ON IT^O FORMULAS FOR JUMP PROCESSES

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Abstract. A well-known Itô formula for finite dimensional processes, given in terms of stochastic integrals with respect to Wiener processes and Poisson random measures, is revisited and is revised. The revised formula, which corresponds to the classical Itô formula for semimartingales with jumps, is then used to obtain a generalisation of an important infinite dimensional Itô formula for continuous semimartingales from Krylov [14] to a class of L_p-valued jump processes. This generalisation is motivated by applications in the theory of stochastic PDEs.

1. Introduction

This is a review paper on some Itô formulas in finite and infinite dimensional spaces. First we consider finite dimensional Itô-Lévy processes, which are $\mathbb{R}^M$-valued stochastic processes $X = (X_t)_{t\geq 0}$ given in terms of stochastic integrals with respect to Wiener processes and Poisson random measures. They play important roles in modelling stochastic phenomena when jumps may occur at random times, see for example, [4] and [5]. Chain rules, called Itô formulas, for their transformations $\phi(X_t)$ by sufficiently smooth functions ϕ are basic tools in the investigations of the stochastic phenomena modelled by Itô-Lévy processes, see, e.g., [13] and the references therein. It is therefore important to have Itô formulas for large classes of processes X and functions ϕ. Note that classical Itô's formula, (2.4) below, holds only under some restrictive conditions, which are not satisfied in important applications, for example in applications to filtering theory of partially observed jump diffusions. Therefore we revisit the chain rule (2.4) for finite dimensional Itô-Lévy processes, discuss its limitations, and derive formula (2.12) from it, which corresponds to a well-known Itô formula for general semimartingales, and is valid without restrictive conditions on the Itô-Lévy processes X and on the functions ϕ.

In the second part of the paper we discuss infinite dimensional generalisations of the Itô formula (2.12) from point of view of applications in stochastic PDEs (SPDEs). In the theory of parabolic SPDEs, arising in nonlinear filtering theory, the solutions $v = v_t(x)$ of SPDEs have the stochastic differentials

$$dv_t(x) = (f_t^0(x) + \sum_{i=1}^{d} \tfrac{\partial}{\partial x^i} f_t^i(x))\,dt + \sum_r g_t^r(x)\,dm_t^r \tag{1.1}$$

with appropriate random functions f^α and g^r of $t \in [0,T]$ and $x = (x^1, ..., x^d) \in \mathbb{R}^d$, and a sequence of martingales $(m^i)_{i=1}^\infty$ is. This stochastic differential is understood in a weak

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sense, i.e., for each smooth function φ with compact support on $\mathbb{R}^d$ we have the stochastic differential

$$d(v_t,\varphi) = ((f_t^0,\varphi) - \sum_i (f_t^i, \tfrac{\partial}{\partial x^i}\varphi))\,dt + \sum_r (g_t^r,\varphi)\,dm_t^r,$$

where (u,v) denotes the Lebesgue integral over $\mathbb{R}^d$ of the product uv for functions u and v of $x \in \mathbb{R}^d$. In the L_2-theory of SPDEs f^α and g^r are $L_2(\mathbb{R}^d,\mathbb{R})$-valued functions of (ω,t), satisfying appropriate measurability conditions, and to get a priori estimates, a suitable formula for $|v|_{L_2}^2$ plays crucial roles. Such a formula in an abstract setting was first obtained in [17] when $(m^i)_{i=1}^\infty$ is a sequence of independent Wiener processes. The proof in [17] is connected with the theory of SPDEs developed in [17]. A direct proof was given in [16], which was generalised in [8] to the case of square integrable martingales $m = (m^i)$. A nice short proof was presented in [15], and further generalisations can be found, for example, in [9] and [18]. The above results on Itô formula are used in the L_2-theory of linear and nonlinear SPDEs to obtain existence, uniqueness and regularity results under various assumptions see, e.g., [7], [16], [17], [18] and [19]. To have a similar tool for studying solvability, uniqueness and regularity problems for solutions in L_p-spaces for $p \neq 2$ one should establish a suitable formula for $|v_t|_{L_p}^p$, which was first achieved in Krylov [14] for $p \geq 2$ when $(m^i)_{i=1}^\infty$ is a sequence of independent Wiener processes.

In section 3 we present a generalisation of the main result from Krylov [14] to the case when the stochastic differential of v_t is of the form

$$dv_t(x) = (f_t^0(x) + \sum_{i=1}^d \tfrac{\partial}{\partial x^i} f_t^i(x))\,dt + \sum_r g_t^r(x)\,dw_t^r + \int_Z h_t(z,x)\tilde\pi(dz,dt), \tag{1.2}$$

where $\tilde\pi(dz,dt)$ is a Poisson martingale measure with a σ-finite characteristic measure μ on a measurable space $(Z,\mathcal{Z})$, and h is a function on $\Omega\times[0,T]\times Z\times\mathbb{R}^d$. This is Theorem 3.1 below, which is a slight generalisation of Theorem 2.2 on Itô's formula from [10] for $|v_t|_{L_p}^p$ for $p \geq 2$. We prove it by adapting ideas and methods from Krylov [14]. In particular, we use the finite dimensional Itô's formula (2.19) below for $|v_t^\varepsilon(x)|^p$ for each $x\in\mathbb{R}^d$, where v_t^ε is an approximation of v_t obtained by smoothing it in x. Hence we integrate both sides of the formula for $|v_t^\varepsilon(x)|^p$ over $\mathbb{R}^d$, change the order of deterministic and stochastic integrals, integrate by parts in terms containing derivatives of smooth approximations of f^i, and finally we let $\varepsilon\to 0$. Though the idea of the proof is simple, there are several technical difficulties to implement it. We sketch the proof of Theorem 3.1 in section 3, further details of the proof can be found in [10]. Theorem 3.1 plays a crucial role in proving existence, uniqueness and regularity results in [11] for solutions to stochastic integro-differential equations. In [11] instead of a single random field $v_t(x)$ we have to deal with a system of random fields $v_t^i(x)$ for $i = 1,2,...,M$, and we need estimates for $||\sum_i |v^i|^2|^{1/2}|_{L_p}$. This is why in Theorem 3.1 we consider a system a random fields v^i, $i=1,2,...,M$.

There are known theorems in the literature on Itô's formula for semimartingales with values in separable Banach spaces, see for example, [3], [20], [21], [22] and [23]. In some directions these results are more general than Theorem 3.1, but they do not cover it. In [3] and [21] only continuous semimartingales are considered and their differential does not contain $D_i f^i\,dt$ terms. In [20], [22] and [23] semimartingales containing stochastic integrals with respect to Poisson random measures and martingale measures are considered, but they do not contain terms corresponding to $D_i f^i$. Thus the Itô formula in these papers cannot be applied to $|v_t|^p_{L_p}$ when the stochastic differential dv_t is given by (1.2).

In conclusion we present some notions and notations. All random elements are given on a fixed complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ contains all P-zero sets of $\mathcal{F}$. The σ-algebra of the predictable subsets of $\Omega\times[0,\infty)$ is denoted by $\mathcal{P}$. We are given a sequence $w = (w^1_t, w^2_t, ...)_{t\geq 0}$ of $\mathcal{F}_t$-adapted independent Wiener processes $w^r = (w^r_t)_{t\geq 0}$, such that $w_t - w_s$ is independent of $\mathcal{F}_s$ for any $0 \leq s \leq t$. For an integer $m \geq 1$ we are given also a sequence of independent Poisson random measures $\pi^k(dz, dt)$ on $[0,\infty)\times Z^k$, with intensity measure $\mu^k(dz)\,dt$ for $k = 1, 2, ..m$, where μ^k is a σ-finite measure on a measurable space $(Z^k, \mathcal{Z}^k)$ with a countably generated σ-algebra $\mathcal{Z}^k$. We assume that the process $\pi^k_t(\Gamma) := \pi^k(\Gamma\times(0,t])$, $t \geq 0$, is $\mathcal{F}_t$-adapted and $\pi^k_t(\Gamma) - \pi^k_s(\Gamma)$ is independent of $\mathcal{F}_s$ for any $0 \leq s \leq t$ and $\Gamma \in \mathcal{Z}^k$ such that $\mu^k(\Gamma) < \infty$. We use the notation $\tilde\pi^k(dz, dt) = \pi^k(dz, dt) - \mu^k(dz)dt$ for the *compensated Poisson random measure*, and set $\tilde\pi^k_t(\Gamma) = \tilde\pi^k(\Gamma\times(0,t]) = \pi^k_t(\Gamma) - t\mu^k(\Gamma)$ for $t \geq 0$ and $\Gamma \in \mathcal{Z}$ such that $\mu^k(\Gamma) < \infty$. If $m = 1$ then we write π, $\tilde\pi$, Z, $\mathcal{Z}$ and μ in place of π^1, $\tilde\pi^1$, Z^1, $\mathcal{Z}^1$ and μ^1, respectively. For basic results concerning stochastic integrals with respect to Poisson random measures and Poisson martingale measures we refer to [1] and [12].

Let $M > 0$ be an integer. The space of sequences $\nu = (\nu^1, \nu^2, ...)$ of vectors $\nu^k \in \mathbb{R}^M$ with finite norm

$$|\nu|_{\ell_2} = \big(\sum_{k=1}^\infty |\nu^k|^2\big)^{1/2}$$

is denoted by $\ell_2 = \ell_2(\mathbb{R}^M)$, and by l_2 when $M = 1$. We use the notation D_i to denote the i-th derivative, i.e.

$$D_i = \frac{\partial}{\partial x_i}, \quad i = 1, 2, ..., M.$$

For vectors v from Euclidean spaces, $|v|$ means the Euclidean norm of v. The space of smooth functions with compact support in $\mathbb{R}^M$ is denoted by $C^\infty_0(\mathbb{R}^M)$. For integers $k \geq 1$ the notation $C^k(\mathbb{R}^M)$ means the space of functions on $\mathbb{R}^M$ whose derivatives up to order k exist and are continuous, and $C^k_b(\mathbb{R}^M)$ denotes the space of functions on $\mathbb{R}^M$ whose derivatives up to order k are bounded continuous functions. When we talk about the derivatives up to order k of a function f then among these derivatives we always consider the “ zero order derivative” of f, i.e., f itself.

