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REMARKS ON THE THIN OBSTACLE PROBLEM AND CONSTRAINED GINIBRE ENSEMBLES

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Abstract. We consider the problem of constrained Ginibre ensemble with prescribed portion of eigenvalues on a given curve $\Gamma\subset\mathbb{R}^2$ and relate it to a thin obstacle problem. The key step in the proof is the H^1 estimate for the logarithmic potential of the equilibrium measure. The coincidence set has two components: one in Γ and another one in $\mathbb{R}^2\setminus\Gamma$ which are well separated. Our main result here asserts that this obstacle problem is well posed in $H^1(\mathbb{R}^2)$ which improves previous results in $H^1_{loc}(\mathbb{R}^2)$.

1. Introduction

Let Γ be a regular curve in $\mathbb{R}^2$ with locally finite length and $\mathcal{M}_a$ the set of all probability measures such that

$$\mu(\Gamma)\ge a,\quad a\in(0,1). \tag{1.1}$$

By an abuse of notation we let $\Gamma:\mathbb{R}\to\mathbb{R}^2$ be the arc-length parametrization of the curve such that

$$|\dot\Gamma(t)|=1,\quad t\in\mathbb{R}.$$

In this paper we consider the minimizers of the energy

$$I[\mu]=\int\int\log\frac{1}{|x-y|}d\mu(x)d\mu(y)+\int Qd\mu \tag{1.2}$$

where $Q(x)$ is a given function such that the weight function $w=e^{-Q}$ on $\mathbb{R}^2$ is admissible (see Definition 1.1 p.26 [8]). This means that w satisfies the following three conditions:

- **(H1)** w is upper semi-continuous;
- **(H2)** $\{w\in\mathbb{R}^2\ s.t.\ w(z)>0\}$ has positive capacity;
- **(H3)** $|z|w(z)\to0$ as $|z|\to\infty$.

In higher dimensions $\mathbb{R}^d, d\ge3$ one can consider more general kernels

$$K(x-y)=\begin{cases}\log\frac{1}{|x-y|}, & d=2,\\ \frac{1}{|x-y|^{d-2}}, & d\ge3,\end{cases} \tag{1.3}$$

with Γ being a Lyapunov surface in $\mathbb{R}^d$ and define the energy as follows

$$I[\mu]=\int\int K(x-y)d\mu(x)d\mu(y)+\int Qd\mu. \tag{1.4}$$

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In this note we mostly confine ourselves with quadratic potentials $Q(x) = |x|^2$ in $\mathbb{R}^2$, although all our results remain valid for more general Q satisfying $(\mathbf{H1}) - (\mathbf{H3})$. Furthermore, our main result on global L^2 estimate of the gradient of the equilibrium potential with kernel $K(x-y) = |x-y|^{-d}$ remains valid in in $\mathbb{R}^d, d \geq 3$, see Theorem 4.1.

The functional $I[\mu]$, with $Q = |x|^2, d = 2$, arises in the description of the convergence of the spectral measure of square $N \times N$ matrices with complex independent, standard Gaussian entries (i.e., the Ginibre ensemble) as $N \to \infty$. In case when there are no constraints imposed on the eigenvalues, it is well known that the eigenvalues spread evenly in the ball of radius $\sqrt{N}$, and after renormalization by a factor $\frac{1}{\sqrt{N}}$ the normalized spectral measure converges to the characteristic function of the unit disc. This is known as the circular law [4], [2]. In this context the functional I is used to prove large deviation principles for the spectral measure.

If one demands that the eigenvalues are real (i.e. when $a = 1, \Gamma = \mathbb{R}$) we get the so called semicircle law. More generally, one can demand that a portion of eigenvalues is contained in a prescribed set Γ. This is considered in [2] when a portion of eigenvalues are contained in an open bounded subset of $\mathbb{R}^2$ and in [4] when Γ is a line. These problems can be related to the thin obstacle and obstacle problems respectively. The key step in proving this is to establish $H^1_{loc}(\mathbb{R}^2)$ estimates for the logarithmic potential

$$U^{\mu_a} = K * \mu_a$$

of the corresponding equilibrium measure. The aim of this note is to show that the thin obstacle problem is well-posed in $H^1(\mathbb{R}^2)$ by showing that in fact $U^{\mu_a} \in H^1(\mathbb{R}^2)$, see Theorem 4.1. This improves the previous results in [2] and [4].

The paper is organized as follows: In the next section we prove the existence and uniqueness of the equilibrium measure μ_a minimizing the energy $I[\mu]$. In section 3 we discuss some basic properties of μ_a. In particular we show that there are two positive constants A_Γ and A_0 such that $2U^{\mu_a} + Q = A_\Gamma$ on $\operatorname{supp} \mu_a \cap \Gamma$ and $2U^{\mu_a} + Q = A_0$ on $\operatorname{supp} \mu_a \setminus \Gamma$. Furthermore, $A_\Gamma > A_0$. This fact will be used later to show that $\operatorname{supp} \mu_a \setminus \Gamma$ and $\operatorname{supp} \mu_a \cap \Gamma$ are disjoint.

