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# TWO RATIONAL NODAL QUARTIC THREEFOLDS

IVAN CHELTSOV AND CONSTANTIN SHRAMOV

ABSTRACT. We prove that the quartic threefolds defined by

$$\sum_{i=0}^5 x_i = \sum_{i=0}^5 x_i^4 - t \left( \sum_{i=0}^5 x_i^2 \right)^2 = 0$$

in  $\mathbb{P}^5$  are rational for  $t = \frac{1}{6}$  and  $t = \frac{7}{10}$ .

## 1. INTRODUCTION

Consider the six-dimensional permutation representation  $\mathbb{W}$  of the group  $\mathfrak{S}_6$ . Choose coordinates  $x_0, \dots, x_5$  in  $\mathbb{W}$  so that they are permuted by  $\mathfrak{S}_6$ . Then  $x_0, \dots, x_5$  also serve as homogeneous coordinates in the projective space  $\mathbb{P}^5 = \mathbb{P}(\mathbb{W})$ .

Let us identify  $\mathbb{P}^4$  with a hyperplane

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 0$$

in  $\mathbb{P}^5$ . Denote by  $X_t$  the quartic threefold in  $\mathbb{P}^4$  that is given by the equation

$$(1.1) \quad x_0^4 + x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 = t \left( x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \right)^2,$$

where  $t$  is an element of the ground field, which we will always assume to be the field  $\mathbb{C}$  of complex numbers. Then  $X_t$  is singular for every  $t \in \mathbb{C}$ . Indeed, denote by  $\Sigma_{30}$  the  $\mathfrak{S}_6$ -orbit of the point  $[1 : 1 : \omega : \omega : \omega^2 : \omega^2]$ , where  $\omega = e^{\frac{2\pi i}{3}}$ . Then  $|\Sigma_{30}| = 30$ , and  $X_t$  is singular at every point of  $\Sigma_{30}$  for every  $t \in \mathbb{C}$  (see, for example, [12, Theorem 4.1]).

The possible singularities of the quartic threefold  $X_t$  have been described by van der Geer in [12, Theorem 4.1]. To recall his description, denote by  $\mathcal{L}_{15}$  the  $\mathfrak{S}_6$ -orbit of the line that passes through the points  $[1 : 0 : -1 : 1 : 0 : -1]$  and  $[0 : 1 : -1 : 0 : 1 : -1]$ , and denote by  $\Sigma_6$ ,  $\Sigma_{10}$ , and  $\Sigma_{15}$  the  $\mathfrak{S}_6$ -orbits of the points  $[-5 : 1 : 1 : 1 : 1 : 1]$ ,  $[-1 : -1 : -1 : 1 : 1 : 1]$ , and  $[1 : -1 : 0 : 0 : 0 : 0]$ , respectively. Then the curve  $\mathcal{L}_{15}$  is a union of fifteen lines, while  $|\Sigma_6| = 6$ ,  $|\Sigma_{10}| = 10$ , and  $|\Sigma_{15}| = 15$ . Moreover, one has

$$\text{Sing}(X_t) = \begin{cases} \mathcal{L}_{15} & \text{if } t = \frac{1}{4}, \\ \Sigma_{30} \cup \Sigma_{15} & \text{if } t = \frac{1}{2}, \\ \Sigma_{30} \cup \Sigma_{10} & \text{if } t = \frac{1}{6}, \\ \Sigma_{30} \cup \Sigma_6 & \text{if } t = \frac{7}{10}, \\ \Sigma_{30} & \text{otherwise.} \end{cases}$$

Furthermore, if  $t \neq \frac{1}{4}$ , then all singular points of the quartic threefold  $X_t$  are isolated ordinary double points (nodes).

The threefold  $X_{\frac{1}{2}}$  is classical. It is the so-called *Burkhardt quartic*. In [3], Burkhardt discovered that the subset  $\Sigma_{30} \cup \Sigma_{15}$  is invariant under the action of the simple group  $\mathrm{PSP}_4(\mathbf{F}_3)$  of order 25920. In [7], Coble proved that  $\Sigma_{30} \cup \Sigma_{15}$  is the singular locus of the threefold  $X_{\frac{1}{2}}$ , and proved that  $X_{\frac{1}{2}}$  is also  $\mathrm{PSP}_4(\mathbf{F}_3)$ -invariant. Later Todd proved in [22] that  $X_{\frac{1}{2}}$  is rational. In [15], de Jong, Shepherd-Barron, and Van de Ven proved that  $X_{\frac{1}{2}}$  is the unique quartic threefold in  $\mathbb{P}^4$  with 45 singular points.

The quartic threefold  $X_{\frac{1}{4}}$  is also classical. It is known as the *Igusa quartic* from its modular interpretation as the Satake compactification of the moduli space of Abelian surfaces with level 2 structure (see [12]). The projectively dual variety of the quartic threefold  $X_{\frac{1}{4}}$  is the so-called *Segre cubic*. Since the Segre cubic is rational,  $X_{\frac{1}{4}}$  is rational as well.

During *Kullfest* conference dedicated to the 60th anniversary of Viktor Kulikov that was held in Moscow in December 2012, Alexei Bondal and Yuri Prokhorov posed

**Problem 1.2.** *Determine all  $t \in \mathbb{C}$  such that  $X_t$  is rational.*

Since  $X_t$  is singular, we cannot apply Iskovskikh and Manin's theorem from [14] to  $X_t$ . Similarly, we cannot apply Mella's [18, Theorem 2] to  $X_t$  either, because the quartic threefold  $X_t$  is not  $\mathbb{Q}$ -factorial by [1, Lemma 2]. Nevertheless, Beauville proved

**Theorem 1.3** ([1]). *If  $t \notin \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{7}{10}\}$ , then  $X_t$  is non-rational.*

Both  $X_{\frac{1}{2}}$  and  $X_{\frac{1}{4}}$  are rational. The goal of this paper is to prove

**Theorem 1.4.** *The quartic threefolds  $X_{\frac{1}{6}}$  and  $X_{\frac{7}{10}}$  are also rational.*

Surprisingly, the proof of Theorem 1.4 goes back to two classical papers of Todd. Namely, we will construct an explicit  $\mathfrak{A}_6$ -birational map  $\mathbb{P}^3 \dashrightarrow X_{\frac{7}{10}}$  that is a special case of Todd's construction from [20]. Similarly, we will construct an explicit  $\mathfrak{S}_5$ -birational map  $\mathbb{P}^3 \dashrightarrow X_{\frac{1}{6}}$  that is a degeneration of Todd's construction from [21]. We emphasize that our proof is self-contained, i.e. it does not rely on the results proved in [20] and [21], but recovers the necessary facts in our particular situation using additional symmetries arising from group actions.

*Remark 1.5.* Todd proved in [22] that the Burkhardt quartic  $X_{\frac{1}{2}}$  is determinantal (see also [19, §5.1]). The constructions of our birational maps  $\mathbb{P}^3 \dashrightarrow X_{\frac{7}{10}}$  and  $\mathbb{P}^3 \dashrightarrow X_{\frac{1}{6}}$  imply that both  $X_{\frac{7}{10}}$  and  $X_{\frac{1}{6}}$  are determinantal (see [19, Example 6.4.2] and [19, Example 6.2.1]). Yuri Prokhorov pointed out that the quartic threefold

$$\det \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_4 & y_0 & y_3 & y_4 \\ y_2 & y_1 & y_1 & y_0 \\ y_0 & y_3 & y_2 & y_4 \end{pmatrix} = 0$$

in  $\mathbb{P}^4$  with homogeneous coordinates  $y_0, \dots, y_4$  has exactly 45 singular points. Thus, it is isomorphic to the Burkhardt quartic  $X_{\frac{1}{2}}$  by [15]. It would be interesting to find similar determinantal equations of the threefolds  $X_{\frac{7}{10}}$  and  $X_{\frac{1}{6}}$ .

The plan of the paper is as follows. In Section 2 we recall some preliminary results on representations of a central extension of the group  $\mathfrak{S}_6$ , and some of its subgroups.

In Section 3 we collect results concerning a certain action of the group  $\mathfrak{A}_5$  on  $\mathbb{P}^3$ , and study  $\mathfrak{A}_5$ -invariant quartic surfaces; the reason we pay so much attention to this group is that it is contained both in  $\mathfrak{A}_6$  and in  $\mathfrak{S}_5$ , and thus the information about its properties simplifies the study of the latter two groups. In Section 4 we collect auxiliary results about the groups  $\mathfrak{S}_6$ ,  $\mathfrak{A}_6$  and  $\mathfrak{S}_5$ , in particular about their actions on curves and their five-dimensional irreducible representations. In Section 5 we construct an  $\mathfrak{A}_6$ -equivariant birational map  $\mathbb{P}^3 \dashrightarrow X_{\frac{7}{10}}$ . Finally, in Section 6 we construct an  $\mathfrak{S}_5$ -equivariant birational map  $\mathbb{P}^3 \dashrightarrow X_{\frac{1}{6}}$  and make some concluding remarks.

Throughout the paper, we denote a cyclic group of order  $n$  by  $\mu_n$ , and we denote a dihedral group of order  $2n$  by  $D_{2n}$ . In particular, one has  $D_{12} \cong \mathfrak{S}_3 \times \mu_2$ . By  $F_{36}$  we denote a group isomorphic to  $(\mu_3 \times \mu_3) \rtimes \mu_4$ , and by  $F_{20}$  we denote a group isomorphic to  $\mu_5 \rtimes \mu_4$ .

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## 2. REPRESENTATION THEORY

Recall that the permutation group  $\mathfrak{S}_6$  has two central extensions  $2^+\mathfrak{S}_6$  and  $2^-\mathfrak{S}_6$  by the group  $\mu_2$  with the central subgroup contained in the commutator subgroup (see [8, p. xxiii] for details). We denote the first of them (i. e. the one where the preimages of a transposition in  $\mathfrak{S}_6$  under the natural projection have order two) by  $2.\mathfrak{S}_6$  to simplify notation. Similarly, for any group  $\Gamma$  we denote by  $2.\Gamma$  a non-split central extension of  $\Gamma$  by the group  $\mu_2$ .

We start with recalling some facts about four- and five-dimensional representations of the group  $2.\mathfrak{S}_6$  we will be working with. A reader who is not interested in details here can skip to Corollary 2.1, or even to Section 4 where we reformulate everything in geometric language. Also, we will see in Section 4 that our further constructions do not depend much on the choice of representations, and all computations one makes for one of them actually apply to all others.

Let  $\mathbb{I}$  and  $\mathbb{J}$  be the trivial and the non-trivial one-dimensional representations of the group  $\mathfrak{S}_6$ , respectively. Consider the six-dimensional permutation representation  $\mathbb{W}$  of  $\mathfrak{S}_6$ . One has

$$\mathbb{W} \cong \mathbb{I} \oplus \mathbb{W}_5 \otimes \mathbb{J}$$

for some irreducible representation  $\mathbb{W}_5$  of  $\mathfrak{S}_6$ . We can regard  $\mathbb{I}$ ,  $\mathbb{J}$  and  $\mathbb{W}_5$  as representations of the group  $2.\mathfrak{S}_6$ . Recall that there is a double cover  $SL_4(\mathbb{C}) \rightarrow SO_6(\mathbb{C})$ , see e.g. [10, Exercise 20.39]. Using it, we conclude that there is an embedding of the group  $2.\mathfrak{S}_6$  into  $SL_4(\mathbb{C})$ . This embedding gives rise to two four-dimensional representations of  $2.\mathfrak{S}_6$  that differ by a tensor product with  $\mathbb{J}$ . We fix one of these two representations  $\mathbb{U}_4$ . Note that

$$\mathbb{I} \oplus \mathbb{W}_5 \cong \Lambda^2(\mathbb{U}_4).$$

Recall that there are coordinates  $x_0, \dots, x_5$  in  $\mathbb{W}$  that are permuted by the group  $\mathfrak{S}_6$ . We will refer to a subgroup of  $2.\mathfrak{S}_6$  fixing one of the corresponding points as a *standard* subgroup  $2.\mathfrak{S}_5$ ; we denote any such subgroup by  $2.\mathfrak{S}_5^{st}$ . A subgroup of  $2.\mathfrak{S}_6$  that is isomorphic to  $2.\mathfrak{S}_5$  but is not conjugate to a standard  $2.\mathfrak{S}_5$  will be called a *non-standard* subgroup  $2.\mathfrak{S}_5$ ; we denote any such subgroup by  $2.\mathfrak{S}_5^{nst}$ . These agree with standard and non-standard subgroups of  $\mathfrak{S}_6$  isomorphic to  $\mathfrak{S}_5$ , although outer automorphisms of  $\mathfrak{S}_6$  do not lift to  $2.\mathfrak{S}_6$ . Any subgroup of  $2.\mathfrak{S}_6$  that is isomorphic to  $2.\mathfrak{A}_5$ ,  $2.\mathfrak{S}_4$  or  $2.\mathfrak{A}_4$  and is contained in  $2.\mathfrak{S}_5^{st}$  is denoted by  $2.\mathfrak{A}_5^{st}$ ,  $2.\mathfrak{S}_4^{st}$  or  $2.\mathfrak{A}_4^{st}$ , respectively. Similarly, any subgroup of  $2.\mathfrak{S}_6$  that is isomorphic to  $2.\mathfrak{A}_5$ ,  $2.\mathfrak{S}_4$  or  $2.\mathfrak{A}_4$  and is contained in  $2.\mathfrak{S}_5^{nst}$  is denoted by  $2.\mathfrak{A}_5^{nst}$ ,  $2.\mathfrak{S}_4^{nst}$  or  $2.\mathfrak{A}_4^{nst}$ , respectively.

The values of characters of important representations of the group  $2.\mathfrak{S}_6$ , and the information about some of its subgroups are presented in Table 1, cf. [8, p. 5]. The first two columns of Table 1 describe conjugacy classes of elements of the group  $2.\mathfrak{S}_6$ . The first column lists the orders of the elements in the corresponding conjugacy class, and the second column, except for the entries in the second and the third row, gives a cycle type of the image of an element under projection to  $\mathfrak{S}_6$  (for example,  $[3, 2]$  denotes a product of two disjoint cycles of lengths 3 and 2). By  $\text{id}$  we denote the identity element of  $2.\mathfrak{S}_6$ , and  $z$  denotes the unique non-trivial central element of  $2.\mathfrak{S}_6$ . Note that the preimages of some of conjugacy classes in  $\mathfrak{S}_6$  split into a union of two conjugacy classes in  $2.\mathfrak{S}_6$ . The next three columns list the values of the characters of the representations  $\mathbb{W}$ ,  $\mathbb{W}_5$  and  $\mathbb{U}_4$  of  $2.\mathfrak{S}_6$ . Note that there is no real ambiguity in the choice of  $\sqrt{-3}$  since we did not specify any way to distinguish the two conjugacy classes in  $2.\mathfrak{S}_6$  whose elements are projected to cycles of length 6 in  $\mathfrak{S}_6$  up to this point (note that the two ways to choose a sign here is exactly a tensor multiplication of the representation with  $\mathbb{J}$ , i.e. the choice between two homomorphisms of  $2.\mathfrak{S}_6$  to  $\text{SL}_4(\mathbb{C})$  having the same image). The remaining columns list the numbers of elements from each of the conjugacy classes of  $2.\mathfrak{S}_6$  in subgroups of certain types. By  $2.F_{36}$  (respectively, by  $2.F_{20}$ , by  $2.D_{12}^{nst}$ ) we denote a subgroup of  $2.\mathfrak{S}_6$  (respectively, of  $2.\mathfrak{S}_6$ , or of  $2.\mathfrak{S}_5^{nst}$ ) isomorphic to a central extension of  $F_{36}$  (respectively, of  $F_{20}$ , or of  $D_{12}$ ) by  $\mu_2$ . A subgroup  $2.F_{20}$  is actually contained in a subgroup  $2.\mathfrak{S}_5^{st}$  and in a subgroup  $2.\mathfrak{S}_5^{nst}$ . Note that the intersection of a conjugacy class in a group with a subgroup may (and often does) split into several conjugacy classes in this subgroup.

