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# FROM BRACES TO HECKE ALGEBRAS & QUANTUM GROUPS

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ABSTRACT. We examine links between the theory of braces and set theoretical solutions of the Yang-Baxter equation, and fundamental concepts from the theory of quantum integrable systems. More precisely, we make connections with Hecke algebras and we identify new quantum groups associated to set-theoretic solutions coming from braces. We also construct a novel class of quantum discrete integrable systems and we derive symmetries for the corresponding periodic transfer matrices.

## 1. INTRODUCTION

The Yang-Baxter equation is a fundamental equation in the theory of quantum integrable models and solvable statistical systems, as well as in the formulation of quantum groups [18, 37, 15]. It was introduced in [53] as a main tool for the investigation of many particle systems with  $\delta$ -type interactions, and in [4] for the study of a two-dimensional solvable statistical model. Since Drinfeld [14] suggested a theory of set-theoretic solutions to the Yang-Baxter equation be developed, set-theoretic solutions have been extensively investigated using braided groups, and more recently by applying the theory of braces and skew-braces.

Set theoretical solutions and Yang-Baxter maps have been also extensively studied in the context of classical discrete integrable systems linked to the notion of Darboux-Bäcklund transformation within the Lax pair formulation [1, 52, 44]. In classical integrable systems usually a Poisson structure exists associated to a classical  $r$ -matrix, which is a solution of the classical Yang-Baxter equation [19]. Also, relevant recent results on Yang-Baxter maps, when the quantum group symmetry is a priori requirement can be found in [5].

It is worth noting that [28] provides one of the first instances of classification of set-theoretical solutions of Yang-Baxter equation. Various connections between the set theoretical Yang-Baxter equation and geometric crystals [17, 6], or soliton cellular automata [50, 27] have been also demonstrated.

The theory of braces was established around 2005, when Wolfgang Rump developed a structure called a brace to describe all finite involutive set-theoretic solutions of the Yang-Baxter equation. Rump showed that every brace yields a solution to the Yang-Baxter equation, and every non-degenerate, involutive set-theoretic solution of

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the Yang-Baxter equation can be obtained from a brace, a structure which generalises nilpotent rings. Subsequently skew-braces were developed by Guarnieri and Vendramin to describe non-involutive solutions [26].

The theory of braces and skew braces has connections with numerous research areas, for example with group theory (Garside groups, regular subgroups, factorised groups— see for example [3, 30, 31, 49]), algebraic number theory, Hopf-Galois extensions [2, 47], non-commutative ring theory [46, 40, 41], Knot theory [39, 42], Hopf algebras, quantum groups [16], universal algebra, groupoids [29], semi-braces [8], trusses [7] and Yang-Baxter maps. Moreover, skew braces are related to non-commutative physics, Yetter-Drinfeld modules and Nichols algebras.

Note that in the present article when we say set theoretic solutions we always mean finite, non-degenerate, involutive, set-theoretic solutions of the Yang-Baxter equation. As was shown by Rump all finite, non-degenerate, involutive, set-theoretic solutions of the Yang-Baxter equation (2.1) are coming from braces (Theorem 2.4), therefore some times we may call such solutions *brace solutions*. Because set-theoretic solutions coming from braces are involutive, it is possible to *Baxterise* them [38] and obtain solutions to the parameter dependent Yang-Baxter equation, which appear in quantum integrable systems.

The aim of this paper is to investigate connections between the theory of braces and selected topics from the theory of quantum integrable systems. More precisely:

- (1) We derive new quantum groups associated to braces.
- (2) We construct a novel class of quantum discrete integrable systems.
- (3) We identify symmetries of the periodic transfer matrices of the novel integrable systems.

Note that in [16] Etingof, Schedler and Soloviev constructed quantum groups associated to set-theoretic solutions, however we use a different construction coming from parameter dependent solutions of the Yang-Baxter equation and our quantum groups differ from these in [16].

Rump showed that every nilpotent ring is a brace, therefore readers who are not familiar with the theory of braces may replace use of the word brace in this paper with the words nilpotent ring. Readers interested in learning more about the theory of braces are referred to [9, 10, 24, 45, 46, 48].

**Structure of the paper.** This paper is divided into four sections:

- (1) Section 1 contains the introduction of the paper.
- (2) Section 2 shows how to construct  $R$ -matrices associated to non-degenerate, involutive, set-theoretic solutions of the Yang-Baxter equation in preparation for Section 3.
- (3) Section 3 contains information on how to go about connecting the theory of quantum integrable systems with  $R$ -matrices constructed from braces as per Section 2. We construct the new quantum algebra associated to braces. We also construct various realizations of the relevant quantum algebras using

classical results from the theory of braces, along with posing some more open questions. Note that the Yangian is a special case within the larger class of quantum algebras emerging from braces.

The section consists of three subsections:

- 3.1: The Yang-Baxter equation &  $A$ -type Hecke algebra.
- 3.2: Quantum algebras from braces.
- 3.3: Representations of quantum algebras.

At the end of each section there are relevant questions and lines of inquiry for further research.

- (4) Section 4, offers information on the construction of a *new class* of integrable quantum spin chain-like systems associated to braces. Spin chain-like systems are typically constructed by means of tensor realizations of the underlying quantum algebra, by introducing the so called transfer matrix. We show that the periodic transfer matrix constructed from Baxterized solutions of the  $A$ -type Hecke algebra  $\mathcal{H}_N(q = 1)$  can be exclusively expressed in terms of the generators of the  $A$ -type Hecke algebras plus some periodic term. This is a universal result that holds for any representation of the  $A$ -type Hecke algebra  $\mathcal{H}_N(q = 1)$ , and is one of the main propositions of this study. We then focus on solutions of the Yang-Baxter equation coming from braces and we employ the theory of braces to construct symmetries of the corresponding periodic transfer matrices.

The section consists of two subsections:

- 4.1: Tensor representations of quantum algebras & integrable systems.
- 4.2: Symmetries of the periodic transfer matrix of novel classes of spin chains.

## 2. BASIC INFORMATION ABOUT BRACES & SET-THEORETIC SOLUTIONS

For a set-theoretic solution of the Braid equation, we will use notation  $(X, \check{r})$ , instead of the usual notation  $(X, r)$ , to be consistent with notations used in quantum integrable systems.

Let  $\mathcal{N}$  be a natural number, and let  $e_{i,j}$  denote the  $\mathcal{N} \times \mathcal{N}$  matrix whose all entries are 0 except for the  $i, j$ -th entry, which is 1.

Let  $X = \{x_1, \dots, x_{\mathcal{N}}\}$  be a set and  $\check{r} : X \times X \rightarrow X \times X$ . Denote

$$\check{r}(x, y) = (\sigma_x(y), \tau_y(x)) = ({}^x y, x^y).$$

We say that  $\check{r}$  is non-degenerate if  $\sigma_x$  and  $\tau_y$  are bijective functions. Suppose that  $(X, \check{r})$  is an involutive, non-degenerate set-theoretic solution of the Braid equation:

$$(\check{r} \times Id_X)(Id_X \times \check{r})(\check{r} \times Id_X) = (Id_X \times \check{r})(\check{r} \times Id_X)(Id_X \times \check{r}).$$

With a slight abuse of notation, let  $\check{r}$  also denote the  $R$ -matrix associated to the linearisation of  $\check{r}$  on  $V = \mathbb{C}X$  (see [48] for more details).

This matrix is also called a *check-matrix*. Then the check-matrix related to  $(X, \check{r})$  is  $\check{r} = \check{r}_{i,j;k,l}$ , where  $\check{r}_{i,j;k,l} = 1$  if and only if  $\check{r}(i, j) = (k, l)$ , and is zero otherwise.

Notice that the matrix  $\check{r} : V \otimes V \rightarrow V \otimes V$  satisfies the (constant) Braid equation:

$$(\check{r} \otimes Id_V)(Id_V \otimes \check{r})(\check{r} \otimes Id_V) = (Id_V \otimes \check{r})(\check{r} \otimes Id_V)(Id_V \otimes \check{r}).$$

Notice that  $\check{r}^2 = I_{V \otimes V}$  the identity matrix, because  $\check{r}$  is involutive.

Let  $r = \tau \check{r}$  be the corresponding solution of the quantum Yang-Baxter equation (where  $\tau(x, y) = (y, x)$ ) and let  $r$  denote the matrix associated to the linearisation of  $r$  on  $V = \mathbb{C}X$ . Notice that the matrix  $r$  satisfies the constant Yang-Baxter equation.

Let  $e_{x,y}$  be the matrix with  $x, y$  entry equal to 1 and all the other entries 0.

**Lemma 2.1.** *Let notation be as above. Then the matrix  $\check{r}$  has the form:*

$$(2.1) \quad \check{r} = \sum_{x,y \in X} e_{x,\sigma_x(y)} \otimes e_{y,\tau_y(x)}.$$

The matrix  $r$  has the form:

$$r = \mathcal{P} \cdot \check{r} = \left( \sum_{x,y \in X} e_{y,x} \otimes e_{x,y} \right) \left( \sum_{x,y \in X} e_{x,\sigma_x(y)} \otimes e_{y,\tau_y(x)} \right),$$

consequently

$$(2.2) \quad r = \sum_{x,y \in X} e_{y,\sigma_x(y)} \otimes e_{x,\tau_y(x)}.$$

Moreover because  $\check{r}$  is involutive we get  $\check{r}(\sigma_x(y), \tau_y(x)) = (x, y)$ , therefore

$$r = \sum_{x,y \in X} e_{\tau_y(x),x} \otimes e_{\sigma_x(y),y}$$

*Proof.* We use a direct calculation using the way in which the  $R$ -matrix associated to a set-theoretic solution  $(X, \check{r})$  is built (for more details, see Definition 2.3 in [48]). It is worth noticing that, in some books on quantum groups, the obtained check-matrix is also transposed. However, for involutive solutions, which we consider in this paper, the obtained  $R$ -matrix is symmetric, so it is the same.  $\square$

In [45, 46] Rump showed that every solution  $(X, r)$  can be in a good way embedded in a brace.

**Definition 2.2** (Proposition 4, [46]). *A left brace is an abelian group  $(A; +)$  together with a multiplication  $\cdot$  such that the circle operation  $a \circ b = a \cdot b + a + b$  makes  $A$  into a group, and  $a \cdot (b + c) = a \cdot b + a \cdot c$ .*

In many papers, the following equivalent definition from [10] is used:

**Definition 2.3** ([10]). *A left brace is a set  $G$  together with binary operations  $+$  and  $\circ$  such that  $(G, +)$  is an abelian group,  $(G, \circ)$  is a group, and  $a \circ (b + c) + a = a \circ b + a \circ c$  for all  $a, b, c \in G$ .*

The additive identity of a brace  $A$  will be denoted by 0 and the multiplicative identity by 1. In every brace  $0 = 1$ . The same notation will be used for skew braces (in every skew brace  $0 = 1$ ).

Some authors use the notation  $\cdot$  instead of  $\circ$  and  $*$  instead of  $\cdot$  (see for example [10, 24, 23]).

In [16], Etingof, Schedler and Soloviev introduced the retract relation for any solution  $(X, r)$ . Denote  $X = \{x_1, \dots, x_N\}$  and  $r(x, y) = (\sigma_x(y), \tau_y(x))$ . Recall that the retract relation  $\sim$  on  $X$  is defined by  $x_i \sim x_j$  if  $\sigma_{x_i} = \sigma_{x_j}$ . The induced solution  $Ret(X, r) = (X/\sim, r^\sim)$  is called the *retraction* of  $X$ . A solution  $(X, r)$  is called a *multi-permutation solution* of level  $m$  if  $m$  is the smallest non-negative integer such that after  $m$  retractions we obtain the solution with one element.

