{-k}D) A_\gamma f|]^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^n)} \lesssim \Big\| \Big(\sum_{k=1}^{\infty} |\beta(2^{-k}D) f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^n)}.$$

This, together with the obvious low frequency L^p estimates, yield the asserted Sobolev bound via Littlewood–Paley inequalities. □

Remark. Using Besov and Triebel-Lizorkin spaces one gets from (A.3) the stronger inequality

$$A_\gamma : B^0_{p,p} \to F^{1/p}_{p,r}$$

for all $r > 0$.

Appendix B. Derivative bounds for implicitly defined functions

The following lemma is a particular instance of a more general lemma on derivative bounds for implicitly defined functions found in [1, Appendix C].

Lemma B.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $I \subseteq \mathbb{R}$ an open interval, $g \colon I \to \mathbb{R}^n$ a C^∞ mapping and $y \colon \Omega \to I$ a C^∞ mapping such that*

$$\langle g \circ y(\boldsymbol{x}), \boldsymbol{x} \rangle = 0 \qquad \text{for all } \boldsymbol{x} \in \Omega.$$

For $\mathbf{e} \in S^{n-1}$ let $\nabla_{\mathbf{e}}$ denote the directional derivative operator with respect to $\boldsymbol{x}$ in the direction of $\mathbf{e}$. Suppose $A, M_1, M_2 > 0$ are constants such that

$$\begin{cases} |\langle g' \circ y(\boldsymbol{x}), \boldsymbol{x} \rangle| & \geqslant \quad AM_2, \\ |\langle g^{(N)} \circ y(\boldsymbol{x}), \boldsymbol{x} \rangle| & \lesssim_N \quad AM_2^N \\ |\langle g^{(N)} \circ y(\boldsymbol{x}), \mathbf{e} \rangle| & \lesssim_N \quad AM_1M_2^N \end{cases} \qquad \text{for all } N \in \mathbb{N} \text{ and all } \boldsymbol{x} \in \Omega \tag{B.1}$$

Then the function y satisfies

$$|\nabla_{\mathbf{e}}^N y(\boldsymbol{x})| \lesssim_N M_1^N M_2^{-1} \qquad \text{for all } \boldsymbol{x} \in \Omega \text{ and all } N \in \mathbb{N}_0. \tag{B.2}$$

Furthermore, for any C^∞ function $h \colon I \to \mathbb{R}^n$ for which there exists some constant $B > 0$ satisfying

$$\begin{cases} |\langle h^{(N)} \circ y(\boldsymbol{x}), \boldsymbol{x} \rangle| & \lesssim_N \quad BM_2^N \\ |\langle h^{(N)} \circ y(\boldsymbol{x}), \mathbf{e} \rangle| & \lesssim_N \quad BM_1M_2^N \end{cases} \qquad \text{for all } N \in \mathbb{N} \text{ and all } \boldsymbol{x} \in \Omega, \tag{B.3}$$

one has

$$\big|\nabla_{\mathbf{e}}^N \langle h \circ y(\boldsymbol{x}), \boldsymbol{x} \rangle\big| \lesssim_N BM_1^N \qquad \text{for all } \boldsymbol{x} \in \Omega \text{ and all } N \in \mathbb{N}. \tag{B.4}$$

The following example illustrates how Lemma B.1 is applied in practice in this article.

Example B.2 (Application to the case $J = 3$)**.** *Let $\gamma \in \mathfrak{G}_4(\delta_0)$, and $\theta : \hat{\mathbb{R}}^4 \backslash \{0\} \to I_0$ satisfying*

$$\langle \gamma'' \circ \theta(\xi), \xi \rangle = 0.$$

We apply the previous result with $g = \gamma''$ and $h = \gamma'$. If $B \leqslant A$ the conditions (B.1) and (B.3) read succinctly as

$$\begin{cases} |\langle \gamma^{(3)} \circ \theta(\xi), \xi \rangle| & \geqslant \quad AM_2, \\ |\langle \gamma^{(1+N)} \circ \theta(\xi), \xi \rangle| & \lesssim_N \quad BM_2^N \\ |\langle \gamma^{(1+N)} \circ \theta(\xi), \mathbf{e} \rangle| & \lesssim_N \quad BM_1M_2^N \end{cases} \qquad \text{for all } N \in \mathbb{N} \text{ and all } \xi \in \Omega \subset \hat{\mathbb{R}}^4 \backslash \{0\},$$

which imply

$$|\nabla_{\boldsymbol{e}}^N \theta(\xi)| \lesssim_N M_1^N M_2^{-1} \qquad \text{and} \qquad |\nabla_{\boldsymbol{e}}^N \langle \gamma' \circ \theta(\xi), \xi \rangle| \lesssim_N BM_1^N.$$

for all $N \in \mathbb{N}$ and all $\xi \in \Omega \subset \hat{\mathbb{R}}^4 \backslash \{0\}$.

