Non-lorentzian spacetimes

NON-LORENTZIAN SPACETIMES

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Abstract. I review some of my recent work on non-lorentzian geometry. I review the classification of kinematical Lie algebras and their associated Klein geometries. I then describe the Cartan geometries modelled on them and their characterisation in terms of their intrinsic torsion.

Contents

1. Introduction

Ever since Minkowski introduced his eponymous spacetime (see, e.g., [1] for his famous 1908 address to the Society of Natural Scientists in Cologne, reproduced in English translation in [2]), lorentzian geometry has played a fundamental rôle in Physics and, until recently, has dominated our attempts to model the universe. Minkowski spacetime replaced the Galilei spacetime of newtonian mechanics as the geometric arena for Einstein's theory of special relativity, conceived to describe Maxwell's theory of electrodynamics. With the advent of quantum mechanics, it became the arena of relativistic quantum mechanics and, unavoidably, of quantum field theory. On the other hand, Minkowski spacetime is the flat model of lorentzian geometry, the Cartan geometry modelled on it, and which is the basis of Einstein's theory of general relativity, which accurately describes a wide range of gravitational phenomena. The attempt at marrying quantum theory and general relativity into a quantum theory of gravity has kept the theoretical/mathematical physics community busy for the best part of the last 75 years. Why then should one bother with non-lorentzian spacetimes, except as a purely mathematical curiosity?

One answer to this question lies precisely in the difficulty to formulate a quantum theory of gravity. It may help to keep the following picture in mind: the Bronstein cube of physical theories.

This cube is a cartoon of physical theories one can obtain from classical mechanics (CM) by turning on certain parameters: the inverse speed of light $1/\mathsf{c}$, Newton's gravitational constant (or, more geometrically, curvature) G and Planck's constant $\hbar$. Of course, these are physical constants and as such they take particular values, but let us pretend that we can change them at will. There are three directions we can go in from classical mechanics: to special relativity (SR) by turning on $1/\mathsf{c}$, to quantum mechanics

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(QM) by turning on $\hbar$, and to newtonian gravity (NG) by turning on G. From special relativity we may go to general relativity (GR) by turning on G and to relativistic quantum mechanics and hence quantum field theory (QFT) by turning on $\hbar$. These two theories are the standard points of departure towards the final goal of a theory of quantum gravity (QG). However as the picture makes clear, there is a third possible line of approach: via the (as yet non-existent) quantisation of newtonian gravity (QNG).

In this review I shall remain in the non-quantum world and concentrate on the bottom side of the Bronstein cube. In geometrical terms, classical mechanics and special relativity are described by Klein geometries: Galilei and Minkowski spacetimes, whereas turning on G corresponds to constructing Cartan geometries modelled on them. I shall therefore start by describing Galilei and Minkowski spacetimes and show that they are Klein geometries of the Galilei and Poincaré groups, respectively. I shall recognise these groups as examples of kinematical Lie groups and recall the classification of kinematical Lie algebras. This will lead to the classification of kinematical Klein geometries, of which Galilei and Minkowski spacetimes are but two of a plethora of examples which nevertheless give rise to a small class of Cartan geometries of spacetimes: lorentzian, galilean (or Newton–Cartan), carrollian and aristotelian. I do not discuss aristotelian geometries in this review and concentrate on the galilean and carrollian geometries. These Cartan geometries are examples of G-structures, which I shall classify further in terms of their intrinsic torsion.

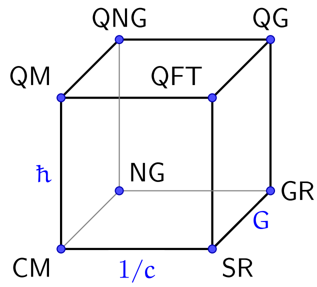

Figure 1. Bronstein cube

This short review is organised as follows. In Section 2 I discuss two classical models of the universe, as motivation for the rest of the geometries in the review: Galilei spacetime is discussed in Section 2.1 and Minkowski spacetime in Section 2.2. Comparison of the relativity groups of Galilei and Minkowski spacetimes, suggests the notion of a kinematical Lie group and in Section 3 I define the notion of a kinematical Lie algebra and review their classification, given in Table 1. This is a necessary first step to the determination of the associated kinematical Klein geometries. These are discussed in Section 4 and their classification (up to coverings) is given in Table 2. After a brief review of the basic notions of Klein geometry in Section 4.1, I discuss the four classes of kinematical Klein geometries in the Table: lorentzian in Section 4.2, riemannian in Section 4.3, galilean in Section 4.4 and carrollian in Section 4.5. I do not dwell on the riemannian case, since riemannian manifolds do not play the rôle of spacetimes. For the lorentzian and carrollian cases, I give explicit geometric realisations of the spacetimes in Table 2, postponing the galilean case until later in the review. In Section 5 I turn my attention to the kinematical Cartan geometries modelled on the kinematical Klein geometries. In Section 5.1 I present a very brief review of the basic notions of Cartan geometry that I shall require, mainly about G-structures since the kinematical Cartan geometries turn out to be G-structures. I also review the notion of the intrinsic torsion of a G-structure, which will allow a refinement of the classification of kinematical Cartan geometries. Section 5.2 briefly recaps, mostly for orientation, the case of lorentzian geometry. In Section 5.3 I discuss Newton–Cartan geometry, which is the Cartan geometry modelled on galilean spacetimes and I summarise the classification in terms of intrinsic torsion. In Section 5.3.1 I discuss how to obtain Newton–Cartan geometries as null reductions of lorentzian geometries and this is used in Section 5.3.2 in order to give the promised geometric realisations of the galilean Klein geometries in Table 2. In Section 5.4 I discuss carrollian geometry and summarise the classification in terms of intrinsic torsion. The nomenclature of carrollian structures is meant to be reminiscent of the classification of hypersurfaces in riemannian geometry. This is made precise in Section 5.4.1, where I show that a null hypersurface in a lorentzian manifold has a carrollian structure and re-interpret the intrinsic torsion in this light using the properties of the null Weingarten map and associated null second fundamental form. In Section 5.4.2 I discuss another source of carrollian structures: namely, the natural structure on the bundle of scales of a riemannian conformal manifold. Finally, in Section 6 I give a brief overview of related topics not covered in this review and some natural extensions of the work described here.

Acknowledgments

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2. Two models of the universe

In this section I review two classical models of the universe: Galilei spacetime of newtonian mechanics and Minkowski spacetime of special relativity. Both spacetimes are described by an affine space, homogeneous under the action of a kinematical Lie group (to be defined below), but their invariant structures differ: whereas Galilei spacetime has a Galilei-invariant clock and ruler, Minkowski spacetime has a Poincaré-invariant proper distance. In the corresponding Cartan geometries, the clock and ruler will be seen as a flat example of a (weak) Newton–Cartan structure, whereas the proper distance is, of course, a flat example of a lorentzian metric.

Although the physically relevant dimension is 4, let us work in $d+1$ dimensions. Let $\mathbb{A}^{d+1}$ denote $(d+1)$-dimensional real affine space. In the present context, points in $\mathbb{A}^{d+1}$ are referred to as **(spacetime) events**. The affine space $\mathbb{A}^{d+1}$ is a torsor over $\mathbb{R}^{d+1}$, thought of as an abelian group under vector addition. The action of $\mathbb{R}^{d+1}$ on $\mathbb{A}^{d+1}$ is via parallel displacements. Given any two points $a, b \in \mathbb{A}^{d+1}$, there exists a unique parallel displacement $v \in \mathbb{R}^{d+1}$ such that $b = a + v$. It is customary to refer to v as $b - a$ and hence to identify parallel displacements with differences of points.

It is often convenient for calculations to use an explicit model for $\mathbb{A}^{d+1}$ as the affine hyperplane in $\mathbb{R}^{d+2}$ with equation $x^{d+2} = 1$. This embeds the affine group $\mathrm{Aff}(d+1,\mathbb{R})$ into the general linear group $\mathrm{GL}(d+2,\mathbb{R})$ as the subgroup which preserves that hyperplane or, more explicitly, as the subgroup consisting of matrices of the form

$$\begin{pmatrix} L & v \\ 0 & 1 \end{pmatrix} \tag{2.1}$$

where $v \in \mathbb{R}^{d+1}$ and $L \in \mathrm{GL}(d+1,\mathbb{R})$. The relativity groups of Galilei and Minkowski spacetimes are subgroups of the affine group consisting of all the parallel displacements but a restricted subgroup of linear transformations consisting of orthogonal transformations and boosts; although of course the notion of boost differs in the galilean and lorentzian worlds.

2.1. Galilei spacetime.

The following formulation of the Galilei spacetime is originally due to Weyl [3]. Galilei spacetime is a triple $(\mathbb{A}^{d+1}, \tau, \lambda)$ where the **clock** $\tau : \mathbb{R}^{d+1} \to \mathbb{R}$ and the **ruler** $\lambda : \ker\tau \to \mathbb{R}$ are defined as follows.

The clock measures the time interval $\tau(b-a)$ between two events $a, b \in \mathbb{A}^{d+1}$. In the explicit model $\mathbb{A}^{d+1} \subset \mathbb{R}^{d+2}$, letting $a = (\boldsymbol{x}, x^{d+1}, 1)$ and $b = (\boldsymbol{y}, y^{d+1}, 1)$, with $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, then $\tau(b-a) = y^{d+1} - x^{d+1}$. Two events $a, b \in \mathbb{A}^{d+1}$ are **simultaneous** if $\tau(b-a) = 0$. Fixing an event a, the set of events simultaneous to a defines a d-dimensional affine subspace

$$\mathbb{A}^d_a := a + \ker\tau = \{a + v \mid \tau(v) = 0\} \tag{2.2}$$

of $\mathbb{A}^{d+1}$. The quotient $\mathbb{A}^{d+1}/\ker\tau$ is an affine line $\mathbb{A}^1$, so that the clock gives a fibration $\pi : \mathbb{A}^{d+1} \to \mathbb{A}^1$ whose fibre at $\pi(a)$ consists of the affine hypersurface $\mathbb{A}^d_a$, as illustrated in Figure 2.

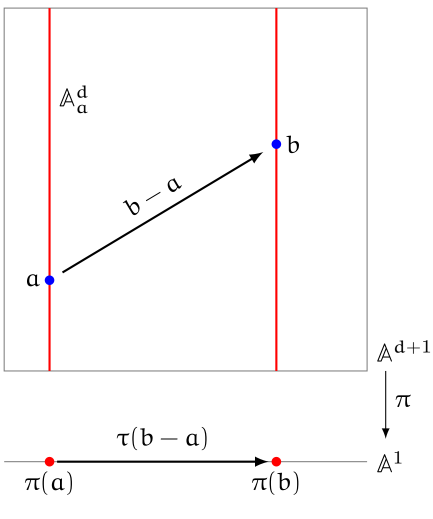

Figure 2. The clock fibration $\pi : \mathbb{A}^{d+1} \to \mathbb{A}^1$

The ruler measures the (euclidean) distance between simultaneous events. If $a, b \in \mathbb{A}^{d+1}$ are simultaneous, $\lambda(b-a)$ is the euclidean length of $b - a \in \ker\tau$. Again, in the explicit model, if $a = (\boldsymbol{x}, x^{d+1}, 1)$ and

$b = (\boldsymbol{y}, y^{d+1}, 1)$, with $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ and $x^{d+1} = y^{d+1}$, are any two simultaneous events, then $\lambda(b-a) = \|\boldsymbol{y}-\boldsymbol{x}\|$, the euclidean distance between $\boldsymbol{x}$ and $\boldsymbol{y}$.

