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# PERIODIC STOCHASTIC KORTEWEG-DE VRIES EQUATION WITH THE ADDITIVE SPACE-TIME WHITE NOISE

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Abstract. We prove the local well-posedness of the periodic stochastic Korteweg-de Vries equation with the additive space-time white noise. In order to treat low regularity of the white noise in space, we consider the Cauchy problem in the Besov-type space $\widehat{b}^s_{p,\infty}(\mathbb{T})$ for $s = -\frac{1}{2}+$, $p = 2+$ such that $sp < -1$. In establishing the local well-posedness, we use a variant of the Bourgain space adapted to $\widehat{b}^s_{p,\infty}(\mathbb{T})$ and establish a nonlinear estimate on the second iteration on the integral formulation. The deterministic part of the nonlinear estimate also yields the local well-posedness of the deterministic KdV in $M(\mathbb{T})$, the space of finite Borel measures on $\mathbb{T}$.

Contents



## 1. Introduction

In this paper, we prove the local well-posedness of the periodic stochastic KdV equation (SKdV) with the additive space-time white noise:

$$\begin{cases} du + (\partial_x^3 u + u\partial_x u)dt = dW \\ u(x,0) = u_0(x) \end{cases} \tag{1}$$

where $u$ is a real-valued function, $(x,t) \in \mathbb{T} \times \mathbb{R}^+$ with $\mathbb{T} = [0, 2\pi)$, and $W(t) = \frac{\partial B}{\partial x}$ is a cylindrical Wiener process on $L^2(\mathbb{T})$. With $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$, we have $W(t) = \beta_0(t)e_0 + \sum_{n\neq 0} \frac{1}{\sqrt{2}}\beta_n(t)e_n(x)$ where $\{\beta_n\}_{n\geq 0}$ is a family of mutually independent complex-valued Brownian motions (here we take $\beta_0$ to be real-valued) in a fixed probability space $(\Omega, \mathcal{F}, P)$ associated with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\beta_{-n}(t) = \overline{\beta_n}(t)$ for $n \geq 1$. Note that $\text{Var}(\beta_n(1)) = 2$ for $n \geq 1$.

In [8], de Bouard-Debussche-Tsutsumi considered

$$\begin{cases} du + (\partial_x^3 u + u\partial_x u)dt = \phi dW \\ u(x,0) = u_0(x), \end{cases} \tag{2}$$

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where $\phi$ is a bounded linear operator in $L^2(\mathbb{T})$. They showed that (2) is locally well-posed when $\phi$ is a Hilbert-Schmidt operator from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$ for $s > -\frac{1}{2}$. See [8] and the references therein for the previous works in the periodic and nonperiodic settings.

In our present work, we consider the case when $\phi$ is the identity operator on $L^2(\mathbb{T})$. i.e. we take the additive noise to be the space-time white noise $\frac{\partial^2 B}{\partial_t \partial_x}$, where $B(x,t)$ is a two parameter Brownian motion on $\mathbb{T} \times \mathbb{R}^+$. Note that $\phi$ is a Hilbert-Schmidt operator from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$ for $s < -\frac{1}{2}$ but not for $s \geq -\frac{1}{2}$.

Suppose that $u$ is the solution to (1), or equivalently to (2) with $\phi = \text{Id}$, the identity operator on $L^2(\mathbb{T})$. Let $v_1(x,t) = u(x + \alpha_0 t, t) - \alpha_0$, where $\alpha_0 =$ the mean of $u_0$. Then, $v_1$ satisfies (1) with the mean 0 initial condition $u_0 - \alpha_0$. Now, let $\mathbb{P}_0$ be the projection onto the spatial frequency 0, and $\mathbb{P}_{n\neq 0} = \text{Id} - \mathbb{P}_0$. Note that $\mathbb{P}_0 W(t) = \beta_0(t) e_0(x) = \frac{1}{\sqrt{2\pi}} \beta_0(t)$. By letting $v_2 = v_1 - \frac{1}{\sqrt{2\pi}} \beta_0(t)$, we see that $u$ satisfies (1) if and only if $v_2$ satisfies

$$\begin{cases} dv_2 + (\partial_x^3 v_2 + (v_2 + \frac{1}{\sqrt{2\pi}} \beta_0(t)) \partial_x v_2) dt = \mathbb{P}_{n\neq 0} dW \\ v_2(x,0) = u_0(x) - \alpha_0 \end{cases}$$

almost surely since $\beta_0(0) = 0$ a.s. By setting $v_3(x,t) = v_2(x + c_\omega(t), t)$ with $c_\omega(t) = \int_0^t \frac{1}{\sqrt{2\pi}} \beta_0(t') dt'$, it follows that $v_3$ satisfies

$$\begin{cases} dv_3 + (\partial_x^3 v_3 + v_3 \partial_x v_3) dt = d\widetilde{W} \\ v_3(x,0) = u_0(x) - \alpha_0, \end{cases}$$

where $\widetilde{W}(x,t) = \sum_{n\neq 0} \frac{1}{\sqrt{2}} \beta_n(t) e_n(x + c_\omega(t)) = \sum_{n \neq 0} \frac{1}{\sqrt{2}} \beta_n(t) e^{inc_\omega(t)} e_n(x)$. i.e. $v_3$ solves (2) where

$$\phi = \text{diag}(\phi_n; n \neq 0) \quad \text{with } \phi_n(t) = e^{inc_\omega(t)} \quad \text{and} \quad c_\omega(t) = \int_0^t \frac{1}{\sqrt{2\pi}} \beta_0(t') dt' \tag{3}$$

(with respect to the basis $\{e_n\}_{n\in\mathbb{Z}}$.) Moreover, note that $v_3$ has the spatial mean 0 (as long as it exists) since $e_0 \notin \text{Range}(\phi)$. Therefore, in the remaining of the paper, we concentrate on studying the local well-posedness of (2) with $\phi$ given by (3) and the mean 0 initial condition $u_0$, (which implies that $u$ has the spatial mean 0 as long as it exists.)

Recall that $u$ is called a (local-in-time) mild solution to (2) if $u$ satisfies

$$u(t) = S(t)u_0 - \tfrac{1}{2} \int_0^t S(t-t') \partial_x u^2(t') dt' + \int_0^t S(t-t') \phi(t') dW(t') \tag{4}$$

at least for $t \in [0,T]$ for some $T > 0$, where $S(t) = e^{-t\partial_x^3}$.

Note that the first two terms in (4) also appear in the deterministic KdV theory. Thus, we briefly review recent well-posedness results of the periodic (deterministic) KdV:

$$\begin{cases} u_t + u_{xxx} + u u_x = 0 \\ u\big|_{t=0} = u_0, \end{cases} \qquad (x,t) \in \mathbb{T} \times \mathbb{R}. \tag{5}$$

In [1], Bourgain introduced a new weighted space-time Sobolev space $X^{s,b}$ whose norm is given by

$$\|u\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} = \|\langle n \rangle^s \langle \tau - n^3 \rangle^b \widehat{u}(n,\tau)\|_{L^2_{n,\tau}(\mathbb{Z}\times\mathbb{R})}, \tag{6}$$

where $\langle\,\cdot\,\rangle = 1 + |\cdot|$. He proved the local well-posedness of (5) in $L^2(\mathbb{T})$ via the fixed point argument, immediately yielding the global well-posedness in $L^2(\mathbb{T})$ thanks to the conservation of the $L^2$ norm. Kenig-Ponce-Vega [11] improved Bourgain's result and established the local well-posedness in $H^{-\frac{1}{2}}(\mathbb{T})$ by establishing the bilinear estimate

$$\|\partial_x(uv)\|_{X^{s,-\frac{1}{2}}} \lesssim \|u\|_{X^{s,\frac{1}{2}}}\|v\|_{X^{s,\frac{1}{2}}}, \tag{7}$$

for $s \geq -\frac{1}{2}$ under the mean 0 assumption on $u$ and $v$. Colliander-Keel-Staffilani-Takaoka-Tao [5] proved the corresponding global well-posedness result via the $I$-method.

There are also results on (5) which exploit its complete integrability. In [2], Bourgain proved the global well-posedness of (5) in the class $M(\mathbb{T})$ of measures $\mu$, assuming that its total variation $\|\mu\|$ is sufficiently small. His proof is based on the trilinear estimate on the second iteration of the integral formulation of (5), assuming an a priori uniform bound on the Fourier coefficients of the solution $u$ of the form

$$\sup_{n\in\mathbb{Z}} |\widehat{u}(n,t)| < C \tag{8}$$

for all $t \in \mathbb{R}$. Then, he established (8) using the complete integrability. More recently, Kappeler-Topalov [9] proved the global well-posedness of the KdV in $H^{-1}(\mathbb{T})$ via the inverse spectral method.

There are also results on the necessary conditions on the regularity with respect to smoothness or uniform continuity of the solution map : $u_0 \in H^s(\mathbb{T}) \to u(t) \in H^s(\mathbb{T})$. Bourgain [2] showed that if the solution map is $C^3$, then $s \geq -\frac{1}{2}$. Christ-Colliander-Tao [4] proved that if the solution map is uniformly continuous, then $s \geq -\frac{1}{2}$. (Also, see Kenig-Ponce-Vega [12].) These results, in particular, imply that we can not hope to have a local-in-time solution of KdV via the fixed point argument in $H^s$, $s < -\frac{1}{2}$. Recall that, for each fixed $t$, the space-time white noise $\frac{\partial^2 B}{\partial_t \partial_x}$ lies in $\cap_{s<-\frac{1}{2}} H^s \setminus H^{-\frac{1}{2}}$ almost surely. Hence, these results for KdV can not be applied to study the local well-posedness of (1).