It is immediate to see from Table 1 that  $\mathbb{U}_4$  is an irreducible representation of the group  $2.\mathfrak{S}_6$ . Using the information provided by Table 1, we immediately obtain the following results.

**Corollary 2.1.** *Let  $\Gamma$  be a subgroup of  $2.\mathfrak{S}_6$ . After restriction to the subgroup  $\Gamma$  the  $2.\mathfrak{S}_6$ -representation  $\mathbb{U}_4$*

- (i) *remains irreducible, if  $\Gamma$  is one of the subgroups  $2.\mathfrak{A}_6$ ,  $2.\mathfrak{S}_5^{nst}$ ,  $2.\mathfrak{A}_5^{nst}$ ,  $2.\mathfrak{S}_4^{nst}$ ,  $2.F_{36}$ , or  $2.F_{20}$ ;*
- (ii) *splits into a sum of two non-isomorphic irreducible two-dimensional representations, if  $\Gamma$  is one of the subgroups  $2.\mathfrak{A}_5^{st}$ ,  $2.\mathfrak{A}_4^{nst}$ , or  $2.D_{12}^{nst}$ .*

*Proof.* Compute inner products of the corresponding characters with themselves, and keep in mind that neither of the groups  $2.\mathfrak{A}_5^{st}$ ,  $2.\mathfrak{A}_4^{nst}$ , and  $2.D_{12}^{nst}$  has an irreducible three-dimensional representation with a non-trivial action of the central subgroup.  $\square$

*Remark 2.2.* By Corollary 2.1(i), the  $2.\mathfrak{S}_5^{nst}$ -representation  $\mathbb{U}_4$  is irreducible. One can check that it is not induced from any proper subgroup of  $2.\mathfrak{S}_5^{nst}$ , i.e. it defines a primitive

TABLE 1. Characters and subgroups of the group  $2.\mathfrak{S}_6$ 

ord	type	$\mathbb{W}$	$\mathbb{W}_5$	$\mathbb{U}_4$	$2.\mathfrak{S}_6$	$2.\mathfrak{A}_6$	$2.\mathfrak{S}_5^{nst}$	$2.\mathfrak{A}_5^{st}$	$2.\mathfrak{A}_5^{nst}$	$2.\mathfrak{S}_4^{nst}$	$2.\mathfrak{A}_4^{nst}$	$2.F_{36}$	$2.F_{20}$	$2.D_{12}^{nst}$
1	id	6	5	4	1	1	1	1	1	1	1	1	1	1
2	$z$	6	5	-4	1	1	1	1	1	1	1	1	1	1
2	[2]	4	-3	0	30	0	0	0	0	0	0	0	0	0
4	[2, 2]	2	1	0	90	90	30	30	30	6	6	18	10	6
4	[2, 2, 2]	0	1	0	30	0	20	0	0	12	0	0	0	8
6	[3]	3	2	2	40	40	0	20	0	0	0	4	0	0
3	[3]	3	2	-2	40	40	0	20	0	0	0	4	0	0
6	[3, 2]	1	0	0	120	0	0	0	0	0	0	0	0	0
6	[3, 2]	1	0	0	120	0	0	0	0	0	0	0	0	0
6	[3, 3]	0	-1	-1	40	40	20	0	20	8	8	4	0	2
3	[3, 3]	0	-1	1	40	40	20	0	20	8	8	4	0	2
8	[4]	2	-1	0	180	0	60	0	0	12	0	0	20	0
8	[4, 2]	0	-1	0	180	180	0	0	0	0	0	36	0	0
10	[5]	1	0	1	144	144	24	24	24	0	0	0	4	0
5	[5]	1	0	-1	144	144	24	24	24	0	0	0	4	0
12	[6]	0	1	$\sqrt{-3}$	120	0	20	0	0	0	0	0	0	2
12	[6]	0	1	$-\sqrt{-3}$	120	0	20	0	0	0	0	0	0	2

subgroup isomorphic to  $2.\mathfrak{S}_5$  in  $\mathrm{GL}_4(\mathbb{C})$ . Note that this subgroup is not present in the list given in [9, §8.5]. It is still listed by some other classical surveys, see e.g. [2, §119].

**Corollary 2.3.** *Let  $\Gamma$  be a subgroup of  $2.\mathfrak{S}_6$ . After restriction to the subgroup  $\Gamma$  the  $2.\mathfrak{S}_6$ -representation  $\mathbb{W}_5$*

- (i) *remains irreducible, if  $\Gamma$  is one of the subgroups  $2.\mathfrak{A}_6$ ,  $2.\mathfrak{S}_5^{nst}$ , or  $2.\mathfrak{A}_5^{nst}$ ;*
- (ii) *splits into a sum of the trivial and an irreducible four-dimensional representation if  $\Gamma$  is a subgroup  $2.\mathfrak{A}_5^{st}$ ;*
- (iii) *splits into a sum of the trivial and two different irreducible two-dimensional representations if  $\Gamma$  is a subgroup  $2.D_{12}^{nst}$ .*

In the sequel we will denote the restrictions of the  $2.\mathfrak{S}_6$ -representations  $\mathbb{U}_4$  and  $\mathbb{W}_5$  to various subgroups by the same symbols for simplicity. The next two corollaries are implied by direct computations (we used GAP software [11] to perform them).

**Corollary 2.4.** *The following assertions hold:*

- (i) *the  $\mathfrak{A}_6$ -representation  $\mathrm{Sym}^2(\mathbb{U}_4^\vee)$  does not contain one-dimensional subrepresentations;*
- (ii) *the  $\mathfrak{A}_6$ -representation  $\mathrm{Sym}^4(\mathbb{U}_4^\vee)$  does not contain one-dimensional subrepresentations;*
- (iii) *the  $\mathfrak{A}_5^{nst}$ -representation  $\mathrm{Sym}^2(\mathbb{U}_4^\vee)$  splits into a sum of two different irreducible three-dimensional representations and one irreducible four-dimensional representation;*
- (iv) *the  $2.\mathfrak{A}_5^{nst}$ -representation  $\mathrm{Sym}^3(\mathbb{U}_4^\vee)$  does not contain one-dimensional subrepresentations;*
- (v) *the  $\mathfrak{A}_5^{nst}$ -representation  $\mathrm{Sym}^4(\mathbb{U}_4^\vee)$  has a unique two-dimensional subrepresentation, and this subrepresentation splits into a sum of two trivial representations of  $\mathfrak{A}_5^{nst}$ .*

Recall that all representations of a symmetric group are self-dual. Therefore, to study invariant hypersurfaces in  $\mathbb{P}(\mathbb{W}_5)$  we will use the following result.

**Corollary 2.5.** *Let  $\Gamma$  be one of the groups  $\mathfrak{S}_6$ ,  $\mathfrak{A}_6$  or  $\mathfrak{S}_5^{nst}$ . Then*

- (i) *the  $\Gamma$ -representation  $\mathrm{Sym}^2(\mathbb{W}_5)$  has a unique one-dimensional subrepresentation;*
- (ii) *the  $\Gamma$ -representation  $\mathrm{Sym}^4(\mathbb{W}_5)$  has a unique two-dimensional subrepresentation, and this subrepresentation splits into a sum of two trivial representations of  $\Gamma$ .*

We conclude this section by recalling some information about several subgroups of  $2.\mathfrak{S}_6$  that are smaller than those listed in Table 1. Namely, we list in Table 2 orders, types and numbers of elements in certain subgroups of  $2.\mathfrak{A}_5^{nst}$ . We keep the notation used in Table 1. By  $2.\mathfrak{S}'_3$  we denote a subgroup of  $2.\mathfrak{A}_5^{nst}$  isomorphic to  $2.\mathfrak{S}_3$ . Note that the preimage in  $2.\mathfrak{S}_6$  of any subgroup  $\mu_5 \subset \mathfrak{S}_6$  is isomorphic to  $\mu_{10}$ .

Looking at Table 2 (and keeping in mind character values provided by Table 1) we immediately obtain the following.

**Corollary 2.6.** *Let  $\Gamma$  be a subgroup of  $2.\mathfrak{A}_5^{nst} \subset 2.\mathfrak{S}_6$ . After restriction to  $\Gamma$  the  $2.\mathfrak{S}_6$ -representation  $\mathbb{U}_4$*

- (i) *splits into a sum of two non-isomorphic irreducible two-dimensional representations if  $\Gamma$  is a subgroup  $2.D_{10}$ ;*
- (ii) *splits into a sum of an irreducible two-dimensional representation and two non-isomorphic one-dimensional representations if  $\Gamma$  is a subgroup  $2.\mathfrak{S}'_3$ ;*

TABLE 2. Subgroups of  $2.\mathfrak{A}_5^{nst}$ 

ord	type	$2.D_{10}$	$2.\mathfrak{S}'_3$	$2.(\mu_2 \times \mu_2)$	$\mu_{10}$
1	id	1	1	1	1
2	$z$	1	1	1	1
4	$[2, 2]$	10	6	6	0
6	$[3, 3]$	0	2	0	0
3	$[3, 3]$	0	2	0	0
10	$[5]$	4	0	0	4
5	$[5]$	4	0	0	4

- (iii) splits into a sum of two isomorphic irreducible two-dimensional representations if  $\Gamma$  is a subgroup  $2.(\mu_2 \times \mu_2)$ ;
- (iv) splits into a sum of four pairwise non-isomorphic one-dimensional representations if  $\Gamma$  is a subgroup  $\mu_{10}$ .

### 3. ICOSAHEDRAL GROUP IN THREE DIMENSIONS

In this section, we consider the action of the group  $\mathfrak{A}_5$  on the projective space  $\mathbb{P}^3$  arising from a non-standard embedding of  $\mathfrak{A}_5 \hookrightarrow \mathfrak{S}_6$ . Namely, we identify  $\mathbb{P}^3$  with the projectivization  $\mathbb{P}(\mathbb{U}_4)$ , where  $\mathbb{U}_4$  is the restriction of the four-dimensional irreducible representation of the group  $2.\mathfrak{S}_6$  introduced in Section 2 to a subgroup  $2.\mathfrak{A}_5^{nst}$  (which we will refer to as just  $2.\mathfrak{A}_5$  in this section). Recall from Corollary 2.1(i) that  $\mathbb{U}_4$  is an irreducible representation of  $2.\mathfrak{A}_5$ .

*Remark 3.1* (see e. g. [8, p. 2]). Let  $\Gamma$  be a proper subgroup of  $\mathfrak{A}_5$  such that the index of  $\Gamma$  is at most 15. Then  $\Gamma$  is isomorphic either to  $\mathfrak{A}_4$ , or to  $D_{10}$ , or to  $\mathfrak{S}_3$ , or to  $\mu_5$ , or to  $\mu_2 \times \mu_2$ . In particular, if  $\mathfrak{A}_5$  acts transitively on the set of  $r < 15$  elements, then  $r \in \{5, 6, 10, 12\}$ .

**Lemma 3.2.** *Let  $\Omega$  be an  $\mathfrak{A}_5$ -orbit of length  $r \leq 15$  in  $\mathbb{P}^3$ . Then either  $r = 10$ , or  $r = 12$ . Moreover,  $\mathbb{P}^3$  contains exactly two  $\mathfrak{A}_5$ -orbits of length 10 and exactly two  $\mathfrak{A}_5$ -orbits of length 12.*

*Proof.* By Remark 3.1 one has  $r \in \{1, 5, 6, 10, 12, 15\}$ . The case  $r = 1$  is impossible since  $\mathbb{U}_4$  is an irreducible  $2.\mathfrak{A}_5$ -representation. Restricting  $\mathbb{U}_4$  to subgroups of  $2.\mathfrak{A}_5$  isomorphic to  $2.\mathfrak{A}_4$ ,  $2.D_{10}$ , and  $2.(\mu_2 \times \mu_2)$ , and applying Corollaries 2.1(ii) and 2.6(i),(iii), we see that  $r \notin \{5, 6, 15\}$ .

Restricting  $\mathbb{U}_4$  to a subgroup of  $2.\mathfrak{A}_5$  isomorphic to  $2.\mathfrak{S}_3$ , applying Corollary 2.6(ii) and keeping in mind that there are ten subgroups isomorphic to  $\mathfrak{S}_3$  in  $\mathfrak{A}_5$ , we see that  $\mathbb{P}^3$  contains exactly two  $\mathfrak{A}_5$ -orbits of length 10.

Finally, restricting  $\mathbb{U}_4$  to a subgroup of  $2.\mathfrak{A}_5$  isomorphic to  $\mu_{10}$ , applying Corollary 2.6(iv) and keeping in mind that there are six subgroups isomorphic to  $\mu_5$  in  $\mathfrak{A}_5$ , we see that  $\mathbb{P}^3$  contains exactly two  $\mathfrak{A}_5$ -orbits of length 12.  $\square$

**Lemma 3.3.** *There are no  $\mathfrak{A}_5$ -invariant surfaces of degree at most three in  $\mathbb{P}^3$ .*

*Proof.* Apply Corollary 2.4(iii),(iv).  $\square$



By Corollary 2.1(ii), the subgroup  $\mathfrak{A}_4 \subset \mathfrak{A}_5$  leaves invariant two disjoint lines in  $\mathbb{P}^3$ , say  $L_1$  and  $L'_1$ . Let  $L_1, \dots, L_5$  be the  $\mathfrak{A}_5$ -orbit of the line  $L_1$ , and let  $L'_1, \dots, L'_5$  be the  $\mathfrak{A}_5$ -orbit of the line  $L'_1$ .

**Lemma 3.4.** *The lines  $L_1, \dots, L_5$  (respectively, the lines  $L'_1, \dots, L'_5$ ) are pairwise disjoint.*

*Proof.* Suppose that some of the lines  $L_1, \dots, L_5$  have a common point. Since the action of  $\mathfrak{A}_5$  on the set  $\{L_1, \dots, L_5\}$  is doubly transitive, this implies that every two of the lines  $L_1, \dots, L_5$  have a common point. Therefore, either all lines  $L_1, \dots, L_5$  are coplanar, or all of them pass through one point. Both of these cases are impossible since the  $2\mathfrak{A}_5$ -representation  $\mathbb{U}_4$  is irreducible by Corollary 2.1(i). Therefore, the lines  $L_1, \dots, L_5$  are pairwise disjoint. A similar argument applies to the lines  $L'_1, \dots, L'_5$ .  $\square$

**Corollary 3.5.** *Any  $\mathfrak{A}_5$ -orbit contained in the union  $L_1 \cup \dots \cup L_5$  has length at least 20.*

*Proof.* Corollary 2.1(ii) implies that the stabilizer  $\Gamma \cong \mathfrak{A}_4$  of the line  $L_1$  acts on  $L_1$  faithfully. Therefore, the length of any  $\Gamma$ -orbit contained in  $L_1$  is at least four. Thus the required assertion follows from Lemma 3.4.  $\square$

We are going to describe the configuration formed by the lines  $L_1, \dots, L_5, L'_1, \dots, L'_5$ .

**Definition 3.6.** Let  $T_1, \dots, T_5, T'_1, \dots, T'_5$  be different lines in a projective space. We say that they form a *double five configuration* if the following conditions hold:

- the lines  $T_1, \dots, T_5$  (respectively, the lines  $T'_1, \dots, T'_5$ ) are pairwise disjoint;
- for every  $i$  the lines  $T_i$  and  $T'_i$  are disjoint;
- for every  $i \neq j$  the line  $T_i$  meets the line  $T'_j$ .

