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CHARACTERIZATIONS OF CATEGORIES OF COMMUTATIVE C*-SUBALGEBRAS

CHRIS HEUNEN

Abstract. We aim to characterize the category of injective $*$-homomorphisms between commutative C*-subalgebras of a given C*-algebra A. We reduce this problem to finding a weakly terminal commutative subalgebra of A, and solve the latter for various C*-algebras, including all commutative ones and all type I von Neumann algebras. This addresses a natural generalization of the Mackey–Piron programme: which lattices are those of closed subspaces of Hilbert space? We also discuss the way this categorified generalization differs from the original question.

1. Introduction

The collection $\mathcal{C}(A)$ of commutative C*-subalgebras of a fixed C*-algebra A can be made into a category under various choices of morphisms. Two natural ones are inclusions and injective $*$-homomorphisms, resulting in categories $\mathcal{C}_{\subseteq}(A)$ and $\mathcal{C}_{\rightarrowtail}(A)$, respectively. The goal of this article is to characterize these categories.

Categories based on $\mathcal{C}(A)$ are interesting for a number of reasons. A first motivation to study such categories is the hope that they could lead to a noncommutative extension of Gelfand duality. It is known that $\mathcal{C}_{\subseteq}(A)$ determines A as a partial C*-algebra [2]. Except when $A \cong \mathbb{C}^2$ or $A \cong \mathbb{M}_2(\mathbb{C})$, equivalently $\mathcal{C}_{\subseteq}(A)$ determines precisely the quasi-Jordan structure of A [11, 12]. Thus, $\mathcal{C}(A)$ in itself is already an interesting invariant of A. Moreover, structures based on $\mathcal{C}(A)$ circumvent obstructions to a noncommutative Gelfand duality that afflict many other candidates [1]. Indeed, for C*-algebras A with enough projections, adding a little more structure to $\mathcal{C}(A)$ fully determines the algebra structure of A [18, 17]. To get a full noncommutative Gelfand duality for such algebras, it suffices to characterize the structures based on $\mathcal{C}(A)$ that arise this way; an important step is clearly to characterize categories of the form $\mathcal{C}(A)$.

Second, there is a physical perspective on $\mathcal{C}(A)$. The underlying idea, due to Bohr, is that one can only empirically access a quantum mechanical system, whose observables are modeled by a (noncommutative) C*-algebra, through its classical subsystems, as modeled by commutative C*-subalgebras [15]. Categories based on $\mathcal{C}(A)$ are of paramount importance in the recent uses of topos theory in research

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in foundations of physics based on this idea that proposes a new form of quantum logic [7, 16]. Knowing which categories are of the form $\mathcal{C}(A)$ also characterizes which toposes are of the form studied in that programme. This should increase insight into the intrinsic structure of such toposes, and hence shed light on the foundations of quantum physics such toposes aim to describe logically.

Third, more generally, a characterization of $\mathcal{C}(A)$ satisfactorily addresses a general theme in research in foundations of quantum mechanics. For example, it addresses (a categorification of) the Mackey–Piron programme. This programme asks the question: which orthomodular lattices are those of closed subspaces of Hilbert space? (See [24, 28, 22].) A characterization of $\mathcal{C}(A)$ would provide an answer, because choosing a commutative C*-subalgebra of the matrix algebra $\mathbb{M}_n(\mathbb{C})$ amounts to choosing an orthonormal subset and hence a closed subspace of $\mathbb{C}^n$, and an appropriate generalization to infinite dimension holds as well (see also Theorem 2.5 below and [14]). Similarly, a characterization of $\mathcal{C}(A)$ has consequences in the study of test spaces. These are defined as collections of orthogonal subsets of a Hilbert space satisfying some conditions, and have been proposed as axioms for operational quantum mechanics. One of the major questions there is again which test spaces arise from propositions on Hilbert spaces [30].

Our main result is to reduce characterizing $\mathcal{C}_{\rightarrowtail}(A)$ to finding a weakly terminal commutative subalgebra of A. This is closely related to analyzing all maximal abelian subalgebras (masas). Explicating the structure of masas of C*-algebras in general is a hard problem, and not much seems to be known systematically outside of the case of factors of type I and type II$_1$; see [5, 27]. Fortunately, finding a weakly terminal commutative subalgebra is generally easier than finding all masas. We prove that the following classes of C*-algebras A possess weakly terminal commutative subalgebras, and therefore we find a full characterization of $\mathcal{C}_{\rightarrowtail}(A)$ for:

- type I von Neumann algebras, including all finite-dimensional C*-algebras;
- commutative C*-algebras.

The strategy behind our characterization is as follows. The key insight is to recognize $\mathcal{C}_{\rightarrowtail}(D)$ for a commutative C*-algebra D as the Grothendieck construction of an action of a monoid M on a partially ordered set P. We characterize such so-called amalgamations. Next, we use known results to characterize the partially ordered set $P = \mathcal{C}_{\subseteq}(D)$, consisting of partitions of the Gelfand spectrum of D. Then, we show that $\mathcal{C}_{\rightarrowtail}(A)$ is equivalent to $\mathcal{C}_{\rightarrowtail}(D)$ for a weakly terminal object D in $\mathcal{C}_{\rightarrowtail}(A)$. Finally, we establish such a weakly terminal object D for the various types of C*-algebras A mentioned, finishing the characterization. This last step is the only one limiting our characterization to C*-algebras A with weakly terminal commutative subalgebras. Summarizing:

(1) show that a C*-algebra A has a weakly terminal abelian subalgebra D;
(2) show that $\mathcal{C}_{\rightarrowtail}(A)$ is equivalent to $\mathcal{C}_{\rightarrowtail}(D)$;
(3) show that $\mathcal{C}_{\rightarrowtail}(D)$ is equivalent to $P(X) \rtimes S(X)$, with X the spectrum of D;
(4) characterize $P(X) \rtimes S(X)$ in terms of $P(X)$ and $S(X)$;
(5) a characterization of $P(X)$ exists;
(6) in the cases in question, X, and hence $S(X)$, is easy to characterize.

Thus we address the Mackey–Piron programme in a different way than the theorems of Piron [24] and Solèr [28], which together form the only characterization of the lattice of closed subspaces of a Hilbert space we are aware of. Piron's theorem states that the lattice should be complete, atomic, irreducible, orthomodular, and satisfy the covering law, from which it follows that it must be the lattice of closed subspaces of some Euclidean space over a skew field. Solèr's theorem says that if additionally this Euclidean space is infinite-dimensional and has the property that any closed subspace is a direct summand, then the skew field must be the reals, complexes or quaternions, and the space must be a Hilbert space. Both Solèr's direct summand condition and Piron's lattice-theoretic axioms relate to our use of partition lattices $P(X)$, but instead of orthomodularity we use the action of $S(X)$. Interestingly, our results apply to arbitrary Hilbert spaces, whereas Solèr's theorem only holds for infinite-dimensional ones.

The paper is structured as follows. We start with Section 2, which introduces the poset $\mathcal{C}_\subseteq(A)$ and the category $\mathcal{C}_\rightarrowtail(A)$ and discusses their basic properties and motivation. A more in-depth analysis of the relationship between the two, again depending on the Grothendieck construction, is made later, in Section 7. Our main results are presented in between. To aid intuition, we first cover the finite-dimensional case, and only then incorporate the subtleties of the infinite-dimensional case. Section 3 characterizes amalgamations of groups and posets, which is then used in Section 4 to establish the characterization in the finite-dimensional case. Then, Section 5 refines the earlier analysis to characterize amalgamations of monoids and posets. This is used in Section 6 to establish the characterization in the infinite-dimensional case. Appendix A records some intermediate results of independent interest. In particular, it discusses an alternative way to investigate the relationship between $\mathcal{C}_\rightarrowtail(A)$ and $\mathcal{C}_\subseteq(A)$.

To end this introduction let us briefly indicate the differences between $\mathcal{C}_\rightarrowtail(A)$ and $\mathcal{C}_\subseteq(A)$. This will be discussed in more depth in Section 7, but it might be helpful to mention them now to set the scene. Any morphism in $\mathcal{C}_\rightarrowtail(A)$ factors uniquely as a $*$-isomorphism followed by a morphism in $\mathcal{C}_\subseteq(A)$. If $\mathcal{C}_\rightarrowtail(A) \cong \mathcal{C}_\rightarrowtail(B)$ are isomorphic categories, then $\mathcal{C}_\subseteq(A) \cong \mathcal{C}_\subseteq(B)$ are isomorphic posets. Therefore, as discussed above, both categories $\mathcal{C}_\rightarrowtail(A)$ and $\mathcal{C}_\subseteq(A)$ are *invariants* of the C*-algebra A, in the sense that both determine the (quasi-)Jordan structure of A, and are hence respected by (quasi-)Jordan homomorphisms. We will mostly be interested in a coarser notion of invariant, namely equivalence of categories, rather than isomorphism of categories. For posetal categories like $\mathcal{C}_\subseteq(A)$, isomorphism and equivalence coincide, but for $\mathcal{C}_\rightarrowtail(A)$ this makes a difference: $\mathcal{C}_\rightarrowtail(A) \simeq \mathcal{C}_\rightarrowtail(B)$ need not imply $\mathcal{C}_\subseteq(A) \cong \mathcal{C}_\subseteq(B)$ (and certainly not $A \cong B$). It turns out that $\mathcal{C}_\rightarrowtail(A) \simeq \mathcal{C}_\rightarrowtail(B)$ are equivalent categories precisely when $\mathcal{C}_\subseteq(A)$ and $\mathcal{C}_\subseteq(B)$ are Morita-equivalent, in the sense that they have equivalent presheaf categories $\mathrm{PSh}(\mathcal{C}_\subseteq(A)) \simeq \mathrm{PSh}(\mathcal{C}_\subseteq(B))$. This explains why equivalence of categories is a more natural invariant from the point of view of category theory and topos theory.

2. Motivation

We do not require C*-algebras to have a unit, and write **Cstar** for the category of C*-algebras and $*$-homomorphisms.

Definition 2.1. Write $\mathcal{C}(A)$, or simply $\mathcal{C}$, for the collection of nonzero commutative C*-subalgebras C of a C*-algebra A. This set of objects can be made into a category by various choices of morphisms, such as:

- inclusions $C \hookrightarrow C'$, given by $c \mapsto c$, yielding a (posetal) category $\mathcal{C}_\subseteq(A)$;
- injective $*$-morphisms $C \rightarrowtail C'$, giving a (left-cancellative) category $\mathcal{C}_\rightarrowtail(A)$.

These two categories are interesting for two related reasons. First, they form a major ingredient in a new attack on a noncommutative extension of Gelfand duality [2, 1, 18]. Essentially, one could think of them as invariants of a C*-algebra. Second, they play an important role in the recent use of topos theory in the foundations of quantum physics. From this perspective, one could think of them as encoding the logic of a quantum-mechanical system whose observables are modeled by the C*-algebra A. We will discuss these two perspectives in turn, but first we consider functoriality of the construction $A \mapsto \mathcal{C}(A)$. Section 7 below discusses the relationship between the two choices of morphisms, $\mathcal{C}_{\rightarrowtail}(A)$ or $\mathcal{C}_{\subseteq}(A)$ in more detail.

Functoriality. The assignment $A \mapsto \mathcal{C}_{\subseteq}(A)$ extends to a functor: given a $*$-homomorphism $\varphi\colon A \to B$, direct images $C \mapsto \varphi(C)$ form a morphism $\mathcal{C}_{\subseteq}(A) \to \mathcal{C}_{\subseteq}(B)$ of posets, for if $C \subseteq C'$, then $\varphi(C) \subseteq \varphi(C')$. Well-definedness relies on the following fundamental fact, that we record as a lemma for future reference.

Lemma 2.2. *The set-theoretic image of a C*-algebra under a $*$-homomorphism is again a C*-algebra.*

Proof. See [20, Theorem 4.1.9]. □ □

The assignment $A \mapsto \mathcal{C}_{\rightarrowtail}(A)$ has to be adapted to be made functorial. Either we only consider injective $*$-homomorphisms $A \rightarrowtail B$, or we restrict the target category $\mathcal{C}_{\rightarrowtail}(A)$ as follows. Write **Cat** for the category of small categories and functors.