Throughout this paper we will use the following result, which is implicit in [45, 46] and explicit in Theorem 4.4 of [10].

**Theorem 2.4.** (*Rump's theorem*, [45, 46, 10]). *Assume  $(B, +, \circ)$  is a brace. If the map  $\check{r}_B : B \times B \rightarrow B \times B$  is defined as  $\check{r}_B(x, y) = (\sigma_x(y), \tau_y(x))$ , where  $\sigma_x(y) = x \circ y - x$ ,  $\tau_y(x) = t \circ x - t$ , and  $t$  is the inverse of  $\sigma_x(y)$  in the circle group  $(B, \circ)$ , then  $(B, \check{r}_B)$  is an involutive, non-degenerate solution of the braid equation. Conversely, if  $(X, \check{r})$  is an involutive, non-degenerate solution of the braid equation, then there exists a brace  $(B, +, \circ)$  (called an underlying brace of the solution  $(X, \check{r})$ ) such that  $B$  contains  $X$ ,  $\check{r}_B(X \times X) \subseteq X \times X$ , and the map  $\check{r}$  is equal to the restriction of  $\check{r}_B$  to  $X \times X$ . Moreover, both the additive  $(B, +)$  and multiplicative  $(B, \circ)$  groups of the brace  $(B, +, \circ)$  are generated by  $X$ .*

We will call the brace  $B$  an underlying brace of the solution  $(X, \check{r})$ , or a brace associated to the solution  $(X, \check{r})$ . We will also say that the solution  $(X, \check{r})$  is associated to brace  $B$ . Notice that this is also related to the formula of set-theoretic solutions associated to the braided group (see [16] and [24]).

The following fact was also discovered by Rump.

**Remark 2.5.** *Let  $(N, +, \cdot)$  be an associative nilpotent ring. If for  $a, b \in N$  we define*

$$a \circ b = a \cdot b + a + b,$$

*then  $(N, +, \circ)$  is a brace.*

**Definition 2.6.** *Let  $(X, \check{r})$  and  $(Y, \check{r}')$  be set-theoretic solutions of the braid equation, and let  $f : X \rightarrow Y$  be a function such that  $\check{r}'(f(x), f(y)) = (f \times f)(\check{r}(x, y))$ , for all  $x, y \in X$ . Then  $f$  is called a homomorphism of solutions. If  $f$  is one-to-one then  $f$  is called an isomorphism of solutions.*

**Lemma 2.7.** *Notice that if a solution  $(X, \check{r})$  comes from brace  $B$ , and  $J$  is an ideal in  $B$  and  $X_J = \{x + J : x \in X\}$  is a subset of the factor brace  $B/J$  then the map  $f : X \rightarrow X_J$  given by  $f(x) = x + J$  is a homomorphism of solutions  $(X, \check{r})$  and  $(X_J, \check{r}_J)$ , where  $(X_J, \check{r}_J)$  is the solution associated to the brace  $B/J$  on the set  $X_J$ .*

*Proof.* It follows immediately from the properties of an ideal in a brace (ideals in braces were defined in [46]. See also [10]).  $\square$

### 3. HECKE ALGEBRAS & QUANTUM GROUPS FROM BRACES

**3.1. The Yang-Baxter equation & Hecke algebras.** In this section we explore various connections between braces, representations of the  $A$ -type Hecke algebras, and quantum algebras. In particular, after showing some fundamental properties for the brace  $R$ -matrices and making the direct connection with  $A$ -type Hecke algebras, we derive new quantum algebras coming from braces. The Yangian  $\mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$  turns out to be a special case within this larger class of quantum algebras.

Before we start our investigation on the aforementioned connections let us first derive some preliminary results, that will be essential especially when proving the integrability of open spin-chain like systems, this issue however is discussed separately in [13]. Recall the Yang-Baxter equation in the braid form in the presence of spectral parameters  $\lambda_1, \lambda_2$  ( $\delta = \lambda_1 - \lambda_2$ ):

$$(3.1) \quad \check{R}_{12}(\delta) \check{R}_{23}(\lambda_1) \check{R}_{12}(\lambda_2) = \check{R}_{23}(\lambda_2) \check{R}_{12}(\lambda_1) \check{R}_{23}(\delta).$$

where  $\check{R} : V \rightarrow V$ , and let in general  $\check{R} = \sum_j a_j \otimes b_j$ , then in the index notation  $\check{R}_{12} = \sum_j a_j \otimes b_j \otimes I_V$ ,  $\check{R}_{23} = \sum_j I_V \otimes a_j \otimes b_j$  and  $\check{R}_{13} = \sum_j a_j \otimes I_V \otimes b_j$ .

We focus here on Baxterized solutions of (3.1) coming from braces, i.e.

$$(3.2) \quad \check{R}(\lambda) = \lambda \check{r} + \mathbb{I},$$

where  $\mathbb{I} = I_X \otimes I_X$  and  $I_X$  is the identity matrix of dimension equal to the cardinality of the set  $X$ . Indeed, (3.2) satisfies (3.1), provided that  $\check{r}$  satisfies the braid equation and  $\check{r}^2 = I_{X \otimes X}$ . Also, we recall the notation introduced in Lemma 2.1 for the matrix  $\check{r}$  (2.1). Let also,  $R = \mathcal{P}\check{R}$ , then

$$(3.3) \quad R(\lambda) = \lambda r + \mathcal{P},$$

where  $r$  is defined in (2.2), and  $R$  is a solution of the Yang-Baxter equation in the familiar form:

$$(3.4) \quad R_{12}(\delta) R_{13}(\lambda_1) R_{23}(\lambda_2) = R_{23}(\lambda_2) R_{13}(\lambda_1) R_{12}(\delta).$$

**Remark 3.1.** *It would be useful for the following Proposition to introduce the notion of partial transposition. Let  $A \in \text{End}(\mathbb{C}^{\mathcal{N}} \otimes \mathbb{C}^{\mathcal{N}})$  expressed as:  $A = \sum_{i,j,k,l=1}^{\mathcal{N}} A_{ij,kl} e_{i,j} \otimes e_{k,l}$ . We define the partial transposition as follows (in the index notation):*

$$(3.5) \quad A_{12}^{t_1} = \sum_{i,j,k,l=1}^{\mathcal{N}} A_{ij,kl} e_{i,j}^t \otimes e_{k,l}, \quad A_{12}^{t_2} = \sum_{i,j,k,l=1}^{\mathcal{N}} A_{ij,kl} e_{i,j} \otimes e_{k,l}^t$$

where  $e_{i,j}^t = e_{j,i}$ .

**Proposition 3.2.** *The brace  $R$ -matrix satisfies the following fundamental properties:*

$$(3.6) \quad R_{12}(\lambda) R_{21}(-\lambda) = (-\lambda^2 + 1)\mathbb{I}, \quad \text{Unitarity}$$

$$(3.7) \quad R_{12}^{t_1}(\lambda) R_{12}^{t_2}(-\lambda - \mathcal{N}) = \lambda(-\lambda - \mathcal{N})\mathbb{I}, \quad \text{Crossing-unitarity}$$

$$(3.8) \quad R_{12}^{t_1 t_2}(\lambda) = R_{21}(\lambda),$$

where  ${}^{t_1,2}$  denotes transposition on the first, second space respectively.

*Proof.* Recall  $R_{21} = \mathcal{P}R_{12}\mathcal{P}$ , the proof of unitarity is straightforward due to  $\check{r}^2 = \mathcal{P}^2 = \mathbb{I}$ . To prove crossing-unitarity (3.7) it suffices to show the following identities:

$$(3.9) \quad (\mathcal{P}_{12}^{t_1})^2 = \mathcal{N}\mathcal{P}_{12}^{t_1}, \quad r_{12}^{t_1}\mathcal{P}_{12}^{t_1} = \mathcal{P}_{12}^{t_1}r_{12}^{t_2} = \mathcal{P}_{12}^{t_1}, \quad r_{12}^{t_1}r_{12}^{t_2} = \mathbb{I}.$$

The above can be easily shown, from the definitions of  $\mathcal{P} = \sum_{x,y} e_{x,y} \otimes e_{y,x}$  and  $r = \mathcal{P}\check{r}$  (2.2). Given (3.9) the crossing-unitarity immediately follows. The last property (3.8) immediately follows from the definitions of  $R_{12}$ ,  $R_{21}$  and the brace representation.  $\square$

We can now state the obvious connection of the brace representation with the  $A$ -type Hecke algebra.

**Definition 3.3.** *The  $A$ -type Hecke algebra  $\mathcal{H}_N(q)$  is defined by the generators  $g_l$ ,  $l \in \{1, 2, \dots, N-1\}$  and the exchange relations:*

$$(3.10) \quad g_l g_{l+1} g_l = g_{l+1} g_l g_{l+1},$$

$$(3.11) \quad [g_l, g_m] = 0, \quad |l - m| > 1$$

$$(3.12) \quad (g_l - q)(g_l + q^{-1}) = 0.$$

**Remark 3.4.** *The brace solution  $\check{r}$  (2.1) is a representation of the  $A$ -type Hecke algebra for  $q = 1$ .*

*Indeed,  $\check{r}$  satisfies the braid relation and  $\check{r}^2 = \mathbb{I}$ , which can be easily shown by using the involution. We can then define  $g_1 = \check{r} \otimes I^{\otimes N}$  and  $g_i = I^{\otimes i-1} \otimes \check{r} \otimes I^{\otimes N-i-1}$ . Let us also show below the braid relation:*

$$(3.13) \quad (\check{r} \otimes I_X) (I_X \otimes \check{r}) (\check{r} \otimes I_X) = (I_X \otimes \check{r}) (\check{r} \otimes I_X) (\check{r} \otimes I_X).$$

*The LHS of the equation above is equal to*

$$(3.14) \quad \sum e_{x, \sigma_{\bar{x}}(\bar{y})} \otimes e_{y, \tau_{\bar{y}}(\bar{x})} \otimes e_{\hat{y}, \tau_{\hat{y}}(\hat{x})}$$

*provided that  $\hat{x} = \tau_y(x)$ ,  $\bar{x} = \sigma_x(y)$ ,  $\bar{y} = \sigma_{\hat{x}}(\hat{y})$ . We can change  $\hat{y}$  to be denoted by  $z$ , to obtain:*

$$(3.15) \quad \sum e_{x, \sigma_{\bar{x}}(\bar{y})} \otimes e_{y, \tau_{\bar{y}}(\bar{x})} \otimes e_{z, \tau_z(\hat{x})}$$

*provided that  $\hat{x} = \tau_y(x)$ ,  $\bar{x} = \sigma_x(y)$ ,  $\bar{y} = \sigma_{\hat{x}}(z)$ .*

*On the other hand, denote  $r_1(x, y, z) = (\sigma_x(y), \tau_y(x), z)$ ,  $r_2(x, y, z) = (x, \sigma_y(z), \tau_z(y))$ . Notice that, using the same notation as in the expression for the LHS, we obtain*

$$r_1 r_2 r_1(x, y, z) = (\sigma_{\bar{x}}(\bar{y}), \tau_{\bar{y}}(\bar{x}), \tau_z(\hat{x})).$$

*Similarly, the RHS is equal to*

$$(3.16) \quad \sum e_{x, \sigma_x(y)} \otimes e_{x'', \sigma_{x'}(y')} \otimes e_{y'', \tau_{y'}(x')}$$

*provided that  $y = \sigma_{x''}(y'')$ ,  $x' = \tau_y(x)$ ,  $y' = \tau_{y''}(x'')$ .*

*We can denote the variable  $y$  by  $t$  and then change the variable  $x''$  to  $y$ , and variable  $y''$  to  $z$  we obtain that the RHS is equal to*

$$(3.17) \quad \sum e_{x, \sigma_x(t)} \otimes e_{y, \sigma_{x'}(y')} \otimes e_{z, \tau_{y'}(x')}$$



provided that  $t = \sigma_y(z)$ ,  $x' = \tau_t(x)$ ,  $y' = \tau_z(y)$ .

