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Axioms for the category of Hilbert spaces

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We provide axioms that guarantee a category is equivalent to that of continuous linear functions between Hilbert spaces. The axioms are purely categorical and do not presuppose any analytical structure. This addresses a question about the mathematical foundations of quantum theory raised in reconstruction programs such as those of von Neumann, Mackey, Jauch, Piron, Abramsky, and Coecke.

Hilbert space | monoidal category | colimit

Quantum mechanics have mathematically been firmly founded on Hilbert spaces and operators between them for nearly a century (1). There has been continuous inquiry into the special status of this foundation ever since (2–4). How are the mathematical axioms to be interpreted physically? Can the theory be reconstructed from a different framework whose axioms can be interpreted physically? Such reconstruction programs involve a mathematical reformulation of (a generalization of) the theory of Hilbert spaces and their operators, such as operator algebras (5), orthomodular lattices (6, 7), and, most recently, categorical quantum mechanics (8, 9). The latter uses the framework of category theory (10) and emphasizes operators more than their underlying Hilbert spaces. It postulates a category with structure that models physical features of quantum theory (11). The question of how “to justify the use of Hilbert space” (ref. 7, p. 1) then becomes, Which axioms guarantee that a category is equivalent to that of continuous linear functions between Hilbert spaces? This article answers that mathematical question. The axioms are purely categorical in nature and do not presuppose any analytical structure such as continuity, complex numbers, or probabilities. The approach is similar to Lawvere’s categorical characterization of the theory of sets (12).

Axioms

We consider six axioms on a (locally small) category \mathcal{C} . The first two impose additional structure: an involution called a dagger and a tensor product. All axioms have to respect the dagger. In particular, the right notion of inclusion is dagger monomorphism, and this notion permeates the last four axioms. Axioms three and four demand finite (co)completeness, roughly, direct sums and equalizers. The last two axioms ask that dagger subobjects behave well: Intuitively, one should be able to divide by them and take directed suprema of them. More precisely,

- (D) The category is equipped with a dagger \dagger . This is an operation that maps each morphism $f: H \rightarrow K$ to a morphism $f^\dagger: K \rightarrow H$ such that $f^{\dagger\dagger} = f$, and $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$, and $\text{id}_H^\dagger = \text{id}_H$. If $f^\dagger \circ f = \text{id}_H$, we call f a dagger monomorphism, and if additionally $f \circ f^\dagger = \text{id}_K$, we call it a dagger isomorphism.
- (T) The category is equipped with a dagger monoidal structure \otimes whose unit I is a simple monoidal separator. The former means that the isomorphisms $(H \otimes K) \otimes L \simeq H \otimes (K \otimes L)$ and $I \otimes H \simeq H \simeq H \otimes I$ are dagger isomorphisms. The latter means that I has exactly two subobjects and that $f = g: H \otimes K \rightarrow L$ as soon as $f \circ (h \otimes k) = g \circ (h \otimes k)$ for all $h: I \rightarrow H$ and $k: I \rightarrow K$.

- (B) Any two objects H and K have a dagger biproduct $H \oplus K$, and there is a zero object 0 . The latter means that for every object H there exist unique morphisms $0 \rightarrow H$ and $H \rightarrow 0$. Hence any two objects H and K have a unique morphism $0_{H,K}: H \rightarrow K$ that factors through 0 . The former means that there are dagger monomorphisms $i: H \rightarrow H \oplus K$ and $j: K \rightarrow H \oplus K$ that make $H \oplus K$ a coproduct of H and K and satisfy $j^\dagger \circ i = 0_{H,K}$ (13).
- (E) Any two morphisms $H \rightarrow K$ have a dagger equalizer $f: E \rightarrow H$, that is, an equalizer f that is a dagger monomorphism.
- (K) Any dagger monomorphism $N \rightarrow H$ is a dagger equalizer of some morphism $H \rightarrow K$ and $0_{H,K}: H \rightarrow K$.
- (C) The wide subcategory of dagger monomorphisms has directed colimits. This means that if (N, \leq) is a directed partially ordered set, H_n are objects, and $H_{m \leq n}: H_m \rightarrow H_n$ are dagger monomorphisms, then there is a universal object H with dagger monomorphisms $H_n \rightarrow H$ that commute with the morphisms $H_{m \leq n}$.

The categories $\mathbf{Hilb}_{\mathbb{R}}$ of real Hilbert spaces and $\mathbf{Hilb}_{\mathbb{C}}$ of complex Hilbert spaces with continuous linear functions satisfy these axioms: (D) is given by adjoints, (T) by tensor product, (B) by direct sum, (E) and (K) by closed subspaces, and (C) by the completion of the directed union. We will show that any category \mathcal{C} that satisfies these axioms is equivalent to one of these two categories.

Context

This characterization combines established research programs on dagger categories and on orthomodular spaces. Dagger categories are categories with a certain self-duality and were first considered by Mac Lane (14); the terminology (D) was introduced by Selinger (15). One research thread has been to find conditions that guarantee that the scalars of a monoidal dagger category form a field in the manner of Mitchell’s embedding theorem for abelian categories (16). Abramsky and Coecke (8)

Significance

Hilbert spaces and their operators are the mathematical foundation of quantum mechanics. The problem of reconstructing this foundation from first principles has been open for nearly a century. What is mathematically special about Hilbert spaces and their operators? This paper gives a categorical axiomatization. Unlike previous partial results of this type, the axioms do not presuppose probabilities, complex amplitudes, or continuity and are not limited to finite dimension.