The relativity group of Galilei spacetime is called the **Galilei group** and it consists of those affine transformations of $\mathbb{A}^{d+1}$ which preserve the clock and the ruler. It embeds in $\mathrm{GL}(d+2, \mathbb{R})$ as those matrices of the form

$$\begin{pmatrix} R & \boldsymbol{v} & \boldsymbol{p} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.3}$$

where $R \in \mathrm{O}(d)$, $\boldsymbol{p}, \boldsymbol{v} \in \mathbb{R}^d$ and $s \in \mathbb{R}$.

The action of the matrix in equation (2.3) on an event $(\boldsymbol{x}, t, 1)$ gives the event $(R\boldsymbol{x} + t\boldsymbol{v} + \boldsymbol{p}, t + s, 1)$ which may be interpreted as the composition of an orthogonal transformation $\boldsymbol{x} \mapsto R\boldsymbol{x}$, a **Galilei boost** $\boldsymbol{x} \mapsto \boldsymbol{x} + t\boldsymbol{v}$, a spatial translation $\boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{p}$ and a temporal translation $t \mapsto t + s$. The stabiliser subgroup of the event $(\boldsymbol{0}, 0, 1)$, isomorphic to what is commonly called the **homogeneous Galilei group**, consists of orthogonal transformations and boosts. As discussed later, the Cartan geometry modelled on Galilei spacetime is an G-structure with G the homogeneous Galilei group.

The Lie algebra of the Galilei group is unsurprisingly called the **Galilei algebra** and it is isomorphic to the subalgebra of $\mathfrak{gl}(d+2, \mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} A & \boldsymbol{v} & \boldsymbol{p} \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.4}$$

where $A \in \mathfrak{so}(d)$, $\boldsymbol{v}, \boldsymbol{p} \in \mathbb{R}^d$ and $s \in \mathbb{R}$. Introduce a basis $L_{ab} = -L_{ba}, B_a, P_a, H$ by

$$\begin{pmatrix} A & \boldsymbol{v} & \boldsymbol{p} \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix} = \tfrac{1}{2} A^{ab} L_{ab} + v^a B_a + p^a P_a + sH \tag{2.5}$$

and in this way easily work out the Lie brackets of the Galilei algebra in this basis. The nonzero brackets are given by

$$\begin{aligned} [L_{ab}, L_{cd}] &= \delta_{bc} L_{ad} - \delta_{ac} L_{bd} - \delta_{bd} L_{ac} + \delta_{bd} L_{ac} \\ [L_{ab}, B_b] &= \delta_{bc} B_a - \delta_{ac} B_b \\ [L_{ab}, P_b] &= \delta_{bc} P_a - \delta_{ac} P_b \\ [B_a, H] &= P_a. \end{aligned} \tag{2.6}$$

This shows that L_{ab} span an $\mathfrak{so}(d)$ subalgebra, relative to which B_a, P_a transform according to the three-dimensional vector representation and H transforms as the one-dimensional scalar representation. These turn out to be the defining properties of a kinematical Lie algebra (with spatial isotropy).

2.2. **Minkowski spacetime.** Minkowski spacetime is described by a pair $(\mathbb{A}^{d+1}, \Delta)$, where the **proper distance** $\Delta : \mathbb{R}^{d+1} \to \mathbb{R}$ is defined as follows. In the explicit model of $\mathbb{A}^{d+1} \subset \mathbb{R}^{d+2}$, if $a = (\boldsymbol{x}, t, 1)$ and $b = (\boldsymbol{y}, s, 1)$,

$$\Delta(b - a) = \|\boldsymbol{y} - \boldsymbol{x}\|^2 - c^2(s - t)^2, \tag{2.7}$$

where c is a parameter interpretable as the speed of light.

There is no longer a separate clock and ruler, or in Minkowski's own words (in the English translation of [2]):

> Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.

In particular, there is no longer an invariant notion of simultaneity between events. Simultaneity is replaced by a notion of causality, geometrised by lightcones at every spacetime event a: the **lightcone** $\mathbb{L}_a$ of a being defined as those events which are a zero proper distance away from a:

$$\mathbb{L}_a = \left\{ b \in \mathbb{A}^{d+1} \;\middle|\; \Delta(b - a) = 0 \right\}. \tag{2.8}$$

Two events $a, b \in \mathbb{A}^{d+1}$ are said to be causally related if $\Delta(b - a) \leqslant 0$. If $a = (\boldsymbol{x}, t, 1)$ and $b = (\boldsymbol{y}, s, 1)$ are causally related, one says that a is in the causal future of b (and hence b is in the causal past of a) if $t - s > 0$.

The relativity group of Minkowski spacetime is the **Poincaré group** and consists of those affine transformations which preserve the proper distance between events. It embeds in $\mathrm{GL}(d+2, \mathbb{R})$ as those matrices

$$\begin{pmatrix} L & v \\ 0 & 1 \end{pmatrix} \tag{2.9}$$

where $L^T\eta L = \eta$ and $v \in \mathbb{R}^{d+1}$. The Poincaré group is thus isomorphic to the semidirect product $O(d,1) \ltimes \mathbb{R}^{d+1}$, where $O(d,1)$ is the **Lorentz group**. The effect of the Poincaré transformation with matrix (2.9) on an event $(x,1)$ is the event $(Lx+v,1)$, which is the effect of a Lorentz transformation $x \mapsto Lx$ and a (spatiotemporal) translation $x \mapsto x+v$. The Lorentz group is the stabiliser subgroup of the event $(\mathbf{0},0,1)$ and, of course, lorentzian geometry is the study of $\mathscr{H}$-structures with $\mathscr{H}$ the Lorentz group.

The Lie algebra of the Poincaré group embeds in $\mathfrak{gl}(d+2,\mathbb{R})$ as those matrices of the form

$$\begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \tag{2.10}$$

where $A^T\eta + \eta A = 0$ and $v \in \mathbb{R}^{d+1}$. Introducing a basis $L_{mn} = -L_{nm}, P_m$, where now $m,n = 0,1,\dots,d$, by

$$\begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} = \tfrac{1}{2}A^{mn}L_{mn} + v^m P_m, \tag{2.11}$$

it is easy to calculate the nonzero Lie brackets:

$$\begin{aligned} [L_{mn}, L_{pq}] &= \eta_{np}L_{mq} - \eta_{mp}L_{nq} - \eta_{nq}L_{mp} + \eta_{mq}L_{np} \\ [L_{mn}, P_p] &= \eta_{np}P_m - \eta_{mp}P_n. \end{aligned} \tag{2.12}$$

To ease comparison with the Galilei algebra (2.6), let $P_m = (P_a, H = P_0)$ and $L_{mn} = (L_{ab}, B_a = L_{0a})$, relative to which the brackets become

$$\begin{aligned} [L_{ab}, L_{cd}] &= \delta_{bc}L_{ad} - \delta_{ac}L_{bd} - \delta_{bd}L_{ac} + \delta_{bd}L_{ac} \\ [L_{ab}, B_b] &= \delta_{bc}B_a - \delta_{ac}B_b \\ [L_{ab}, P_b] &= \delta_{bc}P_a - \delta_{ac}P_b \\ [B_a, B_b] &= c^2 L_{ab} \\ [B_a, P_b] &= \delta_{ab}H \\ [B_a, H] &= c^2 P_a. \end{aligned} \tag{2.13}$$

Again L_{ab} span an $\mathfrak{so}(d)$ subalgebra relative to which B_a, P_a transform according to the vector representation and H transforms according to the one-dimensional scalar representation. What distinguishes the Poincaré and Galilei algebras are the Lie brackets which do not involve the L_{ab}: the last bracket in equation (2.6) and the last three brackets in equation (2.13).

The Lie brackets in (2.13) depend explicitly on the parameter c, the speed of light, which may formally be set to any desired value. For any nonzero value, the resulting Lie algebras are isomorphic: simply rescale $B_a \mapsto c^{-1}B_a$ and $H \mapsto c^{-1}H$, which is an isomorphism for nonzero c, resulting in the brackets with $c = 1$. On the other hand, setting $c = 0$ results in a non-isomorphic Lie algebra, with brackets

$$\begin{aligned} [L_{ab}, L_{cd}] &= \delta_{bc}L_{ad} - \delta_{ac}L_{bd} - \delta_{bd}L_{ac} + \delta_{bd}L_{ac} \\ [L_{ab}, B_b] &= \delta_{bc}B_a - \delta_{ac}B_b \\ [L_{ab}, P_b] &= \delta_{bc}P_a - \delta_{ac}P_b \\ [B_a, P_b] &= \delta_{ab}H. \end{aligned} \tag{2.14}$$

This algebra was first studied by Lévy-Leblond [4], who named it the **Carroll algebra** in honour of Lewis Carroll, the pseudonym of Charles Dodgson, the author of *Alice's Adventures in Wonderland*.

Alternatively, one can obtain the Galilei algebra formally as the limit $c \to \infty$ of the Poincaré algebra; that is, as a Lie algebra contraction. Indeed, let us rescale $B_a \mapsto c^{-2}B_a$, relative to which the Lie brackets become

$$\begin{aligned} [L_{ab}, L_{cd}] &= \delta_{bc}L_{ad} - \delta_{ac}L_{bd} - \delta_{bd}L_{ac} + \delta_{bd}L_{ac} \\ [L_{ab}, B_b] &= \delta_{bc}B_a - \delta_{ac}B_b \\ [L_{ab}, P_b] &= \delta_{bc}P_a - \delta_{ac}P_b \\ [B_a, B_b] &= c^{-2} L_{ab} \\ [B_a, P_b] &= c^{-2}\delta_{ab}H \\ [B_a, H] &= P_a. \end{aligned} \tag{2.15}$$

The rescaling is an isomorphism for any non-zero value of c^{-2} and hence results in an isomorphic Lie algebra. In the limit $c^{-2} \to 0$, the rescaling is singular, but the Lie brackets do have a limit and it is evident that the resulting Lie brackets in this limit agree with those of the Galilei algebra in equation (2.6), showing that the Galilei algebra is a contraction of the Poincaré algebra.

These two limits of the Poincaré algebra: $c \to 0$ and $c \to \infty$ can be understood geometrically according to what they do to the lightcones. In the limit $c \to \infty$, the lightcone $\mathbb{L}_a$ opens up to become the affine hyperplane of events simultaneous to a, where now $\mathbb{A}^{d+1}$ is to be interpreted as Galilei spacetime. In the limit $c \to 0$, the lightcone $\mathbb{L}_a$ closes up to become the affine temporal line based at a, as depicted in Figure 3. Taking the limit $c \to \infty$ says that any characteristic speed in the physics is much smaller than the speed of light and hence this limit is typically known as the non-relativistic limit. In contrast, in the limit $c \to 0$, since no material body can travel faster than the speed light, motion is impossible. This is called the ultra-relativistic (or ultra-local) limit.