Observe now that using the same notation as in the RHS we obtain

$$r_2 r_1 r_2(x, y, z) = (\sigma_x(t), \sigma_{x'}(y'), \tau_{y'}(x')).$$

Equality between the LHS and RHS expressions is guaranteed for the brace solution and follows from Rump's theorem (Theorem 2.4), since  $r_1 r_2 r_1(x, y, z) = r_2 r_1 r_2(x, y, z)$  by Rump's theorem (because the map  $(x, y) \rightarrow (\sigma_x(y), \tau_y(x))$  satisfies the set-theoretic solution of the Braid equation).

**3.2. Quantum algebras from braces.** Given a solution of the Yang-Baxter equation, an associated quantum algebra can be identified, within the so-called FRT construction [18], via the fundamental relation (we have multiplied the familiar RTT relation by the permutation operator):

$$(3.18) \quad \check{R}_{12}(\lambda_1 - \lambda_2) L_1(\lambda_1) L_2(\lambda_2) = L_1(\lambda_2) L_2(\lambda_1) \check{R}_{12}(\lambda_1 - \lambda_2),$$

where  $\check{R}(\lambda) \in \text{End}(\mathbb{C}^{\mathcal{N}} \otimes \mathbb{C}^{\mathcal{N}})$ ,  $L(\lambda) \in \text{End}(\mathbb{C}^{\mathcal{N}}) \otimes \mathfrak{A}$ , and  $\mathfrak{A}$  is the quantum algebra defined by (3.18). Here we have used the ‘‘index notation’’, i.e. we define<sup>1</sup>

$$(3.19) \quad L_1(\lambda) = \sum_{z,w=1}^{\mathcal{N}} e_{z,w} \otimes I_{\mathcal{N}} \otimes L_{z,w}(\lambda)$$

$$(3.20) \quad L_2(\lambda) = \sum_{z,w=1}^{\mathcal{N}} I_{\mathcal{N}} \otimes e_{z,w} \otimes L_{z,w}(\lambda)$$

$$(3.21) \quad \check{R}_{12} = \check{R} \otimes 1_{\mathfrak{A}}$$

where  $I_{\mathcal{N}}$  is the  $\mathcal{N}$  dimensional identity matrix,  $1_{\mathfrak{A}}$  is the unit element of the algebra  $\mathfrak{A}$  and  $L_{z,w}(\lambda)$  are elements of the affine algebra  $\mathfrak{A}$  (defined by (3.18)). The quantum algebra is equipped with a co-product  $\Delta : \mathfrak{A} \rightarrow \mathfrak{A} \otimes \mathfrak{A}$  [18, 15]. Indeed, we define  $T_{1,23}(\lambda) = L_{13}(\lambda)L_{12}(\lambda)$ , which satisfies (3.18) and is expressed as  $T_{1,23}(\lambda) = \sum_{x,y \in X} e_{x,y} \otimes \Delta(L_{x,y}(\lambda))$ .

We shall focus now on solutions of the Yang-Baxter equation coming from braces and will identify the defining relations of the associated quantum algebra via (3.18).

**Proposition 3.5.** *The quantum algebra associated to the solution  $\check{R} = I_{X \otimes X} + \lambda \check{r}$ , where  $\check{r}$  is the brace solution*

$$\check{r} = \sum_{x,y \in X} e_{x,\sigma_x(y)} \otimes e_{y,\tau_y(x)}$$

is defined by generators  $L_{z,w}^{(m)}$ ,  $z, w \in X$ , and defining relations

$$(3.22) \quad \begin{aligned} L_{z,w}^{(n)} L_{\hat{z},\hat{w}}^{(m)} - L_{z,w}^{(m)} L_{\hat{z},\hat{w}}^{(n)} &= L_{z,\sigma_w(\hat{w})}^{(m)} L_{\hat{z},\tau_{\hat{w}}(w)}^{(n+1)} - L_{z,\sigma_w(\hat{w})}^{(m+1)} L_{\hat{z},\tau_{\hat{w}}(w)}^{(n)} \\ &- L_{\sigma_z(\hat{z}),w}^{(n+1)} L_{\tau_{\hat{z}}(z),\hat{w}}^{(m)} + L_{\sigma_z(\hat{z}),w}^{(n)} L_{\tau_{\hat{z}}(z),\hat{w}}^{(m+1)}. \end{aligned}$$

<sup>1</sup>Notice that in  $L$  in addition to the indices 1 and 2 in (3.18) there is also an implicit ‘‘quantum index’’  $n$  associated to  $\mathfrak{A}$ , which for now is omitted, i.e. one writes  $L_{1n}$ ,  $L_{2n}$ .

*Proof.* We express  $L$  as a formal power series expansion  $L_a(\lambda) = \sum_{n=0}^{\infty} \frac{L_a^{(n)}}{\lambda^n}$ ,  $a \in \{1, 2\}$ . Substituting expression (3.2), and the  $\lambda^{-1}$  expansion of  $L_a(\lambda)$  in (3.18) we obtain the defining relations of the quantum algebra associated to a brace  $R$ -matrix (we focus on terms  $\lambda_1^{-n} \lambda_2^{-m}$ ):

$$(3.23) \quad \begin{aligned} & \check{r}_{12} L_1^{(n+1)} L_2^{(m)} - \check{r}_{12} L_1^{(n)} L_2^{(m+1)} + L_1^{(n)} L_2^{(m)} \\ & = L_1^{(m)} L_2^{(n+1)} \check{r}_{12} - L_1^{(m+1)} L_2^{(n)} \check{r}_{12} + L_1^{(m)} L_2^{(n)}. \end{aligned}$$

Equation (3.23) immediately leads to the quantum algebra relations (3.22), after recalling similarly to (3.19)-(3.21):

$$L_1^{(k)} = \sum_{i,j \in X} e_{i,j} \otimes I_X \otimes L_{i,j}^{(k)}, \quad L_2^{(k)} = \sum_{i,j \in X} I_X \otimes e_{i,j} \otimes L_{i,j}^{(k)},$$

$\check{r}_{12} = \check{r} \otimes 1_{\mathfrak{A}}$ ,  $I_X$  is the identity matrix of dimension equal to the cardinality of the set  $X$ , and  $L_{i,j}^{(k)}$  are the generators of the associated quantum algebra.

Indeed, by substituting the above expressions for  $L_1^{(k)}$  and  $L_2^{(k)}$  and  $\check{r} = \sum_{i,j \in X} e_{i,i_j} \otimes e_{j,i}$  in (3.23) and computing both sides we obtain:

$$\sum_{x,j,y,i \in X} e_{x,j} \otimes e_{y,i} \otimes Q_{x,j,y,i}^{(m,n)} = \sum_{x,j,y,i \in X} e_{x,j} \otimes e_{y,i} \otimes P_{x,j,y,i}^{(m,n)}$$

where

$$Q_{x,j,y,i}^{(m,n)} = L_{x,y,j}^{(n+1)} L_{x^y,i}^{(m)} - L_{x,y,j}^{(n)} L_{x^y,i}^{(m+1)} + L_{x,j}^{(n)} L_{y,i}^{(m)},$$

$$P_{x,j,y,i}^{(m,n)} = L_{x,j}^{(m)} L_{y,j^i}^{(n+1)} - L_{x,j}^{(m+1)} L_{y,j^i}^{(n)} + L_{x,j}^{(m)} L_{y,i}^{(n)}.$$

Notice that  $Q_{x,j,y,l}^{(m,n)} = P_{x,j,y,l}^{(m,n)}$  are the defining relations in our quantum algebra.  $\square$

**Definition 3.6.** Let  $(X, \check{r})$  be a set-theoretic solution of the Yang-Baxter equation, with  $\check{r}(x, y) = (x^y, x^y)$ . The quantum algebra associated to the brace  $\check{R}$  matrix  $\check{R}(\lambda) = I_{X \otimes X} + \lambda \check{r}$  is defined by generators  $L_{z,w}^{(m)}$ ,  $z, w \in X$ ,  $m = 0, 1, 2, \dots$  and defining relations

$$\begin{aligned} & L_{x,y,j}^{(n+1)} L_{x^y,i}^{(m)} - L_{x,y,j}^{(n)} L_{x^y,i}^{(m+1)} + L_{x,j}^{(n)} L_{y,i}^{(m)} = \\ & L_{x,j}^{(m)} L_{y,j^i}^{(n+1)} - L_{x,j}^{(m+1)} L_{y,j^i}^{(n)} + L_{x,j}^{(m)} L_{y,i}^{(n)} \end{aligned}$$

for  $x, j, y, i \in X$ . This algebra will be denoted as  $\mathfrak{A}(X, \check{r})$ .

This is the same algebra as in Proposition 3.5.

In this part we recall some basic notions about the  $\mathfrak{gl}_{\mathcal{N}}$  algebra and the Yangian  $\mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$ .

**Definition 3.7.** The  $\mathfrak{gl}_{\mathcal{N}}$  algebra is a Lie algebra (over  $\mathbb{C}$ ) with generators denoted as  $\mathfrak{l}_{i,j}$  for  $i, j \in \{1, \dots, \mathcal{N}\}$ , that satisfy:

$$(3.24) \quad [\mathfrak{l}_{i,j}, \mathfrak{l}_{k,l}] = \mathfrak{l}_{i,l} \delta_{k,j} - \mathfrak{l}_{k,j} \delta_{i,l}.$$

**Remark 3.8.** Recall that  $e_{x,y}$  are  $\mathcal{N} \times \mathcal{N}$  matrices with elements  $(e_{x,y})_{z,w} = \delta_{x,z} \delta_{y,w}$ ,  $x, y \in \{1, \dots, \mathcal{N}\}$ . We call the  $\mathcal{N}$  dimensional representation  $\rho : \mathfrak{gl}_{\mathcal{N}} \rightarrow \text{End}(\mathbb{C}^{\mathcal{N}})$ , such that  $\iota_{x,y} \mapsto e_{x,y}$  the fundamental representation of  $\mathfrak{gl}_{\mathcal{N}}$ , i.e. the matrices  $e_{x,y}$  are the generators of  $\mathfrak{gl}_{\mathcal{N}}$  in the fundamental ( $\mathcal{N}$ -dimensional) representation.