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considered parts of (T) and (B) in observing that the scalars of a monoidal category with biproducts form a commutative semiring. Vicary (17) emphasized the compatibility of the dagger (D) with all other structure, added simplicity to (T), and introduced (E) to show that the scalars embed into a field. Heunen (18) showed that adding (K) implies that the scalars are a field. Vicary (17) and Heunen (18) both also embedded the scalars into \mathbb{C} using additional assumptions that are not needed here. Axioms (D) to (K) together imply that each object may be regarded as a Hermitian space.

Axiom (C) implies that each such Hermitian space \mathcal{H} is orthomodular. Orthomodular spaces were first studied by Kaplansky (19); the main research thread has been to find conditions that guarantee that an infinite-dimensional orthomodular space is a Hilbert space. Amemiya and Araki (20) proved Piron's (21) conjecture that a (real or complex) inner product space is an orthomodular space if and only if it is a Hilbert space. Wilbur (22) found algebraic conditions that imply that an orthomodular space is a Hilbert space. Continuing research in this direction culminated in Solèr's theorem that any orthomodular space with an infinite orthonormal subset is a Hilbert space (23, 24). Solèr's theorem has been applied to characterize quantum logic as an orthomodular lattice (24–27). This research line about orthomodular lattices is of wide interest due to the reconstruction program, which aims to derive the structure of Hilbert space from physical first principles. See ref. 28 for a literature survey of such reconstructions.

Scalars

We begin by looking at the morphisms $z: I \rightarrow I$. These are called scalars, because in any category they form a monoid under composition, with $\text{id}_I: I \rightarrow I$ as multiplicative unit. In a monoidal category, as in axiom (T), this monoid is commutative, because the multiplication of two scalars $z, w: I \rightarrow I$ equals

$$I \simeq I \otimes I \xrightarrow{w \otimes z} I \otimes I \simeq I \quad [1]$$

for which an Eckmann–Hilton argument holds (ref. 11, lemma 2.3).

The dagger of axiom (D) provides an involution on scalars. The biproducts of axiom (B) afford a matrix calculus on morphisms (ref. 11, section 2.2.4). In particular, there are a codiagonal map $(i\ j): I \oplus I \rightarrow I$ and a diagonal map $(i\ j)^\dagger: I \rightarrow I \oplus I$ that let us add scalars $w, z: I \rightarrow I$:

$$I \xrightarrow{(i\ j)^\dagger} I \oplus I \xrightarrow{w \oplus z} I \oplus I \xrightarrow{(i\ j)} I. \quad [2]$$

Together with the zero morphism $0_{I,I}: I \rightarrow I$, the scalars hence form an involutive commutative semiring (ref. 11, lemma 2.15).

Lemma 1. $\mathcal{I} = \mathcal{C}(I, I)$ is a field with involution $z \mapsto z^\dagger$.

Proof (sketch). To see that there are multiplicative inverses, let $z \in \mathcal{I}$. By axiom (E), z factors as an epimorphism $I \rightarrow H$ followed by a dagger monomorphism $f: H \rightarrow I$. It then follows from the simplicity of I in axiom (T) that f is either zero or invertible. If $f = 0$, then $z = 0$. If f is invertible, then z is an epimorphism. But then z^\dagger is monic and hence zero or invertible. In either case z is zero or invertible.

It follows that \mathcal{I} must either have additive inverses or be zerosumfree. The latter term means that $w + z = 0$ implies $w = z = 0$. But this would contradict axiom (K), because the equalizer $\ker(f)$ of $f: H \rightarrow K$ and $0_{H,K}$ is zero if and only if f is a monomorphism, but according to Eq. 2 $\ker(i\ j) = 0$ if $w + z = 0$ whereas $(i\ j) \neq 0$. For details, see ref. 18, lemma 4.5. ■

We will see in Proposition 5 below that in fact \mathcal{I} must be \mathbb{R} (with the trivial involution) or \mathbb{C} .

Furthermore, any morphism $f: H \rightarrow K$ can be multiplied by a scalar $z: I \rightarrow I$ to get

$$H \simeq I \otimes H \xrightarrow{z \otimes f} I \otimes K \simeq K. \quad [3]$$

Similarly, the addition of Eq. 2 works just as well for parallel morphisms $f, g: H \rightarrow K$ of arbitrary type. It follows from Lemma 1 that the morphisms $H \rightarrow K$ always form a vector space over \mathcal{I} . In particular, we may consider morphisms $I \rightarrow H$ to be vectors.

Projections

Now consider projections on an object H , that is, morphisms $p: H \rightarrow H$ satisfying $p^\dagger \circ p = p$. They are partially ordered by $p \leq q$ if and only if $q \circ p = p$. We next show that arbitrary subsets of projections have least upper bounds and there is an orthocomplement p^\perp satisfying $p^{\perp\perp} = p$, and $p \leq q$ if and only if $q^\perp \leq p^\perp$.

Lemma 2. The projections on an object H form a complete ortholattice with $p^\perp = \text{id}_H - p$.

Proof (sketch). Projections on H are in one-to-one correspondence with dagger subobjects of H , that is, dagger isomorphism classes of dagger monomorphisms $f: N \rightarrow H$. These are also partially ordered, by $f \leq g$ when f factors through g . Axiom (K) provides an order isomorphism between projections and dagger subobjects (ref. 29, lemma 5 and proposition 12). By axiom (C), these partially ordered sets furthermore have least upper bounds of directed subsets.

The orthocomplement of a dagger subobject $f: N \rightarrow H$ is given by $\ker(f^\dagger)$. Axioms (D) and (E) ensure that it indeed satisfies $f^{\perp\perp} = f$, and $f \leq g$ if and only if $g^\perp \leq f^\perp$ (ref. 29, lemma 1).