2. Itô formulas in finite dimensions

We consider an $\mathbb{R}^M$-valued semimartigale $X = (X^1_t, ..., X^M_t)_{t\geq 0}$ given by

$$X_t = X_0 + \int_0^t f_s\,ds + \int_0^t g^r_s\,dw^r_s$$
$$+\sum_{k=1}^m \int_0^t \int_{Z^k} \bar h^k_s(z)\,\pi^k(dz, ds) + \sum_{k=1}^m \int_0^t \int_{Z^k} h^k_s(z)\,\tilde\pi^k(dz, ds), \quad \text{for } t \geq 0, \tag{2.1}$$

where X_0 is an $\mathbb{R}^M$-valued $\mathcal{F}_0$-measurable random variable, $f = (f^i_t)_{t\geq 0}$ and $g = (g^{ir}_t)_{t\geq 0}$ are predictable processes with values in $\mathbb{R}^M$ and $\ell_2 = \ell_2(\mathbb{R}^M)$, respectively, $\bar h^k = (\bar h^{ik}_t(z))_{t\in[0,T]}$

and $h^k = (h^{ik}_t(z))_{t\geq 0}$ are $\mathbb{R}^M$-valued $\mathcal{P}\otimes\mathcal{Z}$-measurable functions on $\Omega\times\mathbb{R}_+\times Z$ for every $k = 1,2,...,m$, such that almost surely for every $k = 1,2,...,m$

$$\bar h^{ik}_t(z)h^{jk}_t(z) = 0 \quad \text{for } i,j = 1,2,...,M, \text{ for all } t\geq 0 \text{ and } z\in Z, \tag{2.2}$$

and

$$\sum_{k=1}^m \left(\int_0^T\int_{Z_k} |\bar h^k_t(z)|\,\pi^k(dz,dt) + \int_0^T\int_{Z_k}|h^k_t(z)|^2\mu^k(dz)\,dt\right) < \infty, \quad \int_0^T |f_t| + |g_t|^2_{\ell_2}\,dt < \infty \tag{2.3}$$

for every $T > 0$. Here and later on, unless otherwise indicated, the summation convention with respect to repeated integer-valued indices is used, i.e., $g^r_s\,dw^r_s$ means $\sum_r g^r_s\,dw^r_s$.

The following Itô's formula is well-known for $m = 1$.

Theorem 2.1. *Let conditions* (2.2) *and* (2.3) *hold and assume there is a constant K such that $|h^k|\leq K$ for all $(\omega,t,z)\in\Omega\times\mathbb{R}_+\times Z$ and $k = 1,2,...,m$. Then for any $\phi\in C^2(\mathbb{R}^M)$ the process $(\phi(X_t))_{t\geq 0}$ is a semimartingale such that*

$$\begin{aligned}\phi(X_t) = \phi(X_0) + \int_0^t f^i_s D_i\phi(X_s) + \tfrac12 g^{ir}_s g^{jr}_s D_iD_j\phi(X_s)\,ds + \int_0^t g^{ir}_s D_i\phi(X_s)\,dw^r_s\\ + \sum_{k=1}^m\int_0^t\int_{Z_k}\phi(X_{s-} + \bar h^k_s(z)) - \phi(X_{s-})\,\pi^k(dz,ds)\\ + \sum_{k=1}^m\int_0^t\int_{Z_k}\phi(X_{s-} + h^k_s(z)) - \phi(X_{s-})\,\tilde\pi^k(dz,ds)\\ + \sum_{k=1}^m\int_0^t\int_{Z_k}\Big(\phi(X_s + h^k_s(z)) - \phi(X_s) - h^{ik}_s(z)D_i\phi(X_s)\Big)\,\mu^k(dz)\,ds\end{aligned} \tag{2.4}$$

holds almost surely for all $t\geq 0$.

Proof. This theorem, with a finite dimensional Wiener process $w = (w^1,...,w^{d_1})$ in place of an infinite sequence of independent Wiener processes and for $m = 1$ is proved, for example, in [12], see Theorem 5.1 in chapter II. Following this proof with appropriate changes one can easily prove the above theorem as follows. Since μ^k is σ-finite for $k = 1,2,...,m$, for each k we have an increasing sequence $(Z^k_n)^\infty_{n=1}$ of sets $Z^k_n\in\mathcal{Z}^k$ such that $Z^k = \cup^\infty_{n=1}Z^k_n$ and $\mu^k(Z^k_n) < \infty$ for every n. For a fixed integer $n\geq 1$ let $\rho^k_1 < \rho^k_2 < ...$ denote the increasing sequence of times where the jumps of $N^k := (\pi^k_t(Z^k_n))_{t\geq 0}$ occur. Similarly, let $\tau_1 < \tau_2 < ...$ be the jump times of the process $N = \sum^m_{k=1}N_k$. Then ρ^k_i and τ_i are stopping times for every $k = 1,2,...,m$ and $i\geq 1$, and for almost every $\omega\in\Omega$ the set of time points $\{\tau_i(\omega) : i\geq 1\}$ contains all points of discontinuities of $(X^n_t(\omega))_{t\geq 0}$, where the process X^n is defined by

$$\begin{aligned}X^n_t = X_0 + \int_0^t f_s\,ds + \int_0^t g^r_s\,dw^r_s + \sum_{k=1}^m\int_0^t\int_{Z^k}\bar h^k_s(z)\mathbf{1}_{Z^k_n}(z)\,\pi^k(dz,ds)\\ + \sum_{k=1}^m\int_0^t\int_{Z^k}h^k_s(z)\mathbf{1}_{Z^k_n}(z)\,\pi^k(dz,ds) - V^n_t \quad \text{for } t\geq 0\end{aligned} \tag{2.5}$$

with

$$V_t^n := \sum_{k=1}^m \int_0^t \int_{Z^k} h_s^k(z)\mathbf{1}_{Z_n^k}(z)\,\mu^k(dz)\,ds.$$

Clearly, $\phi(X_t^n) = \phi(X_0^n) + A_t^n + B_t^n$ with

$$A_t^n = \sum_{i\geq 1}\big(\phi(X^n_{\tau_i\wedge t}) - \phi(X^n_{\tau_i\wedge t-})\big), \quad B_t^n = \sum_{i\geq 1}\big(\phi(X^n_{\tau_i\wedge t-}) - \phi(X^n_{\tau_{i-1}\wedge t})\big),$$

where we set $\tau_0 := 0$ and $X^n_{\tau_i\wedge t-} := X^n_{\tau_i-}$ for $t\geq\tau_i$ and $X^n_{\tau_i\wedge t-} := X^n_t$ for $t<\tau_i$. By Itô's formula for Itô processes we have

$$\phi(X^n_{\tau_i\wedge t-}) - \phi(X^n_{\tau_{i-1}\wedge t}) = \int_{\tau_{i-1}\wedge t}^{\tau_i\wedge t-} D_l\phi(X_s^n)f_s^l + \tfrac{1}{2}D_{jl}\phi(X_s^n)g^{jr}g^{lr}\,ds$$
$$+ \int_{\tau_{i-1}\wedge t}^{\tau_i\wedge t-} D_l\phi(X_s^n)g_s^{lr}\,dw_s^r - \int_{\tau_{i-1}\wedge t}^{\tau_i\wedge t-} D_l\phi(X_s^n)\,dV_s^n,$$

which gives

$$B_t^n = \int_0^t D_l\phi(X_s^n)f_s^l + \tfrac{1}{2}D_{jl}\phi(X_s^n)g^{jr}g^{lr}\,ds$$
$$+ \int_0^t D_l\phi(X_s^n)g_s^{lr}\,dw_s^r - \int_0^t D_l\phi(X_s^n)\,dV_s^n. \tag{2.6}$$

Notice that ρ_i^k has a density with respect to the Lebesgue measure for $i\geq 1$, and ρ_i^k and ρ_j^l are independent for $k\neq l$. Hence $P(\rho_i^k = \rho_j^l) = 0$ for $k\neq l$ and positive integers i,j. Consequently, for almost every $\omega\in\Omega$ we have $\{\tau_i(\omega) : i\geq 1\} = \cup_{k=1}^m\{\rho_i^k(\omega) : i\geq 1\}$ such that the sets in the union are almost surely pairwise disjoint. Hence, taking also into account condition (2.2), we get that almost surely

$$A_t^n = \sum_{k=1}^m\sum_{i\geq 1}\big(\phi(X^n_{\rho_i^k\wedge t}) - \phi(X^n_{\rho_i^k\wedge t-})\big) = \bar{A}_t^n + \tilde{A}_t^n$$

for all $t\geq 0$, where

$$\bar{A}_t^n = \sum_{k=1}^m\int_0^t\int_{Z^k}\big(\phi(X^n_{s-}+\bar{h}_s^k(z)) - \phi(X^n_{s-})\big)\mathbf{1}_{Z_k^n}(z)\,\pi^k(dz,ds),$$

$$\tilde{A}_t^n = \sum_{k=1}^m\int_0^t\int_{Z^k}\big(\phi(X^n_{s-}+h_s^k(z)) - \phi(X^n_{s-})\big)\mathbf{1}_{Z_k^n}(z)\,\pi^k(dz,ds)$$
$$= \sum_{k=1}^m\int_0^t\int_{Z^k}\big(\phi(X^n_{s-}+h_s^k(z)) - \phi(X^n_{s-})\big)\mathbf{1}_{Z_k^n}(z)\,\tilde{\pi}^k(dz,ds)$$
$$+ \sum_{k=1}^m\int_0^t\int_{Z^k}\big(\phi(X^n_{s-}+h_s^k(z)) - \phi(X^n_{s-})\big)\mathbf{1}_{Z_k^n}(z)\,\mu^k(dz)\,ds.$$

Combining this with (2.6) we get

$$\phi(X_t^n) = \phi(X_0) + \int_0^t D_l\phi(X_s^n)f_s^l + \tfrac{1}{2}D_{jl}\phi(X_s^n)g^{jr}g^{lr}\,ds + \int_0^t D_l\phi(X_s^n)g_s^{lr}\,dw_s^r$$

$$+\sum_{k=1}^{m}\int_0^t\int_{Z^k}\big(\phi(X^n_{s-}+\bar h^k_s(z))-\phi(X^n_{s-})\big)\mathbf{1}_{Z^n_k}(z)\,\pi^k(dz,ds),$$

$$+\sum_{k=1}^{m}\int_0^t\int_{Z^k}\big(\phi(X^n_{s-}+h^k_s(z))-\phi(X^n_{s-})\big)\mathbf{1}_{Z^n_k}(z)\,\tilde\pi^k(dz,ds)$$

$$+\sum_{k=1}^{m}\int_0^t\int_{Z^k}\big(\phi(X^n_{s-}+h^k_s(z))-\phi(X^n_{s-})-D_l\phi(X^n_s)h^{lk}_s(z)\big)\mathbf{1}_{Z^n_k}(z)\,\mu^k(dz)\,ds.$$

Hence we can finish the proof by letting $n\to\infty$ and using standard facts about convergence of Lebesgue integrals and stochastic integrals with respect to Wiener processes and random measures. □

In some publications only the natural conditions (2.2) and (2.3) are assumed in the formulation of the above theorem, but these conditions are not sufficient for (2.4) to hold, as the following simple example shows.

