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REMARKS ON NONLINEAR SMOOTHING UNDER RANDOMIZATION FOR THE PERIODIC KDV AND THE CUBIC SZEGÖ EQUATION

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Abstract. We consider Cauchy problems of some dispersive PDEs with random initial data. In particular, we construct local-in-time solutions to the mean-zero periodic KdV almost surely for the initial data in the support of the mean-zero Gaussian measures on $H^s(\mathbb{T})$, $s > s_0$ where $s_0 = -\frac{11}{6} + \frac{\sqrt{61}}{6} \approx -0.5316 < -\frac{1}{2}$, by exhibiting nonlinear smoothing under randomization on the second iteration of the integration formulation. We also show that there is no nonlinear smoothing for the dispersionless cubic Szegö equation under randomization of initial data.

Contents

1. Introduction

1.1. **Nonlinear smoothing under randomization on initial data for nonlinear Schrödinger equations.** In studying invariance of the Gibbs measure for the defocusing cubic nonlinear Schrödinger equation (NLS) on $\mathbb{T}^2$, Bourgain [4] considered the following

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Cauchy problem for the 2-d Wick ordered cubic NLS:[1]

$$\begin{cases} iu_t - \Delta u \pm (u|u|^2 - 2u \fint |u|^2 dx) = 0 \\ u|_{t=0} = u_0, \end{cases} \qquad x \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2 \tag{1.1}$$

with *random* initial data u_0 of the form

$$u_0(x) = u_0^\omega(x) = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{\sqrt{1+|n|^2}} e^{in\cdot x}, \tag{1.2}$$

where $\{g_n\}_{n\in\mathbb{Z}^2}$ is a family of independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$. We can regard u_0 in (1.2) as a typical element in the support of the Gaussian part of the Gibbs measure on $\mathbb{T}^2$:[2]

$$d\rho = Z^{-1} \exp\Big(-\frac{1}{2}\int |u|^2 dx - \frac{1}{2}\int |\nabla u|^2 dx\Big) \prod_{x\in\mathbb{T}^2} du(x). \tag{1.3}$$

Note that (1.3) is basically the Wiener measure on $\mathbb{T}^2$. Hence, we see that u_0 of the form (1.2) belongs almost surely (a.s.) to $H^s(\mathbb{T}^2)\setminus L^2(\mathbb{T}^2)$ for any $s<0$.[3]

In [2], Bourgain introduced a new weighted space-time Sobolev space $X^{s,b}(\mathbb{T}^d\times\mathbb{R})$, whose norm is given by

$$\|u\|_{X^{s,b}(\mathbb{T}^d\times\mathbb{R})} = \|\langle n\rangle^s \langle \tau - |n|^2\rangle^b \widehat{u}(n,\tau)\|_{l^2_n L^2_\tau(\mathbb{Z}^d\times\mathbb{R})},$$

where $\langle \cdot \rangle = 1 + |\cdot|$, and proved that (1.1) is locally well-posed in $H^s(\mathbb{T}^2)$ for $s>0$. This result barely misses the scaling-critical space $L^2(\mathbb{T}^2)$ for (1.1) due to a slight loss of derivative in the periodic L^4-Strichartz estimate on $\mathbb{T}^2$.

There are two main components in establishing invariance of Gibbs measures for Hamiltonian PDEs:

(a) construction of the Gibbs measure,
(b) (at least local-in-time) well-posedness on the support of the Gibbs measure.

In [4], Bourgain constructed the Gibbs measure after applying Wick ordering to the nonlinear part of the Hamiltonian (in the defocusing case.) As for (b), one needs to construct a continuous flow of (1.1) on the support of the Gibbs measure, i.e. outside $L^2(\mathbb{T}^2)$. This is exactly the main difficulty in [4], due to the absence of deterministic well-posedness even in $L^2(\mathbb{T}^2)$. In order to resolve this issue, Bourgain considered the Cauchy problem (1.1) with random initial data (1.2) and successfully constructed local-in-time solutions almost surely in ω by exhibiting *nonlinear smoothing under randomization of initial data*. Then, by invariance of the finite dimensional Gibbs measures with approximation argument, such local-in-time solutions were then extended globally in time.

We briefly describe Bourgain's idea in the following. First, write (1.1) in the Duhamel formulation:

$$u(t) = \Gamma u(t) := S(t)u_0 \pm i\int_0^t S(t-t')\mathcal{N}(u)(t')dt', \tag{1.4}$$

where $S(t) = e^{-it\Delta}$, u_0 is as in (1.2), and $\mathcal{N}(u) = u|u|^2 - 2u\fint |u|^2$. Note that the linear part $S(t)u_0$ has the same regularity as u_0 for each fixed $t\in\mathbb{R}$, i.e. $S(t)u_0^\omega \notin L^2(\mathbb{T}^2)$

[1] In [4], Wick ordering is a renormalization needed for constructing the Gibbs measure on $\mathbb{T}^2$. The equation (1.1) appears as an equivalent formulation of the NLS obtained from the Wick ordered Hamiltonian.

[2] We introduced $-\frac{1}{2}\int |u|^2 dx$ to avoid the zero frequency issue. This modification is done implicitly in [4].

[3] This regularity can be easily determined from the basic theory of Gaussian measures on Hilbert and Banach spaces. See Kuo [18] and Zhidkov [27].

a.s. However, by a combination of deterministic PDE theory and probabilistic techniques, Bourgain showed that $\int_0^t S(t-t')\mathcal{N}(u)(t')dt'$ lies almost surely in a smoother space $H^s(\mathbb{T})$ for some small $s > 0$ (namely, nonlinear smoothing under randomization.) Indeed, he showed that for each small $T > 0$, there exists a set Ω_T with complemental measure $< e^{-\frac{1}{T^\delta}}$ such that for $\omega \in \Omega_T$, Γ defined in (1.4) is a contraction on a ball around the linear solution, i.e. on $S(t)u_0^\omega + B$, where B denotes the ball of radius 1 in the usual Bourgain space $X^{s,\frac{1}{2}+,T}$ for some small $s > 0$, where $X^{s,\frac{1}{2}+,T}$ is a local-in-time version of the $X^{s,b}$ space on $[-T, T]$.

Recently, several results appeared in this direction, exhibiting nonlinear smoothing under randomization of initial data. See Burq-Tzvetkov [7] for the cubic nonlinear wave equation on a three-dimensional compact Riemannian manifold and Thomann [25] for NLS with a confining potential on $\mathbb{R}^d$. (In [25], there is a statement for NLS without a potential but the result is stated in terms of the Sobolev space corresponding to the Laplace operator with a confining potential.) Colliander-Oh [10] considered the 1-d Wick ordered cubic NLS (1.1) on $\mathbb{T}$ (both defocusing and focusing) with random initial data u_0 of the form:

$$u_0(x) = u_0^\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\sqrt{1+|n|^{2\alpha}}} e^{inx}. \tag{1.5}$$

Note that u_0 in (1.5) is a.s. in $H^{\alpha-\frac{1}{2}-}(\mathbb{T}) := \bigcap_{s<\alpha-\frac{1}{2}} H^s(\mathbb{T}) \setminus H^{\alpha-\frac{1}{2}}(\mathbb{T})$. In particular, u_0 is almost surely in the negative Sobolev spaces for $\alpha \leq \frac{1}{2}$. Also, recall that u_0 in (1.5) represents a typical element in the support of the Gaussian measure ρ_α on the distributions on $\mathbb{T}$:

$$d\rho_\alpha = Z^{-1} \exp\Big(-\frac{1}{2}\int |u|^2 dx - \frac{1}{2}\int |D^\alpha u|^2 dx\Big) \prod_{x\in\mathbb{T}} du(x), \tag{1.6}$$

where $D = \sqrt{-\partial_x^2}$. In [10], it is shown that (1.1) is locally well-posed almost surely in $H^{\alpha-\frac{1}{2}-}(\mathbb{T})$ for each $\alpha > \frac{1}{6}$, i.e. in $H^s(\mathbb{T})$ for each $s > -\frac{1}{3}$. Moreover, we constructed almost sure global-in-time solutions (1.1) for each $\alpha > \frac{5}{12}$, i.e. in $H^s(\mathbb{T})$ for each $s > -\frac{1}{12}$.

The local-in-time argument in [10] closely follows that of Bourgain [4]. The main goal is to show that for each small $T > 0$, there exists a set Ω_T with $P(\Omega_T^c) < e^{-\frac{1}{T^\delta}}$ such that for $\omega \in \Omega_T$, Γ in (1.4) is a contraction on $S(t)u_0^\omega + B$, where B denotes the ball of radius 1 in $X^{s,\frac{1}{2}+,T}$ for some $s \geq 0$. By the standard linear estimate on the Duhamel term in (1.4), it suffices to prove

$$\|\mathcal{N}(u)\|_{X^{s,-\frac{1}{2}+,T}} \lesssim T^\theta, \tag{1.7}$$

for $u \in S(t)u_0^\omega + B$ with some $\theta > 0$, where $S(t) = e^{-it\partial_x^2}$, u_0^ω is as in (1.5), and $\mathcal{N}(u) = \mathcal{N}(u,u,u) = u|u|^2 - 2u \fint |u|^2$. Since $u \in S(t)u_0^\omega + B$, we can write u as $u = S(t)u_0^\omega + v$ for some v with $\|v\|_{X^{s,\frac{1}{2}+,T}} \leq 1$. Hence, it suffices to show (1.7) for $\mathcal{N}(u_1,u_2,u_3)$, assuming that u_j is either of the type

(I) linear part: *random, less regular*

$$u_j(x,t) = S(t)u_0^\omega = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\sqrt{1+|n|^{2\alpha}}} e^{i(nx+n^2t)}, \quad \text{or}$$

(II) nonlinear part: *deterministic, smoother*

$$u_j = v_j \text{ with } \|v_j\|_{X^{s,\frac{1}{2}+,T}} \leq 1.$$

Then, (1.7) was established by a combination of deterministic multilinear analysis and probabilistic argument; in particular, the hypercontractivity of the Ornstein-Uhlenbeck semigroup related to products of Gaussian random variables played an important role.

In the following subsections, we discuss our main results. The first result is on KdV equation with a derivative nonlinearity. We show that a simple application of the ideas in probabilistic Cauchy theory [4, 10] with a fixed point argument fails (Subsection 4.1.) Nonetheless, we apply the *second iteration argument*[4] in the probabilistic setting to improve known deterministic well-posedness results (without using complete integrability), i.e. $s = -\frac{1}{2}$. See Theorem 1.1. The second result is on the *dispersionless* cubic Szegö equation (See (1.17) below.) Here, we show that even with randomization on initial data, one can not improve the deterministic well-posedness result (Proposition 1.6.) This result in particular indicates that it is important to have both randomization and dispersion together to yield an improvement over deterministic results.

1.2. **Korteweg-de Vries equation.** In this part, we consider the periodic Korteweg-de Vries (KdV) equation:

$$\begin{cases} u_t + u_{xxx} + uu_x = 0 \\ u\big|_{t=0} = u_0, \end{cases} \qquad x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \tag{1.8}$$

with random initial data u_0 of the form

$$u_0(x) = u_0^\omega(x) = \sum_{n\neq 0} \frac{g_n(\omega)}{|n|^\alpha} e^{inx} \in H_0^{\alpha-\frac{1}{2}-}(\mathbb{T}) \ \text{ a.s.}, \tag{1.9}$$

where $\{g_n\}_{n\in\mathbb{N}}$ is a family of independent standard complex-valued Gaussian random variables with $g_{-n} = \overline{g_n}$ for $n \in \mathbb{N}$ and $H_0^s(\mathbb{T})$ denotes a subspace of the Sobolev space $H^s(\mathbb{T})$ consisting of real-valued mean-zero elements. Note that u_0 in (1.9) represents a typical element in the support of the Gaussian measure on the real-valued mean-zero distributions on $\mathbb{T}$:

$$d\rho_{0,\alpha} = Z^{-1} \exp\Big(-\frac{1}{2}\int (D^\alpha u)^2 dx\Big) \prod_{x\in\mathbb{T}} du(x), \quad u, \text{ mean } 0. \tag{1.10}$$

Our main goal is to construct local-in-time solutions in $C([-T,T]; H_0^{\alpha-\frac{1}{2}-}(\mathbb{T}))$ for each $\alpha \in (\alpha_0, 0]$ with some $\alpha_0 < 0$ by exhibiting nonlinear smoothing under randomization on initial data.