The  $\mathfrak{gl}_{\mathcal{N}}$  algebra is equipped with a coproduct  $\Delta : \mathfrak{gl}_{\mathcal{N}} \rightarrow \mathfrak{gl}_{\mathcal{N}} \otimes \mathfrak{gl}_{\mathcal{N}}$ , such that

$$(3.25) \quad \Delta(\iota_{i,j}) = 1 \otimes \iota_{i,j} + \iota_{i,j} \otimes 1, \quad \forall \iota_{i,j} \in \mathfrak{gl}_{\mathcal{N}}.$$

The  $N$ -coproduct is obtained by iteration  $\Delta^{(N)} = (\text{id} \otimes \Delta^{(N-1)})\Delta = (\Delta^{(N-1)} \otimes \text{id})\Delta$  (co-associativity holds):

$$(3.26) \quad \Delta^{(N)}(\iota_{i,j}) := \sum_{n=1}^N (\iota_{i,j})_n = \sum_{n=1}^N 1 \otimes \dots \otimes \underbrace{\iota_{i,j}}_{n^{\text{th}} \text{ position}} \otimes \dots \otimes 1.$$

The element  $(\iota_{i,j})_n$ , in the standard *index notation* above, appears in the  $n^{\text{th}}$  position of the  $N$  co-product, also 1 is the unit element of the algebra and we set  $\Delta^{(2)}(Y) := \Delta(Y)$ . The elements  $\Delta^{(N)}(e_{i,j})$  are tensor representations of  $\mathfrak{gl}_{\mathcal{N}}$ , i.e.  $\mathfrak{gl}_{\mathcal{N}} \rightarrow \text{End}((\mathbb{C}^{\mathcal{N}})^{\otimes N})$ , such that  $\iota_{i,j} \mapsto \Delta^{(N)}(e_{i,j})$ .

We recall two well known results about Yangians. For more information on Yangians we refer the interested reader to [15, 43]. The Yangian for  $\mathfrak{gl}_{\mathcal{N}}$  will be denoted as  $\mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$ .

**Definition 3.9.** The Yangian  $\mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$  is an algebra over  $\mathbb{C}$  (an infinite extension of  $\mathfrak{gl}_{\mathcal{N}}$ ) with generators denoted as  $L_{i,j}^{(n)}$ , for  $i, j \in \{1, 2, \dots, \mathcal{N}\}$ ,  $n \in \mathbb{N}$ , ( $L_{i,j}^{(0)} = \delta_{i,j}$ ) that satisfy the defining relations:

$$(3.27) \quad \left[ L_{i,j}^{(n+1)}, L_{k,l}^{(m)} \right] - \left[ L_{i,j}^{(n)}, L_{k,l}^{(m+1)} \right] = L_{k,j}^{(m)} L_{i,l}^{(n)} - L_{k,j}^{(n)} L_{i,l}^{(m)}.$$

The latter relations are defined up to an overall multiplicative constant that can be absorbed by rescaling the generators.

**Corollary 3.10.** In the special case  $\check{r} = \mathcal{P}$  the  $\mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$  algebra is recovered as the corresponding quantum algebra from (3.23).

*Proof.* We consider the special case where  $\check{r} = \mathcal{P}$  in (3.23), which corresponds to the Yangian  $\mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$ . Recall that we express  $L$  as a formal power series expansion  $L(\lambda) = I_X \otimes 1 + \sum_{n=1}^{\infty} \frac{L^{(n)}}{\lambda^n}$  ( $L^{(0)} = I_X \otimes 1$ ). Then the fundamental relation (3.23) leads to:

$$(3.28) \quad \left[ L_1^{(n+1)}, L_2^{(m)} \right] - \left[ L_1^{(n)}, L_2^{(m+1)} \right] = \mathcal{P}_{12} \left( L_1^{(m)} L_2^{(n)} - L_1^{(n)} L_2^{(m)} \right).$$

Recalling that  $L_1^{(n)} = \sum_{x,y \in X} e_{x,y} \otimes I \otimes L_{x,y}^{(n)}$ ,  $L_2^{(n)} = \sum_{x,y \in X} I \otimes e_{x,y} \otimes L_{x,y}^{(n)}$ , and  $\mathcal{P}_{12} = \sum e_{ij} \otimes e_{ji} \otimes 1$ , we arrive at (3.27) (see also [15]).  $\square$

**Corollary 3.11.** In the case of the Yangian  $\mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$  the finite dimensional subalgebra  $\mathfrak{gl}_{\mathcal{N}}$  emerges from (3.28), and is realized by the elements of  $L^{(1)}$ .

*Proof.* Considering (3.28) for  $n = 0$  and  $m = 1$  we obtain

$$(3.29) \quad \left[ L_1^{(1)}, L_2^{(1)} \right] = \mathcal{P}_{12} \left( L_1^{(1)} - L_2^{(1)} \right).$$

From (3.29) we deduce that the elements  $L_{i,j}^{(1)}$  satisfy the defining relations of  $\mathfrak{gl}_{\mathcal{N}}$ .  $\square$

Note that the choice  $L^{(0)} = I_X \otimes 1$  in the case of the Yangian is compatible with the fact that  $L(\lambda) = \lambda I_X \otimes 1 + \mathfrak{L}$ , where  $\mathfrak{L} = \sum_{i,j=1}^{\mathcal{N}} e_{i,j} \otimes \mathfrak{l}_{i,j}$  and  $\mathfrak{l}_{i,j}$  are the generators of  $\mathfrak{gl}_{\mathcal{N}}$  (Definition 3.7), provides a realization of  $\mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$ , [18, 43] (see also next section, comments in the proof of Corollary 4.3 on tensor representations of the quantum algebra in the special case of Yangian).

After the brief “interlude” regarding the Yangian case we return to quantum algebras associated to general brace solutions.

**Proposition 3.12.** *Let  $(X, \check{r})$  and  $(Y, \check{r}')$  be set-theoretic solutions of the Braid equation. If  $f : X \rightarrow Y$  is a surjective homomorphism of solutions  $(X, \check{r})$  and  $(Y, \check{r}')$ , then the map  $\mathfrak{A}(X, \check{r}) \rightarrow \mathfrak{A}(Y, \check{r}')$ , determined by*

$$L_{x,y}^{(k)} \mapsto L_{f(x),f(y)}^{(k)}$$

*is a homomorphism of algebras.*

*Proof.* We can verify that this function maps the defining relations of the quantum algebra  $\mathfrak{A}(X, \check{r})$  onto defining relations of  $\mathfrak{A}(Y, \check{r}')$ .  $\square$

By a representation of an algebra  $A$  we will mean a factor algebra  $A/I$  where  $I$  is an ideal in the algebra  $A$ .

**Proposition 3.13.** *Let  $B$  be a brace,  $X$  be a subset of  $B$  and  $(X, \check{r})$  be an involutive solution of the Braid relation obtained from this brace as in Lemma 2.7. Let  $J$  be an ideal of the brace  $B$ . Let  $(X_J, \check{r}_J)$  be the solution associated to the brace  $B/J$  on the subset  $X_J$  of the factor brace  $B/J$ . Then the algebra  $\mathfrak{A}(X_J, \check{r}_J)$  is a representation of the algebra  $\mathfrak{A}(X, \check{r})$ .*

*Proof.* Let  $A$  be the free algebra with generators  $u_{i,j}^{(n)}$ , then we can define homomorphism of algebras  $f : A \rightarrow \mathfrak{A}(X, \check{r})$  by  $f(u_{i,j}^{(n)}) = L_{i,j}^{(n)}$  and let  $I$  be the kernel of this map. By the First Isomorphism Theorem  $A/I$  is isomorphic as algebra to  $\mathfrak{A}(X, \check{r})$ . We can also define homomorphism of algebras  $g : A \rightarrow \mathfrak{A}(X_J, \check{r}_J)$  by  $g(u_{i,j}^{(n)}) = L_{i+J,j+J}^{(n)}$  and let  $T$  be the kernel of  $g$ . Observe that  $I \subseteq T$ . By the First Isomorphism Theorem  $A/T$  is isomorphic as algebra to  $\mathfrak{A}(X_J, \check{r}_J)$ . By the Second Isomorphism Theorem for rings the factor algebra  $(A/I)/(T/I)$  is isomorphic to  $A/T$ , therefore  $\mathfrak{A}(X_J, \check{r}_J)$  is a representation  $\mathfrak{A}(X, \check{r})$ .  $\square$

We call a solution  $(Y, \check{r})$  of the Braid equation trivial if and only if  $\check{r}(x, y) = (y, x)$  for all  $x, y \in Y$ . In this case we may denote  $\check{r}$  as  $\tau$ .

**Proposition 3.14.** *A set-theoretic solution  $(X, \check{r})$  which can be homomorphically mapped onto a trivial solution  $(Y, \tau)$  will have  $\mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$  as a representation of its algebra  $\mathfrak{A}(X, \check{r})$ , where  $\mathcal{N}$  is the cardinality of  $Y$ .*

*Proof.* The trivial solution  $(Y, \tau)$  of cardinality  $\mathcal{N}$  has  $\mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$  as its quantum group by Corollary 3.10. By Proposition 3.13, a solution  $(X, \check{r})$  which can be homomorphically mapped onto a trivial solution  $(Y, \tau)$  will have  $\mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$  as a representation of its algebra  $\mathfrak{A}(X, \check{r})$ .  $\square$

Recall that if  $(X, \check{r})$  is a non-degenerate, involutive set-theoretic solution of the Braid equation, and  $Y \subseteq X$ ,  $e \in Y$ , then  $Y$  is an orbit of  $e$  if for  $x \in X, y \in Y$  we have  $\sigma_x(y) \in Y$  and  $\tau_x(y) \in Y$  and  $Y$  is the smallest set with this property.

**Corollary 3.15.** *If  $(X, \check{r})$  is an involutive, non-degenerate solution with  $\mathcal{N}$  orbits, then  $\mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$  is a representation of the algebra  $\mathfrak{A}(X, \check{r})$ .*

*Proof.* We can map  $(X, \check{r})$  onto a trivial solution by mapping each element on its orbit. It is easy to check that this map is a homomorphism of set-theoretic solutions. The result now follows from Proposition 3.14.  $\square$

Some basic information about orbits and examples of orbits can be found in [48], page 90 (just above section 2.2). Solutions which have only one orbit are called indecomposable solutions. Indecomposable involutive non-degenerate set-theoretic solutions of the Yang-Baxter equation can be constructed using one-generator braces [48].

Let  $m \in \mathbb{Z}^+$  and  $X_m = \{1, 2, \dots, m\}$ , define also  $\check{r}_m(i, j) = (j + 1, i - 1)$  where addition and subtraction are taken modulo  $m$ . This is a special type of Lyubashenko's solution [14]. We define  $\mathfrak{A}_m$  to be the algebra  $\mathfrak{A}(X_m, \check{r}_m)$ . The following shows that Lyubashenko's solutions are useful for constructing representations of algebras constructed from braces.

**Proposition 3.16.** *Let  $(X, \check{r})$  be a finite, indecomposable, involutive and non-degenerate set theoretic solution of a finite multipermutation level. Suppose that  $\sigma_x(z) \neq \sigma_y(z)$  for some  $x, y, z \in X$ . Then the algebra  $\mathfrak{A}(X, \check{r})$  associated to the solution  $(X, \check{r})$  can be mapped onto the algebra  $\mathfrak{A}_m$  for some  $m > 1$ .*

*Proof.* Let  $(X_{ret}, \check{r}_{ret})$  be the retraction of  $(X, \check{r})$ . Notice that the retraction of  $(X, \check{r})$  has more than one element, since  $\sigma_x(z) \neq \sigma_y(z)$  for some  $x, y, z \in X$ . Notice that  $(X_{ret}, \check{r}_{ret})$  satisfies the assumptions of Proposition 7.1 from [48], hence it can be mapped onto solution  $(X_m, \check{r}_m)$  for some  $m > 1$ .  $\square$

Proposition 3.16 suggests the following question:

**Question 1.** Investigate representations of the algebra  $\mathfrak{A}_m$  which is associated to Lyubashenko's solution.