The applications in the different cases $J = 4$ are similar, with the choices $(g, h) = (\gamma^{(3)}, \gamma'')$, $(g, h) = (\gamma^{(3)}, \gamma')$ and $(g, h) = (\gamma'', \gamma')$.

Appendix C. Integration-by-parts lemmata

C.1. **Non-stationary phase.** For $a \in C_c^\infty(\mathbb{R})$ supported in an interval $I \subset \mathbb{R}$ and $\phi \in C^\infty(I)$, define the oscillatory integral

$$\mathcal{I}[\phi, a] := \int_{\mathbb{R}} e^{i\phi(s)} a(s)\, \mathrm{d}s.$$

The following lemma is a standard application of integration-by-parts.

Lemma C.1 (Non-stationary phase). *Let $R \geqslant 1$ and ϕ, a be as above. Suppose that for each $j \in \mathbb{N}_0$ there exist constants $C_j \geqslant 1$ such that the following conditions hold on the support of a:*

i) $|\phi'(s)| > 0$,
ii) $|\phi^{(j)}(s)| \leqslant C_j R^{-(j-1)} |\phi'(s)|^j$ *for all* $j \geqslant 2$,
iii) $|a^{(j)}(s)| \leqslant C_j R^{-j} |\phi'(s)|^j$ *for all* $j \geqslant 0$.

Then for all $N \in \mathbb{N}_0$ there exists some constant $C(N)$ such that

$$|\mathcal{I}[\phi, a]| \leqslant C(N) \cdot |\operatorname{supp} a| \cdot R^{-N}.$$

Moreover, $C(N)$ depends on $C_1, \dots, C_N$ but is otherwise independent of ϕ and a and, in particular, does not depend on r.

A detailed proof of this lemma can be found in [1, Appendix D].

C.2. **Kernel estimates.** The following lemma, which is used to obtain L^∞ bounds for the multipliers, is based on integration-by-parts in the ξ variable.

Lemma C.2. *Let $a \in C_c^\infty(\hat{\mathbb{R}}^n \times I_0)$, $\sigma > 0$, $\lambda_j > 0$ for $1 \leqslant j \leqslant n$ and $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Suppose the following conditions hold:*

i) $|\{s \in \mathbb{R} : (\xi; s) \in \operatorname{supp} a$ *for some* $\xi \in \hat{\mathbb{R}}^n\}| \leqslant \sigma$,
ii) $\operatorname{supp}_\xi a \subseteq \{\xi \in \hat{\mathbb{R}}^n : |\langle \xi, v_j \rangle| \leqslant \lambda_j$ *for* $1 \leqslant j \leqslant n\}$,
iii) $|\nabla_{\boldsymbol{v}_j}^N a(\xi; s)| \lesssim_N \lambda_j^{-N}$ *for all* $(\xi; s) \in \hat{\mathbb{R}}^n \times \mathbb{R}$, $1 \leqslant j \leqslant n$ *and* $N \in \mathbb{N}_0$.

Then

$$\|m[a]\|_{M^\infty(\mathbb{R}^n)} \lesssim \sigma.$$

Here $\nabla_{\boldsymbol{v}} := \boldsymbol{v} \cdot \nabla$ denotes the directional derivative with respect to ξ in the direction of $\boldsymbol{v} \in S^{n-1}$.

Proof of Lemma C.2. For $f \in \mathscr{S}(\mathbb{R}^n)$ we have $m[a](D)f = K[a] * f$ where the kernel $K[a]$ is given by

$$K[a](x) = \int_{\mathbb{R}} \mathcal{F}^{-1} a(\,\cdot\,; s)(x + \gamma(s)) \chi(s)\, \mathrm{d}s.$$

Here $\mathcal{F}^{-1}$ denotes the inverse Fourier transform in the ξ variable. Consequently,

$$\|m[a](D)\|_{M^\infty(\mathbb{R}^n)} \leqslant \|K[a]\|_{L^1(\mathbb{R}^n)} \leqslant \int_{\mathbb{R}} \|\mathcal{F}^{-1} a(\,\cdot\,; s)\|_{L^1(\mathbb{R}^n)} \chi(s)\, \mathrm{d}s.$$

By the hypothesis i) on the s-support, the problem is therefore reduced to showing

$$\sup_{s \in \mathbb{R}} \|\mathcal{F}^{-1} a(\,\cdot\,; s)\|_{L^1(\mathbb{R}^n)} \lesssim 1. \tag{C.1}$$

However, the conditions ii) and iii), combined with a standard integration-by-parts argument, imply

$$|\mathcal{F}^{-1} a(\,\cdot\,; s)(x)| \lesssim_N \Big(\prod_{j=1}^n \lambda_j\Big)\Big(1 + \sum_{j=1}^n \lambda_j |\langle x, \boldsymbol{v}_j \rangle|\Big)^{-N} \qquad \text{for all } (x; s) \in \mathbb{R}^n \times \mathbb{R} \text{ and all } N \geqslant 0,$$

from which the desired bound (C.1) follows. ☐

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