Lemma 2.3. *There is a functor* **Cstar** $\to$ **Cat***, sending A to the subcategory of $\mathcal{C}_{\rightarrowtail}(A)$ with morphisms those $i\colon C \to C'$ satisfying*

$$i^{-1}(I \cap C') = I \cap C$$

for all closed (two-sided) ideals I of A.

Proof. Let $\varphi\colon A \to B$ be a $*$-homomorphism, and let i be as in the statement of the lemma. Then i induces a well-defined injective $*$-homomorphism $\varphi(C) \to \varphi(C')$ precisely when $\varphi(c_1) = \varphi(c_2) \iff \varphi(i(c_1)) = \varphi(i(c_2))$. Since φ and i are linear, this comes down to $\varphi(c) = 0 \iff \varphi(i(c)) = 0$, *i.e.* $\ker(\varphi) \cap C = \ker(\varphi \circ i)$. Setting $I = \ker(\varphi)$, this becomes

$$\begin{aligned} I \cap C &= \{c \in C \mid \varphi(c) = 0\} \\ &= \{c \in C \mid \varphi(i(c)) = 0\} \\ &= i^{-1}\big(\{c' \in C' \mid \varphi(c') = 0\}\big) \\ &= i^{-1}(I \cap C') \end{aligned}$$

and is therefore satisfied. □ □

Notice that $*$-homomorphisms satisfying the condition of the previous lemma are automatically injective, as is seen by taking $I = \{0\}$.

Notice also that when A is a topologically simple C*-algebra, such as the algebra $\mathbb{M}_n(\mathbb{C})$ of n-by-n complex matrices, then the subcategory of the previous lemma is actually the whole category $\mathcal{C}_{\rightarrowtail}(A)$.

Invariants. Let us temporarily consider von Neumann algebras A and their von Neumann subalgebras $\mathcal{V}(A)$, giving categories $\mathcal{V}_{\subseteq}$ and $\mathcal{V}_{\rightarrowtail}$. We will show that $\mathcal{V}_{\subseteq}$ contains exactly the same information as the lattice $\mathrm{Proj}(A)$ of projections of A, in the technical sense that they are functors with equivalent images. This lattice has been studied in depth, so from the point of view of (new) invariants of A, the category $\mathcal{V}_{\rightarrowtail}$ is more interesting. See also Remark 7.8 below. By extension, $\mathcal{C}_{\rightarrowtail}$ is possibly more interesting as an invariant than $\mathcal{C}_{\subseteq}$, because $\mathcal{C}(A)$ and $\mathcal{V}(A)$ coincide for finite-dimensional C*-algebras A.

Denote the category of von Neumann algebras and unital normal $*$-homomorphisms by **Neumann**, and write **cNeumann** for the full subcategory of commutative (unital von Neumann) algebras. Denote the category of orthomodular lattices and lattice morphisms preserving the orthocomplement by **Ortho**. The functor $\mathrm{Proj}\colon \mathbf{Neumann} \to \mathbf{Ortho}$ takes A to $\{p \in A \mid p^2 = p = p^*\}$ under the ordering $p \leq q$ iff $pq = p$. On morphisms $f\colon A \to B$ it acts as $p \mapsto f(p)$. Recall that the essential image of a functor F is the smallest subcategory of the target category containing all isomorphisms and all morphisms of the form $F(f)$. Denote the essential image of Proj by $\mathbf{D}$; traditional quantum logic is the study of this subcategory of **Ortho** [25].

Denote by **Poset**[**cNeumann**] the following category: objects are sets of commutative von Neumann algebras partially ordered by inclusion (*i.e.* $C \leq C'$ iff $C \subseteq C'$); morphisms are monotonic functions. We may regard $\mathcal{V}_{\subseteq}$ as a functor $\mathbf{Neumann} \to \mathbf{Poset}[\mathbf{cNeumann}]$. Denote the essential image of $\mathcal{V}_{\subseteq}$ by $\mathbf{C}$; this is a subcategory of **Poset**[**cNeumann**].

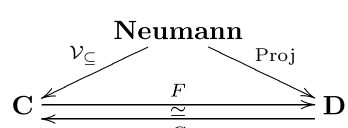
We now define two new functors, $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$. The functor F acts on an object $\mathcal{V}_{\subseteq}(A)$ as follows. For each $C \in \mathcal{V}_{\subseteq}(A)$, we know that $\mathrm{Proj}(C)$ is a Boolean algebra [25, 4.16]. Because additionally the hypothesis of Kalmbach's Bundle lemma, is satisfied, these Boolean algebras unite into an orthomodular lattice $F(\mathcal{V}_{\subseteq}(A))$. This assignment extends naturally to morphisms.

Lemma 2.4 (Bundle lemma)**.** *Let $\{B_i\}$ be a family of Boolean algebras such that $\vee_i = \vee_j$, $\neg_i = \neg_j$, and $0_i = 0_j$ on intersections $B_i \cap B_j$. If $\leq$ on $\bigcup_i B_i$ is transitive and makes it into a lattice, then $\bigcup_i B_i$ is an orthomodular lattice.*

Proof. See [21, 1.4.22]. □ □

The functor G acts on the projection lattice L of a von Neumann algebra as follows. Consider all complete Boolean sublattices B of L as a poset under inclusion. For each B, the continuous functions on its Stone spectrum form a commutative von Neumann algebra. Thus we obtain an object $G(L)$ in $\mathbf{C}$, and this assignment extends naturally to morphisms.

Theorem 2.5. *The functors F and G form an equivalence, and make the following diagram commute.*



Proof. Follows directly from the definitions and the previous lemma. □ □

Indeed, both $\mathcal{V}_{\subseteq}(A)$ and $\mathrm{Proj}(A)$ capture the Jordan algebra structure of A [13], excepting the case where A has summands of type I_2.

Returning to the setting of C*-algebras, notice that the previous theorem fails, because there are C*-algebras without any nontrivial projections. But every C*-algebra has many commutative C*-subalgebras: every self-adjoint element generates one, and every element of a C*-algebra is a complex linear combination of self-adjoint elements. For C*-algebras, $\mathcal{C}_{\subseteq}(A)$ captures precisely the pseudo-Jordan algebra structure of A [11, 12]. In this regard, it is also worth remarking that the functor $\mathcal{C}_{\subseteq}\colon \mathbf{Cstar} \to \mathbf{Poset}[\mathbf{cCstar}]$ factors through the category of partial C*-algebras [2].

Toposes in foundations of physics. The main theorem in the application of topos theory to foundations of quantum physics is the following. The tautological functor $C \mapsto C$ is an internal (possibly nonunital) C*-algebra [16, Theorem 6.4.8]. It holds in both toposes $\mathbf{Set}^{\mathcal{C}_{\subseteq}}$ and $\mathbf{Set}^{\mathcal{C}_{\rightarrowtail}}$ because of the fundamental Lemma 2.2 above. Categorically, $\mathcal{C}_{\rightarrowtail}$ is a more natural choice than $\mathcal{C}_{\subseteq}$.

But to characterize a presheaf category is the same as characterizing the category it is based on, by Morita equivalence; see also Section 7 and Appendix A below. Thus, our main results also characterize toposes of the form $\mathbf{Set}^{\mathcal{C}_{\rightarrowtail}}$. For a more or less practical account of the above folklore knowledge we refer to [4].

3. Poset-group-amalgamations

This section recalls the Grothendieck construction, focusing on the special case of an action of a group on a poset. We will call the resulting categories poset-group-amalgamations. The goal of this section is to characterize such categories. This is interesting in its own right, but even more so because in Section 6 we will see that $\mathcal{C}_{\rightarrowtail}$ is of this form. For that reason, we prefer a practical characterization. Therefore, we will not pursue the highest possible level of generality: the discussion in this section is in elementary terms spelling out what is probably folklore knowledge. In particular, the characterization in this section can be extended to poset-category-amalgamations, and perhaps even to a characterization of Grothendieck constructions of arbitrary indexed categories, but we will not pursue this here. We will use the Grothendieck construct, also called the category of elements, again in Section 7, where it is discussed more abstractly. The main idea in this section is to factor out symmetries into a monoid action, leaving just the partial order.

Definition 3.1. An *action* of a monoid M (in the category of sets) *on* a category $\mathbf{C}$ is a functor $F\colon M \to \mathbf{Cat}(\mathbf{C}, \mathbf{C})$. Write mx for the action of Fm on an object x of $\mathbf{C}$, and mf for the action of Fm on a morphism f of $\mathbf{C}$.

Definition 3.2. If a monoid M acts on a category $\mathbf{C}$, then we can perform the *Grothendieck construction*: we can make a new category $\mathbf{C} \rtimes M$ whose objects are those of $\mathbf{C}$, and whose morphisms $x \to y$ are pairs (m, f) such that $\mathrm{dom}(f) = x$ and $\mathrm{cod}(f) = my$. Composition and identities are inherited from M and $\mathbf{C}$. Explicitly, $\mathrm{id}_x = (1, \mathrm{id}_x)$, and $(n, g) \circ (m, f) = (mn, (mg)f)$.

If the category $\mathbf{C}$ in the previous definition is a partially ordered set P, then $P \rtimes M$ has as objects $p \in P$, and morphisms $p \to q$ are $m \in M$ such that $p \leq mq$, with unit and composition from M^{op}.

An illustrative example to keep in mind is the following. Let M be the group of unitary n-by-n matrices. Let P be the lattice of subspaces of $\mathbb{C}^n$, ordered by inclusion. Then M acts on P by $UV = \{U(v) \mid v \in V\}$ for $U \in M$ and $V \in P$.

Morphisms in $P \rtimes M$ between subspaces $V \subseteq \mathbb{C}^n$ and $W \subseteq \mathbb{C}^n$ are unitary matrices U such that $U^{-1}(v) \in W$ for all $v \in V$.

This section characterizes categories of the form $P \rtimes G$ for an action of a group G on a poset P with a least element. Our characterization will rely on weakly initial objects to recover P from $P \rtimes G$. Categorically, this is trivial, but as we will see in Sections 4 and 6, it is a very important step in our application. An object 0 is *weakly initial* when for any object x there exists a (not necessarily unique) morphism $0 \to x$; notice that such an object is not necessarily unique up to isomorphism, as an initial object would be. If a category $\mathbf{A}$ has a weak initial object 0, we can regard the endohomset monoid $\mathbf{A}(0,0)$ as a one-object category. Recall that a *retraction* of a functor is a left-inverse.

Lemma 3.3. *If a category $\mathbf{A}$ has a weak initial object 0 and a faithful retraction F of the inclusion $\mathbf{A}(0,0) \hookrightarrow \mathbf{A}$, then its objects are preordered by*

$$x \leq y \iff \exists f \in \mathbf{A}(x,y).\, F(f) = 1.$$

Proof. Clearly $\leq$ is reflexive, because $F(\mathrm{id}_x) = 1$. It is also transitive, for if $x \leq y$ and $y \leq z$, then there are $f\colon x \to y$ and $g\colon y \to z$ with $F(f) = 1 = F(g)$, so that $g \circ f\colon x \to z$ satisfies $F(g \circ f) = F(g) \circ F(f) = 1 \circ 1 = 1$ and $x \leq z$. □ □

Thus we can recover the group G from $\mathbf{A} = P \rtimes G$ by looking at $\mathbf{A}(0,0)$. We can also recover the poset P from $\mathbf{A}$ by the previous lemma. What is left is to reconstruct the action of G on P given just $\mathbf{A}$. For $m \in G$ and $p \in P$, we can access the object mq through the morphisms $m\colon p \to q$ in $\mathbf{A}$. There is always at least one such morphism, namely $m\colon mq \to q$, because trivially $mq \leq mq$. In fact, this is always an isomorphism. We will now use this fact to recover the action of G on P from $\mathbf{A}$. We will call this an *amalgamation* by analogy with the use of the term in algebra.