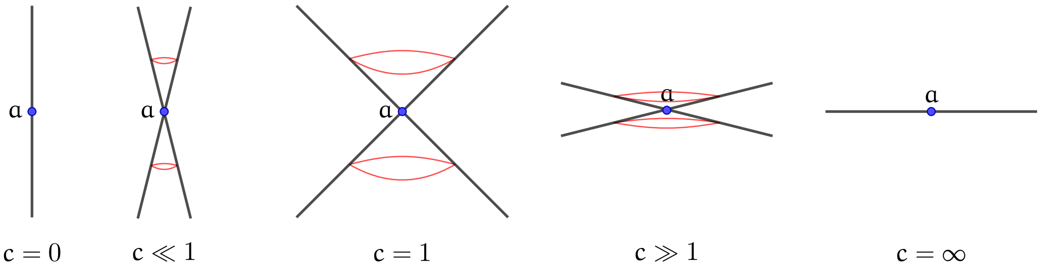

FIGURE 3. Cartoon of the effect of varying c on the lightcone $\mathbb{L}_a$

3. Kinematical Lie algebras

In Section 2 I discussed two models of the universe: the Galilei and Minkowski spacetimes. Both are affine spaces homogeneous under the action of a "kinematical" Lie group: the Galilei and Poincaré groups, respectively. In this section I will define this notion and discuss the classification of kinematical Lie algebras.

More than half a century ago, Bacry and Lévy-Leblond [5] asked themselves the question of which were the possible kinematics, rephrasing the question mathematically as the classification of kinematical Lie algebras. A careful comparison of the Poincaré, Galilei and Carroll algebras encountered in Section 2 suggests the following definition, now for $(d+1)$-dimensional spacetimes.[1]

Definition 1. Let V be a d-dimensional euclidean vector space and $\mathfrak{so}(V)$ the corresponding orthogonal Lie algebra. A **kinematical Lie algebra** (for spatially isotropic $(d+1)$-dimensional spacetimes) is a real Lie algebra $\mathfrak{k}$ with a subalgebra $\mathfrak{r} \cong \mathfrak{so}(V)$ and such that under the restriction to $\mathfrak{r}$ of the adjoint representation of $\mathfrak{k}$,

$$\mathfrak{k} \cong \mathfrak{so}(V) \oplus 2V \oplus \mathbb{R}, \tag{3.1}$$

where $\mathfrak{so}(V)$, V and $\mathbb{R}$ are the adjoint, defining and trivial one-dimensional representations of $\mathfrak{so}(V)$, respectively.

In addition, Bacry and Lévy-Leblond also imposed that $\mathfrak{k}$ should admit two automorphisms: *parity* $P_a \mapsto -P_a$ and *time-reversal* $H \mapsto -H$; although they did point out that those restrictions were "by no means compelling" and indeed twenty years later, Bacry and Nuyts [9] relaxed them to arrive at the classification of four-dimensional kinematical Lie algebras as we understand them today. This classification was recovered using deformation theory in [10] and extended to arbitrary dimension in [11, 12]. The case of $d = 1$ is a recontextualisation of the Bianchi classification of three-dimensional real Lie algebras [13, 14], here re-interpreted as kinematical Lie algebras for two-dimensional spacetimes. The cases of $d = 2$ and $d = 3$ are complicated by the existence of additional $\mathfrak{r}$-invariant tensors in $\wedge^2 V$ and $\wedge^3 V$, respectively, which can contribute to the brackets. And, indeed, there are kinematical Lie algebras in dimension $2+1$ and $3+1$ which have no higher-dimensional analogues. I will refer the interested reader to the papers cited above and will concentrate here on those kinematical Lie algebras which exist in generic dimensions.

[1] Strictly speaking, the definition is for spatially isotropic spacetimes. There are generalisations where the rotational subalgebra $\mathfrak{r}$ in the definition is replaced by a Lorentz subalgebra $\mathfrak{so}(d-1,1)$ or more generally a pseudo-orthogonal subalgebra $\mathfrak{so}(p, d-p)$. Such homogeneous spaces do occur in nature. Indeed, as shown in [6], the blow-up [7] of spatial infinity of Minkowski spacetime is a homogeneous space of the Poincaré group with lorentzian isotropy. There are other homogeneous spaces of the Poincaré group occurring at the asymptotic infinities of Minkowski spacetime, as discussed in [8], which have carrollian or Carroll-like structures.

It is often convenient to choose an orthonormal basis for V and corresponding generators $L_{ab} = -L_{ba}, B_a, P_a, H$, with $a, b = 1, 2, \ldots, d$, for $\mathfrak{k}$ in terms of which, the defining properties of a kinematical Lie algebra are contained in the following brackets:

$$
\begin{aligned}
[L_{ab}, L_{cd}] &= \delta_{bc} L_{ad} - \delta_{ac} L_{bd} - \delta_{bd} L_{ac} + \delta_{bd} L_{ac} \\
[L_{ab}, B_b] &= \delta_{bc} B_a - \delta_{ac} B_b \\
[L_{ab}, P_b] &= \delta_{bc} P_a - \delta_{ac} P_b \\
[L_{ab}, H] &= 0.
\end{aligned}
\tag{3.2}
$$

A naive (but effective) way to approach the classification of kinematical Lie algebras is simply to write down the most general $\mathfrak{r}$-invariant Lie brackets for the generators B_a, P_a, H and impose the Jacobi identity. The Jacobi identity cuts out an algebraic variety $\mathscr{J}$ in $\operatorname{Hom}_{\mathfrak{r}}(\wedge^2 W, \mathfrak{k})$, for $W = 2V \oplus \mathbb{R}$. Two points in $\mathscr{J}$ define isomorphic kinematical Lie algebras if and only if they are in the same orbit of $\mathrm{GL}_{\mathfrak{r}}(W)$, the group of $\mathfrak{r}$-invariant general linear transformation of W. One studies the orbit decomposition of $\mathscr{J}$ and selects a unique representative for each orbit.

This procedure results in Table 1, which lists the kinematical Lie algebras in generic dimension $d+1$. For $d \leqslant 2$, there are some degeneracies (e.g., if $d = 2$, the Galilei algebra $\mathfrak{g}$ is isomorphic to the Carroll algebra $\mathfrak{c}$), but for general d the table below lists non-isomorphic kinematical Lie algebras and for $d > 3$ the table is complete. The table lists the nonzero Lie brackets in addition to the defining ones in equation (3.2). It also uses a shorthand notation omitting indices. The only $\mathfrak{r}$-invariant tensor which can appear is δ_{ab} and hence there is an unambiguous way to add indices. There is no standard notation for all the kinematical Lie algebras, so I have made some choices.

Table 1. Kinematical Lie algebras in generic dimension

Name	Nonzero Lie brackets in addition to those in (3.2)					Comments
$\mathfrak{s}$						
$\mathfrak{g}$	$[H, \mathbf{B}] = -\mathbf{P}$					
$\mathfrak{n}^0$	$[H, \mathbf{B}] = \mathbf{B} + \mathbf{P}$	$[H, \mathbf{P}] = \mathbf{P}$				
$\mathfrak{n}^+_\gamma$	$[H, \mathbf{B}] = \gamma \mathbf{B}$	$[H, \mathbf{P}] = \mathbf{P}$				$\gamma \in [-1, 1]$
$\mathfrak{n}^-_\chi$	$[H, \mathbf{B}] = \chi \mathbf{B} + \mathbf{P}$	$[H, \mathbf{P}] = \chi \mathbf{P} - \mathbf{B}$				$\chi \geqslant 0$
$\mathfrak{c}$				$[\mathbf{B}, \mathbf{P}] = H$		
$\mathfrak{iso}(d,1)$ $\mathfrak{iso}(d+1)$	$[H, \mathbf{B}] = -\varepsilon \mathbf{P}$		$[\mathbf{B}, \mathbf{B}] = \varepsilon \mathbf{L}$	$[\mathbf{B}, \mathbf{P}] = H$		$\varepsilon = \pm 1$
$\mathfrak{so}(d+1, 1)$	$[H, \mathbf{B}] = \mathbf{B}$	$[H, \mathbf{P}] = -\mathbf{P}$		$[\mathbf{B}, \mathbf{P}] = H + \mathbf{L}$		
$\mathfrak{so}(d,2)$ $\mathfrak{so}(d+2)$	$[H, \mathbf{B}] = -\varepsilon \mathbf{P}$	$[H, \mathbf{P}] = \varepsilon \mathbf{B}$	$[\mathbf{B}, \mathbf{B}] = \varepsilon \mathbf{L}$	$[\mathbf{B}, \mathbf{P}] = H$	$[\mathbf{P}, \mathbf{P}] = \varepsilon \mathbf{L}$	$\varepsilon = \pm 1$

Let us now describe each of the algebras in turn:

- The Lie algebra $\mathfrak{s}$ is the **static** kinematical Lie algebra: all additional brackets are zero. Therefore every kinematical Lie algebras is a deformation of $\mathfrak{s}$.
- The Galilei algebra is denoted by $\mathfrak{g}$ and a closely related Lie algebra has been denoted by $\mathfrak{n}^0$. In $\mathfrak{g}$ and $\mathfrak{n}^0$, ad_H is not semisimple, but has a nontrivial Jordan block:

$$
\mathrm{ad}_H^{\mathfrak{g}} \begin{pmatrix} \mathbf{B} \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{B} \\ \mathbf{P} \end{pmatrix} \qquad \text{and} \qquad \mathrm{ad}_H^{\mathfrak{n}^0} \begin{pmatrix} \mathbf{B} \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{B} \\ \mathbf{P} \end{pmatrix}. \tag{3.3}
$$

- There are two one-parameter families of Lie algebras, deforming the **Newton–Hooke** algebras [5, 15]:
  - $\mathfrak{n}^+_\gamma$, with $\gamma \in [-1, 1]$, which for $\gamma = -1$ is the Newton–Hooke algebra $\mathfrak{n}^+$ in the notation of [15]; and
  - $\mathfrak{n}^-_\chi$, with $\chi \geqslant 0$, which for $\chi = 0$ is the other Newton–Hooke algebra $\mathfrak{n}^-$, in the notation of [15].

  These two families correspond to the cases where ad_H is semisimple, with real eigenvalues in $\mathfrak{n}^+_\gamma$ and complex eigenvalues in $\mathfrak{n}^-_\chi$:

$$
\mathrm{ad}_H^{\mathfrak{n}^+} \begin{pmatrix} \mathbf{B} \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{B} \\ \mathbf{P} \end{pmatrix} \qquad \text{and} \qquad \mathrm{ad}_H^{\mathfrak{n}^-} \begin{pmatrix} \mathbf{B} \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} \chi & 1 \\ -1 & \chi \end{pmatrix} \begin{pmatrix} \mathbf{B} \\ \mathbf{P} \end{pmatrix}. \tag{3.4}
$$

- The Carroll algebra [4] is denoted $\mathfrak{c}$.
- The Poincaré algebra is $\mathfrak{iso}(d, 1)$ and the euclidean algebra is $\mathfrak{iso}(d+1)$.

- The remaining Lie algebras are $\mathfrak{so}(d+2)$, $\mathfrak{so}(d+1,1)$ and $\mathfrak{so}(d,2)$, which for $d \geqslant 2$ are semisimple. Finite-dimensional semisimple Lie algebras are rigid, but they do admit contractions. Of the Lie algebras in the Table, those which can be obtained as contractions from these are $\mathfrak{s}$, $\mathfrak{c}$, $\mathfrak{g}$, $\mathfrak{n}^-_{\chi=0}$, $\mathfrak{n}^+_{\gamma=-1}$, $\mathfrak{iso}(d+1)$ and $\mathfrak{iso}(d,1)$, which are precisely the kinematical Lie algebras admitting parity and time-reversal automorphisms and whose four-dimensional avatars were the objects in the first classification of Bacry and Lévy-Leblond [5].

