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AFFINE CONES OVER SMOOTH CUBIC SURFACES

IVAN CHELTSOV, JIHUN PARK AND JOONYEONG WON

ABSTRACT. We show that affine cones over smooth cubic surfaces do not admit non-trivial $\mathbb{G}_a$-actions.

Throughout this article, we assume that all considered varieties are algebraic and defined over an algebraically closed field of characteristic 0.

1. INTRODUCTION

One of the motivations for the present article originates from the articles of H. A. Schwartz ([34]) and G. H. Halphen ([17]) in the middle of 19th century, where they studied polynomial solutions of Brieskorn-Pham polynomial equations in three variables after L. Euler (1756), J. Liouville (1879) and so fourth ([12]). Meanwhile, since the middle of 20th century the study of rational singularities has witnessed great development ([2], [5], [26]). These two topics, one classic and the other modern, encounter each other in contemporary mathematics. For instance, there is a strong connection between the existence of a rational curve on a normal affine surface, i.e., a polynomial solution to algebraic equations, and rational singularities ([15]).

As an additive analogue of toric geometry, unipotent group actions, specially $\mathbb{G}_a$-actions, on varieties are very attractive objects to study. Indeed, $\mathbb{G}_a$-actions have been investigated for their own sake ([3], [18], [29], [35], [40]). We also observe that $\mathbb{G}_a$-actions appear in the study of rational singularities. In particular, the article [15] shows that a Brieskorn-Pham surface singularity is a cyclic quotient singularity if and only if the surface admits a non-trivial regular $\mathbb{G}_a$-action. Considering its 3-dimensional analogue, H. Flenner and M. Zaidenberg in 2003 proposed the following question ([15, Question 2.22]):

> *Does the affine Fermat cubic threefold $x^3 + y^3 + z^3 + w^3 = 0$ in $\mathbb{A}^4$ admit a non-trivial regular $\mathbb{G}_a$-action?*

Even though it is simple-looking, this problem stands open for 10 years. It turns out that this problem is purely geometric and can be considered in a much wider setting ([19], [20], [21], [22], [31]).

To see the problem from a wider view point, we let X be a smooth projective variety with a polarisation H, where H is an ample divisor on X. The *generalized cone* over (X, H) is the affine variety defined by

$$\hat{X} = \mathrm{Spec}\left(\bigoplus_{n \geqslant 0} H^0\left(X, \mathcal{O}_X\left(nH\right)\right)\right).$$

Remark 1.1. The affine variety $\hat{X}$ is the usual cone over X embedded in a projective space by the linear system $|H|$ provided that H is very ample and the image of the variety X is projectively normal.

Let S_d be a smooth del Pezzo surface of degree d and let $\hat{S}_d$ be the generalized cone over $(S_d, -K_{S_d})$. For $3 \leqslant d \leqslant 9$, the anticanonical divisor $-K_{S_d}$ is very ample and the generalized

cone $\hat{S}_d$ is the affine cone in $\mathbb{A}^{d+1}$ over the smooth variety anticanonically embedded in $\mathbb{P}^d$. In particular, for $d = 3$, the cubic surface S_3 is defined by an cubic homogenous polynomial equation $F(x, y, z, w) = 0$ in $\mathbb{P}^3$, and hence the cone $\hat{S}_3$ is the affine hypersurface in $\mathbb{A}^4$ defined by the equation $F(x, y, z, w) = 0$. For $d = 2$, the generalized cone $\hat{S}_2$ is the affine cone in $\mathbb{A}^4$ over the smooth hypersurface in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$ defined by a quasihomogeneous polynomial of degree 4. For $d = 1$, the cone $\hat{S}_1$ is the affine cone in $\mathbb{A}^4$ over the smooth hypersurface in the weighted projective space $\mathbb{P}(1, 1, 2, 3)$ defined by a quasihomogeneous polynomial of degree 6 ([16, Theorem 4.4]).

It is natural to ask whether the affine variety $\hat{S}_d$ admits a non-trivial $\mathbb{G}_a$-action. The problem at the beginning is just a special case of this.

T. Kishimoto, Yu. Prokhorov and M. Zaidenberg have been studying this generalised problem and proved the following:

Theorem 1.2. If $4 \leqslant d \leqslant 9$, then the generalized cone $\hat{S}_d$ admits an effective $\mathbb{G}_a$-action.

Proof. See [19, Theorem 3.19]. $\qquad\qquad\square$

Theorem 1.3. If $d \leqslant 2$, then the generalized cone $\hat{S}_d$ does not admit a non-trivial $\mathbb{G}_a$-action.

Proof. See [22, Theorem 1.1]. $\qquad\qquad\square$

Their proofs make good use of a geometric property called cylindricity, which is worthwhile to be studied for its own sake.

Definition 1.4 ([19]). Let M be a $\mathbb{Q}$-divisor on X. An M-polar cylinder in X is an open subset

$$U = X \setminus \mathrm{Supp}(D)$$

defined by an effective $\mathbb{Q}$-divisor D on X with $D \sim_{\mathbb{Q}} M$ such that U is isomorphic to $Z \times \mathbb{A}^1$ for some affine variety Z.

They show that the existence of an H-polar cylinder on X is equivalent to the existence of a non-trivial $\mathbb{G}_a$-action on the generalized cone over (X, H).

Lemma 1.5. Suppose that the cone $\hat{X}$ is normal. Then the cone $\hat{X}$ admits an effective $\mathbb{G}_a$-action if and only if X contains an H-polar cylinder.

Proof. See [21, Corollary 2.12]. $\qquad\qquad\square$

Remark 1.6. If X is a rational surface, then there always exists an ample Cartier divisor H on X such that $\hat{X}$ is normal and X contains an H-polar cylinder (see [19, Proposition 3.13]), which implies, in particular, that $\hat{X}$ admits an effective $\mathbb{G}_a$-action.

Indeed, what T. Kishimoto, Yu. Prokhorov and M. Zaidenberg proved for their two theorems is that the del Pezzo surface S_d has a $(-K_{S_d})$-polar cylinder if $4 \leqslant d \leqslant 9$ but no $(-K_{S_d})$-polar cylinder if $d \leqslant 2$.

The main result of the present article is

Theorem 1.7. A smooth cubic surface S_3 in $\mathbb{P}^3$ does not contain any $(-K_{S_3})$-polar cylinders.

Together with Theorems 1.2 and 1.3, this makes us reach the following conclusion.

Corollary 1.8. Let S_d be a smooth del Pezzo surface of degree d. Then $\hat{S}_d$ admits a non-trivial regular $\mathbb{G}_a$-action if and only if $d \geqslant 4$.

In particular, we here present a long-expected answer to the question raised by H. Flenner and M. Zaidenberg.

Corollary 1.9. The affine Fermat cubic threefold $x^3 + y^3 + z^3 + w^3 = 0$ in $\mathbb{A}^4$ does not admit a non-trivial regular $\mathbb{G}_a$-action.

In order to show the non-existence of a $(-K_{S_3})$-polar cylinder on a cubic del Pezzo surface S_3, we apply the following statement.

Lemma 1.10. Let S_d be a smooth del Pezzo surface of degree $d \leqslant 4$. Suppose that S_d contains a $(-K_{S_d})$-polar cylinder, i.e., there is an open affine subset $U \subset S_d$ and an effective anticanonical $\mathbb{Q}$-divisor[1] D such that $U = S_d \setminus \mathrm{Supp}(D)$ and $U \cong Z \times \mathbb{A}^1$ for some smooth rational affine curve Z. Then there exists a point P on S_d such that

- the log pair (S_d, D) is not log canonical at the point P;
- if there exists a *unique* divisor T in the anticanonical linear system $|-K_{S_d}|$ such that the log pair (S_d, T) is not log canonical at the point P, then there is an effective anticanonical $\mathbb{Q}$-divisor D' on the surface S_d such that
  - the log pair (S_d, D') is not log canonical at the point P;
  - the support of D' does not contain at least one irreducible component of the support of the divisor T.

Proof. This follows from [19, Lemma 4.11] and the proof of [19, Lemma 4.14] (cf. the proof of [22, Lemma 5.3]). Since the proof is presented implicitly and dispersedly in [19], for the convenience of the reader, we give a detailed proof in Appendix A. $\square$

Applying Lemma 2.2, we easily obtain

Corollary 1.11. Let S_3 be a smooth del Pezzo surface of degree 3. Suppose that S_3 contains a $(-K_{S_3})$-polar cylinder. Then there is an effective anticanonical $\mathbb{Q}$-divisor D on S_3 such that

- the pair (S_3, D) is not log canonical at some point P on S_3;
- the support of D does not contain at least one irreducible component of the tangent hyperplane section T_P of S_3 at the point P.

The lemma above may be one example that shows how important it is to study singularities of effective anticanonical $\mathbb{Q}$-divisors on Fano manifolds. In addition, it shows that the problem proposed at the beginning is strongly related to the log canonical thresholds of effective anticanonical $\mathbb{Q}$-divisors on del Pezzo surfaces.

In this article, we prove the following

Theorem 1.12. Let S_d be a smooth del Pezzo surface of degree $d \leqslant 3$ and let D be an effective anticanonical $\mathbb{Q}$-divisor on S_d. Suppose that the log pair (S_d, D) is not log canonical at a point P. Then there exists a unique divisor T in the anticanonical linear system $|-K_{S_d}|$ such that the log pair (S_d, T) is not log canonical at the point P. Moreover, the support of D contains all the irreducible components of $\mathrm{Supp}(T)$.

Corollary 1.13. Let S_3 be a smooth cubic surface in $\mathbb{P}^3$ and let D be an effective anticanonical $\mathbb{Q}$-divisor on S_3. Suppose that the log pair (S_3, D) is not log canonical at a point P. Then for the tangent hyperplane section T_P at the point P, the log pair (S_3, T_P) is not log canonical at P and $\mathrm{Supp}(D)$ contains all the irreducible components of $\mathrm{Supp}(T_P)$.

Note that Corollary 1.13 contradicts the conclusion of Corollary 1.11. It simply means that the hypothesis of Corollary 1.11 fails to be true. This shows that Theorem 1.12 implies Theorem 1.7. Moreover, we see that Theorem 1.12 recovers Theorem 1.3 through Lemma 1.10 as well.

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[1]An anticanonical $\mathbb{Q}$-divisor on a variety X is a $\mathbb{Q}$-divisor $\mathbb{Q}$-linearly equivalent to an anticanonical divisor of X, meanwhile, an effective anticanonical divisor on X is a member of the anticanonical linear system $|-K_X|$.

Remark 1.14. The condition $d \leqslant 3$ is crucial in Theorem 1.12. Indeed, if $d \geqslant 4$, then the assertion of Theorem 1.12 is no longer true (cf. the proof of [19, Theorem 3.19]). For example, consider the case when $d = 4$. There exists a birational morphism $f \colon S_4 \to \mathbb{P}^2$ such that f is the blow up of $\mathbb{P}^2$ at five points that lie on a unique irreducible conic. Denote this conic by C. Let $\tilde{C}$ be the proper transform of the conic C on the surface S_4 and let $E_1, \ldots, E_5$ be the exceptional divisors of the morphism f. Put

$$D = \frac{3}{2}\tilde{C} + \sum_{i=1}^{5} \frac{1}{2}E_i.$$

It is an effective anticanonical $\mathbb{Q}$-divisor on S_4 and the log pair (S_4, D) is not log canonical at any point P on $\tilde{C}$. Moreover, for any $T \in |-K_{S_4}|$, its support cannot be contained in the support of the divisor D.