Definition 3.4. A category $\mathbf{A}$ is called a *poset-group-amalgamation* when there exist a partial order P and a group G such that:

(A1) there is a weak initial object 0, unique up to isomorphism;
(A2) there is a faithful retraction F of the inclusion $\mathbf{A}(0,0) \hookrightarrow \mathbf{A}$;
(A3) there is an isomorphism $\alpha\colon \mathbf{A}(0,0) \to G^{\mathrm{op}}$ of monoids;
(A4) there is an equivalence $(\mathbf{A}, \leq) \underset{\beta'}{\overset{\beta}{\rightleftarrows}} P$ of preorders;
(A5) for each object x there is an isomorphism $f\colon x \to \beta'(\beta(x))$ with $\alpha F(f) = 1$;
(A6) for each y and m there is an isomorphism $f\colon x \to y$ with $\alpha F(f) = m$.

Example 3.5. If P is a partial order with least element, and G is a group acting on P, then $P \rtimes G$ satisfies (A1)–(A6).

Proof. The least element 0 of P is a weak initial object, satisfying (A1). Conditions (A2)–(A4) are satisfied by definition, and (A5) is vacuous. To verify (A6) for $q \in P$ and $m \in G$, notice that $mq \leq mq$, so $f = 1\colon mq \to mq$ is an isomorphism with $\alpha F(f) = 1$. □ □

Lemma 3.6. *If $\mathbf{A}$ satisfies (A1)–(A6), then it induces an action of G on P given by $mp = \beta(x)$ if $f\colon x \to \beta'(p)$ is an isomorphism with $\alpha(F(f)) = m$.*

Proof. First, notice that for any $p \in P$ and $m \in G$ there exists an isomorphism $f\colon x \to \beta'(p)$ with $\alpha(F(f)) = m$ by (A6). If there is another isomorphism $f'\colon x' \to \beta'(p)$ with $\alpha(F(f')) = m$, then their composition gives $x \cong x'$, and therefore

$\beta(x) \cong \beta(x')$. But because P is a partial order, this means $\beta(x) = \beta(x')$. Thus the action is well-defined on objects.

To see that it is well-defined on morphisms, suppose that $p \leq q$. Then there is a morphism $f\colon \beta'(p) \to \beta'(q)$ with $F(f) = 1$. For any $m\colon 0 \to 0$, axiom (A6) provides isomorphisms $f_p\colon x_p \to \beta'(p)$ and $f_q\colon x_q \to \beta'(q)$ with $\alpha(F(f_p)) = m = \alpha(F(f_q))$. Then $f = f_q^{-1} f f_p\colon x_p \to x_q$ is an isomorphism satisfying $\alpha F(f) = mm^{-1} = 1$. So $mp \leq mq$.

Next, we verify that this assignment is functorial $G \to \mathbf{Cat}(P,P)$. Clearly $\mathrm{id}_{\beta'(p)}$ is an isomorphism $x \to \beta'(p)$ with $F(\mathrm{id}_{\beta'(p)}) = 1$. Therefore $1p = \beta(\beta'(p)) = p$.

Finally, for $m_2, m_1 \in M$ and $p \in P$, we have $m_1 p = \beta(x_1)$ where $f_1\colon x_1 \to \beta'(p)$ is an isomorphism with $\alpha(F(f_1)) = m_1$. So $m_2(m_1 p) = \beta(x_2)$ where $f_2\colon x_2 \to \beta'(\beta(x_1))$ is an isomorphism with $\alpha(F(f_2)) = m_2$. By (A5), there is an isomorphism $h\colon x_1 \to \beta'(\beta(x_1))$ with $F(h) = 1$. So $f = f_1 h^{-1} f_2$ is an isomorphism $x_2 \to \beta'(p)$ with $\alpha(F(f)) = m_2 m_1$. Thus $(m_2 m_1)p = \beta(x_2) = m_2(m_1 p)$. □ □

Theorem 3.7. *If* $\mathbf{A}$ *satisfies (A1)–(A6), then there is an equivalence* $\mathbf{A} \to P \rtimes G$ *given by* $x \mapsto \beta(x)$ *on objects and* $f \mapsto \alpha(F(f))$ *on morphisms.*

Proof. First we verify that the assignment of the statement is well-defined, *i.e.* that $\alpha(F(f))$ is indeed a morphism of $P \rtimes G$. Given $f\colon x \to y$, we need to show that $\beta(x) \leq \alpha(F(f)) \cdot \beta(y)$. Unfolding the definition of action, this means finding an isomorphism $k\colon x' \to \beta'(\beta(y))$ with $\alpha(F(k)) = \alpha(F(f))$ and $\beta(x) \leq \beta(x')$. Unfolding the definition of the preorder, the latter means finding a morphism $h'\colon \beta'(\beta(x)) \to \beta'(\beta(x'))$ with $F(h') = 1$. By (A5), it suffices to find $h\colon x \to x'$ with $F(h) = 1$ instead. But (A6) provides an isomorphism $k\colon x' \to \beta'(\beta(y))$ with $\alpha(F(k)) = \alpha(F(f))$. By (A5) again, there exists an isomorphism $l\colon y \to \beta'(\beta(y))$ with $\alpha(F(l)) = 1$. Finally, we can take $h = k^{-1} l f\colon x \to x'$. This morphism indeed satisfies $\alpha(F(h)) = \alpha(F(f)) \cdot \alpha(F(l)) \cdot \alpha(F(k))^{-1} = \alpha(F(k)) \cdot \alpha(F(k))^{-1} = 1$.

Functoriality follows directly from the previous lemma, so indeed we have a well-defined functor $\mathbf{A} \to P \rtimes G$. Moreover, our functor is essentially surjective because β is an equivalence, and it is faithful because F is faithful.

Finally, to prove fullness, let $m\colon \beta(x) \to \beta(y)$ be a morphism in $P \rtimes G$. This means that $\beta(x) \leq m\beta(y)$, which unfolds to: there are a morphism $f\colon x \to z$ and an isomorphism $h\colon z \to \beta'(\beta(y))$ in $\mathbf{A}$ with $\alpha(F(f)) = 1$ and $\alpha(F(h)) = m$. By (A5), this is equivalent to the existence of a morphism $f\colon x \to z$ with $\alpha(F(f)) = 1$ and an isomorphism $h\colon z \to y$ in $\mathbf{A}$ with $\alpha(F(h)) = m$. Now take $k = hf\colon x \to y$ in $\mathbf{A}$. Then

$$\alpha(F(k)) = \alpha(F(hf)) = \alpha(F(f)) \cdot \alpha(F(h)) = 1 \cdot m = m.$$

Hence our functor is full, and we conclude that it is (half of) an equivalence. □ □

4. The finite-dimensional case

This section uses poset-group-amalgamations to completely characterize the category $\mathcal{C}_{\rightarrowtail}(A)$ for finite-dimensional C*-algebras A. *En passant*, we will also characterize the poset category $\mathcal{C}_{\subseteq}(C)$ for commutative finite-dimensional C*-algebras C.

Finite partition lattices. We start with identifying the appropriate poset P. Recall that a *partition* p of $\{1,\ldots,n\}$ is a family of disjoint subsets $p_1,\ldots,p_k$ of $\{1,\ldots,n\}$ whose union is $\{1,\ldots,n\}$. Partitions are ordered by *refinement*: $p \leq q$

whenever each p_i is contained in a q_j. Ordered this way, the partitions of $\{1, \ldots, n\}$ form a lattice, called the *partition lattice*, that we denote by $P(n)$. It is known when a lattice is (isomorphic to) the partition lattice $P(n)$. We recall such a characterization below, but first we briefly have to recall some terminology.

Recall that a lattice is *semimodular* if $a \vee b$ covers b whenever a covers $a \wedge b$. A finite lattice is *geometric* when it is atomic and semimodular. Any geometric lattice has a well-defined *rank* function: $\mathrm{rank}(x)$ is the length of a(ny) chain from 0 to x in L. An element x in a lattice is *modular* when $a \vee (x \wedge y) = (a \vee x) \wedge y$ for all $a \leq y$. The *Möbius function* of a finite lattice is the unique function $\mu \colon L \to \mathbb{Z}$ satisfying $\sum_{y<x} \mu(x) = \delta_{0,x}$. It can be defined recursively by $\mu(0) = 1$ and $\mu(x) = -\sum_{y \leq x} \mu(x)$ for $x > 0$; see [3]. The *characteristic polynomial* of a finite lattice L is $\sum_{x \in L} \mu(x) \cdot \lambda^{\mathrm{rank}(1) - \mathrm{rank}(x)}$. Finally, we write $\uparrow\! x$ for the principal ideal $\{z \in L \mid x \leq z\}$ of $x \in L$.

Theorem 4.1. *A lattice L is isomorphic to $P(n+1)$ if and only if:*

- *(P1) it is geometric;*
- *(P2) if $\mathrm{rank}(x) = \mathrm{rank}(y)$, then $\uparrow\! x \cong \uparrow\! y$;*
- *(P3) it has a modular coatom;*
- *(P4) its characteristic polynomial is $(\lambda - 1) \cdots (\lambda - n)$.*

Proof. See [31]. □ □

This immediately extends to a characterization of $\mathcal{C}_{\subseteq}(A)$ for finite-dimensional commutative C*-algebras A (which are always unital).

Corollary 4.2. *A lattice L is isomorphic to $\mathcal{C}_{\subseteq}(A)^{\mathrm{op}}$ for a commutative C*-algebra A of dimension $n+1$ if and only if it satisfies (P1)–(P4).*

Proof. The lattice $\mathcal{C}_{\subseteq}(A)$ is that of subobjects of A in the category of finite-dimensional commutative C*-algebras and unital $*$-homomorphisms. Recall that a *subobject* is an equivalence class of monomorphisms into a given object, where two monics are identified when they factor through one another by an isomorphism. The dual notion is a *quotient*: an equivalence class of epimorphisms out of a given object. By Gelfand duality, $\mathcal{C}_{\subseteq}(A)$ is isomorphic to the opposite of the lattice of quotients of the discrete topological space $\mathrm{Spec}(A)$ with $n+1$ points. But the latter is precisely $P(n+1)^{\mathrm{op}}$. □ □

Symmetric group actions. The appropriate group to consider is the symmetric group $S(n)$ of all permutations π of $\{1, \ldots, n\}$. The group $S(n)$ acts on $P(n)$. Explicitly, $\pi p = (\pi p_1, \ldots, \pi p_k)$ for $p = (p_1, \ldots, p_k) \in P(n)$ and $\pi \in S(n)$, where $\pi p_l = \{\pi(i) \mid i \in \pi_l\}$. That is, one works in the quotient group of $S(n)$ by the Young subgroups $S(n_1) \times \cdots \times S(n_k)$, where the n_l ares the cardinality of the parts p_l of the partition p. The following lemma might be considered the main insight of this article.

Lemma 4.3. *If A is a commutative C*-algebra of dimension n, then there is an isomorphism $\mathcal{C}_{\rightarrowtail}(A)^{\mathrm{op}} \cong P(n) \rtimes S(n)$ of categories.*

Proof. We may assume that $A = \mathbb{C}^n$. Objects C of $\mathcal{C}_{\rightarrowtail}(A)$ then are of the form $C = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid \forall k \forall i,j \in p_k \colon x_i = x_j\}$ for some partition $p = (p_1, \ldots, p_l)$ of $\{1, \ldots, n\}$. But these are precisely the objects of $P(n)$, and hence of $P(n) \rtimes S(n)$.

If $f \colon C' \to C$ is a morphism of $\mathcal{C}_{\rightarrowtail}(A)$, *i.e.* an injective $*$-homomorphism, then $f(C') \subseteq C$ is a C*-subalgebra. Say $C' = \{x \in \mathbb{C}^n \mid \forall k \forall i,j \in p'_k \colon x_i = x_j\}$

for a partition $p' = (p'_1, \ldots, p'_{l'})$. Then we see that f must be induced by an injective function $\{1, \ldots, n\} \to \{1, \ldots, n\}$, which we can extend to a permutation $\pi \in S(n)$. Then $C' \to C$ means that $\pi p' \leq p$. But this is precisely a morphism in $(P(n) \rtimes S(n))^{\mathrm{op}}$. □ □

Terminal subalgebras. A maximal abelian subalgebra D of a C*-algebra A is a maximal element in $\mathcal{C}_{\subseteq}(A)$. If A is finite-dimensional, such D are unique up to conjugation with a unitary.