Example 2.1. Consider a one-dimensional semimartingale $(X_t)_{t\in[0,T]}$ given by (2.1) with $f=0$, $g=0$, $\bar h=0$ and $h_t(z)=\mathbf{1}_{t>0}t^{-1/4}$, $t\geq 0$, $z\in Z=\mathbb{R}\setminus\{0\}$, when $\pi(dz,dt)$ is the measure of jumps of a standard Poisson process and $\tilde\pi(dz,dt)=\pi(dz,dt)-\mu(dz)dt$ is its compensated measure, where $\mu=\delta_1$ is the Dirac measure on Z concentrated at 1. Then obviously conditions (2.2) and (2.3) hold, and for $\phi(x)=x^4$ the last integrand in (2.4) is

$$|X_{s-}+h_s(z)|^4-|X_{s-}|^4-4X^3_{s-}h_s(z)=\sum_{i=1}^{3}c_i(s,z)$$

with

$$c_1(s,z)=6|X_{s-}|^2\mathbf{1}_{s>0}s^{-1/2},\quad c_2(s,z)=4X_{s-}\mathbf{1}_{s>0}s^{-3/4},\quad c_3(s,z)=\mathbf{1}_{s>0}s^{-1}.$$

Clearly,

$$\int_0^t\int_Z|c_i(s,z)|\,\mu(dz)\,ds<\infty\quad\text{for } i=1,2,\quad\text{and}\quad\int_0^t\int_Z c_3(s,z)\,\mu(dz)\,ds=\infty$$

for every $t>0$, which shows that the last integral in (2.4) is infinite. Similarly, one can show that almost surely

$$\int_0^t\int_Z(|X_{s-}+h_s(z)|^4-|X_{s-}|^4)^2\,\mu(dz)\,ds=\infty\quad\text{for every } t>0,$$

which means the stochastic integral with respect to $\tilde\pi(dz,ds)$ in (2.4) does not exist.

It is easy to see that the last two integrals in (2.4) are well-defined as Itô and Lebesgue integrals, respectively, under the additional boundedness assumption on h. Instead of this extra condition on h one can make additional assumptions on ϕ to ensure that formula (2.4) holds. It is sufficient to assume that the derivatives of ϕ up to second order are bounded. Such a condition, however, excludes the applicability of Itô's formula to power functions $\phi(x)=|x|^p$ for $p\geq 2$. Notice that for any $\phi\in C^2(\mathbb{R}^M)$ the conditions

$$\sum_{k=1}^{m}\int_0^T\int_{Z_k}|\phi(X_s+h^k_s(z))-\phi(X_s)|^2\,\mu^k(dz)\,ds<\infty \tag{2.7}$$

and

$$\sum_{k=1}^{m}\int_0^T\int_{Z_k}|\phi(X_s+h_s^k(z))-\phi(X_s)-h_s^k(z)\nabla\phi(X_s)|\,\mu^k(dz)\,ds<\infty \quad \text{(a.s.)} \tag{2.8}$$

ensure the existence of the last two integrals in (2.4) respectively. Thus we can expect that under conditions (2.2)-(2.3) and (2.7)-(2.8) formula (2.4) is valid.

Theorem 2.2. *Let conditions* (2.2)-(2.3) *and* (2.7)-(2.8) *hold. Assume* $\phi\in C^2(\mathbb{R}^M)$. *Then* $\phi(X_t)$ *is a semi-martingale such that* (2.4) *holds almost surely for all* $t\geq 0$.

Proof. This theorem is a slight generalisation of Theorem 5.2 in [2]. For the convenience of the reader we deduce this theorem from Theorem 2.1 here. For notational simplicity we assume $m=1$, with additional indices the case $m>1$ can be proved in the same way.

For vectors $a=(a^1,....,a^M)\in\mathbb{R}^M$ and functions $\phi\in C^2(\mathbb{R}^M)$ we define the functions $I^a\phi$ and $J^a\phi$ by

$$I^a\phi(v)=\phi(v+a)-\phi(v),\quad J^a\phi(v)=\phi(v+a)-\phi(v)-a^iD_i\phi(v),\quad v\in\mathbb{R}^M. \tag{2.9}$$

Assume first $\phi\in C_b^2(\mathbb{R}^M)$. Approximate h by $h^{(n)}=(h^{1(n)},...,h^{M(n)})$ and define

$$X_t^{(n)}=X_0+\int_0^t f_s\,ds+\int_0^t g_s^r\,dw_s^r+\int_0^t\int_Z \bar h_s(z)\,\pi(dz,ds)+\int_0^t\int_Z h_s^{(n)}(z)\,\tilde\pi(dz,ds)$$

for integers $n\geq 1$, where $h_t^{i(n)}=-n\vee h_t^i\wedge n$. Then (2.4) holds with $X_t^{i(n)}$ and $h_t^{i(n)}$ in place of X_t^i and h_t^i, respectively, for each $i=1,2,...,M$. Clearly,

$$\int_0^T\int_Z|h_s^{(n)}(z)-h_s(z)|^2\,\mu(dz)\,ds\to 0\quad\text{(a.s.)}\quad\text{for each } T>0,$$

which implies

$$\int_0^t\int_Z h_s^{(n)}(z)\,\tilde\pi(dz,ds)\to\int_0^t\int_Z h_s(z)\,\tilde\pi(dz,ds)$$

in probability uniformly in $t\in[0,T]$. Consequently for each $T>0$ we have

$$\sup_{t\in[0,T]}|X_t^{(n)}-X_t|\to 0$$

in probability. It is easy to see

$$\int_0^t f_s^iD_i\phi(X_s^{(n)})+\tfrac12 g_s^{ir}g_s^{jr}D_iD_j\phi(X_s^{(n)})\,ds\to\int_0^t f_s^iD_i\phi(X_s)+\tfrac12 g_s^{ir}g_s^{jr}D_iD_j\phi(X_s)\,ds,$$

$$\int_0^t g_s^{ir}D_i\phi(X_s^{(n)})\,dw_s^r\to\int_0^t g_s^{ir}D_i\phi(X_s)\,dw_s^r,$$

$$\int_0^t\int_Z I^{\bar h_s(z)}\phi(X_{s-}^{(n)})\,\pi(dz,ds)\to\int_0^t\int_Z I^{\bar h_s(z)}\phi(X_{s-})\,\pi(dz,ds)$$

in probability uniformly in $t\in[0,T]$ for $T>0$. Furthermore, by Taylor's formula we have

$$|J^{h_s^{(n)}(z)}\phi(X_s^{(n)})|\leq\int_0^1(1-\theta)|h_s^{i(n)}(z)h_s^{j(n)}(z)D_{ij}\phi(X_s^{(n)}+\theta h_s^{(n)}(z))|\,d\leq C|h_s(z)|^2$$

$$|I^{h_s^{(n)}(z)}\phi(X_s^{(n)})|^2 \le \int_0^1 |\nabla\phi(X_s^{(n)} + \theta h_s^{(n)}(z))h_s^{(n)}(z)|^2\,d\theta \le C|h_s(z)|^2$$

with a constant C independent of n. Hence by Lebesgue's theorem on dominated convergence for $T > 0$ we have

$$\int_0^T \int_Z |J^{h_s^{(n)}(z)}\phi(X_s^{(n)}) - J^{h_s(z)}\phi(X_s)|\,\mu(dz)\,ds \to 0 \quad \text{for } t \ge 0$$

and

$$\int_0^T \int_Z |I^{h_s^{(n)}(z)}\phi(X_s^{(n)}) - I^{h_s(z)}\phi(X_s)|^2\,\mu(dz)\,ds \to 0 \quad \text{for } T \ge 0$$

in probability, which implies

$$\int_0^t \int_Z I^{h_s^{(n)}(z)}\phi(X_{s-}^{(n)})\,\tilde\pi(dz,ds) \to \int_0^t \int_Z I^{h_s(z)}\phi(X_{s-})\,\tilde\pi(dz,ds)$$

in probability uniformly in $t \in [0,T]$ for each $T > 0$. Hence, letting $n \to \infty$ in (2.4) with $h^{(n)}$ and $X^{(n)}$ in place of h and X, respectively, we prove the theorem for $\phi \in C_b^2(\mathbb{R}^M)$.
For $\phi \in C^2(\mathbb{R}^M)$ we define ϕ_n for integers $n \ge 1$ by $\phi_n(x) = \phi(x)\zeta(x/n)$, $x \in \mathbb{R}^M$, where ζ is a smooth function on $\mathbb{R}^M$ with values in $[0,1]$ such that $\zeta(x) = 1$ for $|x| \le 1$ and $\zeta(x) = 0$ for $|x| \ge 2$. Then $\phi_n \in C_b^2(\mathbb{R}^M)$, and therefore (2.4) holds with ϕ_n in place of ϕ. Thus it remains to take limit as $n \to \infty$ for each term in (2.4) with ϕ_n in place of ϕ. Clearly as $n \to \infty$, we have

$$\phi_n(x) \to \phi(x), \quad D_i\phi_n(x) \to D_i\phi(x), \quad D_{ij}\phi_n(x) \to D_{ij}\phi(x)$$

uniformly on compact subsets of $\mathbb{R}^M$ for $i,j = 1,2,...,M$. Hence it is easy to see

$$\int_0^t f_s^i D_i\phi_n(X_s) + \tfrac{1}{2}g_s^{ir}g_s^{jr}D_iD_j\phi_n(X_s)\,ds \to \int_0^t f_s^i D_i\phi(X_s) + \tfrac{1}{2}g_s^{ir}g_s^{jr}D_iD_j\phi(X_s)\,ds$$

and

$$\int_0^t g_s^{ir}D_i\phi_n(X_s)\,dw_s^r \to \int_0^t g_s^{ir}D_i\phi(X_s)\,dw_s^r$$

in probability, uniformly in $t \in [0,T]$ as $n \to \infty$. Using the simple identity

$$I^a(\varphi\phi)(x) = \phi(x)I^a\varphi(x) + \varphi(x+a)I^a\phi(x), \quad a,x \in \mathbb{R}^M$$

with $\varphi = \phi_n$ and $a = h_s(z)$, we get

$$\begin{aligned}|I^{h_s(z)}\phi_n(X_s) - I^{h_s(z)}\phi(X_s)| &\le |\phi(X_s)||I^{h_s(z)}\zeta_n(X_s)| + |1-\zeta_n(X_s + h_s(z))||I^{h_s(z)}\phi(X_s)| \\ &\le \frac{C}{n}|\phi(X_s)||h_s(z)| + |1-\zeta_n(X_s + h_s(z))||I^{h_s(z)}\phi(X_s)| \\ &\le \frac{C}{n}|\phi(X_s)||h_s(z)| + |I^{h_s(z)}\phi(X_s)| \qquad (2.10)\end{aligned}$$

with a constant C independent of n, and since $\lim_{n\to\infty}|1-\zeta_n(X_s + h_s(z))| = 0$, we have

$$\limsup_{n\to\infty} |I^{h_s(z)}\phi_n(X_s) - I^{h_s(z)}\phi(X_s)| = 0 \quad \text{for every } (\omega,s,z).$$

Hence by (2.10), taking into account conditions (2.3) and (2.7) on h and $I^{h_s(z)}\phi(X_s)$, we can apply Lebesgue's theorem on dominated convergence to obtain

$$\lim_{n\to\infty}\int_0^T\int_Z |I^{h_s(z)}\phi(X_s)-I^{h_s(z)}\phi_n(X_s)|^2\,\mu(dz)\,ds=0\quad\text{(a.s.)},$$

which implies that for $n\to\infty$ we have

$$\int_0^t\int_Z I^{h_s(z)}\phi_n(X_s)\,\tilde\pi(dz,ds)\to\int_0^t\int_Z I^{h_s(z)}\phi(X_s)\,\tilde\pi(dz,ds)$$

in probability uniformly in $t\in[0,T]$ for each $T>0$. Similarly, we get

$$\lim_{n\to\infty}\int_0^T\int_Z |I^{\bar h_s(z)}\phi_n(X_s)-I^{\bar h_s(z)}\phi(X_s)|\,\pi(dz,ds)=0\quad\text{(a.s.)}$$

for every $T\ge 0$. Using the identity

$$J^a(\varphi\phi)(x)=\phi(x)J^a\varphi(x)+\varphi(x)J^a\phi(x)+I^a\varphi I^a\phi,\quad a,x\in\mathbb{R}^M$$

with $\varphi=\phi_n$ and $a=h_s(z)$, we get

$$\begin{aligned}&J^{h_s(z)}\phi(X_s)-J^{h_s(z)}\phi_n(X_s)\\&=(1-\zeta_n(X_s))J^{h_s(z)}\phi(X_s)+\phi(X_s)J^{h_s(z)}\zeta_n(X_s)+I^{h_s(z)}\phi(X_s)I^{h_s(z)}\zeta_n(X_s).\end{aligned}$$