Now, let us discuss the spaces which capture the regularities of the spatial and space-time white noise. Recently, we proved the invariance of the (spatial) white noise for the (deterministic) KdV in [13] (also see [14]) by first establishing the local well-posedness in an appropriate Banach space containing the support of the (spatial) white noise. Define the Besov-type space via the norm

$$\|f\|_{\widehat{b}^s_{p,\infty}} := \|\widehat{f}\|_{b^s_{p,\infty}} = \sup_j \|\langle n\rangle^s \widehat{f}(n)\|_{L^p_{|n|\sim 2^j}} = \sup_j \Big(\sum_{|n|\sim 2^j} \langle n\rangle^{sp}|\widehat{f}(n)|^p\Big)^{\frac{1}{p}}. \tag{9}$$

In [13], using the theory of abstract Wiener spaces, we showed that $\widehat{b}^s_{p,\infty}$ contains the full support of the (spatial) white noise for $sp < -1$. (The statement also holds true for $sp = -1$.)

Let's consider the stochastic convolution $\Phi(t)$ given by

$$\Phi(t) = \int_0^t S(t-t')\phi(t')dW(t'), \tag{10}$$

where $\phi$ is given by (3). Define a variant of the $X^{s,b}$ space adjusted to $\widehat{b}^s_{p,\infty}(\mathbb{T})$. Let $X^{s,b}_{p,q}$ be the completion of the Schwartz class $\mathcal{S}(\mathbb{T}\times\mathbb{R})$ under the norm

$$\|u\|_{X^{s,b}_{p,q}} = \|\langle n\rangle^s \langle \tau - n^3\rangle^b \widehat{u}(n,\tau)\|_{b^0_{p,\infty} L^q_\tau}. \tag{11}$$

Note that $X^{s,b}_{p,q}$ defined in (11) is the space of functions $u$ such that $S(-t)u(\cdot,t) \in (\widehat{b}^s_{p,\infty})_x(\mathcal{F}L^{b,q})_t$, where $\mathcal{F}L^{b,q}$ is defined via the norm

$$\|f\|_{\mathcal{F}L^{b,q}} := \|\langle\tau\rangle^b \widehat{f}(\tau)\|_{L^q}. \tag{12}$$

In [13], we also showed that the local-in-time white noise is supported on $\mathcal{F}L^{c,q}$ for $cq < -1$. This implies that the Brownian motion belongs locally in time to $\mathcal{F}L^{b,q}$ for $(b-1)q < -1$. Hence, with $b < \frac{1}{2}$ and $q = 2$, we see that the local-in-time stochastic convolution $\eta(t)\Phi(t)$ lies in $X^{s,b}_{p,q}$ almost surely, with $sp < -1$, $b < \frac{1}{2}$ and $q = 2$, where $\eta(t)$ is a smooth cutoff supported on $[-1,2]$ with $\eta(t) \equiv 1$ on $[0,1]$.

The argument by de Bouard-Debussche-Tsutsumi [8] is based on the result by Roynette [15] on the endpoint regularity of the Brownian motion. i.e. the Brownian motion $\beta(t)$ belongs to the Besov space $B^{1/2}_{p,q}$ if and only if $q = \infty$ (with $1 \leq p < \infty$.) Then, they proved a variant of the bilinear estimate (7) by Kenig-Ponce-Vega adjusted to their Besov space setting, establishing the local well-posedness via the fixed point theorem. Note that the use of a variant of the bilinear estimate (7) required a slight regularization of the noise in space via $\phi$ so that the smoothed noise has the spatial regularity $s > -\frac{1}{2}$. Thus, they could not treat the space-time white noise, i.e. $\phi = \text{Id}$.

Our result is based on two observations. The first one is that our $l^p_n$-based function spaces $\widehat{b}^s_{p,\infty}$ in (9) and $X^{s,b}_{p,q}$ in (11) capture the regularity of the spatial and space-time white noise for $sp < -1$, $b < \frac{1}{2}$ and $q = 2$. The second is that we can indeed carry out Bourgain's argument in [2], a nonlinear estimate on the second iteration, *without* assuming the a priori bound (8), if we take the initial data $u_0 \in \widehat{b}^s_{p,\infty}$ for $s > -\frac{1}{2}$ with $p > 2$. Then, we construct a solution $u$ as a strong limit of the smooth solutions $u^N$ (with smooth $u_0^N$ and $\phi^N$) of (2). Note that our nonlinear estimate on the second iteration in Section 5 depends on the stochastic term, whereas the bilinear estimate in [8] is entirely deterministic.

Finally, we present our main results.

**Theorem 1.** *Let $\phi$ be as in (3) and $p = 2+$. Then, let $s = -\frac{1}{2} + \delta$ with $\frac{p-2}{4p} < \delta < \frac{p-2}{2p}$. i.e. $sp < -1$. Also, let $u_0$ be $\mathcal{F}_0$-measurable such that it has mean 0 and belongs to $\widehat{b}^s_{p,\infty}(\mathbb{T})$ almost surely. Then, there exists a stopping time $T_\omega > 0$ and a unique process $u \in C([0,T_\omega]; \widehat{b}^s_{p,\infty}(\mathbb{T}))$ satisfying (2) on $[0,T_\omega]$ almost surely.*

As a corollary, we obtain the following:

**Theorem 2.** *The stochastic KdV (1) with the additive space-time white noise is locally well-posed almost surely (with the prescribed mean on $u_0$.)*

**Remark 1.1.** Our argument provides an answer to the question posed by Bourgain in [2, Remark on p.120], at least in the local-in-time setting. The deterministic part of the nonlinear estimate in Section 5 can be used to establish the local well-posedness of (5) for a finite Borel measure $u_0 = \mu \in M(\mathbb{T})$ with $\|\mu\| < \infty$ *without* the complete integrability or the smallness assumption on $\mu$. Note that $\mu \in \widehat{b}^s_{p,\infty}$ for $sp \leq -1$ since $\sup_n |\widehat{\mu}(n)| < \|\mu\| < \infty$. Hence, it can be used to study the Cauchy problem on $M(\mathbb{T})$ for non-integrable KdV-variants. Also, see [14].

**Remark 1.2.** Let $\mathcal{F}L^{s,p}(\mathbb{T})$ be the space of functions on $\mathbb{T}$ defined via the norm $\|f\|_{\mathcal{F}L^{s,p}} = \|\langle n\rangle^s \widehat{f}(n)\|_{L^p_n}$. Recall from [13] that $\mathcal{F}L^{s,p}(\mathbb{T})$ contains the support of the (spatial) white noise when $sp < -1$. Then, Theorems 1 and 2 can also be established in $\mathcal{F}L^{s,p}(\mathbb{T})$ for

$s = -\frac{1}{2}+$, $p = 2+$ with $sp < -1$. The modification is straightforward once we note that $\|f\|_{\mathcal{F}L^{s-\varepsilon,p}} \lesssim \|f\|_{\widehat{b}^s_{p,\infty}}$ for any $\varepsilon > 0$, and thus we omit the details.

This paper is organized as follows: In Section 2, we introduce some notations. In Section 3, we introduce function spaces along with their embeddings and state deterministic linear estimates from [1] and [13]. In Section 4, we study some basic properties of the stochastic convolution. In Section 5, we prove Theorem 1 by establishing the nonlinear estimate on the second iteration of the integral formulation (4).

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## 2. Notation

In the periodic setting on $\mathbb{T}$, the spatial Fourier domain is $\mathbb{Z}$. Let $dn$ be the normalized counting measure on $\mathbb{Z}$. We say $f \in L^p(\mathbb{Z})$, $1 \leq p < \infty$, if

$$\|f\|_{L^p(\mathbb{Z})} = \bigg( \int_{\mathbb{Z}} |f(n)|^p dn \bigg)^{\frac{1}{p}} := \bigg( \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} |f(n)|^p \bigg)^{\frac{1}{p}} < \infty.$$

If $p = \infty$, we have the obvious definition involving the supremum. We often drop $2\pi$ for simplicity. If a function depends on both $x$ and $t$, we use ${}^{\wedge_x}$ (and ${}^{\wedge_t}$) to denote the spatial (and temporal) Fourier transform, respectively. However, when there is no confusion, we simply use ${}^\wedge$ to denote the spatial Fourier transform, the temporal Fourier transform, and the space-time Fourier transform, depending on the context.

For a Banach space $X \subset \mathcal{S}'(\mathbb{T}\times\mathbb{R})$, we use $\widehat{X}$ to denote the space of the Fourier transforms of the functions in $X$, which is a Banach space with the norm $\|f\|_{\widehat{X}} = \|\mathcal{F}^{-1}_{n,\tau} f\|_X$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform (in $n$ and $\tau$.) Also, for a space $Y$ of functions on $\mathbb{Z}$, we use $\widehat{Y}$ to denote the space of the inverse Fourier transforms of the functions in $Y$ with the norm $\|f\|_{\widehat{Y}} = \|\mathcal{F}f\|_Y$. Now, define $\widehat{b}^s_{p,q}(\mathbb{T})$ by the norm

$$\|f\|_{\widehat{b}^s_{p,q}(\mathbb{T})} = \|\widehat{f}\|_{b^s_{p,q}(\mathbb{Z})} := \Big\| \|\langle n\rangle^s \widehat{f}(n)\|_{L^p_{|n|\sim 2^j}} \Big\|_{l^q_j} = \bigg( \sum_{j=0}^{\infty} \Big( \sum_{|n|\sim 2^j} \langle n\rangle^{sp} |\widehat{f}(n)|^p \Big)^{\frac{q}{p}} \bigg)^{\frac{1}{q}} \tag{13}$$

for $q < \infty$ and by (9) when $q = \infty$.