By axioms (B) and (E), pullbacks exist, and so any two dagger subobjects have a greatest lower bound (ref. 29, lemma 2). But because of the orthocomplementation any two dagger subobjects then have a least upper bound. Hence any finite number of projections have a least upper bound.

Now, the least upper bound of an arbitrary subset of projections is the least upper bound of the directed family of least upper bounds of finite subsets and hence exists. Finally, it is clear that $p^\perp = \text{id}_H - p$ is an orthocomplement. ■

Notation. Morphisms out of I will play a dual role, as we will also regard them as vectors in a Hilbert space. To distinguish between these two roles we use the following notation: For an object H , we write \mathcal{H} for the homset $\mathcal{C}(I, H)$. For a morphism $f: H \rightarrow K$, we write $F: \mathcal{H} \rightarrow \mathcal{K}$ for the function that postcomposes with f .

We have already established that \mathcal{H} is a vector space (over \mathcal{I}) and that F is a linear function. In fact, axiom (D) makes \mathcal{H} into a Hermitian space (ref. 30, definition 1.1): The Hermitian form $\langle f | g \rangle$ of $f, g \in \mathcal{H}$ is $g^\dagger \circ f \in \mathcal{I}$; nondegeneracy follows from axiom (E) (ref. 17, lemma 2.5). Hence we can speak about the orthocomplement $V^\perp = \{f \in \mathcal{H} \mid \forall v \in V: \langle f | v \rangle = 0\}$ of a subspace $V \subseteq \mathcal{H}$. A subspace V is called closed when $V^{\perp\perp} = V$. We now justify using the same notation for this orthocomplementation, which ostensibly differs from that of Lemma 2.

Lemma 3. The function $p \mapsto p \circ \mathcal{H} = \{p \circ h \mid h \in \mathcal{H}\}$ is an isomorphism from the ortholattice of projections on H to the ortholattice of closed subspaces of \mathcal{H} .

Proof. For a projection $p: H \rightarrow H$ and a vector $h \in \mathcal{H}$,

$$\begin{aligned} h \in p^\perp \circ \mathcal{H} &\iff (\text{id}_H - p) \circ h = h \iff p \circ h = 0 \\ &\iff \forall f \in \mathcal{H}: (p \circ f)^\dagger \circ h = f^\dagger \circ p \circ h = 0 \\ &\iff h \in (p \circ \mathcal{H})^\perp. \end{aligned}$$

The third equivalence uses that I is a monoidal separator as in axiom T. Thus $p^\perp \circ \mathcal{H} = (p \circ \mathcal{H})^\perp$, and $(p \circ \mathcal{H})^{\perp\perp} = p \circ \mathcal{H}$ is indeed a closed subspace of \mathcal{H} .

Observe that $0_{I,H} \circ \mathcal{H} = \{0_{I,H}\}$, that $\text{id}_H \circ \mathcal{H} = \mathcal{H}$, and that $p \circ \mathcal{H} \subseteq q \circ \mathcal{H}$ when $p \leq q$. Hence $p \mapsto p \circ \mathcal{H}$ is a monotone function that respects orthocomplementation. It remains to show that it is an order embedding and that it is surjective.

Let p and q be projections on H . If $p \circ \mathcal{H} \subseteq q \circ \mathcal{H}$, then $q \circ p \circ f = p \circ f$ for all $f \in \mathcal{H}$, so by axiom (T), $q \circ p = p$; that is, $p \leq q$. Therefore, the function $p \mapsto p \circ \mathcal{H}$ is an order embedding, and in particular injective.

To establish that it is surjective, let V be a closed subspace of \mathcal{H} . For a nonzero vector $h \in \mathcal{H}$, define a projection

$$p_h = (h \circ h^\dagger) \cdot (h^\dagger \circ h)^{-1} : H \rightarrow H,$$

where the dot denotes the scalar multiplication of Eq. 3. Set $p = \bigvee_{v \in V \setminus \{0\}} p_v$. For each vector $v \in V \setminus \{0\}$ then $p_v \circ v = v$ and so $p \circ v = p \circ p_v \circ v = p_v \circ v = v$. Thus $V \subseteq p \circ \mathcal{H}$.

Let h be a nonzero vector in \mathcal{H} , and assume that $h \in V^\perp$. Then $p_h \circ p_v = 0$ for all $v \in V \setminus \{0\}$; in other words, $p_v \leq p_h^\perp$. It follows that $p = \bigvee_{v \in V \setminus \{0\}} p_v \leq p_h^\perp$, and so $p_h \circ p = 0$. Hence $p_h \circ \mathcal{H} \perp p \circ \mathcal{H}$, and in particular $h \perp p \circ \mathcal{H}$. Therefore, any vector $h \in \mathcal{H}$ that is orthogonal to V is also orthogonal to $p \circ \mathcal{H}$. Since both V and $p \circ \mathcal{H}$ are closed subspaces of \mathcal{H} , we conclude that $p \circ \mathcal{H} \subseteq V$. Altogether, $p \circ \mathcal{H} = V$, and the function $p \mapsto p \circ \mathcal{H}$ is surjective. ■

Orthogonality

A Hermitian space \mathcal{H} is called orthomodular when $\mathcal{H} = V \oplus V^\perp$ for all closed subspaces $V \subseteq \mathcal{H}$ (ref. 30, definition 1.2).

Lemma 4. \mathcal{H} is an orthomodular space.