**Lemma 3.7.** *The lines  $L_1, \dots, L_5, L'_1, \dots, L'_5$  form a double five configuration. Moreover, the only line in  $\mathbb{P}^3$  that intersects all lines of  $L_1, \dots, L_5$  but  $L_i$  is the line  $L'_i$ , and the only line in  $\mathbb{P}^3$  that intersects all lines of  $L'_1, \dots, L'_5$  but  $L'_i$  is the line  $L_i$ .*

*Proof.* For any  $i$  the lines  $L_i$  and  $L'_i$  are disjoint by construction. The lines  $L_1, \dots, L_5$  (respectively, the lines  $L'_1, \dots, L'_5$ ) are pairwise disjoint by Lemma 3.4.

Since any three pairwise skew lines in  $\mathbb{P}^3$  are contained in a smooth quadric surface, and an intersection of two different quadric surfaces in  $\mathbb{P}^3$  cannot contain three pairwise skew lines, we see that for any three indices  $1 \leq i < j < k \leq 5$  there is a unique quadric surface  $Q_{ijk}$  in  $\mathbb{P}^3$  passing through the lines  $L_i, L_j$  and  $L_k$ . Moreover, the quadric  $Q_{ijk}$  is smooth. Note also that the quadric  $Q_{ijk}$  is not  $\mathfrak{A}_5$ -invariant by Lemma 3.3. This implies that all five lines  $L_1, \dots, L_5$  are not contained in a quadric.

Therefore, we may assume that the quadric  $Q_{123}$  does not contain the line  $L_4$ . It is well-known that in this case either there is a unique line  $L$  meeting all four lines  $L_1, \dots, L_4$ , or there are exactly two lines  $L$  and  $L'$  meeting  $L_1, \dots, L_4$ . In the latter case the stabilizer  $\Gamma \subset \mathfrak{A}_5$  of the quadruple  $L_1, \dots, L_4$  (i.e. the stabilizer of the line  $L_5$ ) preserves the lines  $L_5, L$  and  $L'$ . On the other hand, the lines  $L$  and  $L'$  are different from  $L_5$  since  $L_5$  meets neither of the lines  $L_1, \dots, L_4$ ; moreover, the group  $\Gamma \cong \mathfrak{A}_4$  fixes both  $L$  and  $L'$ . But  $\Gamma$  cannot fix three different lines in  $\mathbb{P}^3$  by Corollary 2.1(ii). The contradiction shows that there is a unique line  $L$  meeting  $L_1, \dots, L_4$ . Again we see that  $L \neq L_5$ , so that  $L = L'_5$  by Corollary 2.1(ii).

Since the group  $\mathfrak{A}_5$  permutes the lines  $L_1, \dots, L_5$  transitively, we conclude that the only line in  $\mathbb{P}^3$  that intersects all lines of  $L_1, \dots, L_5$  except  $L_i$  is the line  $L'_i$ . Similarly, we see that the only line in  $\mathbb{P}^3$  that intersects all lines of  $L'_1, \dots, L'_5$  except  $L'_i$  is the line  $L_i$ . In particular, the lines  $L_1, \dots, L_5, L'_1, \dots, L'_5$  form a double five configuration.  $\square$

**Lemma 3.8.** *Every  $\mathfrak{A}_5$ -invariant curve of degree at most three in  $\mathbb{P}^3$  is a twisted cubic. Moreover, there are exactly two  $\mathfrak{A}_5$ -invariant twisted cubic curves in  $\mathbb{P}^3$ .*

*Proof.* Let  $C$  be an  $\mathfrak{A}_5$ -invariant curve of degree at most three in  $\mathbb{P}^3$ . Since the  $2\mathfrak{A}_5$ -representation  $\mathbb{U}_4$  is irreducible, we conclude that  $C$  is a twisted cubic.

By Corollary 2.4(iii), one has

$$(3.9) \quad \mathrm{Sym}^2(\mathbb{U}_4) \cong \mathbb{V}_3 \oplus \mathbb{V}'_3 \oplus \mathbb{V}_4,$$

where  $\mathbb{V}_3$ ,  $\mathbb{V}'_3$ , and  $\mathbb{V}_4$ , are irreducible representations of the group  $\mathfrak{A}_5$  of dimensions 3, 3, and 4, respectively. Note that  $\mathbb{V}_3$  and  $\mathbb{V}'_3$  are not isomorphic.

Denote by  $\mathcal{Q}$  and  $\mathcal{Q}'$  the linear systems of quadrics in  $\mathbb{P}^3$  that correspond to  $\mathbb{V}_3$  and  $\mathbb{V}'_3$ , respectively. Since  $\mathbb{P}^3$  does not contain  $\mathfrak{A}_5$ -orbits of lengths less or equal to eight by Lemma 3.2, we see that the base loci of  $\mathcal{Q}$  and  $\mathcal{Q}'$  contain  $\mathfrak{A}_5$ -invariant curves  $\mathcal{C}^1$  and  $\mathcal{C}^2$ , respectively. The degrees of these curves must be less than four, so that they are twisted cubic curves. This also implies that the base loci of  $\mathcal{Q}$  and  $\mathcal{Q}'$  are exactly the curves  $\mathcal{C}^1$  and  $\mathcal{C}^2$ , respectively.

Now take an arbitrary  $\mathfrak{A}_5$ -invariant twisted cubic curve  $C$  in  $\mathbb{P}^3$ . The quadrics in  $\mathbb{P}^3$  passing through  $C$  define a three-dimensional  $\mathfrak{A}_5$ -subrepresentation in  $\mathrm{Sym}^2(\mathbb{U}_4)$ . Moreover, different  $\mathfrak{A}_5$ -invariant twisted cubics give different  $\mathfrak{A}_5$ -subrepresentations of  $\mathrm{Sym}^2(\mathbb{U}_4)$ . Thus, (3.9) implies that  $C$  coincides either with  $\mathcal{C}^1$  or with  $\mathcal{C}^2$ .  $\square$

Keeping in mind Lemma 3.8, we will denote the two  $\mathfrak{A}_5$ -invariant twisted cubic curves in  $\mathbb{P}^3$  by  $\mathcal{C}^1$  and  $\mathcal{C}^2$  throughout this section.

*Remark 3.10.* The curves  $\mathcal{C}^1$  and  $\mathcal{C}^2$  are disjoint. Indeed, otherwise, their intersection would contain at least 12, which is impossible, since a twisted cubic curve is an intersection of quadrics.

The lines in  $\mathbb{P}^3$  that are tangent to the curves  $\mathcal{C}^1$  and  $\mathcal{C}^2$  sweep out quartic surfaces  $\mathcal{S}^1$  and  $\mathcal{S}^2$ , respectively. These surfaces are  $\mathfrak{A}_5$ -invariant. The singular loci of  $\mathcal{S}^1$  and  $\mathcal{S}^2$  are the curves  $\mathcal{C}^1$  and  $\mathcal{C}^2$ , respectively. In particular, the surfaces  $\mathcal{S}^1$  and  $\mathcal{S}^2$  are different. Their singularities along these curves are locally isomorphic to a product of  $\mathbb{A}^1$  and an ordinary cusp.

Denote by  $\mathcal{P}$  the pencil of quartics in  $\mathbb{P}^3$  generated by  $\mathcal{S}^1$  and  $\mathcal{S}^2$ .

**Lemma 3.11.** *All  $\mathfrak{A}_5$ -invariant quartic surfaces in  $\mathbb{P}^3$  are contained in the pencil  $\mathcal{P}$ .*

*Proof.* This follows from Corollary 2.4(v).  $\square$

We proceed by describing the base locus of the pencil  $\mathcal{P}$ . This was done in [4, Remark 2.6], but we reproduce the details here for the convenience of the reader.

**Lemma 3.12.** *The base locus of the pencil  $\mathcal{P}$  is an irreducible curve  $B$  of degree 16. It has 24 singular points, these points are in a union of two  $\mathfrak{A}_5$ -orbits of length 12, and each of them is an ordinary cusp of the curve  $B$ . The curve  $B$  contains a unique  $\mathfrak{A}_5$ -orbit of length 20.*

*Proof.* Denote by  $B$  the base curve of the pencil  $\mathcal{P}$ . Let us show that the curves  $\mathcal{C}^1$  and  $\mathcal{C}^2$  are not contained in  $B$ . Since  $\mathcal{C}^1$  is projectively normal, there is an exact sequence of  $2\mathfrak{A}_5$ -representations

$$0 \rightarrow H^0(\mathcal{I}_{\mathcal{C}^1} \otimes \mathcal{O}_{\mathbb{P}^3}(4)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \rightarrow H^0(\mathcal{O}_{\mathcal{C}^1} \otimes \mathcal{O}_{\mathbb{P}^3}(4)) \rightarrow 0,$$

where  $\mathcal{I}_{\mathcal{C}^1}$  is the ideal sheaf of  $\mathcal{C}^1$ . The  $2\mathfrak{A}_5$ -representation  $H^0(\mathcal{O}_{\mathcal{C}^1} \otimes \mathcal{O}_{\mathbb{P}^3}(4))$  contains a one-dimensional subrepresentation corresponding to the unique  $\mathfrak{A}_5$ -orbit of length 12 in  $\mathcal{C}^1 \cong \mathbb{P}^1$ . This shows that  $\mathcal{P}$  contains a surface that does not pass through  $\mathcal{C}^1$ , so that  $\mathcal{C}^1$  is not contained in  $B$ . Similarly, we see that  $\mathcal{C}^2$  is not contained in  $B$ .

Let  $\rho: \hat{\mathcal{S}}^1 \rightarrow \mathcal{S}^1$  be the normalization of the surface  $\mathcal{S}^1$ , and let  $\hat{\mathcal{C}}^1$  be the preimage of the curve  $\mathcal{C}^1$  via  $\rho$ . Then the action of the group  $\mathfrak{S}_5$ , and in particular of its subgroup  $\mathfrak{A}_5$ , lifts to  $\hat{\mathcal{S}}^1$ . One has  $\hat{\mathcal{S}}^1 \cong \mathbb{P}^1 \times \mathbb{P}^1$ , and  $\rho^*(\mathcal{O}_{\mathcal{S}^1} \otimes \mathcal{O}_{\mathbb{P}^3}(1))$  is a divisor of bi-degree  $(1, 2)$ . This shows that  $\hat{\mathcal{C}}^1$  is of bi-degree  $(1, 1)$ . Thus, the action of  $\mathfrak{A}_5$  on  $\hat{\mathcal{S}}$  is diagonal by [6, Lemma 6.4.3(i)].

Denote by  $\hat{B}$  be the preimage of the curve  $B$  via  $\rho$ . Then  $\hat{B}$  is a divisor of bi-degree  $(4, 8)$ . Hence, the curve  $\hat{B}$  is irreducible by [6, Lemma 6.4.4(i)], so that the curve  $B$  is irreducible as well.

Note that the curve  $\hat{B}$  is singular. Indeed, the intersection  $\mathcal{S}^1 \cap \mathcal{C}^2$  is an  $\mathfrak{A}_5$ -orbit  $\Sigma_{12}$  of length 12, because  $\mathcal{C}^2$  is not contained in  $\mathcal{S}^1$ . Similarly, we see that the intersection  $\mathcal{S}^2 \cap \mathcal{C}^1$  is also an  $\mathfrak{A}_5$ -orbit  $\Sigma'_{12}$  of length 12. These  $\mathfrak{A}_5$ -orbits  $\Sigma_{12}$  and  $\Sigma'_{12}$  are different by Remark 3.10. Since  $B$  is the scheme theoretic intersection of the surfaces  $\mathcal{S}^1$  and  $\mathcal{S}^2$ , it must be singular at every point of  $\Sigma_{12} \cup \Sigma'_{12}$ . Denote by  $\hat{\Sigma}_{12}$  and  $\hat{\Sigma}'_{12}$  the preimages via  $\rho$  of the  $\mathfrak{A}_5$ -orbits  $\Sigma_{12}$  and  $\Sigma'_{12}$ , respectively. Then  $\hat{B}$  is singular in every point of  $\hat{\Sigma}'_{12}$ .

The curve  $\hat{B}$  is smooth away of  $\hat{\Sigma}'_{12}$ , because its arithmetic genus is 21, and the surface  $\hat{\mathcal{S}}^1$  does not contain  $\mathfrak{A}_5$ -orbits of length less than 12. On the other hand, we have

$$\hat{B} \cap \hat{\mathcal{C}}^1 = \hat{\Sigma}_{12},$$

because  $\hat{B} \cdot \hat{\mathcal{C}}^1 = 12$  and  $\hat{\Sigma}_{12} \subset \hat{B}$ . This shows that  $B$  is an irreducible curve whose only singularities are the points of  $\Sigma_{12} \cup \Sigma'_{12}$ , and each such point is an ordinary cusp of the curve  $B$ . In particular, the genus of the normalization of the curve  $B$  is 9. By [6, Lemma 5.1.5], this implies that  $B$  contains a unique  $\mathfrak{A}_5$ -orbit of length 20.  $\square$

The following classification of  $\mathfrak{A}_5$ -invariant quartic surfaces in  $\mathbb{P}^3$  was obtained in [4, Theorem 2.4].

**Lemma 3.13.** *The pencil  $\mathcal{P}$  contains two surfaces  $\mathcal{R}^1$  and  $\mathcal{R}^2$  with ordinary double singularities, such that the singular loci of  $\mathcal{R}^1$  and  $\mathcal{R}^2$  are  $\mathfrak{A}_5$ -orbits of length 10. Every surface in  $\mathbb{P}^3$  different from  $\mathcal{S}^1$ ,  $\mathcal{S}^2$ ,  $\mathcal{R}^1$  and  $\mathcal{R}^2$  is smooth.*

*Proof.* Let  $S$  be a surface in  $\mathcal{P}$  that is different from  $\mathcal{S}^1$  and  $\mathcal{S}^2$ . It follows from Lemma 3.3 that  $S$  is irreducible. Assume that  $S$  is singular.

We claim that  $S$  has isolated singularities. Indeed, suppose that  $S$  is singular along some  $\mathfrak{A}_5$ -invariant curve  $Z$ . Taking a general plane section of  $S$ , we see that the degree of  $Z$  is at most three. Thus, one has either  $Z = \mathcal{C}^1$  or  $Z = \mathcal{C}^2$  by Lemma 3.8. Since neither of these curves is contained in the base locus of  $\mathcal{P}$  by Lemma 3.12, this would imply that either  $S = \mathcal{S}^1$  or  $S = \mathcal{S}^2$ . The latter is not the case by assumption.

We see that the singularities of  $S$  are isolated. Hence,  $S$  contains at most two non-Du Val singular points by [23, Theorem 1] applied to the minimal resolution of singularities of the surface  $S$ . Since the set of all non-Du Val singular points of the surface  $S$  must be  $\mathfrak{A}_5$ -invariant, we see that  $S$  has none of them by Lemma 3.2. Thus, all singularities of  $S$  are Du Val.

By [6, Lemma 6.7.3(iii)], the surface  $S$  has only ordinary double singularities, the set  $\text{Sing}(S)$  consists of one  $\mathfrak{A}_5$ -orbit, and

$$|\text{Sing}(S)| \in \{5, 6, 10, 12, 15\}.$$

Since  $\mathbb{P}^3$  does not contain  $\mathfrak{A}_5$ -orbits of lengths 5, 6, and 15 by Lemma 3.2, we see that  $\text{Sing}(S)$  is either an  $\mathfrak{A}_5$ -orbit of length 10 or an  $\mathfrak{A}_5$ -orbit of length 12.