First, let us briefly review recent well-posedness results on KdV (1.8). By introducing the $X^{s,b}$ space adapted to (the linear part of) KdV with the norm

$$\|u\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} = \|\langle n\rangle^s \langle \tau - n^3\rangle^b \widehat{u}(n,\tau)\|_{l^2_n L^2_\tau(\mathbb{Z}\times\mathbb{R})}, \tag{1.11}$$

Bourgain [3] proved local well-posedness of (1.8) in $L^2(\mathbb{T})$ via the fixed point argument, immediately yielding global well-posedness in $L^2(\mathbb{T})$ thanks to the conservation of the L^2-norm. Kenig-Ponce-Vega [14] (also see [9]) improved Bourgain's result and proved local well-posedness in $H^{-\frac{1}{2}}(\mathbb{T})$ by establishing the bilinear estimate

$$\|\partial_x(uv)\|_{X^{s,-\frac{1}{2}}} \lesssim \|u\|_{X^{s,\frac{1}{2}}} \|v\|_{X^{s,\frac{1}{2}}}, \tag{1.12}$$

[4]The second iteration argument is a name of the method, where we (partially) iterate the Duhamel formulation as in (4.16) and (4.17). This is not to be confused with the *second iterate* as in (1.21) and (4.7).

for $s \geq -\frac{1}{2}$ under the mean zero assumption on u and v. Colliander-Keel-Staffilani-Takaoka-Tao [9] proved the corresponding global well-posedness result via the I-method.

There are also results on (1.8) which exploit its complete integrability. In [5], Bourgain proved global well-posedness of (1.8) in the class $M(\mathbb{T})$ of measures μ, assuming that its total variation $\|\mu\|$ is sufficiently small. His proof is based on the trilinear estimate for the *second iteration* of the integral formulation of (1.8), assuming an a priori uniform bound on the Fourier coefficients of the solution u of the form

$$\sup_{n\in\mathbb{Z}} |\widehat{u}(n,t)| < C \tag{1.13}$$

for all $t \in \mathbb{R}$. Then, he established (1.13) using the complete integrability. More recently, Kappeler-Topalov [12] proved global well-posedness of the KdV in $H^{-1}(\mathbb{T})$ via the inverse spectral method.

Next, we state results regarding the necessary conditions[5] on the regularity with respect to smoothness or uniform continuity of the solution map : $u_0 \in H^s(\mathbb{T}) \to u(t) \in H^s(\mathbb{T})$. Bourgain [5] showed that if the solution map is C^3, then $s \geq -\frac{1}{2}$. Christ-Colliander-Tao [8] proved that if the solution map is uniformly continuous, then $s \geq -\frac{1}{2}$. (Also, see Kenig-Ponce-Vega [15].) These results, in particular, state that it is non-trivial to construct solutions in $C([-T,T];H^s)$ for $s < -\frac{1}{2}$.

In the following, we combine deterministic and probabilistic arguments to construct local-in-time solutions $u \in C([-T,T];H^s)$, $s = \alpha - \frac{1}{2}- < -\frac{1}{2}$, with initial data (1.9), where $T = T(\omega)$. The main difficulty is to lower the differentiability so that $\alpha \leq 0$, i.e. $s < -\frac{1}{2}$. Our focus is to develop a method *without* complete integrability of (1.8), exploiting nonlinear smoothing under randomization. It turns out that when $\alpha \leq 0$, Bourgain's idea in [4] - a fixed point argument around the linear solution discussed in Subsection 1.1 (as in [4, 7, 10, 25]) - does *not* work for (1.8). See Subsection 4.1. Instead, we adapt the *second iteration* argument in the probabilistic setting.

Theorem 1.1. *Let a_0 be the positive root of $a^2 + \frac{5}{3}a - 1 = 0$ given by*

$$a_0 = -\tfrac{5}{6} + \tfrac{\sqrt{61}}{6} \approx 0.4684, \tag{1.14}$$

and let $\alpha_0 := a_0 - \frac{1}{2} \approx -0.0316$. Then, for each $\alpha \in (\alpha_0, 0] \approx (-0.0316, 0]$, KdV (1.8) is locally well-posed almost surely in $H_0^{\alpha-\frac{1}{2}-}(\mathbb{T})$. More precisely, there exist $c, \beta > 0$ such that for each $T \ll 1$, there exists a set $\Omega_T \in \mathcal{F}$ with the following properties:

(i) *$P(\Omega_T^c) = \rho_{0,\alpha} \circ u_0(\Omega_T^c) < e^{-\frac{c}{T^\beta}}$, where $u_0 : \Omega \to H_0^{\alpha-\frac{1}{2}-}(\mathbb{T})$.*
(ii) *For each $\omega \in \Omega_T$ there exists a (unique) solution u of (1.8) in*

$$e^{-t\partial_x^3} u_0 + C([-T,T]; H_0^{-\frac{1}{2}+}(\mathbb{T})) \subset C([-T,T]; H_0^{\alpha-\frac{1}{2}-}(\mathbb{T}))$$

with the initial condition u_0^ω given by (1.9). Here, the uniqueness holds only in a mild sense. See Subsection 4.3.

In particular, we have almost sure local well-posedness of KdV with respect to the Gaussian measure $\rho_{0,\alpha}$ in (1.10) supported on $H^s(\mathbb{T})$ for each $s > s_0$ with

$$s_0 = -\tfrac{11}{6} + \tfrac{\sqrt{61}}{6} \approx -0.5316. \tag{1.15}$$

[5]These results are often referred to as ill-posedness results.

The novelty of Theorem 1.1 is the construction of local-in-time solutions in $C([-T,T];H^s_0(\mathbb{T}))$ with $s<-\frac{1}{2}$ without using complete integrability. This is non-trivial in view of ill-posedness results in $H^s(\mathbb{T})$, $s<-\frac{1}{2}$, described above. The argument combines the second iteration and the probabilistic argument, and does not rely on complete integrability of the equation. Therefore, it can be apply to other non-integrable KdV variants. Lastly, the basic argument in the previous work such as [4, 7, 10, 25] in random Cauchy theory is based on a fixed point argument around the (random) linear solution discussed in Subsection 1.1. (This can be also viewed as a fixed point argument for the nonlinear part, regarding the linear part as a random "forcing term".) As shown in Subsection 4.1, an attempt to follow this argument fails for KdV. The proof of Theorem 1.1 presents a new way (via non-"fixed point" argument - namely the second iteration argument in the probabilistic setting) to construct solution in random Cauchy theory. See Richards [24] for a successful application of this idea to the random Cauchy theory of the quartic KdV (with nonlinearity u^3u_x.)

Remark 1.2. The regularity $s_0\approx -0.5316$ in Theorem 1.1 is by no means sharp. One of the main purposes of this result is to show what one can do when Bourgain's idea on random Cauchy theory in [4] - a fixed point argument around the linear solution - fails. In Section 4, we adapt Bourgain's deterministic second iteration argument [5] in the probabilistic setting. In particular, we use the (deterministic) estimates in [5], that are valid for $s<-\frac{1}{2}$, without modification. However, the values of s and b of the $X^{s,b}$-norm in these estimates from [5] are strongly interrelated, giving a restriction the regularity. See Section 4. We believe that one can lower the regularity s_0 in Theorem 1.1 by improving the deterministic estimates in [5] with the values of s and b "independent". (Namely, keep $b=\frac{1}{2}-$ or $\frac{1}{2}+$ and determine the value of s for which the estimates hold.) We do not pursue this direction. Lastly, note that Theorem 1.1 covers the critical value $\alpha=0$. See Remark 1.3.

Remark 1.3. The value $\alpha=0$, corresponding to the regularity $s=-\frac{1}{2}-$, is the critical value in terms of the know well-posedness/ill-posedness results discussed above. Moreover, when $\alpha=0$, u_0 in (1.9) corresponds to the (mean-zero) Gaussian white noise on $\mathbb{T}$, which is formally invariant in view of the L^2-conservation. Then, one can follow Bourgain's argument in [4] (i) to extend the solutions in Theorem 1.1 globally in time, i.e. in $C(\mathbb{R}_t;H^{-\frac{1}{2}-}(\mathbb{T}))$, and (ii) to establish invariance of the white noise under the flow of KdV. See Quastel-Valkó [23], Oh [19, 21], and Oh-Quastel-Valkó [22] for other proofs of invariance of white noise for KdV. Recently, Richards [24] applied the second iteration argument to the quartic KdV in the probabilistic setting and constructed solutions below the deterministic threshold $s=\frac{1}{2}$ (and established invariance of the Gibbs measure for the quartic KdV.)

Remark 1.4. In [19, 20, 21], we proved local well-posedness of (1.8) in $\mathcal{F}L^{s,p}(\mathbb{T})$ with $s>-\frac{1}{2}, p=2+$, $sp<-1$, where the Fourier-Lebesgue space $\mathcal{F}L^{s,p}(\mathbb{T})$ is defined by the norm

$$\|f\|_{\mathcal{F}L^{s,p}(\mathbb{T})}=\|\langle n\rangle^s\widehat{f}(n)\|_{l^p_n(\mathbb{Z})}. \tag{1.16}$$

The proof is based on the second iteration argument introduced in [5]. It is known [1, 19] that u_0 in (1.9) is almost surely in $\mathcal{F}L^{s,p}$ for $(s-\alpha)p<-1$. Hence, this result shows that there exists $\alpha_0<0$ such that for each $\alpha\in[\alpha_0,0]$, KdV (1.8) is a.s. locally well-posed with initial data (1.9) such that $u\in C([-T,T];\mathcal{F}L^{s,p})$ with $s=\alpha-\frac{1}{p}->-\frac{1}{2}$, where $T=T(\|u_0(\omega)\|_{\mathcal{F}L^{s,p}})$. We point out that this deterministic well-posedness argument in $\mathcal{F}L^{s,p}$ completely fails when $p=2$ and $s<-\frac{1}{2}$. In particular, Theorem 1.1 does not follow from the result in [19, 20, 21].

Remark 1.5. A linear part of a solution constructed in Theorem 1.1 indeed lies in $C([-T,T];B(\mathbb{T}))$ for any Banach space $B(\mathbb{T}) \supset H_0^\alpha(\mathbb{T})$ such that $(H_0^\alpha, B, \rho_{0,\alpha})$ is an abstract Wiener space (roughly speaking, any Banach space B containing H_0^α where the Gaussian measure $\rho_{0,\alpha}$ makes sense as a countable additive probability measure.)[6] In this case, a solution u to (1.8) lies in

$$u = e^{-t\partial_x^3} u_0 + (-\partial_t - \partial_x^3)^{-1} u \in C([-T,T];B(\mathbb{T})) + C([-T,T];H_0^{-\frac{1}{2}+}(\mathbb{T})).$$

As examples of B, we can take the Sobolev spaces $W^{s,p}$ with $s < \alpha - \frac{1}{2}$, and the Fourier-Lebesgue spaces $\mathcal{F}L^{s,p}$ with $s < \alpha - \frac{1}{p}$, where $\mathcal{F}L^{s,p}$ is defined via the norm (1.16). See Bényi-Oh [1] for regularity of $\rho_{0,\alpha}$ in different function spaces. We can also take the Besov spaces $B_{p,\infty}^{\alpha-\frac{1}{2}}$ with $p < \infty$. In [1], we study the regularity of ρ_α in (1.6) with $\alpha = 1$ but it can be easily adjusted for $\rho_{0,\alpha}$ in (1.10) with any α.

1.3. Cubic Szegö equation.

In studying the cubic NLS: $iu_t - \Delta u = |u|^2 u$ on a manifold M, Burq-Gérard-Tzvetkov [6] observed that dispersive properties are strongly influenced by the geometry of the underlying manifold M. Gérard-Grellier [11] further pointed out that dispersion disappears completely when M is a sub-Riemannian manifold. As a toy model to study *non-dispersive* Hamiltonian equations, Gérard-Grellier [11] introduced the *cubic Szegö equation*:

$$\begin{cases} iu_t = \Pi(|u|^2 u) \\ u|_{t=0} = u_0 \end{cases} \qquad x \in \mathbb{T}, \tag{1.17}$$

where Π is the Szegö projector onto the non-negative frequencies, i.e.

$$\Pi(f) := \sum_{n \geq 0} \widehat{f}(n) e^{inx}.$$

It turned out that (1.17) is completely integrable with infinitely many conservation laws and that analysis of (1.17) has a strong connection to the theory of complex variables.

It is shown in [11] that (1.17) is globally well-posed in $H_+^{\frac{1}{2}}(\mathbb{T}) := \Pi(H^{\frac{1}{2}}(\mathbb{T}))$ via the energy method (for $s > \frac{1}{2}$ - the argument for $s = \frac{1}{2}$ is more intricate) and the conservation of the $H_+^{\frac{1}{2}}$-norm. Our interest is to investigate if there is any nonlinear smoothing by considering random initial data u_0 of the form

$$u_0(x) = u_0^\omega(x) = \sum_{n \geq 0} \frac{g_n(\omega)}{\sqrt{1 + |n|^{2\alpha}}} e^{inx} \in H_+^{\alpha - \frac{1}{2}-}(\mathbb{T}) \setminus H_+^{\alpha - \frac{1}{2}}(\mathbb{T}), \quad \text{a.s.} \tag{1.18}$$

for $\alpha \leq 1$. First, write (1.17) in the integral formulation:

$$u(t) = u_0 - i\mathcal{N}(u,u,u)(t), \tag{1.19}$$

where $\mathcal{N}(\cdot,\cdot,\cdot)$ is given by

$$\mathcal{N}(u_1,u_2,u_3)(t) = \int_0^t \Pi(u_1 \overline{u_2} u_3)(t') dt'. \tag{1.20}$$

In the following, we consider the *second iterate*:

$$z(t) = u_0 - i\mathcal{N}(u_0,u_0,u_0)(t), \tag{1.21}$$

where u_0 is as in (1.18).

