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Citation for published version:
Pease, A, Guhe, M & Smaill, A 2009, Analogy formulation and modification in geometry. in *Proceedings of the Second International Conference on Analogy.* pp. 358-364.

Published In:
Proceedings of the Second International Conference on Analogy

ANALOGY FORMULATION AND MODIFICATION IN GEOMETRY

Alison Pease
A.Pease@ed.ac.uk
Markus Guhe
M.Guhe@ed.ac.uk
Alan Smaill
A.Smaill@ed.ac.uk
School of Informatics, University of Edinburgh,
Informatics Forum, 10 Crichton Street,
Edinburgh, EH8 9AB, U.K.

ABSTRACT

We assume that the processes of analogy formulation and modification comprise some combination of finding and representing a fruitful source domain, forming appropriate associations, making predictions and inferences, verifying these new ideas, and learning (not necessarily in this order). We argue that these processes are as ubiquitous and fundamental in mathematics as they are elsewhere in empirical science, and therefore ideas in analogy research apply equally to mathematics. As a case study, we explore the origin and evolution of the Descartes–Euler conjecture, and discuss how geometry has developed via analogies which were used to invent and analyse this conjecture.

1 INTRODUCTION

Analogies in geometry between two and three dimensions have been bearing both healthy and unhealthy fruit for millennia. Dating back to Babylonian times, analogical mappings have been made between area and volume, line and plane, length and area, shape and solid, triangle and pyramid, trapezoid and frustum, parallelogram (formed by two pairs of parallel lines) and parallelepiped (formed by three pairs of parallel planes), and so on. The analogy has been used to suggest new entities, concepts, conjectures and representations, and to help to evaluate or support conjectures and thus motivate a proof attempt. Of particular interest to us, is the role of this and other analogies in the origin and evolution of Descartes--Euler's conjecture. In this example the target domain is geometrical solids and their classification, and different source domains are invoked at different points to suggest, evaluate, develop and prove the Descartes--Euler conjecture. These source domains themselves are open to improvement via analogy forming processes.

This well developed and well documented analogy is a useful example in analogy research since case studies often come from elsewhere in empirical science. Taking a descriptive rather than normative approach, and analysing historical analogies and mappings in different contexts and domains is important for testing the generality of ideas in analogy research. We assume that processes of analogy formulation and modification include finding and representing a fruitful source domain, mapping appropriate associations, making predictions and inferences, verifying these new ideas, and learning from the analogy. We discuss how these processes apply to our case study, and argue that they are as interlinked, messy and essential in mathematical thought as they are elsewhere in science.

2 ANALOGIES BETWEEN DIFFERENT DIMENSIONS IN GEOMETRY

One of the earliest examples of an analogy between two and three dimensional geometry was the Babylonian *negative analogy* (in Hesse's terms [9]) between the area of a trapezoid (formed by cutting off the top of a triangle by a line parallel to the base of the triangle) and the volume of a frustum (formed by cutting off the top of a pyramid by a plane parallel to the base of the pyramid), shown in Figure 1.

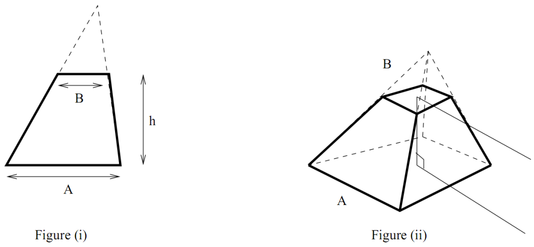

Figure 1. Analogy between a 2-dimensional trapezoid and 3-dimensional frustum in figures (i) and (ii) respectively. The A and B denote the length of the base and top lines in figure (i), and the area of the base and top polygons in figure (ii).

The formula for the area of a trapezoid is h*½(A + B), where A and B are the length of lines A and B, and trapezoids can be seen as analogous to frusta; with area mapping to volume, and length of the base and top lines of a trapezoid mapping to the area of the base and top of a frustum. Hence, the Babylonians made the incorrect analogical inference that the formula for the volume of a frustum would be h*½(A + B), where A and B are the areas of the polygons A and B [6, pp 4–6]. While use of analogy is easier to spot when the analogy fails, Babylonians also knew the formula for calculating the volume of a parallelepided (area*height), and since this can be found using the same mappings and one between a parallelogram (with an area of base*height) and a parallelepided, it is likely that the analogy was also used in this (and other) fruitful ways[1].

While demonstrating the lack of verification after inference (a reflection on the standards of rigour in mathematics at the time), this example also shows that mappings which suggest correct inferences between some elements of a domain may be lead to incorrect inferences for other elements. Such flawed but useful analogies should not be rejected prematurely, but used with caution.