Proof. Let V be a closed subspace of \mathcal{H} , and let h be a vector in \mathcal{H} . Lemma 3 provides a projection p on H such that $V = p \circ \mathcal{H}$. Trivially $p \circ h \in V$. Also $p^\perp \circ h \in V^\perp$ because for all $v \in V$,

$$v^\dagger \circ (p^\perp \circ h) = ((\text{id}_H - p) \circ v)^\dagger \circ h = (v - (p \circ v))^\dagger \circ h = 0.$$

We conclude that $h = (p \circ h) + (p^\perp \circ h) \in V + V^\perp$ and more generally that $\mathcal{H} = V \oplus V^\perp$. ■

The next step uses orthogonal subsets of \mathcal{H} that can be infinite. This requires some preparation. If R is a finite set, axiom (B) shows that a coproduct $\bigoplus_R I$ of copies of I indexed by R exists; denote it by I^R . We may assume that $I^R = I$ when R is a singleton. Write $i_{R,S} : I^R \rightarrow I^S$ for the canonical morphism induced by an inclusion $R \subseteq S$ of finite sets. Then $i_{S,T} \circ i_{R,S} = i_{R,T}$ for all finite sets $R \subseteq S \subseteq T$. Moreover, using the matrix calculus for morphisms afforded by axiom (B) (ref. 11, section 2.2.4), it is clear that $i_{R,S}$ is a dagger monomorphism.

Now let A be an arbitrary set. Write $R \subseteq A$ if R is a finite subset of A . The morphisms $i_{R,S}$ form a diagram of dagger monomorphisms in \mathbf{C} indexed by the category of finite subsets $R \subseteq A$ and inclusions between them. By axiom (C) this diagram has a colimit I^A , with universal dagger monomorphisms $i_{R,A} : I^R \rightarrow I^A$. (If A is finite, the diagram has a terminal object, so both definitions of I^A agree.)

Proposition 5. The involutive field \mathcal{I} is isomorphic to \mathbb{R} (with the trivial involution) or \mathbb{C} , and \mathcal{H} is a Hilbert space for any object H .

Proof. In a first step, we show that $e_a = i_{\{a\},A} : I \rightarrow I^A$ form an orthonormal subset of $\mathbf{C}(I, I^A)$, where a ranges over an arbitrary set A . Because e_a is a dagger monomorphism, $\langle e_a | e_a \rangle = e_a^\dagger \circ e_a = \text{id}_I$. Similarly, for distinct $a, b \in A$,

$$\begin{aligned} \langle e_a | e_b \rangle &= e_a^\dagger \circ e_b = i_{\{a\},\{a,b\}}^\dagger \circ i_{\{a,b\},A}^\dagger \circ i_{\{a,b\},A} \circ i_{\{b\},\{a,b\}} \\ &= i_{\{a\},\{a,b\}}^\dagger \circ i_{\{b\},\{a,b\}} = 0, \end{aligned}$$

because $i_{\{a\},\{a,b\}} : I^{\{a\}} \rightarrow I^{\{a,b\}}$ and $i_{\{b\},\{a,b\}} : I^{\{b\}} \rightarrow I^{\{a,b\}}$ are inclusions of distinct summands of $I^{\{a,b\}} = I^{\{a\}} \oplus I^{\{b\}}$.

Thus $\mathbf{C}(I, I^A)$, which is an orthomodular space over \mathcal{I} by Lemma 4, has an infinite orthonormal subset. It follows from

Solèr's theorem that \mathcal{I} is isomorphic to \mathbb{R} or \mathbb{C} (ref. 24 or ref. 30, theorem 1.3). It is not isomorphic to \mathbb{H} because it is commutative (31).

Let H be an object of \mathbf{C} . The Hermitian space $\mathbf{C}(I, H \oplus I^{\mathbb{N}})$ is an orthomodular space by Lemma 4. It has an infinite orthonormal subset because the inclusion morphism $I^{\mathbb{N}} \rightarrow H \oplus I^{\mathbb{N}}$ is a dagger monomorphism, and thus postcomposition by this inclusion morphism is a function $\mathbf{C}(I, I^{\mathbb{N}}) \rightarrow \mathbf{C}(I, H \oplus I^{\mathbb{N}})$ that preserves the Hermitian form. Appealing again to Solèr's theorem, we conclude that $\mathbf{C}(I, H \oplus I^{\mathbb{N}})$ is a Hilbert space over \mathcal{I} .

Similarly, postcomposing with the inclusion morphism $H \rightarrow H \oplus I^{\mathbb{N}}$ gives a function $\mathcal{H} \rightarrow \mathbf{C}(I, H \oplus I^{\mathbb{N}})$ that preserves the Hermitian form. It follows that the Hermitian form on \mathcal{H} is an inner product. If \mathcal{H} has an infinite orthonormal subset, then it is a Hilbert space over \mathcal{I} by Lemma 4 and Solèr's theorem. If \mathcal{H} does not have an infinite orthonormal subset, then it is simply a finite-dimensional inner product space over \mathcal{I} and thus also a Hilbert space over \mathcal{I} . ■

If there exists a morphism $z : I \rightarrow I$ such that $z^\dagger \neq z$, then $\mathcal{I} = \mathbb{C}$; otherwise $\mathcal{I} = \mathbb{R}$. This gives one way to distinguish between the complex and real cases. Another way is to ask that there is a morphism $z : I \rightarrow I$ with $z \circ z = -1$. There may be more, for example inspired by symmetries (26). We will abbreviate $\text{Hilb}_{\mathcal{I}}$ to Hilb from now on.