To our surprise, Theorem 1.12 has other applications that are interesting for their own sake.

From here to the end of this section, let X be a projective variety with at worst Kawamata log terminal singularities and let H be an ample divisor on X.

Definition 1.15. The α-invariant of the log pair (X, H) is the number defined by

$$\alpha\left(X, H\right) = \sup\left\{\lambda \in \mathbb{Q} \ \middle| \ \begin{array}{l} \text{the log pair } (X, \lambda D) \text{ is log canonical for every} \\ \text{effective } \mathbb{Q}\text{-divisor } D \text{ on } X \text{ with } D \sim_{\mathbb{Q}} H. \end{array} \right\}.$$

The invariant $\alpha(X, H)$ has been studied intensively by many people who used different notations for $\alpha(X, H)$ ([1], [6], [14], [4, § 3.4] [10, Definition 3.1.1], [11, Appendix A], [38, Appendix 2]). The notation $\alpha(X, H)$ is due to G. Tian who defined $\alpha(X, H)$ in a different way ([38, Appendix 2]). However, both the definitions coincide by [11, Theorem A.3]. In the case when X is a Fano variety, the invariant $\alpha(X, -K_X)$ is known as the famous α-invariant of Tian and it is denoted simply by $\alpha(X)$. The α-invariant of Tian plays a very important role in Kähler geometry due to the following.

Theorem 1.16 ([13], [30], [36])**.** Let X be a Fano variety of dimension n with at worst quotient singularities. If $\alpha(X) > \frac{n}{n+1}$, then X admits an orbifold Kähler–Einstein metric.

The exact values of the α-invariants of smooth del Pezzo surfaces, as below, have been obtained in [7, Theorem 1.7]. Those of del Pezzo surfaces defined over a field of positive characteristic are presented in [28, Theorem 1.6] and those of del Pezzo surface with du Val singularities in [8] and [33].

Theorem 1.17. Let S_d be a smooth del Pezzo surface of degree d. Then

$$\alpha(S_d) = \begin{cases} 1/3 \text{ if } d = 9, 7 \text{ or } S_8 = \mathbb{F}_1; \\ 1/2 \text{ if } d = 6, 5 \text{ or } S_8 = \mathbb{P}^1 \times \mathbb{P}^1; \\ 2/3 \text{ if } d = 4; \end{cases}$$

$$\alpha(S_3) = \begin{cases} 2/3 \text{ if } S_3 \text{ is a cubic surface in } \mathbb{P}^3 \text{ with an Eckardt point;} \\ 3/4 \text{ if } S_3 \text{ is a cubic surface in } \mathbb{P}^3 \text{ without Eckardt points;} \end{cases}$$

$$\alpha(S_2) = \begin{cases} 3/4 \text{ if } |-K_{S_2}| \text{ has a tacnodal curve;} \\ 5/6 \text{ if } |-K_{S_2}| \text{ has no tacnodal curves;} \end{cases}$$

$$\alpha(S_1) = \begin{cases} 5/6 \text{ if } |-K_{S_1}| \text{ has a cuspidal curve;} \\ 1 \text{ if } |-K_{S_1}| \text{ has no cuspidal curves.} \end{cases}$$

Remark 1.18. Theorem 1.12 also provides the exact values of the α-invariants for smooth del Pezzo surfaces of degrees $\leqslant 3$. We here show how to extract the values from Theorem 1.12. Let μ be the value in Theorem 1.17 for the α-invariant of S_d. From [32, Proposition 3.2] we can easily obtain an effective anticanonical divisor C on the surface S_d such that $(S_d, \mu C)$ is log canonical but not Kawamata log terminal. This gives us $\alpha(S_d) \leqslant \mu$.

Suppose that $\alpha(S_d) < \mu$. Then there are an effective anticanonical $\mathbb{Q}$-divisor D and a positive rational number $\lambda < \mu$ such that $(S_d, \lambda D)$ is not log canonical at some point P on S_d. Since $\lambda < 1$, the log pair (S_d, D) is not log canonical at the point P either. By Theorem 1.12, there exists a divisor $T \in |-K_{S_d}|$ such that (S_d, T) is not log canonical at P. In addition, $\mathrm{Supp}(D)$ contains all the irreducible components of $\mathrm{Supp}(T)$.

The log pair $(S_d, \lambda T)$ is log canonical ([32, Proposition 3.2]). Put $D_\epsilon = (1 + \epsilon)D - \epsilon T$ for every non-negative rational number ϵ. Then $D_0 = D$ and D_ϵ is effective for $0 < \epsilon \ll 1$ because $\mathrm{Supp}(D)$ contains all the irreducible components of $\mathrm{Supp}(T)$. Choose the biggest ϵ such that D_ϵ is still effective. Then $\mathrm{Supp}(D_\epsilon)$ does not contain at least one irreducible component of $\mathrm{Supp}(T)$.

Since $(S_d, \lambda T)$ is log canonical at P and $(S_d, \lambda D)$ is not log canonical at P, the log pair $(S_d, \lambda D_\epsilon)$ is not log canonical at P either (see Lemma 2.2). In particular, the log pair (S_d, D_ϵ) is not log canonical at P. However, this contradicts Theorem 1.12 since D_ϵ is an effective anticanonical $\mathbb{Q}$-divisor. Therefore, $\alpha(S_d) = \mu$.

Corollary 1.19. Let S_d be a smooth del Pezzo surface of degree $d \leqslant 3$. If $d = 3$, suppose, in addition, that S_3 does not contain an Eckardt point. Then S_d admits a Kähler–Einstein metric.

The problem on the existence of Kähler–Einstein metrics on smooth del Pezzo surfaces is completely solved by G. Tian and S.-T. Yau in [37] and [39]. In particular, Corollary 1.19 follows from [37, Main Theorem].

The invariant $\alpha(X, H)$ has a global nature. It measures the singularities of effective $\mathbb{Q}$-divisors on X in a fixed $\mathbb{Q}$-linear equivalence class. F. Ambro suggested in [1] a function that encodes the local behavior of $\alpha(X, H)$.

Definition 1.20 ([1])**.** The α-function α_X^H of the log pair (X, H) is a function on X into real numbers defined as follows: for a given point $P \in X$,

$$\alpha_X^H(P) = \sup \left\{ \lambda \in \mathbb{Q} \; \middle| \; \begin{array}{l} \text{the log pair } (X, \lambda D) \text{ is log canonical at the point } P \in X \\ \text{for every effective } \mathbb{Q}\text{-divisor } D \text{ on } X \text{ with } D \sim_{\mathbb{Q}} H. \end{array} \right\}.$$

Lemma 1.21. The identity $\alpha(X, H) = \inf_{P \in X} \alpha_X^H(P)$ holds.

Proof. It is easy to check. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the case when X is a Fano variety, we denote the α-function of the log pair $(X, -K_X)$ simply by α_X.

Example 1.22. One can easily see that $\alpha_{\mathbb{P}^n}(P) \leqslant \frac{1}{n+1}$ for every point P on $\mathbb{P}^n$. This implies that the α-function $\alpha_{\mathbb{P}^n}$ is the constant function with the value $\frac{1}{n+1}$ since $\alpha(\mathbb{P}^n) = \frac{1}{n+1}$.

Example 1.23. It is easy to see $\alpha_{\mathbb{P}^1 \times \mathbb{P}^1}(P) \leqslant \frac{1}{2}$ for every point P on $\mathbb{P}^1 \times \mathbb{P}^1$. Since $\alpha(\mathbb{P}^1 \times \mathbb{P}^1) = \frac{1}{2}$ by Theorem 1.17, the α-function $\alpha_{\mathbb{P}^1 \times \mathbb{P}^1}$ is the constant function with the value $\frac{1}{2}$ by Lemma 1.21. Moreover, if X is a Fano variety with at most Kawamata log terminal singularities, then the proof of [11, Lemma 2.21] shows that

$$\alpha_{X \times \mathbb{P}^1}(P) = \min \left\{ \frac{1}{2}, \; \alpha_X \left(\mathrm{pr}_1 (P) \right) \right\}$$

for every point P on $X \times \mathbb{P}^1$, where $\mathrm{pr}_1 : X \times \mathbb{P}^1 \to X$ is the projection on the first factor. Using the similar argument in the proof of [11, Lemma 2.29], one can show that the α-function

of a product of Fano varieties with at most Gorenstein canonical singularities is the point-wise minimum of the pull-backs of the α-functions on the factors.

As shown in Remark 1.18, the following can be obtained from Theorem 1.12 in a similar manner.

Corollary 1.24. Let S_d be a smooth del Pezzo surface of degree $d \leqslant 3$. Then the α-function of S_d is as follows:

$$\alpha_{S_3}(P) = \begin{cases} 2/3 & \text{if the point } P \text{ is an Eckardt point;} \\ 3/4 & \text{if the tangent hyperplane section at } P \text{ has a tacnode at the point } P; \\ 5/6 & \text{if the tangent hyperplane section at } P \text{ has a cusp at the point } P; \\ 1 & \text{otherwise;} \end{cases}$$

$$\alpha_{S_2}(P) = \begin{cases} 3/4 & \text{if there is an effective anticanonical divisor with a tacnode at the point } P; \\ 5/6 & \text{if there is an effective anticanonical divisor with a cusp at the point } P; \\ 1 & \text{otherwise;} \end{cases}$$

$$\alpha_{S_1}(P) = \begin{cases} 5/6 & \text{if there is an effective anticanonical divisor with a cusp at the point } P; \\ 1 & \text{otherwise.} \end{cases}$$

By Lemma 1.21, Corollary 1.24 implies that Theorem 1.17 holds for smooth del Pezzo surfaces of degrees at most 3. Thus, it is quite natural that we should extend Corollary 1.24 to all smooth del Pezzo surfaces in order to obtain a functional generalisation of Theorem 1.17. This will be done in Section 6, where we prove

Theorem 1.25. Let S_d be a smooth del Pezzo surface of degree $d \geqslant 4$. Then the α-function of S_d is as follows:

$$\alpha_{\mathbb{P}^2}(P) = 1/3;$$

$$\alpha_{\mathbb{F}_1}(P) = 1/3; \qquad \alpha_{\mathbb{P}^1 \times \mathbb{P}^1}(P) = 1/2;$$

$$\alpha_{S_7}(P) = \begin{cases} 1/3 & \text{if the point } P \text{ lies on the } -1\text{-curve that intersects two other } -1\text{-curves;} \\ 1/2 & \text{otherwise;} \end{cases}$$

$$\alpha_{S_6}(P) = 1/2;$$

$$\alpha_{S_5}(P) = \begin{cases} 1/2 & \text{if there is } -1\text{-curve passing through the point } P; \\ 2/3 & \text{if there is no } -1\text{-curve passing though the point } P; \end{cases}$$

$$\alpha_{S_4}(P) = \begin{cases} 2/3 & \text{if } P \text{ is on a } -1\text{-curve;} \\ 3/4 & \text{if there is an effective anticanonical divisor that consists of} \\ & \text{two 0-curves intersecting tangentially at the point } P; \\ 5/6 & \text{otherwise.} \end{cases}$$

Let us describe the structure of this article. In Section 2, we describe the results that will be used in the proofs of Theorems 1.12 and 1.25. We also prove Theorem 1.12 for a smooth del Pezzo surface of degree 1 (see Lemma 2.3). In Section 3, we prove two results about singular del Pezzo surfaces of degree 2 that will be used in the proofs of Theorems 1.12 and 1.25. In addition, we verify Theorem 1.12 for a smooth del Pezzo surface of degree 2 (see Lemma 3.5). In Section 4, we prove Theorem 1.12 omitting the proof of Lemma 4.8 that plays a crucial role in the proof of Theorem 1.12. In Section 5, we prove Lemma 4.8. In Section 6, Theorem 1.25 is shown. In Appendix A, we prove Lemma 1.10.