One of Descartes's contributions to mathematics, and as the forerunner of Euler in this field, was to further develop analogies between two and three dimensional geometry. For instance, he suggested a three-dimensional analogue of Pythagoras's theorem, that for any right-angled triangle with sides a, b and c, $a^2 + b^2 = c^2$ where c is the side subtending the right angle. Descartes's formulation in three dimensions was $A^2 + B^2 + C^2 = D^2$, where A, B, C are the areas of the three sides and D is the area of the base which is opposite to the right-angled corner. Even more interesting is his four-dimensional analogue: $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \varepsilon^2$, where α, β, γ and δ are the volumes of the four solids and ε is the volume of the solid "opposed to the right angle". (Attempts to extend Pythagoras's theorem to multi-dimensions are also found in [7].) This latter conjecture is curious, since Descartes did not suggest a mapping between right angle in two dimensions to right angle in four dimensions. Thus, the conjecture is meaningless in the conventional sense, although clearly meaningful as a conjecture full of intuition and potential. Ludwig Schläfli subsequently defined the mapping and extended the Pythagorean theorem to the multi-dimensional case: see [19, p. 193], quoted in [18, p.130]. This suggests that infer-

[1] A parallelogram is formed by two pairs of parallel lines, and includes squares, rectangles, rhombi, etc. and a parallelepiped by three pairs of parallel planes, and includes cubes (six square faces), cuboids (six rectangular faces) and rhombohedra (six rhombus faces), etc.

ence steps are sometimes made *before* mappings have been made.

3 INITIAL MAPPINGS FOR DESCARTES – EULER'S CONJECTURE

A formulation of Descartes–Euler's conjecture states that for any polyhedra, the number of vertices (V) minus the number of edges (E) plus the number of faces (F) is equal to 2. This relationship holds for regular polyhedra; we show this for the cube, the tetrahedron and the octahedron in Figure 2.

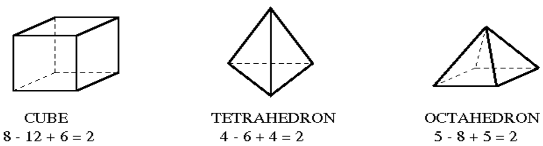

Figure 2: Euler's conjecture: V - E + F = 2 for the cube, the tetrahedron and the octahedron.

While Descartes came very close to formulating the conjecture [4, Part I] Euler was the first to do so explicitly [5, Proposition 4]: "in every solid body bounded by plane faces the sum of the number of solid angles and the number of faces exceeds the number of edges by two". While it is not known precisely how he arrived at the conjecture, the analogy between the polyhedral and polygonal domains and the relationship between the number of sides and angles of a polygon certainly played a role, at least in Euler's own rational reconstruction of the conjecture[2]. In the notation used for some analogies, the question is formed as:

$$V = E : polygons :: ? : polyhedra$$

The mapping between the *angles* and *sides* components of polygons onto today's *vertices*, *edges* and *faces* components of polyhedra was not trivial. It can quickly be seen that there is no straightforward mapping such that *V = E* would hold. Even generalising to "V increases with E" fails in some cases, although holds for most under a standard mapping (a pair of examples where this conjecture fails is a pentagonal prism, which has 10 vertices and 15 edges, and a tower consisting of a pyramid whose base is placed on the upper face of a cube, which has 9 vertices but 16 edges). Hence, careful thought is required to pull out relevant features and produce an appropriate mapping.

In the polygonal conjecture, the component "face" is an irrelevant feature (noise), whereas in polyhedra it is a key feature. One possible mapping of a "face" of a polygon would be to a "face" of a polyhedron. Such a mapping, in which salience is high in the target and low in the base, would contrast Ortony's salience imbalance theory [15]. This states that common features with high salience in the source domain and low salience in the target domain are mapped to each other. The fact that our polygons/polyhedra analogy makes sense in both directions, i.e. information from the 'target' can be used to further knowledge about the 'source' as well as vice versa, may be relevant here: Ortony's examples are unidirectional metaphors (for example, "dew is a veil" and "billboards are warts"). Another potential mapping would be from polygonal "sides" to polyhedral "faces", since in their respective domains they each bound the entity under investigation.

For sides of a face, Euler distinguished between the polygonal concept "side", and the analogy-generated polyhedral concept

[2] In a letter of November, 1750 to Christian Goldbach, he wrote: "Recently it occurred to me to determine the general properties of solids bounded by plane faces, because there is no doubt that general theorems can be found for them, *just as for plane rectilinear figures*, whose properties are: (1) that in every plane figure the number of sides is equal to the number of angles, and (2) that the sum of all the angles is equal to twice as many angles as there are sides, less four.". In his editorial summary of [5] (published later the same month): "*While in plane geometry* polygons could be classified very easily according to the number of their sides, which of course is always equal to the number of their angles, in stereometry the classification of polyhedra represents a much more difficult problem, since the number of faces alone is insufficient for this purpose" (quoted in footnote one [12, p6]) (our italics).