**3.3. Representations of algebras.** We start this subsection with the following observation.

**Proposition 3.17.** *Let  $(X, \check{r})$  be a set theoretic solution of the braid equation and let  $A_{(X, \check{r})}$  be an arbitrary associative algebra generated by elements from the set  $X$  and satisfying relations  $xy = uv$  whenever  $\check{r}(x, y) = (u, v)$ . The following holds:*

- (1)  $A_{(X, \check{r})}$  is a representation of the algebra  $\mathfrak{A}(X, \check{r})$  when we map  $L_{x,y}^{(n)}$  to  $x \in A_{(X, \check{r})}$  for every  $n$ .
- (2)  $A_{(X, \check{r})} \otimes A_{(X, \check{r})}$  is a representation of the algebra  $\mathfrak{A}(X, \check{r})$  when we map  $L_{x,y}^{(n)}$  to  $x \otimes y \in A_{(X, \check{r})} \otimes A_{(X, \check{r})}$  for every  $n$ .
- (3) Let  $R$  be an arbitrary commutative associative algebra over the field  $\mathbb{C}$  generated by elements  $c_1, c_2, \dots$ . Then  $A_{(X, \check{r})} \otimes R$  is a representation of the algebra  $\mathfrak{A}(X, \check{r})$  when we map  $L_{x,y}^{(n)}$  to  $x \otimes c_n \in A_{(X, \check{r})} \otimes R$  for every  $n$ .
- (4) In the above point 3 if  $0 = c_2 = c_3 = \dots$  then we obtain a representation satisfying  $L(\lambda) = L_0 + \lambda L_1$  (where  $L_1, L_0$  are independent of  $\lambda$ ).

*Proof.* The proof is by verifying that the algebra relations will go to zero after applying the above homomorphisms of algebras (related to the above representations).  $\square$

*Examples of algebras  $A_{(X, \check{r})}$ .* Let  $(X, \check{r})$  be an indecomposable, involutive set-theoretic solution of the Yang-Baxter equation. Various types of algebras belonging to the class  $A_{(X, \check{r})}$  were investigated extensively by several authors, and a lot is known about them [33, 34]. *Quantum binomial algebras* were introduced and investigated by Gateva-Ivanova in [20], [22] and [21]. *The monomial algebras of I type* were investigated for involutive solutions in [25], [32], [35], [36], and recently for both involutive and non-involutive solutions in [23] and [30]. *The structure algebras of set-theoretic solutions.* The algebra generated by the set  $X$  and with defining relations  $xy = uv$  if  $\check{r}(x, y) = (u, v)$  was investigated in [30], where in section 5 they study prime ideals in such algebras and hence representations of such algebras which are prime (they showed that under mild assumptions they correspond to prime ideals of a group algebra associated to the same set-theoretic solution). For involutive solutions such algebras were previously investigated in [25, 32, 35, 36]. Let  $G$  be a permutation group of a finite, non-degenerate, involutive set-theoretic solution  $(X, \check{r})$  of the Braid equation. Then  $A_{(X, \check{r})}$  can be taken to be *the group algebra*  $\mathbb{C}[G]$ . Such algebras were investigated in [16].

*Research directions and open questions.*

*Question 1.* Let  $(X, \check{r})$  be an involutive, non-degenerate set-theoretic solution. Does the algebra  $\mathfrak{A}_{(X, \check{r})}$  have a finite Gröbner basis?

*Question 2.* What can be said about the algebra  $\mathfrak{A}_{(X, \check{r})}$  associated to a solution  $(X, \check{r})$  of a finite multipermutation level? Or of an indecomposable solution? Does  $\mathfrak{A}_{(X, \check{r})}$  have any representations of small dimensions?

#### 4. NOVEL CLASS OF QUANTUM INTEGRABLE SYSTEMS & ASSOCIATED SYMMETRIES

In this section we introduce physical spin-chain like systems, with periodic boundary conditions, associated to braces, and we investigate the corresponding symmetries. Specifically, in the next subsection we provide the general setup for constructing integrable quantum spin chains via tensor representations of quantum algebras [18]. The transfer matrix, which will be defined in subsection 4.1, is the generating function of

a family of mutually commuting quantities, which guarantee in principle the quantum integrability of the spin-chain system. For instance, the momentum and Hamiltonian of the system belong to this family of commuting quantities.

In subsection 4.2 we focus on integrable systems constructed from brace solutions of the Yang-Baxter equation and we investigate the existence of classes of symmetries of the corresponding periodic transfer matrix. When we say symmetries of the transfer matrix we mean families of objects ( $\mathcal{N}^N \times \mathcal{N}^N$  matrices in the cases examined here), that commute with the transfer matrix and of course they do not belong to the family of mutually commuting quantities. Interestingly some of these symmetries consist of objects that form certain non-commutative algebras, as will be clear in subsection 4.2. The properties of the braces play a significant role when identifying the new classes of symmetries. Note that the knowledge of symmetries provides invaluable information regarding for instance the multiplicities of the spectrum of the transfer matrix, and this in turn has significant implications on the physical behavior of the system at hand.

Before we derive various new families of symmetries of the transfer matrix we first prove one of the most important propositions of the present investigation. Specifically, we show that the periodic transfer matrix constructed from Baxterized solutions of the  $A$ -type Hecke algebra  $\mathcal{H}_N(q = 1)$  can be exclusively expressed in terms of the generators of the  $A$ -type Hecke algebras plus some periodic term. This is a universal result that holds for any representation of the  $A$ -type Hecke algebra  $\mathcal{H}_N(q = 1)$ .

**4.1. Tensor representations of quantum algebras & integrable systems.** Given any solution of the Yang-Baxter equation we define the so-called monodromy matrix  $T_{0,12\dots N}(\lambda) \in \text{End}(\mathbb{C}^N \otimes (\mathbb{C}^N)^{\otimes N})$ , which is a tensor representation of the quantum group (3.18), [18]

$$(4.1) \quad T_{0,12\dots N}(\lambda) = R_{0N}(\lambda) \dots R_{02}(\lambda) R_{01}(\lambda),$$

recall  $R = \mathcal{P}\tilde{R}$ , (e.g. in the case of brace solutions  $\tilde{R}$  is given by (3.3), (2.2)). We define also the transfer matrix  $\mathfrak{t}_{12\dots N}(\lambda) = \text{tr}_0(T_{0,12\dots N}(\lambda)) \in \text{End}((\mathbb{C}^N)^{\otimes N})$ . The monodromy matrix  $T$  satisfies (3.18), and hence one can show that the transfer matrix provides mutually commuting quantities [18]:  $(\mathfrak{t}(\lambda) = \lambda^N \sum_k \frac{\mathfrak{t}^{(k)}}{\lambda^k})$

$$(4.2) \quad [\mathfrak{t}(\lambda), \mathfrak{t}(\mu)] = 0 \Rightarrow [\mathfrak{t}^{(k)}, \mathfrak{t}^{(l)}] = 0.$$

Note that historically the index 0 is called ‘‘auxiliary’’, whereas the indices  $1, 2, \dots, N$  are called ‘‘quantum’’, and they are usually suppressed for simplicity, as is also done in the next subsection, i.e. we simply write  $T_0(\lambda)$  and  $\mathfrak{t}(\lambda)$ .

The ultimate goal in the context of quantum integrable systems, or any quantum system for that matter, is the identification of the eigenvalues and eigenvectors of the corresponding Hamiltonian. In the frame of quantum integrable systems more specifically there exists a set of mutually commuting ‘‘Hamiltonians’’, guaranteed by the existence of a quantum  $R$ -matrix that satisfies the Yang-Baxter equation. As already discussed this set of mutually commuting objects is generated by the transfer

matrix. Thus the derivation of the eigenvalues and eigenstates of the transfer matrix is the significant problem within quantum integrability. This is in general an intricate task and the typical methodology used is the Bethe ansatz formulation, or suitable generalizations, depending on the problem at hand. A detailed study of this problem for transfer matrices associated to brace solutions will be presented elsewhere.

Here we are focusing primarily on the investigation of possible existing new symmetries of the periodic transfer matrix, as any information regarding the symmetries of the transfer matrix provides for instance valuable insight on the multiplicities occurring in the spectrum. It will be transparent in what follows that the study of the symmetries of the  $R$ -matrices is a first step towards formulating the symmetries of the transfer matrix. A detailed analysis on more generic symmetry algebras and boundary conditions is presented in [13].

**4.2. Symmetries of the periodic transfer matrix of novel classes of spin chains.** The main objective in this subsection is the investigation of the symmetries of the periodic transfer matrix for quantum spin chains constructed for brace solutions of the Yang-Baxter equation. We first present one key proposition, which will have significant implications when studying the symmetries of the period transfer matrix. This result becomes even more prominent when integrable boundary conditions are implemented to integrable systems, especially those coming from braces [13].

The following Proposition 4.1 and Lemma 4.2 are quite general and hold for any  $R(\lambda) = \lambda\mathcal{P}\check{r} + \mathcal{P}$ , where  $\check{r}$  provides a representation of the  $A$ -type Hecke algebra  $\mathcal{H}_N(q = 1)$ , i.e. satisfies the braid relation and  $\check{r}^2 = \mathbb{I}$ , and  $\mathcal{P}$  is the permutation operator.

**Proposition 4.1.** *Consider the  $\lambda$ -series expansion of the monodromy matrix:  $T(\lambda) = \lambda^N \sum_{k=0}^N \frac{T^{(k)}}{\lambda^k}$  for any  $R(\lambda) = \lambda\mathcal{P}\check{r} + \mathcal{P}$ , where  $\check{r}$  provides a representation of the  $A$ -type Hecke algebra  $\mathcal{H}_N(q = 1)$ . Let also  $H^{(k)} = \mathfrak{t}^{(k)}(\mathfrak{t}^{(N)})^{-1}$ ,  $k = 0, \dots, N - 1$  and  $H^{(N)} = \mathfrak{t}^{(N)}$ , where  $\mathfrak{t}^{(k)} = Tr_0(T_0^{(k)})$ . Then the commuting quantities,  $H^{(k)}$  for  $k = 1, \dots, N - 1$ , are expressed exclusively in terms of the elements  $\check{r}_{nn+1}$ ,  $n = 1, \dots, N - 1$ , and  $\check{r}_{N1}$ .*

*Proof.* Let us introduce some useful notation. We define, for  $0 \leq m < n \leq N + 1$ :  $\mathbb{P}_{n-1;m+1} = \mathcal{P}_{0n-1} \dots \mathcal{P}_{0m+1}$ ,  $n > m + 2$ ,  $\mathbb{P}_{n-1;n} = \text{id}$ ,  $\mathbb{P}_{n;n} = \mathcal{P}_{0n}$  and for  $1 \leq n \leq N$ :  $\Pi = \mathcal{P}_{12} \mathcal{P}_{23} \dots \mathcal{P}_{N-1N}$ ,  $\check{\mathfrak{R}}_{n;m} = \check{r}_{n-1n} \check{r}_{n-2n-1} \dots \check{r}_{mm+1}$ ,  $n > m + 1$ ,  $\check{\mathfrak{R}}_{n;n} = \text{id}$ ,  $\check{\mathfrak{R}}_{n+1;n} = \check{r}_{nn+1}$ . Note that  $\log(\Pi)$  is the momentum operator of the system. Let us also define the ordered product:

$$\prod_{1 \leq j \leq k}^{\leftarrow} \check{r}_{n_j n_{j+1}} = \check{r}_{n_1 n_1+1} \check{r}_{n_2 n_2+1} \dots \check{r}_{n_k n_k+1} : \quad n_1 > n_2 > \dots > n_k.$$