4. Kinematical Klein geometries

In this section I review the classification of kinematical Klein geometries arrived at in [16] and further studied in [17].

Definition 2. A **kinematical Lie group** is a real Lie group whose Lie algebra is kinematical (see Definition 1). A **kinematical Klein geometry** is a $(d+1)$-dimensional homogeneous space of such a kinematical Lie group, whose Klein pair $(\mathfrak{k}, \mathfrak{h})$ is such that $\mathfrak{k}$ is a kinematical Lie algebra and $\mathfrak{h}$ is a subalgebra containing the subalgebra $\mathfrak{r} \cong \mathfrak{so}(V)$ and such that under the restriction to $\mathfrak{r}$ of the adjoint representation, $\mathfrak{h} \cong \mathfrak{r} \oplus V$.

Already in the pioneering work of Bacry and Lévy-Leblond [5], for $d = 3$ and the restricted list of kinematical Lie algebras admitting parity and time-reversal automorphisms, a list of eleven possible kinematical Klein pairs are discussed. Upon closer analysis one of their Klein pairs is not effective and describes a static aristotelian spacetime, resulting in ten kinematical Klein geometries: all of which are reductive and symmetric. The classification of (simply-connected) kinematical Klein geometries was arrived at in [16], to where I refer the interested reader for details. The classification for $d \geqslant 3$ is summarised in Table 2, where the Lie brackets of the kinematical Lie algebra $\mathfrak{k}$ are listed in a basis where the subalgebra $\mathfrak{h}$ is spanned by L_{ab}, B_a.

Table 2. $(d+1)$-dimensional kinematical Klein geometries $(d \geqslant 3)$

Name	Klein pair	Nonzero Lie brackets in addition to (3.2)				
Minkowski	$(\mathfrak{iso}(d,1), \mathfrak{so}(d,1))$	$[H, \mathbf{B}] = -\mathbf{P}$		$[\mathbf{B}, \mathbf{B}] = \mathbf{L}$	$[\mathbf{B}, \mathbf{P}] = H$	
de Sitter	$(\mathfrak{so}(d+1,1), \mathfrak{so}(d,1))$	$[H, \mathbf{B}] = -\mathbf{P}$	$[H, \mathbf{P}] = -\mathbf{B}$	$[\mathbf{B}, \mathbf{B}] = \mathbf{L}$	$[\mathbf{B}, \mathbf{P}] = H$	$[\mathbf{P}, \mathbf{P}] = -\mathbf{L}$
anti de Sitter	$(\mathfrak{so}(d,2), \mathfrak{so}(d,1))$	$[H, \mathbf{B}] = -\mathbf{P}$	$[H, \mathbf{P}] = \mathbf{B}$	$[\mathbf{B}, \mathbf{B}] = \mathbf{L}$	$[\mathbf{B}, \mathbf{P}] = H$	$[\mathbf{P}, \mathbf{P}] = \mathbf{L}$
euclidean	$(\mathfrak{iso}(d+1), \mathfrak{so}(d+1))$	$[H, \mathbf{B}] = \mathbf{P}$		$[\mathbf{B}, \mathbf{B}] = -\mathbf{L}$	$[\mathbf{B}, \mathbf{P}] = H$	
sphere	$(\mathfrak{so}(d+2), \mathfrak{so}(d+1))$	$[H, \mathbf{B}] = \mathbf{P}$	$[H, \mathbf{P}] = -\mathbf{B}$	$[\mathbf{B}, \mathbf{B}] = -\mathbf{L}$	$[\mathbf{B}, \mathbf{P}] = H$	$[\mathbf{P}, \mathbf{P}] = -\mathbf{L}$
hyperbolic	$(\mathfrak{so}(d+1,1), \mathfrak{so}(d+1))$	$[H, \mathbf{B}] = \mathbf{P}$	$[H, \mathbf{P}] = \mathbf{B}$	$[\mathbf{B}, \mathbf{B}] = -\mathbf{L}$	$[\mathbf{B}, \mathbf{P}] = H$	$[\mathbf{P}, \mathbf{P}] = \mathbf{L}$
Galilei	$(\mathfrak{g}, \mathfrak{iso}(d))$	$[H, \mathbf{B}] = -\mathbf{P}$				
de Sitter–Galilei	$(\mathfrak{n}^+_{\gamma=-1}, \mathfrak{iso}(d))$	$[H, \mathbf{B}] = -\mathbf{P}$	$[H, \mathbf{P}] = -\mathbf{B}$			
torsional de Sitter–Galilei	$(\mathfrak{n}^+_{\gamma\in(-1,1)}, \mathfrak{iso}(d))$	$[H, \mathbf{B}] = -\mathbf{P}$	$[H, \mathbf{P}] = \gamma\mathbf{B} + (1+\gamma)\mathbf{P}$			
torsional de Sitter–Galilei	$(\mathfrak{n}^0, \mathfrak{iso}(d))$	$[H, \mathbf{B}] = -\mathbf{P}$	$[H, \mathbf{P}] = \mathbf{B} + 2\mathbf{P}$			
anti de Sitter–Galilei	$(\mathfrak{n}^-_{\chi=0}, \mathfrak{iso}(d))$	$[H, \mathbf{B}] = -\mathbf{P}$	$[H, \mathbf{P}] = \mathbf{B}$			
torsional anti de Sitter–Galilei	$(\mathfrak{n}^-_{\chi>0}, \mathfrak{iso}(d))$	$[H, \mathbf{B}] = -\mathbf{P}$	$[H, \mathbf{P}] = (1+\chi^2)\mathbf{B} + 2\chi\mathbf{P}$			
Carroll	$(\mathfrak{c}, \mathfrak{iso}(d))$				$[\mathbf{B}, \mathbf{P}] = H$	
de Sitter–Carroll	$(\mathfrak{iso}(d+1), \mathfrak{iso}(d))$		$[H, \mathbf{P}] = -\mathbf{B}$		$[\mathbf{B}, \mathbf{P}] = H$	$[\mathbf{P}, \mathbf{P}] = -\mathbf{L}$
anti de Sitter–Carroll	$(\mathfrak{iso}(d,1), \mathfrak{iso}(d))$		$[H, \mathbf{P}] = \mathbf{B}$		$[\mathbf{B}, \mathbf{P}] = H$	$[\mathbf{P}, \mathbf{P}] = \mathbf{L}$
lightcone	$(\mathfrak{so}(d+1,1), \mathfrak{iso}(d))$	$[H, \mathbf{B}] = \mathbf{B}$	$[H, \mathbf{P}] = -\mathbf{P}$		$[\mathbf{B}, \mathbf{P}] = H + \mathbf{L}$	

The table is divided into four sections, from top to bottom: lorentzian, riemannian, galilean and carrollian Klein geometries. In order to explain this coarser classification I need to explain these terms, but before doing so, I will review briefly some basic notions of Klein geometry.

4.1. Basic notions of Klein geometry.

Let $\mathfrak{k}$ be a Lie algebra and $\mathfrak{h}$ a Lie subalgebra. The Klein pair $(\mathfrak{k}, \mathfrak{h})$ is said to be

- **effective** if $\mathfrak{h}$ does not contain a nonzero ideal of $\mathfrak{k}$, and
- **geometrically realisable** if there exists a kinematical Lie group $\mathscr{K}$ with Lie algebra $\mathfrak{k}$ such that the connected subgroup $\mathscr{H}$ generated by $\mathfrak{h}$ is closed. The homogeneous space $\mathscr{K}/\mathscr{H}$ is a **geometric realisation** of $(\mathfrak{k}, \mathfrak{h})$.

Two Klein pairs $(\mathfrak{k}_1, \mathfrak{h}_1)$ and $(\mathfrak{k}_2, \mathfrak{h}_2)$ are **isomorphic** if there is a Lie algebra isomorphism $\varphi : \mathfrak{k}_1 \to \mathfrak{k}_2$ with $\varphi(\mathfrak{h}_1) = \mathfrak{h}_2$. The fundamental theorem of Klein geometry states that isomorphism classes of geometrically realisable, effective Klein pairs are in bijective correspondence with isomorphism classes of simply-connected homogeneous spaces. Paraphrasing, effective and geometrically realisable Klein pairs classify homogeneous spaces up to coverings.

Given an effective, geometrically realisable Klein pair $(\mathfrak{k},\mathfrak{h})$, the **linear isotropy representation** $\lambda : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{k}/\mathfrak{h})$ is defined as the representation induced by the restriction to $\mathfrak{h}$ of the adjoint representation of $\mathfrak{k}$. Explicitly, letting $X \in \mathfrak{h}$ and $\overline{Y} \in \mathfrak{k}/\mathfrak{h}$ denote the residue class modulo $\mathfrak{h}$ of $Y \in \mathfrak{k}$,

$$\lambda_X \overline{Y} := \overline{\operatorname{ad}_X Y}. \tag{4.1}$$

The holonomy principle establishes a bijective correspondence between invariant tensors of the linear isotropy representation and tensor fields on the homogeneous space which are invariant under the kinematical Lie group $\mathscr{K}$. It is these invariant tensor fields which determine the type of the Klein geometry.

A Klein pair $(\mathfrak{k},\mathfrak{h})$ is said to be **reductive** if the short exact sequence

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{k}/\mathfrak{h} \longrightarrow 0 \tag{4.2}$$

splits in the category of $\mathfrak{h}$-modules. This is equivalently to the existence of a complement $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{k}$ which is stable under the restriction to $\mathfrak{h}$ of the adjoint action of $\mathfrak{k}$; that is, $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ with $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}$ in the obvious notation. A reductive Klein pair $(\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}, \mathfrak{h})$ is said to be **symmetric** if $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$. All Klein geometries in Table 2 are reductive with the exception of the lightcone and of the reductive ones, all but the "torsional" ones are symmetric.

The "torsional" adjective refers to the torsion of the canonical invariant connection on a reductive Klein geometry. An **invariant connection** on a Klein geometry with Klein pair $(\mathfrak{k},\mathfrak{h})$ is an $\mathfrak{h}$-equivariant linear map $\Lambda : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{k}/\mathfrak{h})$, written $X \mapsto \Lambda_X$, whose restriction to $\mathfrak{h}$ is the linear isotropy representation; that is, for all $X \in \mathfrak{h}$, $\Lambda_X = \lambda_X$.

All Klein geometries in Table 2 admit invariant connections, with the exception of the lightcone (for $d \geqslant 2$). These have been tabulated in [17].

If $(\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}, \mathfrak{h})$ is reductive, then invariant connections are determined by their **Nomizu map** [18], the $\mathfrak{h}$-equivariant bilinear map $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ defined by $\alpha(X,Y) := \Lambda_X Y$ for all $X, Y \in \mathfrak{m}$. The torsion $\Theta : \wedge^2\mathfrak{m} \to \mathfrak{m}$ and curvature $\Omega : \wedge^2\mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ of an invariant connection are given in terms of the Nomizu map by

$$\begin{aligned} \Theta(X,Y) &= \alpha(X,Y) - \alpha(Y,X) - [X,Y]_{\mathfrak{m}} \\ \Omega(X,Y)Z &= \alpha(X,\alpha(Y,Z)) - \alpha(Y,\alpha(X,Z)) - \alpha([X,Y]_{\mathfrak{m}},Z) - [[X,Y]_{\mathfrak{h}},Z], \end{aligned} \tag{4.3}$$

where $X = X_{\mathfrak{h}} + X_{\mathfrak{m}}$ is the decomposition of $X \in \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$.