[6] Strictly speaking, we need that $e^{-t\partial_x^3}$ acts on $B(\mathbb{T})$ continuously. e.g. we need $p < \infty$ for $\mathcal{F}L^{s,p}$.

Proposition 1.6. *Let u_0 be as in* (1.18).
(a) *Let $\alpha \in (\frac{1}{2}, 1]$. Then for $s \geq \alpha - \frac{1}{2}$, we have*

$$\big\|\mathcal{N}(u_0, u_0, u_0)\big\|_{C([-T,T];H^s_+)} = \infty, \quad a.s.$$

for any $T > 0$. In particular, even with $\alpha = 1$, the second iterate $z(t)$ for the cubic Szegö equation (1.17) *is a.s. unbounded in $H^{\frac{1}{2}}_+$.*
(b) *Let $\alpha > \frac{1}{2}$. Then, the second iterate $z(t)$ for the cubic Szegö equation* (1.17) *is a.s. bounded in H^s_+ with $s < \alpha - \frac{1}{2} - \varepsilon$ for any $\varepsilon > 0$.*

Proposition 1.6 (a) shows that there is no gain of regularity, even for $\alpha = 1$. In view of well-posedness of (1.17) in $H^{\frac{1}{2}}_+$, we conclude that there is no smoothing upon randomization of initial data. See also Remark 5.1. This shows that dispersion is closely related to nonlinear smoothing under randomization of initial data. See Section 5 for details.

Remark 1.7. It is interesting to compare Proposition 1.6 with the boundedness of the second iterate for KdV; in Remark 4.1, we show that the nonlinear part of the second iterate for KdV is bounded in $L^2(\mathbb{T})$ even for deterministic mean-zero initial data u_0 in $H^s_0(\mathbb{T})$ as long as $s > -\frac{3}{4}$.

On the one hand, the failure of well-posedness for the cubic Szegö equation (1.17) below $H^{\frac{1}{2}}_+(\mathbb{T})$ is due to the unboundedness of the second iterate below $H^{\frac{1}{2}}_+(\mathbb{T})$ (indeed even in $H^{\frac{1}{2}}_+(\mathbb{T})$ - see [11]), and this problem can not be removed by considering the random initial data of the form (1.18). On the other hand, the failure of well-posedness for KdV below $H^{-\frac{1}{2}}_0(\mathbb{T})$ is not caused by the second iterate, and randomization of initial data with the second iteration argument comes in rescue. We also point out that while the failure of well-posedness of Wick ordered cubic NLS (1.1) below L^2 is due to the unboundedness of the second iterate below L^2, this problem can be removed by a simple fixed point argument (around the linear solution) with random initial data.

Lastly, recall an analogous result for the Benjamin-Ono equation by Tzvetkov [26]. He showed that the second iterate with initial data u_0 of the form (1.9) with $\alpha = \frac{1}{2}$ is in $H^s(\mathbb{T}) \setminus L^2(\mathbb{T})$ for any $s < 0$. (Recall that the threshold regularity of the deterministic well-posedness theory is $L^2(\mathbb{T})$. See Molinet [16, 17].) On the one hand, this result shows that there is no nonlinear smoothing under randomization of initial data. On the other hand, the second iterate is at least as regular as the random initial data. Proposition 1.6 (b) also states that the second iterate is at least as regular as the random initial data. Although there is no nonlinear smoothing under randomization of initial data, these results state that there is still a possibility of constructing solutions below the deterministic threshold regularities. However, such a construction is out of reach at this point.

This paper is organized as follows: We introduce notations in Section 2 and state basic lemmata in Section 3. In Section 4, we prove Theorem 1.1 by establishing the nonlinear estimate on the second iteration of the integral formulation. In Section 5, we show that there is no extra smoothing for the cubic Szegö equation upon randomization of initial data.

2. Notation

Let $X^{s,b}$ denote the periodic Bourgain space defined in (1.11). We often use the shorthand notation $\|\cdot\|_{s,b}$ to denote the $X^{s,b}$ norm. Since the $X^{s,\frac{1}{2}}$ norm fails to control $L^\infty_t H^s_x$ norm, we introduce a smaller space $Z^{s,b}(\mathbb{T}\times\mathbb{R})$ whose norm is given by

$$\|u\|_{Z^{s,b}(\mathbb{T}\times\mathbb{R})} := \|u\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} + \|u\|_{Y^{s,b-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \tag{2.1}$$

where $\langle\,\cdot\,\rangle = 1 + |\cdot|$ and $\|u\|_{Y^{s,b}(\mathbb{T}\times\mathbb{R})} = \|\langle n\rangle^s \langle \tau - n^3\rangle^b \widehat{u}(n,\tau)\|_{l^2_n L^1_\tau(\mathbb{Z}\times\mathbb{R})}$. We also define the local-in-time version $Z^{s,b,T}$ on $\mathbb{T}\times[-T,T]$, by

$$\|u\|_{Z^{s,b,T}} = \inf\big\{\|\widetilde{u}\|_{Z^{s,b}(\mathbb{T}\times\mathbb{R})} : \widetilde{u}|_{[-T,T]} = u\big\}.$$

The local-in-time versions of other function spaces are defined analogously.

If a function depends on both x and t, we use $^{\wedge_x}$ (and $^{\wedge_t}$) to denote the spatial (and temporal) Fourier transform, respectively. However, when there is no confusion, we simply use $^\wedge$ to denote the spatial Fourier transform, the temporal Fourier transform, and the space-time Fourier transform, depending on the context. For simplicity, we often drop 2π in dealing with the Fourier transforms. If a function f is random, we may use the superscript f^ω to show the dependence on ω

Lastly, let $\eta \in C_c^\infty(\mathbb{R})$ be a smooth cutoff function supported on $[-2,2]$ with $\eta \equiv 1$ on $[-1,1]$ and let $\eta_{_T}(t) = \eta(T^{-1}t)$. We use c, C to denote various constants, usually depending only on α and s. If a constant depends on other quantities, we will make it explicit. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$. Similarly, we use $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$ and use $A \ll B$ when there is no general constant C such that $B \leq CA$. We also use $a+$ (and $a-$) to denote $a+\varepsilon$ (and $a-\varepsilon$), respectively, for arbitrarily small $\varepsilon \ll 1$.

3. Basic lemmata and linear estimates

We first state several useful lemmata. See [10] for the proofs. Recall that by restricting the Bourgain spaces onto a small time interval $[-T,T]$, we can gain a small power of T (at a loss of regularity on $\langle \tau - n^3\rangle$.)

Lemma 3.1. *For $b > b' > 0$, we have*

$$\|u\|_{X^{s,b',T}} = \|\eta_{_T} u\|_{X^{s,b',T}} \lesssim T^{b-b'-}\|u\|_{X^{s,b,T}}. \tag{3.1}$$

The proof basically follows from

$$\|\widehat{\eta_{_T}}\|_{L^q_\tau} \sim T^{\frac{q-1}{q}}\|\widehat{\eta}\|_{L^q_\tau} \sim T^{\frac{q-1}{q}}, \tag{3.2}$$

where $\widehat{\eta_{_T}}(\tau) = T\widehat{\eta}(T\tau)$, and interpolation.

Next, we present a probabilistic lemma related to the Gaussian random variables.

Lemma 3.2. *Let $\varepsilon, \beta > 0$. Then, for $T > 0$, we have*

$$|g_n(\omega)| \leq C_\varepsilon T^{-\frac{\beta}{2}}\langle n\rangle^\varepsilon \tag{3.3}$$

for all $n \in \mathbb{Z}$ for ω outside an exceptional set of measure $< e^{-\frac{c}{T^\beta}}$.

Now, we briefly go over the linear estimates related to KdV. Let $S(t) = e^{-t\partial_x^3}$ and $T \leq 1$ in the following. We first present the homogeneous and nonhomogeneous linear estimates. See [2, 13] for details.

Lemma 3.3. *For any $s \in \mathbb{R}$ and $b < \frac{1}{2}$, we have $\|S(t)u_0\|_{X^{s,b,T}} \lesssim T^{\frac{1}{2}-b}\|u_0\|_{H^s}$.*

Lemma 3.4. *For any $s \in \mathbb{R}$ and $b \leq \frac{1}{2}$, we have*

$$\Big\| \int_0^t S(t-t')F(x,t')dt' \Big\|_{X^{s,b,T}} \lesssim \|F\|_{Z^{s,b-1,T}}.$$

Also, we have $\big\| \int_0^t S(t-t')F(x,t')dt' \big\|_{X^{s,b,T}} \lesssim \|F\|_{X^{s,b-1}}$ for $b > \frac{1}{2}$.

The next lemma is the periodic L^4-Strichartz estimate for KdV due to Bourgain [3].

Lemma 3.5. *Let u be a function on $\mathbb{T} \times \mathbb{R}$. Then, we have $\|u\|_{L^4_{x,t}} \lesssim \|u\|_{X^{0,\frac{1}{3}}}$.*

4. On the KdV equation

In this section, we construct local-in-time solutions to (1.8) with random initial data of the form (1.9) for $\alpha \in (\alpha_0, 0]$, where α_0 is as in Theorem 1.1.

4.1. **Overview; Unboundedness of nonlinear term.** By writing KdV (1.8) in the Duhamel formulation, we have

$$u(t) = S(t)u_0 + \mathcal{N}(u,u)(t), \tag{4.1}$$

where $S(t) = e^{-t\partial_x^3}$ and $\mathcal{N}(\cdot,\cdot)$ is given by

$$\mathcal{N}(u_1,u_2)(t) := -\frac{1}{2}\int_0^t S(t-t')\partial_x(u_1u_2)(t')dt'. \tag{4.2}$$

It follows from the conservation of mean that $u(t)$ has the spatial mean 0 for each $t \in \mathbb{R}$ since u_0 has the mean 0. We use (n,τ), (n_1,τ_1), and (n_2,τ_2) to denote the Fourier variables for uu, the first factor, and the second factor u of uu in $\mathcal{N}(u,u)$, respectively. i.e. we have $n = n_1 + n_2$ and $\tau = \tau_1 + \tau_2$. By the mean zero assumption on u and by the fact that we have $\partial_x(uu)$ in the definition of $\mathcal{N}(u,u)$, we may assume $n, n_1, n_2 \neq 0$. We also use the following notation:

$$\sigma_0 := \langle \tau - n^3 \rangle \text{ and } \sigma_j := \langle \tau_j - n_j^3 \rangle.$$

One of the main ingredients is the observation due to Bourgain [3]:

$$n^3 - n_1^3 - n_2^3 = 3nn_1n_2, \text{ for } n = n_1 + n_2, \tag{4.3}$$

which in turn implies that

$$\begin{aligned} \text{MAX} &:= \max(\sigma_0,\sigma_1,\sigma_2) \\ &\gtrsim \langle (\tau - n^3) - (\tau_1 - n_1^3) - (\tau_2 - n_2^3) \rangle \sim \langle nn_1n_2 \rangle. \end{aligned} \tag{4.4}$$

This estimate (4.4) played a crucial role in establishing the bilinear estimate (1.12).

If we were to proceed as in [4, 10], then we would need to estimate $\|\mathcal{N}(u_1,u_2)\|_{Z^{-\frac{1}{2},\frac{1}{2},T}}$, assuming that u_j is either of type

(I) random, less regular:

$$u_j(t) = \eta_{_T}(t)\sum_{n\neq 0} g_n(\omega)e^{i(nx+n^3t)}, \quad \text{or} \tag{4.5}$$

(II) deterministic, smoother: $u_j = v$ with $\|v\|_{Z^{-\frac{1}{2},\frac{1}{2},T}} \leq 1$.

In the following, we show that estimate on $\|\mathcal{N}(u_1, u_2)\|_{Z^{-\frac{1}{2},\frac{1}{2},T}}$ fails to hold by considering the case when both u_1 and u_2 are of type (I).

By computing the Duhamel term explicitly (see (4.19) below), one of the main contributions to $\|\mathcal{N}(u_1, u_2)\|_{Z^{-\frac{1}{2},\frac{1}{2},T}}$ is given by $\|\partial_x(u_1u_2)\|_{X^{-\frac{1}{2},-\frac{1}{2},T}}$. Now, assume that u_1 and u_2 are of type (I). i.e., we have $\widehat{u}_j(n_j, \tau_j) = \widehat{\eta}_T(\tau_j - n_j^3)g_{n_j}$, $j = 1, 2$.