Throughout the paper, $\eta(t)$ denotes a smooth cutoff supported on $[-1, 2]$ with $\eta(t) \equiv 1$ on $[0, 1]$, and let $\eta_{_T}(t) = \eta(T^{-1}t)$. We use $c$, $C$ to denote various constants, usually depending only on $s$, $p$, and $\delta$. If a constant depends on other quantities, we make it explicit. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$. Similarly, we use $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$ and use $A \ll B$ when there is no general constant $C$ such that $B \leq CA$. We also use $a+$ (and $a-$) to denote $a + \varepsilon$ (and $a - \varepsilon$), respectively, for arbitrarily small $\varepsilon \ll 1$.

## 3. Function Spaces and Basic Embeddings

First, let $X^{s,b}$ denote the usual periodic Bourgain space defined in (6). We often use the shorthand notation $\|\cdot\|_{s,b}$ to denote the $X^{s,b}$ norm. Now, define $X^{s,b}_{p,q}$, the Bourgain space adapted to $\widehat{b}^s_{p,\infty}$, to be the completion of the Schwartz functions on $\mathbb{T}\times\mathbb{R}$ with respect to the norm given by

$$\|u\|_{X^{s,b}_{p,q}} = \|\langle n\rangle^s \langle \tau - n^3\rangle^b \widehat{u}(n,\tau)\|_{b^0_{p,\infty} L^q_\tau} = \sup_j \|\langle n\rangle^s \langle \tau - n^3\rangle^b \widehat{u}(n,\tau)\|_{L^p_{|n|\sim 2^j} L^q_\tau}. \tag{14}$$

In the following, we take $p = 2+$ and $s = -\frac{1}{2}+ = -\frac{1}{2} + \delta$ with $\delta < \frac{p-2}{2p}$ (and $\delta > \frac{p-2}{4p}$) such that $sp < -1$. Lastly, given $T > 0$, we define $X^{s,b,T}_{p,q}$ as a restriction of $X^{s,b}_{p,q}$ on $[0,T]$ by

$$\|u\|_{X^{s,b,T}_{p,q}} = \|u\|_{X^{s,b}_{p,q}[0,T]} = \inf\big\{\|\widetilde{u}\|_{X^{s,b}_{p,q}} : \widetilde{u}|_{[0,T]} = u\big\}.$$

We define the local-in-time versions of the other function spaces analogously.

Now, we discuss the basic embeddings. For $p \geq 2$, we have $\|a_n\|_{L^p_n} \leq \|a_n\|_{L^2_n}$. Thus, we have $\|f\|_{\widehat{b}^s_{p,\infty}} \leq \|f\|_{H^s}$, and thus

$$\|u\|_{X^{s,b}_{p,2}} \leq \|u\|_{X^{s,b}}. \tag{15}$$

By Hölder inequality, we have

$$\begin{aligned}\|f\|_{H^{-\frac{1}{2}-\delta}} &= \Big(\sum_j (2^j)^{0-}\|\langle n\rangle^{-\frac{1}{2}-\delta+}\widehat{f}(n)\|^2_{|n|\sim 2^j}\Big)^{\frac{1}{2}}\\ &\leq \sup_j \|\langle n\rangle^{-2\delta+}\|_{L^{\frac{2p}{p-2}}}\|\langle n\rangle^{-\frac{1}{2}+\delta}\widehat{f}(n)\|_{L^p_n} \leq \|f\|_{\widehat{b}^s_{p,\infty}}\end{aligned} \tag{16}$$

for $s = -\frac{1}{2} + \delta$ with $\delta > \frac{p-2}{4p}$. Hence, for $s = -\frac{1}{2} + \delta$ with $\delta > \frac{p-2}{4p}$, we have

$$\|u\|_{X^{-\frac{1}{2}-\delta,b}} \lesssim \|u\|_{X^{s,b}_{p,2}}. \tag{17}$$

Now, we briefly go over the linear estimates. Let $S(t) = e^{-t\partial_x^3}$ and $T \leq 1$ in the following. We first present the homogeneous and nonhomogeneous linear estimates. See [1], [10], [13] for details of the proofs.

**Lemma 3.1.** *For any $s \in \mathbb{R}$ and $b < \frac{1}{2}$, we have $\|S(t)u_0\|_{X^{s,b,T}_{p,2}} \lesssim T^{\frac{1}{2}-b}\|u_0\|_{\widehat{b}^s_{p,\infty}}$.*

**Lemma 3.2.** *For any $s \in \mathbb{R}$ and $b \leq \frac{1}{2}$, we have*

$$\Big\|\int_0^t S(t-t')F(x,t')dt'\Big\|_{X^{s,b,T}_{p,2}} \lesssim \|F\|_{X^{s,b-1}_{p,2}} + \|F\|_{X^{s,-1}_{p,1}}.$$

*Also, we have $\big\|\int_0^t S(t-t')F(x,t')dt'\big\|_{X^{s,b,T}_{p,2}} \lesssim \|F\|_{X^{s,b-1}_{p,2}}$ for $b > \frac{1}{2}$.*

The next lemma is the periodic $L^4$ Strichartz estimate due to Bourgain [1].

**Lemma 3.3.** *Let $u$ be a function on $\mathbb{T}\times\mathbb{R}$. Then, we have $\|u\|_{L^4_{x,t}} \lesssim \|u\|_{X^{0,\frac{1}{3}}}$.*

Lastly, recall that by restricting the Bourgain spaces onto a small time interval $[0,T]$, we can gain a small power of $T$. See Colliander-Oh [6] for the proof.

**Lemma 3.4.** *For $0 \leq b' < b \leq \frac{1}{2}$, we have*

$$\|u\|_{X^{s,b',T}} = \|\eta_T u\|_{X^{s,b',T}} \lesssim T^{b-b'-}\|u\|_{X^{s,b}}.$$

## 4. Stochastic Convolution

In this section, we study basic properties of the stochastic convolution $\Phi(t)$ defined in (10). In particular, we prove that $\eta\Phi$ belongs to $X^{s,b,T}_{p,2}$ and is continuous from $[0,T]$ into $\widehat{b}^s_{p,\infty}$ for $T \leq 1$ almost surely for $sp < -1$ and $(b-1)\cdot 2 < -1$, where $\eta(t)$ is a smooth cutoff supported on $[-1,2]$ with $\eta(t) \equiv 1$ on $[0,1]$.

Before stating the main results, we point out the following. Let $\phi$ be the identity operator on $L^2(\mathbb{T})$ or be as in (3). Then, we know that such $\phi$ is Hilbert-Schmidt from $L^2(\mathbb{T})$ into $H^s(\mathbb{T})$ if and only if $s < -\frac{1}{2}$. In other words, with a slight abuse of notation, define

$$\phi := \sum_{n\in\mathbb{Z}} \phi e_n = \sum_{n\in\mathbb{Z}} \phi_n e_n \tag{18}$$

in view of $\phi = \operatorname{diag}(\phi_n; n \neq 0)$. Then, we have $\phi \in H^s(\mathbb{T})$ if and only if $s < -\frac{1}{2}$. Moreover, we have $\|\phi\|_{HS(L^2;H^s)} = \|\phi\|_{H^s}$, where $\|\cdot\|_{HS(L^2;H^s)}$ denotes the Hilbert-Schmidt norm from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$. For such $\phi$, we also have $\phi \in \widehat{b}^s_{p,\infty}(\mathbb{T})$ if and only if $sp \leq -1$, and we can use $\|\phi\|_{\widehat{b}^s_{p,\infty}}$ to discuss the regularity of $\phi$ in place of the Hilbert-Schmidt norm. This is one of the reasons for using this space. (We need only $sp < -1$ for our purpose since the nonlinear estimate in Section 5 holds for $s = -\frac{1}{2}$ and $p = 2+$ with $sp < -1$.)

**Proposition 4.1.** *Let $0 < T \leq 1$ and $p = 2+$. Moreover, let $s = -\frac{1}{2} + \delta$ and $b = \frac{1}{2} - \delta$ with $\frac{p-2}{4p} < \delta < \frac{p-2}{2p}$. i.e. $sp < -1$ and $(b-1)\cdot 2 < -1$. Then, for the stochastic convolution $\Phi(t)$ defined in* (10) *with $\phi$ as in* (3)*, we have*

$$\mathbb{E}\big[\|\eta\Phi\|_{X^{s,b,T}_{p,2}}\big] \leq C(\eta, s, p) < \infty. \tag{19}$$

*In particular, $\Phi \in X^{-\frac{1}{2}+\delta, \frac{1}{2}-\delta, T}_{p,2}$ almost surely.*

Before going into the proof of Proposition 4.1, recall the following. Let $\beta_1$ and $\beta_2$ be independent real-valued Brownian motions on $(\Omega, \mathcal{F}, P)$, and $f_1(t,\omega)$ and $f_2(t,\omega)$ be real-valued stochastic processes independent of $\beta_1$ and $\beta_2$. Then, we can regard $\beta_j$ and $f_j$ as $\beta_j(t,\omega) = \beta_j(t,\omega_1)$ and $f_j(t,\omega) = f_j(t,\omega_2)$, where $\omega = (\omega_1,\omega_2) \in \Omega_1 \times \Omega_2 = \Omega$. Thus, in taking an expectation, we can first integrate over $\omega_1 \in \Omega_1$. Then, for $m \in \mathbb{N}$, we have

$$\begin{aligned}
&\mathbb{E}\bigg(\Big|\int_a^b f_1(t)d\beta_1(t) + \int_a^b f_2(t)d\beta_2(t)\Big|^{2m}\bigg)\\
&= \mathbb{E}\bigg(\sum_{k=0}^{2m}\binom{2m}{k}\Big(\int_a^b f_1(t)d\beta_1(t)\Big)^k\Big(\int_a^b f_2(t)d\beta_2(t)\Big)^{2m-k}\bigg)\\
&= \mathbb{E}_{\Omega_2}\bigg[\sum_{n=0}^{m}\binom{2m}{2n}\frac{(2n)!}{2^n n!}\|f_1(\cdot,\omega_2)\|^{2n}_{L^2(a,b)}\frac{(2(m-n))!}{2^{m-n}(m-n)!}\|f_2(\cdot,\omega_2)\|^{2(m-n)}_{L^2(a,b)}\bigg].
\end{aligned} \tag{20}$$

In the computation above, we used the fact that, for each fixed $\omega_2$, $\int_a^b f_j(t,\omega_2)d\beta_j(t,\omega_1)$ is a Gaussian random variable on $\Omega_1$ with variance $\|f_j(\cdot,\omega_2)\|^2_{L^2(a,b)}$.