2. Preliminaries

This section presents simple but essential tools for the article. Most of the described results here are well-known and valid in much more general settings (cf. [23], [24] and [25]).

Let S be a projective surface with at most du Val singularities, let P be a smooth point of the surface S and let D be an effective $\mathbb{Q}$-divisor on S.

Lemma 2.1. If the log pair (S, D) is not log canonical at the point P, then $\mathrm{mult}_P(D) > 1$.

Proof. This is a well-known fact. See [25, Proposition 9.5.13], for instance. $\qquad\square$

Write $D = \sum_{i=1}^r a_i D_i$, where D_i's are distinct prime divisors on the surface S and a_i's are positive rational numbers.

Lemma 2.2. Let T be an effective $\mathbb{Q}$-divisor on S such that
- $T \sim_{\mathbb{Q}} D$ but $T \neq D$;
- $T = \sum_{i=1}^r b_i D_i$ for some non-negative rational numbers $b_1, \ldots, b_r$.

For every non-negative rational number ϵ, put $D_\epsilon = (1 + \epsilon)D - \epsilon T$. Then
1. $D_\epsilon \sim_{\mathbb{Q}} D$ for every $\epsilon \geqslant 0$;
2. the set $\{\epsilon \in \mathbb{Q}_{>0} \mid D_\epsilon \text{ is effective.}\}$ attains the maximum μ;
3. the support of the divisor D_μ does not contain at least one component of $\mathrm{Supp}(T)$;
4. if (S, T) is log canonical at P but (S, D) is not log canonical at P, then (S, D_μ) is not log canonical at P.

Proof. The first assertion is obvious. For the rest we put

$$c = \max \left\{ \frac{b_i}{a_i} \;\middle|\; i = 1, \ldots, r \right\}.$$

For some index k we have $c = \frac{b_k}{a_k}$.

Suppose that $c \leqslant 1$. Then $a_i \geqslant b_i$ for every i. It means that the divisor $D - T = \sum_{i=1}^r (a_i - b_i)D_i$ is effective. However, it is impossible since $D - T$ is non-zero and numerically trivial on a projective surface. Thus, $c > 1$, and hence $b_k > a_k$.

Put $\mu = \frac{1}{c-1}$. Then $\mu = \frac{a_k}{b_k - a_k} > 0$ and

$$D_\mu = \frac{b_k}{b_k - a_k}D - \frac{a_k}{b_k - a_k}T = \sum_{i=1}^r \frac{b_k a_i - a_k b_i}{b_k - a_k} D_i,$$

where $b_k a_i - a_k b_i \geqslant 0$ by the choice of k. In particular, the divisor D_μ is effective and its support does not contain the curve D_k. Moreover, for every positive rational number ϵ, $D_\epsilon = \sum_{i=1}^r (a_i + \epsilon a_i - \epsilon b_i)D_i$. If $\epsilon > \mu$, then

$$\epsilon(b_k - a_k) > \mu(b_k - a_k) = \frac{a_k}{b_k - a_k}(b_k - a_k) = a_k,$$

and hence D_ϵ is not effective. This proves the second and the third assertions.

If both (S, T) and (S, D_μ) are log canonical at P, then (S, D) must be log canonical at P because $D = \frac{\mu}{1+\mu}T + \frac{1}{1+\mu}D_\mu$ and $\frac{\mu}{1+\mu} + \frac{1}{1+\mu} = 1$. $\qquad\square$

Despite its naïve appearance, Lemma 2.2 is a very handy tool. To illustrate this, we here verify Theorem 1.12 for a del Pezzo surface of degree 1. This simple case also immediately follows from the proof of [7, Lemma 3.1] or from the proof of [22, Proposition 5.1].

Lemma 2.3. Suppose that S is a smooth del Pezzo surface of degree 1 and D is an effective anticanonical $\mathbb{Q}$-divisor on S. If the log pair (S, D) is not log canonical at the point P, then there exists a unique divisor $T \in |-K_S|$ such that (S, T) is not log canonical at P. Moreover, the support of D contains all the irreducible components of T.

Proof. Let T be a curve in $|-K_S|$ that passes through the point P. Note that T is irreducible. If the log pair (S, T) is log canonical at P, then it follows from Lemma 2.2 that there exists an effective anticanonical $\mathbb{Q}$-divisor D' on the surface S such that the log pair (S, D') is not log canonical at P and $\mathrm{Supp}(D')$ does not contain the curve T. We then obtain $1 = T \cdot D' \geqslant \mathrm{mult}_P(D')$. This is impossible by Lemma 2.1. Thus, the log pair (S, T) is not log canonical at the point P.

Moreover, the divisor T is singular at the point P. Therefore, the point P is not the base point of the pencil $|-K_S|$. Consequently, such a divisor T is unique.

If the curve T is not contained in $\mathrm{Supp}(D)$, then $1 = T \cdot D \geqslant \mathrm{mult}_P(D)$. Therefore, the curve T must be contained in $\mathrm{Supp}(D)$ by Lemma 2.1. $\qquad\square$

The following is a ready-made Adjunction for our situation. See [24, Theorem 5.50] for a more general version.

Lemma 2.4. Suppose that the log pair (S, D) is not log canonical at P. If a component D_j with $a_j \leqslant 1$ is smooth at the point P, then

$$D_j \cdot \left(\sum_{i \neq j} a_i D_i \right) \geqslant \sum_{i \neq j} a_i \mathrm{mult}_P \left(D_j \cdot D_i \right) > 1.$$

Proof. See [28, Lemma 2.5] for a characteristic-free proof in dimension 2. $\qquad\square$

Let $f \colon \tilde{S} \to S$ be the blow up of the surface S at the point P with the exceptional divisor E and let $\tilde{D}$ be the proper transform of D by the blow up f. Then

$$K_{\tilde{S}} + \tilde{D} + \left(\mathrm{mult}_P(D) - 1 \right) E = f^* \left(K_S + D \right).$$

The log pair (S, D) is log canonical at the point P if and only if the log pair $(\tilde{S}, \tilde{D} + (\mathrm{mult}_P(D) - 1)E)$ is log canonical along the curve E.

Remark 2.5. If the log pair (S, D) is not log canonical at P, then there exists a point Q on E at which the log pair $(\tilde{S}, \tilde{D} + (\mathrm{mult}_P(D) - 1)E)$ is not log canonical. Lemma 2.1 then implies

$$\mathrm{mult}_P(D) + \mathrm{mult}_Q(\tilde{D}) > 2.$$

If $\mathrm{mult}_P(D) \leqslant 2$, then the log pair $(\tilde{S}, \tilde{D} + (\mathrm{mult}_P(D) - 1)E)$ is log canonical at every point of the curve E other than the point Q. Indeed, if the log pair $(\tilde{S}, \tilde{D} + (\mathrm{mult}_P(D) - 1)E)$ is not log canonical at another point O on E, then Lemma 2.4 generates an absurd inequality

$$2 \geqslant \mathrm{mult}_P(D) = \tilde{D} \cdot E \geqslant \mathrm{mult}_Q(\tilde{D}) + \mathrm{mult}_O(\tilde{D}) > 2.$$

3. Del Pezzo surfaces of degree 2

Let S be a del Pezzo surface of degree 2 with at most two ordinary double points. Then the linear system $|-K_S|$ is free from base points and induces a double cover $\pi \colon S \to \mathbb{P}^2$ ramified along a reduced quartic curve $R \subset \mathbb{P}^2$. Moreover, the curve R has at most two ordinary double points. In particular, the quartic curve R is irreducible.

Lemma 3.1. For an effective anticanonical $\mathbb{Q}$-divisor D on S, the log pair (S, D) is log canonical outside finitely many points on S.

Proof. Suppose it is not true. Then we may write $D = m_1 C_1 + \Omega$, where C_1 is an irreducible reduced curve, m_1 is a positive rational number strictly bigger than 1 and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curve C_1. Since

$$2 = -K_S \cdot D = -K_S \cdot (m_1 C_1 + \Omega) = -m_1 K_S \cdot C_1 - K_S \cdot \Omega \geqslant -m_1 K_S \cdot C_1 > -K_S \cdot C_1,$$

we have $-K_S \cdot C_1 = 1$. Then $\pi(C_1)$ is a line in $\mathbb{P}^2$. Thus, there exists an irreducible reduced curve C_2 on S such that $C_1 + C_2 \sim -K_S$ and $\pi(C_1) = \pi(C_2)$. Note that $C_1 = C_2$ if and only if the line $\pi(C_1)$ is an irreducible component of the branch curve R. Since the curve R is irreducible, this is not the case. Thus, we have $C_1 \neq C_2$.

Note that $C_1^2 = C_2^2$ because C_1 and C_2 are interchanged by the biregular involution of S induced by the double cover π. Thus, we have

$$2 = (C_1 + C_2)^2 = 2C_1^2 + 2C \cdot C_2,$$

which implies that $C_1 \cdot C_2 = 1 - C_1^2$. Since C_1 and C_2 are smooth rational curves, we can easily obtain $C_1^2 = C_2^2 = -1 + \frac{k}{2}$, where k is the number of singular points of S that lie on the curve C_1.

Now we write $D = m_1 C_1 + m_2 C_2 + \Gamma$, where m_2 is a non-negative rational number and Γ is an effective $\mathbb{Q}$-divisor whose support contains neither the curve C_1 nor the curve C_2. Then

$$1 = C_1 \cdot (m_1 C_1 + m_2 C_2 + \Gamma) = m_1 C_1^2 + m_2 C_1 \cdot C_2 + C_1 \cdot \Gamma \geqslant m_1 C_1^2 + m_2 C_1 \cdot C_2 = m_1 C_1^2 + m_2 (1 - C_1^2),$$

and hence $1 \geqslant m_1 C_1^2 + m_2(1 - C_1^2)$. Similarly, from $C_2 \cdot D = 1$, we obtain $1 \geqslant m_2 C_1^2 + m_1(1 - C_1^2)$. The obtained two inequalities imply that $m_1 \leqslant 1$ and $m_2 \leqslant 1$ since $C_1^2 = -1 + \frac{k}{2}$, $k = 0, 1, 2$. Since $m_1 > 1$ by assumption, it is a contradiction. $\qquad\square$

The following two lemmas can be verified in a similar way as that of [7, Lemma 3.5]. Nevertheless we present their proofs for reader's convenience.

Lemma 3.2. For an effective anticanonical $\mathbb{Q}$-divisor D on S, the log pair (S, D) is log canonical at every point outside the ramification divisor of the double cover π.

Proof. Suppose that (S, D) is not log canonical at a point P whose image by π lies outside R.

