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Perceiving Prospects Properly[*]

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Abstract

When an agent chooses between prospects, noise in information processing generates an effect akin to the winner's curse. Statistically unbiased perception systematically overvalues the chosen action because it fails to account for the possibility that noise is responsible for making the preferred action appear to be optimal. The optimal perception pattern exhibits a key feature of prospect theory, namely, overweighting of small probability events (and corresponding underweighting of high probability events). This bias arises to correct for the winner's curse effect.

1 Introduction

There is considerable evidence that human perception of reality is noisy and biased.[1] While randomness can be understood as a technological limitation of human cognition, systematic behavioral biases, such as those documented in the psychological experiments of Kahneman

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[1]McFadden (1999) summarizes the experimental evidence as follows: "humans fail to retrieve and process information consistently.... These failures may be fundamental, the result of the way human memory is wired. I conclude that perception-rationality fails, and that the failures are systematic, persistent, pervasive, and large in magnitude."

and Tversky (1979), are more puzzling. Since there is no obvious reason why natural or cultural evolution could not remove these biases, their prevalence suggests that they serve a purpose.

This paper argues that perception biases arise as a second-best solution when some noise in information processing is unavoidable. In particular, we show that overweighting of small probability events optimally mitigates errors due to randomness. Our model also provides a framework for conceptualizing errors in decision-making, allowing us to consider, for example, whether overweighting of small probabilities is a mistake or an optimal heuristic. Finally, our results demonstrate how explicitly modelling the structure of decision-making can illuminate patterns of observed behavior.

Our model separates decision-making into two stages. At the first stage, the decision-maker observes the parameters of the decision problem and encodes them using a perception strategy. In our main model, the encoded values are then subject to stochastic noise. At the second stage, the decision-maker chooses an action based on these noisy values of the parameters. The noise can be interpreted as physiological randomness in the functioning of the brain, as failing to remember or keep track of all relevant information during decision-making, or as random computational errors. Each of these cases can be viewed as a loss of information during the decision process. Our main focus is on the optimal design of the perception strategy: given that noise will prevent the use of the true values in the second stage, how should those values be encoded beforehand?[2]

One natural perception strategy to consider is the unbiased one that gives rise, on average, to the correct parameter values after the noise is introduced. We argue that the unbiased strategy suffers from a problem akin to the "winner's curse," making it suboptimal. Just as a bidder in a common value auction should condition her value on winning, the design of the perception strategy should condition on the chosen action. Unbiased perception fails to account for the possibility that noise may be responsible for making an action appear to be optimal. Biases in perception can correct for this winner's curse by generating a more cautious evaluation of actions.

The intuition for our results can be described most simply in a related model capturing a status quo bias.[3] Consider an agent who chooses between the status quo and an alternative

[2]We view the perception strategy as being applied subconsciously and optimized through evolution rather than through conscious reasoning. Kirkpatrick and Epstein (1992) present experimental evidence suggesting that subconscious distortions drive choice even when subjects correctly identify objective probabilities. See also Camerer et al. (2005) for a discussion of conscious and subconscious processing of probabilities.

[3]An earlier version of this paper explores this variation in more detail (Steiner and Stewart, 2014).

action. The agent's perception is chosen from a class of strategies differing only in the degree of status quo bias, i.e., in the extent to which the perception of the status quo reward is exaggerated. In particular, the strategy with no status quo bias yields unbiased perception of rewards, whereas with a nonzero bias, the agent's average perception systematically favors one of the two actions. The perception design problem consists of choosing the degree of bias that maximizes the expected reward the agent receives across all possible realizations of the binary decision problem. It turns out that the unbiased strategy is (generically) suboptimal: the optimal perception strategy is unbiased *conditional on the two options being perceived as equally attractive*, which implies that, *unconditionally*, it is biased.

Suppose that the average status quo is better than typical rewards from the alternative action (as one might expect if the status quo results from previous optimizing choices). Then unbiased perception leads to a winner's curse because, conditional on perceiving the alternative as optimal, the agent overvalues it. As a result, the optimal status quo bias is positive, correcting for the winner's curse; optimal perception makes the agent cautious about the alternative.

This paper focuses on the perception of probabilities. As in the preceding example, unbiased perception of probabilities leads to a winner's curse since errors that increase the relative attractiveness of an action make that action more likely to be chosen. We argue that overweighting small probabilities (and underweighting large ones) mitigates the overoptimism stemming from the winner's curse.

Probability distortions may, at first blush, seem unlikely to help. Biasing probabilities does not, on average, make the agent more pessimistic or optimistic; exaggeration of small probabilities makes the agent less inclined to fly for fear of an accident, but more inclined to play casino games that offer a small probability of a large reward. However, such a bias does tend to make the agent more pessimistic about *attractive* lotteries. The reason is that lotteries perceived as valuable are much less likely to share the structure of a casino game than that of a decision to fly. Compare two lotteries offering the same expected reward: one a "flight lottery" that gives a high probability gain and a low probability loss, and the other a "casino lottery" that gives a low probability gain and a high probability loss. For any given loss, the two lotteries can have the same expected reward only if the low probability of a gain in the casino lottery is compensated with a very high reward. If very high rewards are rare, then attractive lotteries are typically like the flight lottery. Exaggerating low probabilities therefore tends to increase the weight given to losses, reducing the perceived value of the most attractive lotteries.

As for the status quo bias, the optimal perception of probabilities in a lottery is unbiased conditional on the lottery being perceived as equal in value to its opportunity cost—the value of the next best option. Regardless of the opportunity cost, the optimal perception is unconditionally biased. When the opportunity cost is high relative to average rewards available to the agent, the bias takes the form of overweighting small probabilities and underweighting large ones. A high opportunity cost arises naturally when the agent chooses from a rich set of options, and thus we focus on this case. Our interpretation is that perception is not tailored to each choice problem, and problems typically involve a large number of options, including many that individuals scarcely consider because they are clearly suboptimal. The opportunity cost interpretation is formalized in Section 5.

Is overweighting of small probabilities (and underweighting of large ones) a mistake? On the one hand, since such a bias in perception is an optimal response to subsequent information loss, the agent would be worse off on average if he "debiased" his perception across all decision problems. On the other hand, a globally optimal perception strategy may perform poorly in some decision problems. In particular, the ex ante optimal perception strategy performs badly when the agent faces the casino lottery described above. Since the casino lottery is unlikely to be an attractive option, the ex ante optimal strategy introduces relatively large perception errors in lotteries of that form. Ex post, an outside observer who knows that the agent faces such a lottery could reasonably characterize the agent's perception bias as a mistake because it is suboptimal given the observer's information. Section 6 discusses these issues in more detail.

An important feature of our model is that it involves a friction in information processing, as opposed to noise in observation (that is, the noise appears *after* the encoding of the parameters of the problem, not before). With frictionless information processing and noisy observation of probabilities, the optimal perception strategy in our model would encode the expected probability conditional on observed information, leading to behavior identical to that of a conventional Bayesian decision-maker. Thus we focus on the problem in which parameters are observed without noise, where biases in perception arise as a way to mitigate information processing errors. We discuss this distinction in more detail at the end of Section 3.1.

Although our model of the cognitive process is stylized, the neuroscientific literature offers some support for a two-stage, noisy decision process. Glimcher (2009) describes the emerging neuroeconomic consensus that the choice system in primates "involves a two-stage mechanism. The first of these stages is concerned with the valuation of all goods and actions; the second is concerned with choosing. . . [from] the choice set." Bossaerts et al. (2009) discuss evidence that there are at least two imperfectly correlated brain signals involved in the choice process, one for assessing value, the other for the choice itself. More broadly, Glimcher (2005) surveys a body of evidence suggesting fundamental randomness in the activity of the brain. Tobler et al. (2008) document that probabilities are simultaneously encoded in more than one area of the brain, and that neuronal coding of probabilities in areas associated with probabilistic decision-making shows an inverted S-shaped pattern. The combination of separate encoding of probabilities and randomness of neural activity lends support to our approach of modeling encoded probabilities as subject to noise.

Our paper fits into the literature on the principal-agent approach to evolution (see, e.g., Robson, 2001b; Samuelson and Swinkels, 2006; Robson and Samuelson, 2011). Robson (2001a), Rayo and Becker (2007), and Netzer (2009) study the evolutionary design of incentives for agents who cannot process information perfectly. They find that the optimal incentives are steeper at ranges of stimuli that the agent encounters more frequently, which can be interpreted as allocating greater attention to more common problems.[4] Our results can also be understood in terms of optimal attention allocation, but extended to choice under uncertainty and using a different model of information processing.

Several papers study foundations for the biases captured in prospect theory. Herold and Netzer (2010) argue that inverted S-shaped probability weighting is an optimal response to S-shaped valuation of rewards. Similarly, Frenkel et al. (2012) view the endowment effect as a heuristic benefitting agents who suffer from the winner's curse in bilateral trade. In contrast with these two papers, we derive optimal distortions in perception in the absence of frictions in other dimensions of the decision process. Woodford (2012a,b) studies optimal perception using insights from the rational inattention literature. Woodford's analysis focuses on a relatively simple objective (namely, minimization of the mean square error) while allowing for a rich class of perception strategies. In contrast, we focus on maximization of expected rewards, and identify systematic deviations relative to the mean-square-error-minimizing perception.

Compte and Postlewaite (2012) study optimal heuristics for choice under uncertainty, and identify conditions under which a decision-maker exhibits "cautiousness" toward less certain outcomes. While our results can be interpreted similarly as a form of cautiousness,

[4] Friedman (1989) provides an early analysis of the attention allocation problem using a reduced form of evolutionary optimization.

we differ significantly in terms of focus and modelling approach. In particular, in their model, cautiousness is optimal in a class of relative simple strategies, whereas in ours it is a feature of the best response to imperfections in information processing.

Eyster and Rabin (2005) formalize a winner's curse in strategic settings by assuming that players fail to fully account for how others' actions depend on their types. The winner's curse effect that would arise if the agent used the unbiased perception strategy in our model is similar in spirit insofar as it can be thought of as a result of failing to correctly account, at the observation stage, for the strategy used at the decision stage. While Eyster and Rabin focus on the strategic consequences of incorrect beliefs, we focus on the optimal design of perception to alleviate the winner's curse.