*Proof.* By Hölder inequality, we have

$$\|\langle\tau - n^3\rangle^{\frac{1}{2}-\delta}\widehat{u}(n,\tau)\|_{L^2_\tau} \leq \|\langle\tau - n^3\rangle^{-2\delta}\|_{L_\tau^{\frac{2p}{p-2}}}\|\langle\tau - n^3\rangle^{\frac{1}{2}+\delta}\widehat{u}(n,\tau)\|_{L^p_\tau}.$$

i.e. We have $\|\eta\Phi\|_{X^{s,\frac{1}{2}-\delta}_{p,2}} \lesssim \|\eta\Phi\|_{X^{s,\frac{1}{2}+\delta}_{p,p}}$ as long as $\delta > \frac{p-2}{4p}$. Thus, we will work in $X^{s,\frac{1}{2}+\delta}_{p,p}$ in the following.

Let $g(t) = \eta(t)\int_0^t S(-r)\phi(r)dW(r)$. i.e. $\eta(t)\Phi(\cdot,t) = S(t)g(\cdot,t)$. Assume that each $\beta_n$ is extended to a Brownian motion on $\mathbb{R}$ in such a way that the family $\{\beta_n\}_{n\geq 0}$ is still independent. Note that for $t \in [0,T]$, we have

$$\widehat{g}(n,t) = \eta(t)\int_0^t \eta(r)e^{-irn^3}\phi_n(r)\chi_{[0,T]}(r)\tfrac{1}{\sqrt{2}}d\beta_n(r). \tag{21}$$

We have inserted $\eta(r)$ and $\chi_{[0,T]}(r)$ in the integrand since $\eta(r)\chi_{[0,T]}(r) \equiv 1$ for $r \in [0,t] \subset [0,T]$. For notational simplicity, we use $\phi_n(r)$ to denote $\phi_n(r)\chi_{[0,T]}(r)$ in the following. i.e. we assume that $\phi_n$ is supported on $[0,T]$. By (3), we have $|\phi_n(r)| \leq 1$ for $r \in \mathbb{R}$.

Now, we write the left hand side of (19) as

$$\begin{aligned}\mathbb{E}\Big(\|\eta\Phi\|_{X^{s,\frac{1}{2}+\delta,T}_{p,p}}\Big) \lesssim \mathbb{E}\bigg[\sup_j 2^{js}\Big(\sum_{|n|\sim 2^j}\sum_{k=1}^\infty 2^{kp(\frac{1}{2}+\delta)}\int_{|\tau|\sim 2^k}|\widehat{g}(n,\tau)|^p d\tau\Big)^{\frac{1}{p}}\bigg]\\ + \mathbb{E}\bigg[\sup_j 2^{js}\Big(\sum_{|n|\sim 2^j}\int_{|\tau|\leq 2}|\widehat{g}(n,\tau)|^p d\tau\Big)^{\frac{1}{p}}\bigg].\end{aligned} \tag{22}$$

• **Part 1:** First, we estimate the second term in (22). Let

$$G_n(r,\tau) = \eta(r)e^{-irn^3}\phi_n(r)\int_r^\infty \eta(t)e^{-it\tau}dt. \tag{23}$$

Also write $\beta_n = \beta_n^{(r)} + i\beta_n^{(i)}$ where $\beta_n^{(r)} = \operatorname{Re}\beta_n$ and $\beta_n^{(i)} = \operatorname{Im}\beta_n$. Then, by the stochastic Fubini Theorem, we have, for $m \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}\big[|\widehat{g}(n,\tau)|^{2m}\big] &= \mathbb{E}\Big(\Big|\int_{\mathbb{R}}\eta(t)e^{-it\tau}\int_{-\infty}^t \eta(r)e^{-irn^3}\phi_n(r)\tfrac{1}{\sqrt{2}}d\beta_n(r)dt\Big|^{2m}\Big)\\ &= 2^{-m}\mathbb{E}\Big(\Big|\int_{-1}^2 G_n(r,\tau)d\beta_n(r)\Big|^{2m}\Big)\\ &\lesssim \mathbb{E}\Big(\Big|\int_{-1}^2 \operatorname{Re}G_n(r,\tau)d\beta_n^{(r)}(r) - \int_{-1}^2 \operatorname{Im}G_n(r,\tau)d\beta_n^{(i)}(r)\Big|^{2m}\Big)\\ &\quad + \mathbb{E}\Big(\Big|\int_{-1}^2 \operatorname{Im}G_n(r,\tau)d\beta_n^{(r)}(r) + \int_{-1}^2 \operatorname{Re}G_n(r,\tau)d\beta_n^{(i)}(r)\Big|^{2m}\Big).\end{aligned} \tag{24}$$

Note that $|\operatorname{Re}G_n(r,\tau)|$, $|\operatorname{Im}G_n(r,\tau)| \leq |G_n(r,\tau)| \leq \|\eta\|_{L^1}|\phi_n(r)| \lesssim \|\eta\|_{L^1}\chi_{[0,T]}(r)$. Thus, we have $\|\operatorname{Re}G_n(r,\tau)\|^{2k}_{L^2_r}\|\operatorname{Im}G_n(r,\tau)\|^{2(m-k)}_{L^2_r} \lesssim \|\eta\|^{2m}_{L^1}$ for $k = 0, \cdots, m$. Then, by (20) along with the independence of $\phi_n$, $\beta_n^{(r)}$ and $\beta_n^{(i)}$, we have

$$\|\widehat{g}(n,\tau)\|_{L^{2m}(\Omega)} \leq C = C(\eta, m)$$

independent of $n$ and $\tau$. Hence, for $p \in (2,4)$, we have

$$\big(\mathbb{E}\big[|\widehat{g}(n,\tau)|^p\big]\big)^{\frac{1}{p}} \leq \|\widehat{g}(n,\tau)\|^\theta_{L^2(\Omega)}\|\widehat{g}(n,\tau)\|^{1-\theta}_{L^4(\Omega)} \lesssim 1, \tag{25}$$

by interpolation, where $\theta \in (0,1)$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{4}$. Then, the second term in (22) is estimated by

$$\begin{aligned}(22) &\leq \Big(\sum_{j=0}^\infty 2^{jsp}\sum_{|n|\sim 2^j}\int_{|\tau|\leq 2}\mathbb{E}\big[|\widehat{g}(n,\tau)|^p\big]d\tau\Big)^{\frac{1}{p}} \lesssim \Big(\sum_{j=0}^\infty 2^{jsp}\sum_{|n|\sim 2^j} 1\Big)^{\frac{1}{p}}\\ &\sim \Big(\sum_{j=0}^\infty 2^{(sp+1)j}\Big)^{\frac{1}{p}} \leq C < \infty,\end{aligned} \tag{26}$$

since $sp < -1$.

• **Part 2:** Next, we estimate the first term in (22). Let

$$\begin{cases} G_n^{(1)}(r,\tau) = \eta(r)e^{-irn^3}\phi_n(r)\int_r^\infty \eta'(t)\frac{e^{-it\tau}}{i\tau}dt, \\ G_n^{(2)}(r,\tau) = \eta^2(r)e^{-irn^3}\phi_n(r)\frac{e^{-ir\tau}}{i\tau}. \end{cases} \tag{27}$$

Then, by the stochastic Fubini theorem and integration by parts, we have

$$\begin{aligned} \sqrt{2}\widehat{g}(n,\tau) &= \int_{-1}^2 G_n(r,\tau)d\beta_n(r) = \int_{-1}^2 G_n^{(1)}(r,\tau)d\beta_n(r) + \int_{-1}^2 G_n^{(2)}(r,\tau)d\beta_n(r) \\ &=: I_n^{(1)}(\tau) + I_n^{(2)}(\tau). \end{aligned} \tag{28}$$

Thus, we have $|\widehat{g}(n,\tau)|^p \lesssim \big|I_n^{(1)}(\tau)\big|^p + \big|I_n^{(2)}(\tau)\big|^p$.

First, we estimate the contribution from $G_n^{(1)}$. For $|\tau| \sim 2^k$, we have

$$\bigg|\int_r^\infty \eta'(t)\frac{e^{-it\tau}}{i\tau}dt\bigg| \le |\tau^{-2}\eta'(r)| + \bigg|\int_r^\infty \eta''(t)\frac{e^{-it\tau}}{\tau^2}dt\bigg| \le C_\eta 2^{-2k}, \tag{29}$$

by partial integration. Thus, we have $|G_n^{(1)}(r,\tau)| \lesssim 2^{-2k}$. Then, repeating a similar computation as in Part 1, we obtain

$$\big(\mathbb{E}\big[|I_n^{(1)}(\tau)|^p\big]\big)^{\frac{1}{p}} \le \|I_n^{(1)}(\tau)\|_{L^2(\Omega)}^{\theta}\|I_n^{(1)}(\tau)\|_{L^4(\Omega)}^{1-\theta} \lesssim 2^{-2k}, \tag{30}$$

by (20) and interpolation. Hence, the contribution to (22) is estimated by

$$\begin{aligned} (22) &\le \bigg(\sum_{j=0}^\infty 2^{jsp}\sum_{|n|\sim 2^j}\sum_{k=1}^\infty 2^{kp(\frac{1}{2}+\delta)}\int_{|\tau|\sim 2^k}\mathbb{E}\big[|I_n^{(1)}(\tau)|^p\big]d\tau\bigg)^{\frac{1}{p}} \\ &\lesssim \bigg(\sum_{j=0}^\infty 2^{j(sp+1)}\sum_{k=1}^\infty 2^{k(-\frac{3p}{2}+\delta p+1)}\bigg)^{\frac{1}{p}} \le C < \infty, \end{aligned} \tag{31}$$

since $sp < -1$ and $-\frac{3p}{2} + \delta p + 1 < 0$.