The **canonical invariant connection** of a reductive Klein geometry is the unique invariant connection with zero Nomizu map. Its torsion and curvature take particularly simple forms:

$$\Theta(X,Y) = -[X,Y]_{\mathfrak{m}} \qquad \text{and} \qquad \Omega(X,Y)Z = -[[X,Y]_{\mathfrak{h}},Z]. \tag{4.4}$$

It follows that a reductive Klein geometry is torsion-free if and only if it is symmetric.

The holonomy representation of the canonical invariant connection is isomorphic to the ideal $[\mathfrak{m},\mathfrak{m}]_{\mathfrak{h}} \subset \mathfrak{h}$ acting on $\mathfrak{m}$ via the restriction of the linear isotropy representation [18, §12]. It follows that any $\mathscr{K}$-invariant tensor field on the homogeneous space is parallel with respect to to the canonical invariant connection.

4.2. **Lorentzian Klein geometries.** The lorentzian Klein geometries all have Klein pairs $(\mathfrak{k},\mathfrak{h})$ with $\mathfrak{h} \cong \mathfrak{so}(d,1)$. They can be characterised by the existence of an $\mathfrak{h}$-invariant lorentzian inner product on $\mathfrak{k}/\mathfrak{h}$; that is, a nondegenerate bilinear form in $\left(\odot^2(\mathfrak{k}/\mathfrak{h})^*\right)^{\mathfrak{h}}$ with lorentzian signature. The holonomy principle gives rise to a $\mathscr{K}$-invariant lorentzian metric on the homogeneous space. Since $\dim \mathscr{K} = \frac{1}{2}(d+1)(d+2)$, the homogeneous space has constant sectional curvature, so one of Minkowski spacetime (flat), de Sitter spacetime (positive curvature) and anti de Sitter spacetime (negative curvature). Geometrically, there is a one-parameter family of (anti) de Sitter spacetimes, corresponding to the value of the scalar curvature, but all de Sitter spacetimes are isomorphic as homogeneous spaces and similarly for all anti de Sitter spacetimes.

Let us give a geometric realisation for each one. Minkowski spacetime, as described in Section 2.2, is a geometric realisation of the Klein pair $(\mathfrak{iso}(d,1), \mathfrak{so}(d,1))$.

A geometric realisation of $(d+1)$-dimensional de Sitter spacetime is provided by a quadric hypersurface in a lorentzian vector space of dimension $d+2$. Let $x^0, x^1, \dots, x^{d+1}$ be Cartesian coordinates for $\mathbb{R}^{d+2}$ with the flat lorentzian metric

$$-(dx^0)^2 + \sum_{i=1}^{d+1} (dx^i)^2. \tag{4.5}$$

Then for any $\ell \neq 0$, the hypersurface cut out by the quadric

$$- (x^0)^2 + \sum_{i=1}^{d+1} (dx^i)^2 = \ell^2 \tag{4.6}$$

is a geometric realisation of the Klein pair $(\mathfrak{so}(d+1,1), \mathfrak{so}(d,1))$.

A geometric realisation of $(d+1)$-dimensional anti de Sitter spacetime is also provided by a quadric hypersurface, but this time in a pseudo-euclidean vector space with an inner product of signature $(d,2)$. Let $x^0, x^1, \dots, x^{d+1}$ be Cartesian coordinates for $\mathbb{R}^{d+2}$ with the flat metric

$$- (dx^0)^2 + \sum_{i=1}^{d} (dx^i)^2 - (dx^{d+1})^2. \tag{4.7}$$

Then for all $\ell \neq 0$, the hypersurface cut out by the quadric

$$- (x^0)^2 + \sum_{i=1}^{d} (x^i)^2 - (x^{d+1})^2 = -\ell^2 \tag{4.8}$$

is a geometric realisation of the Klein pair $(\mathfrak{so}(d,2), \mathfrak{so}(d,1))$.

4.3. **Riemannian Klein geometries.** The riemannian Klein geometries all have Klein pairs $(\mathfrak{k}, \mathfrak{h})$ with $\mathfrak{h} \cong \mathfrak{so}(d+1)$. They can be characterised by the existence of an invariant euclidean inner product on $\mathfrak{k}/\mathfrak{h}$. The story is very similar to the lorentzian case above. The resulting homogeneous spaces are riemannian symmetric spaces with constant sectional curvature: euclidean space (flat), the round sphere (positive curvature) and hyperbolic space (negative curvature). Although strictly speaking they are Klein geometries of kinematical Lie groups, they play no rôle as spacetimes. One could eliminate them from the discussion by imposing a further condition on a kinematical Klein geometry; namely, that the generic orbits of the one-parameter subgroup corresponding to any $v^a B_a \in \mathfrak{h}$ on the homogeneous space should be non-compact. In the riemannian Klein geometries, these one-parameter subgroups act as rotations and therefore have compact orbits.

4.4. **Galilean Klein geometries.** The galilean Klein geometries all have Klein pairs $(\mathfrak{k}, \mathfrak{h})$ with $\mathfrak{h} \cong \mathfrak{iso}(d)$. They can be characterised by the existence of an invariant covector in the dual $(\mathfrak{k}/\mathfrak{h})^*$ of the linear isotropy representation and an invariant symmetric bivector in $\odot^2(\mathfrak{k}/\mathfrak{h})$. The corresponding invariant tensor fields on the homogeneous space can be interpreted as a clock and ruler, just as in Galilei spacetime in Section 2.1. All galilean Klein geometries are reductive and therefore admit a canonical invariant connection. This connection is torsion-free for the Galilei and (anti) de Sitter–Galilei spacetimes, which explains the adjective "torsional" in the other two classes.

Galilei spacetime, as described in Section 2.1, is a geometric realisation of the Klein pair $(\mathfrak{g}, \mathfrak{iso}(d)$. I will postpone discussion of the geometric realisations of the other galilean Klein geometries until Section 5.3.1 after I discuss how to to obtain Newton–Cartan structures by null reductions of lorentzian manifolds.

4.5. **Carrollian Klein geometries.** The carrollian Klein geometries are in a certain sense dual to the galilean Klein geometries. They all have Klein pairs $(\mathfrak{k}, \mathfrak{h})$ where $\mathfrak{h} \cong \mathfrak{iso}(d)$, but the $\mathfrak{iso}(d)$ subalgebra of $\mathfrak{gl}(\mathfrak{k}/\mathfrak{h})$ in the carrollian and galilean cases are not conjugate under $\mathrm{GL}(\mathfrak{k}/\mathfrak{h})$. Indeed, they have different invariant tensors of the linear isotropy representation. Carrollian Klein geometries are characterised by the existence of an invariant vector in $\mathfrak{k}/h$ and an invariant symmetric bilinear form in $\odot^2(\mathfrak{k}/\mathfrak{h})^*$. Except for the lightcone, which is not reductive, all other carrollian Klein geometries are reductive and symmetric. Anti de Sitter–Carroll is a Klein geometry associated to the Poincaré algebra and is the geometry underlying the blow-up of past and future timelike infinities in Minkowski spacetime [8].

All of the carrollian Klein geometries may be realised geometrically as null hypersurfaces in lorentzian manifolds. As the name suggests, the $(d+1)$-dimensional lightcone is realised geometrically as the future (or past) deleted lightcone in a $(d+2)$-dimensional lorentzian vector space. Let $x^0, x^1, \dots, x^{d+1}$ be cartesian coordinates for $\mathbb{R}^{d+2}$ and consider the flat lorentzian metric given in equation (4.5). Then the null hypersurface

$$- (x^0)^2 + \sum_{i=1}^{d+1} (dx^i)^2 = 0 \tag{4.9}$$

with $x^0 > 0$, say, is a geometric realisation of the Klein pair $(\mathfrak{so}(d+1,1), \mathfrak{iso}(d))$.

The other three carrollian Klein geometries may be realised as null hypersurfaces in Minkowski spacetime (for the Carroll spacetime), de Sitter spacetime (for the de Sitter–Carroll spacetime) and anti de Sitter spacetime (for the anti de Sitter–Carroll spacetime).

Consider $(d+2)$-dimensional Minkowski spacetime with coordinates $x^0, x^1, \dots, x^{d+1}$ and metric

$$-(dx^0)^2 + \sum_{i=1}^{d+1} (dx^i)^2. \tag{4.10}$$

As shown originally in [19], Carroll spacetime embeds here as the null hyperplane $x^0 = x^{d+1}$. Indeed, it is not hard to show (see, e.g., [16, §4.2.5]) that this null hyperplane is a geometric realisation of the Klein pair $(\mathfrak{c}, \mathfrak{iso}(d))$.

Consider now $(d+3)$-dimensional Minkowski spacetime with coordinates $x^0, x^1, \dots, x^{d+2}$ and metric

$$-(dx^0)^2 + \sum_{i=1}^{d+2} (dx^i)^2. \tag{4.11}$$

Let $\mathscr{Q}$ denote the quadric hypersurface defined by

$$-(x^0)^2 + \sum_{i=1}^{d+2} (x^i)^2 = \ell^2, \tag{4.12}$$

which, as seen in Section 4.2 and for every $\ell \neq 0$, is covered by de Sitter spacetime. Let $\mathscr{N}$ denote the hyperplane cut out by $x^0 = x^{d+2}$. Then as shown in [16, §4.2.5], the intersection $\mathscr{Q} \cap \mathscr{N}$ is a geometric realisation of the Klein pair $(\mathfrak{iso}(d+1), \mathfrak{iso}(d))$.

Consider finally a $(d+3)$-dimensional pseudo-euclidean space with coordinates $x^0, x^1, \dots, x^{d+2}$ and metric

$$-(dx^0)^2 + \sum_{i=1}^{d+1} (dx^i)^2 - (dx^{d+2})^2. \tag{4.13}$$

Let $\mathscr{Q}'$ denote the quadric hypersurface defined by

$$-(x^0)^2 + \sum_{i=1}^{d+1} (x^i)^2 - (x^{d+2})^2 = -\ell^2, \tag{4.14}$$

which, as seen in Section 4.2 and for any $\ell \neq 0$, is covered by anti de Sitter spacetime. Let $\mathscr{N}'$ denote the hyperplane cut out by $x^{d+1} = x^{d+2}$. Then as shown in [16, §4.2.5] and [20], the intersection $\mathscr{Q}' \cap \mathscr{N}'$ is a geometric realisation of the Klein pair $(\mathfrak{iso}(d,1), \mathfrak{iso}(d))$. This geometric realisation has been used recently in order to describe the asymptotic geometry of Minkowski spacetime in terms of homogeneous spaces of the Poincaré group [8].

5. Kinematical Cartan geometries

In this section I discuss the Cartan geometries modelled on the kinematical Klein geometries discussed in Section 4, but before doing so, I review the basic notions of Cartan geometry relevant to our discussion. A good treatment of Cartan geometry is given in [21] and there is a growing list of explicit applications of Cartan geometry to gravitation [22, 23, 24, 25].