Let H be a general curve in $|-K_S|$ that passes through the point P. Since $\pi(P) \notin R$, the surface S is smooth at the point P. Then

$$2 = H \cdot D \geqslant \mathrm{mult}_P(H)\mathrm{mult}_P(D) \geqslant \mathrm{mult}_P(D),$$

and hence $\mathrm{mult}_P(D) \leqslant 2$.

Let $f : \tilde{S} \to S$ be the blow up of the surface S at the point P. We have

$$K_{\tilde{S}} + \tilde{D} + (\mathrm{mult}_P(D) - 1)\, E = f^* (K_S + D)\,,$$

where $\tilde{D}$ is the proper transform of the divisor D on the surface $\tilde{S}$ and E is the exceptional curve of the blow up f. Then the log pair $(\tilde{S}, \tilde{D} + (\mathrm{mult}_P(D) - 1)\, E)$ is not log canonical at some point Q on E but log canonical at every point of E other than the point Q by Remark 2.5. In addition, we have

$$(3.3) \qquad\qquad \mathrm{mult}_P(D) + \mathrm{mult}_Q(\tilde{D}) > 2.$$

Since $\pi(P) \notin R$, there exists a unique reduced but possibly reducible curve $C \in |-K_X|$ such that the curve C passes through the point P and its proper transform $\tilde{C}$ by the blow up f passes through the point Q. Note that the curve C is smooth at the point P. Since (S, C) is log canonical at the point P, Lemma 2.2 enables us to assume that the support of D does not contain at least one irreducible component of the curve C.

If the curve C is irreducible, then

$$2 - \mathrm{mult}_P(D) = 2 - \mathrm{mult}_P(C)\mathrm{mult}_P(D) = \tilde{C} \cdot \tilde{D} \geqslant \mathrm{mult}_Q(\tilde{C})\mathrm{mult}_Q(\tilde{D}) = \mathrm{mult}_Q(\tilde{D}).$$

This contradicts (3.3). Thus, the curve C must be reducible.

We may then write $C = C_1 + C_2$, where C_1 and C_2 are irreducible smooth curves that intersect at two points. Without loss of generality we may assume that the curve C_1 is not contained in the support of D. The point P must belong to C_2: otherwise we would have

$$1 = D \cdot C_1 \geqslant \mathrm{mult}_P(D) > 1.$$

We put $D = nC_2 + \Omega$, where n is a non-negative rational number and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curve C_2. Then

$$1 = C_1 \cdot D = (2 - \frac{1}{2}k)n + C_1 \cdot \Omega \geqslant (2 - \frac{1}{2}k)n,$$

where k is the number of singular points of S on C_1. On the other hand, the log pair $(\tilde{S}, n\tilde{C}_2 + \tilde{\Omega} + (\mathrm{mult}_P(D) - 1)E)$ is not log canonical at the point Q, where $\tilde{C}_2$ and $\tilde{\Omega}$ are the proper transforms of C_2 and Ω, respectively, on the surface $\tilde{S}$, and we have $n \leqslant 1$ by Lemma 3.1. We then obtain

$$(2 - \frac{1}{2}k)n = \tilde{C}_2 \cdot (\tilde{\Omega} + (\mathrm{mult}_P(D) - 1)E) > 1$$

from Lemma 2.4. This is a contradiction.   □

Lemma 3.4. For a smooth point P of S with $\pi(P) \in R$, let T_P be the unique divisor in $|-K_S|$ that is singular at the point P. If the log pair (S, T_P) is log canonical at P, then for an effective anticanonical $\mathbb{Q}$-divisor D on S the log pair (S, D) is log canonical at the point P.

Proof. Suppose that (S, D) is not log canonical at the point P. Applying Lemma 2.2 to the log pairs (S, D) and (S, T_P), we may assume that $\mathrm{Supp}(D)$ does not contain at least one irreducible component of the curve T_P. Thus, if the divisor T_P is irreducible, then Lemma 2.1 gives an absurd inequality

$$2 = T_P \cdot D \geqslant \mathrm{mult}_P(T_P)\mathrm{mult}_P(D) \geqslant 2\mathrm{mult}_P(D) > 2$$

since T_P is singular at the point P. Hence, T_P must be reducible.

We may then write $T_P = T_1 + T_2$, where T_1 and T_2 are smooth rational curves. Note that the point P is one of the intersection points of T_1 and T_2. Without loss of generality, we may assume that the curve T_1 is not contained in the support of D. Then

$$1 = T_1 \cdot D \geqslant \mathrm{mult}_P(T_1)\mathrm{mult}_P(D) = \mathrm{mult}_P(D) > 1$$

by Lemma 2.1. The obtained contradiction completes the proof.   □

Lemmas 3.2 and 3.4 prove the following result.

Lemma 3.5. Suppose that the del Pezzo surface S is smooth. Let D be an effective anticanonical $\mathbb{Q}$-divisor on S. Suppose that the log pair (S, D) is not log canonical at a point P. Then there exists a unique divisor $T \in |-K_S|$ such that (S, T) is not log canonical at P. The support of the divisor D contains all the irreducible components of T. In case, the divisor T is either an irreducible rational curve with a cusp at P or a union of two -1-curves meeting tangentially at the point P.

Proof. By Lemma 3.2, the point $\pi(P)$ must lie on R. Then there exists a unique curve $T \in |-K_S|$ that is singular at the point P. By Lemma 3.4, the log pair (S, T) is not log canonical at P.

Suppose that the support of D does not contain an irreducible component of T. Then the proof of Lemma 3.4 works verbatim to derive a contradiction.

The last assertion immediately follows from [32, Proposition 3.2].   □

Consequently, Lemma 3.5 shows that Theorem 1.12 holds for a smooth del Pezzo surface of degree 2.

4. CUBIC SURFACES

In the present section we prove Theorem 1.12. Lemma 2.3 and Lemma 3.5 show that Theorem 1.12 holds for del Pezzo surfaces of degrees 1 and 2, respectively. Thus, to complete the proof, let S be a smooth cubic surface in $\mathbb{P}^3$ and let D be an effective anticanonical $\mathbb{Q}$-divisor of the surface S.

Lemma 4.1. The log pair (S, D) is log canonical outside finitely many points.

Proof. Suppose not. Then we may write $D = mC + \Omega$, where C is an irreducible curve, m is a positive rational number strictly bigger than 1 and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curve C. Then

$$3 = -K_S \cdot (mC + \Omega) = -mK_S \cdot C - K_S \cdot \Omega \geqslant -mK_S \cdot C > -K_S \cdot C.$$

It implies that the curve C is either a line or an irreducible conic.

Suppose that C is a line. Let Z be a general irreducible conic on S such that $Z + C \sim -K_S$. Since Z is general, it is not contained in the support of D. We then obtain

$$2 = Z \cdot D = Z \cdot (mC + \Omega) = 2m + Z \cdot \Omega \geqslant 2m.$$

It contradicts our assumption.

Suppose that C is an irreducible conic. Then there exists a unique line L on S such that $L + C \sim -K_S$. Write $D = mC + nL + \Gamma$, where n is a non-negative rational number and Γ is an effective $\mathbb{Q}$-divisor whose support contains neither the conic C nor the line L. Then

$$1 = L \cdot D = L \cdot (mC + nL + \Gamma) = 2m - n + L \cdot \Gamma \geqslant 2m - n.$$

On the other hand,

$$2 = C \cdot D = C \cdot (mC + nL + \Gamma) = 2n + C \cdot \Gamma \geqslant 2n.$$

Combining two inequalities, we obtain $2m \leqslant 1 + n \leqslant 2$. This contradicts our assumption too. $\square$

For a point P on S, let T_P be the tangent hyperplane section of the surface S at the point P. This is the unique anticanonical divisor that is singular at the point P. The curve T_P is reduced but it may be reducible.

In order to prove Theorem 1.12 we must show that (S, D) is log canonical at the point P provided that one of the following two conditions is satisfied:

- the log pair (S, T_P) is log canonical at P;
- the log pair (S, T_P) is not log canonical at P but $\mathrm{Supp}(D)$ does not contain at least one irreducible component of the curve T_P.

The log pair (S, T_P) is log canonical at P if and only if the point P is an ordinary double point of the curve T_P. Thus, (S, T_P) is log canonical at P if and only if T_P is one of the following curves: an irreducible cubic curve with one ordinary double point, a union of three coplanar lines that do not intersect at one point, a union of a line and a conic that intersect transversally at two points.

Overall, we must consider the following cases:

(a) T_P is a union of three lines that intersect at the point P (Eckardt point);
(b) T_P is a union of a line and a conic that intersect tangentially at the point P;
(c) T_P is an irreducible cubic curve with a cusp at the point P;
(d) T_P is an irreducible cubic curve with one ordinary double point;
(e) T_P is a union of three coplanar lines that do not intersect at one point;
(f) T_P is a union of a line and a conic that intersect transversally at two points.

We consider these cases one by one in separate lemmas, i.e., Lemmas 4.3, 4.5, 4.6, 4.7, 4.8 and 4.9. We however present the detailed proof of Lemma 4.8 in Section 5 to improve the readability of this section. These lemmas altogether imply Theorem 1.12.

Lemma 4.2. If the support of D does not contain a line passing through the point P, then the log pair (S, D) is log canonical at the point P.

Proof. Let L be a line passing through the point P that is not contained in the support of D. Then the inequality $1 = L \cdot D \geqslant \mathrm{mult}_P(D)$ implies that the log pair (S, D) is log canonical at the point P by Lemma 2.1. $\square$

Lemma 4.3. Suppose that the tangent hyperplane section T_P consists of three lines intersecting at the point P. If the support of D does not contain at least one of the three lines, then the log pair (S, D) is log canonical at the point P.

Proof. It immediately follows from Lemma 4.2. □

From now on, let $f : \tilde{S} \to S$ be the blow up of the cubic surface S at the point P. In addition, let $\tilde{D}$ be the proper transform of D by the blow up f and E be the exceptional curve of f. We then have

$$(4.4) \qquad K_{\tilde{S}} + \tilde{D} + (\mathrm{mult}_P(D) - 1) E = f^* (K_S + D).$$

Note that the log pair (S, D) is log canonical at the point P if and only if the log pair

$$(\tilde{S}, \tilde{D} + (\mathrm{mult}_P(D) - 1) E)$$

is log canonical along the exceptional divisor E.

Lemma 4.5. Suppose that the tangent hyperplane section T_P consists of a line and a conic intersecting tangentially at the point P. If the support of D does not contain both of the line and the conic, then the log pair (S, D) is log canonical at the point P.

Proof. Suppose that the log pair (S, D) is not log canonical at the point P. Let L and C be the line and the conic, respectively, such that $T_P = L + C$. By Lemma 4.2, we may assume that the conic C is not contained but the line L is contained in the support of D. We write $D = nL + \Omega$, where n is a positive rational number and Ω is an effective $\mathbb{Q}$-divisor whose support contains neither the line L nor the conic C. We have $\mathrm{mult}_P(D) \leqslant C \cdot D = 2$.

We write $\tilde{D} = n\tilde{L} + \tilde{\Omega}$, where $\tilde{\Omega}$ and $\tilde{L}$ are the proper transforms of the divisor D and the line L, respectively, on the surface $\tilde{S}$. Let $\tilde{C}$ be the proper transform of the conic C on the surface $\tilde{S}$. Note that the three curves $\tilde{L}$, $\tilde{C}$ and E meet at one point.