Now, we consider the contribution from $I_n^{(2)}(\tau)$. With $\beta_n = \beta_n^{(r)} + i\beta_n^{(i)}$, we have $|I_n^{(2)}(\tau)|^2 \lesssim \Big|\int_{-1}^2 G_n^{(2)}(r,\tau)d\beta_n^{(r)}(r)\Big|^2 + \Big|\int_{-1}^2 G_n^{(2)}(r,\tau)d\beta_n^{(i)}(r)\Big|^2$. We only estimate the first term since the second term is estimated in the same way. By Ito formula (c.f. [8]), we have

$$\begin{aligned} \bigg|\int_{-1}^2 G_n^{(2)}(r,\tau)d\beta_n^{(r)}(r)\bigg|^2 &= \int_{-1}^2 \eta^4(t)\frac{|\phi_n(t)|^2}{\tau^2}dt \\ &\quad + 2\mathrm{Re}\int_{-1}^2\int_{-\infty}^t G_n^{(2)}(r,\tau)d\beta_n^{(r)}(r)\overline{G_n^{(2)}}(t,\tau)d\beta_n^{(r)}(t) =: I_n'(\tau) + I_n''(\tau). \end{aligned}$$

The contribution from $I_n'(\tau)$ is at most

$$\begin{aligned} (22) &\lesssim \bigg(\sum_{j=0}^\infty 2^{jsp}\sum_{|n|\sim 2^j}\sum_{k=1}^\infty 2^{kp(\frac{1}{2}+\delta)}\int_{|\tau|\sim 2^k}|\tau|^{-p}d\tau\Big(\int_{-1}^2\eta^4(t)dt\Big)^{\frac{p}{2}}\bigg)^{\frac{1}{p}} \\ &\lesssim \|\eta\|_{L^4}^2\bigg(\sum_{j=0}^\infty 2^{j(sp+1)}\sum_{k=1}^\infty 2^{k(-\frac{p}{2}+\delta p+1)}\bigg)^{\frac{1}{p}} \le C < \infty, \end{aligned} \tag{32}$$

since $sp < -1$ and $\delta < \frac{p-2}{2p}$.

We finally estimate the contribution from $I''_n(\tau)$. Write $I''_n(\tau) = \int_{-1}^2 H_n(t) d\beta_n^{(r)}(t)$, where $H_n(t) = \int_{-\infty}^t \widetilde{H}_n(r,t) d\beta_n^{(r)}(r)$ with

$$\widetilde{H}_n(r,t) = 2\tau^{-2}\mathrm{Re}\big(\eta^2(r)\eta^2(t)e^{i(t-r)n^3}\phi_n(r)\overline{\phi_n(t)}e^{i(t-r)\tau}\big). \tag{33}$$

Then, by Ito isometry and $|\phi_n(w,t)| \leq 1$ for all $(\omega,t) \in \Omega \times \mathbb{R}$, we have

$$\begin{aligned} \mathbb{E}\big[|I''_n(\tau)|^2\big] &= \mathbb{E}\Big[\Big(\int_{-1}^2 H_n(t)d\beta_n^{(r)}(t)\Big)^2\Big] \sim \int_{-1}^2 \mathbb{E}\big[H_n^2(t)\big]dt \\ &= \int_{-1}^2 \mathbb{E}\Big[\Big(\int_{-\infty}^t \widetilde{H}_n(r,t)d\beta_n^{(r)}(r)\Big)^2\Big]dt = \int_{-1}^2\int_{-1}^t \mathbb{E}\big[|\widetilde{H}_n(r,t)|^2\big]drdt \\ &\lesssim \tau^{-4}\int_{-1}^2\int_{-1}^t \eta^4(r)\eta^4(t)drdt \lesssim \tau^{-4}. \end{aligned} \tag{34}$$

Hence, the contribution from $I''_n(\tau)$ is at most

$$\begin{aligned} (22) &\lesssim \Big(\sum_{j=0}^\infty 2^{jsp}\sum_{|n|\sim 2^j}\sum_{k=1}^\infty 2^{kp(\frac{1}{2}+\delta)}\int_{|\tau|\sim 2^k}\mathbb{E}\big[|I''_n(\tau)|^{\frac{p}{2}}\big]d\tau\Big)^{\frac{1}{p}} \\ &\lesssim \Big(\sum_{j=0}^\infty 2^{jsp}\sum_{|n|\sim 2^j}\sum_{k=1}^\infty 2^{kp(\frac{1}{2}+\delta)}\int_{|\tau|\sim 2^k}\big(\mathbb{E}\big[|I''_n(\tau)|^2\big]\big)^{\frac{p}{4}}d\tau\Big)^{\frac{1}{p}} \\ &\lesssim \Big(\sum_{j=0}^\infty 2^{j(sp+1)}\sum_{k=1}^\infty 2^{k(-\frac{p}{2}+\delta p+1)}\Big)^{\frac{1}{p}} \leq C < \infty, \end{aligned} \tag{35}$$

for $p \leq 4$, $sp < -1$, and $\delta < \frac{p-2}{2p}$. □

We state a corollary to the proof of Proposition 4.1 for a general diagonal covariance operator $\phi(t,\omega) = \mathrm{diag}(\phi_n(t,\omega); n \in \mathbb{Z})$, which is independent of $\{\beta_n\}_{n\geq 1}$.

**Corollary 4.2.** *Let $0 < T \leq 1$, $p = 2+$, and $s, s' \in \mathbb{R}$ with $s < s'$. Moreover, let $b = \frac{1}{2} - \delta$ with $\frac{p-2}{4p} < \delta < \frac{p-2}{2p}$. i.e. $(b-1)\cdot 2 < -1$. Then, for the stochastic convolution $\Phi(t)$ defined in (10) with $\phi \in L^p([0,T]\times\Omega;\widehat{b}^{s'}_{p,\infty})$, independent of $\{\beta_n\}_{n\geq 1}$, we have*

$$\mathbb{E}\big[\|\eta\Phi\|_{X^{s,b,T}_{p,2}}\big] \leq C(\eta,s,s',p)\|\phi\|_{L^p([0,T]\times\Omega;\widehat{b}^{s'}_{p,\infty})}. \tag{36}$$

*In particular, $\Phi \in X^{s,\frac{1}{2}-\delta,T}_{p,2}$ almost surely.*

*Proof.* In the proof of Proposition 4.1, we used $|\phi_n(t)| \leq 1$ whenever $\phi_n(t)$ appeared. Now, we briefly go through the proof of Proposition 4.1, keeping track of $\phi_n(t)$. Since $\phi$ is independent of $\{\beta_n\}_{n\geq 1}$, we regard $\beta_n$ and $\phi_n$ as $\beta_n(t,\omega) = \beta_n(t,\omega_1)$ and $\phi_n(t,\omega) = \phi_n(t,\omega_2)$, where $\omega = (\omega_1,\omega_2) \in \Omega_1 \times \Omega_2 = \Omega$.

In (25), we have $\mathbb{E}\big[|\widehat{g}(n,\tau)|^p\big] \lesssim \mathbb{E}_{\Omega_2}\|\phi_n(\cdot,\omega_2)\|^p_{L^2[0,T]}$. Then, in (26), we have

$$\begin{aligned}(22) &\leq \bigg(\sum_{j=0}^\infty 2^{jsp}\sum_{|n|\sim 2^j}\int_{|\tau|\leq 2}\mathbb{E}_{\Omega_2}\|\phi_n(\cdot,\omega_2)\|^p_{L^2[0,T]}d\tau\bigg)^{\frac{1}{p}}\\ &\leq \bigg(\sum_{j=0}^\infty 2^{j(s-s')p}2^{js'p}\sum_{|n|\sim 2^j}\|\phi_n(\cdot,\omega_2)\|^p_{L^p([0,T]\times\Omega_2)}\bigg)^{\frac{1}{p}}\\ &\lesssim \|\phi\|_{L^p([0,T]\times\Omega;\widehat{b}^{s'}_{p,\infty})}\end{aligned}$$

since $s-s'<0$. A similar modification in (30) and (31) (and (32)) takes care of the contribution from $I_n^{(1)}(\tau)$ (and $I_n'(\tau)$, respectively.) Now, as for $I_n''(\tau)$, we first integrate only over $\Omega_1$ in (34) and obtain

$$\mathbb{E}_{\Omega_1}\big[|I_n''(\tau)|^2\big] \lesssim \tau^{-4}\int_{-1}^2\int_{-1}^t \eta^4(r)\eta^4(t)|\phi_n(r)|^2|\phi_n(t)|^2drdt \lesssim \tau^{-4}\|\phi_n\|^4_{L^2[0,T]}.$$