Corollary 6. The assignment of $F : \mathcal{H} \rightarrow \mathcal{K}$ to $f : H \rightarrow K$ is a functor $\mathbf{C}(I, -) : \mathbf{C} \rightarrow \text{Hilb}$.

Proof. If $f : H \rightarrow K$ is a morphism in \mathbf{C} , we have to show that the function $F : \mathcal{H} \rightarrow \mathcal{K}$ is continuous and linear. Linearity follows from axioms (B) and (T) (ref. 11, lemma 2.6 and proposition 2.23). Similarly, postcomposition by $t = f^\dagger \circ f : H \rightarrow H$ is a linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$. For all $g, h \in \mathcal{H}$, we find $\langle h | T(g) \rangle = h^\dagger \circ (t \circ g) = (t \circ h)^\dagger \circ g = \langle T(h) | g \rangle$, so T is self-adjoint. It follows from the Hellinger–Toeplitz theorem that T is continuous (ref. 32, corollary III.12). Because

$$\|F(g)\|^2 = \|f \circ g\|^2 = g^\dagger \circ t \circ g = \langle g | T(g) \rangle \leq \|g\|^2 \|T\|,$$

F has norm at most $\|T\|^{1/2}$ and is therefore continuous. ■

Bases

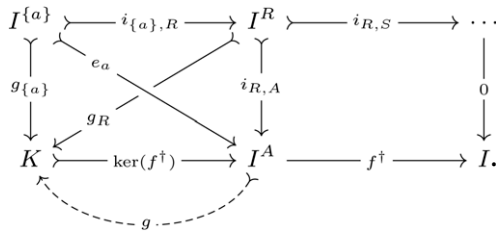
We continue the analysis of $\mathbf{C}(I, I^A)$ by showing that the orthonormal set in the Proof of Proposition 5 is in fact an orthonormal basis. Recall that the dimension of a Hilbert space is defined to be the cardinality of any orthonormal basis.

Lemma 7. For any set A , the dimension of the Hilbert space $\mathbf{C}(I, I^A)$ is the cardinality of A .

Proof. Proposition 5 showed that $e_a = i_{\{a\},A}$, where a ranges over A , is an orthonormal subset of $\mathbf{C}(I, I^A)$. We will show that it is an orthonormal basis. Suppose that a vector $f : I \rightarrow I^A$ is orthogonal to e_a for all $a \in A$. Given $a \in R \subseteq A$, we find $f^\dagger \circ i_{R,A} \circ i_{\{a\},R} = f^\dagger \circ e_a = 0$. Because $i_{\{a\},R} : I \rightarrow I^R$ are the coprojections of the coproduct I^R , we have $f^\dagger \circ i_{R,A} = 0$ for all $R \subseteq A$. The morphism $f^\dagger : I^A \rightarrow I$ has a dagger monomorphism $\ker(f^\dagger) : K \rightarrow I^A$ as kernel. This gives dagger monomorphisms $g_R : I^R \rightarrow K$ such that $\ker(f^\dagger) \circ g_R = i_{R,A}$, for $R \subseteq A$. These form a cocone on the diagram of dagger monomorphisms $i_{R,S}$ for $R, S \subseteq A$: For if $R \subseteq S \subseteq A$, then

$$\ker(f^\dagger) \circ g_S \circ i_{R,S} = i_{S,A} \circ i_{R,S} = i_{R,A} = \ker(f^\dagger) \circ g_R,$$

and because $\ker(f^\dagger)$ is a dagger monomorphism, we have that $g_S \circ i_{S,R} = g_R$:



Therefore there exists a unique mediating morphism $g : I^A \rightarrow K$ with $g \circ i_{R,A} = g_R$ for all $R \in A$. Thus, $\ker(f^\dagger) \circ g \circ i_{R,A} = \ker(f^\dagger) \circ g_R = i_{R,A}$ for all $R \in A$, and by the universal property of the colimit I^A , we conclude $\ker(f^\dagger) \circ g = \text{id}_{I^A}$. It follows that $\ker(f^\dagger)$ is an epimorphism, and being a dagger monomorphism, it is a dagger isomorphism. Therefore, $f^\dagger = 0$.

Overall, the zero vector is the only one in the Hilbert space $\mathbf{C}(I, I^A)$ that is orthogonal to each vector e_a , for $a \in A$. Thus, $\{e_a\}_{a \in A}$ is an orthonormal basis of $\mathbf{C}(I, I^A)$. ■

Equivalence

We are now ready to prove the main result. A functor is a dagger equivalence when it is an equivalence of categories that preserves the dagger of morphisms.

Theorem 8. *The functor $\mathbf{C}(I, -) : \mathbf{C} \rightarrow \mathbf{Hilb}_{\mathcal{I}}$ is an equivalence of dagger categories, where \mathcal{I} is either \mathbb{R} or \mathbb{C} .*

Proof. Let $f : H \rightarrow K$ be a morphism in \mathbf{C} . For all vectors $h \in \mathcal{H}$ and $k \in \mathcal{K}$ axiom (D) shows

$$\langle f^\dagger \circ k | h \rangle = (f^\dagger \circ k)^\dagger \circ h = k^\dagger \circ f \circ h = \langle k | f \circ h \rangle.$$

Thus $\mathbf{C}(I, f^\dagger) = \mathbf{C}(I, f)^\dagger$; that is, $\mathbf{C}(I, -)$ is a dagger functor.

If morphisms $f, g : H \rightarrow K$ satisfy $\mathbf{C}(I, f) = \mathbf{C}(I, g)$, then $f \circ h = g \circ h$ for all vectors $h \in \mathcal{H}$, so $f = g$ by axiom (T). Therefore, $\mathbf{C}(I, -)$ is faithful.