Our main result Theorem 4.1 is contained in section 4. To prove it we study the Fourier transformations of U^{μ_a} and μ_a. It leads to some integral identity involving Bessel functions. This approach is based on a method of L. Carleson [3]. Finally, combining the results obtained, in section 5 we show that U^{μ_a} solves the obstacle problem where the obstacle is given by

$$\psi(x) = \begin{cases} \frac{1}{2}(A_\Gamma - |x|^2) & \text{if } x \in \Gamma, \\ \frac{1}{2}(A_0 - |x|^2) & \text{if } x \in \mathbb{R}^2 \setminus \Gamma. \end{cases} \tag{1.5}$$

2. Existence of minimizers

In this section we show the existence of a unique equilibrium measure.

Theorem 2.1. *Suppose $d = 2, \Gamma \subset \mathbb{R}^2$ is a regular $C^{1,\alpha}$ smooth planar curve without self-intersections. There is a unique minimizer $\mu_a \in \mathcal{M}_a$ of $I[\mu]$ such that*

$$I[\mu_a] = \inf_{\mu \in \mathcal{M}_a} I[\mu].$$

Proof. Observe that the uniqueness follows from the convexity of $\mathcal{M}_a$ and can be proved as in [4]. Moreover, $I[\mu]$ is also semicontinuous. Thus, we have to show that $I[\mu]$ is bounded by below for all $\mu \in \mathcal{M}_a$

and there is at least one μ_0 such that $I[\mu]$ is finite. The lower bound follows as in the proof of Theorem 1.3 (a) p. 27 [8].

It remains show that the $\inf_{\mu\in\mathcal{M}_a} I[\mu]<\infty$. Let χ_D denote the characteristic function of the set D and take

$$\mu = a\frac{1}{L}\mathcal{H}^1\llcorner(\Gamma\cap\Omega)+(1-a)\frac{1}{|B|}\chi_B$$

where $B=B_\rho(z)=\{x\in\mathbb{R}^2 \ : \ |x-z|<\rho\}$ with small ρ such that $B\subset\Omega$, $\Omega\subset\mathbb{R}^2$ is a compact, $L=\mathcal{H}^1(\Gamma\cap\Omega)>0$, and $\operatorname{dist}(\Gamma,B)>0$. Observe that for this choice of μ we have

$$\int_\Omega \log\frac{1}{|x-y|}d\mu(x)=\frac{1}{L}\int_0^L\log\frac{1}{|\Gamma(t)-y|}dt+\frac{1}{|B|}\int_B\log\frac{1}{|x-y|}d\mu(x).$$

Assuming that Γ is given by arc-length parametrization we have for the logarithmic energy

$$\mathcal{L}[\mu]=\frac{a^2}{L^2}\int_0^L\int_0^L\log\frac{1}{|\Gamma(t)-\Gamma(s)|}dtds+\frac{2a(1-a)}{L|B|}\int_0^L\int_B\log\frac{1}{|\Gamma(t)-y|}dtdy+\frac{(1-a)^2}{|B|^2}\int_B\int_B\log\frac{1}{|x-y|}dxdy. \tag{2.1}$$

Since $\operatorname{dist}(\Gamma,B)>0$ then the second integral is bounded. As for the last integral then after change of variables $x-y=\xi$ we have

$$\int_{B\rho(z)}\log\frac{1}{|x-y|}dx=\int_{B_\rho(z-y)}\log\frac{1}{|\xi|}d\xi\le\int_{B_{2\rho}(0)}\log\frac{1}{|\xi|}dx<\infty$$

where we used $|z-y|\le\rho$ and the fact that ρ is small by construction.

It remains to check that the first integral is finite. Let us fix $s\in[0,L]$ Then we have that

$$\begin{aligned}\int_0^L\log\frac{1}{|\Gamma(t)-\Gamma(s)|}dt &= \int_{-s}^{L-s}\log\frac{1}{|\Gamma(\tau+s)-\Gamma(s)|}d\tau=\\ &= \tau\log\frac{1}{|\Gamma(\tau+s)-\Gamma(s)|}\Bigg|_{-s}^{L-s}-\int_{-s}^{L-s}\tau\frac{\dot\Gamma(\tau+s)\cdot(\Gamma(\tau+s)-\Gamma(s))}{|\Gamma(\tau+s)-\Gamma(s)|^2}d\tau=\\ &= (L-s)\log\frac{1}{|\Gamma(L)-\Gamma(s)|}+s\log\frac{1}{|\Gamma(0)-\Gamma(s)|}-I_0\end{aligned}$$