Hence taking into account $|(1-\zeta_n(X_s))|\le 1$,

$$|J^{h_s(z)}\zeta_n(X_s)|\le\int_0^1(1-\theta)|h_s^i(z)h_s^j(z)D_{ij}\zeta_n(X_s+\theta h_s(z))|\,d\theta\le\frac{C}{n}|h_s(z)|^2,$$

$$|I^{h_s(z)}\phi(X_s)I^{h_s(z)}\zeta_n(x)|\le\frac{C}{n}|I^{h_s(z)}\phi(X_s)||h_s(z)|\le\frac{C}{n}(|I^{h_s(z)}\phi(X_s)|^2+|h_s(z)|^2)$$

and $\lim_{n\to\infty}|(1-\zeta_n(X_s))|=0$, we obtain

$$\begin{aligned}&|J^{h_s(z)}\phi(X_s)-J^{h_s(z)}\phi_n(X_s)|\\&\le|J^{h_s(z)}\phi(X_s)|+\frac{C}{n}(|\phi(X_s)||h_s(z)|^2+|I^{h_s(z)}\phi(X_s)|^2+|h_s(z)|^2)\end{aligned}\tag{2.11}$$

with a constant C independent of n, and

$$\lim_{n\to\infty}|J^{h_s(z)}\phi(X_s)-J^{h_s(z)}\phi_n(X_s)|=0\quad\text{for all }(\omega,s,z).$$

Thus by virtue of (2.11) and conditions (2.2), (2.7) and (2.8) on h, $I^h(X_s)$ and $J^h(X_s)$, we can use Lebesgue's theorem on dominated convergence again to get

$$\lim_{n\to\infty}\int_0^T\int_Z|J^{h_s(z)}\phi_n(X_s)-J^{h_s(z)}\phi(X_s)|\,\mu(dz)\,ds\quad\text{(a.s.)}$$

for every $T\ge 0$, which completes the proof of Theorem 2.2. □

Remark 2.1. The above theorem is useful if one can check that conditions (2.7)-(2.8) are satisfied. If $D_i\phi$ and $D_{ij}\phi$ are bounded functions for every $i,j=1,2,...,M$, then conditions (2.7)-(2.8) are always satisfied, since for every $t>0$

$$\int_0^t\int_Z|I^{h_s(z)}\phi(X_s)|^2\,\mu(dz)\,ds=\int_0^t\int_Z\Big|\int_0^1\nabla\phi(X_s+h_s(z))h_s(z)\,d\theta\Big|^2\,\mu(dz)\,ds$$

$$\leq C\int_0^t\int_Z |h_s(z)|^2\,\mu(dz)\,ds<\infty \quad \text{(a.s.)}$$

and

$$\int_0^t\int_Z |J^{h_s(z)}\phi(X_s)|\,\mu(dz)\,ds=\int_0^t\int_Z|\int_0^1(1-\theta)h_s^i(z)h_s^j(z)D_{ij}\phi(X_s+\theta h_s(z))d\theta|\,\mu(dz)\,ds$$
$$\leq C\int_0^t\int_Z|h_s(z)|^2\mu(dz)ds<\infty \quad \text{(a.s.)}$$

with a constant C. Thus by virtue of the above theorem, under the conditions (2.2) and (2.3) Itô formula (2.4) holds if the first and second order derivatives of ϕ are bounded continuous functions. As Example 2.1 shows Theorem 2.2 is not applicable to $\phi(x)=|x|^p$ for $p\geq 2$.

Next we formulate an Itô formula which holds under the natural conditions. (2.2)-(2.3)

Theorem 2.3. *Let conditions* (2.2) *and* (2.3) *hold, and let ϕ from $C^2(\mathbb{R}^M)$. Then $\phi(X_t)$ is a semimartingale such that*

$$\phi(X_t)=\phi(X_0)+\int_0^t D_i\phi(X_s)g_s^{ir}\,dw_s^r+\int_0^t D_i\phi(X_s)f_s^i+\tfrac{1}{2}D_iD_j\phi(X_s)g_s^{ir}g_s^{jr}\,ds$$

$$+\sum_{k=1}^m\int_0^t\int_{Z_k}\phi(X_{s-}+\bar h_s^k(z))-\phi(X_{s-})\,\pi^k(dz,ds)+\sum_{k=1}^m\int_0^t\int_{Z_k}D_i\phi(X_{s-})h_s^{ik}(z)\,\tilde\pi^k(dz,ds)$$
$$+\sum_{k=1}^m\int_0^t\int_{Z_k}\phi(X_{s-}+h_s^k(z))-\phi(X_{s-})-D_i\phi(X_{s-})h_s^{ik}(z)\,\pi^k(dz,ds) \tag{2.12}$$

almost surely for all $t\geq 0$.

Proof. We prove Theorem 2.3 by rewriting Itô formula (2.4) into equation (2.12) under the additional condition that h is bounded, and then we dispense with this condition by approximating h by bounded functions. For notational simplicity we assume $m=1$, for $m>1$ the proof goes in the same way. First in addition to the conditions (2.2) and (2.3) assume there is a constant K such that $|h|\leq K$. By Taylor's formula for $I^a\phi(v)$ and $J^a\phi(v)$, introduced in (2.9), for each $v,a\in\mathbb{R}^M$ we have

$$|I^a\phi(v)|\leq\sup_{|x|\leq|a|+|v|}|D\phi(x)||a|,\quad |J^a\phi(v)|\leq\sup_{|x|\leq|a|+|v|}|D^2\phi(x)||a|^2, \tag{2.13}$$

where $|D\phi|^2:=\sum_{i=1}^M|D_i\phi|^2$ and $|D^2\phi|^2:=\sum_{i=1}^M\sum_{j=1}^M|D_iD_j\phi|^2$. Since $(X_t)_{t\geq0}$ is a cadlag process, $R:=\sup_{t\leq T}|X_t|$ is a finite random variable for each fixed T. Thus we have

$$\int_0^T\int_Z|J^{h_t(z)}\phi(X_{t-})|\mu(dz)\,dt\leq\sup_{|x|\leq R+K}|D^2\phi(x)|\int_0^T\int_Z|h_t(z)|^2\,\mu(dz)\,dt<\infty \tag{2.14}$$

and

$$\int_0^T\int_Z|J^{h_t(z)}\phi(X_{t-})|^2\mu(dz)\,dt\leq\sup_{|x|\leq R+K}|D^2\phi(x)|^2K^2\int_0^T\int_Z|h_t(z)|^2\,\mu(dz)\,dt<\infty \tag{2.15}$$

almost surely. Clearly,

$$\int_0^T \int_Z |D_i\phi(X_{t-})h_t^i(z)|^2\,\mu(dz)\,dt \le \sup_{|x|\le R}|D\phi(x)|^2 \int_0^T \int_Z |h_t(z)|^2\,\mu(dz)\,dt < \infty \text{ (a.s.)}.$$

Hence, by virtue of (2.15) the stochastic Itô integral

$$\int_0^t \int_Z \phi(X_{s-}+h_t(z)) - \phi(X_s)\,\tilde\pi(dz,ds) = \int_0^t \int_Z I^{h_s(z)}\phi(X_{s-})\tilde\pi(dz,ds)$$

can be decomposed as

$$\int_0^t \int_Z I^{h_s(z)}\phi(X_{s-})\tilde\pi(dz,ds) = \int_0^t \int_Z J^{h_s(z)}\phi(X_{s-})\,\tilde\pi(dz,ds) + \int_0^t \int_Z D_i\phi(X_{s-})h_s^i(z)\,\tilde\pi(dz,ds),$$

and by virtue of (2.14) and (2.15),

$$\int_0^t \int_Z J^{h_s(z)}\phi(X_{s-})\,\tilde\pi(dz,ds) + \int_0^t \int_Z J^{h_s(z)}\phi(X_{s-})\,\mu(dz)\,ds = \int_0^t \int_Z J^{h_s(z)}\phi(X_{s-})\,\pi(dz,ds).$$

Hence

$$\begin{aligned}&\int_0^t \int_Z I^{h_s(z)}\phi(X_{s-})\,\tilde\pi(dz,ds) + \int_0^t \int_Z J^{h_s(z)}\phi(X_{s-})\,\mu(dz)\,ds\\ &= \int_0^t \int_Z D_i\phi(X_{s-})h_s^i(z)\,\tilde\pi(dz,ds) + \int_0^t \int_Z J^{h_s(z)}\phi(X_{s-})\,\pi(dz,ds),\end{aligned}$$

which shows that Theorem 2.3 holds under the additional condition that $|h|$ is bounded. To prove the theorem in full generality we approximate h by $h^{(n)} = (h^{1(n)},...,h^{M(n)})$, where $h_t^{in} = -n \vee h_t^i \wedge n$ for integers $n \ge 1$, and define

$$X_t^{(n)} := X_0 + \int_0^t f_s\,ds + \int_0^t g_s^r\,dw_s^r + \int_0^t \int_Z \bar h_s(z)\,\pi(dz,ds) + \int_0^t \int_Z h_s^{(n)}(z)\,\tilde\pi(dz,ds), \quad t \in [0,T].$$

Clearly, for all (ω,t,z)

$$|h^{(n)}| \le \min(|h|, nM) \quad \text{and} \quad h^{(n)} \to h \quad \text{as } n \to \infty.$$

Therefore Theorem 2.3 for $X^{(n)}$ holds, and

$$\lim_{n\to\infty} \int_0^T \int_Z |h_t^{(n)}(z) - h_t(z)|^2\,\mu(dz)\,dt = 0 \text{ (a.s.)},$$

which implies

$$\sup_{t\le T}|X_t^{(n)} - X_t| \to 0 \quad \text{in probability as } n \to \infty.$$