2 Model

An agent faces a binary decision problem in which he chooses between an alternative that delivers payoff $s \in \mathbb{R}$—which we refer to as the opportunity cost—and a lottery that pays a reward r_1 or r_2 in $\mathbb{R}$ with respective probabilities p and $1-p$. While all of these parameters are observable to the agent, he may make suboptimal choices due to errors in information processing.

We distinguish between two stages of decision-making. In the first stage, an observation center observes the probability p and sends a message $m(p) \in [\underline{m}, \overline{m}]$ to a decision center. The message is subject to random noise captured by a term ε drawn from a non-degenerate distribution on an interval; we view ε as resulting from physical noise within the agent's brain, from a failure to retain information, or from computational errors. The message m and noise ε combine to form the *perceived probability* (or simply the *perception*) $q = c(m, \varepsilon) \in [0, 1]$, where c is continuous, increasing in m, and differentiable in m with a continuous partial derivative. The function c captures both the physical properties of the communication channel and the way in which the decision center decodes the arriving stimulus.[5]

[5] Eschewing explicit modeling of the message received by the decision center and its decoding simplifies notation and implicitly allows for considerable generality in communication. One (less general) explicit model of this process that implies all of our assumptions on $c(\cdot)$ is as follows. The message m sent by the observation center together with the noise ε determine a received message $\tilde{m}(m, \varepsilon) \in [\underline{m}, \overline{m}]$, where $\tilde{m}(\cdot)$ is continuous, increasing in both arguments, continuously differentiable in m, and onto $[\underline{m}, \overline{m}]$. The decision center translates the received message into a perceived probability according to an onto function $\tilde{q} : [\underline{m}, \overline{m}] \longrightarrow [0, 1]$ that is increasing and continuously differentiable. In this case, the messages can be thought of as capturing the intensity of the signal between the two centers.

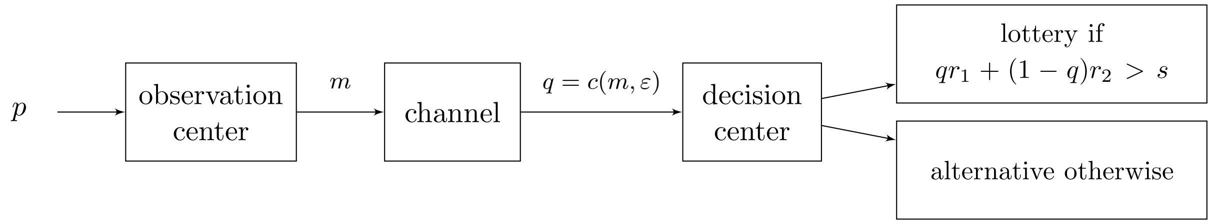


Figure 1: The two-stage decision process with interim noise.

In the second stage, the decision center chooses the lottery if $qr_1 + (1-q)r_2 > s$, and chooses the alternative otherwise.[6] Thus, for any lottery $\ell = (p, r_1, r_2)$ and perception q, the agent receives expected payoff

$$f(\ell, q; s) = \begin{cases} pr_1 + (1-p)r_2 & \text{if } qr_1 + (1-q)r_2 > s, \\ s & \text{otherwise.} \end{cases} \tag{1}$$

Figure 1 summarizes the decision process.

By taking the distribution of ε as fixed, we implicitly assume that Nature cannot reduce the amount of noise. Our interpretation is that Nature has already optimized along this dimension.

The values of the rewards and the alternative are measured in terms of utilities that represent the expected fitness over the agent's lifetime associated with each outcome, and that incorporate risk preferences.[7] Although the function c and the choice rule of the decision center are fixed, we implicitly allow for the possibility that behavior is also optimized at the decision stage as long as, given the perception strategy, the optimal decision rule corresponds to one of the functions c that we consider (see Section 5 for details). Moreover, by not insisting on optimality of c, we allow for—but do not require—the existence of constraints in the evolution of the decision rule.

Draws of p, (r_1, r_2), and ε are independent. In addition, the distribution of (r_1, r_2) is symmetric and p is continuously distributed with a density ψ that is symmetric around 1/2. These symmetry assumptions simplify the characterization of the optimal strategy and capture the idea that the indices of the rewards have no intrinsic meaning.

Nature chooses a *perception strategy* $m(p; s)$, where $m(\cdot; s) : [0, 1] \longrightarrow [\underline{m}, \overline{m}]$, to max-

[6]We make the implicit assumption that rewards are processed without noise only for tractability. In Section 7 we discuss noise in processing of rewards, and speculate about the results of a combined model.

[7]See Robson (2001b) for an elucidation of the connection between utilities and fitness.

imize, for each p, the agent's expected payoff given the distribution over lotteries.[8] An optimal perception strategy $m^*(p;s)$ satisfies

$$m^*(p;s) \in \arg\max_{m\in[\underline{m},\overline{m}]} E\left[f\left((p,r_1,r_2),c(m,\varepsilon)\right)\right] \tag{2}$$

for each p, where the expectation is over the noise ε and the rewards (r_1, r_2). Since the message space is compact and, for each p, the expected payoff is continuous in m, an optimal strategy always exists. If there are multiple optimal strategies then all of our results hold for each such strategy. We therefore ignore potential multiplicity and simply refer to "the" optimal strategy.

Although we take the value s of the alternative to be exogenous here, we argue in Section 5 that it can be thought of as the opportunity cost associated with choosing the lottery. If the agent chooses from a large set of independent lotteries, the opportunity cost—which corresponds to the perceived value of the next-best available lottery—tends to be high relative to the ex ante expected value of any given lottery. We therefore focus primarily on the case in which s is relatively high.

3 Special case

Before analyzing the general model, in this section we illustrate the main result in a relatively simple special case with a particular distribution of rewards and additively separable noise. We relax many of the following assumptions in Section 4.

The rewards r_1 and r_2 are independently drawn from the standard normal distribution. For each message m, the perception is given by $q = m + \varepsilon$, where the noise ε attains values σ and $-\sigma$, each with probability 1/2, and $\sigma \in (0, 1/2)$. To avoid complications due to boundary effects, the density $\psi(p)$ has support $[\sigma, 1-\sigma]$, and the message space is $[\sigma, 1-\sigma]$, ensuring that the perception q is always in $[0, 1]$.

In the absence of noise, the perception optimization problem is trivial: the unbiased perception strategy $m(p) \equiv p$ achieves the first-best. When there is noise, the optimal perception strategy exhibits systematic biases.

Theorem 1. *The optimal perception strategy $m^*(p;s)$ is nondecreasing in p. Furthermore, if $s > 3^{1/4}$, then the agent overstates small probabilities and understates large probabilities;*

[8] When it is not needed, we often drop s from the arguments of $m(\cdot)$ and $f(\cdot)$.

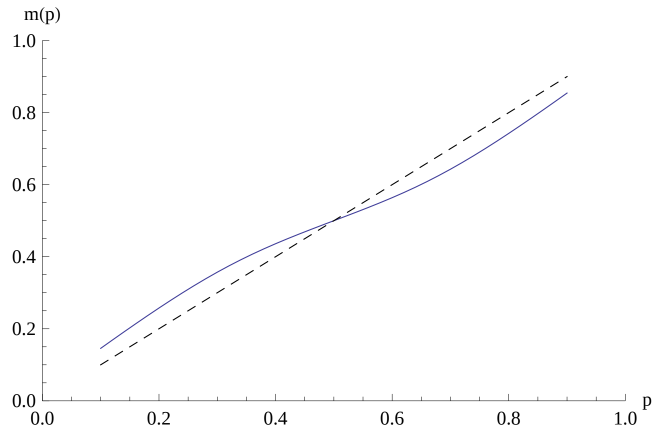


Figure 2: The optimal perception strategy $m^*(p;s)$ (solid curve) for opportunity cost $s=2$ relative to the unbiased strategy $m(p;s) \equiv p$ (dashed line).

that is, for all $p \in [\sigma, 1-\sigma] \setminus \{1/2\}$, $|m^*(p;s) - 1/2| < |p - 1/2|$.

Figure 2 depicts the optimal probability perception for a particular opportunity cost.

Although the strategy $m(p)$ describes internal communication within the agent, the perception is, in principle, observable in an experiment. By varying the rewards, an experimenter can recover the subject's stochastic probability perception $q = m(p) + \varepsilon$ of the objective probability p. The average perception across many repetitions of the experiment is equal to $m(p)$. Theorem 1 therefore indicates that an agent using the optimal strategy m^* will be seen to be overweighting small probabilities and underweighting large ones.[9]

Note that Nature cannot condition the perception of p on the rewards in the lottery. If the perception could depend on rewards, the first-best could be achieved by effectively making the observation center compute the optimal action and then send an extreme message to the decision center indicating which action to take. By requiring the perception strategy $m(p;s)$ to depend only on p and s, we constrain Nature to choose a heuristic that performs well on average across all possible rewards. This approach is consistent with neuroscientific findings that responses to changes in probabilities are associated with activity in regions of the brain different from those that respond to changes in rewards, suggesting that probabilities and rewards are processed separately (see Knutson et al.,

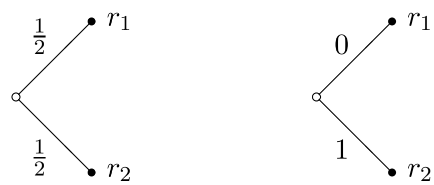
[9] Experimental evidence lends support to idea that probability perception is stochastic. Abdellaoui (2000) finds that, for each objective probability, both under- and over-weighting are present in data; when aggregated, the usual inverted S-shape appears.

2005; Berns et al., 2008; Berns and Bell, 2012).

3.1 Intuition

When does a small change in perception affect choice? If the expected lottery reward and the opportunity cost are far apart, a small perception change does not affect the choice and thus has no impact on outcomes. A marginal change in perception is consequential only when the two alternatives are *perceived to be a tie*: that is, when $qr_1 + (1-q)r_2 = s$. The design of the optimal perception strategy, then, must condition on a tie occurring.

Conditioning on a (perceived) tie tends to increase the weight placed on more extreme probabilities because perceptions q close to 0 or 1 are more likely to lead to a tie than are perceptions close to 1/2. To see this, consider the following two lotteries, labelled with their perceived probabilities:



The perceived value of the first lottery is $(r_1 + r_2)/2$. Ex ante, before the rewards r_1 and r_2 are realized, the value of this lottery is normally distributed with mean 0 and variance 1/2. The perceived value of the second lottery is r_2. Ex ante, the value of the second lottery also has mean 0, but it has a higher variance (equal to 1). When the opportunity cost is high, the higher variance makes a tie with the second lottery more likely than with the first.