5.1. **Basic notions of Cartan geometry.** To specify a Cartan geometry, one needs, in addition to an effective Klein pair $(\mathfrak{k}, \mathfrak{h})$, a Lie group $\mathscr{H}$ with Lie algebra $\mathfrak{h}$ and an action of $\mathscr{H}$ on $\mathfrak{k}$ extending the adjoint action on $\mathfrak{h}$ and denoted Ad. More precisely, a **Cartan geometry** modelled on $(\mathfrak{k}, \mathfrak{h})$ with group $\mathscr{H}$ is a right $\mathscr{H}$-principal bundle $P \to M$ together with a **Cartan connection**: a $\mathfrak{k}$-valued one-form ω on P satisfying the following conditions:

(1) (*non-degeneracy*) for each $p \in P$, $\omega_p : T_pP \to \mathfrak{g}$ is an isomorphism;
(2) (*equivariance*) for every $h \in \mathscr{H}$, $R_h^*\omega = \mathrm{Ad}(h^{-1}) \circ \omega$, where R_h is the diffeomorphism of P induced by the right action of $h \in \mathscr{H}$; and
(3) (*normalisation*) for all $X \in \mathfrak{h}$, $\omega(\xi_X) = X$, where $\xi_X \in \mathscr{X}(P)$ is the corresponding fundamental vector field.

The induced action of $\mathscr{H}$ on $\mathfrak{k}/\mathfrak{h}$ is the linear isotropy representation of $\mathscr{H}$. If faithful, the Cartan geometry is said to be of the **first order** and it then follows that the principal bundle $P \to M$ is a reduction of the frame bundle of M; that is, an G-structure with $G = \mathscr{H}$. This will always be the case for the kinematical Cartan geometries under discussion.

G-structures come with additional structure: a soldering form. In the present context it is given by $\theta \in \Omega^1(P; \mathfrak{k}/\mathfrak{h})$, the projection to $\mathfrak{k}/\mathfrak{h}$ of the Cartan connection. The soldering form is both horizontal and equivariant and hence it defines a section of $\mathrm{Hom}(TM, P \times_{\mathscr{H}} \mathfrak{k}/\mathfrak{h})$. The non-degeneracy condition of the Cartan connection says that the soldering form defines a bundle isomorphism between the tangent bundle TM and the "fake tangent bundle" $P \times_{\mathscr{H}} \mathfrak{k}/\mathfrak{h}$. This allows to identify tensor bundles over M with the corresponding associated vector bundles to the principal bundle $P \to M$, a fact that shall be used tacitly.

Let $\omega \in \Omega^1(P, \mathfrak{h})$ be an Ehresmann connection on P. If the Cartan geometry is reductive in addition to being of the first order, then the $\mathfrak{h}$-component of the Cartan connection is such an Ehresmann connection, but Ehresmann connections exist even if the Cartan geometry is not reductive. An Ehresmann connection defines a Koszul connection on any associated vector bundle to $P \to M$ and, in particular, on the fake tangent bundle. The soldering form transports that Koszul connection to an affine connection on TM. Affine connections obtained in this way are said to be **adapted** to the G-structure.

The difference between two adapted connections is a one-form with values in the adjoint bundle $\operatorname{Ad} P = P \times_{\mathscr{H}} \mathfrak{h}$. The linear isotropy representation $\lambda : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{k}/\mathfrak{h})$ allows us to view $\operatorname{Ad} P$ as a sub-bundle of the endomorphisms of the fake tangent bundle and, via the soldering form, as endomorphisms of the tangent bundle. In summary, whereas the difference between any two affine connections on TM is a one-form with values in $\operatorname{End}(TM)$, if the affine connections are adapted, it is a one-form taking values in the sub-bundle $\operatorname{Ad} P$.

Let us introduce the notation $V := \mathfrak{k}/\mathfrak{h}$ and let us define the **Spencer differential**

$$\partial : \operatorname{Hom}(V, \mathfrak{h}) \to \operatorname{Hom}(\wedge^2 V, V) \qquad \text{by} \qquad (\partial\kappa)(v, w) = \kappa_v w - \kappa_w v \tag{5.1}$$

for all $v, w \in V$. As with every linear map, ∂ fits inside a four-term exact sequence

$$0 \longrightarrow \ker\partial \longrightarrow \mathfrak{h} \otimes V^* \xrightarrow{\partial} V \otimes \wedge^2 V^* \longrightarrow \operatorname{coker}\partial \longrightarrow 0 \tag{5.2}$$

and hence an exact sequence of the corresponding associated vector bundles on M. The sections of the associated vector bundles to these representations can be interpreted as follows:

- $V \otimes \wedge^2 V^*$: torsion of (adapted) affine connections;
- $\mathfrak{h} \otimes V^*$: difference between adapted affine connections;
- $\ker\partial$: differences which do not alter the torsion; and
- $\operatorname{coker}\partial$: **intrinsic torsions** of adapted connections, so called because this is part of the torsion which does not depend on the connection and hence it is intrinsic to the G-structure.

The intrinsic torsion is a coarse yet easy to determine invariant of the kinematical Cartan geometries. A more detailed treatment can be found in [26].

5.2. **Lorentzian geometry.** Let $V = \mathbb{R}^{d+1}$ thought of as the defining representation of $O(d,1)$, which is the subgroup of $GL(d+1,\mathbb{R})$ leaving a lorentzian inner product invariant. If M is a $(d+1)$-dimensional manifold, a $O(d,1)$-structure is a sub-bundle of the frame bundle consisting of the pseudo-orthonormal frames. Every such frame gives isomorphism $T_pM \to \mathbb{R}^{d+1}$ for each point p where it is defined and the lorentzian inner product on $\mathbb{R}^{d+1}$ pulls back to a lorentzian inner product on T_pM. Restricting to pseudo-orthonormal frames, these inner products define a smooth lorentzian metric g on M. The fundamental theorem of riemannian geometry, the existence of a unique torsion-free metric connection, may be re-interpreted in the language of G-structures as saying that any adapted (i.e., metric) connection on the orthonormal frame bundle can be modified to have zero torsion (since $\operatorname{coker}\partial = 0$) and, moreover, that there is a unique such modification (since $\ker\partial = 0$).

5.3. **Newton–Cartan geometry.** Galilean G-structures and their adapted connections were first discussed in [27, 28] and further studied in [29, 30, 31, 26].

Let $V = \mathbb{R}^{d+1}$. It is convenient to choose a suggestive notation for the elementary basis of $\mathbb{R}^{d+1}$: $\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_d$ and for the canonical dual basis $\alpha^0, \alpha^1, \dots, \alpha^d$. Let G denote the subgroup of $GL(V)$ which fixes $\alpha^0 \in V^*$ and $\sum_{a=1}^d \boldsymbol{e}_a \boldsymbol{e}_a \in \odot^2 V$. Explicitly,

$$G = \left\{ \begin{pmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{v} & A \end{pmatrix} \middle| \, \boldsymbol{v} \in \mathbb{R}^d,\ A \in O(d) \right\} < GL(d+1, \mathbb{R}), \tag{5.3}$$

with Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{v} & A \end{pmatrix} \middle| \, \boldsymbol{v} \in \mathbb{R}^d,\ A \in \mathfrak{so}(d) \right\} < \mathfrak{gl}(d+1, \mathbb{R}). \tag{5.4}$$

Let M be an $(d+1)$-dimensional manifold with a G-structure with G the group in equation (5.3). The characteristic tensor fields are now a nowhere-vanishing one-form $\tau \in \Omega^1(M)$ (the **clock one-form**) and a corank-one positive-semidefinite symmetric bivector $\lambda \in \Gamma(\odot^2 TM)$ (the **spatial cometric**) with $\lambda(\tau, -) = 0$. The triple (M, τ, λ) is a **(weak) Newton–Cartan geometry**, which can be promoted to a **Newton–Cartan geometry** by the addition of an adapted affine connection ∇.

Already in [28], the kernel and cokernel of the Spencer differential for a Newton–Cartan G-structure was determined to be isomorphic to $\wedge^2 V^*$ as a G-module and that under the isomorphism $\operatorname{coker}\partial \cong \wedge^2 V^*$, the intrinsic torsion of an adapted connection is sent to $d\tau \in \Omega^2(M)$. It was further shown in [26] that in generic d (here, $d \neq 1,4$) there are three G-submodules of coker ∂ and, correspondingly, three kinds of Newton–Cartan geometries, in the notation of [32, Table I], who first identified these classes:

(1) **torsionless Newton–Cartan geometry** (NC) if $d\tau = 0$;
(2) **twistless torsional Newton–Cartan geometry** (TTNC), if $d\tau \wedge \tau = 0$; and
(3) **torsional Newton–Cartan geometry** (TNC), if $d\tau \wedge \tau \neq 0$.

If $d = 1$, $d\tau \wedge \tau = 0$ on dimensional grounds, hence there are two intrinsic torsion classes of two-dimensional Newton–Cartan geometries: those where $d\tau = 0$ and those where $d\tau \neq 0$. Similarly, if $d = 4$, and if the G-structure reduces further to a G_0-structure, with $G_0 \subset G$ the connected component of the identity – for example, if M were simply-connected – then the subbundle $\ker\tau \subset TM$ is oriented and riemannian, so that one can distinguish between those five-dimensional torsional Newton–Cartan geometries where the restriction of $d\tau$ to $\ker\tau$ is self-dual, anti self-dual or neither.

5.3.1. *Null reductions.* In this section I review the construction of (weak) Newton–Cartan geometries via null reductions of lorentzian geometries [33, 34].

Let (N, g) be a lorentzian manifold with a nowhere-vanishing null Killing vector field $\xi \in \mathscr{X}(N)$. Let us assume that ξ is complete and that it generates a one-parameter subgroup Γ of isometries of (N, g) in such a way that the quotient $M = N/\Gamma$ is a smooth manifold. Let $\pi : N \to M$ denote the resulting principal Γ bundle. As is well known, the pullback $\pi^* : \Omega^\bullet(M) \to \Omega^\bullet(N)$ of differential forms sets up a $C^\infty(M)$-module isomorphism between $\Omega^\bullet(M)$ and the **basic forms**:

$$\Omega^\bullet_\Gamma(N) = \{\omega \in \Omega^\bullet(N) \mid \imath_\xi\omega = 0 \quad \text{and} \quad \imath_\xi d\omega = 0\}. \tag{5.5}$$

The first condition ($\imath_\xi\omega = 0$) says that ω is **horizontal** and the second condition, together with horizontality, says that $\mathscr{L}_\xi\omega = 0$, so that ω is Γ-invariant. Let $\flat : TN \to T^*N$ and $\sharp : T^*N \to TN$ denote the musical isomorphisms induced by the lorentzian metric g. I shall use the same notation for the corresponding $C^\infty(M)$-module isomorphisms between the spaces of sections: $\flat : \mathscr{X}(N) \to \Omega^1(N)$ and $\sharp : \Omega^1(N) \to \mathscr{X}(N)$.

The one-form $\xi^\flat \in \Omega^1(N)$ metrically dual to the null Killing vector field is basic: it is horizontal because ξ is null and it is invariant because ξ is Killing. Therefore $\xi^\flat = \pi^*\tau$ for some nowhere vanishing $\tau \in \Omega^1(M)$.