The prime example is the following: if A is the C*-algebra $\mathbb{M}_n(\mathbb{C})$ of n-by-n complex matrices, then maximal abelian subalgebras D are precisely the subalgebras consisting of all matrices that are diagonal in some fixed basis.

In finite dimension, maximal elements of $\mathcal{C}_{\subseteq}(A)$ are the same as terminal objects of $\mathcal{C}_{\rightarrowtail}(A)$. For the following lemma, weakly terminal objects of $\mathcal{C}_{\subseteq}(A)$ are in fact enough. Recall that an object D is weakly terminal when every object C allows a morphism $C \to D$.

Lemma 4.4. *If $\mathcal{C}_{\rightarrowtail}(A)$ has a weak terminal object D, then there is an equivalence $\mathcal{C}_{\rightarrowtail}(A) \simeq \mathcal{C}_{\rightarrowtail}(D)$ of categories.*

Proof. Clearly the inclusion $\mathcal{C}_{\rightarrowtail}(D) \hookrightarrow \mathcal{C}_{\rightarrowtail}(A)$ is a full and faithful functor, so it suffices to prove that it is essentially surjective. Let $C \in \mathcal{C}_{\rightarrowtail}(A)$. Then there exists an injective $*$-homomorphism $f \colon C \to D$ because D is weakly terminal. Hence $C \cong f(C) \in \mathcal{C}_{\rightarrowtail}(D)$. □ □

The characterization. We can now bring all the pieces together.

Theorem 4.5. *For a category $\mathbf{A}$, the following are equivalent:*

- *the category $\mathbf{A}$ is equivalent to $\mathcal{C}_{\rightarrowtail}(\mathbb{M}_n(\mathbb{C}))^{\mathrm{op}}$;*
- *the category $\mathbf{A}$ is equivalent to $P(n) \rtimes S(n)$;*
- *$\mathbf{A}$ satisfies (A1)–(A6), and*
  $(\mathbf{A}, \leq)$ satisfies (P1)–(P4) for $n-1$, and
  $\mathbf{A}(0,0)^{\mathrm{op}}$ is isomorphic to the symmetric group on n elements.

Proof. Combine the previous two lemmas with Theorem 3.7 and Theorem 4.1. □
□

We can actually do better than characterizing factors $A = \mathbb{M}_n(\mathbb{C})$ of type I_n: the next theorem characterizes $\mathcal{C}_{\rightarrowtail}(A)$ for any finite-dimensional C*-algebra A.

Lemma 4.6. *If $\mathcal{C}_{\rightarrowtail}(A_i)$ has a weak terminal object D_i for each i in a set I, then the C*-direct sum $\bigoplus_{i \in I} D_i$ is a weak terminal object in $\mathcal{C}_{\rightarrowtail}(\bigoplus_{i \in I} A_i)$.*

Proof. Let $C \in \mathcal{C}(\bigoplus_{i \in I} A_i)$. Then C is contained in the commutative subalgebra $\bigoplus_{i \in I} \pi_i(C)$ of $\bigoplus_{i \in I} A_i$. Because each D_i is weakly terminal, there exist morphisms $f_i \colon \pi_i(C) \to D_i$. Therefore $\bigoplus_{i \in I} f_i$ is a morphism $\bigoplus_{i \in I} \pi_i(C) \to \bigoplus_{i \in I} D_i$, and thus the latter is weakly terminal in $\mathcal{C}_{\rightarrowtail}(\bigoplus_{i \in I} A_i)$. □ □

Theorem 4.7. *A category $\mathbf{A}$ is equivalent to $\mathcal{C}_{\rightarrowtail}(A)^{\mathrm{op}}$ for a finite-dimensional C*-algebra A if and only if there are $n_1, \ldots, n_k \in \mathbb{N}$ such that:*

- *$\mathbf{A}$ satisfies (A1)–(A5) and (A6');*
- *$(\mathbf{A}, \leq)$ satisfies (P1)–(P4) for $(\sum_{i=1}^k n_i) - 1$;*
- *$\mathbf{A}(0,0)^{\mathrm{op}}$ is isomorphic to the symmetric group on $\sum_{i=1}^k n_i$ elements;*
- *$\sum_{i=1}^k n_i^2 = \dim(A)$.*

Proof. Every finite-dimensional C*-algebra A is isomorphic to a matrix realization of the form $\bigoplus_{i=1}^{k} \mathbb{M}_{n_i}(\mathbb{C})$ with $n = \sum_{i=1}^{k} n_i^2$ [6, Theorem III.1.1]. By Lemmas 4.4 and 4.6, we have

$$\mathcal{C}_{\rightarrowtail}(A) \simeq \mathcal{C}_{\rightarrowtail}(\bigoplus_{i=1}^{k} \mathbb{C}^{n_i}) \cong \mathcal{C}_{\rightarrowtail}(\mathbb{C}^{(\sum_{i=1}^{k} n_i)}).$$

So by Lemma 4.3, $\mathcal{C}_{\rightarrowtail}(A)^{\mathrm{op}} \simeq P(m) \rtimes S(m)$ for $m = \sum_{i=1}^{k} n_i$. Now the statement follows from Theorem 4.5. □ □

Notice that by Lemma 4.6 we may indeed use the whole partition lattice $P(m)$ in the previous theorem instead of the truncated one $P(n_1) \times \cdots \times P(n_k)$; this is one of the consequences of working with equivalences of categories instead of isomorphisms.

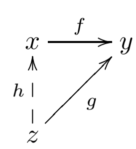
5. Poset-monoid-amalgamations

The main idea of our characterization of $\mathcal{C}_{\rightarrowtail}(A)$ for finite-dimensional C*-algebras A holds unabated in the infinite-dimensional case. However, the technical implementation of the idea needs some adapting. For example, the appropriate monoid is no longer a group. Therefore, we will have to refine axiom (A6) into (A6') and (A7') as follows. We re-list the other axioms for convenience.

Definition 5.1. A category $\mathbf{A}$ is called a *poset-monoid-amalgamation* when there exist a partial order P and a monoid M such that:

- (A1') there is a weak initial object 0, unique up to isomorphism;
- (A2') there is a faithful retraction F of the inclusion $\mathbf{A}(0,0) \hookrightarrow \mathbf{A}$;
- (A3') there is an isomorphism $\alpha\colon \mathbf{A}(0,0) \to M^{\mathrm{op}}$ of monoids;
- (A4') there is an equivalence $(\mathbf{A}, \leq) \underset{\beta'}{\overset{\beta}{\rightleftarrows}} P$ of preorders;
- (A5') for each object x there is an isomorphism $f\colon x \to \beta'(\beta(x))$ with $\alpha F(f) = 1$;
- (A6') for each object y and $m\colon 0 \to 0$, there is $f\colon x \to y$ such that $F(f) = m$, that is universal in the sense that $f' = fg$ with $F(g) = 1$ for any $f'\colon x' \to y$ with $F(f') = m$;
- (A7') if $F(f) = m_2 m_1$ for a morphism f, then $f = f_1 f_2$ with $F(f_i) = m_i$.

The idea behind axiom (A6') is that in $P \rtimes M$, we can access the object mq through the morphisms $m\colon p \to q$. There is always at least one such morphism, namely $m\colon mq \to q$, because trivially $mq \leq mq$. This might not be an isomorphism, but it still has the universal property that all other morphisms $m\colon p \to q$ factor through it. We can rephrase this universality as follows: for each object y of $P \rtimes M$ and $m \in M$, there is a maximal element of the set $\{f\colon x \to y \mid \alpha(F(f)) = m\}$, preordered by $f \leq g$ iff $f = hg$ for some morphism h satisfying $\alpha(F(h)) = 1$.



Also, notice the swap in (A7'). It is caused by the contravariance in the composition of $P \rtimes M$ and (A3'), and is not a mistake, as the following example shows.

Example 5.2. If P is a partial order with least element, and M is a monoid acting on P, then $P \rtimes M$ is a poset-monoid-amalgamation.

Proof. The least element 0 of P is a weak initial object, satisfying (A1'). Conditions (A2')–(A4') are satisfied by definition, and (A5') is vacuous. To verify (A6') for $q \in P$ and $m \in M$, notice that $mq \leq mq$, and if $p \leq mq$, then certainly $p \leq 1mq$. Finally, (A7') means that if $p \leq m_2 m_1 r$, we should be able to provide q such that $p \leq m_2 q$ and $q \leq m_1 r$; taking $q = m_1 r$ suffices. □ □

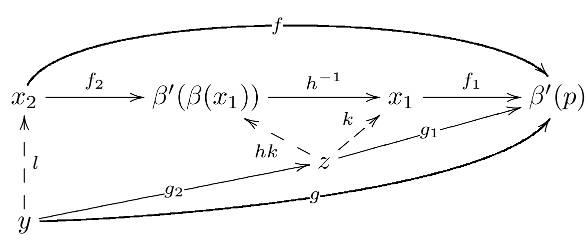
Lemma 5.3. *If* **A** *satisfies (A1')–(A7'), then it induces an action of M on P given by $pm = \beta(x)$ if $f \colon x \to \beta'(p)$ is a maximal element with $\alpha(F(f)) = m$.*

Proof. First, notice that for any $p \in P$ and $m \in M$ there exists a maximal $f \colon x \to \beta'(p)$ with $\alpha(F(f)) = m$ by (A6'). If there is another maximal $f' \colon x' \to \beta'(p)$ with $\alpha(F(f')) = m$, then there are morphisms $g \colon x \to x'$ and $g' \colon x' \to x$ with $F(g) = 1 = F(g')$. Hence $F(gg') = 1 = F(g'g)$, and because F is faithful, g is an isomorphism with g' as inverse. So $x \cong x'$, and therefore $\beta(x) \cong \beta(x')$. But because P is a partial order, this means $\beta(x) = \beta(x')$. Thus the action is well-defined on objects.

To see that it is well-defined on morphisms, suppose that $p \leq q$. Then there is a morphism $f \colon \beta'(p) \to \beta'(q)$ with $F(f) = 1$. For any $m \colon 0 \to 0$, we can find maximal $f_p \colon x_p \to \beta'(p)$ with $F(f_p) = m$, and maximal $f_q \colon x_q \to \beta'(q)$ with $F(f_q) = m$. Now $ff_p \colon x_p \to \beta'(q)$ has $F(ff_p) = m$. Because f_q is a maximal such morphism, ff_p factors through f_q. That is, there is $h \colon x_p \to x_q$ with $f_q h = ff_p$ and $F(h) = 1$. So $mp \leq mq$.

Next, we verify that this assignment is functorial $M \to \mathbf{Cat}(P, P)$. Clearly $\mathrm{id}_{\beta'(p)}$ is maximal among morphisms $f \colon x \to \beta'(p)$ with $F(f) = 1$. Therefore $1p = \beta(\beta'(p)) = p$.

For $m_2, m_1 \in M$ and $p \in P$, we have $m_1 p = \beta(x_1)$ where $f_1 \colon x_1 \to \beta'(p)$ is maximal such that $\alpha(F(f_1)) = m_1$. Therefore $m_2(m_1 p) = \beta(x_2)$, where the morphism $f_2 \colon x_2 \to \beta'(\beta(x_1))$ is maximal such that $\alpha(F(f_2)) = m_2$. By axiom (A5'), there is an isomorphism $h \colon x_1 \to \beta'(\beta(x_1))$ with $F(h) = 1$. This gives $f = f_1 h^{-1} f_2 \colon x_2 \to \beta'(p)$ with $\alpha(F(f)) = \alpha(F(f_2)) \cdot \alpha(F(h))^{-1} \cdot \alpha(F(f_1)) = m_2 m_1$. We will now show that f is universal with this property. If $g \colon y \to \beta'(p)$ also has $\alpha(F(g)) = m_2 m_1$, then (A7') provides $g_2 \colon y \to z$ and $g_1 \colon z \to \beta'(p)$ with $g = g_1 g_2$ and $\alpha(F(g_i)) = m_i$.