We compute all the members of the expansion of the monodromy  $T^{(k)}$ , using the notation introduced above and the definition (4.1):

$$\begin{aligned}
T_0^{(N)} &= \mathbb{P}_{N;1} = \mathcal{P}_{01}\Pi, \\
T_0^{(N-1)} &= \sum_{n=1}^N \mathbb{P}_{N;n+1} r_{0n} \mathbb{P}_{n-1;1} = \left( \sum_{n=1}^{N-1} \check{r}_{n\ n+1} + \check{r}_{N0} \right) \mathcal{P}_{01}\Pi, \quad \dots \\
T_0^{(N-k)} &= \sum_{1 \leq n_k < \dots < n_1 \leq N} \prod_{j=1}^k \mathbb{P}_{n_{j-1}-1; n_j+1} r_{0n_j} \mathbb{P}_{n_k-1;1} = \\
&\left( \sum_{1 \leq n_k < \dots < n_1 < N} \prod_{1 \leq j \leq k} \check{r}_{n_j\ n_{j+1}} + \sum_{1 \leq n_k < \dots < n_2 < N} \prod_{2 \leq j \leq k} \check{r}_{N0} \check{r}_{n_j\ n_{j+1}} \right) \mathcal{P}_{01}\Pi, \quad \dots \\
T_0^{(1)} &= \left( \sum_{n=1}^{N-1} \check{r}_{N0} \check{\mathfrak{A}}_{N;n+1} \check{\mathfrak{A}}_{n;1} + \check{\mathfrak{A}}_{N;1} \right) \mathcal{P}_{01}\Pi \\
T_0^{(0)} &= \mathbb{R}_{N;1} = \check{r}_{N0} \check{\mathfrak{A}}_{N;1} \mathcal{P}_{01}\Pi = \mathbb{P}_{N;1}.
\end{aligned}$$

Recall that we can express the monodromy matrix and consequently all  $T^{(k)}$  in a block form, i.e.  $T^{(k)} = \sum_{x,y \in X} e_{x,y} \otimes T_{x,y}^{(k)}$ , thus  $\mathfrak{t}^{(k)} = \text{tr}_0(T_0^{(k)}) = \sum_{x \in X} T_{x,x}^{(k)}$ .  $\mathfrak{t}^{(k)}$  commute among each other, and hence any combination of them also provides a family of mutually commuting quantities. For instance, we consider the following convenient combination:  $H^{(k)} = \mathfrak{t}^{(k)}(\mathfrak{t}^{(N)})^{-1}$ ,  $k = 1, \dots, N-1$  and  $H^{(N)} = \mathfrak{t}^{(N)} = \Pi$ , then (periodicity is naturally imposed after taking the trace,  $N+1 \equiv 1$ ):

$$\begin{aligned}
H^{(N-1)} &= \sum_{n=1}^N \check{r}_{n\ n+1}, \\
H^{(N-2)} &= \sum_{1 \leq m < n \leq N} \check{r}_{n\ n+1} \check{r}_{m\ m+1} + \sum_{n=1}^{N-2} \check{r}_{n\ n+1} \check{r}_{N1} + \check{r}_{N1} \check{r}_{N-1\ N}, \quad \dots \\
H^{(N-k)} &= \sum_{1 \leq n_k < \dots < n_1 < N} \prod_{1 \leq j \leq k} \check{r}_{n_j\ n_{j+1}} + \sum_{1 \leq n_k < \dots < n_2 < N-1} \prod_{2 \leq j \leq k} \check{r}_{n_j\ n_{j+1}} \check{r}_{N1} \\
&+ \sum_{1 \leq n_k < \dots < n_2 = N-1} \prod_{l+1 \leq j \leq k} \check{r}_{n_j\ n_{j+1}} \check{r}_{N1} \prod_{2 \leq j \leq l} \check{r}_{n_j\ n_{j+1}} \Big|_{c_j=0, c_l > 0}, \quad \dots \\
H^{(1)} &= \sum_{n=1}^{N-1} \check{\mathfrak{A}}_{n;1} \check{r}_{N1} \check{\mathfrak{A}}_{N;n+1} + \check{\mathfrak{A}}_{N;1},
\end{aligned}$$

where we define  $c_j = n_j - n_{j+1} - 1$ . Indeed, the Hamiltonians  $H^{(k)}$  for  $k = 1, \dots, N-1$  are expressed solely in terms of the elements  $\check{r}_{n\ n+1}$  and  $\check{r}_{N1}$  and this essentially concludes our proof. Let us also for the sake of completeness report  $H^{(0)}$ , which takes the simple form  $H^{(0)} = \text{tr}_0(\check{r}_{N0} \check{\mathfrak{A}}_{N;1} \mathcal{P}_{01})$ , but as opposed to the rest of the commuting Hamiltonians  $H^{(k)}$ ,  $k = 1, \dots, N-1$ , it can not be expressed only in terms of  $\check{r}_{n\ n+1}$  and  $\check{r}_{N1}$ .

It will be also instructive for our purposes here, related especially with Proposition 4.11 (presented later in the text), to compute explicitly  $T^{(0)}$  and  $\mathfrak{t}^{(0)}$ . Recalling expression (4.3) and the form of the brace solution (2.1) we have:

$$(4.3) \quad T^{(0)} = \sum_{x_1, \dots, x_N, y_1, \dots, y_N \in X} e_{y_N, \sigma_{x_1}(y_1)} \otimes e_{x_1, \tau_{y_1}(x_1)} \otimes \dots \otimes e_{x_N, \tau_{y_N}(x_N)}$$

and by taking the trace (periodic boundary conditions:  $N + 1 \equiv 1$ )

$$(4.4) \quad \mathfrak{t}^{(0)} = \sum_{x_1, \dots, x_N, y_1, \dots, y_N \in X} e_{x_1, \tau_{y_1}(x_1)} \otimes \dots \otimes e_{x_N, \tau_{y_N}(x_N)},$$

where both expressions above are subject to the constraints:  $y_n = \sigma_{x_{n+1}}(y_{n+1})$ .  $\square$

We show below an interesting property regarding the element  $\check{\mathfrak{R}}_{N-1;1}$  introduced in the proof of the latter Proposition (see also relevant findings in connection to Murphy elements in Hecke algebras in [11]).

**Lemma 4.2.** *The action of  $\check{\mathfrak{R}}_{N;1} = \check{r}_{N-1N} \check{r}_{N-2N-1} \dots \check{r}_{12}$  on the elements of the  $A$ -type Hecke algebra is given by*

$$\begin{aligned} \check{\mathfrak{R}}_{N;1} \check{r}_{nn+1} &= \check{r}_{n-1n} \check{\mathfrak{R}}_{N;1}, \quad \forall n \in \{2, \dots, N-2\} \\ \check{\mathfrak{R}}_{N;1} \check{r}_{12} &= \check{\mathfrak{R}}_{N;2}, \quad \check{r}_{N-1N} \check{\mathfrak{R}}_{N;1} = \check{\mathfrak{R}}_{N-1;1} \end{aligned}$$

*Proof.* The poof is straightforward via the use of the braid relation

$$\check{r}_{n+1} \check{r}_{n+1n+2} \check{r}_{nn+1} = \check{r}_{n+1n+2} \check{r}_{nn+1} \check{r}_{n+1n+2}, \quad \check{r}^2 = I \otimes I, \text{ and the form of } \check{\mathfrak{R}}_{N;1}. \quad \square$$

We showed that the Hamiltonians  $H^{(k)}$ ,  $k = 1, \dots, N-1$  introduced in Proposition 4.1 are expressed exclusively in terms of the  $A$ -type Hecke elements and the periodic element  $\check{r}_{N1}$ . This fact will be exploited later when investigating the symmetries of the conserved quantities for various brace solutions. In the special case where  $\check{r} = \mathcal{P}$ , i.e. the Yangian (see also Corollary 4.3 below), the Hamiltonian is  $\mathfrak{gl}_N$  symmetric (see also (3.26)). However, if we focus on the more general brace solution we conclude that there is no non-commutative algebra as symmetry of the Hamiltonian or in general of the transfer matrix, with the exception of certain special cases that will be examined later in the text. Note that in [13] the existence of a non-commutative algebra that is also a symmetry of the open boundary Hamiltonian is shown (see also [12] and references therein for relevant findings).

The notion of the so-called Murphy elements associated to Hecke algebras, emerging from open boundary transfer matrices [11], is also discussed in [13] for  $R$ -matrices that come from braces. Moreover, for a special class of set theoretical solutions and for a special choice of boundary conditions it is shown [13], that not only the corresponding Hamiltonian is  $\mathfrak{gl}_N$  symmetric, but also the boundary transfer matrix.

Let us recall in the next corollary the known result about the existence of a non-commutative algebra that is a symmetry of the transfer matrix in the Yangian case.

**Corollary 4.3.** *In the case of Yangian ( $\check{r} = \mathcal{P}$ ),  $\mathfrak{gl}_{\mathcal{N}}$  is a symmetry of the periodic transfer matrix, i.e.*

$$(4.5) \quad \left[ \Delta^{(N)}(e_{i,j}), \mathfrak{t}(\lambda) \right] = 0, \quad i, j \in \{1, 2, \dots, \mathcal{N}\}.$$

*Proof.* The monodromy matrix satisfies the RTT relation (3.18), also in this case  $r = I \otimes I$ , then  $T^{(0)} = I^{\otimes(N+1)}$  (recall the expansion  $T(\lambda) = \sum_n \lambda^{-n} T^{(n)}$ ), and from (3.27) for  $n = 1$  it follows that

$$(4.6) \quad \left[ T_1^{(1)}, T_2^{(m)} \right] = T_2^{(m)} \mathcal{P}_{12} - \mathcal{P}_{12} T_2^{(m)}.$$

By taking the trace over the second space we conclude

$$(4.7) \quad \left[ T_1^{(1)}, \mathfrak{t}^{(m)} \right] = 0 \Rightarrow \left[ T_{i,j}^{(1)}, \mathfrak{t}(\lambda) \right] = 0.$$

$T_{i,j}^{(1)} \in \text{End}((\mathbb{C}^{\mathcal{N}})^{\otimes N})$  are the entries of the  $T$  matrix, and are tensor representations of the  $\mathfrak{gl}_{\mathcal{N}}$  algebra (3.10),

$$(4.8) \quad T_{i,j}^{(1)} = \Delta^{(N)}(e_{i,j}), \quad i, j \in \{1, 2, \dots, \mathcal{N}\},$$

where the co-product is defined in (3.26) ( $i_{i,j} \mapsto e_{i,j}$ , recall also Remark 3.8), i.e. the transfer matrix enjoys the  $\mathfrak{gl}_{\mathcal{N}}$  symmetry.  $\square$

Let  $(X, \check{r})$  be a set theoretical solution of the Yang-Baxter equation. The next natural step is to investigate the existence of non-commutative algebras that are symmetries of the general open boundary transfer matrix, i.e. generalize Corollary 4.3 for any brace solution. This is a fundamental problem and is investigated in [13].