For simplicity of the presentation, we remove the time cutoff η in (4.5). Then, we have $\widehat{u}_j(n_j, \tau_j) = \delta(\tau_j - n_j^3)g_{n_j}$, $j = 1, 2$. Hence, from (4.4) with σ_1, $\sigma_2 \sim 1$, we have $\sigma_0 = \text{MAX} \sim \langle nn_1n_2\rangle$. Then, we have

$$\begin{aligned}
\|\partial_x(u_1u_2)\|^2_{X^{-\frac{1}{2},-\frac{1}{2}}} &= \sum_n \int \bigg| \frac{|n|\langle n\rangle^{-\frac{1}{2}}}{\langle \tau - n^3\rangle^{\frac{1}{2}}} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{u}_1(n_1,\tau_1)\widehat{u}_2(n_2,\tau_2)d\tau_1\bigg|^2 d\tau \\
&\sim \sum_n \int \bigg| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{\delta(\tau_1 - n_1^3)g_{n_1}}{|n_1|^{\frac{1}{2}}} \frac{\delta(\tau_2 - n_2^3)g_{n_2}}{|n_2|^{\frac{1}{2}}} d\tau_1 \bigg|^2 d\tau \\
&= \sum_n \int \sum_{\substack{n=n_1+n_2\\=m_1+m_2}} \int_{\tau=\tau_1+\tau_2} \prod_{j=1}^2 \frac{\delta(\tau_j - n_j^3)g_{n_j}}{|n_j|^{\frac{1}{2}}} d\tau_1 \int_{\tau=\widetilde{\tau}_1+\widetilde{\tau}_2} \prod_{k=1}^2 \frac{\delta(\widetilde{\tau}_k - m_k^3)\overline{g_{m_k}}}{|m_k|^{\frac{1}{2}}} d\widetilde{\tau}_1.
\end{aligned} \tag{4.6}$$

We have nontrivial contribution in (4.6) only when $n_1^3+n_2^3 = m_1^3+m_2^3$ and $n_1+n_2 = m_1+m_2$. Hence, we have $\{n_1, n_2\} = \{m_1, m_2\}$. Therefore, we have

$$(4.6) \sim \sum_n \sum_{n=n_1+n_2} \frac{|g_{n_1}|^2}{|n_1|}\frac{|g_{n_2}|^2}{|n_2|} = \sum_{n_1} \frac{|g_{n_1}|^2}{|n_1|} \sum_{n_2} \frac{|g_{n_2}|^2}{|n_2|} = \infty, \quad \text{a.s.}$$

The last equality holds from the following. Let $F_j(\omega) := 2^{-j}\sum_{|n|\sim 2^j} |g_n(\omega)|^2$. Then, F_j converges to $\text{Var}(g_n) = 2$ a.s. by strong law of large numbers. Hence, the tails of the above sums do not converge to 0.

The above computation involving the Dirac delta function is somewhat formal. It can be made rigorous by using a smooth cutoff function η_T. However, we omit details. As a conclusion, we see that a simple application of the ideas from [4, 10] fails for KdV. In the remaining part of this section, we construct local-in-time solutions by adapting the second iteration argument [5, 20] in the probabilistic setting.

Remark 4.1. Consider the second iterate for the Duhamel formulation (4.1) of KdV:

$$z(t) = S(t)u_0 + \mathcal{N}(S(t)u_0, S(t)u_0)(t), \tag{4.7}$$

where $\mathcal{N}(\cdot,\cdot)$ is defined in (4.2) and $u_0 \in H_0^s(\mathbb{T})$. In the following, we show that the nonlinear part of the second iterate is bounded even in $L^2(\mathbb{T})$ as long as u_0 is in $H_0^s(\mathbb{T})$ for $s > -\frac{3}{4}$. In particular, this shows that the failure of (analytic) well-posedness for KdV below $H_0^{-\frac{1}{2}}(\mathbb{T})$ is not due to the second iterate.

Fix $t > 0$. Then, we have

$$\big\|\mathcal{N}(S(t)u_0, S(t)u_0)(t)\big\|_{L^2} \sim \bigg(\sum_n \bigg| \int_0^t e^{-it'n^3} in \sum_{n=n_1+n_2} \prod_{j=1}^2 e^{it'n_j^3}\widehat{u}_0(n_j)dt'\bigg|^2\bigg)^{\frac{1}{2}}.$$

Since u_0 has mean zero on $\mathbb{T}$, we assume $n_1, n_2 \neq 0$ in the following. Moreover, we assume $n \neq 0$ thanks to the derivative in the nonlinearity $\partial_x(u_1u_2)$. First integrate in t' with (4.3).

Then, by Young's inequality followed by Hölder's inequality, we have

$$\begin{aligned}
\big\|\mathcal{N}(S(t)u_0,S(t)u_0)(t)\big\|_{L^2} &\lesssim \bigg(\sum_n \bigg| in \sum_{n=n_1+n_2}\prod_{j=1}^2 \widehat{u}_0(n_j)\int_0^t e^{-it'(n^3-n_1^3-n_2^3)}dt'\bigg|^2\bigg)^{\frac{1}{2}}\\
&\lesssim \bigg(\sum_n \Big| \sum_{n=n_1+n_2}\prod_{j=1}^2 |n_j|^{-1}|\widehat{u}_0(n_j)|\Big|^2\bigg)^{\frac{1}{2}}\\
&\lesssim \prod_{j=1}^2 \big\|\langle n_j\rangle^{-1}|\widehat{u}_0(n_j)|\big\|_{l^{\frac{4}{3}}_{n_j}} \le \prod_{j=1}^2 \|\langle n_j\rangle^{-1-s}\|_{l^4_{n_j}}\|\langle n_j\rangle^s\widehat{u}_0(n_j)\|_{l^2_{n_j}} \lesssim \|u_0\|_{H^s}^2
\end{aligned}$$

as long as $4(-1-s) < -1$, i.e. $s > -\frac{3}{4}$. By considering the random initial data u_0 of the form (1.9), we can also show that the nonlinear part of the second iterate is a.s. bounded in L^2 as long as $\alpha > -\frac{1}{2}$, namely $s > -1$. We omit the details.

4.2. **Nonlinear analysis via second iteration.** First, we briefly go over Bourgain's argument in [5]. Define

$$A_j = \{(n, n_1, n_2, \tau, \tau_1, \tau_2) \in \mathbb{Z}^3 \times \mathbb{R}^3 : \sigma_j = \text{MAX}\}, \tag{4.8}$$

and let $\mathcal{N}_j(u,u)$ denote the contribution of $\mathcal{N}(u,u)$ on A_j. Then, (4.1) can be written as

$$u(t) = S(t)u_0 + \mathcal{N}_0(u,u)(t) + \mathcal{N}_1(u,u)(t) + \mathcal{N}_2(u,u)(t). \tag{4.9}$$

By the standard bilinear estimate with Lemma 3.5 as in [3], [14], we have

$$\|\mathcal{N}_0(u,u)\|_{-\frac{1}{2}+\delta,\frac{1}{2}-\delta} \le o(1)\|u\|^2_{-\frac{1}{2}-\delta,\frac{1}{2}-\delta}, \tag{4.10}$$

where $o(1) = T^\theta$ with $\theta > 0$ by considering the estimate on a short time interval $[-T,T]$. See (2.17), (2.26), and (2.68) in [5]. Here, we abuse the notation and use $\|\cdot\|_{s,b} = \|\cdot\|_{X^{s,b}}$ to denote the local-in-time version as well. Note that the temporal regularity

$$b = \tfrac{1}{2} - \delta < \tfrac{1}{2}.$$

This allowed us to gain the spatial regularity by 2δ in (4.10). Clearly, we can not expect to do the same for $\mathcal{N}_1(u,u)$. (By symmetry, we do not consider $\mathcal{N}_2(u,u)$ in the following.) When $b < \frac{1}{2}$, the bilinear estimate (1.12) is known to fail for any $s \in \mathbb{R}$ due to the contribution from $\mathcal{N}_1(u,u)$. See [14]. Following the notation in [5], let

$$I_{s,b} = \|\mathcal{N}_1(u,u)\|_{X^{s,b}} \quad \text{and} \quad a := \tfrac{1}{2} - \delta < \tfrac{1}{2}. \tag{4.11}$$

Main goal: Estimate the Duhamel term $\mathcal{N}_1(u,u)$ in $X^{s,b}$ with

$$s := -a = -\tfrac{1}{2} + \delta > -\tfrac{1}{2}, \quad \text{and} \quad b := 1 - a = \tfrac{1}{2} + \delta > \tfrac{1}{2}, \tag{4.12}$$

assuming that u is bounded only in $X^{\widetilde{s},\widetilde{b}}$ with

$$\widetilde{s} := -(1-a) = -\tfrac{1}{2} - \delta < -\tfrac{1}{2}, \quad \text{and} \quad \widetilde{b} := a = \tfrac{1}{2} - \delta < \tfrac{1}{2}. \tag{4.13}$$

By Lemma 3.4 and duality with $\|d(n,\tau)\|_{L^2_{n,\tau}} \le 1$, we have

$$\begin{aligned}
I_{-a,1-a} &= \|\mathcal{N}_1(u,u)\|_{-a,1-a} \qquad (4.14)\\
&\lesssim \sum_{\substack{n,n_1\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{\langle n\rangle^{1-a}d(n,\tau)}{\sigma_0^a}\widehat{u}(n_1,\tau_1)\frac{\langle n_2\rangle^{1-a}c(n_2,\tau_2)}{\sigma_2^a},
\end{aligned}$$

where

$$c(n_2,\tau_2) = \langle n_2\rangle^{-(1-a)}\sigma_2^a\,\widehat{u}(n_2,\tau_2) \text{ so that } \|c\|_{L^2_{n,\tau}} = \|u\|_{-(1-a),a} = \|u\|_{-\frac{1}{2}-\delta,\frac{1}{2}-\delta}. \tag{4.15}$$

The main idea here is to consider the second iteration, i.e. substitute (4.1) for $\widehat{u}(n_1,\tau_1)$ in (4.14), thus leading to a trilinear expression, i.e. write $\mathcal{N}_1(u,u)$ as

$$\mathcal{N}_1(u,u) = \mathcal{N}_1(S(t)u_0,u) + \mathcal{N}_1(\mathcal{N}(u,u),u). \tag{4.16}$$

Applying the second iteration on the second argument of $\mathcal{N}_2(u,u)$, we can write (4.1) and (4.9) as

$$\begin{aligned} u(t) = S(t)u_0 &+ \mathcal{N}_0(u,u)(t) + \mathcal{N}_1(S(t)u_0,u)(t)\\ &+ \mathcal{N}_1(\mathcal{N}(u,u),u)(t) + \mathcal{N}_2(u,S(t)u_0)(t) + \mathcal{N}_2(u,\mathcal{N}(u,u))(t). \end{aligned} \tag{4.17}$$

Since $\sigma_1 = \text{MAX} \gtrsim \langle nn_1n_2\rangle \gg 1$ on A_1, we can assume that

$$\widehat{u}(n_1,\tau_1) = \big(\mathcal{N}(u,u)\big)^\wedge(n_1,\tau_1) \sim \frac{|n_1|}{\sigma_1}\sum_{n_1=n_3+n_4}\int_{\tau_1=\tau_3+\tau_4}\widehat{u}(n_3,\tau_3)\widehat{u}(n_4,\tau_4)d\tau_4. \tag{4.18}$$

Namely, we can assume that the contribution to $\widehat{u}(n_1,\tau_1)$ from the linear part $S(t)u_0$ of (4.9) is negligible since we have $\sigma_1 \sim 1$ for the linear part. Moreover, by the standard computation [3], we have

$$\begin{aligned} \mathcal{N}(u,u)(x,t) &= -i\sum_{k=1}^\infty \frac{i^kt^k}{k!}\sum_{n\neq 0} e^{i(nx+n^3t)}\int \eta(\lambda-n^3)(\lambda-n^3)^{k-1}\widehat{\partial_x u^2}(n,\lambda)d\lambda\\ &\quad + i\sum_{n\neq 0}e^{inx}\int\frac{\big(1-\eta\big)(\tau-n^3)}{\tau-n^3}\widehat{\partial_x u^2}(n,\tau)e^{i\tau t}d\tau\\ &\quad + i\sum_{n\neq 0}e^{i(nx+n^3t)}\int\frac{\big(1-\eta\big)(\lambda-n^3)}{\lambda-n^3}\widehat{\partial_x u^2}(n,\lambda)d\lambda\\ &=: \mathcal{M}_1(u,u)(x,t) + \mathcal{M}_2(u,u)(x,t) + \mathcal{M}_3(u,u)(x,t). \end{aligned} \tag{4.19}$$

Note that $(\mathcal{M}_1(u,u))^\wedge(n_1,\tau_1)$ and $(\mathcal{M}_3(u,u))^\wedge(n_1,\tau_1)$ are distributions supported on $\{\tau_1 - n_1^3 = 0\}$. i.e. $\sigma_1 \sim 1$. Hence, the only contribution for the second iteration on A_1 comes from $\mathcal{M}_2(u,u)$ whose Fourier transform is given in (4.18). This shows the validity of the assumption (4.18).