Assume $\dim(\mathcal{H}) \leq \dim(\mathcal{K})$, and let $U : \mathcal{H} \rightarrow \mathcal{K}$ be a linear isometry. Choose an orthonormal basis $\{h_a\}_{a \in A}$ for the Hilbert space \mathcal{H} . Because U is an isometry, the vectors $k_a = U(h_a)$ form an orthonormal subset of the Hilbert space \mathcal{K} .

Extending the dagger monomorphisms $h_a : I \rightarrow H$ and $k_a : I \rightarrow K$ for finite subsets $R \in A$ to dagger monomorphisms $h_R : I^R \rightarrow H$ and $k_R : I^R \rightarrow K$ using axiom (B) gives two cocones on the diagram $i_{R,S}$. Thus by axiom (C) there are unique mediating dagger monomorphisms $h_A : I^A \rightarrow H$ and $k_A : I^A \rightarrow K$ satisfying $h_A \circ e_a = h_a$ and $k_A \circ e_a = k_a$ for all $a \in A$, where $e_a = i_{\{a\},A}$ as before.

Define $u = k_A \circ h_A^\dagger : H \rightarrow K$. For each $a \in A$ then

$$\begin{aligned} \mathbf{C}(I, u)(h_a) &= k_A \circ h_A^\dagger \circ h_a = k_A \circ h_A^\dagger \circ h_A \circ e_a \\ &= k_A \circ e_a = k_a = U(h_a). \end{aligned}$$

Because $\{h_a\}_{a \in A}$ is an orthonormal basis of the Hilbert space \mathcal{H} , we conclude that $\mathbf{C}(I, u) = U$.

Any continuous linear function $\mathcal{H} \rightarrow \mathcal{K}$ is a linear combination of isometric ones, because we have assumed that $\dim(\mathcal{H}) \leq \dim(\mathcal{K})$. It follows that each continuous linear function $F : \mathcal{H} \rightarrow \mathcal{K}$ is the image of a linear combination of morphisms $H \rightarrow K$ under the functor $\mathbf{C}(I, -)$, which is linear on morphisms by axiom (B). Therefore, there exists a morphism $f : H \rightarrow K$ such that $\mathbf{C}(I, f) = F$. Combining this fact for $\dim(\mathcal{H}) \leq \dim(\mathcal{K})$ with the fact that the functor $\mathbf{C}(I, -)$ preserves daggers, we conclude that the functor is full.

Finally, the functor $\mathbf{C}(I, -)$ is essentially surjective by Lemma 7. Being a full, faithful, and essentially surjective dagger functor, it is an equivalence of dagger categories. ■

Because equivalences preserve limits, it follows directly that the functor $\mathbf{C}(I, -) : \mathbf{C} \rightarrow \mathbf{Hilb}$ preserves all the structure concerned in axioms (D), (B), (E), and (K). It also follows that the functor restricts to a full and faithful functor between the wide subcategories of dagger monomorphisms; because in \mathbf{Hilb} any two isomorphic objects are unitarily isomorphic (ref. 13, theorem 8.3 and proposition 8.4), the restricted functor is still an equivalence, and hence also the structure concerned in axiom (C) is preserved.

Tensor Products

Finally, we prove that the equivalence $\mathbf{C}(I, -) : \mathbf{C} \rightarrow \mathbf{Hilb}$ also preserves the tensor products of axiom (T).

For objects H and K , the function $\mathcal{H} \times \mathcal{K} \rightarrow \mathbf{C}(I, H \otimes K)$ given by $(h, k) \mapsto h \otimes k$ is bilinear by axioms (B) and (T) (ref. 11, lemmas 3.22 and 3.6 and corollary 3.20). It is also bounded as a bilinear map because

$$\begin{aligned} \|h \otimes k\|^2 &= \langle h \otimes k | h \otimes k \rangle = (h \otimes k)^\dagger \circ (h \otimes k) \\ &= (h^\dagger \circ h) \otimes (k^\dagger \circ k) = \langle h | h \rangle \cdot \langle k | k \rangle = \|h\|^2 \cdot \|k\|^2, \end{aligned}$$

where we suppressed the isomorphism $I \otimes I \simeq I$. This defines a continuous linear function:

$$\begin{aligned} M_{H,K} : \mathcal{H} \otimes \mathcal{K} &\rightarrow \mathbf{C}(I, H \otimes K) \\ h \otimes k &\mapsto h \otimes k. \end{aligned} \tag{4}$$

Here the tensor product on the left-hand side is that of Hilbert spaces, whereas the tensor product on the right-hand side is the monoidal structure of \mathbf{C} .

Lemma 9. *The continuous linear functions of Eq. 4 are unitary and form the components of a natural transformation from the functor $\mathbf{C}(I, -) \otimes \mathbf{C}(I, -) : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{Hilb}$ to the functor $\mathbf{C}(I, - \otimes -) : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{Hilb}$.*

Proof. If $h, h' \in \mathcal{H}$ and $k, k' \in \mathcal{K}$, then the inner product $\langle h \otimes k | h' \otimes k' \rangle$ in $\mathbf{C}(I, H \otimes K)$ is equal to $\langle h | h' \rangle \cdot \langle k | k' \rangle$. Hence $M_{H,K}$ preserves inner products between elementary tensors, and linearity and continuity then imply that $M_{H,K}$ is an isometry. So its range is a closed subspace of $\mathbf{C}(I, H \otimes K)$. Suppose there were a nonzero vector f orthogonal to $h \otimes k$ for all $h \in \mathcal{H}$ and $k \in \mathcal{K}$. In other words, $f^\dagger \circ (h \otimes k) = \langle f | h \otimes k \rangle = 0$. It would then follow from axiom (T) that $f^\dagger = 0$, contradicting our choice of f . Therefore, $M_{H,K}$ is surjective and hence unitary.