We are now in the position to provide new examples of symmetries of periodic transfer matrices constructed from set-theoretic solutions of the Yang-Baxter equation. We will use the following known fact:

**Lemma 4.4.** *Let  $R$  be a solution of the Yang-Baxter equation, and  $B$  be a  $\mathcal{N} \times \mathcal{N}$  matrix such that*

$$(4.9) \quad (B \otimes B)R(\lambda) = R(\lambda)(B \otimes B).$$

*Then*

$$(4.10) \quad (B \otimes B^{\otimes N})T(\lambda) = T(\lambda)(B \otimes B^{\otimes N}),$$

*where  $T(\lambda)$  is the monodromy matrix.*

*Proof.* To prove this it is convenient to employ the index notation. Indeed, in the index notation expression (4.9) is translated into:  $\forall n \in \{1, \dots, N\}$

$$\begin{aligned} B_0 B_n R_{0n}(\lambda) &= R_{0n}(\lambda) B_0 B_n \Rightarrow \\ B_0 B_1 \dots B_N R_{0N}(\lambda) \dots R_{01}(\lambda) &= R_{0N}(\lambda) \dots R_{01}(\lambda) B_0 B_1 \dots B_N. \end{aligned}$$

The latter expression is equivalent to (4.10).  $\square$

We will also use the following obvious fact:

**Lemma 4.5.** *If  $B$  is an  $\mathcal{N} \times \mathcal{N}$  matrix and  $\mathcal{P} = \sum_{1 \leq i, j \leq \mathcal{N}} e_{i, j} \otimes e_{j, i}$  then*

$$(B \otimes B)\mathcal{P} = \mathcal{P}(B \otimes B).$$

**Proposition 4.6.** *Let  $(X, \check{r})$  be a set-theoretic solution of the braid equation and let  $f : X \rightarrow X$  be an isomorphism of solutions, so  $f(\sigma_x(y)) = \sigma_{f(x)}(f(y))$  and  $f(\tau_y(x)) = \tau_{f(y)}(f(x))$ . If  $M = \sum_{x \in X} \alpha_x e_{x, f(x)}$ , such that  $0 \neq \alpha_x \in \mathbb{C}$  and  $\alpha_x \alpha_y = \alpha_{\sigma_x(y)} \alpha_{\tau_y(x)}$ ,  $\forall x, y \in X$ , then*

$$(4.11) \quad \left[ M^{\otimes N}, \mathfrak{t}(\lambda) \right] = 0,$$

where  $\mathfrak{t}(\lambda)$  is the transfer matrix for  $R(\lambda) = \mathcal{P} + \lambda \mathcal{P} \check{r}$ .

*Proof.* By means of Lemmata 4.4 and 4.5 it suffices to show that

$$(4.12) \quad (M \otimes M)r = r(M \otimes M),$$

where recall  $r = \mathcal{P} \check{r}$ . Indeed, then by direct computation the LHS of (4.12) reduces to:

$$(4.13) \quad (M \otimes M)r = \sum_{x, y \in X} \alpha_x \alpha_y e_{y, f(\sigma_x(y))} \otimes e_{x, f(\tau_y(x))}$$

whereas the RHS gives:

$$(4.14) \quad r(M \otimes M) = \sum_{x, y \in X} \alpha_{\sigma_x(y)} \alpha_{\tau_y(x)} e_{y, f(\sigma_x(y))} \otimes e_{x, f(\tau_y(x))}.$$

Comparing (4.13), (4.14) we arrive at (4.12), provided that  $\alpha_x \alpha_y = \alpha_{\sigma_x(y)} \alpha_{\tau_y(x)}$ .

Having shown (4.12) it then immediately follows from Lemmata 4.4 and 4.5 that

$$(4.15) \quad (M \otimes M^{\otimes N})T(\lambda) = T(\lambda)(M \otimes M^{\otimes N}).$$

From the latter equation we focus on each element of the matrices (LHS vs RHS in (4.15)) on the auxiliary space (recall the notation  $T = \sum_{x, y} e_{x, y} \otimes T_{x, y}$ ):

$$(4.16) \quad \begin{aligned} \alpha_x M^{\otimes N} T_{f(x), f(y)} &= \alpha_y T_{x, y} M^{\otimes N} \Rightarrow \\ M^{\otimes N} T_{f(x), f(x)} &= T_{x, x} M^{\otimes N} \Rightarrow \left[ M^{\otimes N}, \mathfrak{t}(\lambda) \right] = 0. \end{aligned}$$

We can explicitly express  $M^{\otimes N}$  as

$$(4.17) \quad M^{\otimes N} = \sum_{x_1, x_2, \dots, x_N \in X} \prod_{k=1}^N \alpha_{x_k} e_{x_1, f(x_1)} \otimes e_{x_2, f(x_2)} \cdots \otimes e_{x_N, f(x_N)}.$$

□

Notice also, that due to the symmetry of the  $R$ -matrix (4.12), one easily shows that if  $T$  satisfies the RTT relation then so  $MT$  does. If  $M$  is non-singular then  $M^{\otimes N}$  is a similarity transformation that leaves the transfer matrix invariant, and naturally provides information on the multiplicities of the spectrum.

A special case of the proposition above is the obvious choice:  $f(x) = x$ ,  $\forall x \in X$ . Also, we obtain the following as immediate corollaries.

**Corollary 4.7.** *Let  $(X, \check{r})$  be a finite, non-degenerate involutive set-theoretic solution of the Yang-Baxter equation, and let  $Q_1, \dots, Q_k$  be all the orbits of  $X$ . Let  $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ . If  $M = \sum_{j=1}^k \alpha_j M_j$ , where  $M_j = \sum_{i \in Q_j} e_{i,i}$ , then*

$$(4.18) \quad \left[ M^{\otimes N}, \mathfrak{t}(\lambda) \right] = 0,$$

where  $\mathfrak{t}(\lambda)$  is the transfer matrix for  $R(\lambda) = \mathcal{P} + \lambda \mathcal{P} \check{r}$ .

*Proof.* If all  $\alpha_x \neq 0$  then the result follows from Proposition 4.6 (with  $f(x) = x$ ). Observe that

$$M^{\otimes N} = \sum_{i_1, \dots, i_N \in \mathbb{N}} M_{i_1, \dots, i_N} \alpha_1^{i_1} \cdots \alpha_N^{i_N}.$$

We know that for all non-zero choices of  $\alpha_x$  the matrix  $M^{\otimes N}$  commutes with  $\mathfrak{t}(\lambda)$ . By a ‘‘Vandermonde matrix argument’’ each matrix  $M_{i_1, \dots, i_N \in \mathbb{N}}$  commutes with  $\mathfrak{t}(\lambda)$ , which concludes the proof.  $\square$

**Corollary 4.8.** *Let  $(X, \check{r})$  be a finite, non-degenerate involutive set-theoretic solution of the Yang-Baxter equation, and let  $G(X, r)$  be its structure group (i.e. the group generated by elements from  $X$  and their inverses subject to relations  $xy = uv$  whenever  $r(x, y) = (u, v)$ ). Let  $\alpha : G(X, r) \rightarrow \mathbb{C}^*$  be a group homomorphism. If  $M = \sum_{x \in X} \alpha_x e_{x,x}$ , then  $M^{\otimes N}$  commutes with the transfer matrix.*

Similarly, as in Proposition 4.6 we use Lemmata 4.4 and 4.5 in the following proofs.

**Proposition 4.9.** *Let  $(X, \check{r})$  be a finite, non degenerate involutive set-theoretic solution of the braid equation. Let  $\mathcal{N}$  be the cardinality of  $X$ . Let  $x_1, \dots, x_\alpha \in X$  for some  $\alpha \in \{1, \dots, \mathcal{N}\}$ . Assume that  $\check{r}(x_i, y) = (y, x_i)$ ,  $\forall y \in X$  and  $\forall i \in \{1, \dots, \alpha\}$ . Then  $\forall i, j \in \{1, \dots, \alpha\}$ :*

$$(4.19) \quad \left[ \Delta^{(N)}(e_{x_i, x_j}), \mathfrak{t}(\lambda) \right] = 0,$$

where  $\mathfrak{t}(\lambda)$  is the transfer matrix for  $R(\lambda) = \mathcal{P} + \lambda \mathcal{P} \check{r}$ .

*Proof.* We first recall that  $\Delta^{(N)}(e_{x_i, x_j})$  is defined in (3.26). We will show that  $\forall i, j \in \{1, \dots, \alpha\}$ ,

$$(4.20) \quad \Delta(e_{x_i, x_j})r = r\Delta(e_{x_i, x_j}).$$

Indeed, by direct computation:  $\Delta(e_{x_i, x_j})r = \Delta(e_{x_i, x_j}) = r\Delta(e_{x_i, x_j})$ , i.e. (4.20). It then immediately follows from (4.20) that

$$(4.21) \quad \Delta^{(N+1)}(e_{x_i, x_j})T(\lambda) = T(\lambda)\Delta^{(N+1)}(e_{x_i, x_j}).$$

Recalling on the definition of the co-product (3.26), and the notation  $T(\lambda) = \sum_{x, y \in X} e_{x, y} \otimes T_{x, y}(\lambda)$  we conclude that expression (4.21) leads to

$$(4.22) \quad \begin{aligned} & \sum_w e_{x_i, w} \otimes T_{x_j, w} + \sum_{z, w} e_{z, w} \otimes \Delta^{(N)}(e_{x_i, x_j})T_{z, w}(\lambda) = \\ & \sum_w e_{z, x_j} \otimes T_{z, x_i} + \sum_{z, w} e_{z, w} \otimes T_{z, w}(\lambda)\Delta^{(N)}(e_{x_i, x_j}). \end{aligned}$$

We focus on the diagonal entries of the latter expression and we obtain:

$$(4.23) \quad \begin{aligned} \left[ \Delta^{(N)}(e_{x_i, x_j}), T_{x_i, x_i}(\lambda) \right] &= -T_{x_j, x_i}(\lambda) + \delta_{ij} T_{x_j, x_i}(\lambda), \\ \left[ \Delta^{(N)}(e_{x_i, x_j}), T_{x_j, x_j}(\lambda) \right] &= T_{x_j, x_i}(\lambda) - \delta_{ij} T_{x_j, x_i}(\lambda), \\ \left[ \Delta^{(N)}(e_{x_i, x_j}), T_{z, z}(\lambda) \right] &= 0, \quad z \neq x_i, x_j. \end{aligned}$$

Summing up all the terms above we arrive at (4.19),  $\left[ \sum_{x \in X} T_{x, x}(\lambda), \Delta^{(N)}(e_{x_i, x_j}) \right] = 0$ ,  $\forall i, j \in \{1, \dots, \alpha\}$ , i.e. we conclude that the transfer matrix is  $\mathfrak{gl}_\alpha$  symmetric.  $\square$

**Lemma 4.10.** *If  $r = \mathcal{P}\tilde{r}$  and*

$$(4.24) \quad \Delta(e_{x_i, x_j})r = r\Delta(e_{x_i, x_j})$$

for some  $x_i, x_j \in X$ , then

$$(4.25) \quad e_{x_i, x_j} \otimes e_{x_i, x_j} r = r e_{x_i, x_j} \otimes e_{x_i, x_j}$$

and also,  $\forall n \in \{1, \dots, N\}$ :

$$(4.26) \quad \left[ \sum_{m_1 < m_2 < \dots < m_n = 1}^N (e_{x_i, x_j})_{m_1} \dots (e_{x_i, x_j})_{m_n}, \mathfrak{t}(\lambda) \right] = 0,$$

where we use the standard index notation,

$$(4.27) \quad (e_{x_i, x_j})_m = I \otimes \dots \otimes \underbrace{e_{x_i, x_j}}_{m^{\text{th}} \text{ position}} \otimes I \dots \otimes I,$$

where  $I$  appears  $N - 1$  times.

*Proof.* The proof is straightforward: from the definition of the co-product  $\Delta$ , the fact that  $e_{x, y}^2 = e_{x, y} \delta_{xy}$  and also  $(\Delta(e_{x_i, x_j}))^2 r = r(\Delta(e_{x_i, x_j}))^2$ , we arrive at (4.25).