Then, in (35), we have

$$\mathbb{E}\big[|I_n''(\tau)|^{\frac{p}{2}}\big] = \mathbb{E}_{\Omega_2}\big[\|I_n''(\tau)\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(\Omega_1)}\big] \leq \mathbb{E}_{\Omega_2}\big[\|I_n''(\tau)\|^{\frac{p}{2}}_{L^2(\Omega_1)}\big] \lesssim \tau^{-p}\,\mathbb{E}_{\Omega_2}\|\phi_n(\cdot,\omega_2)\|^p_{L^2[0,T]}$$

for $p\in[2,4]$. The rest follows as before. □

Now, we discuss the continuity of the stochastic convolution. In the remaining of this section, we show that the stochastic convolution $\Phi(t)$ defined in (10) belongs to $C([0,T];\widehat{b}^s_{p,\infty}(\mathbb{T}))$ almost surely. With $\beta_n = \beta_n^{(r)} + i\beta_n^{(i)}$, we have

$$\Phi(t) = \frac{1}{\sqrt{2}}\sum_{n\neq 0}\int_0^t S(t-r)\phi_n(r)e_n d\beta_n^{(r)}(r) + i\frac{1}{\sqrt{2}}\sum_{n\neq 0}\int_0^t S(t-r)\phi_n(r)e_n d\beta_n^{(i)}(r), \tag{37}$$

since $\phi e_0 = 0$ and $\phi e_n = \phi_n e_n$, $n\neq 0$. In the following, we only show the continuity of the first stochastic convolution in (37), which we shall denote by $\Phi^{(r)}(t)$. Also, let $W^{(r)}(t) = \frac{1}{\sqrt{2}}\sum_n \beta_n^{(r)}(t)e_n$. As in Da Prato [7], we use the factorization method based on the elementary identity

$$\int_r^t (t-t')^{\alpha-1}(t'-r)^{-\alpha}dt' = \frac{\pi}{\sin\pi\alpha}, \tag{38}$$

with $\alpha\in(0,1)$ for $0\leq r\leq t'\leq t$. Using (38), we can write the first term in (37) as

$$\Phi^{(r)}(t) = \frac{\sin\pi\alpha}{\pi}\int_0^t S(t-t')(t-t')^{\alpha-1}Y(t')dt', \tag{39}$$

where

$$Y(t') = \int_0^{t'} S(t'-r)(t'-r)^{-\alpha}\phi(r)dW^{(r)}(r). \tag{40}$$

First, we present the following lemma which provides a criterion for the continuity of (39) in terms of the $L^{2m}$-integrability of $Y(t')$.

**Lemma 4.3** (Lemma 2.7 in [7]). *Let $T > 0$, $\alpha \in (0,1)$, and $m > \frac{1}{2\alpha}$. For $f \in L^{2m}([0,T]; \widehat{b}^s_{p,\infty}(\mathbb{T}))$, let*

$$F(t) = \int_0^t S(t-t')(t-t')^{\alpha-1} f(t')dt', \quad 0 \le t \le T.$$

*Then, $F \in C([0,T]; \widehat{b}^s_{p,\infty}(\mathbb{T}))$. Moreover, there exists $C = C(m,T)$ such that*

$$\|F(t)\|_{\widehat{b}^s_{p,\infty}} \le C\|f\|_{L^{2m}([0,T];\widehat{b}^s_{p,\infty})}, \quad 0 \le t \le T.$$

**Remark 4.4.** Although Lemma 2.7 in [7] is stated for a Hilbert space $H$, its proof makes no use of the Hilbert space structure of $H$. Thus the same result holds for $\widehat{b}^s_{p,\infty}(\mathbb{T})$ as well.

In view of Lemma 4.3, it suffices to show that $Y(t') \in L^{2m}([0,T]; \widehat{b}^s_{p,\infty}(\mathbb{T}))$ a.s.

**Proposition 4.5.** *Let $T > 0$, $m \ge 2$, $s = -\frac{1}{2}+$, and $p = 2+$ such that $sp < -1$. Let $\phi$ be as in (3). Then, the stochastic convolution $\Phi^{(r)}(t)$ is continuous from $[0,T]$ into $\widehat{b}^s_{p,\infty}$ almost surely. Moreover, there exists*

$$\mathbb{E}\Big( \sup_{t\in[0,T]} \|\Phi^{(r)}(t)\|^{2m}_{\widehat{b}^s_{p,\infty}} \Big) \le C(m,T,s,p) < \infty.$$

*Proof.* Let $\alpha \in \big(\frac{1}{2m}, \frac{1}{2}\big)$ and $Y$ be as in (40). First, note that $Y$ is real-valued since $\phi_{-n}(s)e_{-n} = \overline{\phi_n(s)e_n}$ and $\beta^{(r)}_{-n} = \beta^{(r)}_n$. Note that $\{\beta^{(r)}_n\}_{n\neq 0}$ and $\phi$ are independent since $\phi$ depends only on $\beta_0$. Thus, we can regard $\beta^{(r)}_n$ and $\phi$ as $\beta^{(r)}_n(\omega) = \beta^{(r)}_n(\omega_1)$ and $\phi(\omega) = \phi(\omega_2)$, where $\omega = (\omega_1,\omega_2) \in \Omega_1 \times \Omega_2 = \Omega$. Then, for each fixed $\omega_2$ and $t' \in [0,t]$, $\widehat{Y(t')}(n)$ is a Gaussian random variable on $\Omega_1$ with $\text{Var}_{\Omega_1}\big(\widehat{Y(t')}(n)\big) = \mathbb{E}_{\Omega_1}\big[|\widehat{Y(t')}(n)|^2\big]$.

Let $G_n(r,\omega_2) = (t'-r)^{-\alpha} e^{i(t'-r)n^3}\phi_n(r,\omega_2)$. Note that $|G_n(r,\omega_2)| = (t'-r)^{-\alpha}$ for $0 < r < t'$ and $n \neq 0$. By Ito isometry, we have

$$\begin{aligned}
\mathbb{E}_{\Omega_1}\big[|\widehat{Y(t')}(n)|^2\big] &= \frac{1}{2}\,\mathbb{E}_{\Omega_1}\Big[\Big|\int_0^{t'} G_n(r,\omega_2)d\beta(r,\omega_1)\Big|^2\Big] \\
&= \frac{1}{2}\int_0^{t'} |G_n(r,\omega_2)|^2 dr \sim \int_0^{t'} (t'-r)^{-2\alpha}dr.
\end{aligned}$$

Then, by Minkowski integral inequality (with $p = 2+ < 2m$) after replacing $\sup_j$ by $\sum_j$, we have

$$\begin{aligned}
\mathbb{E}_{\Omega_1}\big(\|Y(t',\cdot,\omega_2)\|^{2m}_{\widehat{b}^s_{p,\infty}}\big) &= \mathbb{E}_{\Omega_1}\Big[\Big(\sup_j \sum_{|n|\sim 2^j} \langle n\rangle^{sp}|\widehat{Y(t')}(n)|^p\Big)^{\frac{2m}{p}}\Big] \\
&\lesssim \Big(\sum_{j=0}^\infty \sum_{|n|\sim 2^j} 2^{jsp}\Big(\mathbb{E}_{\Omega_1}\big[|\widehat{Y(t')}(n)|^{2m}\big]\Big)^{\frac{p}{2m}}\Big)^{\frac{2m}{p}} \\
&\sim \Big(\sum_{j=0}^\infty 2^{j(sp+1)}\Big)^{\frac{2m}{p}} \Big(\int_0^{t'} (t'-r)^{-2\alpha}dr\Big)^m \lesssim \Big(\frac{(t')^{1-2\alpha}}{1-2\alpha}\Big)^m,
\end{aligned}$$

since $sp < -1$. Therefore, we have

$$\int_0^T \mathbb{E}\big(\|Y(t')\|^{2m}_{\widehat{b}^s_{p,\infty}}\big)dt' = \int_0^T \mathbb{E}_{\Omega_2}\mathbb{E}_{\Omega_1}\big(\|Y(t')\|^{2m}_{\widehat{b}^s_{p,\infty}}\big)dt'$$
$$\lesssim \int_0^T \left(\frac{(t')^{1-2\alpha}}{1-2\alpha}\right)^m dt' \lesssim T^{(1-2\alpha)m+1} < C(m,T,s,p) < \infty.$$

In particular, it follows that $Y(\cdot,\omega) \in L^{2m}([0,T];\widehat{b}^s_{p,\infty})$ almost surely. Then, the desired result follows from Lemma 4.3. □

## 5. Nonlinear Estimate on the Second Iteration

Now, we present the crucial nonlinear analysis. First, we briefly go over Bourgain's argument in [2]. By writing the integral equation, the deterministic KdV (5) is equivalent to

$$u(t) = S(t)u_0 - \tfrac{1}{2}\mathcal{N}(u,u)(t), \tag{41}$$

where $\mathcal{N}(\cdot,\cdot)$ is given by

$$\mathcal{N}(u_1,u_2)(t) := \int_0^t S(t-t')\partial_x(u_1u_2)(t')dt'. \tag{42}$$

In the following, we assume that the initial condition $u_0$ has the mean 0, which implies that $u(t)$ has the spatial mean 0 for each $t \in \mathbb{R}$. We use $(n,\tau)$, $(n_1,\tau_1)$, and $(n_2,\tau_2)$ to denote the Fourier variables for $uu$, the first factor, and the second factor $u$ of $uu$ in $\mathcal{N}(u,u)$, respectively. i.e. we have $n = n_1+n_2$ and $\tau = \tau_1+\tau_2$. By the mean 0 assumption on $u$ and by the fact that we have $\partial_x(uu)$ in the definition of $\mathcal{N}(u,u)$, we assume $n, n_1, n_2 \neq 0$. We also use the following notation:

$$\sigma_0 := \langle \tau - n^3\rangle \text{ and } \sigma_j := \langle \tau_j - n_j^3\rangle.$$