If $f : H_1 \rightarrow H_2$ and $g : K_1 \rightarrow K_2$ are morphisms of \mathbf{C} , then the following diagram commutes:

$$\begin{array}{ccc} \mathbf{C}(I, H_1) \otimes \mathbf{C}(I, K_1) & \xrightarrow{M_{H_1, K_1}} & \mathbf{C}(I, H_1 \otimes K_1) \\ \downarrow \mathbf{C}(I, f) \otimes \mathbf{C}(I, g) & & \downarrow \mathbf{C}(I, f \otimes g) \\ \mathbf{C}(I, H_2) \otimes \mathbf{C}(I, K_2) & \xrightarrow{M_{H_2, K_2}} & \mathbf{C}(I, H_2 \otimes K_2). \end{array}$$

Indeed, for all $h \in \mathcal{H}_1$ and $k \in \mathcal{K}_1$,

$$\begin{aligned} &\mathbf{C}(I, f \otimes g)(M_{H_1, K_1}(h \otimes k)) \\ &= \mathbf{C}(I, f \otimes g)(h \otimes k) \\ &= (f \circ h) \otimes (g \circ k) \\ &= \mathbf{C}(I, f)(h) \otimes \mathbf{C}(I, g)(k) \\ &= M_{H_2, K_2}(\mathbf{C}(I, f)(h) \otimes \mathbf{C}(I, g)(k)) \\ &= M_{H_2, K_2}((\mathbf{C}(I, f) \otimes \mathbf{C}(I, g))(h \otimes k)). \end{aligned}$$

Therefore, the unitary linear functions $M_{H,K}$ are natural in H and K . ■

A (dagger) functor is monoidal when it preserves (the dagger and) the tensor product up to a natural (dagger) isomorphism that respects the (dagger) isomorphisms $(H \otimes K) \otimes L \simeq H \otimes (K \otimes L)$ and that preserves the tensor unit up to (dagger) isomorphism (ref. 33, section 2.4). If a monoidal functor is an equivalence, then its adjoint is automatically also monoidal.

Theorem 10. *The functor $\mathbf{C}(I, -) : \mathbf{C} \rightarrow \mathbf{Hilb}$ is a monoidal dagger equivalence.*

Proof. After Theorem 8, it suffices to show that the functor $\mathbf{C}(I, -)$ together with the natural transformation M of Lemma 8 and the identity operator E on \mathcal{I} is a monoidal functor. By construction, the Hilbert space $\mathbf{C}(I, I)$ is just the field \mathcal{I} considered as a Hilbert space over itself, and so E is a dagger isomorphism from the value of the functor $\mathbf{C}(I, -)$ at I to the tensor unit of $\mathbf{Hilb}_{\mathcal{I}}$.

Writing $a_{H,K,L} : (H \otimes K) \otimes L \rightarrow H \otimes (K \otimes L)$ and $A_{\mathcal{H},\mathcal{K},\mathcal{L}} : (\mathcal{H} \otimes \mathcal{K}) \otimes \mathcal{L} \rightarrow \mathcal{H} \otimes (\mathcal{K} \otimes \mathcal{L})$ for the associators of \mathbf{C} and \mathbf{Hilb} , the diagram

$$\begin{array}{ccc}
 (\mathcal{H} \otimes \mathcal{K}) \otimes \mathcal{L} & \xrightarrow{A_{\mathcal{H},\mathcal{K},\mathcal{L}}} & \mathcal{H} \otimes (\mathcal{K} \otimes \mathcal{L}) \\
 M_{H,K} \otimes \text{id}_{\mathcal{L}} \downarrow & & \downarrow \text{id}_{\mathcal{H}} \otimes M_{K,L} \\
 \mathbf{C}(I, H \otimes K) \otimes \mathcal{L} & & \mathcal{H} \otimes \mathbf{C}(I, K \otimes L) \\
 M_{H \otimes K, \mathcal{L}} \downarrow & & \downarrow M_{H, K \otimes L} \\
 \mathbf{C}(I, (H \otimes K) \otimes L) & \xrightarrow{\mathbf{C}(I, a_{H,K,L})} & \mathbf{C}(I, H \otimes (K \otimes L)).
 \end{array}$$

commutes because for all $h \in \mathcal{H}$, $k \in \mathcal{K}$, and $l \in \mathcal{L}$,

$$\begin{aligned}
 & M_{H,K \otimes L}(\text{id} \otimes M_{K,L}(A_{\mathcal{H} \otimes \mathcal{K}, \mathcal{L}}((h \otimes k) \otimes l))) \\
 &= M_{H,K \otimes L}(\text{id} \otimes M_{K,L}(h \otimes (k \otimes l))) \\
 &= h \otimes (k \otimes l) = a_{H,K,L} \circ ((h \otimes k) \otimes l) \\
 &= \mathbf{C}(I, a_{H,K,L})((h \otimes k) \otimes l) \\
 &= \mathbf{C}(I, a_{H,K,L})(M_{H \otimes K, L}((M_{H,K} \otimes \text{id})((h \otimes k) \otimes l))).
 \end{aligned}$$

Thus $\mathbf{C}(I, -)$ is a monoidal functor. ■

If the category \mathbf{C} is equipped with a dagger braiding, then the functor $\mathbf{C}(I, -)$ is dagger braided by a similar argument. This means that $\mathbf{C}(I, -)$ respects the dagger isomorphisms $H \otimes K \simeq K \otimes H$. Furthermore, the braiding on \mathbf{C} is symmetric as a consequence of the assumption that I is a monoidal separator in axiom (T).