Similarly as in the proof of Proposition 4.9 we know that if  $\Delta(e_{x_i, x_j})r = r\Delta(e_{x_i, x_j})$ , then  $\left[ \Delta^{(N)}(e_{x_i, x_j}), \mathfrak{t}(\lambda) \right] = 0$ , and  $\left[ (\Delta^{(N)}(e_{x_i, x_j}))^n, \mathfrak{t}(\lambda) \right] = 0$ ,  $n \in \{1, \dots, N\}$ , which together with  $e_{x, y}^2 = e_{x, y} \delta_{xy}$  lead to (4.26).  $\square$

We obtain in what follows a general class of symmetries, associated to solutions endowed with some extra special properties. Recall that solutions  $(X, r)$  such that  $r(x, x) = (x, x)$ ,  $\forall x \in X$  are called square free, and they were introduced by Gateva-Ivanova. There was a famous conjecture by Gateva-Ivanova as to whether or not these solutions need to have a finite multi-permutation level. Vendramin subsequently showed that this was not necessary [51]. Later Cedó, Jespers and Okniński investigated these solutions using wreath products of groups and also gave many interesting examples of square-free solutions.

**Proposition 4.11.** *Let  $(X, \tilde{r})$  be a finite non degenerate involutive set-theoretic solution of the braid equation. Let  $\mathcal{N}$  be the cardinality of  $X$ . Let  $x_1, \dots, x_\alpha \in X$  for some  $\alpha \in \{1, \dots, \mathcal{N}\}$  be such that  $\tilde{r}(x_j, x_j) = (x_j, x_j)$ ,  $\forall j \in \{1, \dots, \alpha\}$ . Let  $\mathfrak{t}(\lambda) =$*

$\lambda^N \sum_{k=0}^N \mathbf{t}^{(k)} \lambda^{-k}$  be the transfer matrix for  $R(\lambda) = \mathcal{P} + \lambda \mathcal{P} \check{r}$ . Then  $\forall i, j \in \{1, \dots, \alpha\}$  and  $k = 1, \dots, N$ ,

$$\left[ e_{x_i, x_j}^{\otimes N}, \mathbf{t}^{(k)} \right] = 0.$$

*Proof.* We first show by direct computation that:

$$(4.28) \quad e_{x_i, x_j} \otimes e_{x_i, x_j} \check{r} = \check{r} e_{x_i, x_j} \otimes e_{x_i, x_j} \Rightarrow$$

$$e_{x_i, x_j}^{\otimes N} \check{r}_{n \ n+1} = \check{r}_{n \ n+1} e_{x_i, x_j}^{\otimes N}, \quad n = 1, \dots, N-1,$$

$$(4.29) \quad e_{x_i, x_j}^{\otimes N} \check{r}_{1 \ N} = \check{r}_{1 \ N} e_{x_i, x_j}^{\otimes N}$$

By multiplying equation (4.29) with  $\mathcal{P}_{1 \ N}$  we conclude that:  $\left[ e_{x_i, x_j}^{\otimes N}, \check{r}_{N \ 1} \right] = 0$ .

Then recall from Proposition 4.1 that  $H^{(k)} = \mathbf{t}^{(k)} (\mathbf{t}^{(N)})^{-1}$  for  $k = 1, 2, \dots, N-1$  are expressed exclusively in terms of  $\check{r}_{n \ n+1}$ ,  $n = 1, \dots, N-1$  and  $\check{r}_{N \ 1}$ . Recall also from the proof of Proposition 4.1 that  $H^{(N)} = \mathbf{t}^{(N)} = \mathcal{P}_{1 \ 2} \mathcal{P}_{2 \ 3} \dots \mathcal{P}_{N-1 \ N}$ , which immediately leads to (from the definition of  $\mathcal{P}$ ):  $\left[ e_{x_i, x_j}^{\otimes N}, H^{(N)} \right] = 0$ , which in turn together with expressions (4.28), (4.29) and Proposition 4.1 lead to:  $\left[ e_{x_i, x_j}^{\otimes N}, H^{(k)} \right] = 0$ ,  $k = 1, \dots, N$ . And due to  $H^{(k)} = \mathbf{t}^{(k)} (\mathbf{t}^{(N)})^{-1}$  for  $k = 1, \dots, N-1$  and  $H^{(N)} = \mathbf{t}^{(N)}$ , we conclude that  $\left[ e_{x_i, x_j}^{\otimes N}, \mathbf{t}^{(k)} \right] = 0$  for  $k = 1, \dots, N$  and  $i, j = 1, \dots, \alpha$ .  $\square$

Recall that by Remark 2.5 every nilpotent ring is a brace when we define  $a \circ b = ab + a + b$  (we call this brace the corresponding brace).

We restrict our attention in what follows in the case where  $N$  is odd. Let  $(B, +, \circ)$  be a brace. We say that  $a \in B$  is central if  $a \circ b = b \circ a$ ,  $\forall b \in B$ .

**Proposition 4.12.** *Let  $(B, +, \cdot)$  be a nilpotent ring and  $(B, +, \circ)$  be the corresponding brace. Let  $a$  be a central element in  $B$ , and  $a + a = 0$ ,  $a \circ a = 0$ . Let  $X \subseteq B$  and let  $\check{r}_B$  be defined as in Theorem 2.4. Let  $x, y \in X$  and  $x = \sigma_b(a) = ba + a$ ,  $y = \sigma_c(a) = ca + a$  for some  $b, c \in B$ . Then*

$$(4.30) \quad \left[ e_{x, y}^{\otimes N}, \mathbf{t}(\lambda) \right] = 0,$$

*Proof.* First recall that  $\mathbf{t}(\lambda) = \lambda^N \sum_{k=0}^N \frac{\mathbf{t}^{(k)}}{\lambda^k}$ , recall also that we can express the monodromy matrix and consequently all  $T^{(k)}$  in a block form, i.e.  $T^{(k)} = \sum_{z, w \in X} e_{z, w} \otimes T_{z, w}^{(k)}$ , thus  $\mathbf{t}^{(k)} = \text{tr}_0(T_0^{(k)}) = \sum_{z \in X} T_{z, z}^{(k)}$ . Via Proposition 4.11 it suffices to show that  $e_{x, y}^{\otimes N}$  commutes with  $\mathbf{t}^{(0)}$ .

Denote

$$W_{p_1, \dots, p_N} = Q_{p_1, p_2} \otimes Q_{p_2, p_3} \otimes \dots \otimes Q_{p_{N-1}, p_N} \otimes Q_{p_N, p_1},$$

where  $Q_{i, p} = \sum_{j \in W_{i, p}} e_{j, j^i}$  and  $W_{i, p} = \{j : j^i = p\}$ . Recall also the form of  $\mathbf{t}^{(0)}$  from the proof of Proposition 4.1, expressed as

$$\mathbf{t}^{(0)} = \sum_{p_1, \dots, p_N \in \mathcal{N}} W_{p_1, \dots, p_N} = Q_{p_1, p_2} \otimes Q_{p_2, p_3} \otimes \dots \otimes Q_{p_{N-1}, p_N} \otimes Q_{p_N, p_1},$$

where  $\mathcal{N}$  is the cardinality of  $X$ . We will show that

$$e_{x, y}^{\otimes N} W_{p_1, \dots, p_N} = W_{p_1, \dots, p_N} e_{x, y}^{\otimes N},$$

for all  $p_1, \dots, p_N \leq \mathcal{N}$ .

**Part 1.** We will first calculate  $e_{x,y}^{\otimes N} W_{p_1, \dots, p_N}$ .

Notice that  $e_{x,y}^{\otimes N} W_{p_1, \dots, p_N} = e_{x,y} Q_{p_1, p_2} \otimes e_{x,y} Q_{p_2, p_3} \otimes \dots \otimes e_{x,y} Q_{p_N, p_1}$ . If it is non zero then  $e_{x,y} Q_{p_i, p_{i+1}} \neq 0$  for every  $i$ .

Notice that  $e_{x,y} Q_{p_i, p_{i+1}} = \sum_{j \in W_{p_i, p_{i+1}}} e_{x,y} e_{j, j^{p_i}}$ . If  $e_{x,y} e_{j, j^{p_i}} \neq 0$  then  $j = y$ , and  $j^{p_i} = y^{p_i}$ , hence

$$e_{x,y} Q_{p_i, p_{i+1}} = e_{x,y^{p_i}}.$$

Notice that  $y = j \in W_{p_i, p_{i+1}}$ , hence  ${}^y p_i = p_{i+1}$ . Similarly  ${}^y p_{i+1} = p_{i+2}$ , this implies  ${}^{y \circ y} p_i = p_{i+2}$ . Observe that  $y \circ y = 0$ , and so  $p_i = p_{i+2}$  for every  $i$  (where  $p_{i+N} = p_i$ ). Since  $N$  is odd it follows that  $p_1 = p_2 = \dots = p_N$  and  ${}^y p_i = p_i, \forall i$ .

Now  ${}^y p_i = p_{i+1}$  implies  ${}^y p_i = p_i$ , hence  ${}^a p_i = p_i$ . It follows that  $y^{p_i} = y$  (since  $a$  is central in  $B$ ). Therefore,  $e_{x,y}^{\otimes N} W_{p_1, \dots, p_N} = e_{x,y}^{\otimes N}$  provided that  $p_1 = \dots = p_N$  and  ${}^a p_1 = p_1$ , and otherwise it is zero.

**Part 2.** We will now calculate  $W_{p_1, \dots, p_N} e_{x,y}^{\otimes N}$ .

Notice that if  $W_{p_1, \dots, p_N} e_{x,y}^{\otimes N} \neq 0$  then  $Q_{p_i, p_{i+1}} e_{x,y} \neq 0$ , for every  $i$ . Observe that  $Q_{p_i, p_{i+1}} e_{x,y} = \sum_{j \in W_{p_i, p_{i+1}}} e_{j, j^{p_i}} e_{x,y}$ . If  $e_{j, j^{p_i}} e_{x,y} \neq 0$  then  $j^{p_i} = x$ . Hence  $j = x^{q_i}$  where  $q_i$  is the inverse of  $p_i$  in the group  $(B, \circ)$ . It follows that  $j = q_i a + a$ . Therefore, for given  $p_i$  element  $j_i$  is uniquely determined. Consequently,  $Q_{p_i, p_{i+1}} e_{x,y} = e_{j_i, y}$  and  $j_i = j \in W_{p_i, p_{i+1}}$ . This implies  ${}^{j_i} p_i = p_{i+1}$ , similarly  ${}^{j_{i+1}} p_{i+1} = p_{i+2}$ , etc. It follows that  ${}^j p_i = p_i$  where  $j = j_{i+N-1} \circ \dots \circ j_{i+1} \circ j_i$ . Observe that  $j = a + qa$  for some  $q \in B$  by assumptions on  $x$ . Previously it was shown that  ${}^{j_i} p_i = p_{i+1} = p_i$ , hence  ${}^x p_i = p_i$ . Therefore  ${}^a p_i = p_i$  for  $i = 1, \dots, N$ . This implies  $x^{p_i} = x$ . Observe that  $j_i = x^{q_i}$  notice that  $q_i = p_i \circ \dots \circ p_i$  and since  $x^{p_i} = x$  it follows that  $j_i = x$ .

Therefore,  $W_{p_1, \dots, p_N} e_{x,y}^{\otimes N} = e_{x,y}^{\otimes N}$  provided that  $p_1 = \dots = p_N$  and  ${}^a p_1 = p_1$ , and otherwise it is zero. Consequently,  $e_{x,y}^{\otimes N} W_{p_1, \dots, p_N} = W_{p_1, \dots, p_N} e_{x,y}^{\otimes N}$ .  $\square$

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