"edge" (Euler used the Latin terms 'latus' for side and 'acies' for edge) – "the joints where two faces come together side to side, which, for lack of an accepted word I call `edges'" (letter to Christian Goldbach, November, 1750). "Angles" is ambiguous in three dimensions: they could map to solid angles, plane angles (of all the faces of the polyhedron), or dihedral angles (the internal angle at which two adjacent faces meet). To select an appropriate mapping, Euler fluctuated between two analogous relationships both in the domain of polygons: $V=E$ and $\Sigma\alpha= (V–2)\pi$ (where α runs through all the interior angles of a convex polygon). In the latter relationship, translated to the polyhedra domain, counterexamples can be found in all three mappings of the interior angle (dihedral or solid or plane angles). In this case the counterexamples themselves, and their mappings back to the source domain, need to be examined. For instance, if angle is mapped to dihedral angle, then $\Sigma\alpha$ varies for tetrahedra according to their shape (whether there is a right angle between two planes), whereas in their polygonal counterparts, triangles, $\Sigma\alpha$ does not vary according to whether it is a right-angled triangle or otherwise (it is always 180° in Euclidean geometry). Thus, the mapping from angle to dihedral angle is rejected. For further details on hypothesised mappings between the two domains, see [20, pp. 150–154]. Following such negotiations, Euler mapped "faces" to "faces", "sides" to "edges" and "angles" to "solid angles" (which later become vertices) to get his famous conjecture.

4 POLYA'S MATHEMATICS AND *ANALOGICAL* REASONING

Euler implied that analogy with polygons inspired his *search* for an equation that related key features in polyhedra. Polya, in contrast, in his study on heuristic methods in mathematics, suggests that the analogy was only induced to *test* the Descartes—Euler conjecture to see whether it was worthy of a proof attempt. Polya introduces analogy at a point where Euler's conjecture has been suggested by scientific induction, has passed several increasingly difficult tests but failed on the picture frame example. As a consequence, the conjecture has been modified to "all convex polyhedra" and a new test is required with a new kind of support, rather than further supporting examples, in order to restore confidence in the conjecture. This motivation lies some way between Euler's motivation of using an analogy to *generate* a *new* problem, and the traditional view that analogy can help to *solve* an *existing* problem.

Having made the mappings between the two domains (see Section 3), and achieved the analogous relations: "for all polygons, $V = E$" and "for all convex polyhedra, $V - E + F = 2$", Polya suggests that we now adjust the representation in order to bring these relations closer. Noting that the polyhedron is 3-dimensional, its faces 2-dimensional, its edges 1-dimensional and its vertices 0-dimensional, we may rewrite these equations in order of the increasing dimensions. Thus, $V = E$ is written in the form $V - E + 1 = 1$. (We could go one step further and rewrite the 1 as the number of faces, assuming that all polygons have one face.) We can similarly rewrite our polyhedral relation as $V - E + F - 1 = 1$. In both of these, the number of dimensions starts at zero on the left hand side of the equation, increases by one and has alternating signs. The right hand side is the same in both cases.

Polya then suggests that since the two relations are very close and the first relation, for polygons is true, then we have reason to think that the second relation may be true, and is therefore worthy of a serious proof effort. This technique is encapsulated in Polya's advice when faced with a difficult problem to solve: "Is there a simpler problem?" as well as the surprising "Is there a more general problem?" [17].

5 LAKATOS'S PROOFS AND *ASSOCIATIONS*

Lakatos's characterisation of informal mathematics [12] fairly abounds with metaphors and analogies, which leap out from the page to excite the mind of the analogy researcher. These are delightfully sarcastic ("A woman with a child in her womb is not a counterexample to the thesis that human beings have one head" p.15), insightful ("Columbus did not reach India but he discovered something quite interesting" p. 14) and provocative ("if you want to know the normal healthy body, study it when it is abnormal, when it is ill" p. 23). He describes a world in which 'hopeful monsters' clash with 'ordinary decent polyhedra', polygons have limbs which should be free to stretch out into space, polyhedra have a lunatic fringe, and hang in teratological museums, mathematicians build strongholds into which they strategically retreat, wicked anarchists battle against dogmatic conservatives, mathematical method is an evolutionary struggle between monstrous mutants and harmonious order, discovery is a zip that goes up or down or may follow a zig-zag path, heuristics can be sterile, and rival interpretations of objects are seen as 'distorted imprints on a sick mind, twisting in pain'.