By maximality of f_1, there exists k with $g_1 = f_1 k$ and $\alpha(F(k)) = 1$. And by maximality of f_2, there is exists l with $hkg_2 = f_2 l$ and $\alpha(F(l)) = 1$. Hence

$$g = g_1 g_2 = f_1 k g_2 = f_1 h^{-1} h k g_2 = f_1 h^{-1} f_2 l = fl.$$

So f is maximal with $F(f) = m_2 m_1$. Thus $(m_2 m_1)p = \beta(x_2) = m_2(m_1 p)$. □ □

Theorem 5.4. *If* **A** *satisfies (A1')–(A7'), then there is an equivalence $\mathbf{A} \to P \rtimes M$ given by $x \mapsto \beta(x)$ on objects and $f \mapsto \alpha(F(f))$ on morphisms.*

Proof. First, it follows from (A6') that the assignment of the statement is well-defined, *i.e.* that $\alpha(F(f))$ is indeed a morphism of $P \rtimes M$. Indeed, if $f\colon x \to y$, then we need to show that $\beta(x) \leq \alpha(F(f)) \cdot \beta(y)$. Unfolding the definition of the action, this means we need to find a maximal $k\colon x' \to \beta'(\beta(y))$ with $F(f) = F(k)$, such that $\beta(x) \leq \beta(x')$. Unfolding the definition of the preorder, this means we need to find a morphism $h'\colon \beta'(\beta(x)) \to \beta'(\beta(x'))$ with $F(h') = 1$. By (A5'), it suffices to find $h\colon x \to x'$ with $F(h) = 1$ instead. But by (A6'), there exists maximal $k\colon x' \to \beta'(\beta(y))$ with $F(k) = F(f)$. By its maximality, there exists $h\colon x \to x'$ with $F(h) = 1$ and $f = kh$. In particular, $\beta(x) \leq \beta(x')$.

Functoriality follows directly from the previous lemma, so indeed we have a well-defined functor $\mathbf{A} \to P \rtimes M$. Moreover, our functor is essentially surjective because β is an equivalence, and it is faithful because F is faithful.

Finally, to prove fullness, let $m\colon \beta(x) \to \beta(y)$ be a morphism in $P \rtimes M$. This means that $\beta(x) \leq \beta(y)m$, which unfolds to: there are morphisms $f\colon x \to z$ and $h\colon z \to \beta'(\beta(y))$ with $F(f) = 1$ and h maximal with $\alpha(F(h)) = m$. By (A5'), this is equivalent to the existence of a morphism $f\colon x \to z$ with $F(f) = 1$ and a morphism $h\colon z \to y$ maximal with $\alpha(F(h)) = m$. Now take $k = hf\colon x \to y$. Then

$$\alpha(F(k)) = \alpha(F(hf)) = \alpha(F(f))\alpha(F(h)) = 1 \cdot m = m.$$

Hence our functor is full, and we conclude that it is (half of) an equivalence. □ □

6. The infinite-dimensional case

To adapt Theorem 4.5 to the infinite-dimensional case, we have to make three more adaptations. First, the poset P now becomes a lattice of partitions of a (locally) compact Hausdorff space. Second, the symmetric group gets replaced by symmetric monoids on (locally) compact Hausdorff spaces. Third, we have to be more careful about maximal abelian subalgebras.

Infinite partition lattices. For arbitrary (locally) compact Hausdorff spaces, it is more convenient to talk about equivalence relations than about partitions. An equivalence relation $\sim$ on a (locally) compact Hausdorff space X is called *closed* when the set $\{x \in X \mid \exists u \in U.\, x \sim u\}$ is closed for every closed $U \subseteq X$. Closed equivalence relations on X are also called *partitions*, and form a partial order $P(X)$ under *refinement*:

$$\sim \leq \approx \qquad \iff \qquad \big(\forall x, y \in X.\, x \sim y \implies x \approx y\big).$$

Notice that quotients of a (locally) compact Hausdorff space by an equivalence relation are again (locally) compact Hausdorff if and only if the equivalence relation is closed.

Fortunately, a characterization of $P(X)$ is known, due to Firby [8, 9]. This also gives a characterization of $\mathcal{C}_\subseteq(A)$ for commutative C*-algebras A. As in Section 4, we first briefly recall the necessary terminology. An element b of a lattice is called *bounding* when (i) it is zero or an atom; or (ii) it covers an atom and dominates exactly three atoms; or (iii) for distinct atoms p, q there exists an atom $r \leq b$ such that there are exactly three atoms less than $r \vee p$ and exactly three atoms less than $r \vee q$. A collection of atoms of a lattice with at least four elements is called *single* when it is a maximal collection of atoms of which the join of any two dominates exactly three atoms (not necessarily in the collection). A collection B of nonzero bounding elements of a lattice is called a *1-point* when (i) its atoms form a single collection; and (ii) if a is bounding and $a \geq b \in B$, then $a \in B$; and (iii) any $a \in B$ dominates an atom $p \in B$.

Theorem 6.1. *A lattice L with at least four elements is isomorphic to $P(X)$ for a compact Hausdorff space X if and only if:*

- *(P1') L is complete and atomic;*
- *(P2') the intersection of any two 1-points contains exactly one atom, and any atom belongs to exactly two 1-points;*
- *(P3') for bounding $a, b \in L$ that are contained in a 1-point,*

$$\{p \in \mathrm{Atoms}(L) \mid p \leq a \vee b\}$$
$$= \{p \in \mathrm{Atoms}(L) \mid \text{ if } x \text{ is a 1-point with } p \in x \text{ then } a \in x \text{ or } b \in x\};$$

  for bounding $a, b \in L$ that are not contained in a 1-point,

$$\{p \in \mathrm{Atoms}(L) \mid p \leq a \vee b\} = \{p \in \mathrm{Atoms}(L) \mid p \leq a \text{ or } p \leq b\};$$

- *(P4') for 1-points $x \neq y$ there are bounding a, b with $a \notin x$, $b \notin y$, and $a \vee b = 1$;*
- *(P5') joins of bounding elements are bounding;*
- *(P6') for nonzero $a \in L$, the collection B of bounding elements equal to or covered by a is the unique one satisfying:*
  - *$\bigvee B = a$;*
  - *no 1-point contains two members of B;*
  - *if c is bounding, $b_1 \in B$, and no 1-point contains b_1 and c, then there is a bounding $b \geq c$ such that (i) there is no 1-point containing both b and b_1, and (ii) whenever there is a 1-point containing both b and $b_2 \in B$, then $b \geq b_2$;*
- *(P7') any collection of nonzero bounding elements that is not contained in a 1-point has a finite subcollection that is not contained in a 1-point;*

and X is (homeomorphic to) the set of 1-points of L, where a subset is closed if it is a singleton 1-point or it is the set of 1-points containing a fixed bounding element.

Proof. See [9]. □ □

Remark 6.2. The axiom responsible for compactness of X is (P7'). The previous theorem holds for locally compact Hausdorff spaces X when we replace (P7') by

(P7") every 1-point contains a bounding b such that $\{l \in L \mid l \geq b\}$ satisfies (P7').

Indeed, because (P1')–(P6') already guarantee Hausdorffness, we may take local compactness to mean that every point has a compact neighbourhood that is closed. And closed sets correspond to sets of 1-points containing a fixed bounding element.

As before, this directly leads to a characterization of $\mathcal{C}_{\subseteq}(A)$ for commutative C*-algebras A.

Corollary 6.3. *A lattice L is isomorphic to $\mathcal{C}_{\subseteq}(A)^{\mathrm{op}}$ for a commutative C*-algebra A of dimension at least three if and only if it satisfies (P1')–(P6') and (P7"). The C*-algebra A is unital if and only if L additionally satisfies (P7').*

Proof. The lattice $\mathcal{C}_{\subseteq}(A)$ is that of subobjects of A in the category of commutative (unital) C*-algebras and (unital) nondegenarate $*$-homomorphisms. Recall that a *subobject* is an equivalence class of monomorphisms into a given object, where two monics are identified when they factor through one another by an isomorphism. The dual notion is a *quotient*: an equivalence class of epimorphisms out of a given object.

By Gelfand duality, $\mathcal{C}_{\subseteq}(A)$ is isomorphic to the opposite of the lattice of quotients of $X = \mathrm{Spec}(A)$. But the latter is precisely $P(X)^{\mathrm{op}}$, because categorical quotients in the category of (locally) compact Hausdorff spaces are quotient spaces. □ □

Symmetric monoid actions. We write $S(X)$ for the monoid of continuous functions $f\colon X \twoheadrightarrow X$ with dense image on a locally compact Hausdorff space X, called the *symmetric monoid* on X.

The monoid $S(X)$ acts on $P(X)$, as described in the following lemma. We stick to the notation mx for the action of an element m of a monoid M on an object x of a category as in Definition 3.1. For $f \in S(X)$ and $\sim \in P(X)$ this becomes $f\sim$.

Proposition 6.4. *For any locally compact Hausdorff space X, the monoid $S(X)$ acts on $P(X)$ by*

$$(f\sim) \;=\; (f\times f)^{-1}(\sim).$$

Proof. First of all, notice that $f\sim$ is reflexive, symmetric and transitive, so indeed is a well-defined equivalence relation on X, which is closed because f is continuous. Concretely, $x(f\sim)y$ if and only if $f(x) \sim f(y)$. Moreover, clearly $\mathrm{id}\sim\, = \,\sim$, and $g(f\sim) = (gf)\sim$, so the above is a genuine action. □ □

As before, this directly leads to a characterization of $\mathcal{C}_{\rightarrowtail}(A)$ for commutative C*-algebras A.

Lemma 6.5. *If $A = C(X)$ for a locally compact Hausdorff space X, there is an isomorphism $\mathcal{C}_{\rightarrowtail}(A)^{\mathrm{op}} \cong P(X) \rtimes S(X)$ of categories.*

Proof. By definition, objects C of $\mathcal{C}_{\rightarrowtail}(A)$ are subobjects of $C(X)$ in the category of commutative C*-algebras. By Gelfand duality, these correspond to quotients of X in the category of locally compact Hausdorff spaces. But these, in turn, correspond to closed equivalence relations on X, establishing a bijection between the objects of $\mathcal{C}_{\rightarrowtail}(A)$ and $P(X)$.

Through Gelfand duality, a morphism $C \rightarrowtail C'$ in $\mathcal{C}_{\rightarrowtail}(A)$ correspondsto an epimorphism $g\colon Y' \twoheadrightarrow Y$ between the corresponding spectra. Writing the quotients as $Y = X/\!\sim$ and $Y' = X/\!\approx$ for closed equivalence relations $\sim$ and $\approx$, we find that g corresponds to a continuous $f\colon X \twoheadrightarrow X$ with dense image respecting equivalence:

$$x \approx y \implies f(x) \sim f(y).$$

But this just means $\approx\; \leq (f\sim)$. In other words, this is precisely a morphism $f\colon \approx \to \sim$ in $P(X) \rtimes S(X)$. □ □

Weakly terminal subalgebras. In the infinite-dimensional case, it is no longer true that all maximal abelian subalgebras of a C*-algebra A are isomorphic. However, it suffices if there exists a commutative subalgebra into which all others embed by an injective $*$-homomorphism. To be precise, a commutative C*-subalgebra D of a C*-algebra A is *weakly terminal* when each $C \in \mathcal{C}(A)$ allows an injective $*$-homomorphisms $C \to D$ (that is not necessarily an inclusion, and not necessarily unique). Equivalently, every masa is isomorphic to a subalgebra of D. Weakly terminal commutative subalgebras D are maximal up to isomorphism, in the sense that if D can be extended to a larger commutative C*-subalgebra E, then $D \cong E$. This does not imply that $D = E$, *i.e.* that D is maximal. For a counterexample, take $A = E = C([0,1])$ and $D = \{f \in E \mid f \text{ constant on } [0,\frac{1}{2}]\}$. Then $D \subsetneq E$, but $D \cong C([\frac{1}{2},1]) \cong E$.