Suppose that the singular locus of  $S$  is an  $\mathfrak{A}_5$ -orbit  $\Sigma_{12}$  of length 12. Then  $S$  does not contain other  $\mathfrak{A}_5$ -orbits of length 12 by [6, Lemma 6.7.3(iv)]. Since  $\mathcal{C}^1$  is not contained in the base locus of  $\mathcal{P}$  by Lemma 3.12, and  $\mathcal{C}^1$  is contained in  $\mathcal{S}^1$ , we see that  $\mathcal{C}^1 \not\subset S$ . Since

$$S \cdot \mathcal{C}^1 = 12$$

and  $\Sigma_{12}$  is the only  $\mathfrak{A}_5$ -orbit of length at most 12 in  $\mathcal{C}^1 \cong \mathbb{P}^1$ , we have  $S \cap \mathcal{C}^1 = \Sigma_{12}$ . Thus,

$$12 = S \cdot \mathcal{C}^1 \geq \sum_{P \in \Sigma_{12}} \text{mult}_P(S) = 2|\Sigma_{12}| = 24,$$

which is absurd.

Therefore, we see that the singular locus of  $S$  is an  $\mathfrak{A}_5$ -orbit  $\Sigma_{12}$  of length 10. Vice versa, if an  $\mathfrak{A}_5$ -invariant quartic surface passes through an  $\mathfrak{A}_5$ -orbit of length 10, then it is singular by [6, Lemma 6.7.1(ii)]. We know from Lemma 3.2 that there are exactly two  $\mathfrak{A}_5$ -orbits of length 10 in  $\mathbb{P}^3$ , and it follows from Lemma 3.12 that they are not contained in the base locus of  $\mathcal{P}$ . Thus there are two surfaces  $\mathcal{R}^1$  and  $\mathcal{R}^2$  that are singular exactly at the points of these two  $\mathfrak{A}_5$ -orbits, respectively. The above argument shows that every surface in  $\mathcal{P}$  except  $\mathcal{S}^1$ ,  $\mathcal{S}^2$ ,  $\mathcal{R}^1$  and  $\mathcal{R}^2$  is smooth.  $\square$

Keeping in mind Lemma 3.13, we will denote by  $\mathcal{R}^1$  and  $\mathcal{R}^2$  the two nodal surfaces contained in the pencil  $\mathcal{P}$  until the end of this section.

**Lemma 3.14.** *There is a unique  $\mathfrak{A}_5$ -invariant quartic surface in  $\mathbb{P}^3$  that contains the lines  $L_1, \dots, L_5$  (respectively, the lines  $L'_1, \dots, L'_5$ ). Moreover, this surface is smooth, and it does not contain the lines  $L'_1, \dots, L'_5$  (respectively,  $L_1, \dots, L_5$ ).*

*Proof.* Put  $\mathcal{L} = \sum_{i=1}^5 L_i$  and  $\mathcal{L}' = \sum_{i=1}^5 L'_i$ . Corollary 2.1(ii) implies that the stabilizer in  $\mathfrak{A}_5$  of a general point of  $L_1$  is trivial. Therefore, there exists a surface  $S \in \mathcal{P}$  that contains all lines  $L_1, \dots, L_5$ . By Lemma 3.12 such surface  $S$  is unique.

We claim that  $S \neq \mathcal{S}^1$ . Indeed, all lines contained in  $\mathcal{S}^1$  are tangent to the curve  $\mathcal{C}^1$ , and there are no  $\mathfrak{A}_5$ -orbits of length five in  $\mathcal{C}^1 \cong \mathbb{P}^1$ . Similarly, one has  $S \neq \mathcal{S}^2$ .

We claim that  $S$  is not one of the two nodal surfaces  $\mathcal{R}^1$  and  $\mathcal{R}^2$  contained in the pencil  $\mathcal{P}$ . Indeed, suppose that  $S = \mathcal{R}^1$ . Since the singular locus of  $\mathcal{R}^1$  is an  $\mathfrak{A}_5$ -orbit of length 10 by Lemma 3.13, we see that the lines  $L_1, \dots, L_5$  are contained in the smooth locus of  $\mathcal{R}^1$  by Corollary 3.5. On the other hand, one has  $\mathcal{L}^2 = -10$  by Lemma 3.4. This means that  $\text{rk Pic}(S)^{\mathfrak{A}_5} \geq 2$ , which is impossible by [6, Lemma 6.7.3(i),(ii)].

We see that the surface  $S$  is different from  $\mathcal{R}^1$ . A similar argument shows that  $S$  is different from  $\mathcal{R}^2$ . Hence,  $S$  is smooth by Lemma 3.13.

Let us show that  $S$  does not contain the lines  $L'_1, \dots, L'_5$ . Suppose that it does. By Lemma 3.4 one has

$$\mathcal{L} \cdot \mathcal{L} = \mathcal{L}' \cdot \mathcal{L}' = -10.$$

By [6, Lemma 6.7.1(i)], we have  $\text{rk Pic}(S)^{\mathfrak{A}_5} = 2$ . Let  $\Pi_S$  be the class of a plane section of  $S$ . Then the determinant of the matrix

$$\begin{pmatrix} \mathcal{L} \cdot \mathcal{L} & \mathcal{L} \cdot \mathcal{L}' & \Pi_S \cdot \mathcal{L} \\ \mathcal{L} \cdot \mathcal{L}' & \mathcal{L}' \cdot \mathcal{L}' & \Pi_S \cdot \mathcal{L}' \\ \Pi_S \cdot \mathcal{L} & \Pi_S \cdot \mathcal{L}' & \Pi_S \cdot \Pi_S \end{pmatrix} = \begin{pmatrix} -10 & 20 & 5 \\ 20 & -10 & 5 \\ 5 & 5 & 4 \end{pmatrix}$$

must vanish. This is a contradiction, because it equals 12.

Applying similar arguments, we see that the lines  $L'_1, \dots, L'_5$  are contained in a unique  $\mathfrak{A}_5$ -invariant quartic surface, this surface is smooth and does not contain the lines  $L_1, \dots, L_5$ .  $\square$

*Remark 3.15.* One can use the properties of the pencil  $\mathcal{P}$  to give an alternative proof of Lemma 3.7. Namely, we know from Lemma 3.14 that there are two (different) smooth  $\mathfrak{A}_5$ -invariant quartic surfaces  $S$  and  $S'$  containing the lines  $L_1, \dots, L_5$  and  $L'_1, \dots, L'_5$ , respectively. By Lemma 3.12, the base locus of the pencil  $\mathcal{P}$  is an irreducible curve  $B$  that contains a unique  $\mathfrak{A}_5$ -orbit  $\Sigma$  of length 20. By Corollary 3.5, this implies that  $\Sigma$  is contained in the union  $L_1 \cup \dots \cup L_5$ , because

$$B \cdot (L_1 + \dots + L_5) = 20$$

on the surface  $S$ . Similarly, we see that  $\Sigma$  is contained in  $L'_1 \cup \dots \cup L'_5$ . These facts together with Lemma 3.4 easily imply that the lines  $L_1, \dots, L_5$  and  $L'_1, \dots, L'_5$  form a double five configuration.

Now we will obtain some restrictions on low degree  $\mathfrak{A}_5$ -invariant curves in  $\mathbb{P}^3$ .

**Lemma 3.16.** *Let  $C$  be an irreducible  $\mathfrak{A}_5$ -invariant curve in  $\mathbb{P}^3$  of degree  $d \leq 10$ . Denote by  $g$  the genus of the normalization of the curve  $C$ . Then*

$$g \leq \frac{d^2}{8} + 1 - |\text{Sing}(C)|.$$

*Proof.* Since  $\mathbb{U}_4$  is an irreducible  $2\mathfrak{A}_5$ -representation, the curve  $C$  is not contained in a plane in  $\mathbb{P}^3$ . This implies that a stabilizer in  $\mathfrak{A}_5$  of a general point of the curve  $C$  is trivial. In particular, the  $\mathfrak{A}_5$ -orbit of a general point of  $C$  has length  $|\mathfrak{A}_5| = 60$ .

Let  $S$  be a surface in the pencil  $\mathcal{P}$  that passes through a general point of  $C$ . Then the curve  $C$  is contained in  $S$ , because otherwise one would have

$$60 \leq |S \cap C| \leq S \cdot C = 4d \leq 40,$$

which is absurd. Since the assertion of the lemma clearly holds for the twisted cubic curves  $\mathcal{C}^1$  and  $\mathcal{C}^2$ , we may assume that  $C$  is different from these two curves.

Suppose that  $S = \mathcal{S}^1$ . Let us use the notation of the proof of Lemma 3.12. Denote by  $\hat{C}$  the preimage of the curve  $C$  via  $\rho$ . Then  $\hat{C}$  is a divisor of bi-degree  $(a, b)$  for some non-negative integers  $a$  and  $b$  such that  $d = 2a + b$ . On the other hand, one has

$$|\hat{C} \cap \hat{\mathcal{C}}^1| \leq \hat{C} \cdot \hat{\mathcal{C}}^1 = a + b \leq 2a + b = d \leq 10,$$

which is impossible, since the curve  $\hat{\mathcal{C}}^1 \cong \mathcal{C}^1 \cong \mathbb{P}^1$  does not contain  $\mathfrak{A}_5$ -orbits of length less than 12.

We see that  $S \neq \mathcal{S}^1$ . Similarly, we see that  $S \neq \mathcal{S}^2$ . By Lemma 3.13, either  $S$  is a smooth quartic  $K3$  surface, or  $S$  is one of the surfaces  $\mathcal{R}^1$  and  $\mathcal{R}^2$ . Denote by  $\Pi_S$  a plane

section of  $S$ . Then

$$\det \begin{pmatrix} \Pi_S^2 & \Pi_S \cdot C \\ \Pi_S \cdot C & C^2 \end{pmatrix} = \det \begin{pmatrix} 4 & d \\ d & C^2 \end{pmatrix} = 4C^2 - d^2 \leq 0$$

by the Hodge index theorem.

Suppose that  $C$  is contained in the smooth locus of the surface  $S$ . Denote by  $p_a(C)$  the arithmetic genus of the curve  $C$ . Then

$$C^2 = 2p_a(C) - 2.$$

by the adjunction formula. Thus, we get

$$p_a(C) \leq \frac{d^2}{8} + 1.$$

Since  $g \leq p_a(C) - |\text{Sing}(C)|$ , this implies the assertion of the lemma.

To complete the proof, we may assume that  $C$  contains a singular point of the surface  $S$ . By Lemma 3.13, this means that either  $S = \mathcal{R}^1$  or  $S = \mathcal{R}^2$ . The singularities of the surface  $S$  are ordinary double points, and its singular locus is an  $\mathfrak{A}_5$ -orbit of length 10. In particular, the curve  $C$  contains the whole singular locus of  $S$ . By [6, Theorem 6.7.1], one has  $\text{Pic}(S)^{\mathfrak{A}_5} \cong \mathbb{Z}$ . Since  $\Pi_S^2 = 4$  and the self-intersection of any Cartier divisor on the surface  $S$  is even, we see that the group  $\text{Pic}(S)^{\mathfrak{A}_5}$  is generated by  $\Pi_S$ .

Suppose that  $C$  is a Cartier divisor on  $S$ . Then either  $C \sim \Pi_S$  or  $C \sim 2\Pi_S$ , because  $d \leq 10$ . Since the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(n)) \rightarrow H^0(\mathcal{O}_S(n\Pi_S))$$

is an isomorphism for  $n \leq 3$ , we conclude that there is an  $\mathfrak{A}_5$ -invariant quadric in  $\mathbb{P}^3$ . This is not the case by Lemma 3.3.

Therefore, we see that  $C$  is not a Cartier divisor on  $S$ . Since  $S$  has only ordinary double points, the divisor  $2C$  is Cartier. Thus

$$2C \sim l\Pi_S,$$

for some odd positive integer  $l$ . Since

$$2d = 2C \cdot \Pi_S = l\Pi_S \cdot \Pi_S = 4l,$$

we see that  $l = \frac{d}{2}$ . In particular,  $d$  is even and  $l \leq 5$ .

Let  $\theta: \tilde{S} \rightarrow S$  be the minimal resolution of singularities of the surface  $S$ . Denote by  $\tilde{C}$  the proper transform of the curve  $C$  on the surface  $\tilde{S}$ , and denote by  $\Theta_1, \dots, \Theta_{10}$  the exceptional curves of  $\theta$ . Then

$$2\tilde{C} \sim \theta^*(l\Pi_S) - m \sum_{i=1}^{10} \Theta_i,$$

for some positive integer  $m$ . Moreover,  $m$  is odd, because  $C$  is not a Cartier divisor. We have

$$4\tilde{C}^2 = \Pi_S^2 l^2 - 20m^2 = 4l^2 - 20m^2,$$

which implies that  $\tilde{C}^2 = l^2 - 5m^2$ . Since  $\tilde{C}^2$  is even,  $m$  is odd and  $l \leq 5$ , we see that either  $l = 3$  or  $l = 5$ .

Denote by  $p_a(\tilde{C})$  the arithmetic genus of the curve  $\tilde{C}$ . Then

$$l^2 - 5m^2 = \tilde{C}^2 = 2p_a(\tilde{C}) - 2.$$

by the adjunction formula. In particular, we have

$$25 - 5m^2 \geq l^2 - 5m^2 \geq -2,$$

so that  $l \in \{3, 5\}$  and  $m = 1$ . The latter means that  $C$  is smooth at every point of  $\text{Sing}(S)$ , so that

$$|\text{Sing}(\tilde{C})| = |\text{Sing}(C)|.$$

If  $l = \frac{d}{2} = 3$ , then  $p_a(\tilde{C}) = 3$ . This gives

$$g \leq p_a(\tilde{C}) - |\text{Sing}(\tilde{C})| = 3 - |\text{Sing}(C)| \leq \frac{d^2}{8} + 1 - |\text{Sing}(C)|.$$

Similarly, if  $l = \frac{d}{2} = 5$ , then  $p_a(\tilde{C}) = 11$ . This gives

$$g \leq p_a(\tilde{C}) - |\text{Sing}(\tilde{C})| = 11 - |\text{Sing}(C)| \leq \frac{d^2}{8} + 1 - |\text{Sing}(C)|.$$

□

Recall from [6, Lemma 5.4.1] that there exists a unique smooth irreducible curve of genus 4 with a faithful action of the group  $\mathfrak{A}_5$ . This curve is known as the Bring's curve. Its canonical model is a complete intersection of a quadric and a cubic in a three-dimensional projective space. However, this sextic curve does not appear in our  $\mathbb{P}^3 = \mathbb{P}(\mathbb{U}_4)$  by

**Lemma 3.17.** *Let  $C$  be a smooth irreducible  $\mathfrak{A}_5$ -invariant curve in  $\mathbb{P}^3$  of degree  $d \leq 6$  and genus  $g$ . Then  $g \neq 4$ .*

*Proof.* Suppose that  $g = 4$ . Denote by  $\Pi_C$  the plane section of the curve  $C$ . Then

$$h^0(\mathcal{O}_C(\Pi_C)) = d - 3 + h^0(\mathcal{O}_C(K_C - \Pi_C))$$

by the Riemann–Roch theorem. Since  $C$  is not contained in a plane, this implies that  $\Pi_C \sim K_C$ . Therefore, the projective space  $\mathbb{P}^3$  is identified with a projectivization of an  $\mathfrak{A}_5$ -representation  $H^0(\mathcal{O}_C(K_C))^\vee$ , i.e. of a representation of the group  $2\mathfrak{A}_5$  where the center of  $2\mathfrak{A}_5$  acts trivially. The latter is not the case by construction of  $\mathbb{U}_4$ . □

**Lemma 3.18.** *Let  $C$  be an irreducible smooth  $\mathfrak{A}_5$ -invariant curve in  $\mathbb{P}^3$  of degree  $d = 10$  and genus  $g$ . Then  $g \neq 10$ .*

*Proof.* Suppose that  $g = 10$ . By Lemma 3.12, the base locus of the pencil  $\mathcal{P}$  is an irreducible curve  $B$  of degree 16. In particular, there exists a surface  $S \in \mathcal{P}$  that does not contain  $C$ . Thus, the intersection  $S \cap C$  is an  $\mathfrak{A}_5$ -invariant set that consists of

$$C \cdot S = 4d = 40$$

points (counted with multiplicities). On the other hand, by [6, Lemma 5.1.5], any  $\mathfrak{A}_5$ -orbit in  $C$  has length 12, 30, or 60. □

#### 4. LARGE SUBGROUPS OF $\mathfrak{S}_6$

In this section we collect some auxiliary results about the groups  $\mathfrak{S}_6$ ,  $\mathfrak{A}_6$  and  $\mathfrak{S}_5$ . We start with recalling some general properties of the group  $\mathfrak{A}_6$ .