where I_0 is the last integral. Using the crude estimate

$$\begin{aligned}|I_0| &\le \int_{-s}^{L-s}|\tau|\frac{|\dot\Gamma(\tau+s)|}{|\Gamma(\tau+s)-\Gamma(s)|}d\tau=\int_{-s}^{L-s}\frac{|\tau|}{|\Gamma(\tau+s)-\Gamma(s)|}d\tau=\\ &= \int_{[-s,L-s]\setminus(-\delta,\delta)}\frac{|\tau|}{|\Gamma(\tau+s)-\Gamma(s)|}d\tau+\int_{-\delta}^{\delta}\frac{|\tau|}{|\Gamma(\tau+s)-\Gamma(s)|}d\tau\\ &\le \frac{4L^2}{C_\delta}+\int_{-\delta}^{\delta}\frac{|\tau|}{|\Gamma(\tau+s)-\Gamma(s)|}d\tau\end{aligned} \tag{2.2}$$

because $|\Gamma(\tau+s)-\Gamma(s)|\ge C_\delta$ if $|\tau|\ge\delta$. Finally, from $C^{1,\alpha}$ regularity of Γ we get

$$\begin{aligned}|\Gamma(\tau+s)-\Gamma(s)| &= |\tau|\left|\int_0^1\dot\Gamma(\sigma\tau+s)d\sigma\right|\ge\\ &\ge |\tau|\left(|\dot\Gamma(s)|-\int_0^1|\dot\Gamma(\sigma\tau+s)-\dot\Gamma(s)|d\sigma\right)\\ &\ge |\tau|\left(1-\delta^\alpha\right).\end{aligned} \tag{2.3}$$

Combining (2.3) with (2.2) we get

$$|I_0| \leq \frac{4L^2}{C_\delta} + 2\delta\left(1-\delta^\alpha\right) < \infty.$$

Returning to the first integral in (2.1) we infer

$$\begin{aligned}
\int_0^L\int_0^L \log\frac{1}{|\Gamma(t)-\Gamma(s)|}dtds &\leq \int_0^L \left\{(L-s)\log\frac{1}{|\Gamma(L)-\Gamma(s)|} + s\log\frac{1}{|\Gamma(0)-\Gamma(s)|} + \frac{4L^2}{C_\delta} + 2\delta\left(1-\delta^\alpha\right)\right\}ds\\
&\leq L\left[\frac{4L^2}{C_\delta} + 2\delta\left(1-\delta^\alpha\right)\right] + L\log\frac{1}{C_\delta} + \\
&\quad + \int_\delta^{L-\delta}\left\{(L-s)\log\frac{1}{|\Gamma(L)-\Gamma(s)|} + s\log\frac{1}{|\Gamma(0)-\Gamma(s)|}\right\}ds\\
&\leq C(\delta, L)
\end{aligned}$$

if we choose $\delta > 0$ suitably small. This finishes the proof for $d = 2$. □

Remark 2.2. *If* $d \geq 3, Q(x) = |x|^2$ *then clearly* $I[\mu] \geq 0$. *The upper estimate for* $I[\mu]$ *follows from a similar argument if we assume that* Γ *is a Lyapunov surface and take* $\mu = a\frac{1}{L}\mathcal{H}^{d-1}\llcorner(\Gamma\cap\Omega) + (1-a)\frac{1}{|B|}\chi_B$ *with* $L = \mathcal{H}^{d-1}(\Gamma\cap\Omega)$ *and* $\operatorname{dist}(B,\Gamma) > 0$. *Therefore Theorem 2.1 remains valid for* $d \geq 3$.

3. Basic properties of minimizers

In this section we prove some basic properties of the equilibrium measure. The arguments are along the line of those in [2]. Therefore, we mostly focus on those aspects of the proofs which are new or differ essentially. The results to follow are valid in $\mathbb{R}^d, d \geq 2$ unless otherwise stated.

Lemma 3.1. *Let* μ_a *be as in Theorem 2.1. Then* $\mu_a(\Gamma) = a$.

Proof. If the claim fails then $\mu_a(\Gamma) > a$. Fix $\delta \in (0, a)$ and let $\mu_{a-\delta}$ be the minimizer of $I[\cdot]$ over $\mathcal{M}_{a-\delta} \supset \mathcal{M}_a$. Form $\mu = (1-\varepsilon)\mu_a + \varepsilon\mu_{a-\delta}, \varepsilon \in [0,1]$. Clearly, $\mu \in \mathcal{M}_a$ if we choose $\varepsilon\delta$ sufficiently small because

$$\mu(\Gamma) > a + [\mu_a(\Gamma) - a] - \varepsilon\delta.$$

Consequently, we have from the strict convexity of I

$$\begin{aligned}
I[(1-\varepsilon)\mu_a + \varepsilon\mu_{a-\delta}] &< (1-\varepsilon)I[\mu_a] + \varepsilon I[\mu_{a-\delta}] = I[\mu_a] + \varepsilon\left(I[\mu_{a-\delta}] - I[\mu_a]\right)\\
&\leq I[\mu_a]
\end{aligned}$$