The log pair $(\tilde{S}, n\tilde{L} + \tilde{\Omega} + (\mathrm{mult}_P(D) - 1)E)$ is not log canonical at some point Q on E. However, it is log canonical at every point on E except the point Q by Remark 2.5 since $\mathrm{mult}_P(D) \leqslant 2$. We also obtain $\mathrm{mult}_P(D) + \mathrm{mult}_Q(\tilde{D}) > 2$ from Remark 2.5. This implies that the point Q does not belong to $\tilde{C}$, and hence not to $\tilde{L}$ either. Indeed, if so, then

$$2 - \mathrm{mult}_P(D) = \tilde{C} \cdot \left(n\tilde{L} + \tilde{\Omega} \right) \geqslant n + \mathrm{mult}_Q(\tilde{\Omega}) = \mathrm{mult}_Q(\tilde{D}).$$

This contradicts the inequality from Remark 2.5.

Let $g \colon \tilde{S} \to \bar{S}$ be the contraction of the -2-curve $\tilde{L}$. Then $\bar{S}$ is a del Pezzo surface of degree 2 with one ordinary double point. In particular, the linear system $|-K_{\bar{S}}|$ is free from base points and induces a double cover $\pi \colon \bar{S} \to \mathbb{P}^2$ ramified along an irreducible singular quartic curve $R \subset \mathbb{P}^2$. Note that the point $g(\tilde{L})$ is the ordinary double point of the surface $\bar{S}$. Put $\bar{\Omega} = g(\tilde{\Omega})$, $\bar{E} = g(E)$, $\bar{C} = g(\tilde{C})$ and $\bar{Q} = g(Q)$. Then $\pi(\bar{E}) = \pi(\bar{C})$ since $\bar{E} + \bar{C}$ is an anticanonical divisor on $\bar{S}$. The point $\pi(\bar{Q})$ lies outside R because the point Q lies outside $\tilde{C}$. Since the divisor $\bar{\Omega} + \left(\mathrm{mult}_P(D) - 1 \right)\bar{E}$ is $\mathbb{Q}$-linearly equivalent to $-K_{\bar{S}}$ by construction, Lemma 3.2 shows that the log pair $(\bar{S}, \bar{\Omega} + (\mathrm{mult}_P(D) - 1)\bar{E})$ is log canonical at $\bar{Q}$. However, it is not log canonical at the point $\bar{Q}$ since g is an isomorphism in a neighborhood of the point Q. It is a contradiction. □

Lemma 4.6. Suppose that the tangent hyperplane section T_P is an irreducible cubic curve with a cusp at the point P. If the curve T_P is not contained in the support of D, then the log pair (S, D) is log canonical at the point P.

Proof. First, from the inequality

$$3 = T_P \cdot D \geqslant \mathrm{mult}_P(T_P)\mathrm{mult}_P(D) = 2\mathrm{mult}_P(D),$$

we obtain $\mathrm{mult}_P(D) \leqslant \frac{3}{2}$. Suppose that (S, D) is not log canonical at P. Then the log pair $(\tilde{S}, \tilde{D} + (\mathrm{mult}_P(D) - 1)E)$ is not log canonical at some point Q on E. However, Remark 2.5

shows that it is log canonical at every point on E except the point Q since $\text{mult}_P(D) \leqslant \frac{3}{2}$. We also obtain

$$\text{mult}_P(D) + \text{mult}_Q(\tilde{D}) > 2$$

from Remark 2.5.

The surface $\tilde{S}$ is a smooth del Pezzo surface of degree 2. The linear system $|-K_{\tilde{S}}|$ induces a double cover $\pi \colon \tilde{S} \to \mathbb{P}^2$ ramified along a smooth quartic curve $R \subset \mathbb{P}^2$. Let $\tilde{T}_P$ be the proper transform of the curve T_P on the surface $\tilde{S}$. Then the integral divisor $E + \tilde{T}_P$ is linearly equivalent to $-K_{\tilde{S}}$, and hence $\pi(E) = \pi(\tilde{T}_P)$ is a line in $\mathbb{P}^2$. Moreover, the curve $\tilde{T}_P$ tangentially meet the curve E at a single point. Thus the point $\pi(Q)$ lies on R if and only if the point Q is the intersection point of E and $\tilde{T}_P$.

Applying Lemma 3.2 to the log pair $(\tilde{S}, \tilde{D} + (\text{mult}_P(D) - 1)E)$, we see that the point $\pi(Q)$ belongs to R because the log pair $(\tilde{S}, \tilde{D} + (\text{mult}_P(D) - 1)E)$ is not log canonical at the point Q and the divisor $\tilde{D} + (\text{mult}_P(D) - 1)E$ is $\mathbb{Q}$-linearly equivalent to $-K_{\tilde{S}}$. The point Q therefore lies on the curve $\tilde{T}_P$. Then

$$3 - 2\text{mult}_P(D) = \tilde{T}_P \cdot \tilde{D} \geqslant \text{mult}_Q(\tilde{D}) > 2 - \text{mult}_P(D).$$

This contradicts Lemma 2.1. $\qquad\square$

For the remaining three cases, we show that the hypothesis of Theorem 1.12 is never fulfilled, so that Theorem 1.12 is true.

Lemma 4.7. If the tangent hyperplane section T_P is an irreducible cubic curve with a node at the point P, then the log pair (S, D) is log canonical at the point P.

Proof. Suppose that (S, D) is not log canonical at P. The surface $\tilde{S}$ is a smooth del Pezzo surface of degree two. Since $\tilde{D} + (\text{mult}_P(D) - 1)E \sim_{\mathbb{Q}} -K_Y$ and the log pair $(\tilde{S}, \tilde{D} + (\text{mult}_P(D) - 1)E)$ is not log canonical at some point Q on E, it follows from Lemma 3.5 that there must be an anticanonical divisor H on the surface $\tilde{S}$ such that has either a tacnode or a cusp at the point Q.

If the divisor H has a tacnode at the point Q, then it consists of the exceptional divisor E and another -1-curve L meeting E tangentially at Q. Then the divisor $f(H)$ is an effective anticanonical divisor on S such that it has a cusp at the point P and it is distinct from the divisor T_P. This is impossible.

If the divisor H has a cusp at the point Q, then it must be irreducible. However, it is impossible since H is singular at the point Q and $E \cdot H = 1$. $\qquad\square$

Lemma 4.8. Suppose that the tangent hyperplane section T_P consists of three lines one of which does not pass through the point P. Then the log pair (S, D) is log canonical at P.

Proof. The proof of this lemma is the central and the most beautiful part of the proof of Theorem 1.12. Since the proof is a bit lengthy, we present the proof in a separate section. See Section 5. $\qquad\square$

Lemma 4.9. Suppose that the tangent hyperplane section T_P consists of a line and a conic intersecting transversally. Then the log pair (S, D) is log canonical at the point P.

Proof. We write $T_P = L + C$, where L is a line and C is an irreducible conic that intersect L transversally at the point P. Suppose that (S, D) is not log canonical at the point P.

By Lemmas 2.2 and 4.2, we may assume that the conic C is not contained but the line L is contained in the support of D. We write $D = nL + \Omega$, where n is a positive rational number and Ω is an effective $\mathbb{Q}$-divisor whose support contains neither the line L nor the conic C.

The log pair $(\tilde{S}, \tilde{D} + (\text{mult}_P(D) - 1)E)$ is not log canonical at some point Q on E. However, Remark 2.5 shows that it is log canonical at every point on E except the point Q since $\text{mult}_P(D) \leqslant D \cdot C = 2$.

Let $\tilde{\Omega}$, $\tilde{L}$ and $\tilde{C}$ be the proper transforms of the divisor Ω, the line L and the conic C by the blow up f, respectively.

Suppose that the point Q does not belong to the -2-curve $\tilde{L}$. Let $g\colon \tilde{S} \to \bar{S}$ be the contraction of the curve $\tilde{L}$. Then $\bar{S}$ is a del Pezzo surface of degree 2 with only one ordinary double point at the point $g(\tilde{L})$. In particular, the linear system $|-K_{\bar{S}}|$ induces a double cover $\pi\colon \bar{S} \to \mathbb{P}^2$ ramified along an irreducible singular quartic curve $R \subset \mathbb{P}^2$.

Put $\bar{\Omega} = g(\tilde{\Omega})$, $\bar{E} = g(E)$, $\bar{C} = g(\tilde{C})$ and $\bar{Q} = g(Q)$. Then $\pi(\bar{E}) = \pi(\bar{C})$ since $\bar{E} + \bar{C}$ is an anticanonical divisor on $\bar{S}$. The point $\pi(\bar{Q})$ lies on R if and only if the point Q lies on $\tilde{C}$. The log pair $(\bar{S}, \bar{\Omega} + (\mathrm{mult}_P(D) - 1)\bar{E})$ is not log canonical at $\bar{Q}$ since g is an isomorphism in a neighborhood of the point Q. Since the divisor $\bar{\Omega} + (\mathrm{mult}_P(D) - 1)\bar{E}$ is $\mathbb{Q}$-linearly equivalent to $-K_{\bar{S}}$ by construction, Lemma 3.2 shows that the point Q belongs to $\tilde{C}$.

Note that $\bar{C} + \bar{E}$ is the unique curve in $|-K_{\bar{S}}|$ that is singular at the point $\bar{Q}$. But the log pair $(\bar{S}, \bar{C} + \bar{E})$ is log canonical at the point $\bar{Q}$. Hence, it follows from Lemma 3.4 that the log pair $(\bar{S}, \bar{\Omega} + (\mathrm{mult}_P(D) - 1)\bar{E})$ is log canonical at the point $\bar{Q}$. This is a contradiction. Therefore, the point Q belongs to the -2-curve $\tilde{L}$.

Now we can apply [8, Theorem 1.28] to the log pair $(\tilde{S}, n\tilde{L} + (\mathrm{mult}_P(D) - 1)E + \tilde{\Omega})$ at the point Q to obtain a contradiction immediately. Indeed, it is enough to put $M = 1$, $A = 1$, $N = 0$, $B = 2$, and $\alpha = \beta = 1$ in [8, Theorem 1.28] and check that all the conditions of [8, Theorem 1.28] are satisfied. However, there is a much simpler way to obtain a contradiction. Let us take this simpler way.

There exists another line M on the surface S that intersects L at a point. The line M does not intersect the conic C since $1 = T_P \cdot M = (L + C) \cdot M = L \cdot M$. In particular, the point P does not lie on the line M. Let $h\colon \tilde{S} \to \check{S}$ be the contraction of the proper transform of the line M on the surface $\tilde{S}$. Since M is a -1-curve and the point P does not lie on the line M, the surface $\check{S}$ is a smooth cubic surface in $\mathbb{P}^3$.

Put $\check{\Omega} = h(\tilde{\Omega})$, $\check{E} = h(E)$, $\check{L} = h(\tilde{L})$, $\check{C} = h(\tilde{C})$, $\check{P} = h(Q)$ and $\check{D} = h(\tilde{D})$. Then $(\check{S}, \check{D})$ is not log canonical at the point $\check{P}$ since h is an isomorphism in a neighborhood of the point Q. On the other hand, the divisor $\check{L} + \check{C} + \check{E}$ is an anticanonical divisor of the surface $\check{S}$. Since the point $\check{P}$ is the intersection point of $\check{L}$ and $\check{E}$ and the divisor $\check{D}$ is $\mathbb{Q}$-linearly equivalent to $-K_{\check{S}}$, Lemma 4.8 implies that $(\check{S}, \check{D})$ is log canonical at the point $\check{P}$. This is a contradiction. $\square$

As we already mentioned, Theorem 1.12 follows from Lemmas 4.3, 4.5, 4.6, 4.7, 4.8 and 4.9. Thus Theorem 1.12 has been proved under the assumption that Lemma 4.8 is valid. This will be shown in the following section.