More generally, for any given q, the perceived expected reward from the lottery, $qr_1 + (1-q)r_2$, is normally distributed with mean 0 and variance $q^2 + (1-q)^2$. Given q, the likelihood that the agent perceives a tie is $\phi_q(s)$, where ϕ_q is the density for the normal distribution $N(0, q^2 + (1-q)^2)$. Viewed as a function of q and suppressing s from the notation, we define the *weighting function* $w(q)$ to be equal to $\phi_q(s)$. For each q that a message m may lead to, the effect on fitness of a marginal change in m is larger when q is more likely to lead to a tie, and hence greater weight—captured by $w(q)$—must be accorded to those values of q.[10] When $s > 1$, the weight $w(q)$ is U-shaped, as depicted in

[10] In the next subsection, where we derive the optimal perception strategy, we find that the correct weight given to various values of q differs from $w(q)$ by a factor that does not affect the direction of the distortions.

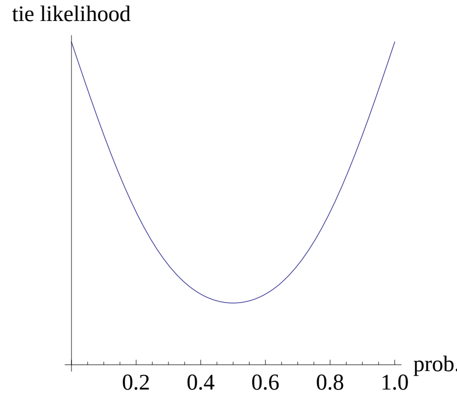


Figure 3: The likelihood of a tie between a lottery and the alternative as a function of the perceived probability for $s = 2$.

Figure 3.

How should the agent distort probabilities in light of the U-shaped weighting function? We show that increasing the steepness of the perception function tends to reduce the effect of errors in perception. One can view this as focusing greater attention on probabilities at which the perception is steeper.[11] For a U-shaped weighting function, more attention should be focused on extreme probabilities than on intermediate ones, suggesting that probabilities should be distorted according to an inverted S-shape, as in Figure 2. In the next subsection, we clarify the trade-off between attentiveness and correctness of perception that determines the optimal perception strategy.

Alternatively, the optimal distortion can be understood by an analogy to the winner's curse. Consider the naïve perception strategy $m(p) \equiv p$. For each p, this strategy leads to unbiased perception of the expected lottery reward in the sense that

$$E[r(p+\varepsilon) - r(p)] = 0,$$

where $r(p) = pr_1 + (1-p)r_2$ and the expectation is over the noise ε. Although the perception is unbiased unconditionally, it *is* biased conditional on the agent perceiving a tie between the lottery and the alternative. In particular, when the opportunity cost is high, one can

[11] See Tversky and Kahneman (1992) for a similar interpretation.

show that, for each p,

$$E[r(p+\varepsilon) - r(p) \mid r(p+\varepsilon) = s] > 0.^{12}$$

In case of a tie, the naïve perception strategy tends to overvalue the lottery because equality with s is more likely to occur if the error ε increases the perceived value of the lottery than if it decreases it.

Relative to the naïve strategy, the optimal strategy decreases the perceived value of the lottery conditional on a tie (this is loosely analogous to bid-shading in common value auctions). It turns out that exaggerating small probabilities (and underreporting large ones) does exactly that. To see how, consider the typical structure of lotteries that the agent perceives as a tie when the opportunity cost is high. One possibility is that the higher probability branch is associated with a large reward, while the lower probability branch has a smaller reward, as in the flight lottery described in the Introduction. Alternatively, as in some casino games, the lottery can have a low probability of a very large reward coupled with a higher probability of a lower reward. Lotteries like the flight lottery are much more common (among ties) because very large rewards are rare. In case of a tie, reducing the perception of high probabilities and exaggerating small ones therefore tends to reduce the perceived value of the lottery, helping to overcome the winner's curse.

An important assumption of our model is that the decision maker loses information about the probability p during the decision process (through the addition of noise); the effects we identify do not arise if instead the noise occurs only *prior to* the initial observation of p. For the sake of comparison, consider an alternative model in which the agent observes a signal $x = p + \varepsilon$ of the true probability p, encodes the signal as a perceived probability q, and then chooses the lottery (p, r_1, r_2) over the alternative s if and only if $qr_1 + (1-q)r_2 > s$. In this case, the optimal strategy is simply to take q to be the posterior expected value of p, that is, $q = E[p \mid x]$.

Why does noise in information processing lead to biased perception while noise in observation does not? In both models, a marginal change in perception of the probability p affects choice only in the event of a tie between the lottery and the alternative. Therefore, in both cases, the optimal perception strategy must condition on a tie occurring; the

[12] Following Shiryaev (1996), given random variables X and Y, we distinguish between the random variable $E[X \mid Y]$ and the function $E[X \mid Y = y]$ of y. In this case, although $E[r(p+\varepsilon) - r(p) \mid r(p+\varepsilon) = s]$ conditions on a zero-probability event, one can define it to be equal to $\lim_{\eta \to 0^+} E[r(p+\varepsilon) - r(p) \mid s - \eta \leq r(p+\varepsilon) \leq s + \eta]$.

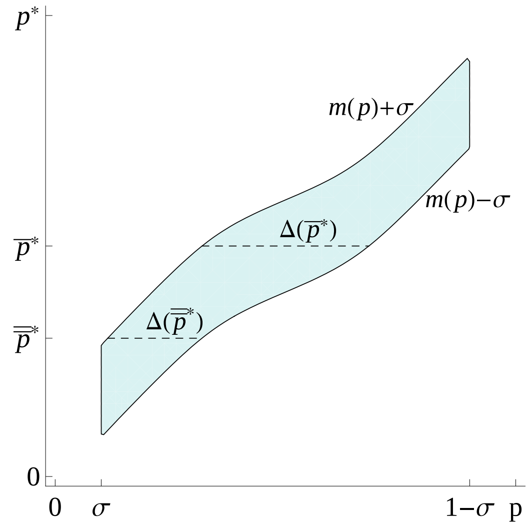


Figure 4: The set D of parameters at which a suboptimal choice may occur. An inverted S-shaped perception strategy m makes D narrower at values of p^* farther from 1/2 (such as $\overline{\overline{p}}^*$) at the expense of making it wider at values close to 1/2 (such as $\overline{p}^*$).

difference between them lies in how this conditioning affects the distribution of noise. In our model, the message is chosen when the agent knows p but not $q = p + \varepsilon$, and the event $qr_1 + (1-q)r_2 = s$ is not independent of ε conditional on p. In the alternative model, the message is chosen when the agent knows $x = p + \varepsilon$, and the event of a tie *is* independent of ε conditional on x.

3.2 Outline of proof

To make the result as transparent as possible, we outline a direct proof of Theorem 1 in this subsection (as opposed to proving it as a corollary of the analogous result for the general model).

Fix s. We begin by identifying those decision problems in which, for a given perception strategy m satisfying $m(p) \in [p-\sigma, p+\sigma]$,[13] the agent may end up choosing suboptimally. Intuitively, this will occur when the value of the lottery is close to the opportunity cost.

Given $r_1 \neq r_2$, let $p^* \in \mathbb{R}$ be the solution to $pr_1 + (1-p)r_2 = s$; that is, given r_1 and r_2, the lottery is the optimal choice whenever p lies on one side of the threshold p^*, and the

[13]Messages outside of $[p-\sigma, p+\sigma]$ are never optimal.

alternative is optimal on the other side (which side depends on which of r_1 or r_2 is greater). A decision problem is difficult in the sense that the agent can choose suboptimally if the parameters (p, p^*) lie in the set

$$D = \{(p, p^*) : p^* \in [m(p) - \sigma, m(p) + \sigma]\} .$$

To see this, consider p^* outside of $[m(p) - \sigma, m(p) + \sigma]$. Since p is within that interval and $q \in \{m(p) - \sigma, m(p) + \sigma\}$, q and p lie on the same side of the threshold p^*, implying that the choice based on q is optimal. On the other hand, if $p^* \in (m(p) - \sigma, m(p) + \sigma)$, then a suboptimal choice occurs for one of the two realizations of ε. Figure 4 illustrates the set D.

Given a strategy $m(\cdot)$, define the ex ante expected loss

$$L = E\left[\max\left\{pr_1 + (1-p)r_2, s\right\} - f\left(\ell, m(p) + \varepsilon\right)\right],$$

where the expectation is over the lottery $\ell = (p, r_1, r_2)$ and the noise ε. The loss L measures how much the agent's expected reward falls below the first-best that can be attained in the absence of noise. The following lemma expresses the loss L as a weighted integral over the set D. For the correct weights, we must adjust the weighting function from the preceding subsection to account for the magnitude of loss due to perception errors, which depends on the size of the gap between the values of the two rewards in the lottery. The larger $|r_1 - r_2|$ is, the more sensitive the expected value of the lottery is to changes in the probability p, making errors in perception of the probability more costly. This effect can be accounted for by viewing the rewards as being drawn from a modified density that assigns greater weight to pairs that are farther apart. Accordingly, let $\tilde{\rho}(\cdot, \cdot)$ be the probability density defined by $\tilde{\rho}(r_1, r_2) = (r_1 - r_2)^2 \phi(r_1)\phi(r_2)/2$, where $\phi(\cdot)$ is the standard normal density. For each $q \in [0, 1]$, let $d_q(\cdot)$ be the density of $r(q) = qr_1 + (1-q)r_2$, where the pair (r_1, r_2) is drawn according to $\tilde{\rho}(r_1, r_2)$. The weighting function is defined to be $\pi(q; s) = d_q(s)$; that is, $\pi(q; s)$ is the likelihood of a tie between the lottery and the alternative when the rewards are drawn according to $\tilde{\rho}$.[14]

Lemma 1. *The expected loss satisfies* $L = \frac{1}{2}\int_D |p^* - p|\psi(p)\pi(p^*)dpdp^*$.