Lemma 6.6. *If $A = B(H)$ for an infinite-dimensional separable Hilbert space H, then $\mathcal{C}_{\rightarrowtail}(A)$ has a weak terminal object, $*$-isomorphic to $L^\infty(0,1) \oplus \ell^\infty(\mathbb{N})$.*

Proof. Let $C \in \mathcal{C}_{\rightarrowtail}(A)$. By Zorn's lemma, C is a C*-subalgebra of a maximal element of $\mathcal{C}_{\subseteq}(A)$. A maximal element in $\mathcal{C}_{\subseteq}(A)$ for a von Neumann algebra A is itself a von Neumann algebra, because it must equal its weak closure. It is known that maximal abelian von Neumann subalgebras of $A = B(H)$ for an infinite-dimensional separable Hilbert space H are unitarily equivalent to one of the following: $L^\infty(0,1)$, $\ell^\infty(\{0,\ldots,n\})$ for $n \in \mathbb{N}$, $\ell^\infty(\mathbb{N})$, $L^\infty(0,1) \oplus \ell^\infty(\{0,\ldots,n\})$ for $n \in \mathbb{N}$, or $L^\infty(0,1) \oplus \ell^\infty(\mathbb{N})$ (see [20, Theorem 9.4.1]). Each of these allows an injective $*$-homomorphism into the latter one, $D = L^\infty(0,1) \oplus \ell^\infty(\mathbb{N})$. Therefore, there exists a morphism $C \to D$ in $\mathcal{C}_{\rightarrowtail}(A)$ for each C in $\mathcal{C}_{\rightarrowtail}(A)$, so that D is weakly terminal in $\mathcal{C}_{\rightarrowtail}(A)$. □ □

If $\dim(H)$ is uncountable, the situation becomes a bit more involved. A complete classification of (maximal) abelian subalgebras of $B(H)$ is known [26, 27], and we will use this to establish a weakly terminal commutative subalgebra in the following lemma. Before doing so, let us explain the intuitition behind the use of cardinal numbers α and β in the statement. For any cardinal number α, the C*-algebra $B(H)$ has a commutative subalgebra $L^\infty(0,1)^\alpha$ that needs to be accounted for in a weakly terminal commutative subalgebra, as in the previous lemma. Because there are $\dim(H)$ many of those, a sum over a second cardinal $\beta \leq \dim(H)$ is called for.

Lemma 6.7. *If $A = B(H)$ for an infinite-dimensional Hilbert space H, then $\mathcal{C}_{\rightarrowtail}(A)$ has a weak terminal object, $*$-isomorphic to $\bigoplus_{\alpha,\beta \leq \dim(H)} L^\infty\big((0,1)^\alpha\big)$, where α, β are cardinals, and $(0,1)^\alpha$ is the product measure space of Lebesgue unit intervals.*

Proof. Maximal abelian subalgebras C of $B(H)$ are isomorphic to direct sums of $L^\infty\big((0,1)^\alpha\big)$ ranging over cardinal numbers α [26]. We must show that $D = \bigoplus_{\alpha,\beta \leq \dim(H)} L^\infty\big((0,1)^\alpha\big)$ can be identified with a subalgebra of $B(H)$ that embeds any such C. A commutative algebra $L^\infty\big((0,1)^\alpha\big)$ acts on the Hilbert space $L^2\big((0,1)^\alpha\big)$. Observe that $L^2(0,1)$ is separable. Hence $\dim\big(L^2\big((0,1)^\alpha\big)\big) = \max(\alpha, \aleph_0)$ unless $\alpha = 0$, in which case the dimension vanishes. Therefore $\dim\big(L^2\big((0,1)^\alpha\big)\big) \leq \dim(H)$ if and only if $\alpha \leq \dim(H)$. Because H is infinite-dimensional, we have the equation $\dim(H) = \dim(H)^3$ of cardinal numbers. Thus any maximal abelian subalgebra C embeds into D, and D itself embeds as a maximal abelian subalgebra of $B(H)$. □ □

The following infinite-dimensional analogue of Lemma 4.6 will allow us to deduce from the previous lemma that arbitrary type I von Neumann algebras possess weakly terminal commutative subalgebras. (For direct integrals, see [20, Chapter 14].)

Lemma 6.8. *Let (M,μ) be a measure space, and for each $t \in M$ let A_t be a von Neumann algebra. If $\mathcal{C}_{\rightarrowtail}(A_t)$ has a weakly terminal object D_t for almost every t, then the direct integral $\int_M^\oplus D_t \mathrm{d}\mu(t)$ is a weak terminal object in $\mathcal{C}_{\rightarrowtail}(\int_M^\oplus A_t \mathrm{d}\mu(t))$.*

Proof. Let $C \in \mathcal{C}_{\rightarrowtail}(\int_M^\oplus A_t \mathrm{d}\mu(t))$. Supposing A_t acts on a Hilbert space H_t, then C is contained in $\int_M^\oplus C_t \mathrm{d}\mu(t)$, where C_t is the von Neumann subalgebra of $B(H_t)$ generated by $\{a_t \mid a \in C\}$. But because almost every D_t is weakly terminal, this in turn embeds into $\int_M^\oplus D_t \mathrm{d}\mu(t)$, which is therefore weakly terminal. □ □

Corollary 6.9. *Every type I von Neumann algebra A possesses a weakly terminal commutative subalgebra D. More precisely: if $A = \int_M^\oplus A_t \mathrm{d}\mu(t)$ for a measure space (M, μ) and type I factors A_t acting on Hilbert spaces H_t, then D is $*$-isomorphic to* $\mathrm{Spec}\left(\int_M^\oplus \bigoplus_{\alpha,\beta\leq\dim(H_t)} L^\infty((0,1)^\alpha)\mathrm{d}\mu(t)\right)$.

Proof. Every type I von Neumann algebra is a direct integral of type I factors [20, Section 14.2]. Since the latter have weakly terminal commutative subalgebras by Lemma 6.7, we can deduce that the original algebra has a weakly terminal commutative subalgebra by Lemma 6.8. □ □

Much less is known about the structure of (maximal) abelian subalgebras of von Neumann algebras of type II and III; see [5, 27]. The results of [29] indicate that the previous lemma might extend to show that $\mathcal{C}_\rightarrowtail(A)$ has a weak terminal object for *any* von Neumann algebra A. It would also be interesting to see if the previous corollary implies that type I C*-algebras have weakly terminal commutative subalgebras.

The characterization. We now arrive at our main result: the characterization $\mathcal{C}_\rightarrowtail$ for infinite-dimensional type I von Neumann algebras.

Theorem 6.10. *For a category $\mathbf{A}$ and an infinite-dimensional type I von Neumann algebra $A = \int_M^\oplus B(H_t)\mathrm{d}\mu(t)$ for a measure space (M,μ) and Hilbert spaces H_t, the following are equivalent:*

- *the category $\mathbf{A}$ is equivalent to $\mathcal{C}_\rightarrowtail(A)^{\mathrm{op}}$;*
- *the category $\mathbf{A}$ is equivalent to $P(X) \rtimes S(X)$,*
  where X is the topological space $\mathrm{Spec}\left(\int_M^\oplus \bigoplus_{\alpha,\beta\leq\dim(H_t)} L^\infty((0,1)^\alpha)\right)$*;*
- $\mathbf{A}$ *satisfies (A1')–(A7'), and*
  $(\mathbf{A}, \leq)$ satisfies (P1')–(P6'),(P7"), giving a topological space X, and
  $\mathbf{A}(0,0)^{\mathrm{op}}$ is isomorphic to the monoid $S(X)$, and
  X is homeomorphic to $\mathrm{Spec}\left(\int_M^\oplus \bigoplus_{\alpha,\beta\leq\dim(H_t)} L^\infty((0,1)^\alpha)\mathrm{d}\mu(t)\right)$.

When $A = B(H)$ for an infinite-dimensional Hilbert space H, the space X simplifies to $\bigsqcup_{\alpha,\beta\leq\dim(H)} \mathrm{Spec}\left(L^\infty\left((0,1)^\alpha\right)\right)$*. When H is separable, X further simplifies to* $\mathrm{Spec}(L^\infty(0,1)) \sqcup \mathrm{Spec}(\ell^\infty(\mathbb{N}))$.

Proof. Combine the previous four lemmas with Theorems 5.4 and 6.1. For the last condition, remember that Gelfand duality turns direct sums of commutative C*-algebras into coproducts of Hausdorff spaces. □ □

The Gelfand spectrum of $\ell^\infty(\mathbb{N})$ is the Stone-Čech compactification of the discrete topology of $\mathbb{N}$. In other words, $\mathrm{Spec}(\ell^\infty(\mathbb{N}))$ consists of the ultrafilters on $\mathbb{N}$. A topological space is homeomorphic to $\mathrm{Spec}(L^\infty(0,1))$ if and only if it is compact, Hausdorff, totally disconnected, and its clopen subsets are isomorphic to the Boolean algebra of (Borel) measurable subsets of the interval $(0,1)$ modulo (Lebesgue) negligible ones. Since both spaces are compact, we could have used (P7') instead of (P7") in the previous theorem for the case $A = B(H)$ with H separable.

7. Inclusions versus injections

This section compares $\mathcal{C}_\subseteq$ to $\mathcal{C}_\rightarrowtail$. We will show for C*-algebras A and B that:

- if $\mathcal{C}_\rightarrowtail(A)$ and $\mathcal{C}_\rightarrowtail(B)$ are isomorphic, $\mathcal{C}_\subseteq(A)$ and $\mathcal{C}_\subseteq(B)$ are isomorphic;

- if $\mathcal{C}_{\rightarrowtail}(A)$ and $\mathcal{C}_{\rightarrowtail}(B)$ are equivalent, $\mathcal{C}_{\subseteq}(A)$ and $\mathcal{C}_{\subseteq}(B)$ are Morita-equivalent.

Here we call two categories $\mathbf{C}$ and $\mathbf{D}$ Morita-equivalent when they have equivalent presheaf categories $\mathrm{PSh}(\mathbf{C}) \simeq \mathrm{PSh}(\mathbf{D})$.

For any category $\mathbf{C}$, recall that the category $\int_{\mathbf{C}} P$ of elements of a presheaf $P \in \mathrm{PSh}(\mathbf{C})$ is defined as follows. Objects are pairs (C, x) of $C \in \mathbf{C}$ and $x \in P(C)$. A morphism $(C, x) \to (D, y)$ is a morphism $f\colon C \to D$ in $\mathbf{C}$ satisfying $x = P(f)(y)$. Recall that, for any presheaf $P \in \mathrm{PSh}(\mathbf{C})$, objects of the slice category $\mathrm{PSh}(\mathbf{C})/P$ are natural transformations $\alpha\colon Q \Rightarrow P$ from some presheaf $Q \in \mathrm{PSh}(\mathbf{C})$ to P.

Lemma 7.1. *For any $P \in \mathrm{PSh}(\mathbf{C})$, the toposes $\mathrm{PSh}(\mathbf{C})/P$ and $\mathrm{PSh}(\int_{\mathbf{C}} P)$ are equivalent.*

Proof. See [23, Exercise III.8(a)]; we write out a proof for the sake of explicitness. Define a functor $F\colon \mathrm{PSh}(\mathbf{C})/P \to \mathrm{PSh}(\int_{\mathbf{C}} P)$ by

$$F\big(Q \overset{\alpha}{\Rightarrow} P\big)(C, x) = \alpha_C^{-1}(x),$$

$$F\big(Q \overset{\alpha}{\Rightarrow} P\big)\big((C, x) \overset{f}{\to} (D, y)\big) = Q(f),$$

$$F(Q \overset{\beta}{\Rightarrow} Q')_{(C,x)} = \beta_C.$$

Define a functor $G\colon \mathrm{PSh}(\int_{\mathbf{C}} P) \to \mathrm{PSh}(\mathbf{C})/P$ by $G(R) = (Q \overset{\alpha}{\Rightarrow} P)$ where

$$Q(C) = \coprod_{x \in P(C)} R(C, x),$$

$$Q(C \overset{f}{\to} D) = R\big((C, P(f)(y)) \overset{f}{\to} (D, y)\big),$$

$$\alpha_C(\kappa_x(r)) = x,$$

where $\kappa_x\colon R(C, x) \to \coprod_{x \in P(C)} R(C, x)$ is the coproduct injection. The functor G acts on morphisms as

$$G(R \overset{\beta}{\Rightarrow} R')_C = \coprod_{x \in P(C)} \beta_{(C,x)}.$$

Then one finds that $GF(Q \overset{\alpha}{\Rightarrow} P) = (Q \overset{\alpha}{\Rightarrow} P)$, and $FG(R) = \hat{R}$, where

$$\hat{R}(C, x) = \{x\} \times R(C, x),$$

$$\hat{R}\big((C, x) \overset{f}{\to} (D, y)\big) = \mathrm{id} \times R\big((C, P(f)(y)) \overset{f}{\to} (D, y)\big).$$

Thus there is a natural isomorphism $R \cong \hat{R}$, and F and G form an equivalence. □ □

Definition 7.2. Define a presheaf $\mathrm{Aut} \in \mathrm{PSh}(\mathcal{C}_{\rightarrowtail})$ by

$$\mathrm{Aut}(C) = \{i\colon C \overset{\cong}{\to} C' \mid C' \in \mathcal{C}\},$$

$$\mathrm{Aut}\big(C \overset{k}{\rightarrowtail} D\big)\big(j\colon D \overset{\cong}{\to} D'\big) = j\big|_{k(C)} \circ k\colon C \overset{\cong}{\to} j(k(C)) = D'.$$

Notice that $\mathrm{Aut}(C)$ contains the automorphism group of C. Also, any automorphism of A induces an element of $\mathrm{Aut}(C)$.

