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A remark on Leclerc's Frobenius categories

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Abstract. Leclerc recently studied certain Frobenius categories in connection with cluster algebra structures on coordinate rings of intersections of opposite Schubert cells. We show that these categories admit a description as Gorenstein projective modules over an Iwanaga-Gorenstein ring of virtual dimension at most two. This is based on a Morita type result for Frobenius categories.

1 Motivation

Let G be a complex simple Lie group of type $Q = A, D$ or E (eg $G = \mathrm{SL}_{n+1}(\mathbb{C})$ for $Q = A_n$) with Borel subgroup $B \subset G$ (eg $B = \{\text{upper triangular matrices}\}$) and Weyl group W (eg $W \cong S_{n+1}$ given by permutation matrices).

For a Weyl group element $w \in W$ there are associated subvarieties C_w (*Schubert cell*) and C^w (*opposite Schubert cell*) in the flag variety G/B . On the other hand, there is a torsion pair $(\mathcal{C}_w, \mathcal{C}^w)$ in the category of finite dimensional modules over the preprojective algebra $\Pi := \Pi(Q)$ and the categories $\mathcal{C}_w, \mathcal{C}^w$ are Frobenius and have projective generators (in fact, the latter statements may be deduced from Proposition 5). These Frobenius categories were used by Geiß, Leclerc & Schröer to categorify cluster algebra structures on coordinate rings of the corresponding (opposite) Schubert cells [3].

Let $v \in W$. The intersections $\mathcal{C}_{v,w} := \mathcal{C}^v \cap \mathcal{C}_w$ are known as *open Richardson varieties* and have been studied by Kazhdan-Lusztig in connection with KL-polynomials. Generalizing the aforementioned work [3], Leclerc [5] categorifies a cluster subalgebra of the coordinate rings of $\mathcal{C}_{v,w}$ using the intersection $\mathcal{C}_{v,w}$ of a torsion free part \mathcal{C}^v with a torsion part \mathcal{C}_w of two torsion pairs mentioned above. Under some finiteness assumptions he obtains a cluster algebra structure on the whole coordinate ring and he conjectures that this holds in general.

The subcategories $\mathcal{C}_{v,w} \subseteq \mathbf{mod} \Pi$ inherit an exact structure which is again Frobenius.

Aim Explain this in a more abstract setting and give equivalent descriptions of $\mathcal{C}_{v,w}$.

This is summarized in the following Proposition which is a special case of Proposition 5.

Proposition 1 *Let $\mathcal{C}_{v,w} := \mathcal{C}_w \cap \mathcal{C}^v \subseteq \mathbf{mod} \Pi$. Then*

- (a) $\mathcal{C}_{v,w}$ is a Frobenius category with $\mathrm{proj} \mathcal{C}_{v,w} = \mathrm{add} f_v t_w(\Pi) = \mathrm{add} t_w f_v(\Pi) =: \mathrm{add} P_{v,w}$. Where $t_u(-)$ denotes the torsion radical and $f_u(-) := (-)/t_u(-)$ for a torsion pair $(\mathcal{C}_u, \mathcal{C}^u)$.
- (b) $\mathcal{C}_{v,w} \xrightarrow{\mathrm{Hom}_{\mathcal{C}_{v,w}}(P_{v,w}, -)} \mathrm{GP}(\Pi_{v,w})$ is an exact equivalence, where $\Pi_{v,w} := \mathrm{End}_{\mathcal{C}_{v,w}}(P_{v,w})$ is an Iwanaga-Gorenstein ring of virtual dimension at most two.
- (c) In particular, $\mathcal{C}_{v,w}$ is equivalent to the subcategory of second syzygies of finite dimensional $\Pi_{v,w}$ -modules.
- (d) The functors f_v and t_w induce ring homomorphisms $\Pi_w := \mathrm{End}_{\mathcal{C}_w}(t_w(\Pi)) \rightarrow \Pi_{v,w}$ and $\Pi^v := \mathrm{End}_{\mathcal{C}^v}(f_v(\Pi)) \rightarrow \Pi_{v,w}$. These are surjective if $\mathcal{C}_v \subseteq \mathcal{C}_w$. In turn, this condition is equivalent to $w = v'v$ with $l(w) = l(v') + l(v)$, called condition (P) in Leclerc [5, 5.1].