*Remark 4.1* (see e.g. [8, p. 4]). Let  $\Gamma$  be a proper subgroup of  $\mathfrak{A}_6$  such that the index of  $\Gamma$  is at most 15. Then  $\Gamma$  is isomorphic either to  $\mathfrak{A}_5$ , or to  $F_{36}$ , or to  $\mathfrak{S}_4$ . In particular, if  $\mathfrak{A}_6$  acts transitively on the set of  $r < 15$  elements, then either  $r = 6$  or  $r = 10$ .

We will need the following result about possible actions of the group  $\mathfrak{A}_6$  on curves of small genera (cf. [5, Theorem 2.18] and [6, Lemma 5.1.5]).

**Lemma 4.2.** *Suppose that  $C$  is a smooth irreducible curve of genus  $g \leq 15$  with a non-trivial action of the group  $\mathfrak{A}_6$ . Then  $g = 10$ .*

*Proof.* Let  $\Omega \subset C$  be an  $\mathfrak{A}_6$ -orbit. Then a stabilizer of a point in  $\Omega$  is a cyclic subgroup of  $\mathfrak{A}_6$ , which implies that

$$|\Omega| \in \{72, 90, 120, 180, 360\}.$$

From the classification of finite subgroups of  $\text{Aut}(\mathbb{P}^1) \cong \text{PGL}_2(\mathbb{C})$  we know that  $g \neq 0$ . Also, it follows from the non-solvability of the group  $\mathfrak{A}_6$  that  $g \neq 1$ .

Put  $\bar{C} = C/\mathfrak{A}_6$ . Then  $\bar{C}$  is a smooth curve. Let  $\bar{g}$  be the genus of the curve  $\bar{C}$ . The Riemann–Hurwitz formula gives

$$2g - 2 = 360(2\bar{g} - 2) + 180a_{180} + 240a_{120} + 270a_{90} + 288a_{72},$$

where  $a_k$  is the number of  $\mathfrak{A}_6$ -orbits in  $C$  of length  $k$ .

Since  $a_k \geq 0$  and  $2 \leq g \leq 15$ , one has  $\bar{g} = 0$ . Thus, we obtain

$$2g - 2 = -720 + 180a_{180} + 240a_{120} + 270a_{90} + 288a_{72}.$$

Going through the values  $2 \leq g \leq 15$ , and solving this equation case by case we see that the only possibility is  $g = 10$ .  $\square$

We proceed by recalling some general properties of the group  $\mathfrak{S}_5$ .

*Remark 4.3* (see e. g. [8, p. 2]). Let  $\Gamma$  be a proper subgroup of  $\mathfrak{S}_5$  such that the index of  $\Gamma$  is less than 12. Then  $\Gamma$  is isomorphic either to  $\mathfrak{A}_5$ , or to  $\mathfrak{S}_4$ , or to  $F_{20}$ , or to  $\mathfrak{A}_4$ , or to  $D_{12}$ . In particular, if  $\mathfrak{S}_5$  acts transitively on the set of  $r < 12$  elements, then  $r \in \{2, 5, 6, 10\}$ .

**Lemma 4.4.** *The group  $\mathfrak{S}_5$  cannot act faithfully on a smooth irreducible curve of genus 5.*

*Proof.* Suppose that  $C$  is a curve of genus 5 with a faithful action of  $\mathfrak{S}_5$ . Considering the action of the subgroup  $\mathfrak{A}_5 \subset \mathfrak{S}_5$  on  $C$  and applying [6, Lemma 5.4.3], we see that  $C$  is hyperelliptic. This gives a natural homomorphism

$$\theta: \mathfrak{S}_5 \rightarrow \text{Aut}(\mathbb{P}^1) \cong \text{PGL}_2(\mathbb{C})$$

whose kernel is either trivial or isomorphic to  $\mu_2$ . Thus  $\theta$  is injective, which gives a contradiction.  $\square$

Now we will prove some auxiliary facts about actions of the groups  $\mathfrak{S}_6$ ,  $\mathfrak{A}_6$  and  $\mathfrak{S}_5$  on the four-dimensional projective space.

*Remark 4.5.* The group  $\mathfrak{S}_6$  has exactly four irreducible five-dimensional representations (see e. g. [8, p. 5]). Starting from one of them, one more can be obtained by a twist by an outer automorphism of  $\mathfrak{S}_6$ , and two remaining ones are obtained from these two by a tensor product with the sign representation. Although these four representations are not isomorphic, the images of  $\mathfrak{S}_6$  in  $\text{PGL}_5(\mathbb{C})$  under them are the same. Every irreducible five-dimensional representation of  $\mathfrak{S}_6$  restricts to an irreducible representation of the subgroup  $\mathfrak{A}_6 \subset \mathfrak{S}_6$ , and restricts to an irreducible representation of the *some* of the subgroups  $\mathfrak{S}_5 \subset \mathfrak{S}_6$ . The group  $\mathfrak{A}_6$  has exactly two irreducible five-dimensional representations, each of them arising this way (see e. g. [8, p. 5]). Similarly, the group  $\mathfrak{S}_5$  has exactly two irreducible five-dimensional representations, each of them arising this



way (see e.g. [8, p. 2]). Note also that every five-dimensional representation of a group  $\mathfrak{A}_6$  or  $\mathfrak{S}_5$  that does not contain one-dimensional subrepresentations is irreducible.

Let  $\mathbb{V}_5$  be an irreducible five-dimensional representation of the group  $\mathfrak{S}_6$ . Put  $\mathbb{P}^4 = \mathbb{P}(\mathbb{V}_5)$ . Keeping in mind Remark 4.5, we see that the image of the corresponding homomorphism  $\mathfrak{S}_6$  to  $\mathrm{PGL}_5(\mathbb{C})$  is the same for any choice of  $\mathbb{V}_5$ , and thus the  $\mathfrak{S}_6$ -orbits and  $\mathfrak{S}_6$ -invariant hypersurfaces in  $\mathbb{P}^4$  do not depend on  $\mathbb{V}_5$  either.

Remark 4.5 implies that there are six linear forms  $x_0, \dots, x_5$  on  $\mathbb{P}^4$  that are permuted by the group  $\mathfrak{S}_6$  (cf. Sections 1 and 2). Indeed, up to a twist by an outer automorphism of  $\mathfrak{S}_6$  and a tensor product with the sign representation,  $\mathbb{V}_5$  is a subrepresentation of the six-dimensional representation  $\mathbb{W}$  of  $\mathfrak{S}_6$ , so that one can take restrictions of the natural coordinates in  $\mathbb{W}$  to be these linear forms. Let  $Q$  be the three-dimensional quadric in  $\mathbb{P}^4$  given by equation

$$(4.6) \quad x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0.$$

The quadric  $Q$  is smooth and  $\mathfrak{S}_6$ -invariant. Note also that equation (1.1) makes sense in our  $\mathbb{P}^4$ .

We will use the notation introduced above until the end of the paper.

**Lemma 4.7.** *Let  $\Gamma$  be either the group  $\mathfrak{S}_6$ , or its subgroup  $\mathfrak{A}_6$ , or a subgroup  $\mathfrak{S}_5$  of  $\mathfrak{S}_6$  such that  $\mathbb{V}_5$  is an irreducible representation of  $\Gamma$ . Then the only  $\Gamma$ -invariant quadric threefold in  $\mathbb{P}^4$  is the quadric  $Q$ . Similarly, every (reduced)  $\Gamma$ -invariant quartic threefold in  $\mathbb{P}^4$  is given by equation (1.1) for some  $t \in \mathbb{C}$ .*

*Proof.* Apply Corollary 2.5. □

By a small abuse of notation we will refer to the points in  $\mathbb{P}^4$  using  $x_i$  as if they were homogeneous coordinates, i.e. a point in  $\mathbb{P}^4$  will be encoded by a ratio of six linear forms  $x_i$ . As in Section 1, let  $\Sigma_6$  and  $\Sigma_{10}$  are the  $\mathfrak{S}_6$ -orbits of the points  $[-5 : 1 : 1 : 1 : 1 : 1]$  and  $[-1 : -1 : -1 : 1 : 1 : 1]$ , respectively. Looking at equation (4.6), we obtain

**Corollary 4.8.** *The quadric  $Q$  does not contain the  $\mathfrak{S}_6$ -orbits  $\Sigma_6$  and  $\Sigma_{10}$ .*

Now we will have a look at the action of the group  $\mathfrak{A}_6$  on  $\mathbb{P}^4$ . Note that  $\mathbb{V}_5$  is an irreducible  $\mathfrak{A}_6$ -representation by Remark 4.5.

**Lemma 4.9.** *There are no  $\mathfrak{A}_6$ -orbits of length less than six in  $\mathbb{P}^4$ . Moreover, the only  $\mathfrak{A}_6$ -orbit of length six in  $\mathbb{P}^4$  is  $\Sigma_6$ .*

*Proof.* The only subgroup of  $\mathfrak{A}_6$  of index less than six is  $\mathfrak{A}_6$  itself (cf. Remark 4.1), so that the first assertion of the lemma follows from irreducibility of the  $\mathfrak{A}_6$ -representation  $\mathbb{V}_5$ . Also, the only subgroups of  $\mathfrak{A}_6$  of index six are  $\mathfrak{A}_5^{st}$  and  $\mathfrak{A}_5^{nst}$ , so that the second assertion of the lemma also follows from Corollary 2.3. □

**Lemma 4.10.** *Let  $X$  be an  $\mathfrak{A}_6$ -invariant quartic threefold in  $\mathbb{P}^4$  that contains an  $\mathfrak{A}_6$ -orbit of length at most six. Then  $X = X_{\frac{7}{10}}$ .*

*Proof.* By Lemma 4.7, one has  $X = X_t$  for some  $t \in \mathbb{C}$ , and by Lemma 4.9 the  $\mathfrak{A}_6$ -orbit  $\Sigma_6$  is contained in  $X_t$ . Since  $\Sigma_6$  is not contained in the quadric  $Q$  by Corollary 4.8, we see that there is a unique  $t \in \mathbb{C}$  such that  $\Sigma_6$  is contained in a quartic given by equation (1.1). Therefore, we conclude that  $t = \frac{7}{10}$ . □

Now we will make a couple of observations about the action of the group  $\mathfrak{S}_5$  on  $\mathbb{P}^4$ . We choose  $\mathfrak{S}_5$  to be a subgroup of  $\mathfrak{S}_6$  such that  $\mathbb{V}_5$  is an irreducible  $\mathfrak{S}_5$ -representation (cf. Remark 4.5 and Corollary 2.3).

**Lemma 4.11.** *Let  $P \in \mathbb{P}^4$  be a point such that its stabilizer in  $\mathfrak{S}_5$  contains a subgroup isomorphic to  $D_{12}$ . Then the  $\mathfrak{S}_5$ -orbit of  $P$  is  $\Sigma_{10}$ .*

*Proof.* By Corollary 2.3(iii), the point in  $\mathbb{P}^4$  fixed by a subgroup  $D_{12} \subset \mathfrak{S}_5$  is unique. On the other hand, it is straightforward to check that a stabilizer in  $\mathfrak{S}_5$  of a point of  $\Sigma_{10}$  contains a subgroup isomorphic to  $D_{12}$ . It remains to notice that the latter stabilizer is actually isomorphic to  $D_{12}$ , since the only subgroups of  $\mathfrak{S}_5$  that contain  $D_{12}$  are  $D_{12}$  and  $\mathfrak{S}_5$  itself, while  $\mathfrak{S}_5$  has no fixed points on  $\mathbb{P}^4$ .  $\square$

**Lemma 4.12.** *Let  $X$  be an  $\mathfrak{S}_5$ -invariant quartic threefold in  $\mathbb{P}^4$  that contains  $\Sigma_{10}$ . Then  $X = X_{\frac{1}{6}}$ .*

*Proof.* By Lemma 4.7, one has  $X = X_t$  for some  $t \in \mathbb{C}$ . Since  $\Sigma_{10}$  is not contained in the quadric  $Q$  by Corollary 4.8, we see that there is a unique  $t \in \mathbb{C}$  such that  $\Sigma_{10}$  is contained in a quartic given by equation (1.1). Therefore, we conclude that  $t = \frac{1}{6}$ .  $\square$

## 5. RATIONALITY OF THE QUARTIC THREEFOLD $X_{\frac{7}{10}}$

In this section we will construct an explicit  $\mathfrak{A}_6$ -equivariant birational map  $\mathbb{P}^3 \dashrightarrow X_{\frac{7}{10}}$ . Implicitly, the construction of this map first appeared in the proof of [5, Theorem 1.20]. Here we will present a much simplified proof of its existence.

We identify  $\mathbb{P}^3$  with the projectivization  $\mathbb{P}(\mathbb{U}_4)$ , where  $\mathbb{U}_4$  is the restriction of the four-dimensional irreducible representation of the group  $2.\mathfrak{S}_6$  introduced in Section 2 to the subgroup  $2.\mathfrak{A}_6$ . By Corollary 2.1(i), the  $2.\mathfrak{A}_6$ -representation  $\mathbb{U}_4$  is irreducible.

**Lemma 5.1.** *There are no  $\mathfrak{A}_6$ -invariant surfaces of odd degree in  $\mathbb{P}^3$ , and no  $\mathfrak{A}_6$ -invariant pencils of surfaces of odd degree in  $\mathbb{P}^3$ . Moreover, there are no  $\mathfrak{A}_6$ -invariant quadric and quartic surfaces in  $\mathbb{P}^3$ .*

*Proof.* Recall that the only one-dimensional representation of the group  $2.\mathfrak{A}_6$  is the trivial representation. Therefore, any  $\mathfrak{A}_6$ -invariant surface of odd degree  $d$  in  $\mathbb{P}^3$  gives rise to a trivial  $2.\mathfrak{A}_6$ -subrepresentation in  $R_d = \text{Sym}^d(\mathbb{U}_4)$ . On the other hand, the non-trivial central element  $z$  of  $2.\mathfrak{A}_6$  acts on  $R_d$  by a scalar matrix with diagonal entries equal to  $-1$ , which shows that  $R_d$  does not contain trivial  $2.\mathfrak{A}_6$ -representations. Also, since the only two-dimensional representation of  $2.\mathfrak{A}_6$  is the sum of two trivial representations, this implies that there are no  $\mathfrak{A}_6$ -invariant pencils of surfaces of odd degree in  $\mathbb{P}^3$ .

The last assertion of the lemma follows from Corollary 2.4(i),(ii).  $\square$

**Lemma 5.2.** *Let  $\Omega$  be an  $\mathfrak{A}_6$ -orbit in  $\mathbb{P}^3$ . Then  $|\Omega| \geq 16$ .*

*Proof.* Lemma 5.1 implies that there are no  $\mathfrak{A}_6$ -orbits of odd length in  $\mathbb{P}^3$ . Thus, if  $\Omega$  is an  $\mathfrak{A}_6$ -orbit in  $\mathbb{P}^3$  of length at most 15, then by Remark 4.1 a stabilizer of its general point is isomorphic either to  $\mathfrak{A}_5$  or to  $F_{36}$ . Both of these cases are impossible by Corollary 2.1.  $\square$

Actually, the minimal degree of an  $\mathfrak{A}_6$ -invariant surface in  $\mathbb{P}^3$  equals 8 (see [5, Lemma 3.7]), and the minimal length of an  $\mathfrak{A}_6$ -orbit in  $\mathbb{P}^3$  equals 36 (see [5, Lemma 3.8]), but we will not need this here.