The σ_1 appearing in the denominator allows us to cancel $\langle n\rangle^{1-a}$ and $\langle n_2\rangle^{1-a}$ in the numerator in (4.14). Then, $I_{-a,1-a}$ can be estimated by

$$\lesssim \sum_{\substack{n=n_1+n_2\\ n_1=n_3+n_4}}\int_{\substack{\tau=\tau_1+\tau_2\\ \tau_1=\tau_3+\tau_4}}\frac{\langle n\rangle^{1-a}d(n,\tau)}{\sigma_0^a}\frac{|n_1|}{\sigma_1}\widehat{u}(n_3,\tau_3)\widehat{u}(n_4,\tau_4)\frac{\langle n_2\rangle^{1-a}c(n_2,\tau_2)}{\sigma_2^a}. \tag{4.20}$$

The argument was then divided into several cases, depending on the sizes of $\sigma_0,\cdots,\sigma_4$. Here, the key algebraic relation is

$$n^3 - n_2^3 - n_3^3 - n_4^3 = 3(n_2+n_3)(n_3+n_4)(n_4+n_2), \quad \text{with } n = n_2+n_3+n_4. \tag{4.21}$$

Then, Bourgain proved -see (2.69) in [5]-

$$I_{-a,1-a} \le o(1)\|u\|_{-(1-a),a}I_{-a,1-a} + o(1)\|u\|^3_{-(1-a),a} + o(1)\|u\|_{-(1-a),a}, \tag{4.22}$$

assuming the a priori estimate (1.13): $|\widehat{u}(n,t)| < C$ for all $n\in\mathbb{Z}$, $t\in\mathbb{R}$. Indeed, the estimates involving the first two terms on the right hand side of (4.22) were obtained without (1.13), and *only* the last term in (4.22) required (1.13), -see "Estimation of (2.62)" in [5]-, which was then used to deduce

$$\|\widehat{u}(n,\cdot)\|_{L^2_\tau} < C. \tag{4.23}$$

The a priori estimate (1.13) is derived via the isospectral property of the KdV flow and is false for a general function in $X^{-(1-a),a}$. (It is here that the smallness of the total variation $\|\mu\|$ is used in [5].)

Our goal is to carry out a similar analysis on the second iteration *without* the a priori estimates (1.13) and (4.23) coming from the complete integrability of KdV. We achieve this goal by exhibiting nonlinear smoothing under randomization on initial data. In the following, we take

$$\alpha > -\delta = a - \tfrac{1}{2}, \tag{4.24}$$

where $\delta > 0$ is as in (4.11), (4.12), and (4.13). Then, the initial data

$$u_0(x) = u_0^\omega(x) = \sum_{n\neq 0} \frac{g_n(\omega)}{|n|^\alpha} e^{inx} \tag{4.25}$$

belongs to $H^{\widetilde{s}}(\mathbb{T}) = H^{-\frac{1}{2}-\delta}(\mathbb{T})$, a.s.

Note that we can use the estimates on $\mathcal{N}_1(\mathcal{N}(u,u),u)$ from [5] *except* when the a priori bound (1.13) was assumed. i.e. we need to estimate the contribution from (2.62) in [5]:

$$R_a(u_2,u_3,u_4) := \sum_n \int_{\tau=\tau_2+\tau_3+\tau_4} \chi_B \frac{d(n,\tau)}{\langle n\rangle^{1+a}\sigma_0^a} \widehat{u}_2(-n,\tau_2)\widehat{u}_3(n,\tau_3)\widehat{u}_4(n,\tau_4) d\tau_2 d\tau_3 d\tau_4, \tag{4.26}$$

where $\|d(n,\tau)\|_{L^2_{n,\tau}} \leq 1$ and $B = \{\sigma_0,\sigma_2,\sigma_3,\sigma_4 < |n|^\gamma\}$ with some small parameter $\gamma > 0$. This corresponds to the case $n_2 = -n$ and $n_3 = n_4 = n$ in (4.20) after some reduction. In our analysis, we directly estimate $R_a(u_2,u_3,u_4)$, assuming that u_j is either of type

(I) linear part: random, less regular

$$u_j(t) = \eta_T(t) \sum_{n\neq 0} \frac{g_n(\omega)}{|n|^\alpha} e^{i(nx+n^3t)}, \quad \text{or}$$

(II) nonlinear part: deterministic, and (expected to be) smoother

$$u_j = \mathcal{N}(u) := \mathcal{N}(u,u) \quad \text{(to be bounded in } X^{-\frac{1}{2}+\delta,\frac{1}{2}-\delta,T}).$$

In [5], this parameter $\gamma = \gamma(a)$, subject to the conditions (2.43) and (2.60) in [5], played a certain role in estimating R_a along with the a priori bound (1.13). However, it plays no role in our analysis. Before proceeding further, we record the conditions on the values of a and γ from [5]:[7]

$$a > \tfrac{7}{18}, \quad \gamma > \frac{2(1-2a)}{a-1/3}, \quad \text{and} \quad 2(1-a)+\gamma < 2. \tag{4.27}$$

From the last two conditions, we obtain the quadratic inequality $a^2 + \frac{5}{3}a - 1 > 0$. By choosing $a > a_0$, where $a_0 = -\frac{5}{6} + \frac{\sqrt{61}}{6} \approx 0.4684$ is as in (1.14) in the statement of Theorem 1.1, we guarantee that all the three conditions in (4.27) are satisfied. Hence, we assume that $a \in (a_0, \frac{1}{2})$ in the following. In particular, this implies that

$$\alpha > \alpha_0 := a_0 - \tfrac{1}{2} \approx -0.0316 \tag{4.28}$$

[7] Note that α in [5] is a in this paper.

and $\delta < -\alpha_0 \approx 0.0316$ such that (4.24) holds.

• **Case 1:** all type (II). By Cauchy-Schwarz and Young's inequalities, we have

$$(4.26) \le \sum_n \|d(n,\cdot)\|_{L^2_\tau} \langle n\rangle^{-1-a} \|\widehat{\mathcal{N}(u)}(-n,\tau_2)\|_{L^{\frac{6}{5}}_{\tau_2}} \|\widehat{\mathcal{N}(u)}(n,\tau_3)\|_{L^{\frac{6}{5}}_{\tau_3}} \|\widehat{\mathcal{N}(u)}(n,\tau_4)\|_{L^{\frac{6}{5}}_{\tau_4}}$$

By Hölder inequality (with appropriate $\pm$ signs) and the fact that $-1-a \le -3a$,

$$\begin{aligned} &\le \sum_n \|d(n,\cdot)\|_{L^2_\tau} \prod_{j=2}^4 \langle n\rangle^{-a} \|\sigma_j^{-\frac{1}{3}-}\|_{L^3_{\tau_j}} \|\sigma_j^{\frac{1}{3}+}\widehat{\mathcal{N}(u)}(\pm n,\tau_j)\|_{L^2_{\tau_j}} \\ &\le T^{\frac{1}{2}-3\delta-} \|d(\cdot,\cdot)\|_{L^2_{n,\tau}} \|\mathcal{N}(u)\|^3_{X^{-a,a}_{6,2}} \le T^{\frac{1}{2}-3\delta-} \|\mathcal{N}(u)\|^3_{X^{-\frac{1}{2}+\delta,\frac{1}{2}-\delta}}, \end{aligned} \tag{4.29}$$

where the last two inequalities follows from Lemma 3.1 by choosing $a > \frac{1}{3}$.

In the following, fix small $\varepsilon > 0$ and $\beta > 0$.

• **Case 2:** all type (I). By Lemma 3.2, we have $|g_n(\omega)| \le CT^{-\frac{\beta}{2}}\langle n\rangle^{\varepsilon}$ outside an exceptional set of measure $< e^{-\frac{c}{T^\beta}}$. Then, by Cauchy-Schwarz, Young's inequalities, and Lemma 3.1 with (4.24), we have

$$\begin{aligned} (4.26) &\lesssim T^{-\frac{3\beta}{2}} \sum_n \int d(n,\tau)\langle n\rangle^{-\frac{3}{2}+\delta+3\varepsilon-3\alpha} \\ &\qquad \times \bigg(\int_{\tau=\tau_2+\tau_3+\tau_4} \widehat{\eta}_T(\tau_2+n^3)\widehat{\eta}_T(\tau_3-n^3)\widehat{\eta}_T(\tau_4-n^3)d\tau_2 d\tau_3 \bigg) d\tau \\ &\le T^{-\frac{3\beta}{2}} \sum_n \|d(n,\cdot)\|_{L^2_\tau} \langle n\rangle^{-\frac{3}{2}+4\delta+3\varepsilon} \|\widehat{\eta}_T * \widehat{\eta}_T * \widehat{\eta}_T\|_{L^2_\tau} \lesssim T^{\frac{1}{2}-\frac{3\beta}{2}-}, \end{aligned} \tag{4.30}$$

as long as $\delta < \frac{1}{4} - \frac{3}{4}\varepsilon$.

• **Case 3:** two type (I), one type (II). Without loss of generality, assume that u_2, u_3 are of type (I), and that u_4 is of type (II). By Lemma 3.2, Cauchy-Schwarz, Young, Hölder inequalities, and Lemma 3.1, we have

$$\begin{aligned} (4.26) &\le T^{-\beta} \sum_n \|d(n,\cdot)\|_{L^2_\tau} \langle n\rangle^{-\frac{3}{2}+\delta+2\varepsilon-2\alpha} \\ &\qquad \times \bigg\| \int_{\tau=\tau_2+\tau_3+\tau_4} \widehat{\eta}_T(\tau_2+n^3)\widehat{\eta}_T(\tau_3-n^3)\widehat{\mathcal{N}(u)}(n,\tau_4)d\tau_2 d\tau_3 \bigg\|_{L^2_\tau} \\ &\le T^{-\beta}\|d\|_{L^2_{n,\tau}} \|\widehat{\eta}_T\|^2_{L^{\frac{6}{5}}} \Big\| \langle n\rangle^{-\frac{3}{2}+3\delta+2\varepsilon} \|\widehat{\mathcal{N}(u)}(n,\tau)\|_{L^{\frac{6}{5}}_\tau} \Big\|_{L^2_n} \\ &\lesssim T^{\frac{1}{3}-\beta-} \Big\| \langle n\rangle^{-\frac{3}{2}+3\delta+2\varepsilon} \|\sigma_4^{-\frac{1}{3}-}\|_{L^3_\tau} \|\sigma_4^{\frac{1}{3}+}\widehat{\mathcal{N}(u)}(n,\tau)\|_{L^2_\tau} \Big\|_{L^2_n} \\ &\lesssim T^{\frac{1}{2}-\delta-\beta-} \|\mathcal{N}(u)\|_{X^{-\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \end{aligned} \tag{4.31}$$

outside an exceptional set of measure $< e^{-\frac{c}{T^\beta}}$ as long as $\delta \le \frac{1}{2} - \varepsilon$.

• **Case 4:** one type (I), two type (II). Without loss of generality, assume that u_2 is of type (I), and that u_3, u_4 are of type (II). By Lemma 3.2, Cauchy-Schwarz, Young, Hölder inequalities, and Lemma 3.1, we have

$$(4.26) \leq T^{-\frac{\beta}{2}} \sum_n \|d(n,\cdot)\|_{L^2_\tau} \langle n\rangle^{-\frac{3}{2}+\delta+\varepsilon-\alpha}$$
$$\times \bigg\| \int_{\tau=\tau_2+\tau_3+\tau_4} \widehat{\eta}_T(\tau_2+n^3)\widehat{\mathcal{N}(u)}(n,\tau_3)\widehat{\mathcal{N}(u)}(n,\tau_4)d\tau_2 d\tau_3 \bigg\|_{L^2_\tau}$$
$$\leq T^{-\frac{\beta}{2}} \|d\|_{L^2_{n,\tau}} \|\widehat{\eta}_T\|_{L^{\frac{6}{5}}} \bigg\| \langle n\rangle^{-\frac{3}{2}+2\delta+\varepsilon} \prod_{j=3}^4 \|\sigma_j^{-\frac{1}{3}-}\|_{L^3_\tau} \|\sigma_j^{\frac{1}{3}+}\widehat{\mathcal{N}(u)}(n,\tau_j)\|_{L^2_{\tau_j}} \bigg\|_{L^2_n}$$
$$\lesssim T^{\frac{1}{6}-\frac{\beta}{2}-} \|\mathcal{N}(u)\|^2_{X^{-\frac{1}{2}+\delta,\frac{1}{3}+}} \lesssim T^{\frac{1}{2}-2\delta-\frac{\beta}{2}-} \|\mathcal{N}(u)\|^2_{X^{-\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \tag{4.32}$$

outside an exceptional set of measure $< e^{-\frac{c}{T^\beta}}$ as long as $\varepsilon \leq \frac{1}{2}$.

Lastly, we point out that the estimates hold even if we restrict the initial data to be supported on $\{|n| \leq N\}$, independent of N. Moreover, we can gain extra power of N^{0-} in Cases 2–4, if we restrict the initial data to be supported on $\{|n| > N\}$.