which is in contradiction with the fact that μ_a is a minimizer. □

Observe that the Fréchet derivative of $I[\mu]$ is $2U^{\mu_a} + Q$ where

$$U^{\mu_a}(y) = \int K(x-y)d\mu_a(x).$$

It is convenient to consider variations of the equilibrium measure in terms of affine combinations. More precisely, let $\mu_\varepsilon = (1-\varepsilon)\mu_a + \varepsilon\nu, \nu \in \mathcal{M}_a, \varepsilon \in [0,1]$, then by direct computation we have that

$$\begin{aligned} I[\mu_\varepsilon] &= (1-\varepsilon)^2 \int\int K(x-y) d\mu_a(x) d\mu_a(y) \\ &\quad + 2\varepsilon(1-\varepsilon)\int\int K(x-y) d\mu_a(x) d\nu(y) + \varepsilon^2 \int\int K(x-y) d\nu(x) d\nu(y) \\ &\quad + (1-\varepsilon)\int Q d\mu_a + \varepsilon \int Q d\nu \\ &= I[\mu_a] + \varepsilon \left(2\int\int K(x-y) d\mu_a(x) d(\nu(y) - \mu_a) + \int Q d(\nu - \mu_a)\right) + O(\varepsilon^2) = \\ &= I[\mu_a] + \varepsilon \int (2U^{\mu_a} + Q) d(\nu - \mu_a) + O(\varepsilon^2). \end{aligned} \tag{3.1}$$

Since μ_a is the minimizer then $I[\mu_a] \le I[\mu]$, and after sending $\varepsilon \to 0$ it follows that

$$\int (2U^{\mu_a} + Q) d(\nu - \mu_a) \ge 0, \quad \forall \nu \in \mathcal{M}_a. \tag{3.2}$$

Lemma 3.2. *Let* $A_\Gamma = \frac{1}{a}\int_\Gamma (2U^{\mu_a} + Q) d\mu_a$ *then quasi everywhere*

$$2U^{\mu_a} + Q \begin{array}{ll} = A_\Gamma & \text{on} \;\; \Gamma \cap \operatorname{supp} \mu_a, \\ \ge A_\Gamma & \text{on} \;\; \Gamma. \end{array} \tag{3.3}$$

Similarly, let us denote $A_0 = \frac{1}{1-a}\int_{\mathbb{R}^2 \setminus \Gamma} (2U^{\mu_a} + Q) d\mu_a$ *then*

$$2U^{\mu_a} + Q \begin{array}{ll} = A_0 & \text{on} \;\; \operatorname{supp} \mu_a \setminus \Gamma, \\ \ge A_0 & \text{on} \;\; \mathbb{R}^2 \setminus (\operatorname{supp} \mu_a \setminus \Gamma). \end{array} \tag{3.4}$$

Furthermore,

$$A_\Gamma > A_0. \tag{3.5}$$

Proof. We first prove (3.3). Suppose that there is a set capacitable E of positive capacity such that $\Gamma \cap E$ has zero capacity and

$$2U^{\mu_a} + Q < A_\Gamma - \delta \;\; \text{q.e. on} \;\; E$$

for some positive δ. Let μ_E be the equilibrium measure of E and form $\nu = \mu_a \llcorner (\mathbb{R}^2 \setminus \Gamma) + a\mu_E$. Clearly $\nu \in \mathcal{M}_a$. Therefore, in view of (3.1) for the measure $\mu_\varepsilon = \varepsilon\mu_a + (1-\varepsilon)\nu \in \mathcal{M}_a$ we get

$$\begin{aligned} I[\mu_\varepsilon] &= I[\mu_a] + \varepsilon \left(2\int\int K(x-y) d\mu_a(x) d(\nu(y) - \mu_a) + \int Q d(\nu - \mu_a)\right) + O(\varepsilon^2) \\ &= I[\mu_a] + \varepsilon \int_\Gamma (2U^{\mu_a} + Q) d(a\mu_E - \mu_a) + O(\varepsilon^2) \\ &= I[\mu_a] + \varepsilon \left(a\int_\Gamma (2U^{\mu_a} + Q) d\mu_E - aA_\Gamma\right) + O(\varepsilon^2) \\ &< I[\mu_a] - a\varepsilon\delta + O(\varepsilon^2) \\ &< I[\mu_a] \end{aligned} \tag{3.6}$$

if ε and δ are sufficiently small. This will be in contradiction with the fact that μ_a is the minimizer. Thus we have proved that $2U^{\mu_a}+Q\ge A_\Gamma$ q.e. on Γ.

Next we show that on $\operatorname{supp}\mu_a\cap\Gamma$ we have $2U^{\mu_a}+Q=A_\Gamma$ q.e. Indeed, from the definition of A_Γ it follows

$$aA_\Gamma=\int_\Gamma(2U^{\mu_a}+Q)d\mu_a\ge aA_\Gamma$$

where the last inequality follows from the first inequality in (3.3). The proof of (3.4) is similar. In order to prove the last claim $A_\Gamma>A_0$ we first observe that there exists a measure $\nu\in\mathcal{M}_a$ such that

- $a>\nu(\Gamma)$,
- $I[\nu]\le I[\mu_a]$.