Given a threshold probability p^*, the agent suffers a large loss when (i) p^* is likely to

[14]To see how this relates to the function $w(q)$ from the last subsection, note that $\pi(q) = w(q)E[(r_1 - r_2)^2 \mid r(q) = s]$.

generate a tie, and (ii) r_1 and r_2 tend to be far apart, making the value of the lottery sensitive to the probabilities. Both of these effects are built in to the weighting function, the first through the $\phi(r_1)\phi(r_2)$ term, and the second through the $(r_1 - r_2)^2$ term in the density $d_q(\cdot)$. Combining these effects, the lemma indicates that the loss tends to be small when the set D is both narrow and adheres closely to the diagonal.

Figure 4 illustrates how the slope of the perception strategy affects the loss L. The set D is narrow precisely when $m(p)$ is steep. Thus the inverted S-shaped perception strategy depicted in Figure 4 performs well toward the extremes at the expense of poorer performance at intermediate probabilities. If perception errors at intermediate probabilities generate smaller losses than those at more extreme probabilities then this leads to an overall gain. The following lemma confirms that this is indeed the case.

Lemma 2. *If $s > 3^{1/4}$, then the weighting function $\pi(q)$ is U-shaped: it is decreasing for $q < 1/2$, increasing for $q > 1/2$, and symmetric with respect to $q = 1/2$.*

To derive the optimal perception strategy, note that the integral in Lemma 1 can be minimized pointwise with respect to p. Thus, for each $p \in [\sigma, 1-\sigma]$, the optimal message satisfies

$$m^*(p) \in \arg\min_m \int_{m-\sigma}^{m+\sigma} |p^* - p| \pi(p^*)\, dp^*.$$

Taking the first-order condition with respect to m proves the following characterization of the optimal strategy.

Lemma 3. *The optimal perception error $q - p$, weighted by $\pi(q)$, is unbiased; that is, for each p,*

$$E[(q-p)\pi(q)] = \sum_{\varepsilon \in \{-\sigma, \sigma\}} (m^*(p) + \varepsilon - p)\pi(m^*(p) + \varepsilon) = 0. \tag{3}$$

Theorem 1 follows from Lemmas 2 and 3. To see that the U-shaped weight implies that it is optimal to exaggerate small probabilities, consider $p < 1/2$ and $s > 3^{1/4}$. Suppose the observation center sends the unbiased message $m = p$, so that the perception is either $p - \sigma$ or $p + \sigma$. A marginal increase in m increases the loss by $\sigma\pi(p + \sigma)$ if the error is σ, and decreases the loss by $\sigma\pi(p - \sigma)$ if the error is $-\sigma$. Since $\pi(p - \sigma) > \pi(p + \sigma)$, increasing the message reduces the expected loss.

By symmetry, for any s, the optimal perception $m^*(p; -s)$ is identical to $m^*(p; s)$.[15] The agent therefore exhibits an inverted S-shaped perception bias whenever $s \notin [-3^{1/4}, 3^{1/4}]$.

[15] To see this, note that $d_p(\cdot)$ is symmetric around 0, and hence $\pi(p; s) \equiv \pi(p; -s)$ for each s. Lemma 3 therefore implies that $m^*(p; s) \equiv m^*(p; -s)$.

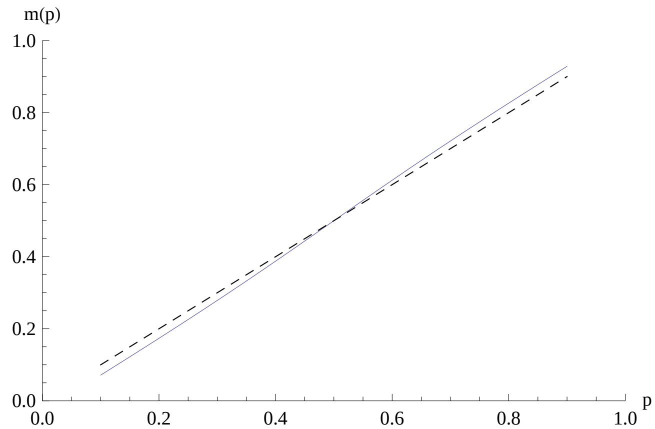


Figure 5: The optimal perception strategy $m^*(p;s)$ (solid curve) for opportunity cost $s = 0$ relative to the unbiased strategy $m(p;s) \equiv p$ (dashed line).

For intermediate opportunity costs, although the optimal perception may not exhibit the inverted S-shape, it always differs from the unbiased strategy; whatever the opportunity cost, conditioning on a tie leads to a nontrivial weighting function. Figure 5 illustrates this point for $s = 0$. We focus on the case when s is large because, as highlighted in Section 5, the opportunity cost tends to be high when the agent faces a large choice set.

4 The general case

In this section, we return to the general model from Section 2. Compared to the special case of Section 3, we now allow for a general continuous joint distribution of rewards with continuous density $\rho(r_1, r_2)$ and finite third moments, and for general perception formation $q = c(m, \varepsilon)$. The perception is nontrivially stochastic in the sense that for every m and q, $\Pr(c(m, \varepsilon) = q) < 1$.

The additional generality in the perception formation demonstrates that the pattern of distortions identified in the special case is not driven by the naïveté of the decision center. In Section 3, the decision center interprets the received message $m + \varepsilon$ at face value, failing to take into account the messaging strategy $m(\cdot)$ employed by the observation center. The general model allows for (but does not require) a decision center that correctly interprets the message, taking into account how the observation center codes the probability. See Section 5 for a detailed example in a closely related setup.

In the general formulation of the model, messages are no longer directly comparable to probabilities. Instead, we compare the optimal perception under two objectives. In the *reward maximization problem*, defined in (2), the optimal strategy $m^*(p; s)$ maximizes the agent's ex ante expected reward. We use as a benchmark the *precision maximization problem*, in which the optimal strategy $\hat{m}(p)$ minimizes the mean square error in perception;[16] that is, for each $p \in [0,1]$,

$$\hat{m}(p) \in \arg\min_m E\left[(c(m,\varepsilon) - p)^2\right].$$

The precision-maximizing perception is a natural generalization of the unbiased perception strategy that we use as the benchmark in Section 3: when noise in communication is additive, the mean square error is minimized by unbiased perception.[17]

The analysis in Section 3 makes use of the weight $\pi(q)$ that measures the importance of precision at each perceived probability q. We extend the construction of the weighting function to the general case as follows. Without loss of generality, we normalize $E[(r_1 - r_2)^2]$ to 1, and define a density $\tilde{\rho}(r_1, r_2) = (r_1 - r_2)^2 \rho(r_1, r_2)$. For each $q \in [0,1]$, let $r(q) = qr_1 + (1-q)r_2$, where the pair (r_1, r_2) is drawn according to $\tilde{\rho}$, and let $d_q(\cdot)$ be the density of $r(q)$. The weighting function is defined to be $\pi(q; s) = d_q(s)$.

As in Section 3, the shape of the optimal perception strategy is closely connected to the shape of the weighting function. Since $r(q)$ is a weighted average of rewards with weights q and $1-q$, its distribution is relatively concentrated when q is close to 1/2 in a sense made precise by the following lemma.

Lemma 4. *If $|q - 1/2| > |q' - 1/2|$ then $r(q)$ is a mean-preserving spread of $r(q')$.*[18,19]

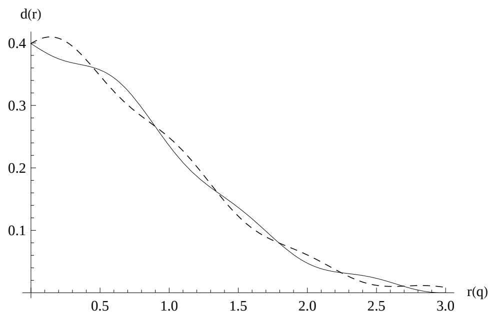
Intuitively, Lemma 4 suggests that, if the opportunity cost is sufficiently high, it should be more likely to tie with $r(q)$ than with $r(q')$ (at least for "well-behaved" distributions), thus giving rise to a U-shaped weighting function. Formalizing this intuition requires some additional technical assumptions.

[16] Because the message space is compact and the objective functions are continuous, solutions to the two problems are guaranteed to exist. However, there may be multiple optima, making the functions m^* and $\hat{m}$ not uniquely defined. Our results hold for any optimal pair of m^* and $\hat{m}$.

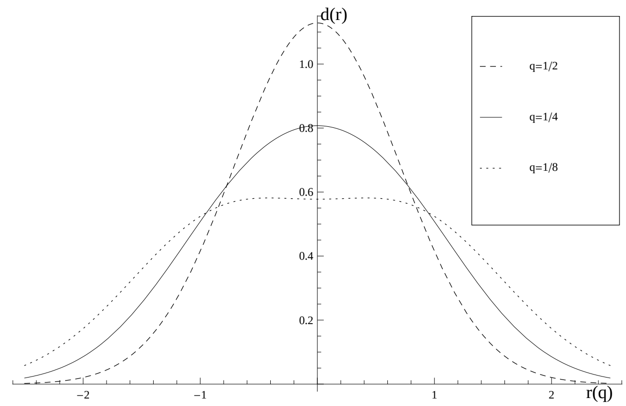
[17] In a related setting, Woodford (2012a,b) takes precision maximization as the objective of perception.

[18] See Appendix B for the definition of mean-preserving spread that is appropriate in this context.

[19] This lemma is related to a result in Gossner and Kuzmics (2015). In studying the performance of choice rules when the decision-maker is ignorant about payoffs, they show that for any choice rule that cannot be supported by strict preferences there exists a mixture over choice rules supported by strict preferences delivering gains that are a mean-preserving spread of the gains under the first rule.



(a) A possible violation of A2: the solid and dashed curves represent the densities of $r(q)$ and $r(q')$, respectively, and intersect infinitely often above any r.



(b) The distribution of $r(q)$ for independent standard normal rewards. For each pair q and q', the densities intersect twice, and the intersection points are bounded above across all such pairs (by $3^{1/4}$).

Figure 6: Examples illustrating A2.

For each message m, let $\overline{q}(m) = \sup_\varepsilon c(m, \varepsilon)$ and $\underline{q}(m) = \inf_\varepsilon c(m, \varepsilon)$ denote the most extreme perceptions. We assume:

A1 The extremes cover the full range from 0 to 1, that is, $\underline{q}(\underline{m}) = 0$ and $\overline{q}(\overline{m}) = 1$.