**Lemma 5.3** (cf. [5, Lemma 4.26]). *Let  $C$  be a smooth irreducible  $\mathfrak{A}_6$ -invariant curve of degree 9 and genus  $g$  in  $\mathbb{P}^3$ . Then  $g \neq 10$ .*

*Proof.* Suppose that  $g = 10$ . Then it follows from [13, Example 6.4.3] that either  $C$  is contained in a unique quadric surface in  $\mathbb{P}^3$ , or  $C$  is a complete intersection of two cubic surfaces in  $\mathbb{P}^3$ . The former case is impossible, since there are no  $\mathfrak{A}_6$ -invariant quadrics in  $\mathbb{P}^3$  by Lemma 5.1. The latter case is impossible, because there are no  $\mathfrak{A}_6$ -invariant pencils of cubic surfaces in  $\mathbb{P}^3$  by Lemma 5.1.  $\square$

Recall that the group  $\mathfrak{A}_6$  contains six standard subgroups isomorphic to  $\mathfrak{A}_5$  and six non-standard subgroups isomorphic to  $\mathfrak{A}_5$  (see the conventions made in Section 2). Denote the former ones by  $H'_1, \dots, H'_6$ , and denote the latter ones by  $H_1, \dots, H_6$ . By Corollary 2.1(ii), each group  $H'_i$  leaves invariant two lines  $L_i^1$  and  $L_i^2$  in  $\mathbb{P}^3$ . Note that each group  $H_i$  permutes transitively the lines  $L_1^1, \dots, L_6^1$  (respectively,  $L_1^2, \dots, L_6^2$ ).

Put  $\mathcal{L}^1 = L_1^1 + \dots + L_6^1$  and  $\mathcal{L}^2 = L_1^2 + \dots + L_6^2$ . Then the curves  $\mathcal{L}^1$  and  $\mathcal{L}^2$  are  $\mathfrak{A}_6$ -invariant, and the curve  $\mathcal{L}^1 + \mathcal{L}^2$  is  $\mathfrak{S}_6$ -invariant.

**Lemma 5.4.** *The lines  $L_1^1, \dots, L_6^1$  (respectively, the lines  $L_1^2, \dots, L_6^2$ ) are pairwise disjoint. Moreover, the curves  $\mathcal{L}^1$  and  $\mathcal{L}^2$  are disjoint.*

*Proof.* We use an argument similar to one in the proof of Lemma 3.4. Suppose that some of the lines  $L_1^1, \dots, L_6^1$  have a common point. Since the action of  $\mathfrak{A}_6$  on the set  $\{L_1^1, \dots, L_6^1\}$  is doubly transitive, this implies that any two of the lines  $L_1^1, \dots, L_6^1$  have a common point. Therefore, either all lines  $L_1^1, \dots, L_6^1$  are coplanar, or all of them pass through one point. Both of these cases are impossible since  $\mathbb{U}_4$  is an irreducible  $2\mathfrak{A}_6$ -representation (see Corollary 2.1(i)). Therefore, the lines  $L_1^1, \dots, L_6^1$  are pairwise disjoint. A similar argument applies to the lines  $L_1^2, \dots, L_6^2$ .

Suppose that some of the lines  $L_1^1, \dots, L_6^1$ , say,  $L_1^1$ , intersects some of the lines  $L_1^2, \dots, L_6^2$ . Since the lines  $L_1^1$  and  $L_2^1$  are disjoint by construction, we may assume that  $L_1^1$  intersects  $L_2^2$ . Since the stabilizer  $H'_1 \subset \mathfrak{A}_6$  of  $L_1^1$  acts transitively on the lines  $L_2^2, \dots, L_6^2$ , we conclude that all five lines  $L_2^2, \dots, L_6^2$  intersect  $L_1^1$ . Therefore, the line  $L_1^1$  contains a subset of at most five points that is invariant with respect to the group  $H'_1 \cong \mathfrak{A}_5$ , which is a contradiction. Thus,  $\mathcal{L}^1$  and  $\mathcal{L}^2$  are disjoint.  $\square$

**Lemma 5.5.** *Let  $C$  be an  $\mathfrak{A}_6$ -invariant curve in  $\mathbb{P}^3$  of degree  $d \leq 10$ . Then either  $C = \mathcal{L}^1$  or  $C = \mathcal{L}^2$ .*

*Proof.* Suppose first that  $C$  is reducible. We may assume that  $\mathfrak{A}_6$  permutes the irreducible components of  $C$  transitively. Thus,  $C$  has either 6 or 10 irreducible components by Remark 4.1, and these irreducible components are lines. By Remark 4.1 and Corollary 2.1 the latter case is impossible, and in the former case one has either  $C = \mathcal{L}^1$  or  $C = \mathcal{L}^2$ .

Therefore, we assume that the curve  $C$  is irreducible. Let  $g$  be the genus of the normalization of the curve  $C$ . We have

$$(5.6) \quad g \leq \frac{d^2}{8} + 1 - |\text{Sing}(C)| \leq 13 - |\text{Sing}(C)|$$

by Lemma 3.16. This implies that the curve  $C$  is smooth, because  $\mathbb{P}^3$  does not contain  $\mathfrak{A}_6$ -orbits of length less than 16 by Lemma 5.2.

If  $d \leq 8$ , then (5.6) gives  $g \leq 9$ . This is impossible by Lemma 4.2

If  $d = 9$ , then (5.6) gives  $g \leq 11$ , so that  $g = 10$  by Lemma 4.2. This is impossible by Lemma 5.3.

Therefore, we see that  $d = 10$ . Thus, (5.6) gives  $g \leq 13$ , so that  $g = 10$  by Lemma 4.2. The latter is impossible by Lemma 3.18.  $\square$

Denote by  $\mathcal{M}$  the linear system on  $\mathbb{P}^3$  consisting of all quartic surfaces passing through the lines  $L_1^1, \dots, L_6^1$ . Then  $\mathcal{M}$  is not empty. In fact, its dimension is at least four by parameter count. Moreover, the linear system  $\mathcal{M}$  does not have base components by Lemma 5.1.

**Lemma 5.7.** *The base locus of  $\mathcal{M}$  does not contain curves except the lines  $L_1^1, \dots, L_6^1$ . Moreover, a general surface in  $\mathcal{M}$  is smooth at a general point of each of the lines  $L_1^1, \dots, L_6^1$ .*

*Proof.* Denote by  $Z$  the union of the curves that are contained in the base locus of  $\mathcal{M}$  and are different from the lines  $L_1^1, \dots, L_6^1$ . Then  $Z$  is a (possibly empty)  $\mathfrak{A}_6$ -invariant curve. Denote its degree by  $d$ . Pick two general surfaces  $M_1$  and  $M_2$  in  $\mathcal{M}$ . Then

$$M_1 \cdot M_2 = Z + m\mathcal{L}^1 + \Delta,$$

where  $m$  is a positive integer, and  $\Delta$  is an effective one-cycle on  $\mathbb{P}^3$  that contains none of the lines  $L_1^1, \dots, L_6^1$ . Note that  $\Delta$  may contain irreducible components of the curve  $Z$ . Let  $\Pi$  be a plane in  $\mathbb{P}^3$ . Then

$$16 = \Pi \cdot M_1 \cdot M_2 = \Pi \cdot Z + m\Pi \cdot \mathcal{L}^1 + \Pi \cdot \Delta = d + 6m + \Pi \cdot \Delta \leq d + 6m,$$

which implies that  $m \leq 2$  and  $d \leq 10$ . By Lemma 5.5, we have  $d = 0$ , so that  $Z$  is empty. Since

$$2 \geq m \geq \text{mult}_{L_i^1}(M_1)\text{mult}_{L_i^1}(M_2),$$

we see that a general surface in  $\mathcal{M}$  is smooth at a general point of  $L_i^1$ .  $\square$

Let  $\alpha: U \rightarrow \mathbb{P}^3$  be a blow up along the lines  $L_1^1, \dots, L_6^1$ . Then  $-K_U^3 = 4$ , and the action of  $\mathfrak{A}_6$  lifts to  $U$ . Denote by  $E_1, \dots, E_6$  the  $\alpha$ -exceptional surfaces that are mapped to  $L_1^1, \dots, L_6^1$ , respectively.

**Lemma 5.8.** *The action of the stabilizer  $H'_i \cong \mathfrak{A}_5$  in  $\mathfrak{A}_6$  of the line  $L_i^1$  on the surface  $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$  is twisted diagonal, i.e.,  $E_i$  is identified with  $\mathbb{P}(\mathbb{U}_2) \times \mathbb{P}(\mathbb{U}'_2)$ , where  $\mathbb{U}_2$  and  $\mathbb{U}'_2$  are different two-dimensional irreducible representations of the group  $2.\mathfrak{A}_5$ .*

*Proof.* This follows from Corollary 2.1(ii).  $\square$

Let us denote by  $\mathcal{M}_U$  the proper transform of the linear system  $\mathcal{M}$  on the threefold  $U$ . Then  $\mathcal{M}_U \sim -K_U$  by Lemma 5.7.

**Lemma 5.9.** *The linear system  $\mathcal{M}_U$  is base point free.*

*Proof.* Let us first show that  $\mathcal{M}_U$  is free from base curves. Suppose that the base locus of the linear system  $\mathcal{M}_U$  contains some curves. Then each of these curves is contained in some of the  $\alpha$ -exceptional surfaces by Lemma 5.7. Denote by  $Z$  the union of all such curves that are contained in  $E_1$ . Then  $Z$  is an  $H'_1$ -invariant curve. For some non-negative integers  $a$  and  $b$ , one has

$$Z \sim as + bl,$$

where  $s$  is a section of the natural projection  $E_1 \rightarrow L_1^1$  such that  $s^2 = 0$  on  $E_1$ , and  $l$  is a fiber of this projection. On the other hand, we have

$$\mathcal{M}_U|_{E_1} \sim -K_U|_{E_1} \sim s + 3l.$$

This gives  $a \leq 1$  and  $b \leq 3$ . Since the action of  $H'_1$  on the surface  $E_1$  is twisted diagonal by Lemma 5.8, the latter is impossible by [6, Lemma 6.4.2(i)] and [6, Lemma 6.4.11(o)].

We see that  $\mathcal{M}_U$  is free from base curves. Since  $\mathcal{M}_U \sim -K_U$ , the linear system  $\mathcal{M}_U$  cannot have more than  $-K_U^3 = 4$  base points. By Lemma 5.2, this implies that  $\mathcal{M}_U$  is base point free.  $\square$

**Corollary 5.10.** *The base locus of the linear system  $\mathcal{M}$  consists of the lines  $L_1^1, \dots, L_6^1$ .*

By Lemma 5.9, the divisor  $-K_U$  is nef. Since  $-K_U^3 = 4$ , it is also big. Thus, we have

$$h^1(\mathcal{O}_U(-K_U)) = h^2(\mathcal{O}_U(-K_U)) = 0$$

by the Kawamata–Viehweg vanishing theorem (see [17]). Hence, the Riemann–Roch formula gives

$$(5.11) \quad h^0(\mathcal{O}_U(-K_U)) = 5.$$

In particular, we see that  $|-K_U| = \mathcal{M}_U$ .

**Lemma 5.12.** *The  $\mathfrak{A}_6$ -representation  $H^0(\mathcal{O}_U(-K_U))$  is irreducible.*

*Proof.* By Lemma 5.1, there are no  $\mathfrak{A}_6$ -invariant quartic surfaces in  $\mathbb{P}^3$ . This implies that  $H^0(\mathcal{O}_U(-K_U))$  does not contain one-dimensional subrepresentations. Hence it is irreducible by Remark 4.5.  $\square$

Lemma 5.9 together with (5.11) implies that there is an  $\mathfrak{A}_6$ -equivariant commutative diagram

$$(5.13) \quad \begin{array}{ccc} & U & \\ \alpha \swarrow & & \searrow \beta \\ \mathbb{P}^3 & \text{---} & \mathbb{P}^4, \\ & \phi & \end{array}$$

where  $\phi$  is a rational map given by  $\mathcal{M}$ , and  $\beta$  is a morphism given by the anticanonical linear system  $|-K_U|$ . By Lemma 5.12 the projective space  $\mathbb{P}^4$  in (5.13) is a projectivization of an irreducible  $\mathfrak{A}_6$ -representation.

Recall from Lemma 3.8 that  $\mathbb{P}^3$  contains exactly two  $H_1$ -invariant twisted cubic curves  $\mathcal{C}_1^1$  and  $\mathcal{C}_1^2$ .

**Lemma 5.14.** *The curve  $\mathcal{L}^1$  intersects exactly one curve among  $\mathcal{C}_1^1$  and  $\mathcal{C}_1^2$ . Moreover, each line among  $L_1^1, \dots, L_6^1$  contains two points of this intersection. Similarly, the curve  $\mathcal{L}^2$  intersects exactly one curve among  $\mathcal{C}_1^1$  and  $\mathcal{C}_1^2$ , and this curve is different from the one that intersects  $\mathcal{L}^1$ .*

*Proof.* By Remark 3.1, the stabilizer in  $H_1$  of the curve  $L_1^1$  is isomorphic to  $D_{10}$ , and thus it has an orbit of length 2 on  $L_1^1$ . Thus, the curve  $\mathcal{L}^1$  contains an  $H_1$ -orbit  $\Sigma_{12}^1$  of length 12 by Lemma 3.2. Similarly, the curve  $\mathcal{L}^2$  contains an  $H_1$ -orbit  $\Sigma_{12}^2$  of length 12. By Lemma 5.4, one has  $\Sigma_{12}^1 \neq \Sigma_{12}^2$ . Moreover,  $\Sigma_{12}^1$  and  $\Sigma_{12}^2$  are the only  $H_1$ -orbits in  $\mathbb{P}^3$  of length 12 by Lemma 3.2. Since  $\mathcal{C}_1^1$  and  $\mathcal{C}_1^2$  are disjoint by Remark 3.10, and each of them contains an  $H_1$ -orbit of length 12, we see that either  $\Sigma_{12}^1 \subset \mathcal{C}_1^1$  and  $\Sigma_{12}^2 \subset \mathcal{C}_1^2$ , or  $\Sigma_{12}^2 \subset \mathcal{C}_1^1$  and  $\Sigma_{12}^1 \subset \mathcal{C}_1^2$ . Since a line cannot have more than two common points with a twisted cubic, this also implies the last assertion of the lemma.  $\square$

Without loss of generality, we may assume that the curve  $\mathcal{L}^1$  intersects  $\mathcal{C}_1^1$ , and the curve  $\mathcal{L}^2$  intersects  $\mathcal{C}_1^2$ . Let  $\mathcal{C}_1^1, \dots, \mathcal{C}_6^1$  be the  $\mathfrak{A}_6$ -orbit of the curve  $\mathcal{C}_1^1$ , and let  $\mathcal{C}_1^2, \dots, \mathcal{C}_6^2$  be the  $\mathfrak{A}_6$ -orbit of the curve  $\mathcal{C}_1^2$ . By Lemma 3.8, the curves  $\mathcal{C}_i^1$  and  $\mathcal{C}_i^2$  are the only twisted cubic curves in  $\mathbb{P}^3$  that are  $H_i$ -invariant. By Lemma 5.14, we have

**Corollary 5.15.** *Every twisted cubic curve  $\mathcal{C}_i^1$  intersects each line among  $L_1^1, \dots, L_6^1$  by two points. Similarly, every twisted cubic curve  $\mathcal{C}_i^2$  intersects each line among  $L_1^2, \dots, L_6^2$  by two points.*

Denote by  $\tilde{\mathcal{C}}_1^1, \dots, \tilde{\mathcal{C}}_6^1$  the proper transforms on  $U$  of the curves  $\mathcal{C}_1^1, \dots, \mathcal{C}_6^1$ , respectively.