First notice that $\mathcal{M}_a\subset\mathcal{M}_{a-\delta}$ for $\delta\in(0,a)$. Fix such $\delta>0$ and let $\mu_{a-\delta}$ be the minimizer of $I[\cdot]$ over $\mathcal{M}_{a-\delta}$. Then by Lemma 3.1 $\mu_{a-\delta}(\Gamma)=a-\delta<a$ and $I[\mu_{a-\delta}]=\inf_{\mathcal{M}_{a-\delta}}I[\mu]\le I[\mu_a]=\inf_{\mathcal{M}_a}I[\mu]$. Therefore one can take $\nu=\mu_{a-\delta}$.

From the strict convexity of I it follows that

$$I[\nu]>I[\mu_a]+\langle DI[\mu_a],\nu-\mu_a\rangle$$

where $DI[\mu]=2U^\mu+Q$ is the Fréchet derivative of $I[\mu]$. Therefore, from the properties of ν we infer

$$0\ge I[\nu]-I[\mu_a]>\langle DI[\mu_a],\nu-\mu_a\rangle \tag{3.7}$$

or equivalently

$$\langle 2U^{\mu_a}+Q,\nu-\mu_a\rangle<0.$$

On the other hand

$$\int(2U^{\mu_a}+Q)d\mu_a=aA_\Gamma+(1-a)A_0 \tag{3.8}$$

while

$$\int(2U^{\mu_a}+Q)d\nu=\int_\Gamma(2U^{\mu_a}+Q)d\nu+\int_{\mathbb{R}^2\setminus\Gamma}(2U^{\mu_a}+Q)d\nu\ge\nu(\Gamma)A_\Gamma+\nu(\mathbb{R}^2\setminus\Gamma)A_0.$$

This together with (3.8), (3.7) yields

$$aA_\Gamma+(1-a)A_0>\nu(\Gamma)A_\Gamma+(1-\nu(\Gamma))A_0\Rightarrow A_0(\nu(\Gamma)-a)>A_\Gamma(\nu(\Gamma)-a).$$

Finally, the property $\nu(\Gamma)<a$ implies that $A_\Gamma>A_0$. □

Corollary 3.3. $\operatorname{supp}\mu_a$ *is compact.*

Proof. If $d\ge 3$ then $K(x-y)\ge 0$, hence by Lemma 3.2 for $x\in\operatorname{supp}\mu_a$ we have

$$\max(A_\Gamma,A_0)\ge 2U^{\mu_a}(x)+Q(x)\ge Q(x)\to\infty\quad\text{if}\quad |x|\to\infty \tag{3.9}$$

which is a contradiction. If $d=2$ then from the triangle inequality we get that

$$K(x-y)\ge-\log|x|-\log\left(1+\frac{|y|}{|x|}\right). \tag{3.10}$$

Consequently, for $x\in\operatorname{supp}\mu_a$

$$\begin{aligned}\max(A_\Gamma,A_0)&\ge 2U^{\mu_a}(x)+Q(x)\ge Q(x)-2\log|x|-\int\log\left(1+\frac{|y|}{|x|}\right)d\mu_a\\&=Q(x)-2\log|x|+O(1)\to\infty\quad\text{if}\quad|x|\to\infty\end{aligned}$$

for sufficiently large $|x|$, where the last inequality follows from (4.12) and $\int Q d\mu_a < I[\mu_a] < \infty$. Since $Q = |x|^2$ (of for the general case from the hypotheses on Q (**H1**) – (**H3**)) it again follows that $\operatorname{supp}\mu_a$ is bounded. □

4. GLOBAL L^2 ESTIMATES FOR U^{μ_a} AND ∇U^{μ_a}

Our main result is contained in the following

Theorem 4.1. *Let $U^{\mu_a}(y) = \int K(x-y)d\mu_a$, if $d \geq 3$ then $\nabla U^{\mu_a} \in L^2(\mathbb{R}^d)$. If $d = 2$ then $U^{\mu_a} \in H^1(\mathbb{R}^2)$. Furthermore, there holds*

$$\|U^{\mu_a}\|_{H^1(\mathbb{R}^2)} \leq C\mathcal{E}[\mu_a]. \tag{4.1}$$

Here $\mathcal{E}[\mu]$ is the energy of μ defined as $\int\int K(x-y)d\mu(x)d\mu(y)$.