A2 There exists a finite upper bound r^* such that for any $q, q' \in [0, 1/2]$ such that $q \neq q'$, the densities of the random variables $r(q)$ and $r(q')$ do not intersect above r^*.

Assumption A1, together with the continuity of $c(\cdot)$, implies that all perceived probabilities can occur for some combination of m and ε. This assumption simplifies the statement of our main result, but is otherwise unimportant; without it, Theorem 2 would hold except that the interval of probabilities on which we obtain a strict comparison between the reward-maximizing and precision-maximizing perceptions would be smaller.

Assumption A2 is a regularity condition ensuring that the tails of the densities of $r(q)$ are well ordered across different values of q, which in turn guarantees that the weighting function $\pi(q; s)$ is U-shaped for all $s > r^*$. The condition—which is needed only if the distribution of rewards has unbounded support—rules out densities like those depicted in Figure 6a that, for some pair q and q', alternate infinitely often to the right of any point.

In addition, it requires that there is an upper bound on intersections that is uniform across q and q'. Although the uniformity requirement is difficult to interpret, we believe that it is a mild restriction; we have verified that it holds for any bivariate normal distribution of rewards, and for independent rewards that have an exponential or Pareto distribution.[20] We have not found a distribution that violates the condition. Figure 6b illustrates the condition for normally distributed rewards.

Define the reward-maximizing perception $q^* = c\left(m^*(p; s), \varepsilon\right)$ and the precision-maximizing perception $\hat{q} = c\left(\hat{m}(p), \varepsilon\right)$. The following result extends Theorem 1 to the general setting, and indicates that, when the opportunity cost is sufficiently high, the optimal perception of small probabilities is biased upward and that of large probabilities is biased downward relative to the precision-maximizing perception.

Let $\sigma = \sup_m \overline{q}(m) - \underline{q}(m)$, which can be viewed as a measure of the degree of noise in information processing.

Theorem 2. *For any opportunity cost s, the optimal message function $m^*(p; s)$ is nondecreasing in p. Furthermore, if $s > r^*$ and $p \in [0, 1]\setminus(1/2-\sigma, 1/2+\sigma)$, then $|m^*(p; s) - 1/2| \leq |\hat{m}(p) - 1/2|$, with a strict inequality if, in addition, $p \in (\sigma, 1 - \sigma)$.*

The set of probabilities for which the theorem identifies a nonzero bias is largest when the noise in information processing is small, with the bias being nonzero on a nonempty set of probabilities as long as $\sigma < 1/4$.

The remainder of this section outlines two lemmas that form the main steps in the proof of Theorem 2. The next lemma shows that the first-order conditions of the two optimization problems differ only in the weight attributed to various perception errors. Let $c_m = \frac{\partial c}{\partial m}$.

Lemma 5. *For any $p \in [0, 1]$ such that $m^*(p)$, $\hat{m}(p) \in (\underline{m}, \overline{m})$, we have the first-order conditions*

$$\begin{aligned} E\left[\pi(q^*)(p - q^*)c_m(m^*(p), \varepsilon)\right] &= 0, \\ \textit{and} \quad E\left[(p - \hat{q})c_m(\hat{m}(p), \varepsilon)\right] &= 0. \end{aligned}$$

As in Section 3, the weighting function is U-shaped.

[20] In addition, we note that the uniformity requirement would not be necessary if q could take on only finitely many values.

Lemma 6. *For all $s > r^*$, $\pi(q; s)$ is decreasing for $q < 1/2$ and increasing for $q > 1/2$.*

The last lemma generalizes the intuition from Section 3.1 based on the observation that, when the opportunity cost s is high, extreme probabilities are more likely to generate ties with s. In that section, we show that the density of the expected reward $r(q)$ becomes more spread out as q moves farther from $1/2$, which extends more generally to the mean-preserving spread comparison in Lemma 4. Combined with the regularity condition A2, this implies that, if $|q - 1/2| > |q' - 1/2|$, then $r(q)$ has a thicker tail than $r(q')$, making $r(q)$ more likely to tie with the alternative when the opportunity cost is high, which in turn implies the lemma and hence Theorem 2.

5 Endogenous opportunity cost

In this section, we analyze a variant of the model that differs from the main model in several respects. First, instead of choosing between a single lottery and a fixed alternative, the agent chooses a subset from a large set of lotteries (with no alternative). This difference highlights that, consistent with our terminology, the value of the alternative in the main model can be thought of as the opportunity cost associated with choosing the lottery—in this case, the value of the marginal lottery that is not chosen. Second, instead of taking the decision rule as fixed, we optimize at both the observation and decision stages. This difference illustrates that our main result does not rely on constraints at the decision stage. Finally, for the sake of tractability, we replace noise in perception with finiteness of the message space. At the end of this section, we discuss the difficulty involved in obtaining this result in the model with noise, and describe a monotonicity restriction under which it holds in that case.

The individual chooses from the set of all lotteries $\ell = (p, r_1, r_2) \in [0, 1] \times \mathbb{R}^2$, which is equipped with the product measure, χ, associated with a uniformly distributed probability p and independent standard normal distributions of rewards r_1 and r_2. The agent's problem is to choose a fixed fraction $\kappa \in (0, 1)$ of the available lotteries; more precisely, the agent selects a measurable subset W of lotteries satisfying $\chi(W) = \kappa$. To approximate the choice of one lottery out of a large set of options, we focus on the case where κ is small.[21]

As in the main model, decision-making occurs in two stages. At the first stage, the observation center associates to each probability p a message from a finite set M (with

[21] Modeling the choice as a fraction of a continuum (as opposed to a single option from a large finite set) simplifies the analysis by eliminating stochasticity in the choice set.

$|M| \geq 3$) to be sent to the decision center. Accordingly, a strategy for the observation center is a measurable function $m : [0,1] \longrightarrow M$, with $m(p)$ interpreted as the message sent to the decision center about each lottery having probability p. The decision center observes, for each lottery (p, r_1, r_2), the values of the rewards r_1 and r_2, and the message $m(p)$ (but not the probability p itself). A strategy for the decision center consists of a measurable function $g : M \times \mathbb{R}^2 \longrightarrow \{0,1\}$ satisfying $\chi(g^{-1}(1)) = \kappa$, with the interpretation that $g(m(p), r_1, r_2) = 1$ if and only if the lottery (p, r_1, r_2) is in the chosen set. We refer to any such function $g(\cdot)$ as a *choice function*.

Nature chooses $m(\cdot)$ and $g(\cdot)$ to maximize the agent's expected reward over the set of lotteries that are chosen. That is, Nature solves

$$\begin{aligned} \max_{m(\cdot), g(\cdot)} \; & E\left[pr_1 + (1-p)r_2 \mid g(m(p), r_1, r_2) = 1\right] \\ \text{s.t.} \quad & \chi(g^{-1}(1)) = \kappa. \end{aligned} \tag{4}$$

Given a message function $m(\cdot)$, we say that the choice function $g(\cdot)$ is *Bayesian* if

$$g(m, r_1, r_2) = \begin{cases} 1 & \text{if } E[p \mid m]r_1 + (1 - E[p \mid m])r_2 \geq s, \\ 0 & \text{otherwise,} \end{cases}$$

where s solves

$$\Pr(E[p \mid m(p)]r_1 + (1 - E[p \mid m(p)])r_2 \geq s) = \kappa. \tag{5}$$

Thus a Bayesian choice function selects the subset of lotteries having the highest expected value conditional on the information conveyed by the message. Note that s is equal to the perceived expected value of the marginal lottery.

As in Section 4, we measure biases in perception relative to the precision-maximizing strategy for the observation center (keeping the strategy of the decision center fixed). In the present framework, given any strategy for the observation center, there is only a finite set of Bayesian posterior expectations of p that the decision center could have (one for each message it could receive). We measure precision relative to that set of posterior expectations, which we denote by Q; thus

$$Q = \{E[p \mid m(p) = m] : m \in M\}.$$

For each p, let

$$\hat{q}(p) \in \arg\min_{q \in Q} (q-p)^2$$

denote the precision-maximizing perception. With this benchmark, any bias in perception—that is, any difference between $\hat{q}(p)$ and $E[p \mid m^*(p)]$—could be eliminated through a change only in the strategy of the observation center.

Proposition 1. *There exists a solution* $(m^*(\cdot), g^*(\cdot))$ *of problem* (4) *for which* $E[p \mid m^*(p)]$ *is non-decreasing in* p*. In every solution, the choice function* $g^*(\cdot)$ *is Bayesian. Moreover, if* κ *is sufficiently small, the individual overvalues small probabilities and undervalues large probabilities. That is, there exists* $\kappa^* > 0$ *such that whenever* $\kappa < \kappa^*$*,*

$$E[p \mid m^*(p)] \geq \hat{q}(p) \tag{6}$$

for almost every $p \in [0, 1/2]$*, with a strict inequality almost everywhere on an open subset of* $[0, 1/2]$*. The symmetric statements hold for* $p \in [1/2, 1]$*.*

Numerical results indicate that a value of κ^* approximately equal to 0.05 is sufficiently small for the conclusion of the proposition to hold, in other words, inverted S-shaped distortions arise when the agent is choosing less than 5% of the available options.

This proposition shows how the value of the alternative in the main model can be interpreted as an opportunity cost in a problem that involves choosing from multiple lotteries: the optimal strategy for the decision center selects a lottery if and only if its perceived expected value exceeds the opportunity cost s, and consequently distortions take on the same inverted S-shape as in the main model.

Although the choice function in Proposition 1 differs from the simple non-Bayesian rule considered in Section 3, the intuition in terms of attention allocation carries over. In both cases, the observation center anticipates errors in perception of probabilities and focuses on marginal lotteries (what we refer to as "ties" in the binary choice framework of Sections 3 and 4). When κ is small, these marginal lotteries have relatively high expected rewards, and therefore tend to have probabilities close to 0 and 1 for the same reason as in the binary case for lotteries that are perceived to tie with a high opportunity cost. Because the agent focuses on distinguishing among these high value lotteries, the optimal perception strategy allocates greater attention to more extreme probabilities. In this context, that means messages sent for probabilities close to 0 and 1 are associated with smaller intervals of probabilities than those sent for probabilities closer to 1/2. This in turn leads to overestimation of small probabilities and underestimation of large ones: for example, a small probability just above the threshold between two messages will be closer to the expected probability associated with the lower message than that with the message it generates.