Let $\alpha, \beta \in \Omega^1(M)$ and let $X = (\pi^*\alpha)^\sharp, Y = (\pi^*\beta)^\sharp \in \mathscr{X}(N)$. Let $f = g(X, Y) \in C^\infty(N)$. Since ξ is Killing, it follows that $\xi(f) = 0$, so that $f = \pi^*h$ for some $h \in C^\infty(M)$. Define $\lambda \in \Gamma(\odot^2 M)$ by $\lambda(\alpha, \beta) = h$. In other words, λ is defined by

$$\pi^*\lambda(\alpha, \beta) = g((\pi^*\alpha)^\sharp, (\pi^*\beta)^\sharp). \tag{5.6}$$

It is easy to see that λ is well-defined, obeys $\lambda(\tau, -) = 0$, is positive-semidefinite and has corank 1. In other words, (M, τ, λ) is a (weak) Newton–Cartan geometry.

5.3.2. *Another look at the galilean Klein geometries.* As promised, I now describe geometric realisations of the galilean Klein geometries in Section 4.4 as null reductions of lorentzian manifolds. There results are joint with Stefan Prohazka and Ross Grassie and will appear in a forthcoming paper on Bargmann spacetimes. In [16], their geometric realisability was shown non-constructively.

Let us consider Klein pairs $(\mathfrak{b}, \mathfrak{h})$, where $\mathfrak{b}$ is a **generalised Bargmann algebra**, namely a one-dimensional extension (not necessarily central) of a kinematical Lie algebra $\mathfrak{k}$:

$$0 \longrightarrow \mathbb{R}Z \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{k} \longrightarrow 0, \tag{5.7}$$

with Z the additional generator. The name of the Lie algebras is due to the fact that the Bargmann algebra is the universal central extension of the Galilei algebra, with additional bracket:

$$[B_a, P_b] = \delta_{ab} Z. \tag{5.8}$$

Every generalised Bargmann algebra has a basis L_{ab}, B_a, P_a, H, Z and shares the kinematical Lie brackets (3.2) in addition to (5.8). It follows by $\mathfrak{r}$-equivariance that Z transforms under the trivial one-dimensional representation of $\mathfrak{so}(V)$. The Newton–Hooke algebras also admit central extensions, whereas the kinematical Lie algebras $\mathfrak{n}^0$, $\mathfrak{n}^+_\gamma$, for $\gamma \in (-1, 1]$ and $\mathfrak{n}^-_\xi$ for $\xi > 0$ admit non-central extensions.

Let us define the generalised Bargmann algebras $\mathfrak{b}^+_\gamma$, for $\gamma \in [-1, 1)$, $\mathfrak{b}^-_\chi$, for $\chi \geqslant 0$ and $\mathfrak{b}^0$ as in Table 3, which lists all nonzero Lie brackets in addition to those in (3.2) and (5.8). The first three Lie algebras in the Table are the universal central extensions of the Galilei algebra $\mathfrak{g}$ and the Newton–Hooke algebras $\mathfrak{n}^\pm$, whereas the last three are non-central extensions of $\mathfrak{n}^+_{\gamma\in(-1,1)}$, $\mathfrak{n}^0$ and $\mathfrak{n}^-_{\chi>0}$, respectively. Notice that even when $\mathfrak{b}^0$ agrees with $\mathfrak{b}^+_{\gamma=1}$, it is an extension of $\mathfrak{n}^0$ and not of $\mathfrak{n}^+_{\gamma=1}$.

TABLE 3. Some relevant generalised Bargmann algebras

Name	Lie brackets in addition to (3.2) and (5.8)		
$\widehat{\mathfrak{g}}$	$[\mathbf{B}, H] = \mathbf{P}$		
$\widehat{\mathfrak{n}}^+ = \mathfrak{b}^+_{\gamma=-1}$	$[\mathbf{B}, H] = \mathbf{P}$	$[H, \mathbf{P}] = -\mathbf{B}$	
$\widehat{\mathfrak{n}}^- = \mathfrak{b}^-_{\chi=0}$	$[\mathbf{B}, H] = \mathbf{P}$	$[H, \mathbf{P}] = \mathbf{B}$	
$\mathfrak{b}^+_{\gamma\in(-1,1)}$	$[\mathbf{B}, H] = \mathbf{P}$	$[H, \mathbf{P}] = \gamma\mathbf{B} + (1+\gamma)\mathbf{P}$	$[H, Z] = (1+\gamma)Z$
$\mathfrak{b}^0$	$[\mathbf{B}, H] = \mathbf{P}$	$[H, \mathbf{P}] = \mathbf{B} + 2\mathbf{P}$	$[H, Z] = 2Z$
$\mathfrak{b}^-_{\chi>0}$	$[\mathbf{B}, H] = \mathbf{P}$	$[H, \mathbf{P}] = (1+\chi^2)\mathbf{B} + 2\chi\mathbf{P}$	$[H, Z] = 2\chi Z$

The Klein pairs $(\mathfrak{b}, \mathfrak{h})$ are such that $\mathfrak{h} \cong \mathfrak{iso}(d)$ is the Lie subalgebra spanned by L_{ab}, B_a. All Klein pairs are reductive, with complementary subspace $\mathfrak{m}$ spanned by P_a, H, Z. Let π^a, η, ζ be the canonical dual basis for $\mathfrak{m}^*$. All Klein pairs $(\mathfrak{b}, \mathfrak{h})$ share the same $\mathfrak{h}$-invariant tensors in the linear isotropy representation. Up to scale they are given by the vector $Z \in \mathfrak{m}$, the covector $\eta \in \mathfrak{m}^*$ and the lorentzian inner product $h := \delta_{ab}\pi^a\pi^b - 2\eta\zeta \in \odot^2\mathfrak{m}^*$. Notice that Z is null relative to the inner product. The vector field corresponding to Z is not only Killing but actually parallel with respect to the Levi-Civita connection of the metric corresponding to h. The Klein pairs $(\mathfrak{b}, \mathfrak{h})$ are geometrically realisable and correspond to $(d+2)$-dimensional homogeneous lorentzian manifolds and, by construction, their null reduction along Z is $(\mathfrak{k}, \mathfrak{h})$, where $\mathfrak{k} = \mathfrak{b}/\mathbb{R}Z$, abusing notation slightly and denoting by $\mathfrak{h}$ both the subalgebra of $\mathfrak{b}$ and its isomorphic image in $\mathfrak{k}$. Comparing with Table 2, it is evident that the reduced Klein pairs $(\mathfrak{k}, \mathfrak{h})$ are precisely the galilean Klein pairs. It should be remarked, however, that in the torsional cases, the galilean structure obtained via the null reduction is not the invariant one, since in those cases the null vector is not invariant.

I conclude this section with an observation: the lorentzian metrics on the geometric realisations of $(\widehat{\mathfrak{g}}, \mathfrak{h})$ and $(\mathfrak{b}^+_{\gamma=0}, \mathfrak{h})$ are flat and one can show that as Newton–Cartan geometries, torsional de Sitter–Galilei spacetime (for $\gamma = 0$) and Galilei spacetime itself are isomorphic, although their descriptions as kinematical Klein geometries are not.

5.4. **Carrollian geometry.** Carrollian geometry is in a sense dual to Newton–Cartan geometry. Again let $V = \mathbb{R}^{d+1}$ with elementary basis $\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_d$ for V and canonically dual basis $\alpha^0, \alpha^1, \dots, \alpha^d$ for V^*. Let $G \subset \mathrm{GL}(V)$ denote the subgroup which leaves invariant $\boldsymbol{e}_0 \in V$ and $\sum_{a=1}^d \alpha^a\alpha^a \in \odot^2 V^*$. Explicitly,

$$G = \left\{ \begin{pmatrix} 1 & \boldsymbol{v}^T \\ \boldsymbol{0} & A \end{pmatrix} \middle| \boldsymbol{v} \in \mathbb{R}^d,\ A \in O(d) \right\} < \mathrm{GL}(d+1, \mathbb{R}), \tag{5.9}$$

with Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & \boldsymbol{v}^T \\ \boldsymbol{0} & A \end{pmatrix} \middle| \boldsymbol{v} \in \mathbb{R}^d,\ A \in \mathfrak{so}(d) \right\} < \mathfrak{gl}(d+1, \mathbb{R}). \tag{5.10}$$

The group of a carrollian structure is abstractly isomorphic to that of a Newton–Cartan structure, both being isomorphic to the euclidean group $\mathrm{ISO}(d) \cong O(d) \ltimes \mathbb{R}^d$. Crucially, however, they are not conjugate inside $\mathrm{GL}(V)$, so that they lead to different geometries with different characteristic tensor fields.

The characteristic tensor fields of a carrollian geometry are a nowhere-vanishing vector field $\xi \in \mathscr{X}(M)$ (the **carrollian vector field**) and a corank-one positive-semidefinite symmetric tensor field $h \in \Gamma(\odot^2 T*M)$ (the **spatial metric**), with $h(\xi, -) = 0$. The triple $(M, \xi, \mathfrak{h})$ defines a **(weak) carrollian geometry**, which can be promoted to a **carrollian geometry** by the addition of an adapted affine connection ∇. If the structure group reduces to the connected component G_0 of G – e.g., if M is simply connected – then there is an addition a nowhere-vanishing top form $\mu \in \Omega^{d+1}(M)$.

As shown in [26], the kernel and cokernel of the Spencer differential are isomorphic as G-modules to $\odot^2 \operatorname{Ann} \boldsymbol{e}_0$, with $\operatorname{Ann} \boldsymbol{e}_0 \subset V^*$ the annihilator of $\boldsymbol{e}_0$. The isomorphism $\operatorname{coker} \partial \cong \odot^2 \operatorname{Ann} \boldsymbol{e}_0$ is induced from a G-equivariant linear map $\mathrm{Hom}(\wedge^2 V, V) \to \odot^2 \operatorname{Ann} \boldsymbol{e}_0$ which induces in turn a bundle map $\Omega^2(M, TM) \to \Gamma(\odot^2 \operatorname{Ann} \xi)$ under which the torsion of an adapted connection is mapped to $\mathscr{L}_\xi h$, the Lie derivative of the spatial metric along the carrollian vector field.

For $d > 1$, there are four G-subbundles of $\odot^2 \operatorname{Ann} \xi$ and hence four classes of carrollian geometries depending on the intrinsic torsion of adapted connections. For reasons which will only become clear after discussing the geometric realisation of carrollian geometries as null hypersurfaces in lorentzian manifolds, I shall refer to them as follows:

(1) **totally geodesic**, if $\mathscr{L}_\xi h = 0$;
(2) **minimal**, if $\mathscr{L}_\xi \mu = 0$;

(3) **totally umbilical**, if $\mathscr{L}_\xi h = fh$ for some $f \in C^\infty(M)$; and
(4) **generic**, otherwise.

If $d = 1$ there are only two submodules and hence two carrollian structures: either $\mathscr{L}_\xi h = 0$ or not.

5.4.1. *Null hypersurfaces.* As shown in [19, 35], a null hypersurface in a lorentzian manifold admits a carrollian structure. Let us review these results here. For more details on null hypersurfaces, see [36, 37].

Let (N, g) be a lorentzian manifold and $M \subset N$ an embedded hypersurface such that the restriction of g to M is degenerate; in other words, M is a **null hypersurface** of N. Because of the lorentzian signature of the metric, the restriction of g to M must be positive-semidefinite and of corank 1. This implies that there exists a future-directed, nowhere-vanishing null vector $\xi \in \Gamma(TM)$ such that for all $p \in M$, $T_pM = \xi_p^\perp$. Integral curves of ξ can be reparametrised in such a way that they are null geodesics. They are called the **null geodesic generators** of the null hypersurface M. The null vector ξ defines a (trivial) line bundle $L \subset TM$, which is independent of the choice of ξ: indeed, $L = TM^\perp \cap TM$ without reference to ξ. The metric g defines a riemannian structure h on the quotient vector bundle TM/L over M via $h(\overline{X}, \overline{Y}) = g(X, Y)$ where $\overline{X} = X \mod L$. It is easy to see that h is well-defined since $g(X, Y)$ only depends on the residue classes of X, Y modulo L.