One of the main ingredients is the observation due to Bourgain [1]:

$$n^3 - n_1^3 - n_2^3 = 3nn_1n_2, \text{ for } n = n_1+n_2, \tag{43}$$

which in turn implies that

$$\text{MAX} := \max(\sigma_0,\sigma_1,\sigma_2) \gtrsim \langle nn_1n_2\rangle. \tag{44}$$

Now, define

$$A_j = \{(n,n_1,n_2,\tau,\tau_1,\tau_2) \in \mathbb{Z}^3\times\mathbb{R}^3 : \sigma_j = \text{MAX}\}, \tag{45}$$

and let $\mathcal{N}_j(u,u)$ denote the contribution of $\mathcal{N}(u,u)$ on $A_j$. By the standard bilinear estimate as in [1], [11], we have

$$\|\mathcal{N}_0(u,u)\|_{-\frac{1}{2}+\delta,\frac{1}{2}-\delta} \leq o(1)\|u\|^2_{-\frac{1}{2}-\delta,\frac{1}{2}-\delta}, \tag{46}$$

where $o(1) = T^\theta$ with some $\theta > 0$ by considering the estimate on a short time interval $[-T,T]$ (e.g. Lemma 3.4). See (2.17), (2.26), and (2.68) in [2]. Here, we abuse the notation and use $\|\cdot\|_{s,b} = \|\cdot\|_{X^{s,b}}$ to denote the local-in-time version as well. Note that the temporal regularity $b = \frac{1}{2}-\delta < \frac{1}{2}$. This allowed us to gain the spatial regularity by $2\delta$. Clearly, we can not expect to do the same for $\mathcal{N}_1(u,u)$. (By symmetry, we do not consider $\mathcal{N}_2(u,u)$ in the following.) The bilinear estimate (7) is known to fail for any $s\in\mathbb{R}$ if $b<\frac{1}{2}$ due to the contribution from $\mathcal{N}_1(u,u)$. See [11]. Following the notation in [2], let

$$I_{s,b} = \|\mathcal{N}_1(u,u)\|_{X^{s,b}} \text{ and } \alpha := \frac{1}{2}-\delta < \frac{1}{2}. \tag{47}$$

Then, by Lemma 3.2 and duality with $\|d(n,\tau)\|_{L^2_{n,\tau}} \leq 1$, we have

$$I_{-\alpha,1-\alpha} = \|\mathcal{N}_1(u,u)\|_{-\alpha,1-\alpha} \lesssim \sum_{\substack{n,n_1\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{\langle n\rangle^{1-\alpha} d(n,\tau)}{\sigma_0^\alpha} \widehat{u}(n_1,\tau_1) \frac{\langle n_2\rangle^{1-\alpha} c(n_2,\tau_2)}{\sigma_2^\alpha}, \tag{48}$$

where

$$c(n_2,\tau_2) = \langle n_2\rangle^{-(1-\alpha)} \sigma_2^\alpha\, \widehat{u}(n_2,\tau_2) \text{ so that } \|c\|_{L^2_{n,\tau}} = \|u\|_{-(1-\alpha),\alpha} = \|u\|_{-\frac{1}{2}-\delta,\frac{1}{2}-\delta}. \tag{49}$$

The main idea here is to consider the second iteration, i.e. substitute (41) for $\widehat{u}(n_1,\tau_1)$ in (48), thus leading to a trilinear expression. Since $\sigma_1 = \text{MAX} \gtrsim \langle n n_1 n_2\rangle \gg 1$ on $A_1$, we can assume that

$$\widehat{u}(n_1,\tau_1) = \big(\mathcal{N}(u,u)\big)^\wedge(n_1,\tau_1) \sim \frac{|n_1|}{\sigma_1} \sum_{n_1=n_3+n_4} \int_{\tau_1=\tau_3+\tau_4} \widehat{u}(n_3,\tau_3)\widehat{u}(n_4,\tau_4) d\tau_4. \tag{50}$$

Note that $\widehat{u}(n_1,\tau_1)$ can not come from $S(t)u_0$ of (41) since we have $\sigma_1 \sim 1$ for the linear part. Moreover, by the standard computation [1], we have

$$\begin{aligned}\mathcal{N}(u,u)(x,t) = & -i \sum_{k=1}^\infty \frac{i^k t^k}{k!} \sum_{n\neq 0} e^{i(nx+n^3t)} \int \eta(\lambda - n^3) \widehat{\partial_x u^2}(n,\lambda) d\lambda \\ & + i \sum_{n\neq 0} e^{inx} \int \frac{\big(1-\eta\big)(\tau-n^3)}{\tau - n^3} \widehat{\partial_x u^2}(n,\tau) e^{i\tau t} d\tau \\ & + i \sum_{n\neq 0} e^{i(nx+n^3t)} \int \frac{\big(1-\eta\big)(\lambda-n^3)}{\lambda - n^3} \widehat{\partial_x u^2}(n,\lambda) d\lambda \\ =: & \,\mathcal{M}_1(u,u)(x,t) + \mathcal{M}_2(u,u)(x,t) + \mathcal{M}_3(u,u)(x,t).\end{aligned} \tag{51}$$

Note that $(\mathcal{M}_1(u,u))^\wedge(n_1,\tau_1)$ and $(\mathcal{M}_3(u,u))^\wedge(n_1,\tau_1)$ are distributions supported on $\{\tau_1 - n_1^3 = 0\}$. i.e. $\sigma_1 \sim 1$. Hence, the only contribution for the second iteration on $A_1$ comes from $\mathcal{M}_2(u,u)$ whose Fourier transform is given in (50). This shows the validity of the assumption (50).

Note that the $\sigma_1$ appearing in the denominator allows us to cancel $\langle n\rangle^{1-\alpha}$ and $\langle n_2\rangle^{1-\alpha}$ in the numerator in (48). Then, $I_{-\alpha,1-\alpha}$ can be estimated by

$$\lesssim \sum_{\substack{n=n_1+n_2\\ n_1=n_3+n_4}} \int_{\substack{\tau=\tau_1+\tau_2\\ \tau_1=\tau_3+\tau_4}} \frac{\langle n\rangle^{1-\alpha} d(n,\tau)}{\sigma_0^\alpha} \frac{|n_1|}{\sigma_1} \widehat{u}(n_3,\tau_3)\widehat{u}(n_4,\tau_4) \frac{\langle n_2\rangle^{1-\alpha} c(n_2,\tau_2)}{\sigma_2^\alpha}. \tag{52}$$

Then, Bourgain divided the argument into several cases, depending on the sizes of $\sigma_0, \cdots, \sigma_4$. Here, the key algebraic relation is

$$n^3 - n_2^3 - n_3^3 - n_4^3 = 3(n_2+n_3)(n_3+n_4)(n_4+n_2), \quad \text{with } n = n_2+n_3+n_4. \tag{53}$$

Then, Bourgain proved -see (2.69) in [2]-

$$I_{-\alpha,1-\alpha} \leq o(1)\|u\|_{-(1-\alpha),\alpha} I_{-\alpha,1-\alpha} + o(1)\|u\|^3_{-(1-\alpha),\alpha} + o(1)\|u\|_{-(1-\alpha),\alpha}, \tag{54}$$

*assuming* the a priori estimate (8): $|\widehat{u}(n,t)| < C$ for all $n \in \mathbb{Z}$, $t \in \mathbb{R}$. Indeed, the estimates involving the first two terms on the right hand side of (54) were obtained without (8), and *only* the last term in (54) required (8), -see "Estimation of (2.62)" in [2]-, which was then used to deduce

$$\|\widehat{u}(n,\cdot)\|_{L^2_\tau} < C. \tag{55}$$

The a priori estimate (8) is derived via the isospectral property of the KdV flow and is false for a general function in $X^{-(1-\alpha),\alpha}$. (It is here that the smallness of the total variation $\|\mu\|$ is used.)

Our goal is to carry out a similar analysis for SKdV (2) on the second iteration *without* the a priori estimates (8) and (55) coming from the complete integrability of KdV. We achieve this goal by considering the estimate in $X^{-\alpha,\alpha}_{p,2} = X^{-\frac{1}{2}+\delta,\frac{1}{2}-\delta}_{p,2}$, where $p = 2+$ and $\frac{p-2}{4p} < \delta < \frac{p-2}{2p}$. By (15) and (17) (recall $-\alpha = -\frac{1}{2}+\delta$ and $-(1-\alpha) = -\frac{1}{2}-\delta$), we have

$$\|u\|_{X^{-\alpha,\alpha}_{p,2}} \leq \|u\|_{X^{-\alpha,\alpha}}, \text{ and } \|u\|_{X^{-(1-\alpha),\alpha}} \lesssim \|u\|_{X^{-\alpha,\alpha}_{p,2}}. \tag{56}$$

Then, it follows from (46) and (56) that

$$\|\mathcal{N}_0(u,u)\|_{X^{-\alpha,\alpha}_{p,2}} \leq o(1)\|u\|^2_{X^{-\alpha,\alpha}_{p,2}}. \tag{57}$$

Now, we consider the estimate on $\|\mathcal{N}_1(u,u)\|_{X^{-\alpha,\alpha}_{p,2}}$. From (56) and $\alpha < 1-\alpha$, it suffices to control $I_{-\alpha,1-\alpha}$. As in the deterministic case, we consider the second iteration, and substitute (4) for $\widehat{u}(n_1,\tau_1)$ in (48). As before, there is no contribution from $S(t)u_0$, or $\mathcal{M}_1(u,u)$, $\mathcal{M}_3(u,u)$ defined in (51). Now, there are two contributions:

(i) $\mathcal{N}_1(\mathcal{M}_2(u,u),u)$ from the deterministic nonlinear part: In this case, we can use the estimates from [2] *except* when the a priori bound (8) was assumed. i.e. we need to estimate the contribution from (2.62) in [2]:

$$R_\alpha := \sum_n \int_{\tau=\tau_2+\tau_3+\tau_4} \chi_B \frac{d(n,\tau)}{\langle n\rangle^{1+\alpha}\sigma_0^\alpha} \widehat{u}(-n,\tau_2)\widehat{u}(n,\tau_3)\widehat{u}(n,\tau_4)d\tau_2 d\tau_3 d\tau_4, \tag{58}$$

where $\|d(n,\tau)\|_{L^2_{n,\tau}} \leq 1$ and $B = \{\sigma_0,\sigma_2,\sigma_3,\sigma_4 < |n|^\gamma\}$ with some small parameter $\gamma > 0$. Note that this corresponds to the case $n_2 = -n$ and $n_3 = n_4 = n$ in (52) after some reduction. In our analysis, we directly estimate $R_\alpha$ in terms of $\|u\|_{X^{-\alpha,\alpha}_{p,2}}$. The key observation is that we can take the spatial regularity $s = -\alpha$ to be greater than $-\frac{1}{2}$ by choosing $p > 2$.