Thus there is a strictly increasing subsequence of positive integers $(n_k)_{k=1}^\infty$ such that

$$\lim_{k\to\infty}\sup_{t\le T}|X_t^{(n_k)} - X_t| = 0 \quad \text{(a.s.)},$$

which implies

$$\rho := \sup_{k\ge1}\sup_{t\le T}|X_t^{(n_k)}| < \infty \quad \text{(a.s.)}.$$

Hence it is easy to pass to the limit $k \to \infty$ in $\phi(X_t^{(n_k)})$ and in the first two integral terms in the equation for $\phi(X_t^{(n_k)})$ in Theorem 2.3. To pass to the limit in the other terms in this equation notice that since $\pi(dz, dt)$ is a counting measure of a point process, from the condition for $\bar{h}$ in (2.3) we get

$$\xi := \pi\text{-ess sup}\, |\bar{h}| < \infty \text{ (a.s.)}, \tag{2.16}$$

where π-ess sup denotes the essential supremum operator with respect to the measure $\pi(dz, dt)$ over $Z \times [0, T]$. Similarly, from the condition for h we have

$$\eta := \pi\text{-ess sup}\, |h| < \infty \text{ (a.s.)}. \tag{2.17}$$

This can be seen by noting that for the sequence of predictable stopping times

$$\tau_j = \inf\left\{t \in [0, T] : \int_0^t \int_Z |h_s(z)|^2 \,\mu(dz)\, ds \geq j\right\}, \quad j = 1, 2, ...,$$

we have

$$E \int_0^T \int_Z \mathbf{1}_{t \leq \tau_j} |h_t(z)|^2 \,\pi(dz, dt) = E \int_0^T \int_Z \mathbf{1}_{t \leq \tau_j} |h_t(z)|^2 \,\mu(dz)\, dt \leq j < \infty,$$

which gives

$$\int_0^T \int_Z |h_t(z)|^2 \,\pi(dz, dt) < \infty \quad \text{almost surely on } \Omega_j = \{\omega \in \Omega : \tau_j \geq T\} \text{ for each } j \geq 1.$$

Since $(\tau_j)_{j=1}^\infty$ is an increasing sequence converging to infinity, we have $P(\cup_{j=1}^\infty \Omega_j) = 1$, i.e.,

$$\int_0^T \int_Z h_t^2(z)\, \pi(dz, dt) < \infty \text{ (a.s.)}, \tag{2.18}$$

which implies (2.17). By (2.16) and the first inequality in (2.13), we have

$$|I^{\bar{h}_t(z)} \phi(X_{t-}^{(n_k)})| + |I^{\bar{h}_t(z)} \phi(X_{t-})| \leq 2 \sup_{|x| \leq \rho + \xi} |D\phi(x)||\bar{h}_t(z)| < \infty$$

almost surely for $\pi(dz, dt)$-almost every $(z, t) \in Z \times [0, T]$. Hence by Lebesgue's theorem on dominated convergence we get

$$\lim_{k \to \infty} \int_0^T \int_Z |I^{\bar{h}_s(z)} \phi(X_{s-}^{(n_k)}) - I^{\bar{h}_s(z)} \phi(X_{s-})| \,\pi(dz, ds) = 0 \quad \text{(a.s.)},$$

which implies that for $k \to \infty$

$$\int_0^t \int_Z I^{\bar{h}_s(z)} \phi(X_{s-}^{(n_k)})\, \pi(dz, ds) \to \int_0^t \int_Z I^{\bar{h}_s(z)} \phi(X_{s-})\, \pi(dz, ds)$$

almost surely, uniformly in $t \in [0, T]$. Clearly,

$$|D_i \phi(X_{t-}^{(n_k)}) h_t^{i(n_k)}(z)|^2 + |D_i \phi(X_{t-}) h_t^i(z)|^2 \leq 2 \sup_{|x| \leq \rho} |D\phi(x)|^2 |h_t(z)|^2$$

almost surely for all $(z, t) \in Z \times [0, T]$. Hence by Lebesgue's theorem on dominated convergence

$$\lim_{k \to \infty} \int_0^T \int_Z |D_i \phi(X_{t-}^{(n_k)}) h_t^{i(n_k)}(z) - D_i \phi(X_{t-}) h_t^i(z)|^2 \,\mu(dz)\, dt = 0 \quad \text{(a.s.)},$$

which implies that for $k \to \infty$

$$\int_0^t \int_Z D_i\phi(X_{s-}^{(n_k)})h_s^{i(n_k)}(z)\,\tilde{\pi}(dz,ds) \to \int_0^t \int_Z D_i\phi(X_{s-})h_s^i(z)\,\tilde{\pi}(dz,ds)$$

in probability, uniformly in $t \in [0,T]$. Finally note that by using the second inequality in (2.13) together with (2.17) we have

$$|J^{h_t^{(n_k)}(z)}\phi(X_{t-}^{(n_k)})| + |J^{h_t(z)}\phi(X_{t-})| \le 2 \sup_{|x|\le\rho+\eta} |D^2\phi(x)||h_t(z)|^2$$

almost surely for $\pi(dz,dt)$-almost every $(z,t) \in Z \times [0,T]$. Hence, taking into account (2.18), by Lebesgue's theorem on dominated convergence we obtain

$$\lim_{k\to\infty} \int_0^T \int_Z |J^{h_t^{(n_k)}(z)}\phi(X_{t-}^{(n_k)}) - J^{h_t(z)}\phi(X_{t-})|\,\pi(dz,dt) = 0 \quad \text{(a.s.)},$$

which implies that for $k \to \infty$

$$\int_0^t \int_Z J^{h_s^{(n_k)}(z)}\phi(X_{s-}^{(n_k)})\,\pi(dz,ds) \to \int_0^t \int_Z J^{h_s(z)}\phi(X_{s-})\,\pi(dz,ds)$$

almost surely, uniformly in $t \in [0,T]$ for every $T > 0$, that finishes the proof of the theorem. □

Remark 2.2. One can give a different proof of Theorem 2.3 by showing that for finite measures μ^k, the Itô formula for general semimartingales, Theorem VIII.27 in [6], applied to $(X_t)_{t\ge0}$, can be rewritten as equation (2.12). Hence by an approximation procedure one can get the general case of σ-finite measures μ^k.

Corollary 2.4. *Let conditions* (2.2) *and* (2.3) *hold. Then for any* $p \ge 2$ *the process* $|X_t|^p$ *is a semimartingale such that*

$$\begin{aligned}
|X_t|^p =& |\psi|^p + p\int_0^t |X_s|^{p-2}X_s^i g_s^{ir}\,dw_s^r \\
&+ \tfrac{p}{2}\int_0^t \left(2|X_s|^{p-2}X_s^i f_s^i + (p-2)|X_s|^{p-4}|X_s^i g_s^{i\cdot}|_{l_2}^2 + \sum_{i=1}^M |X_s|^{p-2}|g_s^{i\cdot}|_{l_2}^2\right) ds \\
&+ \sum_{k=1}^m p\int_0^t \int_{Z_k} |X_{s-}|^{p-2}X_{s-}^i h_s^{ik}(z)\,\tilde{\pi}^k(dz,ds) \\
&+ \sum_{k=1}^m \int_0^t \int_{Z_k} (|X_{s-} + \bar{h}_s^k|^p - |X_{s-}|^p)\,\pi^k(dz,ds) \\
&+ \sum_{k=1}^m \int_0^t \int_{Z_k} \left(|X_{s-} + h_s^k|^p - |X_{s-}|^p - p|X_{s-}|^{p-2}X_{s-}^i h_s^{ik}\right)\pi^k(dz,ds) \qquad (2.19)
\end{aligned}$$

almost surely for all $t \ge 0$, *where, and through the paper, the convention* $0/0 := 0$ *is used whenever it occurs.*

Proof. Notice that $\phi(x) = |x|^p$ for $p \geq 2$ belongs to $C^2(\mathbb{R}^M)$ with

$$D_i|x|^p = p|x|^{p-2}x^i, \quad D_jD_i|x|^p = p(p-2)|x|^{p-4}x^ix^j + p|x|^{p-2}\delta_{ij},$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. Hence it is easy to see that Theorem 2.3 applied to $\phi(x) = |x|^p$ gives the corollary. □

The above corollary will be used to obtain an Itô's formulas for jump processes in L_p-spaces presented in the next section.

3. Itô formula in L_p spaces

Itô formulas in infinite dimensional spaces play important roles in studying stochastic PDEs. Our theorem below is motivated by applications in the theory of stochastic integro-differential equations arising in nonlinear filtering theory of jump diffusions. To present it first we need to introduce some notations, where T is a fixed positive number, and $d \geq 1$ and $M \geq 1$ are fixed integers.

The Borel σ-algebra of a topological space V is denoted by $\mathcal{B}(V)$. For $p, q \geq 1$ we denote by $L_p = L_p(\mathbb{R}^d, \mathbb{R}^M)$ and $\mathcal{L}_q = \mathcal{L}_q(Z, \mathbb{R}^M)$ the Banach spaces of $\mathbb{R}^M$-valued Borel-measurable functions of $f = (f^i(x))_{i=1}^M$ and $\mathcal{Z}$-measurable functions $h = (h^i(z))_{i=1}^M$ of $x \in \mathbb{R}^d$ and $z \in Z$, respectively such that

$$|f|_{L_p}^p = \int_{\mathbb{R}^d} |f(x)|^p \, dx < \infty \quad \text{and} \quad |h|_{\mathcal{L}_q}^q = \int_Z |h(z)|^q \, \mu(dz) < \infty.$$

The notation $\mathcal{L}_{p,q}$ means the space $\mathcal{L}_p \cap \mathcal{L}_q$ with the norm

$$|v|_{\mathcal{L}_{p,q}} = \max(|v|_{\mathcal{L}_p}, |v|_{\mathcal{L}_q}) \quad \text{for } v \in \mathcal{L}_p \cap \mathcal{L}_q.$$

As usual, W_p^1 denotes the space of functions $u \in L_p$ such that $D_iu \in L_p$ for every $i = 1, 2, ..., d$, where D_iv means the generalised derivative of v in x^i for locally integrable functions v on $\mathbb{R}^d$. The norm of $u \in W_p^1$ is defined by

$$|u|_{W_p^1} = |u|_{L_p} + \sum_{i=1}^d |D_iu|_{L_p}.$$

We use the notation $L_p = L_p(\ell_2)$ for $L_p(\mathbb{R}^d, \ell_2)$, the space of Borel-measurable functions $g = (g^{ir})$ on $\mathbb{R}^d$ with values in ℓ_2 such that

$$|g|_{L_p}^p = \int_{\mathbb{R}^d} |g(x)|_{\ell_2}^p \, dx < \infty.$$

For $p, q \in [0, \infty)$ we denote by $L_p = L_p(\mathcal{L}_{p,q})$ and $L_p = L_p(\mathcal{L}_q)$ the Banach spaces of Borel-measurable functions $h = (h^i(x, z))$ and $\tilde{h} = (\tilde{h}^i(x, z))$ of $x \in \mathbb{R}^d$ with values in $\mathcal{L}_{p,q}$ and $\mathcal{L}_q$, respectively, such that

$$|h|_{L_p}^p = \int_{\mathbb{R}^d} |h(x, \cdot)|_{\mathcal{L}_{p,q}}^p \, dx < \infty \quad \text{and} \quad |\tilde{h}|_{L_p}^p = \int_{\mathbb{R}^d} |\tilde{h}(x, \cdot)|_{\mathcal{L}_q}^p \, dx < \infty.$$

For $p \geq 2$ and a separable real Banach space V we denote by $\mathbb{L}_p = \mathbb{L}_p(V)$ the space of predictable V-valued functions $f = (f_t)$ of $(\omega, t) \in \Omega \times [0, T]$ such that

$$|f|^p_{\mathbb{L}_p} = E \int_0^T |f_t|^p_V \, dt < \infty.$$

In the sequel V will be $L_p(\mathbb{R}^d, \mathbb{R}^M)$, $L_p(\mathbb{R}^d, \ell_2)$, or $L_p(\mathbb{R}^d, \mathcal{L}_{p,2})$. When $V = L_p(\mathbb{R}^d, \mathcal{L}_{p,2})$ then for $\mathbb{L}_p(V)$ the notation $\mathbb{L}_{p,2}$ is also used. For $\varepsilon \in (0, 1)$ and locally integrable functions v of $x \in \mathbb{R}^d$ we use the notation $v^{(\varepsilon)}$ for the mollifications of v,

$$v^{(\varepsilon)}(x) = \int_{\mathbb{R}^d} v(x - y) k_\varepsilon(y) \, dy, \quad x \in \mathbb{R}^d, \tag{3.1}$$

where $k_\varepsilon(y) = \varepsilon^{-d} k(y/\varepsilon)$ for $y \in \mathbb{R}^d$ with a fixed function $k \in C_0^\infty$ of unit integral. If v is a locally Bochner-integrable function on $\mathbb{R}^d$, taking values in a Banach space, then the mollification of v is defined as (3.1) in the sense of Bochner integral.