With some abuse of notation, let h stand for the restriction of g to M as well for the induced riemannian structure on TM/L. The triple (M, ξ, h) is a (weak) carrollian structure. The classification of carrollian structures via their intrinsic torsion corresponds to the analogue for null hypersurfaces of the classification of hypersurfaces in riemannian geometry, as I now explain.

The Levi-Civita connection of (N, g) defines a **null Weingarten map** $b : TM/L \to TM/L$ defined by

$$b(\overline{X}) = \overline{\nabla_X \xi}. \tag{5.11}$$

It manifestly depends on the choice of ξ, but notice that if $f \in C^\infty(M)$ is a positive smooth function so that $\widetilde{\xi} := f\xi$ is another generator of L, then $\nabla_X(f\xi) = f\nabla_X\xi \mod L$. In particular, the null Weingarten map at p depends only on the value of ξ at p.

Notice that if $X, Y \in \Gamma(TM)$, $[X, Y] \in \Gamma(TM)$ and hence $g(\xi, [X, Y]) = 0$. Therefore,

$$\begin{aligned} g(\nabla_X\xi, Y) - g(X, \nabla_Y\xi) &= Xg(\xi, Y) - g(\xi, \nabla_XY) - Yg(X, \xi) + g(\xi, \nabla_YX) \\ &= -g(\xi, \nabla_XY - \nabla_YX) \\ &= -g(\xi, [X, Y]) = 0. \end{aligned} \tag{5.12}$$

Hence the **null second fundamental form** $B(\overline{X}, \overline{Y}) := h(b(\overline{X}), \overline{Y}) = g(\nabla_X\xi, Y)$ is a well-defined symmetric form on TM/L. By analogy with the theory of hypersurfaces in riemannian geometry, the null hypersurface M is said to be

- **totally geodesic**, if $B = 0$;
- **minimal**, if $\operatorname{tr} b = 0$;
- **totally umbilical**, if $B = fh$ for some $f \in C^\infty(M)$; and
- **generic**, otherwise.

Notice that although B depends on ξ, it does so via multiplication by a positive function and hence the above conditions are independent on the choice of ξ.

A short calculation shows that the restriction of $\mathscr{L}_\xi g$ to M, denoted $\mathscr{L}_\xi h$, agrees (up to an inconsequential factor of $\frac{1}{2}$) with the null second fundamental form of the hypersurface, showing that the above classification of null hypersurfaces corresponds to the classification of carrollian structures via their intrinsic torsion and explains the names given to the carrollian structures in Section 5.4.

5.4.2. *Bundles of scales of conformal structures.* Another natural source of carrollian geometries are bundles of scales of conformal structures. (See, e.g., [38] for a recent review.) The fundamental example is the conformal sphere, whose bundle of scales can be identified with the future (or past) deleted lightcone, which is a null hypersurface in Minkowski spacetime.

Let (S^{d-1}, g) denote the round $(d-1)$-sphere, thought of as the unit sphere in $\mathbb{R}^d$ with the euclidean metric. Let $[g]$ denote the set of metrics on S^{d-1} conformal to the round metric:

$$[g] = \left\{ \Omega^2 g \mid \Omega \in C^\infty(S^{d-1}) \right\}. \tag{5.13}$$

Pick a point $x \in S^{d-1}$. The value $g_x \in \odot^2 T_x^* S^{d-1}$ at x of the round metric, defines a ray $Q_x = \{\lambda^2 g_x \mid \lambda \in \mathbb{R}^+\} \subset \odot^2 T_x^* S^{d-1}$. The collection $Q = \sqcup_{x \in S^{d-1}} Q_x$ of all such rays can be given a differentiable structure making $\pi : Q \to S^{d-1}$ into a smooth principal $\mathbb{R}^+$-bundle, with $\sigma \in \mathbb{R}^+$ acting as $\lambda^2 g_x \mapsto \sigma^2\lambda^2 g_x$. Let G be the connected component of $\mathrm{SO}(d-1, 1)$. Then G acts transitively on S^{d-1} via conformal transformations and it therefore acts on Q. This action is also transitive and the stabiliser of $g_x \in Q$ is isomorphic to the euclidean group $\mathrm{ISO}(d-1)$. In fact, Q is G-equivariantly diffeomorphic to the future deleted lightcone

$\mathbb{L} \subset \mathbb{R}^{d,1}$, and the diffeomorphism sends $\lambda^2 g_x \in Q$ to $(\lambda, \lambda x) \in \mathbb{L}$, where x is a unit-norm vector in $\mathbb{R}^d$, giving rise to the following commutative triangle:

$$\begin{array}{ccc} Q & \xrightarrow{\cong} & \mathbb{L} \\ & \searrow \quad \swarrow & \\ & S^{d-1} & \end{array} \tag{5.14}$$

which exhibits the diffeomorphism $Q \to \mathbb{L}$ as a bundle isomorphism in addition to as an isomorphism of homogeneous G-spaces. The carrollian structure on Q corresponds to the carrollian structure on $\mathbb{L}$: the carrollian vector field is the fundamental vector field of the $\mathbb{R}^+$-action and the corank-one degenerate metric is the pullback via the projection $\pi : Q \to S^{d-1}$ of the round metric on S^{d-1}.

Now consider a riemannian conformal manifold $(N, [g])$, with $[g] = \left\{\Omega^2 g \;\middle|\; \Omega \in C^\infty(N) \text{ nowhere zero}\right\}$ the conformal class of g. Let $p \in N$ and let $M_p = \left\{\lambda^2 g_p \middle| \lambda \in \mathbb{R}^+\right\} \subset \odot^2 T^*_p N$ be the ray in $\odot^2 T^*_p N$ defined by g_p. Then $M = \sqcup_{p \in N} M_p$ is the total space of a smooth principal $\mathbb{R}^+$-bundle $\pi : M \to N$ called the **bundle of scales of the conformal manifold** N. Let $h = \pi^* g$ and let ξ be the fundamental vector field of the free right $\mathbb{R}^+$-action on M. Then (M, ξ, h) is a (weak) carrollian geometry. This construction of carrollian structures has played a rôle in some recent work [23, 24, 8, 25].

6. Conclusions, omissions and outlook

Despite the prominent rôle played by lorentzian geometry in both general relativity and quantum field theory, it is not the only possible geometrical description of space and time. There are phenomenological reasons for considering non-lorentzian geometries (e.g., condensed matter physics, hydrodynamics,...), but they are also the geometries relevant to a largely unexplored edge of the Bronstein cube in Figure 1 which might provide a new approach to constructing a quantum theory of gravity.

In this short review, I have tried to give a flavour of some of the better studied non-lorentzian geometries: galilean (or Newton–Cartan) and carrollian. I reviewed the classification of kinematical Lie algebras (with spatial isotropy) and their associated Klein geometries. I observed that despite the plethora of Klein geometries, they belong to a small class of Cartan geometries: lorentzian, riemannian, galilean and carrollian. We concentrated on the galilean and carrollian geometries and defined them as G-structures, identified their characteristic tensors and refined the classification according to their intrinsic torsions.

I have omitted several topics. One topic I did not cover is that of the automorphisms of these non-lorentzian geometries. For example, let (M, τ, λ) be a weak Newton–Cartan geometry. Its automorphism group is the subgroup of diffeomorphisms of M preserving τ and λ. Contrary to what happens in lorentzian geometry, the automorphism group need not be finite-dimensional. For example, as shown in [39] and revisited in [17], the Lie algebra $\mathfrak{a}$ of infinitesimal automorphisms of the galilean Klein geometries is an infinite-dimensional Lie algebra known as the **Coriolis algebra**. It is a split extension

$$0 \longrightarrow C^\infty(\mathbb{R}_t, \mathfrak{iso}(d)) \longrightarrow \mathfrak{a} \longrightarrow \mathbb{R}D \longrightarrow 0, \tag{6.1}$$

where $C^\infty(\mathbb{R}_t, \mathfrak{iso}(d))$ is the Lie algebra of smooth functions from the real line (with parameter t) to the euclidean Lie algebra under the pointwise Lie bracket on which D acts as the derivation $\frac{d}{dt}$. Of course, "strengthening" the structure by the addition of an adapted connection reduces the size of the Lie algebra of infinitesimal automorphisms to a finite-dimensional Lie algebra. This is the well-known fact (see, e.g., [40]) that the automorphism group of a Cartan geometry is finite-dimensional. Something similar, but more interesting, happens with carrollian structures. The Lie algebra of infinitesimal (conformal) automorphisms of weak carrollian structures can be infinite-dimensional and are, in fact, intimately linked with the asymptotic symmetries of asymptotically flat lorentzian manifolds, the so-called BMS group [41, 42], as shown originally in [43, 44] and further discussed in [17].

Another omission is supersymmetry. I have stayed here in the realm of classical geometry, but of course there is a notion of non-lorentzian supersymmetry and supergeometry. The results here are far from complete. There are classifications of certain four-dimensional kinematical Lie superalgebras and their associated Klein supergeometries [45, 46, 47], extending earlier work on contractions of the Poincaré and anti de Sitter superalgebras and referred to in those papers.

As for future work, an obvious next step is the study of natural conditions which can be imposed on the curvature of the Cartan connection of a kinematical Cartan geometry. Some of these conditions could have a variational origin, just like Ricci-flatness in lorentzian geometry arises as the Euler–Lagrange equation of the Einstein–Hilbert action. Closer in spirit to the approach outlined in this review is the construction of Cartan geometries via the "gauging procedure". This is the Physics version of the construction of a Cartan geometry from local data (i.e., from an atlas of Cartan gauges, in the language of [21]). Doing so for the Cartan geometry modelled on Minkowski spacetime leads to the Hilbert–Palatini action and results in Ricci-flatness or, more generally, the Einstein condition (in the presence of a cosmological constant). As shown in [22], doing so for the Cartan geometry modelled on (anti) de Sitter spacetime leads to the MacDowell–Mansouri [48] formulation of Einstein gravity. Work is in progress with Emil Have, Stefan Prohazka and Jakob Salzer to "gauge" some of the four-dimensional kinematical Klein geometries of interest.

Another extension of the work reviewed here is to study geodesic motion on the non-Lorentzian geometry. Work is in progress with Can Görmez and Dieter Van den Bleeken studying dynamics on the galilean Klein geometries by studying the geodesics of the invariant connections. One could also study dynamics on these geometries via Souriau's method of coadjoint orbits [49].

The unitary representation theory of the kinematical Lie groups is largely unexplored, with the notable exceptions of the classic work of Wigner and Bargmann [50, 51] for the Poincaré group and of Lévy-Leblond [52] for the Galilei group. This is an important problem which could benefit from the attention of representation theorists.

To conclude, we would not like to finish without mentioning other geometries which are closely related to the kinematical geometries treated here: not just aristotelian (as already mentioned), but also conformal, Lifshitz, Bargmann,... which are finding applications in an expanding set of research areas.

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