Remark 4.2. *It is shown in [3] that $\mathcal{E}[\mu] > 0$ for any probability measure μ and $d \geq 2$. In fact, this can be seen from the proof to follow (see also Corollary 4.3).*

Proof. The case $d \geq 3$ follows from Lemma 1.6 p. 92 [7] (see also Lemma 17 p. 95), which assert that

$$\frac{\partial U^{\mu_a}(x)}{\partial x_i} = \int \frac{\partial K(x-y)}{\partial x_i} d\mu_a$$

almost everywhere and morover

$$\frac{1}{4\pi^2}\int_{\mathbb{R}^d} |\nabla U^{\mu_a}|^2 \leq \int\int K(x-y)d\mu_a(x)d\mu_a(y) = \mathcal{E}[\mu_a].$$

The case of the logarithmic potential follows from a modification of the argument by L. Carleson [3] Lemma 3 page 22. We begin with computing the Fourier transformation of K. Note that since $\operatorname{supp}\mu_a$ is compact we can assume that $K(r) = 0$ for $r \geq r_0$ for some fixed $r_0 > 0$. We have

$$\begin{aligned}\widehat{K}(\xi) &= \int K(x)e^{-2\pi i\langle x,\xi\rangle}dx = \int K(x)e^{-2\pi i\langle x|\xi|, \frac{\xi}{|\xi|}\rangle}dx \\ &= \frac{1}{4\pi^2|\xi|^2}\int K\left(\frac{y}{2\pi|\xi|}\right)e^{i\langle y, \frac{\xi}{|\xi|}\rangle}dx.\end{aligned}$$

Let us denote $K_0(y) = K\left(\frac{y}{2\pi|\xi|}\right)$ and define

$$F(\eta) = \int K_0(y)e^{i\pi\langle y,\eta\rangle}, \quad \eta = \frac{\xi}{|\xi|}.$$

From Lemma 2 p. 21 [3] it follows that there is a universal constant c_1 such that

$$F(\eta) = c_1\int_0^\infty K_0(r)J(r)rdr, \quad |\eta| = 1$$

where J is the Bessel function

$$J(r) = -J''(r) - \frac{J'(r)}{r}, \quad J(0) = 1, J'(0) = 0, \quad J(r) < 1, r \neq 0. \tag{4.2}$$

Therefore $F(\eta)$ can be further simplified as follows

$$\begin{aligned} F(\eta) &= -c_1\int_0^\infty K_0(r)(rJ(r))'dr = \\ &= c_1\int_0^{2\pi|\xi|r_0} rJ'(r)K_0'(r)dr \end{aligned} \tag{4.3}$$

because from the definition of K_0 we have $\operatorname{supp} K_0 \subset [0, 2\pi|\xi|r_0]$. Moreover, $K_0'(r) = -\frac{1}{r}$ hence

$$F(\eta) = c_1(1 - J(2\pi|\xi|r_0)). \tag{4.4}$$

Consequently,

$$\widehat{K}(\xi) = \frac{c_1}{4\pi^2|\xi|^2}(1 - J(2\pi|\xi|r_0)). \tag{4.5}$$

Next we restrict $\mu_1 = \mu_a \llcorner \mathcal{C}$ where $\mathcal{C} \subset \operatorname{supp}\mu_a$ is a compact such that U^{μ_1} is continuous. Observe that $\int U^{\mu_a} d\mu_a$ is finite hence U^{μ_a} is finite μ_a almost everywhere. By Theorem 1.8 p. 70 [7] for every $\varepsilon > 0$ small there is a restriction of μ_a such that

$$0 \le \int \mu_a - \int \mu_1 < \varepsilon.$$

Note that if $\tau = \mu_a - \mu_1$ then we have

$$|\mathcal{E}[\mu_a] - \mathcal{E}[\mu_1]| = \left|\int U^{\mu_a-\mu_1}d\mu_a + \int U^{\mu_a-\mu_1}d\mu_1\right| = \left|\int (U^{\mu_a} + U^{\mu_1})d\tau\right| = O(\varepsilon).$$

Let $\phi_n(y) = n^{\frac{d}{2}}e^{-n\pi|y|^2}$ be the sequence of normalised Gaussian kernels. It is well-known that ϕ_n is a mollification kernel for every $n \in \mathbb{N}$ and moreover $\widehat{\phi_n} = e^{-\frac{\phi|\xi|^2}{n}}$. From the Parseval relation

$$\int (\phi_n * U^{\mu_1})d\mu_1 = \int \widehat{\phi_n}\widehat{K}|\widehat{\mu_1}|^2. \tag{4.6}$$

If we first send $n \to \infty$ and then $\varepsilon \to 0$ to conclude the identity

$$\mathcal{E}[\mu_a] = \int \widehat{K}|\widehat{\mu_a}|^2. \tag{4.7}$$

On the other hand $\widehat{U^{\mu_a}} = \widehat{K}\widehat{\mu_a}$, which yields

$$\begin{aligned} \mathcal{E}[\mu_a] &= \int \widehat{K}(\xi)\frac{|\widehat{U^{\mu_a}}(\xi)|^2}{|\widehat{K}(\xi)|^2}d\xi \\ &= \int \frac{4\pi^2|\xi|^2}{c_1(1 - J(2\pi r_0|\xi|))}|\widehat{U^{\mu_a}}(\xi)|^2 d\xi \\ &= \int_{|\xi|<\delta} + \int_{|\xi|\ge\delta}. \end{aligned} \tag{4.8}$$