5. The proof of Lemma 4.8

To prove Lemma 4.8, we keep the notations used in Section 4. We write $T_P = L + M + N$, where L, M, and N are three coplanar lines on S. We may assume that the point P is the intersection point of the two lines L and M, while it does not lie on the line N. We also write $D = a_0 L + b_0 M + c_0 N + \Omega_0$, where a_0, b_0, and c_0 are non-negative rational numbers and Ω_0 is an effective $\mathbb{Q}$-divisor on S whose support contains none of the lines L, M and N. Put $m_0 = \mathrm{mult}_P(\Omega_0)$.

Suppose that the log pair (S, D) is not log canonical at the point P. Let us seek for a contradiction.

By Lemma 4.1, the log pair (S, D) is log canonical outside finitely many points. In particular, we have $0 \leqslant a_0, b_0, c_0 \leqslant 1$. Also, Lemma 2.1 implies $m_0 + a_0 + b_0 > 1$.

Lemma 5.1. The inequality $m_0 + a_0 + b_0 > c_0 + 1$ holds.

Proof. Since the log pair $(S, a_0L + b_0M + \Omega_0)$ is not log canonical at the point P either, it follows from Lemma 2.4 that

$$1 + a_0 - c_0 = L \cdot (D - a_0L - c_0N) = L \cdot (b_0M + \Omega_0) > 1,$$

which implies $a_0 > c_0$. Similarly, $b_0 > c_0$.

The log pair $(S, L + M + N)$ is log canonical. Since the log pair $(S, a_0L + b_0M + c_0N + \Omega_0)$ is not log canonical at P, it follows from Lemma 2.2 that the log pair

$$\left(S, \frac{1}{1 - c_0}D - \frac{c_0}{1 - c_0}T_P \right)$$

is not log canonical at the point P. Then Lemma 2.1 shows

$$\mathrm{mult}_P\left(\frac{1}{1 - c_0}D - \frac{c_0}{1 - c_0}T_P \right) = \mathrm{mult}_P\left(\frac{a_0 - c_0}{1 - c_0}L + \frac{b_0 - c_0}{1 - c_0}M + \frac{1}{1 - c_0}\Omega_0 \right)$$

$$= \frac{a_0 - c_0}{1 - c_0} + \frac{b_0 - c_0}{1 - c_0} + \frac{m_0}{1 - c_0} > 1.$$

It verifies $m_0 + a_0 + b_0 > c_0 + 1$. □

Since the rational numbers a_0, b_0, c_0 are at most 1 and the log pair $(S, L + M + N)$ is log canonical, the effective $\mathbb{Q}$-divisor Ω_0 cannot be the zero-divisor. Let r be the number of the irreducible components of the support of the $\mathbb{Q}$-divisor Ω_0. Then we write

$$\Omega_0 = \sum_{i=1}^{r} e_i C_{i0},$$

where e_i's are positive rational numbers and C_{i0}'s are irreducible reduced curves of degrees d_{i0} on the surface S. We then see

$$(5.2) \qquad 3 = -K_S \cdot \left(a_0L + b_0M + c_0N + \sum_{i=1}^{r} e_i C_{i0} \right) = a_0 + b_0 + c_0 + \sum_{i=1}^{r} e_i d_{i0}.$$

Denote by $\tilde{L}$, $\tilde{M}$ and $\tilde{N}$ the proper transforms of the lines L, M and N, respectively, on the surface $\tilde{S}$. For each i, denote by $\tilde{C}_{i0}$ the proper transform of the curve C_{i0} on the surface $\tilde{S}$. Then

$$K_{\tilde{S}} + a_0\tilde{L} + b_0\tilde{M} + c_0\tilde{N} + (a_0 + b_0 + m_0 - 1)E + \sum_{i=1}^{r} e_i\tilde{C}_{i0} = f^*(K_S + D).$$

Recall that $a_0 + b_0 + m_0 = \mathrm{mult}_P(D)$.

Lemma 5.3. The inequality $\mathrm{mult}_P(D) = a_0 + b_0 + m_0 \leqslant 2$ holds.

Proof. It immediately follows from the three inequalities

$$1 = L \cdot (a_0L + b_0M + c_0N + \Omega_0) = -a_0 + b_0 + c_0 + L \cdot \Omega_0 \geqslant -a_0 + b_0 + c_0 + m_0,$$
$$1 = M \cdot (a_0L + b_0M + c_0N + \Omega_0) = a_0 - b_0 + c_0 + M \cdot \Omega_0 \geqslant a_0 - b_0 + c_0 + m_0,$$
$$1 = N \cdot (a_0L + b_0M + c_0N + \Omega_0) = a_0 + b_0 - c_0 + N \cdot \Omega_0 \geqslant a_0 + b_0 - c_0.$$

□

The log pair

$$(5.4) \qquad \left(\tilde{S}, a_0\tilde{L} + b_0\tilde{M} + c_0\tilde{N} + (a_0 + b_0 + m_0 - 1)E + \sum_{i=1}^{r} e_i\tilde{C}_{i0} \right)$$

is not log canonical at some point Q on E. Since $\mathrm{mult}_P(D) = a_0 + b_0 + m_0 \leqslant 2$, it follows from Remark 2.5 that the log pair (5.4) is log canonical at every point of the curve E other than the point Q.

Let $g\colon \tilde{S} \to \bar{S}$ be the contraction of the -2-curves $\tilde{L}$ and $\tilde{M}$. Then $\bar{S}$ is a del Pezzo surface of degree 2 with two ordinary double points at the points $g(\tilde{L})$ and $g(\tilde{M})$. The linear system $|-K_{\bar{S}}|$ induces a double cover $\pi\colon \bar{S} \to \mathbb{P}^2$ ramified along an irreducible singular quartic curve $R \subset \mathbb{P}^2$.

Lemma 5.5. *The point Q on the exceptional curve E belongs to either the -2-curve $\tilde{L}$ or the -2-curve $\tilde{M}$.*

Proof. Suppose that the point Q lies on neither $\tilde{L}$ nor $\tilde{M}$. Put $\bar{E} = g(E)$, $\bar{N} = g(\tilde{N})$ and $\bar{Q} = g(Q)$. In addition, we put $\bar{C}_{i0} = g(\tilde{C}_{i0})$ for each i. Then $\pi(\bar{E}) = \pi(\bar{N})$. The point $\pi(\bar{Q})$ lies outside R since the point $\bar{Q}$ is a smooth point of the anticanonical divisor $\bar{E} + \bar{N}$ on $\bar{S}$.

Since g is an isomorphism in a neighborhood of the point Q, the log pair

$$(5.6) \qquad \left(\bar{S}, c_0\bar{N} + (a_0 + b_0 + m_0 - 1)\,\bar{E} + \sum_{i=1}^{r} e_i \bar{C}_{i0} \right)$$

is not log canonical at the point $\bar{Q}$. The divisor $c_0\bar{N} + (a_0 + b_0 + m_0 - 1)\bar{E} + \sum_{i=1}^{r} e_i\bar{C}_{i0}$ is an effective anticanonical $\mathbb{Q}$-divisor on the surface $\bar{S}$. Hence, we are able to apply Lemma 3.2 to the log pair (5.6) to obtain a contradiction. $\qquad\square$

From now on we may assume that the point Q is the intersection point of the -2-curve $\tilde{L}$ and the -1-curve E without loss of generality.

Let $\rho\colon S \dashrightarrow \mathbb{P}^2$ be the linear projection from the point P. Then ρ is a generically 2-to-1 rational map. Thus the map ρ induces a birational involution τ_P of the cubic surface S. The involution τ_P is classically known as the Geiser involution associated to the point P (see [27]).

Remark 5.7. By construction, the involution τ_P is biregular outside the union $L \cup M \cup N$. In fact, one can show that τ_P is biregular outside the point P and the line N. Moreover, one can show that $\tau_P(L) = L$ and $\tau_P(M) = M$.

For each i, put $C_{i1} = \tau_P(C_{i0})$ and denote by d_{i1} the degree of the curve C_{i1}. We then employ new effective $\mathbb{Q}$-divisors

$$\Omega_1 = \sum_{i=1}^{r} e_i C_{i1};$$
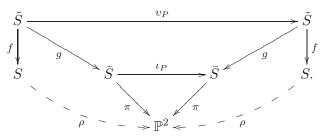$$D_1 = a_1 L + b_1 M + c_1 N + \Omega_1,$$

where $a_1 = a_0$, $b_1 = b_0$ and $c_1 = a_0 + b_0 + m_0 - 1$. Note that $a_0 + b_0 + m_0 - 1 > 0$ by Lemma 2.1 (cf. Lemma 5.1).

Lemma 5.8. *The divisor D_1 is an effective anticanonical $\mathbb{Q}$-divisor on the surface S. The log pair (S, D_1) is not log canonical at the intersection point of L and N.*

Proof. Let $h\colon \tilde{S} \to S'$ be the contraction of the -1-curve $\tilde{N}$. Then S' is a smooth cubic surface in $\mathbb{P}^3$. Put $E' = h(E)$, $L' = h(\tilde{L})$, $M' = h(\tilde{M})$, $Q' = h(Q)$ and $C'_{i0} = h(\tilde{C}_{i0})$ for each i. Then the integral divisor $L' + M' + E'$ is an anticanonical divisor of the cubic surface S'. In particular, the curves L', M' and E' are coplanar lines on S'. Moreover, the point Q' is the intersection point of L' and E' by the assumption right after Lemma 5.5. It does not lie on the line M'.

Let ι_P be the biregular involution of the surface $\bar{S}$ induced by the double cover π. Then ι_P induces a biregular involution υ_P of the surface $\tilde{S}$ since the surface $\tilde{S}$ is the minimal resolution

of singularities of the surface $\bar{S}$. Thus, we have a commutative diagram

This shows $\tau_P = f \circ \upsilon_P \circ f^{-1}$. On the other hand, we have $\upsilon_P(E) = \tilde{N}$ since $\pi \circ g(E) = \pi \circ g(\tilde{N})$. This means that there exists an isomorphism $\sigma \colon S \to S'$ that makes the diagram

commute. By construction, $\sigma(L) = L'$, $\sigma(M) = M'$, $\sigma(N) = E'$, and $\sigma(C_{i1}) = C'_{i0}$ for every i. Recall the point Q' is the intersection point of L' and E'.