- (e) (see [1, 5.16]) If condition (P) holds, then $\Pi_{v,w}$ is Morita equivalent to $\Pi_{v'}$. Therefore, $\Pi_{v,w}$ has the same virtual dimension as $\Pi_{v'}$ which is at most 1, [2].

Remark 2 Let $\Lambda_w := \Pi/I_w$ be the algebra considered in [2]. Then there are algebra isomorphisms $\Lambda_w \cong \Pi^{w_0 w^{-1}} \cong \Pi_{w^{-1}}^{\text{op}}$, where w_0 denotes the longest Weyl group element.

2 A Morita type result for Frobenius categories

Definition/Proposition 3 A two-sided Noetherian ring R is called *Iwanaga-Gorenstein*, if $\text{inj. dim}_R R < \infty$ and $\text{inj. dim } R_R < \infty$. It is well-known that this implies $\text{inj. dim}_R R = d = \text{inj. dim } R_R$. We call $d =: \text{vir. dim } R$ the *virtual dimension* of R .

In this case the category of *Gorenstein-projective* R -modules

$$\text{GP}(R) := \{M \in \text{mod } R \mid \text{Ext}_R^i(M, R) = 0 \text{ for all } i > 0\}$$

is a Frobenius category with subcategory of projective-injective objects $\text{proj } R$. Equivalently, $\text{GP}(R)$ is the subcategory of d -th syzygies of finitely generated R -modules

$$\text{GP}(R) \cong \Omega^d(\text{mod } R) := \{\Omega^d(M) \mid M \in \text{mod } R\}.$$

If R is a local commutative Noetherian ring, Gorenstein projective R -modules are precisely maximal Cohen-Macaulay R -modules and $\text{inj. dim}_R R = \text{kr. dim } R$.

Aim Characterize the categories of Gorenstein projective modules $\text{GP}(R)$ over Iwanaga-Gorenstein rings R among all Frobenius categories.

Notation For an additive category \mathcal{B} , we denote by $\text{mod } \mathcal{B}$ the category of finitely presented contravariant additive functors $\mathcal{B} \rightarrow \text{Ab}$.

We first list properties of the categories $\mathcal{E} := \text{GP}(R)$ for R Iwanaga-Gorenstein.

- (i) $\text{proj } \mathcal{E} = \text{add } P (= \text{proj } R)$ for some $P \in \mathcal{E}$ and $\text{End}_{\mathcal{E}}(P) (\cong \text{End}_R(R))$ is two-sided noetherian.
- (ii) \mathcal{E} is idempotent complete (since $\mathcal{E} \subseteq \text{mod } R$ closed under direct summands).
- (iii) \mathcal{E} is Frobenius (use exact duality $\text{Hom}_R(-, R): \text{GP}(R) \rightarrow \text{GP}(R^{\text{op}})$).
- (iv) \mathcal{E} has weak kernels and cokernels (use Auslander-Buchweitz approximation).
- (v) $\text{gl. dim mod } \mathcal{E}, \text{ gl. dim mod } \mathcal{E}^{\text{op}} \leq n (= \max\{2, \text{inj. dim } R\})$.

The following result may be interpreted as an analogue of Morita theory for Frobenius categories. The implication (b) \Rightarrow (a) is well-known. The converse is the special case $\text{proj } \mathcal{E} = \text{add } \mathcal{P}, \mathcal{M} = \mathcal{E}$ of [4, 2.8], which is due to Iyama and inspired by a stable version of Dong Yang and the author [4, 2.15].

Proposition 4 Let \mathcal{E} be an exact category and let $P \in \mathcal{E}$. TFAE

- (a) \mathcal{E} and P satisfy the conditions (i)-(v) above.
- (b) Set $R = \text{End}_{\mathcal{E}}(P)$. $\text{Hom}_{\mathcal{E}}(P, -): \mathcal{E} \rightarrow \text{GP}(R)$ is an exact equivalence and R is Iwanaga-Gorenstein with $\text{vir. dim } R \leq \text{gl. dim mod } \mathcal{E}$.