The category $\int_{\mathcal{C}_{\rightarrowtail}} \mathrm{Aut}$ of elements of Aut unfolds explicitly to the following. Objects are pairs (C, i) of $C \in \mathcal{C}$ and a $*$-isomorphism $i\colon C \overset{\cong}{\to} C'$. A morphism $(C, i) \to (D, j)$ is an injective $*$-homomorphism $k\colon C \rightarrowtail D$ such that $i = j \circ k$.

Proposition 7.3. *The categories $\mathcal{C}_{\subseteq}$ and $\int_{\mathcal{C}_{\rightarrowtail}} \mathrm{Aut}$ are equivalent.*

Proof. Define a functor $F\colon \mathcal{C}_{\subseteq} \to \int_{\mathcal{C}_{\rightarrowtail}} \mathrm{Aut}$ by $F(C) = (C, \mathrm{id}_C)$ on objects and $F(C \subseteq D) = (C \hookrightarrow D)$ on morphisms. Define a functor $G\colon \int_{\mathcal{C}_{\rightarrowtail}} \mathrm{Aut} \to \mathcal{C}_{\subseteq}$ by $G(C,i) = i(C) = \mathrm{cod}(i)$ on objects and $G\big(k\colon (C,i) \to (D,j)\big) = (i(C) \subseteq j(D))$ on morphisms. Then $GF(C) = C$, and $FG(C,i) = (i(C), \mathrm{id}_{i(C)}) \cong (C,i)$, so that F and G implement an equivalence. □ □

Theorem 7.4. *The toposes* $\mathrm{PSh}(\mathcal{C}_{\subseteq})$ *and* $\mathrm{PSh}(\mathcal{C}_{\rightarrowtail})/\mathrm{Aut}$ *are equivalent.*

Proof. Combining the previous two lemmas, the equivalence is implemented explicitly by the functor $F\colon \mathrm{PSh}(\mathcal{C}_{\rightarrowtail})/\mathrm{Aut} \to \mathrm{PSh}(\mathcal{C}_{\subseteq})$ defined by

$$F\big(P \overset{\alpha}{\Rightarrow} \mathrm{Aut}\big)(C) = \alpha_C^{-1}(\mathrm{id}_C)$$
$$F\big(P \overset{\alpha}{\Rightarrow} \mathrm{Aut}\big)(C \subseteq D) = P(C \hookrightarrow D)$$

and the functor $G\colon \mathrm{PSh}(\mathcal{C}_{\subseteq}) \to \mathrm{PSh}(\mathcal{C}_{\rightarrowtail})/\mathrm{Aut}$ defined by $G(R) = \big(P \overset{\alpha}{\Rightarrow} \mathrm{Aut}\big)$,

$$P(C) = \coprod_{i\colon C \overset{\cong}{\to} C'} R(i(C)),$$
$$P\big(C \overset{k}{\rightarrowtail} D\big) = \coprod_{j\colon D \overset{\cong}{\to} D'} R\big(j(k(C)) \subseteq j(D)\big),$$
$$\alpha_C(\kappa_i(r)) = i.$$

This proves the theorem. □ □

Hence the topos $\mathbf{T} = \mathrm{PSh}(\mathcal{C}_{\rightarrowtail})$ is an *étendue*, which means it is "locally like a space"; more precisely, it contains an object E such that the unique map from E to the terminal object is an epimorphism and the slice category $\mathbf{T}/E$ is (equivalent to) a localic topos. In this case, the object E is the presheaf Aut.

Lemma 7.5. *If* $F\colon \mathbf{C} \to \mathbf{D}$ *is (half of) an equivalence,* X *is any object of* $\mathbf{C}$ *and* $Y \cong F(X)$*, then the slice categories* $\mathbf{C}/X$ *and* $\mathbf{D}/Y$ *are equivalent.*

Proof. Let $G\colon \mathbf{D} \to \mathbf{C}$ be the other half of the given equivalence, and pick an isomorphism $i\colon Y \to F(X)$. Define a functor $H\colon \mathbf{C}/X \to \mathbf{D}/Y$ by $H(a\colon A \to X) = (i \circ Fa\colon FA \to Y)$ and $H(f\colon a \to b) = Ff$. Define a functor $K\colon \mathbf{D}/Y \to \mathbf{C}/X$ by $K(a\colon A \to Y) = (\eta_X^{-1} \circ Gi \circ Ga\colon GA \to X)$ and $K(f\colon a \to b) = Gf$. By naturality of η^{-1} we then have $KH(a) \cong a$. And because $G\varepsilon = \eta^{-1}$ we also have $HK(a) \cong a$. □ □

Lemma 7.6. *If the categories* $\mathbf{C}$ *and* $\mathbf{D}$ *are equivalent, then the toposes* $\mathrm{PSh}(\mathbf{C})$ *and* $\mathrm{PSh}(\mathbf{D})$ *are equivalent.*

Proof. Given functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ that form an equivalence, one directy verifies that $(-) \circ G\colon \mathrm{PSh}(\mathbf{C}) \to \mathrm{PSh}(\mathbf{D})$ and $(-) \circ F\colon \mathrm{PSh}(\mathbf{D}) \to \mathrm{PSh}(\mathbf{C})$ also form an equivalence. □ □

Theorem 7.7. *If* $\mathcal{C}_{\rightarrowtail}(A)$ *and* $\mathcal{C}_{\rightarrowtail}(B)$ *are equivalent categories, then* $\mathcal{C}_{\subseteq}(A)$ *and* $\mathcal{C}_{\subseteq}(B)$ *are Morita-equivalent posets, i.e. the toposes* $\mathrm{PSh}(\mathcal{C}_{\subseteq}(A))$ *and* $\mathrm{PSh}(\mathcal{C}_{\subseteq}(B))$ *are equivalent.*

Proof. If $\mathcal{C}_{\rightarrowtail}(A) \simeq \mathcal{C}_{\rightarrowtail}(B)$, then $\mathrm{PSh}(\mathcal{C}_{\rightarrowtail}(A)) \simeq \mathrm{PSh}(\mathcal{C}_{\rightarrowtail}(B))$ by Lemma 7.6. Moreover, the object Aut_B is (isomorphic to) the image of the object Aut_A under this equivalence. Hence

$$\mathrm{PSh}(\mathcal{C}_{\subseteq}(A)) \simeq \mathrm{PSh}(\mathcal{C}_{\rightarrowtail}(A))/\mathrm{Aut}_A \simeq \mathrm{PSh}(\mathcal{C}_{\rightarrowtail}(B))/\mathrm{Aut}_B \simeq \mathrm{PSh}(\mathcal{C}_{\subseteq}(B))$$

by Theorem 7.4. □ □

Remark 7.8. Hence $\mathcal{C}_{\rightarrowtail}(A)$ is an invariant of the topos $\mathrm{PSh}(\mathcal{C}_{\subseteq}(A))$ as well as of the C*-algebra A. It is not a complete invariant for the latter, however, as shown by Lemma 4.4. For example, $\mathcal{C}_{\rightarrowtail}(\mathbb{M}_n(\mathbb{C})) \simeq \mathcal{C}_{\rightarrowtail}(\mathbb{C}^n)$, but $\mathcal{C}_{\subseteq}(\mathbb{M}_n(\mathbb{C})) \not\cong \mathcal{C}_{\subseteq}(\mathbb{C}^n)$, and certainly $\mathbb{M}_n(\mathbb{C}) \not\cong \mathbb{C}^n$.

We have relied heavily on equivalences of categories, and indeed a logical formula holds in the topos $\mathrm{PSh}(\mathbf{C})$ if and only if it holds in $\mathrm{PSh}(\mathbf{D})$ for equivalent categories $\mathbf{C}$ and $\mathbf{D}$. Therefore one might argue that $\mathcal{C}_{\rightarrowtail}$ has too many morphisms, as compared to $\mathcal{C}_{\subseteq}$, for toposes based on it to have internal logics that are interesting from the point of view of foundations of quantum mechanics. Instead of equivalences, one could consider isomorphisms of categories. This also resembles the original Mackey–Piron question more closely. After all, an equivalence of partial orders is automatically an isomorphism. The following theorem shows that $\mathcal{C}_{\rightarrowtail}(A)$ is a weaker invariant of A than $\mathcal{C}_{\subseteq}(A)$, in this sense.

Theorem 7.9. *If $\mathcal{C}_{\rightarrowtail}(A)$ and $\mathcal{C}_{\rightarrowtail}(B)$ are isomorphic categories, then $\mathcal{C}_{\subseteq}(A)$ and $\mathcal{C}_{\subseteq}(B)$ are isomorphic posets.*

Proof. Let $K\colon \mathcal{C}_{\rightarrowtail}(A) \to \mathcal{C}_{\rightarrowtail}(B)$ be an isomorphism. Suppose that $C, D \in \mathcal{C}_{\rightarrowtail}(A)$ satisfy $C \subseteq D$. Consider the subcategory $\mathcal{C}_{\rightarrowtail}(D)$ of $\mathcal{C}_{\rightarrowtail}(A)$. On the one hand, by Lemma 6.5 it is isomorphic to $P(X) \rtimes S(X)$ for $X = \mathrm{Spec}(D)$, and therefore has a faithful retraction F_A of the inclusion $\mathcal{C}_{\rightarrowtail}(D) \to \mathcal{C}_{\rightarrowtail}(D)(0,0)$ by Theorem 5.4. On the other hand, K maps it to $\mathcal{C}_{\rightarrowtail}(K(D))$, which is isomorphic to $P(Y) \rtimes S(Y)$ for $Y = \mathrm{Spec}(K(D))$, and therefore similarly has a retraction F_B. Moreover, we have $KF_A = F_B K$. Now, by Theorem 5.4, inclusions in $\mathcal{C}_{\rightarrowtail}$ are characterized among all morphisms f by $F(f) = 1$. Hence $F_B(K(C \hookrightarrow D)) = KF_A(C \hookrightarrow D) = K(1) = 1$, and therefore $K(C) \subseteq K(D)$. □ □

It remains open whether existence of an isomorphism $\mathcal{C}_{\subseteq}(A) \cong \mathcal{C}_{\subseteq}(B)$ implies existence of an isomorphism $\mathcal{C}_{\rightarrowtail}(A) \cong \mathcal{C}_{\rightarrowtail}(B)$. This question can be reduced as follows, at least in finite dimension, because every injective *-morphism factors uniquely as a $*$-isomorphism followed by an inclusion. Write $\mathcal{C}_{\cong}(A)$ for the category with $\mathcal{C}(A)$ for objects and $*$-isomorphisms as morphisms. Supposing an isomorphism $F\colon \mathcal{C}_{\subseteq}(A) \to \mathcal{C}_{\subseteq}(B)$, we have $\mathcal{C}_{\rightarrowtail}(A) \cong \mathcal{C}_{\rightarrowtail}(B)$ if and only if there is an isomorphism $G\colon \mathcal{C}_{\cong}(A) \to \mathcal{C}_{\cong}(B)$ that coincides with F on objects. Now, in case A is (isomorphic to) $\mathbb{M}_n(\mathbb{C})$, (so is B, and) if $C, D \in \mathcal{C}(A)$ are isomorphic then so are $F(C)$ and $F(D)$: if $C \cong D$, then $\dim(C) = \dim(D)$, so $\dim(F(C)) = \dim(F(D))$ because F preserves maximal chains, and hence $F(C) \cong F(D)$. However, it is not clear whether this behaviour is functorial, *i.e.* extends to a functor G, or generalizes to infinite dimension.

