Entropy metrics for graph signals

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1 Introduction

In the analysis of time series, entropy is a common tool used to describe the probability distribution of the states of a system. Based on this concept. the seminal paper [1] introduced the so-called permutation entropy (PE) as a measure to quantify irregularity in time series, a fundamental challenge in data analysis. This entropy involves calculating of permutation patterns, i.e., permutations defined by comparing neighbouring values of the time series. In the last years, PE has been applied in different field as biomedicine [2], physical systems [3] and economics [4].

A time series can be considered as a one-dimensional data (1D), while an image can be regarded as a two-dimensional data (2D). In the field of image processing, some entropies algorithms have been proposed to quantify the irregularity, most of them are generalisations of its one-dimensional analogous. For example, 2D sample entropy [5], 2D dispersion entropy [6], 2D distribution entropy [7] and 2D permutation entropy [8]. Most of the methods are straightforwardly generalised to higher-dimensional structures. The generalisations comes from the fact that the underlying structure (the lattice, for example) is a periodic structure, then all these algorithms use the symmetry from the structure to compare the values of the signal. However, it is unclear how to generalise the one-dimensional methods to a general irregular domains.

The study of data defined on irregular graphs is the main interest in the graph signal processing (GSP), an active research area in recent years [9]. This is motivated by the fact that, new technological advances have enabled the recording of data from complex systems. In some cases, the signal domain is not a set of equidistant time points (time series) or a regular grid (image). Graphs can model such data and complex interactions, and these new relations may be included in the data processing techniques as: filtering and subsampling [9].

Of note, the concept of graph entropy has been defined in previous literature [10]. However, this definition involves the computation of the Laplacian eigenvalues, its probability distribution and the Shannon entropy. Therefore, it measures the irregularity of the geometric structure, but not of the signals on the graph itself. In the paper [12], we introduce a measure of the complexity of a signal over a graph, combining the signal values with the topology of the graph.

2 Results

The original PE for time series is based in the comparison between values of consecutive samples. However, consecutive values cannot be defined straightforwardly in irregular graphs. Instead, we will consider the topology of the graph encoded in the adjacency matrix to define the algorithm. In particular, we compare between values on a fixed vertex and its neighbourhoods. This allows us to extend the concept of permutation patterns for signals [1] and images [8] to data on undirected graphs.

Let $G = (\mathscr{V}, \mathscr{E}, \mathbf{A})$ be a graph and $\mathbf{X}$ be a signal on the graph, the permutation entropy for the graph signals $\mathrm{PE_G}$ is defined as follow: For $2 \leq m \in \mathbb{N}$ the *embedding dimension* and $L \in \mathbb{N}$ the *delay time*, for all $i = 1, 2, \ldots, N$, we construct the embedding vector $\mathbf{y}_i^{m,L} \in \mathbb{R}^m$ given by

$$\mathbf{y}_i^{m,L} = \left(y_i^{kL}\right)_{k=0}^{m-1} = \left(y_i^0, y_i^L, \ldots y_i^{(m-1)L}\right) \ , \quad \text{where}$$

$$y_i^{kL} = \frac{1}{|\mathscr{N}_{kL}(i)|} \sum_{j \in \mathscr{N}_{kL}(i)} x_j = \frac{1}{|\mathscr{N}_{kL}(i)|} (\mathbf{A}^{kL}\mathbf{X})_i \ .$$

Recalling that $y_i^0 = x_i$ and $y_i^1 = (I - \Delta)x_i$. Once graph permutation patterns are defined, the rest of the algorithm for permutation entropy can be applied straightforwardly. For a fuller treatment, we refer the reader to [12].

We find that the entropy of the graph signal depends on the signal numerical values $\mathbf{X}$ and the graph topology G. In particular, if we consider $\mathbf{X}$ a time series and G a directed path, then our algorithm reduces to the classical 1D algorithm [1]. Similarly, if $\mathbf{X}$ is an image and G is a 2D lattice, then our algorithm has similar results to the 2D algorithm [8]. We also study the impact on the permutation entropy of the signal $\mathbf{X}$ under some geometrical perturbation on the graph (adding or deleting edges [11]).

We also considerer several graph structures G with the same graph signal $\mathbf{X}$. The entropy has different values depending on the graph underlying, even if it is the same signal $\mathbf{X}$. The permutation entropy for graph algorithm is able to detect different graph structures, and hence a signal could be more regular depending on the underlying graph, as the next example shows.

Example Let $\mathbf{X} = \{x_i\}_{i=1}^N$ be a set of white Gaussian noise samples.

1) We consider the following regular graphs on N vertices: complete, bipartite and cycle graph. Let $\mathbf{X}$ be the white noise signal on the regular graphs. For $N = 500$, we compute the permutation entropy of the signal $\mathbf{X}$ over the different graphs structures and different embedding lengths m. In Fig 1(left), we show the mean and standard deviation from entropy measures on a random signal $\mathbf{X}$ (for 20 realisations) for values $2 \leq m \leq 8$ and the different underlying graphs.

2) We consider the Erdős–Rényi graph $G(N, p)$ for several values of p, then we compute its permutation entropy values for the signal $\mathbf{X}$ and different graph structures $G(N, p)$. Even if it is the same random signal in all the graphs, the values of its entropies are different because the underlying graph has different topologies and edge connections. In Fig 1(right), we show the entropy values for p equal to $0.1, 0.3, 0.6$ and 0.9 with $N = 2000$ and $2 \leq m \leq 8$. The mean and standard deviations are shown for 20 simulations, and in all the cases (except for $m = 2$) there is not overlapping.

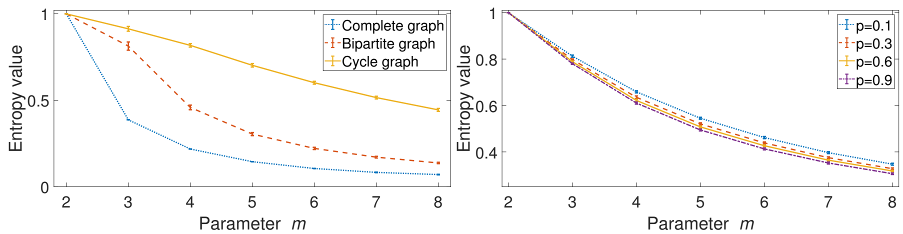

Fig. 1. Left: Permutation entropy measures of the random signal $\mathbf{X}$ on several underlying graphs G on 500 vertices. Right: Erdős–Rényi model for values p equal to $0.1, 0.3, 0.6$ and 0.9 with $N = 2000$ and 20 simulations. Error bars indicate standard deviation.

Summary. For the first time, we extended the concept of a nonlinear entropy metric - permutation entropy- from unidimensional time series to data residing on the vertices of undirected graphs [12]. We also explored how the permutation entropy of graph signals depends on both the signal and the graph. In addition, we showed that it generalises the behaviour of the unidimensional PE [1] and the recently introduced two-dimensional permutation entropy [8].

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