Lakatos's story is presented as a rationally reconstructed dialogue between students and their teacher. It begins at the point when having used the two-dimensional analogy to suggest the Descartes–Euler conjecture for classifying regular and other simple polyhedra, a new source domain is introduced, by Cauchy, for attempting a proof. The extension, but not intension, of polyhedra is roughly known and agreed. The intension is debated later on, after interactions with properties that hold for different explicit definitions of polyhedron. Cauchy's proof embeds the original conjecture (about crystals, or solids) in the theory of rubber sheets. This is Lakatos's most explicit use of analogy: "I propose to use the time-honoured technical term 'proof' for a thought-experiment – or 'quasi-experiment' – which suggests a decomposition of the original conjecture into subconjectures or lemmas, thus embedding it in a possibly quite distant body of knowledge" [12, p. 9]. He describes exactly the sorts of interlinked processes of discovery, justification and representation as proposed by Chalmers *et al.* in the context of high-level perception, representation and analogy [3].

In writing the story as dialogue, Lakatos allows different goals and beliefs about the value of the conjecture, the validity of the proof, mathematical methodology, etc. to affect which analogies and mappings are chosen. This is very much in line with cognitive scientific and psychological thought that perception may be influenced by belief (see, for example [1]), by goals (see [11]), and can be radically reshaped where necessary (see [14], also [3, pp. 171–172]). Examples of a shift in high-level perception, when we re-perceive something in a different way, include Lakatos's methods of monster-barring and monster-adjusting. The former consists of re-perceiving the boundaries of a source or target domain so that particular objects fall outside, rather than inside the boundaries. Monster-adjusting consists of a Gestalt shift in perception, where features of an object are suddenly perceived in a very different way. Lakatos gives the example of the star polyhedron, initially perceived as having 12 vertices, 30 edges and 12 faces, as a counterexample. Others object to this perception, saying that this is the case only if each face is a pentagon. This interpretation can be adjusted to seeing each face as a triangle, in which case the equation becomes $32 - 90 + 60 = 2$ and therefore it is no longer a counterexample. Likewise, polyhedra are re-perceived each time the conjecture is embedded in a different body of knowledge (including vector algebra, projective geometry, algebraic topology, etc.). As highlighted by Chalmers [3], and Holyoak and Thagard [10], amongst others, context and goals play an important role in determining which domains we associate and which mappings we form, when making, interpreting or evaluating an analogy. Thus, by showing different tasks that mathematicians perform, *i.e.*, forming, representing and modifying axioms, entities, concepts and conjectures and proofs, as well as differing motivations within these tasks, Lakatos presents a fertile domain for discussing interactions between different processes in analogical reasoning. (Many of these tasks have typically been given short shrift by philosophers of mathematics, mathematical educators and other researchers in how mathematics is or should be done. One other notable exception is work by cognitive scientists Lakoff and Núñez [13], who place metaphor at the heart of mathematics and suggest how a large number of concepts in arithmetic and other domains have been developed via metaphor.)

After much jostling and negotiating over the conjecture and concept definitions and its analogy to rubber sheets and after effort has been expended on showing that the conjecture "V – E + F = 2" is (or is not) a good mapping (whether it was a positive or negative analogy), the students come back to the problem and a new analogy is suggested. This involves reshaping the perception of a polyhedron from a rubber sheet to a matrix of edges, faces and vertices, and the conjecture reconfigured as:

$$V - E + F = 2 - 2(n-1) + \sum_{k=1}^{F} e_k$$

for *n*-spheroid polyhedra with e_k edges. [12, p. 79]. During the course of the conjecture's development, polyhedra (solids) are seen as analogous to polygons (to formulate the initial problem and conjecture), rubber sheets (to understand the conjecture and the concepts in it, and to find an initial proof), vector algebra (to find a formal proof), projective geometry [8], analytical topology [2], and algebraic topology [16].

The fact that the initial analogy between polygons and polyhedra makes sense in both directions becomes apparent when questions raised in the 'target' domain of polyhedra are then translated back to the polygonal 'source' domain and used to develop polygonal notions and to expand this domain [12, pp 70–73].

6 CONCLUSION

It is important that components of analogical reasoning which have been identified in specific domains are considered in other domains. This will both test the generality of such components and exploit ideas from research in analogy to inform ways in which people reason in domains which have received less attention. The case study of analogies in the Descartes–Euler conjecture explores the complex interaction between different processes of analogy formulation and modification, as well as emphasising the role that analogy has played in the origin and development of concepts, conjecture, proof and associated fields.

ACKNOWLEDGEMENTS

We would like to thank Ross Fraser and our two anonymous reviewers for their useful comments. This work was funded by EPSRC grant EP/F035594/1.

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