Appendix A. Inverse semigroups and étendues

The direct proof of Theorem 7.4 follows from [19, A.1.1.7], but it can also be arrived at through a detour via inverse semigroups, based on results due to Funk [10].

This appendix describes the latter intermediate results, which might be of independent interest. For the rest of this appendix, we fix a unital C*-algebra A, and may therefore write $\mathcal{C}_{\subseteq}$ for $\mathcal{C}_{\subseteq}(A)$ and $\mathcal{C}_{\rightarrowtail}$ for $\mathcal{C}_{\rightarrowtail}(A)$.

Definition A.1. Define a set T with functions $T \times T \xrightarrow{\cdot} T$ and $T \xrightarrow{*} T$ by:

$$T = \left\{ C \overset{i}{\rightarrowtail} A \mid C \in \mathcal{C},\ i \text{ is an injective } *\text{-homomorphism} \right\},$$

$$(C' \overset{i'}{\rightarrowtail} A) \cdot (C \overset{i}{\rightarrowtail} A) = (i^{-1}(C') \overset{i' \circ i}{\rightarrowtail} A),$$

$$(C \overset{i}{\rightarrowtail} A)^* = (i(C) \overset{i^{-1}}{\rightarrowtail} A).$$

The multiplication is well-defined, because the inverse image of a *-algebra under a $*$-homomorphism is again a *-algebra, and the inverse image of a closed set is again a closed set, so that $i^{-1}(C)$ is indeed a commutative C*-algebra. The operation * is well-defined because of Lemma 2.2; and on the image, i^{-1} is a well-defined injective $*$-homomorphism. One can verify that together, these data form an inverse semigroup; that is, multiplication is associative, and i^* is the unique element with $ii^*i = i$ and $i^*ii^* = i^*$.

Lemma A.2. *For $(C \overset{i}{\rightarrowtail} A) \in T$, we have $i^*i = (C \hookrightarrow A)$ and $ii^* = (i(C) \hookrightarrow A)$.*

Proof. For the former claim:

$$(C \overset{i}{\rightarrowtail} A)^* \cdot (C \overset{i}{\rightarrowtail} A) = (i(C) \overset{i^{-1}}{\rightarrowtail} A) \cdot (C \overset{i}{\rightarrowtail} A) = (i^{-1}(i(C)) \overset{i^{-1} \circ i}{\rightarrowtail} A) = (C \hookrightarrow A).$$

For the latter claim:

$$\begin{aligned}(C \overset{i}{\rightarrowtail} A) \cdot (C \overset{i}{\rightarrowtail} A)^* &= (C \overset{i}{\rightarrowtail} A) \cdot (i(C) \overset{i^{-1}}{\rightarrowtail} A) \\ &= ((i^{-1})^{-1}(C) \overset{i \circ i^{-1}}{\rightarrowtail} A) = (i(C) \hookrightarrow A).\end{aligned}$$

This proves the lemma. □ □

Definition A.3. For any inverse semigroup T, one can define the groupoid $G(T)$ whose objects are the idempotents of T, i.e. the elements $e \in T$ with $e^2 = e$. A morphism $e \to f$ is an element $t \in T$ satisfying $e = t^*t$ and $tt^* = f$.

Proposition A.4. *The groupoids $G(T)$ and $\mathcal{C}_{\cong}$ are isomorphic.*

Proof. An element $(C \overset{i}{\rightarrowtail} A)$ of T is idempotent when $i^{-1}(C) = C$ and $i^2 = i$ on C. That is, the objects of $G(T)$ are the inclusions $(C \hookrightarrow A)$ of commutative C*-subalgebras; we can identify them with $\mathcal{C}$.

A morphism $C \to C'$ in $G(T)$ is an element $(D \overset{j}{\rightarrowtail} A)$ of T such that $(C \hookrightarrow A) = j^*j = (D \hookrightarrow A)$ and $(C' \hookrightarrow A) = jj^* = (j(D) \hookrightarrow A)$, *i.e.* $C = D$ and $C' = j(D)$. That is, a morphism $C \to C'$ is an injective $*$-homomorphism $j \colon C \rightarrowtail C'$ that satisfies $j(D) = C'$, *i.e.* that is also surjective. In other words, a morphism $C \to C'$ is a $*$-isomorphism $C \to C'$. □ □

Definition A.5. For any inverse semigroup T, one can define a partial order on the set $E(T) = \{e \in T \mid e^2 = e\}$ of idempotents by $e \leq f$ iff $e = fe$.

In fact, $G(T)$ is an ordered groupoid, with $G(T)_0 = E(T)$.

Proposition A.6. *The posets $E(T)$ and $\mathcal{C}_{\subseteq}$ are isomorphic.*

Proof. As with $G(T)$, objects of $E(T)$ can be identified with $\mathcal{C}$. Moreover, there is an arrow $C \to C'$ if and only if

$$(C \hookrightarrow A) = (C' \hookrightarrow A) \cdot (C \hookrightarrow A) = (C \cap C' \hookrightarrow A),$$

i.e. when $C \cap C' = C$. That is, there is an arrow $C \to C'$ iff $C \subseteq C'$. □ □

Also, $G(T)$ is always a subcategory of the following category $L(T)$.

Definition A.7. For any inverse semigroup T, one can define the left-cancellative category $L(T)$ whose objects are the idempotents of T. A morphism $e \to f$ is an element $t \in T$ satisfying $e = t^*t$ and $t = ft$.

Proposition A.8. *The categories $L(T)$ and $\mathcal{C}_{\rightarrowtail}$ are isomorphic.*

Proof. As with $G(T)$, objects of $L(T)$ can be identified with $\mathcal{C}$. A morphism $C \to C'$ in $L(T)$ is an element $(j \colon D \rightarrowtail A)$ of T such that $(C \hookrightarrow A) = j^*j = (D \hookrightarrow A)$ and

$$(D \overset{j}{\rightarrowtail} A) = (C' \hookrightarrow A) \cdot (D \overset{j}{\rightarrowtail} A) = (j^{-1}(C') \overset{j}{\rightarrowtail} A).$$

That is, a morphism $C \to C'$ is an injective $*$-homomorphism $j \colon C \rightarrowtail A$ such that $C = j^{-1}(C')$. Hence we can identify morphisms $C \to C'$ with injective $*$-homomorphisms $j \colon C \rightarrowtail C'$. □ □

Every ordered groupoid G has a classifying topos $\mathcal{B}(G)$. We now describe the topos $\mathcal{B}(G(T))$ explicitly, unfolding the definitions on [10, page 487].

For a presheaf $P \colon \mathcal{C}_{\subseteq}^{\mathrm{op}} \to \mathbf{Set}$, define another presheaf $P^* \colon \mathcal{C}_{\subseteq}^{\mathrm{op}} \to \mathbf{Set}$ by

$$P^*(C) = \{(j, x) \mid j \in \mathcal{C}_{\cong}(A)(C, C'), x \in P(C')\}.$$

On a morphism $C \subseteq D$, the presheaf $P^* \colon P^*(D) \to P^*(C)$ acts as

$$(k \colon D' \overset{\cong}{\to} D, y \in P(D')) \longmapsto \big(k\big|_C \colon C \overset{\cong}{\to} k(C),\, P(k(C) \subseteq D')(y)\big).$$

An object of $\mathcal{B}(G(T))$ is a pair (P, θ) of a presheaf $P \colon \mathcal{C}_{\subseteq}^{\mathrm{op}} \to \mathbf{Set}$ and a natural transformation $\theta \colon P^* \Rightarrow P$. A morphism $(P, \theta) \to (Q, \xi)$ is a natural transformation $\alpha \colon P \Rightarrow Q$ satisfying $\alpha \circ \theta = \xi \circ \alpha^*$, where the natural transformation $\alpha^* \colon P^* \Rightarrow Q^*$ is defined by $\alpha^*_C(j, x) = (j, \alpha_C(x))$.

Lemma A.9. *The toposes $\mathrm{PSh}(\mathcal{C}_{\rightarrowtail})$ and $\mathcal{B}(G(T))$ are equivalent.*

Proof. Combine Proposition A.8 with [10, Proposition 1.12]. Explicitly, (P, θ) in $\mathcal{B}(G(T))$ gets mapped to $F \colon \mathcal{C}_{\rightarrowtail}(A)^{\mathrm{op}} \to \mathbf{Set}$ defined by $F(C) = P(C)$ and

$$F(k \colon C \rightarrowtail D)(y) = \theta_C(k \colon C \overset{\cong}{\to} k(C),\, P(k(C) \subseteq D)(y)).$$

Conversely, F in $\mathrm{PSh}(\mathcal{C}_{\rightarrowtail})$ gets mapped to (P, θ), where

$$P(C) = F(C),$$
$$P(C \subseteq D) = F(C \hookrightarrow D),$$
$$\theta_C(j \colon C \overset{\cong}{\to} C', x \in F(C')) = F(C' \overset{j^{-1}}{\to} C \subseteq D)(x).$$

This finishes the proof. □ □

There is a canonical object $\mathbf{S} = (S, \theta)$ in $\mathcal{B}(G(T))$, defined as follows.

$$S(C) = \{i \colon C \rightarrowtail A\},$$
$$S(C \subseteq D)(j \colon D \rightarrowtail A) = (j\big|_C \colon C \rightarrowtail A).$$

In this case S^* becomes

$$S^*(C) = \{(j, i) \mid j\colon C \overset{\cong}{\to} C', i\colon C' \rightarrowtail A\},$$

$$S^*(C \subseteq D)(j, i) = (j\mid_C\colon C \overset{\cong}{\to} j(C),\, i\big|_{j(C)}\colon j(C) \rightarrowtail A).$$

Hence we can define a natural transformation $\theta\colon S^* \Rightarrow S$ by

$$\theta_C(j, i) = i \circ j.$$

The equivalence of the previous lemma maps $\mathbf{S}$ in $\mathcal{B}(G(T))$ to $\mathbf{D}$ in $\mathrm{PSh}(\mathcal{C}_{\rightarrowtail})$:

$$\mathbf{D}(C) = \{i\colon C \rightarrowtail A\},$$

$$\mathbf{D}(k\colon C \rightarrowtail D)(j\colon D \rightarrowtail A) = (j \circ k\colon C \rightarrowtail A).$$

Technically, the topos $\mathcal{B}(G(T))$ is an étendue: the unique morphism from some object $\mathbf{S}$ to the terminal object is epic, and the slice topos $\mathcal{B}(G(T))/\mathbf{S}$ is (equivalent to) a localic topos. The following lemma makes the latter equivalence explicit.

Lemma A.10. *The toposes* $\mathcal{B}(G(T))/\mathbf{S}$ *and* $\mathrm{PSh}(\mathcal{C}_{\subseteq})$ *are equivalent.*

Proof. Combine Proposition A.6 with equation (1) in [10, page 488]. □ □

Combining the previous two lemmas, we find:

Theorem A.11. *The toposes* $\mathrm{PSh}(\mathcal{C}_{\rightarrowtail})/\mathbf{D}$ *and* $\mathrm{PSh}(\mathcal{C}_{\subseteq})$ *equivalent.* □

In our specific application, we have more information and it is helpful to reformulate things slightly. By Lemma 2.2, giving an injective $*$-homomorphism $i\colon C \rightarrowtail A$ is the same as giving a $*$-isomorphism $C \cong C'$ for some $C' \in \mathcal{C}$ (by taking $C' = i(C)$). Hence $\mathbf{S}$ is isomorphic to the object $\mathbf{Aut} = (\mathrm{Aut}, \theta)$ in $\mathcal{B}(G(T))$ with $\theta_C(j, i) = i \circ j$. This leads to Theorem 7.4.

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University of Oxford
E-mail address: heunen@cs.ox.ac.uk