Since h is an isomorphism locally around the point Q, the log pair

$$\left(S', a_0 L' + b_0 M' + (a_0 + b_0 + m_0 - 1) E' + \sum_{i=1}^{r} e_i C'_{i0} \right)$$

is not log canonical at the point Q'. Since $a_0 \tilde{L} + b_0 \tilde{M} + c_0 \tilde{N} + (a_0 + b_0 + m_0 - 1) E + \sum_{i=1}^{r} e_i \tilde{C}_{i0} \sim_{\mathbb{Q}} -K_{\tilde{S}}$, we have $a_0 L' + b_0 M' + (a_0 + b_0 + m_0 - 1) E' + \sum_{i=1}^{r} e_i C'_{i0} \sim_{\mathbb{Q}} -K_{S'}$. Therefore, it follows that

$$a_0 L + b_0 M + (a_0 + b_0 + m_0 - 1)N + \sum_{i=1}^{r} e_i C_{i1} \sim_{\mathbb{Q}} -K_S,$$

and the log pair $(S, a_0 L + b_0 M + (a_0 + b_0 + m_0 - 1)N + \sum_{i=1}^{r} e_i C_{i1})$ is not log canonical at the intersection the point of L and N. $\qquad\square$

Now we are able to replace the original effective $\mathbb{Q}$-divisor D by the new effective $\mathbb{Q}$-divisor D_1. By Lemma 5.8, both the $\mathbb{Q}$-divisors have the same properties that we have been using so far. However, the new $\mathbb{Q}$-divisor Ω_1 is *slightly better* than the original one Ω_0 in the sense of the following lemma.

Lemma 5.9. The degree of the $\mathbb{Q}$-divisor Ω_1 is strictly smaller than the degree of Ω_0, i.e.,

$$\sum_{i=1}^{r} e_i d_{i1} < \sum_{i=1}^{r} e_i d_{i0}.$$

Proof. Since $D_1 \sim_{\mathbb{Q}} -K_S$ by Lemma 5.8, we obtain

$$3 = -K_S \cdot \left(a_0 L + b_0 M + (a_0 + b_0 + m_0 - 1)N + \sum_{i=1}^{r} e_i C_{i1} \right) = 2a_0 + 2b_0 + m_0 - 1 + \sum_{i=1}^{r} e_i d_{i1}.$$

On the other hand, we have $a_0 + b_0 + c_0 + \sum_{i=1}^{r} e_i d_{i0} = 3$ by (5.2). Thus, we obtain

$$\sum_{i=1}^{r} e_i d_{i1} = \sum_{i=1}^{r} e_i d_{i0} - (a_0 + b_0 + m_0 - 1 - c_0) < \sum_{i=1}^{r} e_i d_{i0}$$

because $a_0 + b_0 + m_0 - 1 - c_0 > 0$ by Lemma 5.1. $\qquad\square$

Repeating this process, we can obtain a sequence of the effective anticanonical $\mathbb{Q}$-divisors

$$D_k = a_k L + b_k M + c_k N + \Omega_k$$

on the surface S such that each log pair (S, D_k) is not log canonical at one of the three intersection points $L \cap M$, $L \cap N$ and $M \cap N$. Note that

$$\Omega_k = \sum_{i=1}^{r} e_i C_{ik},$$

where C_{ik}'s are irreducible reduced curves of degrees d_{ik}. We then obtain a strictly decreasing sequence of rational numbers

$$\sum_{i=1}^{r} e_i d_{i0} > \sum_{i=1}^{r} e_i d_{i1} > \cdots > \sum_{i=1}^{r} e_i d_{ik} > \cdots$$

by Lemma 5.9. This is a contradiction since the subset

$$\left\{ \sum_{i=1}^{r} e_i n_i \ \middle|\ n_1, n_2, \ldots, n_r \in \mathbb{N} \right\} \subset \mathbb{Q}$$

is discrete and bounded below. It completes the proof of Lemma 4.8.

6. α-FUNCTIONS ON SMOOTH DEL PEZZO SURFACES

In this section, we prove Theorem 1.25. Let S_d be a smooth del Pezzo surface of degree d. Before we proceed, we here make a simple but useful observation.

Lemma 6.1. Let $f : S_d \to S$ be the blow down of a -1-curve E on the del Pezzo surface S_d. Then S is a smooth del Pezzo surface and $\alpha_{S_d}(P) \geqslant \alpha_S(f(P))$ for a point P of S_d outside the curve E.

Proof. This is obvious. $\qquad\square$

We already show that the α-function $\alpha_{\mathbb{P}^2}$ of the projective plane is the constant function with the value $\frac{1}{3}$ (see Example 1.22) and the α-function $\alpha_{\mathbb{P}^1 \times \mathbb{P}^1}$ of the quadric surface is the constant function with the value $\frac{1}{2}$ (see Example 1.23).

Lemma 6.2. The α-function $\alpha_{\mathbb{F}_1}$ on the blow-up $\mathbb{F}_1$ of $\mathbb{P}^2$ at one point is the constant function with the value $\frac{1}{3}$.

Proof. Let P be a given point on $\mathbb{F}_1$. Let $\pi \colon \mathbb{F}_1 \to \mathbb{P}^1$ be the $\mathbb{P}^1$-bundle morphism onto $\mathbb{P}^1$. Let C be its section with $C^2 = -1$ and let L_P be the fiber of the morphism π over the point $\pi(P)$. Since $2C + 3L_P \sim -K_{\mathbb{F}_1}$, we have $\alpha_{\mathbb{F}_1}(P) \leqslant \frac{1}{3}$. But $\alpha(\mathbb{F}_1) = \frac{1}{3}$ by Theorem 1.17. Thus, $\alpha_{\mathbb{F}_1}$ is the constant function with the value $\frac{1}{3}$ by Lemma 1.21. $\qquad\square$

The surface S_7 is the blow-up of $\mathbb{P}^2$ at two distinct points Q_1 and Q_2. Let E be the proper transform of the line passing through the points Q_1 and Q_2 by the two-point blow up $f : S_7 \to \mathbb{P}^2$ with the exceptional curves E_1 and E_2.

Lemma 6.3. The α-function on the del Pezzo surface S_7 of degree 7 has the following values

$$\alpha_{S_7}(P) = \begin{cases} 1/2 & \text{if } P \notin E \\ 1/3 & \text{if } P \in E. \end{cases}$$

Proof. Let P be a point on S. Then $\alpha_{S_7}(P) \geqslant \alpha(S) = \frac{1}{3}$ by Theorem 1.17 and Lemma 1.21.

If the point P belongs to E, then $\alpha_{S_7}(P) \leqslant \frac{1}{3}$ since $2E_1 + 2E_2 + 3E \sim -K_S$. Therefore, $\alpha_{S_7}(P) = \frac{1}{3}$.

Suppose that the point P lies outside E. Let L be a line on $\mathbb{P}^2$ whose proper transform by the blow up f passes through the point P. Since $f^*(2L) + E$ is an effective anticanonical divisor passing through the point P, we have $\alpha_{S_7}(P) \leqslant \frac{1}{2}$.

Let $g \colon S \to \mathbb{P}^1 \times \mathbb{P}^1$ be the birational morphism obtained by contracting the -1-curve E. Then this morphism is an isomorphism around the point P. Then $\alpha_{S_7}(P) \geqslant \alpha_{\mathbb{P}^1 \times \mathbb{P}^1}(g(P))$ by Lemma 6.1. Since $\alpha_{\mathbb{P}^1 \times \mathbb{P}^1}$ is the constant function with the value $\frac{1}{2}$, we obtain $\alpha_{S_7}(P) = \frac{1}{2}$. $\square$

Lemma 6.4. *The α-function α_{S_6} on the del Pezzo surface S_6 of degree 6 is the constant function with the value $\frac{1}{2}$.*

Proof. Let P be a given point on the del Pezzo surface S_6. One can easily check $\alpha_{S_6}(P) \leqslant \frac{1}{2}$. One the other hand, we have a birational morphism $h : S_6 \to S_7$, where S_7 is a del Pezzo surface of degree 7, such that the morphism h is an isomorphism around the point P and the point $h(P)$ is not on the -1-curve of S_7 connected to two different -1-curves. Then $\alpha_{S_6}(P) \geqslant \frac{1}{2}$ by Lemmas 6.1 and 6.3. $\square$

Lemma 6.5. *The α-function on a del Pezzo surface S_5 of degree 5 has the following values*

$$\alpha_{S_5}(P) = \begin{cases} 1/2 & \text{if there is a } -1\text{-curve passing through the point } P; \\ 2/3 & \text{if there is no } -1\text{-curve passing though the point } P. \end{cases}$$

Proof. Let P be a point on S_5. Suppose that P lies on a -1-curve. Then there exists an effective anticanonical divisor not reduced at P. Thus, $\alpha_{S_5}(P) \leqslant \frac{1}{2}$. Meanwhile, we have $\frac{1}{2} = \alpha(S_5) \leqslant \alpha_{S_5}(P)$ by Lemma 1.21 and Theorem 1.17. Therefore, $\alpha_{S_5}(P) = \frac{1}{2}$.

Suppose that the point P is not contained in any -1-curve. Then there exist exactly five irreducible smooth rational curves $C_1, \ldots, C_5$ passing through the point P with $-K_S \cdot C_i = 2$ for each i (cf. the proof of [7, Lemma 5.8]). Moreover, for every C_i, there are four irreducible smooth rational curves E_1^i, E_2^i, E_3^i and E_4^i such that $3C_i + E_1^i + E_2^i + E_3^i + E_4^i$ belongs to the bi-anticanonical linear system $|-2K_{S_5}|$ (cf. Remark 1.14). Therefore, $\alpha_{S_5}(P) \leqslant \frac{2}{3}$.

Suppose that $\alpha_{S_5}(P) < \frac{2}{3}$. Then there is an effective anticanonical $\mathbb{Q}$-divisor D such that $(S, \lambda D)$ is not log canonical at the point P for some positive rational number $\lambda < \frac{2}{3}$. Then $\mathrm{mult}_P(D) > \frac{1}{\lambda}$ by Lemma 2.1. Let $f \colon S_4 \to S_5$ be the blow up of the surface S_5 at the point P with the exceptional curve E and let $\tilde{D}$ be the proper transform of the divisor D on the surface S_4. Then the surface S_4 is a smooth del Pezzo surface of degree 4. We have

$$K_{S_4} + \lambda \tilde{D} + (\lambda \mathrm{mult}_P(D) - 1) E = f^* (K_{S_5} + \lambda D),$$

which implies that the log pair $(S_4, \lambda \tilde{D} + (\lambda \mathrm{mult}_P(D) - 1)E)$ is not log canonical.

On the other hand, the log pair $(S_4, \lambda \tilde{D} + \lambda(\mathrm{mult}_P(D) - 1)E)$ is log canonical because the divisor $\tilde{D} + (\mathrm{mult}_P(D) - 1)E$ is an effective anticanonical $\mathbb{Q}$-divisor of S_4 and $\alpha(S_4) = \frac{2}{3}$ by Theorem 1.17. However, it is absurd because $\lambda(\mathrm{mult}_P(D) - 1) > \lambda \mathrm{mult}_P(D) - 1$. $\square$

Lemma 6.6. *The α-function on a del Pezzo surface S_4 of degree 4 has the following values*

$$\alpha_{S_4}(P) = \begin{cases} 2/3 & \text{if } P \text{ is on a } -1\text{-curve;} \\ 3/4 & \text{if there is an effective anticanonical divisor that consists of} \\ & \text{two 0-curves intersecting tangentially at } P; \\ 5/6 & \text{otherwise.} \end{cases}$$

Proof. Let P be a point on S_4. If the point P lies on a -1-curve L, then there are mutually disjoint five -1-curves $E_1, \ldots, E_5$ that intersect L. Let $h : S_4 \to \mathbb{P}^2$ be the contraction of all E_i's. Since $h(L)$ is a conic in $\mathbb{P}^2$, we see that $3L + \sum_{1 \leqslant i \leqslant 5} E_i$ is a member in the linear system $|-2K_{S_4}|$ (cf. Remark 1.14). This means that $\alpha_{S_4}(P) \leqslant \frac{2}{3}$. Therefore, $\alpha_{S_4}(P) = \frac{2}{3}$ since $\alpha(S_4) \leqslant \alpha_{S_4}(P)$ by Lemma 1.21 and $\alpha(S_4) = \frac{2}{3}$ by Theorem 1.17.