4.3. **Local well-posedness.** Consider initial data u_0^N of the form

$$u_0^N(x) = \sum_{1\leq |n| \leq N} \frac{g_n(\omega)}{|n|^\alpha} e^{inx}. \tag{4.33}$$

Then, for each N, there exists a unique global solution $u^N \in C(\mathbb{R}; H^s(\mathbb{T}))$ for any $s \geq -\frac{1}{2}$ a.s. in ω. Define $\Gamma^N = \Gamma^N_{u_0^N}$ by

$$\Gamma^N v = \Gamma^N_{u_0^N} v := S(t)u_0^N + \mathcal{N}(v,v).$$

Then, $u^N = \Gamma^N u^N$. We set $\mathcal{N}^N := \mathcal{N}(u^N) = \mathcal{N}(u^N, u^N)$.

Now, we put all the *a priori* estimates together. Note that all the implicit constants are independent of N. Also, when there is no superscript N, it means that $N = \infty$. In the following, C_j, θ_j, and ε_j denote positive constants.

Fix $a = \frac{1}{2} - \delta > a_0$ as in (4.11), where a_0 is defined in (1.14). From Lemma 3.3, we have

$$\|S(t)u_0^N\|_{X^{s,b,T}} \leq C_1 \|u_0^N\|_{H^s} \tag{4.34}$$

for any $s, b \in \mathbb{R}$ with $C_1 = C_1(b)$. In particular, by taking $b > \frac{1}{2}$, we see that $S(t)u_0$ is continuous on $[-T,T]$ with values in H^s. From the definition of $\mathcal{N}_j(\cdot,\cdot)$, (4.10), and (4.11), we have

$$\|\mathcal{N}(u^N,u^N)\|_{X^{-a,a,T}} \leq C_2 T^{\theta_1} \|u^N\|^2_{X^{-(1-a),a,T}} + 2I^N_{-a,a}. \tag{4.35}$$

From (4.22) and (4.29)–(4.32), there exists a set $\Omega_T^{(1)}$ with $P\big((\Omega_T^{(1)})^c\big) < e^{-\frac{c}{T^\beta}}$ such that we have

$$I^N_{-a,1-a} \leq C_3\big(T^{\theta_2}\|u^N\|_{X^{-(1-a),a,T}} I^N_{-a,1-a} + T^{\theta_3}\|u^N\|^3_{X^{-(1-a),a,T}} + T^{\theta_4}\|\mathcal{N}^N\|^3_{X^{-a,a,T}}$$
$$+ T^{\theta_5}\|\mathcal{N}^N\|^2_{X^{-a,a,T}} + T^{\theta_6}\|\mathcal{N}^N\|_{X^{-a,a,T}} + T^{\theta_7}\big)$$

on $\Omega_T^{(1)}$. Note that the choice of $\Omega_T^{(1)}$ is independent of N. For fixed $R > 0$, choose $T > 0$ sufficiently small such that $C_3 T^{\theta_2} R \leq \frac{1}{2}$. Then, we have

$$I^N_{-a,1-a} \leq 2C_3\big(T^{\theta_3}\|u^N\|^3_{X^{-(1-a),a,T}} + T^{\theta_4}\|\mathcal{N}^N\|^3_{X^{-a,a,T}}$$
$$+ T^{\theta_5}\|\mathcal{N}^N\|^2_{X^{-a,a,T}} + T^{\theta_6}\|\mathcal{N}^N\|_{X^{-a,a,T}} + T^{\theta_7}\big) \tag{4.36}$$

for $\|u^N\|_{X^{-(1-a),a,T}} \leq R$. From (4.34)–(4.36), we have

$$\begin{aligned} \|u^N\|_{X^{-(1-a),a,T}} \leq C_1\|u_0^N\|_{H^{-(1-a)}} &+ \tfrac{1}{2}C_2T^{\theta_1}\|u^N\|^2_{X^{-(1-a),a,T}} \\ &+ 2C_3\big(T^{\theta_3}\|u^N\|^3_{X^{-(1-a),a,T}} + T^{\theta_4}\|\mathcal{N}^N\|^3_{X^{-a,a,T}} \\ &+ T^{\theta_5}\|\mathcal{N}^N\|^2_{X^{-a,a,T}} + T^{\theta_6}\|\mathcal{N}^N\|_{X^{-a,a,T}} + T^{\theta_7}\big) \end{aligned} \tag{4.37}$$

and

$$\begin{aligned} \|\mathcal{N}^N\|_{X^{-a,a,T}} \leq \tfrac{1}{2}C_2T^{\theta_1}&\|u^N\|^2_{X^{-(1-a),a,T}} \\ &+ 2C_3\big(T^{\theta_3}\|u^N\|^3_{X^{-(1-a),a,T}} + T^{\theta_4}\|\mathcal{N}^N\|^3_{X^{-a,a,T}} \\ &+ T^{\theta_5}\|\mathcal{N}^N\|^2_{X^{-a,a,T}} + T^{\theta_6}\|\mathcal{N}^N\|_{X^{-a,a,T}} + T^{\theta_7}\big). \end{aligned} \tag{4.38}$$

Moreover, for $N > M$, we have

$$\begin{aligned} \|u^N - u^M\|_{X^{-(1-a),a,T}} &= \|\Gamma^N u^N - \Gamma^M u^M\|_{X^{-(1-a),a,T}} \\ &\leq C_1\|u_0^N - u_0^M\|_{H^{-(1-a)}} \\ &\quad + \tfrac{1}{2}C_2T^{\theta_1}\big(\|u^N\|_{X^{-(1-a),a,T}} + \|u^M\|_{X^{-(1-a),a,T}}\big)\|u^N - u^M\|_{X^{-(1-a),a,T}} \\ &\quad + C_4T^{\theta_3}\big(\|u^N\|^2_{X^{-(1-a),a,T}} + \|u^M\|^2_{X^{-(1-a),a,T}}\big)\|u^N - u^M\|_{X^{-(1-a),a,T}} \\ &\quad + C_5T^{\theta_4}\big(\|\mathcal{N}^N\|^2_{X^{-a,a,T}} + \|\mathcal{N}^M\|^2_{X^{-a,a,T}}\big)\|\mathcal{N}^N - \mathcal{N}^M\|_{X^{-a,a,T}} \\ &\quad + C_6M^{-\varepsilon_1}T^{\theta_5}\|\mathcal{N}^N - \mathcal{N}^M\|^2_{X^{-a,a,T}} + C_6M^{-\varepsilon_1}T^{\theta_5}\|\mathcal{N}^N\|^2_{X^{-a,a,T}} \\ &\quad + C_7M^{-\varepsilon_2}T^{\theta_6}\|\mathcal{N}^N - \mathcal{N}^M\|_{X^{-a,a,T}} + C_7M^{-\varepsilon_2}T^{\theta_6}\|\mathcal{N}^N\|_{X^{-a,a,T}} + C_8M^{-\varepsilon_3}T^{\theta_7}. \end{aligned} \tag{4.39}$$

Also, we have

$$\begin{aligned} &\|\mathcal{N}^N - \mathcal{N}^M\|_{X^{-a,a,T}} \\ &\quad \leq \tfrac{1}{2}C_2T^{\theta_1}\big(\|u^N\|_{X^{-(1-a),a,T}} + \|u^M\|_{X^{-(1-a),a,T}}\big)\|u^N - u^M\|_{X^{-(1-a),a,T}} \\ &\quad + C_4T^{\theta_3}\big(\|u^N\|^2_{X^{-(1-a),a,T}} + \|u^M\|^2_{X^{-(1-a),a,T}}\big)\|u^N - u^M\|_{X^{-(1-a),a,T}} \\ &\quad + C_5T^{\theta_4}\big(\|\mathcal{N}^N\|^2_{X^{-a,a,T}} + \|\mathcal{N}^M\|^2_{X^{-a,a,T}}\big)\|\mathcal{N}^N - \mathcal{N}^M\|_{X^{-a,a,T}} \\ &\quad + C_6M^{-\varepsilon_1}T^{\theta_5}\|\mathcal{N}^N - \mathcal{N}^M\|^2_{X^{-a,a,T}} + C_6M^{-\varepsilon_1}T^{\theta_5}\|\mathcal{N}^N\|^2_{X^{-a,a,T}} \\ &\quad + C_7M^{-\varepsilon_2}T^{\theta_6}\|\mathcal{N}^N - \mathcal{N}^M\|_{X^{-a,a,T}} + C_7M^{-\varepsilon_2}T^{\theta_6}\|\mathcal{N}^N\|_{X^{-a,a,T}} + C_8M^{-\varepsilon_3}T^{\theta_7}. \end{aligned} \tag{4.40}$$

Note that in estimating the difference $\Gamma^N u^N - \Gamma^M u^M$ on A_1, one needs to consider

$$\widetilde{I}_{-a,1-a} := \|\mathcal{N}_1(u^N, u^N) - \mathcal{N}_1(u^M, u^M)\|_{-a,1-a} \tag{4.41}$$

as in [5]. We can follow the argument on pp.135-136 in [5], yielding the third term in (4.39), except for R_a defined in (4.26). As for R_a, we can write

$$\mathcal{N}(\mathcal{N}(u,u),u) - \mathcal{N}(\mathcal{N}(v,v),v) = \mathcal{N}(\mathcal{N}(u+v,u-v),u) + \mathcal{N}(\mathcal{N}(v,v),u-v) \tag{4.42}$$

as in (3.4) in [5], and then we can repeat the computation done for R_a, yielding the last six terms in (4.39).

Recall the large deviation estimate: $P(\|u_0(\omega)\|_{H^s} > K) < e^{-cK^2}$ for $s < -\frac{1}{2}$ and sufficiently large K. Given small $T > 0$, let $K = (2C_1C_3T^{\theta_2})^{-1}$. Then, defining $\Omega_T^{(2)}$ by

$$\Omega_T^{(2)} := \{\omega : \|u_0(\omega)\|_{H^{-(1-a)}} \leq K\}$$

we have $P\big((\Omega_T^{(2)})^c\big) < e^{-\frac{c}{T^\beta}}$ for some $c, \beta > 0$. Moreover, by letting $R = 2C_1\|u_0\|_{H^{-(1-a)}}$, we have $C_3T^{\theta_2}R \leq \frac{1}{2}$ on $\Omega_T^{(2)}$. Finally, let $\Omega_T = \Omega_T^{(1)} \cap \Omega_T^{(2)}$. Then, by choosing T sufficiently small, we see that for $\omega \in \Omega_T$, smooth global solutions $u^N(\omega)$ (and $\mathcal{N}^N$) with initial data $u_0^N(\omega)$ converge in $X^{-(1-a),a,T}$ (in $X^{-a,a,T}$, respectively.) For example, if we choose T by

$$T = \inf \big\{ t > 0 : \tfrac{1}{2} C_2 t^{\theta_1} R + 2C_3 (t^{\theta_3} R^3 + t^{\theta_4} R^3 + t^{\theta_5} R^2 + t^{\theta_6} R + t^{\theta_7}) \geq \tfrac{1}{2} R \big\},$$

then (4.37) and (4.38) along with continuity argument show that $\|u\|_{X^{-(1-a),a,T}} \leq R$ and $\|\mathcal{N}(u,u)\|_{X^{-a,a,T}} \leq R$. From (4.39) and (4.40), one obtains a different condition for T. We point out that the nonlinear part $\mathcal{N}_j(u^N, u^N)$, $j = 1, 2$, converges in a stronger space $X^{-a,1-a,T}$. See (4.11) and (4.36).

Let u denote the limit. We still need to show, for $\omega \in \Omega_T$,

(i) u is indeed a solution to (1.8) with $u_0 \in H^{-(1-a)}(\mathbb{T})$ given by (4.25).
(ii) $u \in C([-T,T]; H^{-(1-a)})$.
(iii) uniqueness of solutions.

The argument for (i) and (ii) exactly follows the corresponding argument in [20], and thus we omit details. It follows from (4.34) with $b = \frac{1}{2} + \delta$, (4.36), and symmetry between σ_1 and σ_2 that

$$\begin{cases} S(t)u_0 \in X^{-(1-a),\frac{1}{2}+\delta,T} \subset C([-T,T]; H^{-(1-a)}) \\ \mathcal{N}_1(u,u) + \mathcal{N}_2(u,u) \in X^{-a,\frac{1}{2}+\delta,T} \subset C([-T,T]; H^{-a}). \end{cases}$$

As for $\mathcal{N}_0(u,u)$, i.e. $\sigma_0 = \text{MAX}$, we can repeat the argument in [20] by separately estimating the contributions on $A := \{\max(\sigma_1,\sigma_2) \gtrsim \langle n n_1 n_2 \rangle^{\frac{1}{100}}\}$ and A^c.