**Lemma 5.16.** *One has  $-K_U \cdot \tilde{\mathcal{C}}_1^1 = \dots = -K_U \cdot \tilde{\mathcal{C}}_6^1 = 0$ .*

*Proof.* This follows from Corollary 5.15.  $\square$

We see that each curve  $\tilde{\mathcal{C}}_i^1$  is contracted by  $\beta$  to a point. Since the  $\mathfrak{A}_6$ -orbit of  $\tilde{\mathcal{C}}_1^1$  consists of six curves, we also obtain the following.

**Corollary 5.17.** *The image of the morphism  $\beta$  contains an  $\mathfrak{A}_6$ -orbit of length at most six.*

Since  $-K_U^3 = 4$ , the image of  $\beta$  is either an  $\mathfrak{A}_6$ -invariant quartic threefold or an  $\mathfrak{A}_6$ -invariant quadric threefold. Using results of [24], one can show that the latter case is impossible. However, this immediately follows from Corollary 5.17. Indeed, an  $\mathfrak{A}_6$ -orbit of length at most six cannot be contained in the  $\mathfrak{A}_6$ -invariant quadric by Corollary 4.8 and Lemma 4.9.

**Corollary 5.18.** *The morphism  $\beta$  is birational onto its image, and its image is a quartic threefold.*

Now Lemma 4.10 implies that the image of  $\beta$  is the quartic  $X_{\frac{7}{10}}$ . This proves

**Corollary 5.19.** *The threefold  $X_{\frac{7}{10}}$  is rational.*

Let us conclude this section by recalling two related results proved in [5, §4]. The commutative diagram (5.13) can be extended to an  $\mathfrak{A}_6$ -equivariant commutative diagram

$$(5.20) \quad \begin{array}{ccccc} & U & \xrightarrow{\rho} & U & \\ & \searrow \gamma & & \swarrow \gamma & \\ & X_{\frac{7}{10}} & \xrightarrow{\sigma} & X_{\frac{7}{10}} & \\ \alpha \swarrow & & & & \searrow \alpha \\ \mathbb{P}^3 & \xrightarrow{\phi} & & & \xrightarrow{\phi} \mathbb{P}^3 \\ & & \psi & & \end{array}$$

Here  $\sigma$  is an automorphism of the quartic threefold  $X_{\frac{7}{10}}$  given by an odd permutation in  $\mathfrak{S}_6$  acting on  $\mathbb{P}^4$ , cf. Remark 4.5. The birational map  $\rho$  is a composition of Atiyah flops in 36 curves contracted by  $\gamma$ , and the birational map  $\psi$  is not regular.

The diagram (5.20) is a so-called  $\mathfrak{A}_6$ -Sarkisov link. The subgroup  $\mathfrak{A}_6 \subset \text{Aut}(\mathbb{P}^3)$  together with  $\psi \in \text{Bir}^{\mathfrak{A}_6}(\mathbb{P}^3)$  generates a subgroup isomorphic to  $\mathfrak{S}_6$ . Moreover, the subgroup

$$\text{Aut}^{\mathfrak{A}_6}(\mathbb{P}^3) \subset \text{Bir}^{\mathfrak{A}_6}(\mathbb{P}^3)$$

is also isomorphic to  $\mathfrak{S}_6$ . By [5, Theorem 1.24], the whole group  $\text{Bir}^{\text{al}}(\mathbb{P}^3)$  is a free product of these two copies of  $\mathfrak{S}_6$  amalgamated over the original  $\mathfrak{A}_6$ .

## 6. RATIONALITY OF THE QUARTIC THREEFOLD $X_{\frac{1}{6}}$

In this section we will construct an explicit  $\mathfrak{S}_5$ -equivariant birational map  $\mathbb{P}^3 \dashrightarrow X_{\frac{1}{6}}$ . We identify  $\mathbb{P}^3$  with the projectivization  $\mathbb{P}(\mathbb{U}_4)$ , where  $\mathbb{U}_4$  is the restriction of the four-dimensional irreducible representation of the group  $2.\mathfrak{S}_6$  introduced in Section 2 to a subgroup  $2.\mathfrak{S}_5^{\text{nst}}$ , and denote the latter subgroup simply by  $2.\mathfrak{S}_5$ . By Corollary 2.1(i), the  $2.\mathfrak{S}_5$ -representation  $\mathbb{U}_4$  is irreducible.

**Lemma 6.1.** *Let  $\Omega$  be an  $\mathfrak{S}_5$ -orbit in  $\mathbb{P}^3$ . Then  $|\Omega| \geq 12$ .*

*Proof.* Apply Remark 4.3 together with Corollary 2.1. □

**Lemma 6.2.** *Let  $C$  be an  $\mathfrak{S}_5$ -invariant curve in  $\mathbb{P}^3$  of degree  $d$ . Then  $d \geq 6$ .*

*Proof.* Suppose that  $d \leq 5$ . To start with, assume that  $C$  is reducible and denote by  $r$  the number of its irreducible components. We may assume that  $\mathfrak{S}_5$  permutes the irreducible components of  $C$  transitively. Thus, either  $r = 2$  or  $r = 5$  by Remark 4.3. If  $r = 5$ , the irreducible components of  $C$  are lines, so that this case is impossible by Remark 4.3 and Corollary 2.1(i). Hence, we have  $r = 2$ , and the stabilizer of each of the two irreducible components  $C_1$  and  $C_2$  of  $C$  is the subgroup  $\mathfrak{A}_5 \subset \mathfrak{S}_5$ . Moreover, in this case one has

$$\deg(C_1) = \deg(C_2) \leq 2,$$

which is impossible by Lemma 3.8.

Therefore, we assume that the curve  $C$  is irreducible. Let  $g$  be the genus of the normalization of the curve  $C$ . Then

$$g \leq \frac{d^2}{8} + 1 - |\text{Sing}(C)|$$

by Lemma 3.16, so that  $g \leq 5 - |\text{Sing}(C)|$ . This implies that  $C$  is smooth, because there are no  $\mathfrak{S}_5$ -orbits of length less than 12 by Lemma 6.1.

Since  $\mathfrak{S}_5$  does not act faithfully on  $\mathbb{P}^1$ , we see that  $g \neq 0$ . Thus, either  $g = 4$  or  $g = 5$  by [6, Lemma 5.1.5]. The former case is impossible by Lemma 3.17, while the latter case is impossible by Lemma 4.4. □

Recall from Section 3 that the subgroup  $\mathfrak{A}_4 \subset \mathfrak{A}_5 \subset \mathfrak{S}_5$  fixes two disjoint lines  $L_1$  and  $L'_1$ . As before, we consider the  $\mathfrak{A}_5$ -orbit  $L_1, \dots, L_5$  of the line  $L_1$  and the  $\mathfrak{A}_5$ -orbit  $L'_1, \dots, L'_5$  of the line  $L'_1$ . By Lemma 3.7 the lines  $L_1, \dots, L_5, L'_1, \dots, L'_5$  form a double five configuration (see Definition 3.6). Corollary 2.1(i) implies that the  $\mathfrak{S}_5$ -orbit of the line  $L_1$  is  $L_1, \dots, L_5, L'_1, \dots, L'_5$ .

*Remark 6.3.* Any subgroup  $F_{20} \subset \mathfrak{S}_5$  permutes the ten lines  $L_1, \dots, L_5, L'_1, \dots, L'_5$  transitively. Indeed, let  $c \in F_{20}$  be an element of order five. Then  $c$  is not contained in a stabilizer of the line  $L_1$ , so that the orbit of  $L_1$  with respect to the group  $\Gamma \cong \mu_5$  generated by  $c$  is  $L_1, \dots, L_5$ . Similarly, the  $\Gamma$ -orbit of the line  $L'_1$  is  $L'_1, \dots, L'_5$ . Also, the group  $F_{20}$  is not contained in  $\mathfrak{A}_5$ , so that the  $F_{20}$ -orbit of  $L_1$  contains some of the lines  $L'_1, \dots, L'_5$ , and thus contains all the ten lines  $L_1, \dots, L_5, L'_1, \dots, L'_5$ .

Let  $\mathcal{M}$  be the linear system on  $\mathbb{P}^3$  consisting of all quartic surfaces passing through all lines  $L_1, \dots, L_5$  and  $L'_1, \dots, L'_5$ . Then  $\mathcal{M}$  is not empty. In fact, Lemma 3.7 and parameter count imply that its dimension is at least four. Moreover, the linear system  $\mathcal{M}$  does not have base components by Lemma 3.3.

**Lemma 6.4.** *The base locus of  $\mathcal{M}$  does not contain curves that are different from the lines  $L_1, \dots, L_5, L'_1, \dots, L'_5$ . Moreover, general surface in  $\mathcal{M}$  is smooth in a general point of each of these lines. Furthermore, two general surfaces in  $\mathcal{M}$  intersect transversally at a general point of each of these lines.*

*Proof.* Denote by  $Z$  the union of all curves that are contained in the base locus of the linear system  $\mathcal{M}$  and are different from the lines  $L_1, \dots, L_5, L'_1, \dots, L'_5$ . Then  $Z$  is a (possibly empty)  $\mathfrak{S}_5$ -invariant curve. Denote its degree by  $d$ . Pick two general surfaces  $M_1$  and  $M_2$  in  $\mathcal{M}$ . Then

$$M_1 \cdot M_2 = Z + m \sum_{i=1}^5 L_i + m \sum_{i=1}^5 L'_i + \Delta,$$

where  $m$  is a positive integer, and  $\Delta$  is an effective one-cycle on  $\mathbb{P}^3$  that contains none of the lines  $L_1, \dots, L_5$  and  $L'_1, \dots, L'_5$ . Note that  $\Delta$  may contain irreducible components of the curve  $Z$ . Note also that  $\Delta \neq 0$ , because  $\mathcal{M}$  is not a pencil.

Let  $\Pi$  be a plane in  $\mathbb{P}^3$ . Then

$$16 = \Pi \cdot Z + m \sum_{i=1}^5 \Pi \cdot L_i + m \sum_{i=1}^5 \Pi \cdot L'_i + \Pi \cdot \Delta = d + 10m + \Pi \cdot \Delta > d + 10m,$$

which implies that  $m = 1$  and  $d \leq 5$ . By Lemma 6.2, we have  $d = 0$ , so that  $Z$  is empty. Since

$$1 \geq m \geq \text{mult}_{L_i}(M_1 \cdot M_2) \geq \text{mult}_{L_i}(M_1) \text{mult}_{L_i}(M_2),$$

we see that a general surface in  $\mathcal{M}$  is smooth at a general point of  $L_i$ , and two general surfaces in  $\mathcal{M}$  intersect transversally at a general point of  $L_i$ . Similarly, we see that a general surface in  $\mathcal{M}$  is smooth at a general point of  $L'_i$ , and two general surfaces in  $\mathcal{M}$  intersect transversally at a general point of  $L'_i$ .  $\square$

Let  $g: W \rightarrow \mathbb{P}^3$  be a blow up along the lines  $L_1, \dots, L_5$ , and let  $g': W' \rightarrow \mathbb{P}^3$  be a blow up along the lines  $L'_1, \dots, L'_5$ . Denote by  $\tilde{L}'_1, \dots, \tilde{L}'_5$  (respectively,  $\tilde{L}_1, \dots, \tilde{L}_5$ ) the proper transforms of the lines  $L'_1, \dots, L'_5$  on the threefold  $W$  (respectively, on the threefold  $W'$ ). Let  $h: V \rightarrow W$  be a blow up along the curves  $\tilde{L}'_1, \dots, \tilde{L}'_5$ , and let  $h': V' \rightarrow W'$  be a blow up along the curves  $\tilde{L}_1, \dots, \tilde{L}_5$ . Finally, let  $\alpha: U \rightarrow \mathbb{P}^3$  be a blow up of the (singular) curve that is a union of all lines  $L_1, \dots, L_5, L'_1, \dots, L'_5$ . Then  $U$  has twenty nodes by Lemma 3.7, and there exists a commutative diagram

$$\begin{array}{ccccc}
 & & V & \overset{\tau}{\dashrightarrow} & V' \\
 & & \swarrow & & \searrow \\
 & & W & & W' \\
 & & \swarrow & & \searrow \\
 & & U & & U \\
 & & \downarrow & & \downarrow \\
 & & \mathbb{P}^3 & & \mathbb{P}^3
 \end{array}$$



where  $v$  and  $v'$  are small resolutions of singularities of the threefold  $U$ , and  $\tau$  is a composition of twenty Atiyah flops.

*Remark 6.5.* By construction, the action of group  $\mathfrak{A}_5$  lifts to the threefolds  $W$ ,  $W'$ ,  $V$ ,  $V'$ , and  $U$ . Similarly, the action of the group  $\mathfrak{S}_5$  lifts to the threefold  $U$ , but this action does not lift to  $W$  and  $W'$ . On the threefolds  $V$  and  $V'$ , the group  $\mathfrak{S}_5$  acts biregularly outside of the curves flopped by  $\tau$  and  $\tau^{-1}$ , respectively.

Denote by  $E_1, \dots, E_5$  the  $g$ -exceptional surfaces that are mapped to  $L_1, \dots, L_5$ , respectively. Similarly, denote by  $E'_1, \dots, E'_5$  the  $g'$ -exceptional surfaces that are mapped to  $L'_1, \dots, L'_5$ , respectively. Then all surfaces  $E_1, \dots, E_5, E'_1, \dots, E'_5$  are isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ .

Denote by  $\hat{E}'_1, \dots, \hat{E}'_5$  the  $h$ -exceptional surfaces that are mapped to the curves  $\tilde{L}'_1, \dots, \tilde{L}'_5$ , respectively. Similarly, denote by  $\check{E}_1, \dots, \check{E}_5$  the  $h'$ -exceptional surfaces that are mapped to the curves  $\tilde{L}_1, \dots, \tilde{L}_5$ , respectively. Denote by  $\hat{E}_1, \dots, \hat{E}_5$  the proper transforms on  $V$  of the surfaces  $E_1, \dots, E_5$ , respectively. Finally, denote by  $\check{E}'_1, \dots, \check{E}'_5$  the proper transforms on  $V'$  of the surfaces  $E'_1, \dots, E'_5$ , respectively. Then  $\tau$  maps the surfaces  $\hat{E}_1, \dots, \hat{E}_5$  to the surfaces  $\check{E}_1, \dots, \check{E}_5$ , respectively, and it maps the surfaces  $\hat{E}'_1, \dots, \hat{E}'_5$  to the surfaces  $\check{E}'_1, \dots, \check{E}'_5$ , respectively.

Denote by  $\mathcal{M}_W, \mathcal{M}_V, \mathcal{M}_{W'}, \mathcal{M}_{V'}$ , and  $\mathcal{M}_U$  the proper transforms of the linear system  $\mathcal{M}$  on the threefolds  $W, V, W', V'$ , and  $U$ , respectively. Then it follows from Lemma 6.4 that

$$\mathcal{M}_W \sim -K_W, \quad \mathcal{M}_V \sim -K_V, \quad \mathcal{M}_{W'} \sim -K_{W'}, \quad \mathcal{M}_{V'} \sim -K_{V'},$$

and  $\mathcal{M}_U \sim -K_U$ .

**Lemma 6.6.** *The base locus of the linear system  $\mathcal{M}_W$  does not contain curves that are different from the curves  $\tilde{L}'_1, \dots, \tilde{L}'_5$ . Similarly, the base locus of  $\mathcal{M}_{W'}$  does not contain curves that are different from the curves  $\tilde{L}_1, \dots, \tilde{L}_5$ .*

*Proof.* It is enough to prove the first assertion of the lemma. Suppose that the base locus of the linear system  $\mathcal{M}_W$  contains an irreducible curve  $Z$  that is different from the curves  $\tilde{L}'_1, \dots, \tilde{L}'_5$ . Then  $Z$  is contained in one of the surfaces  $E_1, \dots, E_5$  by Lemma 6.4.