Suppose that the point P does not lie on a -1-curve. Put $\omega = \frac{3}{4}$ in the case when there is an effective anticanonical divisor that consists of two 0-curves intersecting tangentially at the point P and put $\omega = \frac{5}{6}$ otherwise.

One can easily find an effective anticanonical divisor F on the surface S_4 such that $(S_4, \lambda F)$ is not log canonical at the point P for every positive rational number $\lambda > \omega$ (see [32, Proposition 3.2]). This shows that $\alpha_{S_4}(P) \leqslant \omega$. Moreover, it is easy to check that the log pair $(S_4, \omega C)$ is log canonical at the point P for each $C \in |-K_{S_4}|$.

Suppose $\alpha_{S_4}(P) < \omega$. Then there is an effective anticanonical $\mathbb{Q}$-divisor D such that $(S, \omega D)$ is not log canonical at the point P. Note that there are only finitely many effective anticanonical divisors $C_1, \ldots, C_k$ such that each (S_4, C_i) is not log canonical at the point P. Applying Lemma 2.2, we may assume that for each i at least one irreducible component of $\mathrm{Supp}(C_i)$ is not contained in the support of D.

Let $f : S_3 \to S_4$ be the blow up of the surface S_4 at the point P with the exceptional curve E and let $\tilde{D}$ be the proper transform of the divisor D on the surface S_3. Then S_3 is a smooth cubic surface in $\mathbb{P}^3$ and the curve E is a line in S_3. Moreover, the log pair $(S_3, \tilde{D} + (\mathrm{mult}_P(D) - 1)E)$ is not log canonical at some point Q on E because the log pair (S_4, D) is not log canonical at the point P.

Let T_Q be the tangent hyperplane section of the cubic surface S_3 at the point Q. Note that the divisor T_Q contains the line E. Since $\tilde{D} + (\mathrm{mult}_P(D) - 1)E$ is an effective anticanonical $\mathbb{Q}$-divisor on S_3, it follows from Corollary 1.13 that the log pair (S_3, T_Q) is not log canonical at the point Q and the support of $\tilde{D}$ contains all the irreducible components of T_Q. In fact, it follows that the divisor T_Q is either a union of three lines meeting at the point Q or a union of a line and a conic intersecting tangentially at the point Q. The divisor $f(T_Q)$ is an effective anticanonical divisor on S_4 such that the log pair $(S_4, f(T_Q))$ is not log canonical at the point P. This contradicts our assumption since the support of D contains all the irreducible components of the divisor $f(T_Q)$. $\qquad\square$

Consequently, Theorem 1.25 follows from Examples 1.22 and 1.23, and Lemmas 6.2, 6.3, 6.4, 6.5 and 6.6.

Appendix A.

This appendix is devoted to the proof of Lemma 1.10.

Let S be a smooth del Pezzo surface of degree at most 4. Suppose that S contains a $(-K_S)$-polar cylinder, i.e. there is an open affine subset $U \subset S$ and an effective anticanonical $\mathbb{Q}$-divisor D such that $U = S \setminus \mathrm{Supp}(D)$ and $U \cong Z \times \mathbb{A}^1$ for some smooth rational affine curve Z. Put $D = \sum_{i=1}^{r} a_i D_i$, where each D_i is an irreducible reduced curve and each a_i is a positive rational number.

Lemma A.1 ([22, Lemma 4.6]). *The number of the irreducible components of the divisor D is not smaller than the rank of the Picard group of S, i.e., $r \geqslant \mathrm{rk}\,\mathrm{Pic}(S) = 10 - K_S^2 \geqslant 6$.*

Proof. This immediately follows from the exact sequence

$$\bigoplus_{i=1}^{r} \mathbb{Z}[D_i] \longrightarrow \mathrm{Pic}(S) \longrightarrow \mathrm{Pic}(U) \longrightarrow 0,$$

since $\mathrm{Pic}(U) = 0$. $\qquad\square$

To prove Lemma 1.10, we must show that there exists a point $P \in S$ such that

- the log pair (S, D) is not log canonical at the point P;
- if there exists a *unique* divisor T in the anticanonical linear system $|-K_S|$ such that the log pair (S, T) is not log canonical at the point P, then there is an effective anticanonical $\mathbb{Q}$-divisor D' on the surface S such that
  - the log pair (S, D') is not log canonical at the point P;
  - the support of D' does not contain at least one irreducible component of the support of the divisor T.

The natural projection $U \cong Z \times \mathbb{A}^1 \to Z$ induces a rational map $\pi \colon S \dashrightarrow \mathbb{P}^1$ given by a pencil $\mathcal{L}$ on the surface S. Then either $\mathcal{L}$ is base-point-free or its base locus consists of a single point.

Lemma A.2 ([22, Lemma 4.4]). The pencil $\mathcal{L}$ is not base-point-free.

Proof. Suppose that the pencil $\mathcal{L}$ is base-point-free. Then π is a morphism, which implies that there exists exactly one irreducible component of $\mathrm{Supp}(D)$ that does not lie in a fiber of π. Moreover, this component is a section. Without loss of generality, we may assume that this component is D_r. Let L be a sufficiently general curve in $\mathcal{L}$. Then

$$2 = -K_S \cdot L = D \cdot L = \sum_{i=1}^{r} a_i D_i \cdot L = a_r D_r \cdot L,$$

and hence $a_r = 2$. It implies $\alpha(S) \leqslant \frac{1}{2}$. However, it contradicts Theorem 1.17 since the degree of the surface S is at most 4. $\qquad\square$

Denote the unique base point of the pencil $\mathcal{L}$ by P. Let us show that the point P is the point we are looking for. Resolving the base locus of the pencil $\mathcal{L}$, we obtain a commutative diagram

$$
\begin{array}{ccc}
 & W & \\
 {}^{f}\swarrow & & \searrow^{g} \\
S & \dashrightarrow\!\!\!\!\!\!\xrightarrow{\ \pi\ } & \mathbb{P}^1,
\end{array}
$$

where f is a composition of blow ups at smooth points over the point P and g is a morphism whose general fiber is a smooth rational curve. Denote by $E_1, \ldots, E_n$ the exceptional curves of the birational morphism f. Then there exists exactly one curve among them that does not lie in the fibers of the morphism g. Without loss of generality, we may assume that this curve is E_n. Then E_n is a section of the morphism g.

For every D_i, denote by $\tilde{D}_i$ its proper transform on the surface W. Then every curve $\tilde{D}_i$ lies in a fiber of the morphism g.

Lemma A.3. For every effective anticanonical $\mathbb{Q}$-divisor H with $\mathrm{Supp}(H) \subseteq \mathrm{Supp}(D)$, the log pair (S, H) is not log canonical at the point P.

Proof. Write $H = \sum_{i=1}^{k} \epsilon_i \Delta_i$, where each ϵ_i is a non-negative rational number and each Δ_i is an irreducible reduced curve. Denote by $\tilde{\Delta}_i$ the proper transform of Δ_i on the surface W for each i and put $\tilde{H} = \sum_{i=1}^{k} \epsilon_i \tilde{\Delta}_i$. Then

$$K_W + \tilde{H} = f^*\big(K_S + H\big) + \sum_{i=1}^{n} \delta_i E_i \sim_{\mathbb{Q}} \sum_{i=1}^{n} \delta_i E_i$$

for some rational numbers $\delta_1, \ldots, \delta_n$. For a sufficiently general fiber L of the morphism g,

$$-2 = K_W \cdot L = K_W \cdot L + \sum_{i=1}^{r} \epsilon_i \Delta_i \cdot L = \sum_{i=1}^{n} \delta_i E_i \cdot L = \delta_n,$$

because E_n is a section of the morphism g, every curve $\tilde{\Delta}_i$ lies in a fiber of the morphism g and every curve E_i with $i < n$ also lies in a fiber of the morphism g. Hence, the log pair (S, H) is not log canonical at the point P. $\square$

Applying Lemma A.3 to (S, D), we see that the log pair (S, D) is not log canonical at P. Thus, if there exists no anticanonical divisor T such that (S, T) is not log canonical at P, then we are done. Hence, to complete the proof of Lemma 1.10, we assume that there exists a *unique* divisor $T \in |-K_S|$ such that (S, T) is not log canonical at P. Then Lemma 1.10 follows from the lemma below.

Lemma A.4. There exists an effective anticanonical anticanonical $\mathbb{Q}$-divisor D' on S such that the log pair (S, D') is not log canonical at the point P and $\mathrm{Supp}(D')$ does not contain at least one irreducible component of $\mathrm{Supp}(T)$.

Proof. If $\mathrm{Supp}(D)$ does not contain at least one irreducible component of $\mathrm{Supp}(T)$, then we can simply put $D = D'$. Suppose that it is not the case, i.e., we have $\mathrm{Supp}(T) \subseteq \mathrm{Supp}(D)$. Then $T \neq D$. Indeed, the number of the irreducible components of $\mathrm{Supp}(D)$ is at least 6 by Lemma A.1. On the other hand, the number of the irreducible components of $\mathrm{Supp}(T)$ is at most 4 because $-K_S \cdot T = K_S^2$ and $-K_S$ is ample.

Since $T \neq D$, there exists a positive rational number μ such that the $\mathbb{Q}$-divisor $(1 + \mu)D - \mu T$ is effective and its support does not contain at least one irreducible component of $\mathrm{Supp}(T)$. Put $D' = (1 + \mu)D - \mu T$. Note that D' is also an effective anticanonical $\mathbb{Q}$-divisor on S. By our construction, $\mathrm{Supp}(D') \subseteq \mathrm{Supp}(D)$. Thus, the log pair (S, D') is not log canonical at the point P by Lemma A.3. This completes the proof. $\square$

Note that $U \neq S \setminus \mathrm{Supp}(D')$, which implies that the number of the irreducible components of $\mathrm{Supp}(D')$ may be less than $\mathrm{rk}\,\mathrm{Pic}(S)$. Because of this, we can apply Lemma 2.2 only once here. This shows that we really need to use the *uniqueness* of the divisor T in the anticanonical linear system $|-K_S|$ such that (S, T) is not log canonical at P in the proof of Lemma A.4. Indeed, if there is another divisor T' in $|-K_S|$ such that (S, T') is not log canonical at P either, then we would not be able to apply Lemma 2.2 since we may have $D' = T'$.

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Ivan Cheltsov
School of Mathematics, The University of Edinburgh, Edinburgh EH9 3JZ, UK.
`I.Cheltsov@ed.ac.uk`

Jihun Park
IBS Center for Geometry and Physics, Pohang, Gyungbuk 790-784, Korea.
Department of Mathematics, POSTECH, Pohang, Gyungbuk 790-784, Korea.
`wlog@postech.ac.kr`

Joonyeong Won
KIAS, 85 Hoegiro Dongdaemun-gu, Seoul 130-722, Korea.
`leonwon@kias.re.kr`