As noted above, working with a finite message space instead of noisy communication simplifies the analysis. The main difficulty with the noisy model lies in how the distributions of posterior beliefs q vary with the message m. The proof of Proposition 1 relies on a monotone comparative statics result that is applicable only if these distributions can be ordered by first-order stochastic dominance. With a finite message space and no noise, such an ordering obtains trivially because the posterior is a deterministic function of m. With noise, although we conjecture that this property holds quite generally, it appears to be difficult to prove. However, our proof of Proposition 1 extends to the case of noisy communication if we restrict the observation center's set of strategies so as to ensure that the distribution of posteriors is increasing in m (in the sense of first-order stochastic dominance).[22]

6 Debiasing

The gap between the normative basis of expected utility theory and the descriptive origins of prospect theory (Thaler, 2000) has spurred an ongoing debate on "debiasing". As Fischhoff (1982) writes:

> Once judgmental biases are identified, researchers start trying to eliminate them using one of two strategies. The first accepts the existence of the bias and concentrates on devising schemes, such as training programs, that will reduce it.

Jolls and Sunstein (2006) argue that many laws are designed to counteract behavioral biases.

This paper offers a normative foundation for biased perception of probabilities as an optimal response to constraints in information processing. The optimizing role of biased perception suggests that caution is warranted when considering whether deviations from

[22]We have succeeded in verifying the necessary monotonicity property in a variant of the model with endogenous noise, where the observation center chooses, for each p, the distribution of the message m that will be observed by the decision center, and pays a cost proportional to the mutual information between m and p.

expected utility should be eliminated; removing biases across all decision problems would harm the decision-maker.

Our theory suggests that probability biases are helpful in certain settings. For example, Barseghyan et al. (2013) document overvaluation of small probabilities using data on insurance deductible choices; clients who overweight low-probability losses prefer smaller deductibles than would unbiased decision-makers. De Giorgi and Legg (2012) explain the equity premium puzzle by pointing out that agents with prospect theory preferences overvalue the probability of rare market crashes. If errorless perception were possible, probability biases would be harmful in these cases (relative to correct perception). If, however, perception is noisy, then the observed biases can be beneficial in problems with a low probability of generating a loss. Overvaluation of small probabilities in those problems can be understood as a kind of cautiousness, without which the agent would select the risky action too often when perception errors reduce the perceived likelihood of a loss.

This is not to say that decision making cannot be improved upon. Using horse-race data, Thaler and Ziemba (1988) and Snowberg and Wolfers (2010) document excessive betting on low probability events that pay large rewards, leading to expected losses. While our theory suggests that overweighting of small probabilities can be helpful overall, it can also be harmful in settings where the distribution of lotteries differs from that faced by the decision-maker across all problems.

Why then does overvaluation of small probabilities in settings like the racetrack persist? When probabilities and rewards are processed separately, Nature must design the optimal bias across all types of problems. From the ex ante perspective, the distortion is optimal; ex post, it can be harmful. Put differently, debiasing may be beneficial in certain circumstances, but only in those that, from an evolutionary perspective, rarely result in a tie.

7 Extensions

In this paper, we examine noise in evaluation of probabilities, while assuming for tractability that rewards are processed perfectly. In a previous version of the paper (Steiner and Stewart, 2014), we examine two variations of a model involving a choice between a random reward and a status quo option in which information about the reward is processed with noise. To focus more attention on rewards that are more likely to lead to a tie with the status quo, the optimal perception strategy is steeper around the average status quo, leading to an S-shaped perception of rewards (similar to the S-shaped value function used in prospect theory). If the observation center is restricted to a simple class of strategies that introduce a constant bias, the optimal strategy favors the status quo when the status quo is better than the average reward; in other words, perception introduces a status quo bias. Combining noise in both probabilities and rewards within a single model may lead to interesting interaction effects but appears to be intractable. Since Herold and Netzer (2010) show that inverted S-shaped probability weights are an optimal response to an S-shaped value function, we conjecture that a combined model would strengthen the degree of probability weighting.

Our analysis precludes the possibility that perception of probabilities depends on the value of rewards. At the other extreme, where perception can depend on the exact realized rewards, the first-best can be attained. But what if perception depends on imperfect information about realized rewards? We conjecture that this would introduce a difference between perception of probabilities of gains and those of losses. In addition to the inverted S-shape, the agent would tend to put less weight on the probability of gains (and more on the probability of losses) since doing so would help to correct for the winner's curse. This conjecture is broadly consistent with experimental results on reference-dependent probability weights (Tversky and Kahneman, 1992).

Another possibility is that noise in perception could be tailored to decision problem; when the expected rewards are larger, it seems natural to think that the agent would devote more mental resources to getting the choice right, corresponding in our model to a reduction in noise. We conjecture that such a reduction leads to less distortion in probability weights relative to the unbiased perception strategy. In keeping with this conjecture, Fehr-Duda et al. (2010) find that experimental subjects exhibit more conservative probability weights at higher stakes.

Our restriction to binary prospects is again due to issues of tractability. Allowing for more outcomes leads to interdependencies in the optimal perception that significantly complicate the problem. We expect that optimal perception in the general case would share some features of cumulative prospect theory, with inverted S-shaped probability weights that depend on the relative magnitudes of the probabilities in the lottery.

The problems we study in this paper can be viewed as special cases of a cheap talk game between a sender and receiver with common interests in which the receiver has private information and communication is subject to noise.[23] Our results indicate that the optimal equilibrium in such problems is generally nontrivial; the sender's choice must condition on the message being pivotal given the receiver's private information. We leave further exploration of this class of problems to future research.

A Proofs for Section 3

Proof of Lemma 1. Write the expected loss L as

$$\frac{1}{2}\int_{\sigma}^{1-\sigma}\int_{\{(r_1,r_2):(p,p^*)\in D\}}|p^*-p||r_1-r_2|\phi(r_1)\phi(r_2)dr_1dr_2\psi(p)dp.$$

For each p, the inner integral is over the set of pairs (r_1, r_2) for which a suboptimal choice can occur. When a suboptimal choice occurs, the difference in expected reward between the lottery and the alternative is $|p^*-p||r_1-r_2|$. The factor $1/2$ reflects that the suboptimal choice occurs only for one of the two possible realizations of ε.

Consider the substitution $(p^*,\Delta)=(p^*,r_1-r_2)=\left(\frac{s-r_2}{r_1-r_2},r_1-r_2\right)$ in the inner integral. The absolute value of the determinant of the Jacobian matrix associated with this substitution is $\frac{1}{|r_1-r_2|}=\frac{1}{|\Delta|}$. Therefore, the expression for L becomes

$$\frac{1}{2}\int_{\sigma}^{1-\sigma}\int_{m(p)-\sigma}^{m(p)+\sigma}|p^*-p|\pi(p^*)dp^*\psi(p)dp,$$

as needed. □

Proof of Lemma 2. First note that $\pi(q)=\lambda(q)w(q)$, where $\lambda(q)=E[(r_1-r_2)^2 \mid qr_1+(1-q)r_2=s]$. We begin by computing $\lambda(q)$. Write the random variable r_1-r_2 as $a\zeta+b\zeta^{\perp}$ where $\zeta=qr_1+(1-q)r_2$ and $\zeta^{\perp}=-(1-q)r_1+qr_2$. Note that ζ and $\zeta^{\perp}$ are independent. Comparing the coefficients, we obtain

$$a=-\frac{1-2q}{(1-q)^2+q^2},\; b=-\frac{1}{(1-q)^2+q^2}.$$

Conditional on $\zeta=s$, the random variable r_1-r_2 has mean as and variance $V=$

[23] We thank Andy Postlewaite for highlighting this connection. See Blume et al. (2007) for an analysis of cheap talk with a different form of noise.

$b^2 \mathrm{Var}(\zeta^{\perp}) = b^2((1-q)^2 + q^2)$. Thus we have

$$\lambda(q) = E\left[(r_1 - r_2)^2 \mid \zeta = s\right] = (as)^2 + V = \frac{q^2 + (1-q)^2 + s^2(1-2q)^2}{\left(q^2 + (1-q)^2\right)^2}.$$

Multiplying the last expression by $w(q) = \phi_q(s)$ gives

$$\pi(q; s) = \frac{q^2 + (1-q)^2 + s^2(1-2q)^2}{\left(q^2 + (1-q)^2\right)^2} \frac{1}{\sqrt{q^2 + (1-q)^2}} \phi\left(\frac{s}{\sqrt{q^2 + (1-q)^2}}\right),$$

which is symmetric around $q = 1/2$.

Let $y = \frac{1}{\sqrt{q^2+(1-q)^2}}$ and note that y is increasing in q and attains values in $(1, \sqrt{2}]$ if $q \in (0, 1/2]$. Therefore, it suffices to prove that

$$\pi(q(y); s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2 y^2}{2}} y^3 (1 + 2s^2 - s^2 y^2)$$

is decreasing in y on $(1, \sqrt{2}]$. Differentiating gives

$$\frac{\partial \pi(q(y); s)}{\partial y} = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2 y^2}{2}} y^2 \left[s^4 y^4 - 2(s^2 + 3)s^2 y^2 + 3(2s^2 + 1)\right].$$

This derivative is negative if the expression in the square brackets is negative, which is the case whenever

$$y^2 \in \left(\frac{s^2 + 3 - \sqrt{s^4 + 6}}{s^2}, \frac{s^2 + 3 + \sqrt{s^4 + 6}}{s^2}\right). \tag{7}$$

For $s > 3^{1/4}$, this interval contains $[1, 2]$, and therefore (7) holds for $y \in (1, \sqrt{2}]$. □

Proof of Theorem 1. We first prove monotonicity of $m^*(p; s)$. For $m \in [p - \sigma, p + \sigma]$, define the loss $l(m, p) = \int_{m-\sigma}^{m+\sigma} |p^* - p| \pi(p^*) dp^*$. Then $m^*(p)$ maximizes $-l(m, p)$. Given $m, m' \in [p - \sigma, p + \sigma]$ such that $m' > m$, we have

$$\begin{aligned} -l(m', p) + l(m, p) &= -\int_{m+\sigma}^{m'+\sigma} |p^* - p| \pi(p^*) dp^* + \int_{m-\sigma}^{m'-\sigma} |p^* - p| \pi(p^*) dp^* \\ &= -\int_{m+\sigma}^{m'+\sigma} (p^* - p) \pi(p^*) dp^* - \int_{m-\sigma}^{m'-\sigma} (p^* - p) \pi(p^*) dp^*, \end{aligned}$$

where the latter equation follows because $m' - \sigma \leq p \leq m + \sigma$. The partial derivative of this expression with respect to p is positive, and therefore, by Theorem 1 in Van Zandt (2002), $m^*(p; s)$ is nondecreasing in p.