3 From pairs of torsion pairs to Frobenius categories

Notation Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an abelian category \mathcal{A} . In particular, there is a short exact sequence $0 \rightarrow t(X) \rightarrow X \rightarrow f(X) \rightarrow 0$ for all X in \mathcal{A} . This gives rise to functors $t: \mathcal{A} \rightarrow \mathcal{T}$ and $f: \mathcal{A} \rightarrow \mathcal{F}$, which are right (respectively left) adjoint to the canonical inclusions.

Proposition 5 *Let \mathcal{A} be an abelian category with torsion pairs $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ and set $\mathcal{C}_{12} := \mathcal{T}_1 \cap \mathcal{F}_2$. Then the following statements hold:*

- (a) \mathcal{C}_{12} is extension closed and idempotent complete, since \mathcal{T}_1 and \mathcal{F}_2 are. In particular, \mathcal{C}_{12} inherits a natural exact structure from \mathcal{A} .
- (b) \mathcal{C}_{12} has kernels and cokernels. In other words, \mathcal{C}_{12} is a preabelian category. In particular, the categories of finitely presented additive functors $\text{mod } \mathcal{C}_{12}$ and $\text{mod } \mathcal{C}_{12}^{\text{op}}$ are abelian and have global dimension at most 2.

For example, the composition of the canonical inclusions

$$t_1(\ker f) \hookrightarrow \ker f \hookrightarrow X \xrightarrow{f} Y$$

is a kernel of f . Here $\ker f$ denotes the kernel of f in \mathcal{A} .

- (c) *If \mathcal{T}_1 has enough projectives and \mathcal{F}_2 has enough injectives, then \mathcal{C}_{12} has enough injectives ($= t_1(\text{inj } \mathcal{F}_2)$) and projectives ($= f_2(\text{proj } \mathcal{T}_1)$).*
- (d) *If additionally $\text{Ext}_{\mathcal{C}_{12}}^1(X, Y) = 0 \Leftrightarrow \text{Ext}_{\mathcal{C}_{12}}^1(Y, X) = 0$, then \mathcal{C}_{12} is Frobenius. For example, this is satisfied if $\underline{\mathcal{A}}$ or $\mathcal{D}^b(\mathcal{A})$ are 2-Calabi-Yau. This in turn is known to hold for $\mathcal{A} = \text{fdmod}(\widehat{\Pi(Q)})$, where Q is a quiver without loops and $\widehat{\Pi(Q)}$ is the m -adic completion of its preprojective algebra, where m denotes the ideal generated by all arrows.*
- (e) *Assume additionally that $\text{proj } \mathcal{T}_1 = \text{add } P$ and $\text{inj } \mathcal{F}_2 = \text{add } I$, then $\text{proj } \mathcal{C}_{12} = \text{add } f_2(P) = \text{add } t_1(I)$. We assume that $\Pi_{12} := \text{End}_{\mathcal{C}_{12}}(f_2(P))$ is two-sided noetherian. Then there is an exact equivalence*

$$\mathcal{C}_{12} \xrightarrow{\text{Hom}_{\mathcal{C}_{12}}(f_2(P), -)} \text{GP}(\Pi_{12}),$$

and Π_{12} is Iwanaga-Gorenstein of virtual dimension at most 2.

- (f) *In the situation of (e) the functors f_2 and t_1 induce ring homomorphisms $\varphi_2: \text{End}_{\mathcal{T}_1}(P) \rightarrow \Pi_{12}$ and $\tau_1: \text{End}_{\mathcal{F}_2}(I) \rightarrow \Pi_{12}$ with kernels given by the ideals of morphisms factoring over $t_2(P)$ and $f_1(I)$, respectively. The ring homomorphisms are surjective if $\mathcal{T}_2 \subseteq \mathcal{T}_1$. In Example 7, φ_2 is injective but not surjective.*

Remark 6 This is an analogue of Buan, Iyama, Reiten & Scott's [2] dual description of Geiß, Leclerc & Schröer's categories \mathcal{C}_w [3] as categories of submodules of projective modules over the algebra Λ_w , see also [3, Theorem 2.8]. Since Λ_w is Iwanaga-Gorenstein of virtual dimension 1, Gorenstein projective modules are first syzygies, which in turn are just submodules of projective modules. See also [4, Section 6] for a further discussion.