By Lemma 6.4, the curve  $Z$  is a fiber of some of the natural projections  $E_i \rightarrow L_i$ , because otherwise two general surfaces in  $\mathcal{M}_W$  would be tangent in a general point of  $L_i$ . In particular, the only curves in the base locus of the linear system  $\mathcal{M}_W$  are  $\tilde{L}'_i$  and possibly some fibers of the projections  $E_i \rightarrow L_i$ . This shows that  $-K_W$  is nef. Indeed,  $-K_W$  has positive intersections with the fibers of the projections  $E_i \rightarrow L_i$ , it has trivial intersection with all curves  $\tilde{L}'_1, \dots, \tilde{L}'_5$ , and  $-K_W \sim \mathcal{M}_W$  has non-negative intersection with any other curve.

Let  $Z_1 = Z, Z_2, \dots, Z_r$  be the  $\mathfrak{A}_5$ -orbit of the curve  $Z$ . Then  $r \geq 20$  by Corollary 3.5. Pick two general surfaces  $M_1$  and  $M_2$  in the linear system  $\mathcal{M}_W$ . By Lemma 6.4, one has

$$M_1 \cdot M_2 = \sum_{i=1}^5 \tilde{L}'_i + m \sum_{i=1}^r Z_i + \Delta$$

for some positive integer  $m$  and some effective one-cycle  $\Delta$  whose support contains none of the curves  $\tilde{L}'_1, \dots, \tilde{L}'_5$  and  $Z_1, \dots, Z_r$ . Hence

$$\begin{aligned} 14 &= -K_W^3 = -K_W \cdot M_1 \cdot M_2 = -K_W \cdot \left( \sum_{i=1}^5 \tilde{L}'_i + m \sum_{i=1}^r Z_i + \Delta \right) = \\ &= -5K_W \cdot \tilde{L}'_1 - mrK_W \cdot Z - K_W \cdot \Delta = mr - K_W \cdot \Delta \geq mr \geq r \geq 20, \end{aligned}$$

which is absurd.  $\square$

**Lemma 6.7.** *The linear system  $\mathcal{M}_V$  is base point free.*

*Proof.* It is enough to show that  $\mathcal{M}_V$  is free from base curves. Indeed, if the base locus of the linear system  $\mathcal{M}_V$  does not contain base curves, then  $\mathcal{M}_V$  cannot have more than  $-K_V^3 = 4$  base points, because  $\mathcal{M}_V \sim -K_V$ . On the other hand,  $V$  does not contain  $\mathfrak{S}_5$ -orbits of length less than 12, because there are no  $\mathfrak{S}_5$ -orbits of such length on  $\mathbb{P}^3$  by Lemma 6.1.

Suppose that the base locus of the linear system  $\mathcal{M}_V$  contains an irreducible curve  $Z$ . If  $Z$  is not contained in any of the surfaces  $\hat{E}'_1, \dots, \hat{E}'_5$ , then the curve  $h(Z)$  is a base curve of the linear system  $\mathcal{M}_W$  and  $h(Z)$  is different from the curves  $\tilde{L}'_1, \dots, \tilde{L}'_5$ . This is impossible by Lemma 6.6. Similarly, if  $Z$  is not contained in any of the surfaces  $\hat{E}_1, \dots, \hat{E}_5$ , then the curve  $h' \circ \tau(Z)$  is a base curve of the linear system  $\mathcal{M}_{W'}$  that is different from the curves  $\tilde{L}_1, \dots, \tilde{L}_5$ . This is again impossible by Lemma 6.6. Thus,  $Z$  is contained in one of the surfaces  $\hat{E}_1, \dots, \hat{E}_5$ , and in one of the surfaces  $\hat{E}'_1, \dots, \hat{E}'_5$ . In particular, the curves flopped by  $\tau$  are not contained in the base locus of  $\mathcal{M}_V$ .

Without loss of generality, we may assume that  $Z = \hat{E}_1 \cap \hat{E}'_2$ . Let  $C$  be the curve flopped by  $\tau$  that is contained in  $\hat{E}_1$  and intersects  $\hat{E}'_2$ . Then  $C$  intersects  $Z$  by one point. On the other hand, we have  $-K_V \cdot C = 0$ . Since  $\mathcal{M}_V \sim -K_V$ , this implies that  $C$  is disjoint from a general surface in  $\mathcal{M}_V$ . This is impossible, because  $C \cap Z \neq \emptyset$ , while  $Z$  is contained in the base locus of the linear system  $\mathcal{M}_V$ .  $\square$

**Corollary 6.8.** *The linear systems  $\mathcal{M}_{V'}$ , and  $\mathcal{M}_U$  are also base point free.*

*Proof.* Recall that  $\mathcal{M}_V \sim -K_V$ . Thus, the general surface of  $\mathcal{M}_V$  is disjoint from all curves flopped by  $\tau$ , because  $\mathcal{M}_V$  is base point free by Lemma 6.7.  $\square$

**Corollary 6.9.** *The base locus of  $\mathcal{M}$  consists of the lines  $L_1, \dots, L_5, L'_1, \dots, L'_5$ .*

By Lemma 6.7 and Corollary 6.8, the divisors  $-K_V$ ,  $-K_{V'}$ , and  $-K_U$  are nef. Since

$$-K_V^3 = -K_{V'}^3 = -K_U^3 = 4,$$

these divisors are also big. Thus, the Kawamata–Viehweg vanishing theorem and the Riemann–Roch formula give

$$(6.10) \quad h^0(\mathcal{O}_V(-K_V)) = h^0(\mathcal{O}_{V'}(-K_{V'})) = h^0(\mathcal{O}_U(-K_U)) = 4.$$

In particular, one has  $|-K_V| = \mathcal{M}_V$ ,  $|-K_{V'}| = \mathcal{M}_{V'}$ , and  $|-K_U| = \mathcal{M}_U$ .

**Lemma 6.11.** *The  $\mathfrak{S}_5$ -representation  $H^0(\mathcal{O}_U(-K_U))$  is irreducible.*

*Proof.* By Lemma 3.14, there are no  $\mathfrak{S}_5$ -invariant quartic surfaces in  $\mathbb{P}^3$  that pass through the ten lines  $L_1, \dots, L_5, L'_1, \dots, L'_5$ . This implies that  $H^0(\mathcal{O}_U(-K_U))$  does not contain one-dimensional subrepresentations. Hence it is irreducible by Remark 4.5.  $\square$

Lemma 6.7 together with (6.10) implies that there is an  $\mathfrak{S}_5$ -equivariant commutative diagram

$$(6.12) \quad \begin{array}{ccc} & U & \\ \alpha \swarrow & & \searrow \beta \\ \mathbb{P}^3 & \dashrightarrow \phi & \mathbb{P}^4, \end{array}$$

where  $\phi$  is a rational map given by  $\mathcal{M}$ , and  $\beta$  is a morphism given by the anticanonical linear system  $|-K_U|$ . By Lemma 6.11 the projective space  $\mathbb{P}^4$  in (6.12) is a projectivization of an irreducible  $\mathfrak{S}_5$ -representation.

For  $1 \leq i < j \leq 5$ , let  $\Lambda_{ij}$  be the intersection line of the plane spanned by  $L_i$  and  $L'_j$  with the plane spanned by  $L'_i$  and  $L_j$ . Note that the stabilizer of  $\Lambda_{ij}$  in  $\mathfrak{S}_5$  contains a subgroup isomorphic to  $D_{12}$ . Actually, this implies that the stabilizer of  $\Lambda_{ij}$  in  $\mathfrak{S}_5$  is isomorphic to  $D_{12}$ , since  $D_{12}$  is a maximal proper subgroup in  $\mathfrak{S}_5$  (see Remark 4.3) and there are no  $\mathfrak{S}_5$ -invariant lines in  $\mathbb{P}^3$  by Corollary 2.1(i). Denote by  $\hat{\Lambda}_{ij}$  the proper transform of the line  $\Lambda_{ij}$  on the threefold  $V$ , and denote by  $\bar{\Lambda}_{ij}$  its proper transform on  $U$ . Then

$$-K_V \cdot \hat{\Lambda}_{ij} = 0.$$

Since  $v$  is a small birational morphism, we also obtain  $-K_U \cdot \bar{\Lambda}_{ij} = 0$ .

We see that each curve  $\bar{\Lambda}_{ij}$  is contracted by  $\beta$  to a point. Note that the stabilizer of  $\Lambda_{ij}$  in  $\mathfrak{S}_5$  is isomorphic to  $D_{12}$ . Since  $-K_U^3 = 4$ , the image of  $\beta$  is either an  $\mathfrak{S}_5$ -invariant quartic threefold or an  $\mathfrak{S}_5$ -invariant quadric threefold. Applying Corollary 4.8 together with Lemma 4.11, we obtain the following.

**Corollary 6.13.** *The morphism  $\beta$  is birational on its image, and its image is a quartic threefold.*

Now Lemmas 4.11 and 4.12 imply that the image of  $\beta$  is the quartic  $X_{\frac{1}{6}}$ . This proves

**Corollary 6.14.** *The threefold  $X_{\frac{1}{6}}$  is rational.*

*Remark 6.15.* An alternative approach to the rationality of the quartic threefold  $X_{\frac{1}{6}}$  was suggested in [16]. Unfortunately, its implementation seems to contradict the existence of the commutative diagram (6.12). Indeed, the paper [16] studies the action of the subgroup  $F_{20} \subset \mathfrak{S}_6$  on the threefold  $X_{\frac{1}{6}}$ . Since all such subgroups in  $\mathfrak{S}_6$  are conjugate, one may identify  $F_{20}$  with a subgroup of our  $\mathfrak{S}_5$ . By Remark 6.3, the group  $F_{20}$  permutes the ten lines  $L_1, \dots, L_5, L'_1, \dots, L'_5$  transitively. This means that  $\gamma$  is an  $F_{20}\mathbb{Q}$ -factorialization of the quartic threefold  $X_{\frac{1}{6}}$ . Thus, the application of  $F_{20}$ -Minimal Model Program to  $U$  must give the birational map  $\alpha: U \rightarrow \mathbb{P}^3$ . However, [16, Lemma 2.10], [16, Lemma 2.12] and [16, Lemma 2.13] exclude this possibility.

Ten curves  $\bar{\Lambda}_{ij}$  are mapped by  $\gamma$  to ten singular points of the threefold  $X_{\frac{1}{6}}$ . Twenty singular points of  $U$  are mapped by  $\gamma$  to another twenty singular points of  $X_{\frac{1}{6}}$ . Let us describe the curves in  $U$  that are contracted by  $\gamma$  to the remaining ten singular points of the threefold  $X_{\frac{1}{6}}$ . To do this, we need

**Lemma 6.16.** *Let  $\ell_1, \ell_2, \ell_3$  and  $\ell_4$  be pairwise skew lines in  $\mathbb{P}^3$ . Suppose that there is a unique line  $\ell \subset \mathbb{P}^3$  that intersects  $\ell_1, \ell_2, \ell_3$  and  $\ell_4$ . Let  $\pi: Y \rightarrow \mathbb{P}^3$  be a blow up of the*

line  $\ell$ , and  $E \cong \mathbb{P}^1 \times \mathbb{P}^1$  be the exceptional divisor of  $\pi$ . Denote by  $\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3$  and  $\tilde{\ell}_4$  the proper transforms on  $Y$  of the lines  $\ell_1, \ell_2, \ell_3$  and  $\ell_4$ , respectively. Then there exists a unique curve  $C \subset E$  of bi-degree  $(1, 1)$  that intersects the curves  $\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3$  and  $\tilde{\ell}_4$ .

*Proof.* The lines  $\ell_1, \ell_2$ , and  $\ell_3$  are contained in a unique quadric surface  $S \subset \mathbb{P}^3$ . Note that  $S$  is smooth, because  $\ell_1, \ell_2$ , and  $\ell_3$  are disjoint. Furthermore, the line  $\ell$  is contained in  $S$ , because  $\ell$  intersects the lines  $\ell_1, \ell_2$ , and  $\ell_3$  by assumption. Moreover, the line  $\ell_4$  is tangent to  $S$ , since otherwise there would be either two or infinitely many lines in  $\mathbb{P}^3$  that intersect  $\ell_1, \ell_2, \ell_3$  and  $\ell_4$ . Denote by  $\tilde{S}$  the proper transform on  $Y$  of the quadric surface  $S$ . Then  $\tilde{S}$  contains the curves  $\tilde{\ell}_1, \tilde{\ell}_2$ , and  $\tilde{\ell}_3$ . Moreover,  $\tilde{S}$  intersects the curve  $\tilde{\ell}_4$ . Thus  $\tilde{S}|_E$  is the required curve  $C$ .  $\square$

By Lemmas 3.7 and 6.16, each surface  $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$  contains a unique smooth rational curve  $C_i$  of bi-degree  $(1, 1)$  that passes through all four points of the intersection of  $E_i$  with the curves  $\tilde{L}'_1, \dots, \tilde{L}'_5$  (recall that  $E_i \cap \tilde{L}'_i = \emptyset$ ). Similarly, each surface  $E'_i \cong \mathbb{P}^1 \times \mathbb{P}^1$  contains a unique smooth rational curve  $C'_i$  of bi-degree  $(1, 1)$  that passes through all four points of the intersection of  $E'_i$  with the curves  $\tilde{L}_1, \dots, \tilde{L}_5$ . Denote by  $\hat{C}_1, \dots, \hat{C}_5$  the proper transforms on the threefold  $V$  of the curves  $C_1, \dots, C_5$ , respectively. Similarly, denote by  $\check{C}'_1, \dots, \check{C}'_5$  the proper transforms on the threefold  $V'$  of the curves  $C'_1, \dots, C'_5$ , respectively. Then

$$-K_V \cdot \hat{C}_i = -K_{V'} \cdot \check{C}'_i = 0.$$

This implies that the proper transforms of the curves  $\hat{C}_1, \dots, \hat{C}_5$  on the threefold  $V'$  are  $(-2)$ -curves on the surfaces  $\check{E}_1, \dots, \check{E}_5$ , respectively. Similarly, the proper transforms of the curves  $\check{C}'_1, \dots, \check{C}'_5$  on the threefold  $V$  are  $(-2)$ -curves on the surfaces  $\hat{E}'_1, \dots, \hat{E}'_5$ , respectively. Thus, all surfaces  $\hat{E}'_1, \dots, \hat{E}'_5, \check{E}_1, \dots, \check{E}_5$  are isomorphic to the Hirzebruch surface  $\mathbb{F}_2$ .

Denote by  $\bar{C}_1, \dots, \bar{C}_5, \bar{C}'_1, \dots, \bar{C}'_5$  the images of the curves  $\hat{C}_1, \dots, \hat{C}_5, \check{C}'_1, \dots, \check{C}'_5$  on the threefold  $U$ , respectively. Then

$$-K_U \cdot \bar{C}_i = -K_U \cdot \bar{C}'_i = 0,$$

because  $-K_V \cdot \hat{C}_i = -K_{V'} \cdot \check{C}'_i = 0$ , and  $v$  and  $v'$  are small birational morphisms. Thus, the ten curves  $\bar{C}_1, \dots, \bar{C}_5, \bar{C}'_1, \dots, \bar{C}'_5$  are contracted by the morphism  $\beta$  to ten singular points of  $X_{\frac{1}{6}}$ .

It would be interesting to extend the commutative diagram (6.12) to an  $\mathfrak{S}_5$ -Sarkisov link similar to the  $\mathfrak{A}_6$ -Sarkisov link (5.20).

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