Using the expansion $J(t) = \sum_{s=0}^\infty \frac{(-1)^s}{(s!)^2}\left(\frac{t}{2}\right)^{2s} = 1 - \frac{t^2}{4} + \frac{t^4}{64} + \dots$ we see that

$$\frac{4\pi^2|\xi|^2}{c_1(1 - J(2\pi r_0|\xi|))} = \frac{1}{r_0^2 c_1}\frac{4}{(1 - \frac{(2\pi r_0|\xi|)^2}{16} + \dots)}$$

hence the first integral is bounded below by $C(\delta)\frac{1}{r_0^2 c_1}\int_{|\xi|<\delta}|\widehat{U^{\mu_a}}(\xi)|^2 d\xi$ for sufficiently small $\delta > 0$. As for the second integral, we have

$$\int_{|\xi|\geq\delta}\frac{4\pi^2|\xi|^2}{c_1(1-J(2\pi r_0|\xi|))}|\widehat{U^{\mu_a}}(\xi)|^2 d\xi \geq \frac{4\pi^2\delta^2}{c_1}\int_{|\xi|\geq\delta}|\widehat{U^{\mu_a}}(\xi)|^2 d\xi. \tag{4.9}$$

Combining we see that $\widehat{U^{\mu_a}} \in L^2(\mathbb{R}^2)$ which, after we apply Parseval's relation again, yields $U^{\mu_a} \in L^2(\mathbb{R}^2)$ and

$$\|U^{\mu_a}\|_{L^2(\mathbb{R}^2)} \leq C\mathcal{E}[\mu_a]. \tag{4.10}$$

To finish the proof we use that $4\pi^2|\xi|^2|\widehat{U^{\mu_a}}|^2 = |\widehat{\nabla U^{\mu_a}}|^2$ which together with (4.8) implies that

$$\mathcal{E}[\mu_a] = \int \frac{1}{c_1(1-J(2\pi r_0|\xi|))}|\widehat{\nabla U^{\mu_a}}(\xi)|^2 d\xi \geq \frac{1}{c_1}\int|\widehat{\nabla U^{\mu_a}}(\xi)|^2 d\xi \tag{4.11}$$

which finishes the proof. □

Corollary 4.3. *Let μ_a be as in Theorem 2.1. Then there holds*

$$\mathcal{E}[\mu_a] = \int U^{\mu_a} d\mu_a > 0. \tag{4.12}$$

5. THE THIN OBSTACLE PROBLEM

From the $H^1(\mathbb{R}^2)$ estimate for U^{μ_a} it follows that U^{μ_a} is a solution to some variational inequality, and hence U^{μ_a} can be interpreted as a solution to an obstacle problem with a combination of both thin (on Γ) and "thick" obstacles (on $\mathbb{R}^2 \setminus \Gamma$). It is convenient to define the obstacle as follows

$$\psi(x) = \begin{cases} \frac{1}{2}(A_\Gamma - |x|^2) & \text{if } x \in \Gamma, \\ \frac{1}{2}(A_0 - |x|^2) & \text{if } x \in \mathbb{R}^2 \setminus \Gamma. \end{cases} \tag{5.1}$$

Lemma 5.1. *Let U^{μ_a} be the logarithmic potential of μ_a and define*

$$\mathcal{K} = \left\{v \in H^1_{loc}(\mathbb{R}^2) \text{ s.t. } v - U^{\mu_a} \text{ has bounded support in } \mathbb{R}^2,\ v \geq \psi\right\}.$$

Then U^{μ_a} solves the following obstacle problem:

$$\int \nabla U^{\mu_a}\nabla(v - U^{\mu_a}) \geq 0, \quad \forall v \in \mathcal{K}.$$

The proof is the same as in [2].

Corollary 5.2. $\operatorname{dist}(\Gamma, \operatorname{supp}(\mu_a \setminus \Gamma)) > 0$.

Proof. This follows from the estimate $A_\Gamma > A_0$. Indeed, let us assume that $x_0 \in \Gamma \cap \operatorname{supp}\mu_a$ and there is a sequence $\{x_k\}_{k=1}^\infty, x_k \in \operatorname{supp}\mu_a \setminus \Gamma$ such that $\lim_{k\to\infty} x_k \to x_0$. Using the lower semicontinuity of U^{μ_a} (see Lemma 1 p.15 [3]) we see that

$$\frac{1}{2}(A_0 - |x_0|^2) = \liminf_{x_k \to x_0} U^{\mu_a}(x_k) \geq U^{\mu_a}(x_0). \tag{5.2}$$