Recall that the summation convention with respect to integer valued indices is used throughout the paper.

Assumption 3.1. Let ψ^i be an $L_p(\mathbb{R}^d, \mathbb{R})$-valued $\mathcal{F}_0$-measurable random variable, $(u^i_t)_{t \in 0,T}$ be a progressively measurable L_p-valued process and let $f^{i\alpha}$, $g^i = (g^{ir})_{r=1}^\infty$ and h^i be predictable functions on $\Omega \times [0, T] \times Z$ with values in $L_p(\mathbb{R}^d, \mathbb{R})$, $L_p(\mathbb{R}^d, l_2)$ and $L_p(\mathbb{R}^d, \mathcal{L}_{p,2})$, respectively, for each $i = 1, 2, ..., M$ and $\alpha = 0, 1, ..., d$, such that the following conditions are satisfied for each $i = 1, 2, ..., M$.

(i) We have $u^i_t \in W^1_p$ for $P \otimes dt$-a.e. $(\omega, t) \in \Omega \times [0, T]$ such that

$$\int_0^T |u^i_t|^p_{W^1_p} \, dt < \infty \quad \text{(a.s.)}. \tag{3.2}$$

(ii) Almost surely

$$\mathcal{K}^p_p(T) := \sum_{i=1}^M \int_0^T \int_{\mathbb{R}^d} \sum_\alpha |f^{i\alpha}_t(x)|^p + |g^i_t(x)|^p_{l_2} + |h^i_t(x)|^p_{\mathcal{L}_{p,2}} \, dx \, dt < \infty \tag{3.3}$$

(iii) For every $\varphi \in C_0^\infty(\mathbb{R}^d)$ we have

$$(u^i_t, \varphi) = (\psi, \varphi) + \int_0^t (f^{i\alpha}_s, D^*_\alpha \varphi) \, ds + \int_0^t (g^{ir}_s, \varphi) \, dw^r_s + \int_0^t \int_Z (h^i_s(z), \varphi) \, \tilde{\pi}(dz, ds) \tag{3.4}$$

for $P \otimes dt$-almost every $(\omega, t) \in \Omega \times [0, T]$.

In equation (3.4), and later on, we use the notation (v, ϕ) for the Lebesgue integral over $\mathbb{R}^d$ of the product $v\phi$ for functions v and ϕ on $\mathbb{R}^d$ when their product and its integral are well-defined. Below u stands for $(u^1, ..., u^M)$.

Theorem 3.1. *Let Assumption 3.1 hold with $p \geq 2$. Then there is an $L_p(\mathbb{R}^d, \mathbb{R}^M)$-valued adapted cadlag process $\bar{u} = (\bar{u}^i_t)_{t \in [0,T]}$ such that equation (3.4), with $\bar{u}$ in place of u, holds for each $\varphi \in C_0^\infty(\mathbb{R}^d)$ almost surely for all $t \in [0, T]$. Moreover, $u = \bar{u}$ for $P \otimes dt$-almost every $(\omega, t) \in \Omega \times [0, T]$, and almost surely*

$$|\bar{u}_t|^p_{L_p} = |\psi|^p_{L_p} + p \int_0^t \int_{\mathbb{R}^d} |\bar{u}_s|^{p-2} \bar{u}^i_s g^{ir}_s \, dx \, dw^r_s$$

$$+\frac{p}{2}\int_0^t\int_{\mathbb{R}^d} 2|u_s|^{p-2}\bar u_s^i f_s^{i0} - 2|u_s|^{p-2}D_k u_s^i f_s^{ik} - (p-2)|u_s|^{p-4}u_s^i f_s^{ik} D_k|u_s|^2\,dx\,ds$$

$$+\frac{p}{2}\int_0^t\int_{\mathbb{R}^d}(p-2)|u_s|^{p-4}|u_s^i g_s^{i\cdot}|_{l_2}^2 + |u_s|^{p-2}\sum_{i=1}^M |g_s^{i\cdot}|_{l_2}^2\,dx\,ds$$

$$+p\int_0^t\int_Z\int_{\mathbb{R}^d}|u_{s-}|^{p-2}u_{s-}^i h_s^i\,dx\,\tilde\pi(dz,ds)$$

$$+\int_0^t\int_Z\int_{\mathbb{R}^d}(|\bar u_{s-}+h_s|^p - |\bar u_{s-}|^p - p|\bar u_{s-}|^{p-2}\bar u_{s-}^i h_s^i)\,dx\,\pi(dz,ds) \tag{3.5}$$

for all $t\in[0,T]$, where $\bar u_{s-}$ means the left-hand limit in L_p of $\bar u$ at s. If $f^i=0$ for $i=1,2,...,d$ then the above statements hold if Assumption 3.1 is satisfied with (i) replaced in it with the weaker condition that

$$\int_0^T |u_t^i|_{L_p}^p\,dt<\infty \quad \text{(a.s.)}. \tag{3.6}$$

Notice that for $M=1$ equation (3.5) has the simpler form

$$|\bar u_t|_{L_p}^p = |\psi|_{L_p}^p + p\int_0^t\int_{\mathbb{R}^d}|u_s|^{p-2}u_s g_s^r\,dx\,dw_s^r$$

$$+\frac{p}{2}\int_0^t\int_{\mathbb{R}^d}\big(2|u_s|^{p-2}u_s f_s^0 - 2(p-1)|u_s|^{p-2}f_s^i D_i u_s + (p-1)|u_s|^{p-2}|g_s|_{l_2}^2\big)\,dx\,ds$$

$$+p\int_0^t\int_Z\int_{\mathbb{R}^d}|\bar u_{s-}|^{p-2}\bar u_{s-}h_s\,dx\,\tilde\pi(dz,ds)$$

$$+\int_0^t\int_Z\int_{\mathbb{R}^d}\big(|\bar u_{s-}+h_s|^p - |\bar u_{s-}|^p - p|\bar u_{s-}|^{p-2}\bar u_{s-}h_s\big)\,dx\,\pi(dz,ds). \tag{3.7}$$

Theorem 3.1 generalises Theorem 2.1 from [14], and we use ideas and methods from [14] to prove it. The basic idea in [14] adapted to our situation can be explained as follows. Assume first that $f^{i\alpha}=0$ for $\alpha=1,2,...,d$, and suppose from (3.4) we could show the existence of a random field $\bar u=\bar u(t,x)$ and suitable modifications of the integrals of $f^i:=f_s^{i0}(x)$, $g=g_s^{ir}(x)$ and $h_s^i(x,z)$ against ds, dw_s^r and $\tilde\pi(dz,ds)$, respectively, satisfying appropriate measurability conditions such that the equation

$$\bar u_t^i(x) = \psi^i(x) + \int_0^t f_s^i(x)\,ds + \int_0^t g_s^{ir}(x)\,dw_s^r + \int_0^t\int_Z h_s^i(x,z)\,\tilde\pi(dz,ds) \tag{3.8}$$

holds for every $x\in\mathbb{R}^d$ and $i=1,2,...,M$. Then applying Itô's formula (2.19) from Corollary 2.4 to $|\bar u_t(x)|^p=(\sum_i|\bar u_t^i(x)|^2)^{p/2}$ for every $x\in\mathbb{R}^d$, then integrating over $\mathbb{R}^d$, and finally using suitable stochastic Fubini theorems, we could obtain (3.5) when $f^{i\alpha}=0$ for $\alpha\ge1$. When $f^{i\alpha}\ne0$ we could take

$$u^{i(\varepsilon)},\quad \psi^{i(\varepsilon)},\quad f^{i(\varepsilon)} := f^{i0(\varepsilon)} + \sum_{k=1}^d D_k f^{ik(\varepsilon)},\quad g^{ir(\varepsilon)}\quad\text{and}\quad h^{i(\varepsilon)}$$

instead of u^i, ψ^i, f^i, g^{ir} and h^i above, respectively to apply the theorem in the special case, and let $\varepsilon \to 0$ in the corresponding Itô formula after integrating by parts in the terms containing $D_k f^{ik(\varepsilon)}$ for $k = 1,....,d$. Notice that we can formally obtain equation (3.8) from (3.4) with $f^{i1} = ... = f^{id} = 0$ and a suitable process $\bar{u}$ in place of u, by substituting δ_x, the Dirac delta at x, in place of φ. Clearly, we cannot substitute δ_x, but we can substitute approximations $k_\varepsilon(x - \cdot)$ of it to get

$$\bar{u}_t^{i(\varepsilon)}(x) = \psi^{i(\varepsilon)}(x) + \int_0^t f_s^{i(\varepsilon)}(x)\,ds + \int_0^t g_s^{ir(\varepsilon)}(x)\,dw_s^r + \int_0^t \int_Z h_s^{i(\varepsilon)}(x,z)\,\tilde{\pi}(dz,ds) \tag{3.9}$$

in place of (3.8). Therefore the above strategy is modified as follows. One chooses suitable representative of the stochastic integrals in (3.9) so that one could apply Itô's formula (2.19) to $|\bar{u}_t^{(\varepsilon)}(x)|^p$ for each $x \in \mathbb{R}^d$, integrate the obtained formula over $\mathbb{R}^d$, then interchange the order of the integrals, and finally let $\varepsilon \to 0$ to prove equation (3.5) when $f^{ik} = 0$ for $i = 1, 2, ..., M$ and $k = 1, 2, ..., d$.

To implement the above idea we fix a $p \geq 2$ and introduce a class of functions $\mathcal{U}_p$, the counterpart of the class $\mathcal{U}_p$ given in [14]. Let $\mathcal{U}_p$ denote the set of $\mathbb{R}^M$-valued functions $u = u_t(x) = u_t(\omega, x)$ on $\Omega \times [0,T] \times \mathbb{R}^d$ such that

(i) u is $\mathcal{F} \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable,
(ii) for each $x \in \mathbb{R}^d$, $u_t(x)$ is $\mathcal{F}_t$-adapted,
(iii) $u_t(x)$ is cadlag in $t \in [0,T]$ for each (ω, x),
(iv) $u_t(\omega, \cdot)$ as a function of (ω, t) is L_p-valued, $\mathcal{F}_t$-adapted and cadlag in t for every $\omega \in \Omega$.