Now, let us discuss uniqueness. In the following, fix $\alpha \in (\alpha_0, 0]$, where α_0 is as in (4.28). Consider a "nice" initial condition $u_0^* := u_0(\omega^*)$ for some $\omega^* \in \Omega_T$ so that a solution u^* exists on $[-T,T]$ for the initial condition u_0^* (along with the estimate in Subsection 4.2.) Then, for some $\varepsilon, \beta > 0$, we have $\sup_{n \neq 0} \langle n \rangle^{-\varepsilon} |g_n(\omega^*)| \leq C T^{-\frac{\beta}{2}}$. Now, let

$$A_{\gamma,T} = \{ v_0 : \sup_n \langle n \rangle^{-\varepsilon} \big| \widehat{v}_0(n) - |n|^{-\alpha} g_n(\omega^*) \big| \leq \gamma C T^{-\frac{\beta}{2}} \}.$$

Then, we have $\sup_n \langle n \rangle^{-\varepsilon} |\widehat{v}_0(n)| \leq (1+\gamma) C T^{-\frac{\beta}{2}}$ on $A_{\gamma,T}$. Moreover, for $v_0 \in A_{\gamma,T}$, we have

$$\begin{aligned} \|v_0 - u_0^*\|_{H^{-(1-a)}} &= \bigg(\sum_{n \neq 0} \langle n \rangle^{-1-2\delta} \big| \widehat{v}_0(n) - |n|^{-\alpha} g_n(\omega^*) \big|^2 \bigg)^{\frac{1}{2}} \\ &\lesssim \sup_n \langle n \rangle^{-\varepsilon} \big| \widehat{v}_0(n) - |n|^{-\alpha} g_n(\omega^*) \big| \leq \gamma C T^{-\frac{\beta}{2}} \leq \tfrac{1}{10} \|u_0^*\|_{H^{-(1-a)}} \end{aligned}$$

by choosing $\varepsilon < \delta$ and γ sufficiently small. Hence, proceeding as before, we can construct solutions v with initial data $v_0 \in A_{\gamma,T}$, satisfying (4.37) and (4.38) (after making a time interval slightly shorter.) Consider the difference between v and $u^* = u(\omega^*)$. From a slight modification of the argument in Subsection 4.2, we have

$$\begin{aligned}
&\|v-u^*\|_{X^{-(1-a),a,T}}\\
&\quad\le C_1\|v_0-u_0^*\|_{H^{-(1-a)}}\\
&\qquad+\tfrac{1}{2}C_2T^{\theta_1}\big(\|v\|_{X^{-(1-a),a,T}}+\|u^*\|_{X^{-(1-a),a,T}}\big)\|v-u^*\|_{X^{-(1-a),a,T}}\\
&\qquad+C_4T^{\theta_3}\big(\|v\|^2_{X^{-(1-a),a,T}}+\|u^*\|^2_{X^{-(1-a),a,T}}\big)\|v-u^*\|_{X^{-(1-a),a,T}}\\
&\qquad+C_5T^{\theta_4}\big(\|\mathcal{N}\|^2_{X^{-a,a,T}}+\|\mathcal{N}^*\|^2_{X^{-a,a,T}}\big)\|\mathcal{N}-\mathcal{N}^*\|_{X^{-a,a,T}}\\
&\qquad+C_6T^{\theta_5}\|\mathcal{N}-\mathcal{N}^*\|^2_{X^{-a,a,T}}+C_6T^{\theta_5}\|\mathcal{N}^*\|^2_{X^{-a,a,T}}\sup_n\langle n\rangle^{-\varepsilon}\big|\widehat{v}_0(n)-|n|^{-\alpha}g_n(\omega^*)\big|\\
&\qquad+C_7T^{\theta_6}\|\mathcal{N}-\mathcal{N}^*\|_{X^{-a,a,T}}+C_7T^{\theta_6}\|\mathcal{N}^*\|_{X^{-a,a,T}}\sup_n\langle n\rangle^{-\varepsilon}\big|\widehat{v}_0(n)-|n|^{-\alpha}g_n(\omega^*)\big|\\
&\qquad+C_8T^{\theta_7}\sup_n\langle n\rangle^{-\varepsilon}\big|\widehat{v}_0(n)-|n|^{-\alpha}g_n(\omega^*)\big|,
\end{aligned}$$

where $\mathcal{N}=\mathcal{N}(v,v)$ and $\mathcal{N}^*=\mathcal{N}(u^*,u^*)$. A similar estimate holds for $\|\mathcal{N}^N-\mathcal{N}^M\|_{X^{-a,a,T}}$. As a consequence, we obtain

$$\|v-u^*\|_{X^{-(1-a),a,T}}\lesssim C(T,R)\sup_n\langle n\rangle^{-\varepsilon}\big|\widehat{v}_0(n)-|n|^{-\alpha}g_n(\omega^*)\big|=o(1)$$

as $\gamma\to 0$. Note that $u_0^*\in A_{\gamma,T}$ for any $\gamma>0$. Hence, u^* is a unique solution. One can similarly obtain

$$\|v(t)-u^*(t)\|_{C([-T,T];H^{-(1-a)})}\lesssim C(T,R)\sup_n\langle n\rangle^{-\varepsilon}\big|\widehat{v}_0(n)-|n|^{-\alpha}g_n(\omega^*)\big|.$$

This provides a weak form of continuous dependence. Lastly, when $\alpha=0$, u_0 in (1.9) corresponds to the mean-zero Gaussian white noise. In this case, one can extend local-in-time solutions to global ones by invariance of the (finite dimensional) white noise. See [4, 21] for this part of discussion.

5. On the Szegö equation

In this section, we consider the *dispersionless* cubic Szegö equation (1.17) and present the proof of Proposition 1.6. In particular, we show that unlike [4] and [10], there is no gain of regularity even if we take initial data to be random of the form (1.18).

If we were to proceed as in [4] and [10], we would need to estimate the $C([-T,T];H^s_+)$-norm of $\mathcal{N}(u_1,u_2,u_3)$ defined in (1.20) for some $s\ge\frac{1}{2}$, assuming u_j is either of type

(I) random, less regular:

$$u_j(x,t)=\eta_T(t)u_0^\omega=\eta_T(t)\sum_{n\ge 0}\frac{g_n(\omega)}{\sqrt{1+|n|^{2\alpha}}}e^{inx},\quad\text{or}$$

(II) deterministic, smoother:[8]

$$u_j=v_j\text{ with }\|v_j\|_{C([-T,T];H^s_+)}\le 1,$$

In view of the well-posedness result in $H^{\frac{1}{2}}_+(\mathbb{T})$, we consider $\alpha\le 1$ in the following. (See (1.18).) We show that the contribution from all type (I) is infinite a.s. for $\alpha\le 1$. For notational simplicity, we use $\langle n\rangle^\alpha$ for $\sqrt{1+|n|^{2\alpha}}$. Note that all the summations take place over non-negative indices, i.e. $n\ge 0$ and $n_j\ge 0$, $j=1,2,3$.

[8]We could consider a different norm for type (II). However, it is not relevant for the following discussion.

Now, assume u_j is of type (I), $j = 1, 2, 3$. Then, by separating the spatial and temporal components, we have

$$\begin{aligned} \big\|\mathcal{N}(u_1,u_2,u_3)\big\|^2_{C([-T,T];H^s_+)} &= C_T \bigg\| \langle n\rangle^s \bigg| \sum_{n=n_1-n_2+n_3} \frac{g_{n_1}}{\langle n_1\rangle^\alpha}\frac{\overline{g_{n_2}}}{\langle n_2\rangle^\alpha}\frac{g_{n_3}}{\langle n_3\rangle^\alpha}\bigg|\bigg\|^2_{l^2_n(\mathbb{Z}_{\geq 0})} \\ &= C_T \sum_n \langle n\rangle^{2s} \sum_{\substack{n=n_1-n_2+n_3\\ =m_1-m_2+m_3}} \frac{g_{n_1}}{\langle n_1\rangle^\alpha}\frac{\overline{g_{n_2}}}{\langle n_2\rangle^\alpha}\frac{g_{n_3}}{\langle n_3\rangle^\alpha}\frac{\overline{g_{m_1}}}{\langle m_1\rangle^\alpha}\frac{g_{m_2}}{\langle m_2\rangle^\alpha}\frac{\overline{g_{m_3}}}{\langle m_3\rangle^\alpha}. \end{aligned} \tag{5.1}$$

In the following, we first prove Proposition 1.6 (a), showing that (5.1) is infinite a.s. for $\alpha \in (\frac{1}{2}, 1]$ and $s \in [\alpha - \frac{1}{2}, \alpha - \frac{1}{4}]$.

We say that we have a *pair* if we have $n_j = m_j$ for some $j = 1, 2, 3$, (or if we have $n_1 = m_3$ or $n_3 = m_1$.) Then, we can separate the sum in (5.1) into three cases: (a) 3 pairs, (b) 1 pair, (c) no pair. We estimate the contribution from each case in the following.

• **Case (a):** 3 pairs. In this case, the contribution to (5.1) is bounded from below by

$$\sum_n \langle n\rangle^{2s} \sum_{n=n_1-n_2+n_3} \frac{|g_{n_1}|^2}{\langle n_1\rangle^{2\alpha}}\frac{|g_{n_2}|^2}{\langle n_2\rangle^{2\alpha}}\frac{|g_{n_3}|^2}{\langle n_3\rangle^{2\alpha}}. \tag{5.2}$$

Now, consider the contribution from $n = n_1$ and $n_2 = n_3$. For $\alpha > \frac{1}{4}$, we have

$$c_\omega := \sum_{n_2} \langle n_2\rangle^{-4\alpha} |g_{n_2}(\omega)|^4 < \infty, \quad \text{a.s.}$$

since $\mathbb{E}\big[\sum_{n_2}\langle n_2\rangle^{-4\alpha}|g_{n_2}|^4\big] \sim \sum_{n_2}\langle n_2\rangle^{-4\alpha} < \infty$. Note that $c_\omega > 0$ a.s. Let $F_j(\omega) := 2^{-j}\sum_{|n|\sim 2^j}|g_n(\omega)|^2$. Then, $F_j(\omega)$ converges to a positive constant a.s. by strong law of large numbers. Hence, for $\alpha \leq s + \frac{1}{2}$, we have

$$\sum_{|n|\sim 2^j} \langle n\rangle^{2s-2\alpha}|g_n(\omega)|^2 \sim 2^{j(2s-2\alpha+1)}F_j(\omega) \not\to 0, \quad \text{a.s.}$$

Therefore, we have

$$(5.2) \geq c_\omega \sum_n \langle n\rangle^{2s-2\alpha}|g_n(\omega)|^2 = \infty, \quad \text{a.s.}$$

for $\alpha \leq s + \frac{1}{2}$. In particular, when $s = \frac{1}{2}$, the contribution from this case is divergent for $\alpha \leq 1$. This already shows that there is no nonlinear smoothing even if we consider random initial data of the form (1.18).

Suppose that we have one pair $n_1 = m_1$. Moreover, assume $n_2 = n_3$. Then, we have $n = n_1 = m_1$ and thus $m_2 = m_3$. Proceeding in a similar manner as above, we see that the contribution to (5.1) is given by

$$\sum_n \langle n\rangle^{2s}\frac{|g_n|^2}{\langle n\rangle^{2\alpha}} \sum_{n_2,m_2} \frac{|g_{n_2}|^2}{\langle n_2\rangle^{2\alpha}}\frac{|g_{m_2}|^2}{\langle m_2\rangle^{2\alpha}} = \infty, \quad \text{a.s.} \tag{5.3}$$

for $\alpha \leq s + \frac{1}{2}$.

For completeness of the argument, we give a brief discussion to show that the contributions from other cases are finite (at least for $\alpha > \frac{1}{2}$ and $s \leq \alpha - \frac{1}{4}$.)