Let $\rho > 0$ be such that $\{x_k\} \subset B_\rho(x_0)$. If ρ is small then Γ divides $B_\rho(x_0)$ into two parts D^+ and D^-. To fix the ideas let us suppose that D^+ contains a subsequence $\{x_k\}$. Let h be the harmonic function in D^+

such that $h=\psi$ on ∂D^+. Observe that h is continuous at x_0 because $\Gamma\in C^{1,\alpha}$. Since U^{μ_a} is superharmonic and on ∂D^+ we have $U^{\mu_a}\ge\psi=h$ then the comparison principle implies that

$$U^{\mu_a}(x_0)\ge h(x_0)=\frac{1}{2}(A_\Gamma-|x_0|^2). \tag{5.3}$$

Combining (5.2) and (5.3) we see that $A_0\ge A_\Gamma$ which is a contradiction in view of (3.5). □

From Corollary 5.2 it follows that near Γ the potential U^{μ_a} is a solution to a thin obstacle problem in the following sense, see [5] p. 108:

$$\left.\begin{array}{l} U^{\mu_a}\ge\frac{1}{2}(A_\Gamma-Q)\\ \frac{\partial U^{\mu_a}}{\partial n^+}+\frac{\partial U^{\mu_a}}{\partial n^-}\ge 0\\ \left(u-\frac{1}{2}(A_\Gamma-Q)\right)\left(\frac{\partial U^{\mu_a}}{\partial n^+}+\frac{\partial U^{\mu_a}}{\partial n^-}\right)=0\end{array}\right\}\text{on }\Gamma \tag{5.4}$$

where $n^\pm$ are the outward normals on the Γ corresponding to the domains that Γ separates. In particular, if Γ is C^3 regular then U^{μ_a} is $C^{1,\alpha}$ up to Γ from each of its side, see Theorem 11.4 p.111 [5].

A particular case is $\Gamma=\mathbb{R}$ [4]. Using a simple symmetrization argument (see e.g. [6] p. 119 Theorem 4.6) we can show that the potential U^{μ_a} is symmetric w.r.t. the real line and hence we get the Signorini problem near $\mathbb{R}$ [5] p. 111.

One can make the connections with the obstacle problem more explicit by using the $H^1(\mathbb{R}^2)$ estimate in Theorem 4.1 and transforming the energy $I[\mu_a]$. Let $R>0$ be fixed then using the divergence theorem

$$\begin{aligned}\int_{B_R}U^{\mu_a}d\mu_a &= -\frac{1}{2\pi}\int_{B_R}U^{\mu_a}\Delta U^{\mu_a}=\\ &=\frac{1}{2\pi}\int_{B_R}|\nabla U^{\mu_a}|^2-\frac{1}{2\pi}\int_{\partial B_R}U^{\mu_a}\partial_n U^{\mu_a}.\end{aligned} \tag{5.5}$$

For a.e. $R>0$ the last integral can be estimated as follows

$$\left|\int_{\partial B_R}U^{\mu_a}\partial_n U^{\mu_a}\right|\le\int_{\partial B_R}|U^{\mu_a}||\nabla U^{\mu_a}|\le\int_{\partial B_R}|U^{\mu_a}|^2+|\nabla U^{\mu_a}|^2.$$

From Theorem 4.1 and Fubini's theorem it follows that

$$\int_{R^2}(|U^{\mu_a}|^2+|\nabla U^{\mu_a}|^2)=\int_0^\infty\int_{\partial B_R}(|U^{\mu_a}|^2+|\nabla U^{\mu_a}|^2)dR.$$

Consequently,

$$\int_{\partial B_R}|U^{\mu_a}|^2+|\nabla U^{\mu_a}|^2\to 0\quad R\to\infty$$

and we infer from (5.5) that

$$\int_{\mathbb{R}^2}U^{\mu_a}d\mu_a=\frac{1}{2\pi}\int_{\mathbb{R}^2}|\nabla U^{\mu_a}|^2.$$

Recalling that by Corollary 3.3 $\operatorname{supp}\mu_a\subset B_{r_0}$ for some $r_0>0$ and using the divergence theorem again we conclude

$$\begin{aligned}\int_{B_{r_0}}|x|^2d\mu_a &= -\frac{1}{2\pi}\int_{B_{r_0}}|x|^2\Delta U^{\mu_a}=-\frac{1}{2\pi}\int_{B_{r_0}}U^{\mu_a}\Delta|x|^2+\frac{1}{2\pi}\int_{\partial B_{r_0}}(2r_0U^{\mu_a}-r_0^2\partial_n U^{\mu_a})\\ &= -\frac{2}{\pi}\int_{B_{r_0}}U^{\mu_a}+\frac{r_0}{\pi}\int_{\partial B_{r_0}}U^{\mu_a}+r_0^2.\end{aligned} \tag{5.6}$$

Combining these we have that the energy can be rewritten in terms of U^{μ_a} in the following form

$$I[\mu_a] = \frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla U^{\mu_a}|^2 - \frac{2}{\pi}\int_{B_{r_0}} U^{\mu_a} + \frac{r_0}{\pi}\int_{\partial B_{r_0}} U^{\mu_a} + r_0^2.$$

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