The following lemmas present suitable versions of Lebesgue and Itô integrals with values in L_p. The first two of them are obvious corollaries of Lemmas 4.3 and 4.4 in [14].

Lemma 3.2. *Let $f \in \mathbb{L}_p(V)$ for $V = L_p(\mathbb{R}^d, \mathbb{R}^M)$. Then there exists a function $m \in \mathcal{U}_p$ such that for each $\varphi \in C_0^\infty$ almost surely*

$$(m_t, \varphi) = \int_0^t (f_s, \varphi)\,ds$$

holds for all $t \in [0,T]$. Furthermore, we have

$$E\int_{\mathbb{R}^d} \sup_{t\leq T} |m_t(x)|^p\,dx \leq NT^{p-1} E\int_0^T |f_s|_{L_p}^p\,ds,$$

with a constant $N = N(p, M)$.

Lemma 3.3. *Let g be from $\mathbb{L}_p(V)$ for $V = L_p(\mathbb{R}^d, \ell_2)$. Then there exists a function $a \in \mathcal{U}_p$ such that for each $\varphi \in C_0^\infty$ almost surely*

$$(a_t, \varphi) = \sum_{r=1}^\infty \int_0^t (g_s^r, \varphi)\,dw_s^r$$

holds for all $t \in [0,T]$. Furthermore, we have

$$E\int_{\mathbb{R}^d} \sup_{t\leq T} |a_t(x)|^p\,dx \leq NT^{(p-2)/2} E\int_0^T |g_s|_{L_p}^p\,ds$$

with a constant $N = N(p, M)$.

The proof of the following lemma can be found in [10].

Lemma 3.4. *Let $h \in \mathbb{L}_{p,2}$. Then there exists a function $b \in \mathcal{U}_p$ such that for each real-valued $\varphi \in L_q(\mathbb{R}^d)$ with $q = p/(p-1)$, almost surely*

$$(b_t, \varphi) = \int_0^t \int_Z (h_s, \varphi)\, \tilde{\pi}(dz, ds) \tag{3.10}$$

for all $t \in [0, T]$, and

$$E \sup_{t \le T} |(b_t, \varphi)| \le 3T^{(p-2)/(2p)} |\varphi|_{L_q} \left(E \int_0^T |h_t|^p_{L_p(\mathcal{L}_2)}\, dt \right)^{1/p}. \tag{3.11}$$

Furthermore

$$E \int_{\mathbb{R}^d} \sup_{t \le T} |b_t(x)|^p\, dx \le NE \int_0^T |h_t|^p_{L_p(\mathcal{L}_p)}\, dt + NT^{(p-2)/2} E \int_0^T |h_t|^p_{L_p(\mathcal{L}_2)}\, dt \le N' |h|^p_{\mathbb{L}_{p,2}} \tag{3.12}$$

with constants $N = N(p, M)$ and $N' = N'(p, M, T)$.

We are now in the position to sketch the proof of Theorem 3.1. Technical details can be found in [10].

Proof of Theorem 3.1(Sketch). By using standard stopping time arguments we may assume $E|\psi^i|^p_{L_p} < \infty$ and that

$$E \int_0^T |u^i_t|^p_{W^1_p}\, dt < \infty, \quad E\mathcal{K}^p_p(T) < \infty \quad \text{and} \quad E \int_0^T |u^i_t|^p_{L_p}\, dt < \infty$$

hold in place of (3.2), (3.3) and (3.6), respectively for every $i = 1, 2, ..., M$. We prove first the last sentence of the theorem. We have $f^{ik} = 0$ for $i = 1, 2, ..., M$, $k = 1, 2, ..., d$ and use the notation $f^i := f^{i0}$. By Lemmas 3.2, 3.3 and 3.4 there exist $a = (a^i)$ and $b = (b^i)$ and $m = (m^i)$ in $\mathcal{U}_p$ such that for each $\varphi \in C_0^\infty$ almost surely

$$(a^i_t, \varphi) = \int_0^t (f^i_s, \varphi) ds, \quad (b^i_t, \varphi) = \int_0^t (g^{ir}_s, \varphi) dw^r_s$$

and

$$(m^i_t, \varphi) = \int_0^t \int_Z (h^i_s, \varphi)\, \tilde{\pi}(dz, ds)$$

for all $t \in [0, T]$ and $i = 1, ..., M$. Thus $a + b + m$ is an L_p-valued adapted cadlag process such that for $\bar{u}_t := \psi + a_t + b_t + m_t$ we have $(\bar{u}_t, \varphi) = (u_t, \varphi)$ for each $\varphi \in C_0^\infty$ for $P \otimes dt$ almost every $(\omega, t) \in \Omega \times [0, T]$. Hence, by taking a countable set $\Phi \subset C_0^\infty$ such that Φ is dense in L_q, we get that $\bar{u} = u$ for $P \otimes dt$ almost everywhere as L_p-valued functions. Moreover, for each $\varphi \in C_0^\infty$

$$(\bar{u}^i_t, \varphi) = (\psi, \varphi) + \int_0^t (f^i_s, \varphi)\, ds + \int_0^t (g^{ir}_s, \varphi)\, dw^r_s + \int_0^t \int_Z (h^i_s(z), \varphi)\, \tilde{\pi}(dz, ds) \tag{3.13}$$

almost surely for all $t\in[0,T]$, $i=1,2,...,M$, since on both sides we have cadlag processes. By the estimates of Lemmas 3.2, 3.3 and 3.4,

$$E\int_{\mathbb{R}^d}\sup_{t\le T}|u_t(x)|^p\,dx$$
$$\le N\Big(E|\psi|^p_{L_p}+|f|^p_{\mathbb{L}_p}+|g_s|^p_{\mathbb{L}_p}+|h|^p_{\mathbb{L}_{p,2}}\Big)<\infty, \tag{3.14}$$

where $N=N(p,M,T)$ is a constant. Substituting $k_\varepsilon(x-\cdot)$ in place of φ in equation (3.13), for $\varepsilon>0$ and $x\in\mathbb{R}^d$ we have (3.9) almost surely for all $t\in[0,T]$ for $i=1,2,...,M$. Hence by Corollary 2.4 for each $x\in\mathbb{R}^d$ we have almost surely

$$|\bar u^{(\varepsilon)}_t(x)|^p=|\psi^{(\varepsilon)}(x)|^p+\int_0^t p|\bar u^{(\varepsilon)}_{s-}(x)|^{p-2}\bar u^{i(\varepsilon)}_{s-}(x)g^{ir(\varepsilon)}_s(x)\,dw^r_s$$
$$+\int_0^t p|\bar u^{(\varepsilon)}_{s-}(x)|^{p-2}\bar u^{(\varepsilon)i}_{s-}f^{i(\varepsilon)}_s(x)\,ds$$
$$+\tfrac{p}{2}\int_0^t\big((p-2)|\bar u^{(\varepsilon)}_{s-}(x)|^{p-4}|\bar u^{i(\varepsilon)}_{s-}(x)g^{i\cdot(\varepsilon)}_s(x)|^2_{l_2}+|\bar u^{(\varepsilon)}_{s-}(x)|^{p-2}|g^{(\varepsilon)}_s(x)|^2_{\ell_2}\big)\,ds$$
$$+\int_0^t\int_Z p|\bar u^{(\varepsilon)}_{s-}(x)|^{p-2}\bar u^{(\varepsilon)i}_{s-}(x)h^{(\varepsilon)i}_s(x)\,\tilde\pi(dz,ds)+\int_0^t\int_Z J^{h^{(\varepsilon)}_s(x,z)}|\bar u^{(\varepsilon)}_{s-}(x)|^p\,\pi(dz,ds), \tag{3.15}$$

for all $t\in[0,T]$, where the notation

$$J^a|v|^p=|v+a|^p-|v|^p-a^iD_i|v|^p=|v+a|^p-|v|^p-pa^i|v|^{p-2}v^i$$

is used for vectors $a=(a^1,...,a^M):=\bar u^{(\varepsilon)}_{s-}(x)$ and $(v^1,...,v^M):=h^{(\varepsilon)}_s(x,z)\in\mathbb{R}^M$. Furthermore, integrating (3.15) over $\mathbb{R}^d$ and using deterministic and stochastic Fubini theorems, see in [10], we get

$$|\bar u^{(\varepsilon)}_t|^p_{L_p}=|\psi^{(\varepsilon)}|^p_{L_p}+\int_0^t\int_{\mathbb{R}^d}p|u^{(\varepsilon)}_s|^{p-2}u^{i(\varepsilon)}_sg^{ir(\varepsilon)}_s\,dx\,dw^r_s$$
$$+\tfrac{p}{2}\int_0^t\int_{\mathbb{R}^d}2|u^{(\varepsilon)}_s|^{p-2}u^{i(\varepsilon)}_sf^{i(\varepsilon)}_s+(p-2)|u^{(\varepsilon)}_s|^{p-4}|u^{i(\varepsilon)}_sg^{i\cdot(\varepsilon)}_s|^2_{l_2}+|u^{(\varepsilon)}_s|^{p-2}|g^{(\varepsilon)}_s|^2_{l_2}\,dx\,ds$$
$$+\int_0^t\int_Z\int_{\mathbb{R}^d}p|u^{(\varepsilon)}_{s-}|^{p-2}u^{i(\varepsilon)}_{s-}h^{i(\varepsilon)}_s\,dx\,\tilde\pi(dz,ds)+\int_0^t\int_Z\int_{\mathbb{R}^d}J^{h^{(\varepsilon)}_s}|u^{(\varepsilon)}_{s-}|^p\,dx\,\pi(dz,ds) \tag{3.16}$$

almost surely for all $t\in[0,T]$. Finally, by taking $\varepsilon\to0$ in (3.16), we obtain (3.5) with $f^{ik}=0$ for $i=1,2,...,M$ and $k=1,2,...,d$.