Recall that H and H^* are dagger dual objects when there is a morphism $f : I \rightarrow H^* \otimes H$, making the following diagram commute (ref. 11, definition 3.46):

$$\begin{array}{ccccc}
 H \otimes I & \simeq & H & \simeq & I \otimes H \\
 \text{id}_H \otimes f \downarrow & & & & \uparrow f^* \otimes \text{id}_H \\
 H \otimes (H^* \otimes H) & \simeq & (H \otimes H^*) \otimes H & \simeq & (H^* \otimes H) \otimes H.
 \end{array}$$

In \mathbf{Hilb} , an object has a dagger dual if and only if it is finite dimensional (ref. 11, corollary 3.65). Because monoidal dagger functors preserve dagger dual objects (ref. 11, theorem 3.14), it follows from Theorem 10 that the finite-dimensional Hilbert spaces can be categorically axiomatized within \mathbf{C} as the dagger dual objects.

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1. J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, 1955).
2. M. Rédei, Why John von Neumann did not like the Hilbert space formalism of quantum mechanics (and what he liked instead). *Stud. Hist. Philos. Mod. Phys.* **27**, 493–510 (1996).
3. L. Hardy, Quantum theory from five reasonable axioms. arXiv [Preprint] (2001). <https://arxiv.org/abs/quant-ph/0101012v4> (Accessed 10 February 2022).
4. A. Kapustin, Is quantum mechanics exact? *J. Math. Phys.* **54**, 062107 (2013).
5. F. J. Murray, J. von Neumann, On rings of operators. *Ann. Math.* **37**, 116–229 (1936).
6. J. M. Jauch, *Foundations of Quantum Mechanics* (Addison-Wesley, 1968).
7. C. Piron, *Foundations of Quantum Physics* (W. A. Benjamin, 1976).
8. S. Abramsky, B. Coecke, “A categorical semantics of quantum protocols” in *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, 2004* (2004), pp. 415–425.
9. P. Ghez, R. Lima, J. E. Roberts, W^* -categories. *Pac. J. Math.* **120**, 79–109 (1985).
10. S. Mac Lane, *Categories for the Working Mathematician* (Springer, ed. 2, 1971).
11. C. Heunen, J. Vicary, *Categories for Quantum Theory* (Oxford University Press, 2019).
12. F. W. Lawvere, An elementary theory of the category of sets. *Proc. Natl. Acad. Sci. U.S.A.* **52**, 1506–1511 (1964).
13. C. Heunen, M. Karvonen, Limits in dagger categories. *Theory Appl. Categ.* **34**, 468–513 (2019).
14. S. Mac Lane, Duality for groups. *Bull. Am. Math. Soc.* **56**, 485–516 (1950).
15. P. Selinger, “Dagger compact closed categories and completely positive maps” in *Quantum Physics and Logic*, P. Selinger, Ed. (Electronic Notes in Theoretical Computer Sciences, Elsevier B.V., 2007), vol. **170**, pp. 139–163.
16. B. Mitchell, *The Full Embedding Theorem* (Johns Hopkins University Press, 1964).
17. J. Vicary, Completeness of \dagger -categories and the complex numbers. *J. Math. Phys.* **42**, 082104 (2011).
18. C. Heunen, An embedding theorem for Hilbert categories. *Theory Appl. Categ.* **13**, 321–344 (2009).
19. I. Kaplansky, Forms in infinite-dimensional spaces. *An. Acad. Bras. Cienc.* **22**, 1–17 (1950).
20. I. Amemiya, H. Araki, A remark on Piron’s paper. *Publ. Res. Inst. Math. Sci.* **2**, 423–427 (1966).
21. C. Piron, Axiomatique quantique. *Helv. Phys. Acta* **37**, 439–468 (1964).
22. W. J. Wilbur, On characterizing the standard quantum logics. *Trans. Am. Math. Soc.* **233**, 265–282 (1977).
23. H. Gross, Hilbert lattices: New results and unsolved problems. *Found. Phys.* **20**, 529–559 (1990).
24. M. P. Solèr, Characterization of Hilbert spaces by orthomodular spaces. *Commun. Algebra* **1**, 219–243 (1995).
25. F. Maeda, S. Maeda, *Theory of Symmetric Lattices* (Springer, 1970).
26. R. Mayet, Some characterizations of the underlying division ring of a Hilbert lattice by automorphisms. *Int. J. Theor. Phys.* **37**, 109–114 (1998).
27. T. Vetterlein, Orthogonality spaces arising from infinite-dimensional complex Hilbert spaces. *Int. J. Theor. Phys.* **60**, 727–738 (2021).
28. A. Grinbaum, Reconstruction of quantum theory. *Br. J. Philos. Sci.* **58**, 387–408 (2007).
29. C. Heunen, B. Jacobs, Quantum logic in dagger kernel categories. *Order* **27**, 177–212 (2010).
30. S. S. Holland Jr., Orthomodularity in infinite dimensions; a theorem of M. Solèr. *Bull. Am. Math. Soc.* **32**, 205–234 (1995).
31. J. C. Baez, Division algebras and quantum theory. *Found. Phys.* **42**, 819–855 (2012).
32. M. Reed, B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis* (Academic Press, 1980).
33. P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, *Tensor Categories* (American Mathematical Society, 2015).