Consider $p \in [\sigma, 1/2)$. Suppose for contradiction that $m^* \leq p$ (here we suppress the argument of $m^*(p)$ from the notation). Then, by Lemma 2, $\pi(m^* - \sigma) > \pi(m^* + \sigma)$, and hence

$$\begin{aligned} E[\pi(m^* + \varepsilon)(m^* + \varepsilon - p)] &= \frac{1}{2}\pi(m^* + \sigma)(m^* + \sigma - p) + \frac{1}{2}\pi(m^* - \sigma)(m^* - \sigma - p) \\ &< \pi(m^* + \sigma)\left(\frac{1}{2}(m^* + \sigma - p) + \frac{1}{2}(m^* - \sigma - p)\right) \\ &= \pi(m^* + \sigma)\,(m^* - p) \leq 0, \end{aligned}$$

violating (3) in Lemma 3. Therefore, $m^* > p$. The argument for $p \in (1/2, 1 - \sigma]$ is analogous. ☐

B Proofs for Section 4

The proofs in this section follow the order in which the results appear in the main text. In particular, the proofs of Lemmas 5 and 6 appear below the proof of Theorem 2.

Definition 3. Let X_1 and X_2 be real-valued random variables with distribution functions F_1 and F_2, respectively. We say that X_1 is a *mean-preserving spread* of X_2 if

1. $E[X_1] = E[X_2]$, and
2. for every $x \in \mathbb{R}$,

$$\int_{-\infty}^{x} F_1(X)dX \geq \int_{-\infty}^{x} F_2(X)dX.\text{[24]} \tag{8}$$

Proof of Lemma 4. For each r_1 and r_2, define $r^+(p) = pr_1 + (1-p)r_2$, and $r^-(p) = (1-p)r_1 + pr_2$. Let $\hat{\ell}(p, r_1, r_2)$ be the binary lottery $(r^+(p), 1/2; r^-(p), 1/2)$. The symmetry of $\tilde{\rho}$ implies that $r(p)$ is equivalent in distribution to the compound lottery in which (r_1, r_2) is drawn from $\tilde{\rho}$, and then $r(p)$ is drawn from $\hat{\ell}(p, r_1, r_2)$.

Note that, for each draw of (r_1, r_2), the binary lotteries $\hat{\ell}(p, r_1, r_2)$ and $\hat{\ell}(p', r_1, r_2)$ have the same mean $(r_1+r_2)/2$, and, if $|p-1/2| > |p'-1/2|$, $r^+(p')$ and $r^-(p')$ lie in between $r^+(p)$

[24] There is some disagreement in the literature on this terminology. Rothschild and Stiglitz (1970) and Müller (1998) use the term "mean-preserving spread" more narrowly; what we call a mean-preserving spread, Müller (1998) calls a mean-preserving ϕ-increase in risk.

and $r^{-}(p)$. Hence $\hat{\ell}(p, r_1, r_2)$ is a mean-preserving spread of $\hat{\ell}(p', r_1, r_2)$. Thus it suffices to prove the following general claim. Let X be a random variable with distribution function $F(\cdot)$. Let $Y_1(X)$ and $Y_2(X)$ be random variables with distribution functions $G_1(X)(\cdot)$ and $G_2(X)(\cdot)$, respectively, such that, for each X, $Y_1(X)$ is a mean-preserving spread of $Y_2(X)$. Finally, let Z_i be the random variable obtained by composition of X and Y_i (so that Z_i has distribution function $H_i(Z) = \int_{-\infty}^{\infty} G_i(X)(Z)dF(X)$). Then Z_1 is a mean-preserving spread of Z_2.

To prove this claim we need to show that (i) Z_1 and Z_2 have the same mean, and (ii) $\int_{-\infty}^{t} H_1(Z)dZ \geq \int_{-\infty}^{t} H_2(Z)dZ$ for every t.

For (i),

$$E[Z_1] = \int_{-\infty}^{\infty} E[Y_1(X)]dF(X) = \int_{-\infty}^{\infty} E[Y_2(X)]dF(X) = E[Z_2].$$

For (ii),

$$\begin{aligned}
\int_{-\infty}^{t} H_1(Z)dZ &= \int_{-\infty}^{t} \int_{-\infty}^{\infty} G_1(X)(Z)dF(X)dZ \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{t} G_1(X)(Z)dZdF(X) \\
&\geq \int_{-\infty}^{\infty} \int_{-\infty}^{t} G_2(X)(Z)dZdF(X) \\
&= \int_{-\infty}^{t} \int_{-\infty}^{\infty} G_2(X)(Z)dF(X)dZ \\
&= \int_{-\infty}^{t} H_2(Z)dZ,
\end{aligned}$$

where the inequality follows from $Y_1(X)$ being a mean-preserving spread of $Y_2(X)$ for each X, and the equations reversing the order of integration follow from Tonelli's Theorem. □

Proof of Theorem 2. We first prove the second sentence of the theorem (the comparison between $m^*(p; s)$ and $\hat{m}(p; s)$). The statement is vacuous if $\sigma > 1/2$. Accordingly, suppose that $\sigma \leq 1/2$, and consider $p \in [0, 1/2 - \sigma]$. The argument for $p > 1/2 + \sigma$ is analogous.

Fixing $p \in [0, 1/2 - \sigma]$ and omitting the arguments from the notation for $m^*(p; s)$ and $\hat{m}(p; s)$, it follows from Lemma 5 together with the corresponding inequalities for corner solutions that

$$E\left[(p-c(m^*,\varepsilon))c_m(m^*,\varepsilon)\pi(c(m^*,\varepsilon))\right] \begin{cases}\leq 0 & \text{if } m^* = \underline{m},\\ = 0 & \text{if } m^* \in (\underline{m},\overline{m}),\\ \geq 0 & \text{if } m^* = \overline{m},\end{cases}$$

$$E\left[(p-c(\hat{m},\varepsilon))c_m(\hat{m},\varepsilon)\right] \begin{cases}\leq 0 & \text{if } \hat{m} = \underline{m},\\ = 0 & \text{if } \hat{m} \in (\underline{m},\overline{m}),\\ \geq 0 & \text{if } \hat{m} = \overline{m}.\end{cases}$$

Let $m_{1/2} = \sup\{m : \overline{q}(m) \leq 1/2\}$, and recall that $\sigma := \sup_m \overline{q}(m) - \underline{q}(m)$. For any $m > m_{1/2}$, since $c(m,\varepsilon)$ is increasing in m, it follows that, for every ε, $c(m,\varepsilon) \geq \overline{q}(m) - \sigma > 1/2 - \sigma \geq p$. Hence m cannot satisfy the first-order condition for either problem, and we must have $\hat{m}, m^* \leq m_{1/2}$. Therefore, we can restrict attention to messages m satisfying $m \leq m_{1/2}$. For this range, since $c(m,\varepsilon)$ is increasing in m,

$$c(m,\varepsilon) < c(m_{1/2},\varepsilon) \leq \overline{q}(m_{1/2}) \leq 1/2,$$

and hence by Lemma 6, $\pi(c(m,\varepsilon))$ is decreasing in m.

Let

$$\Delta^*(m',m) = E\left[f(\ell, c(m',\varepsilon)) - f(\ell, c(m,\varepsilon))\right]$$

and

$$\hat{\Delta}(m',m) = \frac{\pi(p)}{2}\left(-\hat{L}(m',p) + \hat{L}(m,p)\right),$$

where $\ell = (p, r_1, r_2)$ and $\hat{L}(m,p) = E\left[(c(m,\varepsilon)-p)^2\right]$. These two expressions may be interpreted as the payoff difference between messages m' and m under fitness and precision maximization, respectively, with the caveat that we have rescaled the payoffs in the precision maximization problem by $\frac{\pi(p)}{2}$.

We claim that

$$\Delta^*(m',m) > \hat{\Delta}(m',m)$$

whenever $m' \in (m, m_{1/2})$. This implies that $m^*(p) \geq \hat{m}(p)$ by Theorem 1 in Van Zandt (2002).

To prove the claim, first define

$$\delta^*(q',q) = E\left[f(\ell,q') - f(\ell,q)\right]$$

and

$$\hat{\delta}(q',q) = \frac{\pi(p)}{2}\left(-(q'-p)^2 + (q-p)^2\right).$$

Note that $\Delta^*(m',m) = E\left[\delta^*(c(m',\varepsilon), c(m,\varepsilon))\right]$ and $\hat{\Delta}(m',m) = E\left[\hat{\delta}(c(m',\varepsilon), c(m,\varepsilon))\right]$. Since $c(m,\varepsilon)$ is increasing in m, we have $1/2 \geq c(m_{1/2},\varepsilon) > c(m',\varepsilon) > c(m,\varepsilon)$, and hence it suffices to show that

$$\delta^*(q',q) > \hat{\delta}(q',q) \tag{9}$$

for all $q', q \in [0, 1/2]$ such that $q' > q$.

Note first that

$$\delta^*(q',q) = \int_q^{q'} (p-p^*)\pi(p^*)dp^*. \tag{10}$$

This follows from the same computation as for (14) below, except with $c(m,\varepsilon) = m$, from which we obtain $\frac{\partial}{\partial p^*}E\left[f(\ell,p^*)\right] = (p-p^*)\pi(p^*)$.[25] Second,

$$\hat{\delta}(q',q) = \int_q^{q'} (p-p^*)\pi(p)dp^*; \tag{11}$$

which can be verified directly by integration. Inequality (9) holds whenever $q < q' \leq 1/2$ because, by Lemma 6, π is decreasing in that interval, and hence the integrand in (10) strictly exceeds that in (11) for all $p^* \in [q,q'] \setminus \{p\}$.