Let us prove now the other statements of the theorem. By taking $\varphi^{(\varepsilon)}$ in place of φ in equation (3.4) we get

$$(u^{i(\varepsilon)}_t,\varphi)=(\psi^{i(\varepsilon)},\varphi)+\int_0^t(f^{(i\varepsilon)}_s,\varphi)\,ds+\int_0^t(g^{ir(\varepsilon)}_s,\varphi)\,dw^r_s+\int_0^t\int_Z(h^{i(\varepsilon)}_s,\varphi)\,\tilde\pi(dz,ds) \tag{3.17}$$

for $P\otimes dt$ almost every $(\omega,t)\in\Omega\times[0,T]$ for each $\varphi\in C_0^\infty$, $i=1,2,...,m$, where

$$f^{i(\varepsilon)}_s:=\sum_{k=1}^dD_kf^{ik(\varepsilon)}_s+f^{i0(\varepsilon)}_s,\quad i=1,2,...,M,\quad k=1,2,....,d.$$

Hence by virtue of what we have proved above we have an L_p-valued adapted cadlag process $\bar u^\varepsilon=(\bar u^{i\varepsilon})$ such that for each $\varphi\in C_0^\infty$ almost surely (3.17) holds with $\bar u^{i\varepsilon}$ in place of $u^{i(\varepsilon)}$ for all $t \in [0,T]$. In particular, for each $\varphi \in C_0^\infty$ we have $(u^{(\varepsilon)}, \varphi) = (\bar{u}^\varepsilon, \varphi)$ for $P \otimes dt$-almost every $(\omega, t) \in \Omega \times [0,T]$. Thus $u^{(\varepsilon)} = \bar{u}^\varepsilon$, as L_p-valued functions, for $P \otimes dt$-almost every $(\omega, t) \in \Omega \times [0,T]$, and almost surely (3.16) holds for all $t \in [0,T]$. Moreover, using that by integration by parts

$$\int_{\mathbb{R}^d} |u_s^{(\varepsilon)}|^{p-2} u_s^{i(\varepsilon)} D_k f_s^{ik(\varepsilon)}\, dx = -\int_{\mathbb{R}^d} |u_s^{(\varepsilon)}|^{p-2} f_s^{ik(\varepsilon)} D_k u_s^{i(\varepsilon)}\, dx$$
$$-\tfrac{p-2}{2} \int_{\mathbb{R}^d} |u_s^{(\varepsilon)}|^{p-4} D_k |u_s^{(\varepsilon)}|^2 f_s^{ik(\varepsilon)} u_s^{i(\varepsilon)}\, dx$$

for $P \otimes dt$-almost every $(\omega, t) \in \Omega \times [0,T]$, we get

$$|\bar{u}_t^\varepsilon|_{L_p}^p = |\psi^{(\varepsilon)}|_{L_p}^p + p \int_0^t \int_{\mathbb{R}^d} |\bar{u}_s^\varepsilon|^{p-2} \bar{u}_s^\varepsilon g_s^{ir(\varepsilon)}\, dx\, dw_s^r$$
$$+p \int_0^t \int_{\mathbb{R}^d} |\bar{u}_s^\varepsilon|^{p-2} \bar{u}_s^\varepsilon f_s^{0(\varepsilon)} - |\bar{u}_s^\varepsilon|^{p-2} f_s^{ik(\varepsilon)} D_k u_s^{i(\varepsilon)}\, dx\, ds$$
$$-\tfrac{p}{2} \int_0^t \int_{\mathbb{R}^d} (p-2) |u_s^{(\varepsilon)}|^{p-4} D_k |u_s^{(\varepsilon)}|^2 f_s^{ik(\varepsilon)} u_s^{i(\varepsilon)}\, dx\, ds$$
$$+\int_0^t \int_{\mathbb{R}^d} (p-2) |\bar{u}_{s-}^{(\varepsilon)}|^{p-4} |\bar{u}_{s-}^{i(\varepsilon)} g_s^{i\cdot(\varepsilon)}|_{l_2}^2 + |\bar{u}_{s-}^{(\varepsilon)}|^{p-2} |g_s^{(\varepsilon)}|_{\ell_2}^2\, dx\, ds$$
$$+p \int_0^t \int_Z \int_{\mathbb{R}^d} |\bar{u}_{s-}^\varepsilon|^{p-2} \bar{u}_{s-}^\varepsilon h_s^{(\varepsilon)}\, dx\, \tilde{\pi}(dz, ds) + \int_0^t \int_Z \int_{\mathbb{R}^d} J^{h^{(\varepsilon)}} |\bar{u}_{s-}^\varepsilon|^p\, dx\, \pi(dz, ds) \tag{3.18}$$

almost surely for all $t \in [0,T]$. Hence by Davis', Minkowski and Hölder inequalities, using standard estimates we obtain

$$E \sup_{t \le T} |\bar{u}_t^\varepsilon|_{L_p}^p \le 2\, E |\psi^{(\varepsilon)}|_{L_p}^p + NE \int_0^T |h_t^{(\varepsilon)}|_{L_p(\mathcal{L}_p)}^p\, dt + NT^{p-1} |f^{0(\varepsilon)}|_{\mathbb{L}_p}^p$$
$$+NT^{(p-2)/2} \left(|g^{(\varepsilon)}|_{\mathbb{L}_p}^p + E \int_0^T |h_t^{(\varepsilon)}|_{L_p(\mathcal{L}_2)}^p\, dt + \sum_{\alpha=1}^d |f^{\alpha(\varepsilon)}|_{\mathbb{L}_p}^p + \sum_{\alpha=1}^d |D_\alpha u^{(\varepsilon)}|_{\mathbb{L}_p}^p \right) \tag{3.19}$$

with a constant $N = N(p,d)$, where $f^{\alpha(\varepsilon)} := (f^{1\alpha(\varepsilon)}, ..., f^{M\alpha(\varepsilon)})$, and recall that $|v|_{L_p}$ means the L_p-norm of $|(\sum_{i=1}^M |v^i|^2)^{1/2}|$ for $\mathbb{R}^M$-valued functions $v = (v^1, ..., v^M)$ on $\mathbb{R}^d$. Hence

$$E \sup_{t \le T} |\bar{u}_t^\varepsilon - \bar{u}_t^{\varepsilon'}|_{L_p}^p \to 0 \quad \text{as } \varepsilon,\, \varepsilon' \to 0.$$

Consequently, there is an L_p-valued adapted cadlag process $\bar{u} = (\bar{u}_t)_{t \in [0,T]}$ such that

$$\lim_{\varepsilon \to 0} E \sup_{t \le T} |\bar{u}_t^\varepsilon - \bar{u}|_{L_p}^p = 0.$$

Thus for each $\varphi \in C_0^\infty(\mathbb{R}^d)$ we can take $\varepsilon \to 0$ in

$$(\bar{u}_t^{i\varepsilon}, \varphi) = (\psi^{i(\varepsilon)}, \varphi) + \int_0^t (f_s^{i(\varepsilon)}, \varphi)\, ds + \int_0^t (g_s^{ir(\varepsilon)}, \varphi)\, dw_s^r + \int_0^t \int_Z (h_s^{i(\varepsilon)}, \varphi)\, \tilde{\pi}(dz, ds)$$
$$= (\psi^{i(\varepsilon)}, \varphi) + \int_0^t (f_s^{i0(\varepsilon)}, \varphi)\, ds - \int_0^t (f_s^{ik(\varepsilon)}, D_k \varphi)\, ds + \int_0^t (g_s^{ir(\varepsilon)}, \varphi)\, dw_s^r$$

$$+\int_0^t \int_Z (h_s^{i(\varepsilon)}, \varphi)\, \tilde{\pi}(dz, ds)$$

and it is easy to see that we get

$$(\bar{u}_t^i, \varphi) = (\psi^i, \varphi) + \int_0^t (f_s^{i\alpha}, D_\alpha^* \varphi)\, ds + \int_0^t (g_s^{ir}, \varphi)\, dw_s^r + \int_0^t \int_Z (h_s^i, \varphi)\, \tilde{\pi}(dz, ds)$$

almost surely for all $t \in [0, T]$. Hence $\bar{u} = u$ for $P \otimes dt$-almost every $(\omega, t) \in \Omega \times [0, T]$. Finally letting $\varepsilon \to 0$ in (3.18) we obtain (3.7). □

References

[1] D. Applebaum, *Lévy Processes and Stochastic Calculus*, Cambridge University Press, 2009.

[2] A. Bensoussan and J.L. Lions, *Impulse control and quasi-variational inequalities*, Bordas, 1984.

[3] Z. Brzeźniak, J.M.A.M. van Neerven, M.C. Veraar, L. Weis: Ito's formula in UMD Banach spaces and regularity of solutions of the Zakai equation, *J. Differ. Equ.*, 245 (2008), 30-58.

[4] R. Cont and P. Tankov, *Financial modeling with jump processes*, Chapman and Hall/CRC, 2003.

[5] K. Debicki and M. Mandjes, *Queues and Lévy Fluctuation Theory*, Universitext, Springer, 2015.

[6] C. Dellacherie and P-A. Meyer, *Probability and Potential B: Theory of Martingales*, North-Holland Mathematics Studies 72, Elsevier Science Ltd, 1982.

[7] I. Gyöngy, On Stochastic Equations with Respect to Semimartingales III., *Stochastics*, 7 (1982), 231–254.

[8] I. Gyöngy and N.V. Krylov, On Stochastic Equations with Respect to Semimartingales II. Itô Formula in Banach Spaces, *Stochastics*, 6 (1982), 153–173.

[9] I. Gyöngy and D. Šiška, Itô Formula for Processes Taking Values in Intersection of Finitely Many Banach Spaces, *Stochastic Partial Differential Equations: Analysis and Computations*, 5 (2017), no. 3, 428–455.

[10] I. Gyöngy and S. Wu, Itô's formula for jump processes in L_p-spaces, arXiv:1904.12898.

[11] I. Gyöngy and S. Wu, On L_p-solvability of stochastic integro-differential equations, arXiv:1907.04876v1.

[12] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, North-Holland, 2011.

[13] T. Konstantopoulos, G. Last and S. Lin, On a Class of Lévy Stochastic Networks, *Queueing Systems*, 46 (2004), 409-437.

[14] N.V. Krylov, Itô's formula for the L_p-norm of stochastic W_p^1-valued processes, *Probab. Theory Relat. Fields*, 147 (2010), 583–605.

[15] N.V. Krylov, A relatively short proof of Itô's formula for SPDEs and its applications, *Stochastic Partial Differential Equations: Analysis and Computations*, 1 (2013), 152–174.

[16] N.V. Krylov and B.L. Rozovskii. Stochastic evolution equations. *J. Sov. Math.*, 16:1233–1277, 1981.

[17] E. Pardoux, *Équations aux dériveés partielles stochastiques non linéaries monotones. Étude des solution forte de type Itô.* Thèse Université de Paris Sud, Orsay 1975.

[18] C. Prévôt and M. Röckner, *A concise course on stochastic partial differential equations.* Lecture Notes in Mathematics, 1905. Springer, Berlin, 2007.

[19] B.L. Rozovskii, *Stochastic Evolution Systems. Linear Theory and Applications to Nonlinear Filtering*, Kluwer, Dordrecht (1990).

[20] B. Rüdiger, Itô formula for stochastic integrals w.r.t. compensated Poisson random measures on separable Banach spaces, *Stochastics: An International Journal of Probability and Stochastics Processes*, Vol. 78, No. 6, December 2006, 377–410.

[21] J.M.A.M. van Neerven, M. Veraar, On the stochastic Fubini theorem in infinite dimensions. In: *Stochastic Partial Differential Equations and Applications–VII.* Volume 245 of Lect. Notes Pure Appl. Math., pp. 323–336. Chapman & Hall/CRC, Boca Raton, FL 2006.

[22] J. Zhu, Z. Brzeźniak and E. Hausenblas, Maximal inequalities for stochastic convolutions driven by compensated Poisson random measures in Banach spaces, *Annales de l'Institut Henri Poincaré-Probabilitś et Statistiques*, Vol. 53, No. 2 (2017), 937-956.

[23] J. Zhu, Z. Brzeźniak and W. Liu, Maximal inequalities and exponential estimates for stochastic convolutions driven by Lévy-type processes in Banach spaces with applications to stochastic quasi-geostrophic equations, *Siam J. Math. Anal.*, Vol. 51, No. 3, 2121-2167.

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