4 Examples, remarks and questions

Example 7 We consider the situation of [5, 3.16], i.e. Q is of type A_3 , $w = s_1 s_3 s_2 s_1 s_3$ and $v = s_2$. Then $\varphi_2: \Pi_w := \text{End}_{\mathcal{C}_w}(t_w(\Pi)) \rightarrow \Pi_{v,w}$ is injective and its cokernel in the category of vectorspaces is isomorphic to \mathbb{C} . Moreover, $\Pi_{v,w}$ is the Auslander algebra of the preprojective algebra of type A_2 and therefore is of global (and virtual) dimension 2.

Remark 8 (Duality) Let Q be a Dynkin quiver and let $D := \text{Hom}_k(-, k)$ be the standard duality. It is well-known that there is an algebra isomorphism $\psi: \Pi \cong \Pi^{\text{op}}$, which gives rise to a duality $\Phi: \text{mod } \Pi \xrightarrow{D} \text{mod } \Pi^{\text{op}} \xrightarrow{\psi_*} \text{mod } \Pi$. Using the notation in Leclerc [5, §3.2], one can check that $\Phi(P_{v,w}) \cong P_{w_0^{-1}w, w_0v}$ holds, where w_0 denotes the longest Weyl

group element. In particular, Φ induces an algebra isomorphism $\Pi_{v,w} \cong \Pi_{w_0^{-1}w, w_0v}^{\text{op}}$. Thus $\Pi^v \cong \Pi_{v,w_0} \cong \Pi_{\text{id}, w_0v}^{\text{op}} \cong \Pi_{w_0v}^{\text{op}}$ for the algebras appearing in Proposition 1 (d).

Open Problem 9 Give a 'combinatorial description' of $\Pi_{v,w}$, eg as quiver with relations.

Remark 10 The number of isoclasses of indecomposable projective $\Pi_{v,w}$ -modules seems to be unknown in general. It is not always bounded above by $|Q_0|$, see Example 7.

Question 11 (Leclerc) How does the virtual dimension of $\Pi_{v,w}$ depend on Q, v, w and (how) is this number related to the geometry of the open Richardson variety $C_{v,w}$?

Partial Answer 12 By Remark 2 and [2], $\text{vir. dim } \Pi^v, \Pi_w \leq 1$. They are zero iff \mathcal{C}^v (respectively, \mathcal{C}_w) are exact abelian subcategories of $\text{mod } \Pi$, which are then equivalent to $\text{mod } \Pi/e$ ($e \in \Pi$ idempotent). Thus if $\text{vir. dim } \Pi^v, \Pi_w = 0$, then $\text{vir. dim } \Pi_{v,w} = 0$ (since $\mathcal{C}_{v,w}$ is abelian). If one of Π^v and Π_w has virtual dimension zero, then $\mathcal{C}_{v,w}$ is the torsion (or torsion-free) part of a torsion pair in $\text{mod } \Pi/e$. By Mizuno [6] and [2], $\Pi_{v,w} \cong (\Pi/e)^{v'}$ or $\cong (\Pi/e)_{w'}$ and is therefore of virtual dimension ≤ 1 . Also $\text{gl. dim } \Pi_v = n \leq 1$ (or $\text{gl. dim } \Pi_w = m \leq 1$) implies $\text{gl. dim } \Pi_{v,w} \leq \min\{n, m\}$. If both Π_v, Π_w have infinite global dimension and virtual dimension 1, then virtual dimensions 0, 1, 2 occur for $\Pi_{v,w}$.

Remark 13 (Commutativity) It follows from work of Mizuno [6], that all torsion pairs in $\text{mod } \Pi$ are of the form $(\mathcal{C}_w, \mathcal{C}^w)$ for some Weyl group element w . In particular, there are only finitely many torsion pairs, which is very surprising given the size of $\text{mod } \Pi$. The explicit description of the associated functors t_w and f_w (see eg Leclerc [5, §3.2]) shows that $f_v t_w(M) \cong t_w f_v(M)$ for Weyl group elements $v, w \in W$ and $M \in \text{mod } \Pi$. This seems very unusual for a pair of torsion pairs in general abelian categories and fails already for $\text{mod } U_2(k)$, where $U_2(k)$ denotes the ring of 2×2 upper triangular matrices.

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