So far we have established only a weak inequality between m^* and $\hat{m}$; we will show that the inequality must be strict for all $p \in (\sigma, 1-\sigma) \setminus [1/2-\sigma, 1/2+\sigma]$ (which is nonempty if $\sigma < 1/4$). Note first that if $p > \sigma$ then, by assumption A1 and the definition of σ, $c(\underline{m},\varepsilon) \leq \sigma < p$ for all ε, and hence $\underline{m}$ cannot solve the optimality condition for either problem. Similarly, $\overline{m}$ cannot be optimal, and we can restrict attention to interior solutions.

Consider $p \in (\sigma, 1/2-\sigma)$ (a symmetric argument applies for $p \in (1/2+\sigma, 1-\sigma)$). Suppose that $\hat{m}(p) = m \in (\underline{m}, \overline{m})$. By the first-order condition for the precision maximization problem,

$$E\left[(p - c(m,\varepsilon))c_m(m,\varepsilon)\pi(p)\right] = 0. \tag{12}$$

[25] For a more intuitive argument, note that if perceptions q and q' are on the same side of the critical probability p^* then they generate the same choice behavior, and thus the payoff difference is 0. If $p^* \in [q,q']$, then only one of the two perceptions leads to an error in choice, with an associated payoff loss of $\pi(p^*)|p^*-p|$.

Since

$$(p - c(m,\varepsilon))c_m(m,\varepsilon)\pi(c(m,\varepsilon)) \geq (p - c(m,\varepsilon))c_m(m,\varepsilon)\pi(p) \tag{13}$$

for every ε, the left-hand side of (12) is at most $E\left[(p - c(m,\varepsilon))c_m(m,\varepsilon)\pi(c(m,\varepsilon))\right]$. Moreover, the inequality in (13) is strict unless $c(m,\varepsilon) = p$, which happens with probability less than 1. Therefore, $E\left[(p - c(m,\varepsilon))c_m(m,\varepsilon)\pi(c(m,\varepsilon))\right] > 0$, and hence m does not solve the reward maximization problem.

To prove that $m^*(p)$ is nondecreasing, note first that $\delta^*(q', q)$ is increasing in p when $q' > q$. Since $\Delta^*(m', m) = E\left[\delta^*(c(m',\varepsilon), c(m,\varepsilon))\right]$ and $c(m,\varepsilon)$ is increasing in m, $\Delta^*(m', m)$ must be increasing in p whenever $m' > m$. The result then follows from Theorem 1 in Van Zandt (2002). □

Proof of Lemma 5. The second equation follows directly from the first-order condition for the precision maximization problem.

For the first equation, since $qr_1 + (1-q)r_2 = s$ if and only if $\frac{s-qr_1}{1-q}$,

$$\begin{aligned}
\pi(q;s) &= \frac{\partial}{\partial s}\int_{-\infty}^{\infty}\int_{-\infty}^{\frac{s-qr_1}{1-q}} (r_1 - r_2)^2 \rho(r_1, r_2)\,dr_2\,dr_1 \\
&= \int_{-\infty}^{\infty} (r_1 - r_2(r_1;q))^2 \frac{\rho\left(r_1, r_2\left(r_1;q\right)\right)}{1-q}\,dr_1,
\end{aligned}$$

where $r_2(r_1;q) := \frac{s-qr_1}{1-q}$.

Given any p, let $F = E[f((p, r_1, r_2), c(m,\varepsilon))]$ be the expected reward when the observation center observes p and sends message m. We have

$$\begin{aligned}
F = E\Bigg[&\int_{c(m,\varepsilon)r_1 + (1-c(m,\varepsilon))r_2 < s} s\rho(r_1, r_2)\,dr_1\,dr_2 \\
&+ \int_{c(m,\varepsilon)r_1 + (1-c(m,\varepsilon))r_2 > s} (pr_1 + (1-p)r_2)\rho(r_1, r_2)\,dr_1\,dr_2\Bigg],
\end{aligned}$$

where the expectation is with respect to the noise ε, as are all other expectations for the remainder of this proof.

The expected reward F is continuous in m and the message space is compact. Therefore, an optimal message exists. Moreover, since F is continuously differentiable in m, the optimal message must, for each p, satisfy the first-order condition $\frac{\partial}{\partial m}F = 0$ whenever it is interior. Computing the derivative, we obtain

$$
\begin{aligned}
\frac{\partial F}{\partial m} &= E\left[c_m(m,\varepsilon)\int_{-\infty}^{\infty}\left(s-(pr_1+(1-p)r_2(r_1;c(m,\varepsilon)))\right)\frac{s-r_1}{(1-c(m,\varepsilon))^2}\rho(r_1,r_2(r_1;c(m,\varepsilon)))dr_1\right] \\
&= E\left[c_m(m,\varepsilon)\int_{-\infty}^{\infty}\left(c(m,\varepsilon)r_1+(1-c(m,\varepsilon))r_2(r_1;c(m,\varepsilon))-(pr_1+(1-p)r_2(r_1;c(m,\varepsilon)))\right)\right. \\
&\qquad \left.\cdot\frac{c(m,\varepsilon)r_1+(1-c(m,\varepsilon))r_2(r_1;c(m,\varepsilon))-r_1}{(1-c(m,\varepsilon))^2}\rho(r_1,r_2(r_1;c(m,\varepsilon)))dr_1\right] \\
&= E\left[c_m(m,\varepsilon)\int_{-\infty}^{\infty}\frac{p-c(m,\varepsilon)}{1-c(m,\varepsilon)}\left(r_1-r_2(r_1;c(m,\varepsilon))\right)^2\rho(r_1,r_2(r_1;c(m,\varepsilon)))dr_1\right] \\
&= E\left[(p-c(m,\varepsilon))c_m(m,\varepsilon)\pi(c(m,\varepsilon))\right], \qquad (14)
\end{aligned}
$$

as needed. □

Proof of Lemma 6. Consider p and p' such that $|p-1/2| > |p'-1/2|$. Let $X_1 = -r(p)$ and $X_2 = -r(p')$. By Lemma 4, X_1 is a mean-preserving spread of X_2. Let $F_1(x) = \int_{-x}^{+\infty} d_p(x')dx'$ and $F_2(x) = \int_{-x}^{+\infty} d_{p'}(x')dx'$ be the distribution functions of X_1 and X_2, respectively.

By the regularity condition A2, either $d_p(x) > d_{p'}(x)$ for all $x > r^*$, or $d_p(x) < d_{p'}(x)$ for all $x > r^*$. For the sake of contradiction, suppose the latter. Then $F_1(x) < F_2(x)$ for all $x < -r^*$ and hence

$$\int_{-\infty}^{-r^*} F_1(x')dx' < \int_{-\infty}^{-r^*} F_2(x')dx',$$

which contradicts that X_1 is a mean preserving spread of X_2. □

C Proof of Proposition 1

Proof of Proposition 1. First note that, given any $m(\cdot)$, the objective in problem (4) is maximized when the choice function is Bayesian. It follows that if there exists a solution to (4), then the same strategy $m(\cdot)$ coupled with the corresponding Bayesian choice function also solves (4).

For any given Bayesian choice function $g(\cdot)$, we can think of the observation center as choosing a perception $q(p) = E[p \mid m(p)] \in Q$ for each probability p. Given a Bayesian choice function associated with some (not necessarily optimal) strategy $m(\cdot)$, any optimal strategy for the observation center solves, for almost every p,

$$\max_{q \in Q} E\left[f(\ell, q; s)\right], \tag{15}$$

where $\ell = (p, r_1, r_2)$, $f(\ell, q; s)$ is as in (1), and s solves (5) given the message strategy $m(\cdot)$. The expectation is with respect to the rewards r_1 and r_2 in the lottery ℓ. Note that this problem has a solution since Q is finite. Note also that this problem is essentially the same as that in (2) except that there is now a finite set Q of feasible perceptions, and there is no noise.

A calculation that is essentially identical to that in the proof of Lemma 5 (and which we therefore omit) shows that $q(p)$ solves (15) if and only if it solves

$$\min_{q \in Q} \int_p^q (p^* - p)\pi(p^*; s)dp^*, \tag{16}$$

where $\pi(\cdot)$ is the weighting function defined in Section 3.

The objective in (16) has increasing differences in (p, q), and hence, by Theorem 1 in Van Zandt (2002), the solution $q^*(p)$ is nondecreasing in p. Therefore, the optimal message strategy is measurable with respect to a partition of the interval $[0, 1]$ of probabilities into (at most) $|M|$ subintervals. Fixing an assignment of messages to elements of the partition, such perception strategies are characterized by $|M|-1$ thresholds in the interval $[0, 1]$, and hence can be identified with a compact subset of $[0, 1]^{|M|-1}$. Moreover, within this class of strategies $m(\cdot)$, together with Bayesian choice functions with respect to strategies $\tilde{m}(\cdot)$ in this class, there exists a pair maximizing the objective in (16) since it is continuous with respect to changes in the thresholds characterizing both $m(\cdot)$ and $\tilde{m}(\cdot)$. This solution must involve $\tilde{m}(\cdot) = m(\cdot)$ (a.e.) since that defines the optimal choice function given $m(\cdot)$.

For κ sufficiently small, $s > 3^{1/4}$ since the distribution of $q(m)r_1 + (1-q(m))r_2$ first-order stochastically dominates that of $\min\{r_1, r_2\}$, which has unbounded support. Therefore, the weighting function π is U-shaped for sufficiently small κ. Inequality (6) now follows by the same monotone comparative statics argument as in the proof of Theorem 2. Moreover, the thresholds for the two problems must differ, making the inequality strict for those values of p lying between the thresholds corresponding to a given pair of values of q.

All that remains is to show that the solution within the class of threshold strategies also solves the original problem. If not, there exists a pair $(m_0(\cdot), g_0(\cdot))$ giving a higher value of the objective function. Letting $g_1(\cdot)$ denote the Bayesian choice function given $m_0(\cdot)$,

$m_1(\cdot)$ the optimal message function given $g_1(\cdot)$, and $g_2(\cdot)$ the Bayesian choice function given $m_1(\cdot)$, the pair $(m_1(\cdot), g_2(\cdot))$ lies in the class described by threshold strategies and gives at least as high a value of the objective function as $(m_0(\cdot), g_0(\cdot))$, a contradiction. □

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