• **Case (b):** 1 pair. We only discuss the case when $n_2 = m_2$ and $\{n_1, n_3\} \neq \{m_1, m_3\}$. Other cases follow in a similar manner (except for the case discussed above.) In this case, the contribution to (5.1) is given by

$$R_1(\omega) := \sum_n \langle n\rangle^{2s} \sum_{n_2} \frac{|g_{n_2}|^2}{\langle n_2\rangle^{2\alpha}} \sum_{\substack{n+n_2=n_1+n_3\\ =m_1+m_3}} \frac{g_{n_1}}{\langle n_1\rangle^\alpha}\frac{g_{n_3}}{\langle n_3\rangle^\alpha}\frac{\overline{g_{m_1}}}{\langle m_1\rangle^\alpha}\frac{\overline{g_{m_3}}}{\langle m_3\rangle^\alpha}.$$

By computing the second moment, we have

$$\begin{aligned}\mathbb{E}\big[|R_1|^2\big] = \mathbb{E}\bigg[&\sum_n \langle n\rangle^{2s} \sum_{n_2} \frac{|g_{n_2}|^2}{\langle n_2\rangle^{2\alpha}} \sum_{\substack{n+n_2=n_1+n_3\\ =m_1+m_3}} \frac{g_{n_1}}{\langle n_1\rangle^\alpha}\frac{g_{n_3}}{\langle n_3\rangle^\alpha}\frac{\overline{g_{m_1}}}{\langle m_1\rangle^\alpha}\frac{\overline{g_{m_3}}}{\langle m_3\rangle^\alpha}\\ &\times \sum_{\widetilde{n}} \langle \widetilde{n}\rangle^{2s} \sum_{\widetilde{n}_2} \frac{|g_{\widetilde{n}_2}|^2}{\langle \widetilde{n}_2\rangle^{2\alpha}} \sum_{\substack{\widetilde{n}+\widetilde{n}_2=\widetilde{n}_1+\widetilde{n}_3\\ =\widetilde{m}_1+\widetilde{m}_3}} \frac{\overline{g_{\widetilde{n}_1}}}{\langle \widetilde{n}_1\rangle^\alpha}\frac{\overline{g_{\widetilde{n}_3}}}{\langle \widetilde{n}_3\rangle^\alpha}\frac{g_{\widetilde{m}_1}}{\langle \widetilde{m}_1\rangle^\alpha}\frac{g_{\widetilde{m}_3}}{\langle \widetilde{m}_3\rangle^\alpha}\bigg]. \end{aligned} \tag{5.4}$$

We have nontrivial contribution in (5.4) when $(n_1, n_3, \widetilde{m}_1, \widetilde{m}_3) = (\widetilde{n}_1, \widetilde{n}_3, m_1, m_3)$ (up to permutation.) By assumption, we have $\{n_1, n_3\} \neq \{m_1, m_3\}$ and $\{\widetilde{n}_1, \widetilde{n}_3\} \neq \{\widetilde{m}_1, \widetilde{m}_3\}$. Then, it follows that $(n_1, n_3) = (\widetilde{n}_1, \widetilde{n}_3)$ and $(m_1, m_3) = (\widetilde{m}_1, \widetilde{m}_3)$ (up to permutation.) Hence, we have

$$\begin{aligned}(5.4) \sim \sum_n \langle n\rangle^{2s} \sum_{n_2} \frac{1}{\langle n_2\rangle^{2\alpha}} &\sum_{n+n_2=\widetilde{n}+\widetilde{n}_2} \langle \widetilde{n}\rangle^{2s} \frac{1}{\langle \widetilde{n}_2\rangle^{2\alpha}}\\ &\times \sum_{\substack{n+n_2=n_1+n_3\\ =m_1+m_3}} \frac{1}{\langle n_1\rangle^{2\alpha}}\frac{1}{\langle n_3\rangle^{2\alpha}}\frac{1}{\langle m_1\rangle^{2\alpha}}\frac{1}{\langle m_3\rangle^{2\alpha}}.\end{aligned} \tag{5.5}$$

Without loss of generality, assume $n_1 \gtrsim n$, since $n_1 + n_3 = n + n_2 \geq n$. Then, for fixed n and n_2, we have

$$\langle n\rangle^{2s} \sum_{n+n_2=n_1+n_3} \frac{1}{\langle n_1\rangle^{2\alpha}}\frac{1}{\langle n_3\rangle^{2\alpha}} \lesssim \sum_{n+n_2=n_1+n_3} \frac{1}{\langle n_1\rangle^{2(\alpha-s)}}\frac{1}{\langle n+n_2-n_1\rangle^{2\alpha}} \lesssim \frac{1}{\langle n+n_2\rangle^{\frac{1}{2}+}}$$

for $2(\alpha - s) \geq \frac{1}{2}$ and $\alpha > \frac{1}{2}$, where the last inequality follows from a slight modification of [26, Lemma 2.2]. The same inequality holds when n and n_j are replaced by $\widetilde{n}$ and $\widetilde{n}_j$. Hence, we have

$$(5.5) \lesssim \sum_n \langle n\rangle^{-1-} \sum_{n_2} \langle n_2\rangle^{-2\alpha} \sum_{\widetilde{n}_2} \langle \widetilde{n}_2\rangle^{-2\alpha} < \infty. \tag{5.6}$$

Therefore, we have $R_1(\omega) < \infty$ a.s. for $\alpha > \frac{1}{2}$ and $s \leq \alpha - \frac{1}{4}$.

• **Case (c):** no pair. In this case, the contribution to (5.1) is given by

$$R_2(\omega) := \sum_n \langle n\rangle^{2s} \sum_{*} \frac{g_{n_1}}{\langle n_1\rangle^\alpha}\frac{\overline{g_{n_2}}}{\langle n_2\rangle^\alpha}\frac{g_{n_3}}{\langle n_3\rangle^\alpha}\frac{\overline{g_{m_1}}}{\langle m_1\rangle^\alpha}\frac{g_{m_2}}{\langle m_2\rangle^\alpha}\frac{\overline{g_{m_3}}}{\langle m_3\rangle^\alpha}, \tag{5.7}$$

where $* = \{n = n_1 - n_2 + n_3 = m_1 - m_2 + m_3,\ \text{no pair}\}$. First, suppose $n = n_1$, and thus $n_2 = n_3$. Then, the contribution to (5.7) is given by

$$\sum_n \langle n\rangle^{2s} \sum_{n_2} \frac{|g_{n_2}|^2}{\langle n_2\rangle^{2\alpha}} \sum_{n=m_1-m_2+m_3} \frac{g_n}{\langle n\rangle^\alpha}\frac{\overline{g_{m_1}}}{\langle m_1\rangle^\alpha}\frac{g_{m_2}}{\langle m_2\rangle^\alpha}\frac{\overline{g_{m_3}}}{\langle m_3\rangle^\alpha}.$$

By computing the second moment as before, we have

$$\mathbb{E}\big[|R_2|^2\big] \sim \sum_n \langle n\rangle^{2s} \max(\langle n\rangle^{2s}, \langle m_2\rangle^{2s}) \sum_{n_2} \frac{1}{\langle n_2\rangle^{2\alpha}} \sum_{\widetilde{n}_2} \frac{1}{\langle \widetilde{n}_2\rangle^{2\alpha}}$$
$$\times \sum_{n=m_1-m_2+m_3} \frac{1}{\langle n\rangle^{2\alpha}} \frac{1}{\langle m_1\rangle^{2\alpha}} \frac{1}{\langle m_2\rangle^{2\alpha}} \frac{1}{\langle m_3\rangle^{2\alpha}}.$$

Now, we can follow the argument in Case (b) to show that $\mathbb{E}\big[|R_2|^2\big] < \infty$.

In the following, assume that $n_1, n_3, m_1, m_3 \neq n$. In this case, we have

$$\mathbb{E}\big[|R_2|^2\big] = \mathbb{E}\bigg[\sum_n \langle n\rangle^{2s} \sum_{*} \frac{g_{n_1}}{\langle n_1\rangle^{\alpha}} \frac{\overline{g_{n_2}}}{\langle n_2\rangle^{\alpha}} \frac{g_{n_3}}{\langle n_3\rangle^{\alpha}} \frac{\overline{g_{m_1}}}{\langle m_1\rangle^{\alpha}} \frac{g_{m_2}}{\langle m_2\rangle^{\alpha}} \frac{\overline{g_{m_3}}}{\langle m_3\rangle^{\alpha}}$$
$$\times \sum_{\widetilde{n}} \langle \widetilde{n}\rangle^{2s} \sum_{\widetilde{*}} \frac{\overline{g_{\widetilde{n}_1}}}{\langle \widetilde{n}_1\rangle^{\alpha}} \frac{g_{\widetilde{n}_2}}{\langle \widetilde{n}_2\rangle^{\alpha}} \frac{\overline{g_{\widetilde{n}_3}}}{\langle \widetilde{n}_3\rangle^{\alpha}} \frac{g_{\widetilde{m}_1}}{\langle \widetilde{m}_1\rangle^{\alpha}} \frac{\overline{g_{\widetilde{m}_2}}}{\langle \widetilde{m}_2\rangle^{\alpha}} \frac{g_{\widetilde{m}_3}}{\langle \widetilde{m}_3\rangle^{\alpha}}\bigg] \tag{5.8}$$
$$\sim \sum_n \langle n\rangle^{2s} \sum_{*} \langle \widetilde{n}\rangle^{2s} \prod_{j=1}^{3} \frac{1}{\langle n_j\rangle^{2\alpha}} \frac{1}{\langle m_j\rangle^{2\alpha}}$$

where $\widetilde{*} = \{\widetilde{n} = \widetilde{n}_1 - \widetilde{n}_2 + \widetilde{n}_3 = \widetilde{m}_1 - \widetilde{m}_2 + \widetilde{m}_3, \text{ no pair }, \widetilde{n}_1, \widetilde{n}_3, \widetilde{m}_1, \widetilde{m}_3 \neq \widetilde{n}\}$. Note that there is no summation for $\widetilde{n}$ since it is determined by the values n_j and m_j. Without loss of generality, assume $n_1 \gtrsim n$. Then, by (a slight modification of) [26, Lemma 2.2], we have

$$\langle n\rangle^{2s} \sum_{*} \prod_{j=1}^{3} \frac{1}{\langle n_j\rangle^{2\alpha}} \lesssim \sum_{n_2} \frac{1}{\langle n_2\rangle^{2\alpha}} \sum_{n_1} \frac{1}{\langle n_1\rangle^{2(\alpha-s)}} \frac{1}{\langle n+n_2-n_1\rangle^{2\alpha}}$$
$$\lesssim \sum_{n_2} \frac{1}{\langle n_2\rangle^{2\alpha}} \frac{1}{\langle n+n_2\rangle^{\frac{1}{2}+}} \lesssim \frac{1}{\langle n\rangle^{\frac{1}{2}+}}$$

for $2(\alpha - s) \geq \frac{1}{2}$ and $\alpha > \frac{1}{2}$. Hence, we have $\mathbb{E}\big[|R_2|^2\big] < \infty$ and thus $R_2(\omega) < \infty$ a.s. for $\alpha > \frac{1}{2}$ and $s \leq \alpha - \frac{1}{4}$.

In general, i.e. if $s > \alpha - \frac{1}{4}$, then we can choose $\widetilde{s} < s$ such that $\widetilde{s} \in [\alpha - \frac{1}{2}, \alpha - \frac{1}{4}]$. Then, from the previous computation with $\widetilde{s}$ instead of s, we have

$$\big\|\mathcal{N}(u_1,u_2,u_3)\big\|_{C([-T,T];H^s_+)} \geq \big\|\mathcal{N}(u_1,u_2,u_3)\big\|_{C([-T,T];H^{\widetilde{s}}_+)} = \infty, \quad \text{a.s.}$$

This proves Part (a) of Proposition 1.6.

Part (b) of of Proposition 1.6 follows easily by taking an expectation of (5.1). After taking an expectation, only the case with 3 pairs remains. Assume $n_1 = \max(n_1, n_2, n_3)$ in the following. Then, we have $n_1 \gtrsim n$. With $s = \alpha - \frac{1}{2} - \varepsilon$, we have

$$\mathbb{E}\Big[\big\|\mathcal{N}(u_1,u_2,u_3)\big\|^2_{C([-T,T];H^s_+)}\Big] \lesssim \sum_n \langle n\rangle^{2s} \sum_{n=n_1-n_2+n_3} \frac{1}{\langle n_1\rangle^{2\alpha}} \frac{1}{\langle n_2\rangle^{2\alpha}} \frac{1}{\langle n_3\rangle^{2\alpha}}$$
$$\lesssim \sum_n \sum_{n=n_1-n_2+n_3} \frac{1}{\langle n_1\rangle^{1+\varepsilon}} \frac{1}{\langle n_2\rangle^{2\alpha}} \frac{1}{\langle n_3\rangle^{2\alpha}} < \infty,$$

as long as $\alpha > \frac{1}{2}$.

Remark 5.1. In studying nonlinear smoothing under randomization for the Wick ordered cubic NLS in [10], we needed to control a term similar to (5.1) when we estimate the contribution from all type (I) to the nonlinearity in the $X^{s,-\frac{1}{2}+}$ norm. On the one hand, if $\sigma_0 := \langle \tau - n^2 \rangle$ is large, the estimate was trivial. On the other hand, if $\sigma_0 := \langle \tau - n^2 \rangle$ is small, then the relation:

$$\sigma_0 - \sigma_1 + \sigma_2 - \sigma_3 = -n^2 + n_1^2 - n_2^2 + n_3^2 = -2(n_2 - n_1)(n_2 - n_3),$$

where $\sigma_j := \langle \tau_j - n_j^2 \rangle \lesssim 1$ with $n = n_1 - n_2 + n_3$ and $\tau = \tau = \tau_1 - \tau_2 + \tau_3$, imposed a restriction on the summation. See [10] for details. However, due to the lack of dispersion for (1.17), there is no restriction on the summation (5.1).

In the case of KdV, although a direct estimate on the integral formulation failed to show any nonlinear smoothing (see Subsection 4.1), we could show that there is a gain of regularity by considering the second iteration. This is due to the fact that $\sigma_1 = \langle \tau_1 - n_1^3 \rangle \gtrsim \langle n n_1 n_2 \rangle$ appears in the denominator in the second iteration. See (4.20). However, even if we consider the second iteration for the cubic Szegö equation (1.17), it seems that we do not have any gain due to the lack of dispersion.

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