(ii) $\mathcal{N}_1(\Phi,u)$ from the stochastic convolution $\Phi$ in (10): In view of (56), we estimate

$$\mathbb{E}\big[\|\mathcal{N}_1(\eta\Phi,u)\|_{X^{-\alpha,1-\alpha}}\big] \tag{59}$$

via the stochastic analysis from Section 4.

**Remark 5.1.** In fact, we do not need to take an expectation in (59) since we establish local well-posedness pathwise in $\omega$, i.e. for almost every *fixed* $\omega$. Nonetheless, we estimate (59) with the expectation since it shows how $F_1^N$ and $F_2^N$ defined in (71) arise along with their estimates.

• **Estimate on (i):** In [2], the parameter $\gamma = \gamma(\alpha)$, subject to the conditions (2.43) and (2.60) in [2], played a certain role in estimating $R_\alpha$ along with the a priori bound (8). However, it plays no role in our analysis. By Cauchy-Schwarz and Young's inequalities, we have

$$(58) \leq \sum_n \|d(n,\cdot)\|_{L^2_\tau}\langle n\rangle^{-1-\alpha}\|\widehat{u}(-n,\tau_2)\|_{L^{\frac{6}{5}}_{\tau_2}}\|\widehat{u}(n,\tau_3)\|_{L^{\frac{6}{5}}_{\tau_3}}\|\widehat{u}(n,\tau_4)\|_{L^{\frac{6}{5}}_{\tau_4}}$$

By Hölder inequality (with appropriate $\pm$ signs) and the fact that $-1-\alpha < -3\alpha$,

$$\leq \sum_n \|d(n,\cdot)\|_{L^2_\tau} \prod_{j=2}^4 \langle n\rangle^{-\alpha-} \|\sigma_j^{-\alpha}\|_{L^3_{\tau_j}} \|\sigma_j^\alpha \widehat{u}(\pm n, \tau_j)\|_{L^2_{\tau_j}}$$
$$\leq \|d(\cdot,\cdot)\|_{L^2_{n,\tau}} \|u\|^3_{X^{-\alpha,\alpha}_{6,2}} \leq \|u\|^3_{X^{-\alpha,\alpha}_{p,2}},$$

where the last two inequalities follow by choosing $\alpha > \frac{1}{3}$ and $p = 2+ < 6$.

• **Estimate on (ii):** We use the notation from the proof of Proposition 4.1. It follows from (28) and $\eta(t)\Phi(\cdot,t) = S(t)g(\cdot,t)$ that

$$(\eta\Phi)^\wedge(n_1,\tau_1) = \widehat{g}(n_1,\tau_1 - n_1^3) = \tfrac{1}{\sqrt{2}} \mathrm{I}^{(1)}_{n_1}(\tau_1 - n_1^3) + \tfrac{1}{\sqrt{2}} \mathrm{I}^{(2)}_{n_1}(\tau_1 - n_1^3).$$

Recall that $\sigma_1 = \langle \tau_1 - n_1^3\rangle \gtrsim \langle n n_1 n_2\rangle$. Also, recall from the proof of Proposition 4.1 that $|\phi_{n_1}(r)| = \chi_{[0,T]}(r)$ is independent of $\omega$.

◦ Contribution from $\mathrm{I}^{(1)}_{n_1}(\tau_1 - n_1^3)$: From (48) with (27), (28), and (29), we estimate (59) by

$$\lesssim \mathbb{E}\Big[ \sum_{\substack{n,n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{\langle n\rangle^{1-\alpha} d(n,\tau)}{\sigma_0^\alpha} \frac{1}{\sigma_1^2} \int_0^T |\phi_{n_1}(r)| d\beta_{n_1}(r) \frac{\langle n_2\rangle^{1-\alpha} c(n_2,\tau_2)}{\sigma_2^\alpha}\Big] \tag{60}$$

By Cauchy-Schwarz inequality in $\omega$ and Ito isometry,

$$\lesssim \sum_{\substack{n,n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{d(n,\tau)}{\sigma_0^\alpha} \frac{\|\phi_{n_1}\|_{L^2[0,T]}}{\sigma_1^{\frac{3}{2}-\delta}\langle n_1\rangle^{\frac{1}{2}+\delta}} \frac{\|c(n_2,\tau_2)\|_{L^2(\Omega)}}{\sigma_2^\alpha} \tag{61}$$

By $L^4_{x,t}, L^2_{x,t}, L^4_{x,t}$-Hölder inequality along with Lemma 3.3, (16), (18), (49), and (56)

$$\lesssim T^\theta \|d\|_{L^2_{n,\tau}} \|\phi\|_{L^2([0,T];H^{-\frac{1}{2}-\delta})} \|c\|_{L^2(\Omega;L^2_{n,\tau})} \leq T^\theta \|\phi\|_{L^p([0,T];\widehat{b}^{-\alpha}_{p,\infty})} \|u\|_{L^2(\Omega;X^{-(1-\alpha),\alpha})}$$
$$\lesssim T^\theta \|\phi\|_{L^p([0,T];\widehat{b}^{-\alpha}_{p,\infty})} \|u\|_{L^2(\Omega;X^{-\alpha,\alpha}_{p,2})}.$$

**Remark 5.2.** Strictly speaking, we need to take the supremum over $\{\|d\|_{L^2_{n,\tau}} = 1\}$ *inside* the expectation in (60). However, we do not worry about this issue for simplicity of the presentation, since we have

$$(59) \leq \|\mathcal{N}_1(\eta\Phi,u)\|_{L^2(\Omega;X^{-\alpha,1-\alpha})}$$
$$\leq \bigg( \sum_n \int \frac{\langle n\rangle^{2-2\alpha}}{\sigma_0^{2\alpha}} \mathbb{E}\Big| \int_0^T |\phi_{n_1}(r)| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{\langle n_2\rangle^{1-\alpha} c(n_2,\tau_2)}{\sigma_1^2 \sigma_2^\alpha} d\tau_1 d\beta_{n_1}(r) \Big|^2 d\tau \bigg)^{\frac{1}{2}}$$
$$= \sup_{\|d\|_{L^2_{n,\tau}}=1} (61)$$

by Ito isometry. Also, recall that we have $\mathrm{I}^{(1)}_{n_1}(\tau_1 - n_1^3) = \int_0^T G^{(1)}_{n_1}(r,\tau_1 - n_1^3) d\beta_{n_1}(r)$ where $G^{(1)}_n(r,\tau)$ is defined in (27). Hence, strictly speaking, we should replace $G^{(1)}_{n_1}(r,\tau_1 - n_1^3)$ by $\sigma_1^{-2}|\phi_{n_1}(r)|$ in (60) only after the application of Ito isometry. Once again, we do not worry about this issue for simplicity of the presentation. The same remark applies in the following as well.

∘ Contribution from $\mathrm{I}^{(2)}_{n_1}(\tau_1 - n_1^3)$:

First, suppose that $\max(\sigma_0, \sigma_2) \gtrsim \langle nn_1n_2\rangle^{\frac{1}{100}}$. Say $\sigma_0 \geq \langle nn_1n_2\rangle^{\frac{1}{100}}$. Then, (59) is estimated by

$$\lesssim \mathbb{E}\Big[\sum_{\substack{n,n_1\\n=n_1+n_2}}\int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{\langle n\rangle^{1-\alpha}d(n,\tau)}{\sigma_0^\alpha}\frac{1}{\sigma_1}\int_0^T |\phi_{n_1}(r)|d\beta_{n_1}(r)\frac{\langle n_2\rangle^{1-\alpha}c(n_2,\tau_2)}{\sigma_2^\alpha}\Big]$$

$$\lesssim \sum_{\substack{n,n_1\\n=n_1+n_2}}\int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{d(n,\tau)}{\sigma_0^{\alpha-200\delta}}\frac{\|\phi_{n_1}\|_{L^2[0,T]}}{\sigma_1^{\frac{1}{2}+\delta}\langle n_1\rangle^{\frac{1}{2}+\delta}}\frac{\|c(n_2,\tau_2)\|_{L^2(\Omega)}}{\sigma_2^\alpha} \tag{62}$$

Then, we can conclude this case as before by $L^4_{x,t}, L^2_{x,t}, L^4_{x,t}$-Hölder inequality as long as $\alpha - 200\delta > \frac{1}{3}$, which can be guaranteed by taking $\delta > 0$ sufficiently small, or equivalently, taking $p > 2$ sufficiently close to 2.

Hence, assume $\max(\sigma_0, \sigma_2) \ll \langle nn_1n_2\rangle^{\frac{1}{100}}$. Recall the following lemma from [5, (7.50) and Lemma 7.4].

**Lemma 5.3.** *Let*

$$\Omega(n) = \{\eta \in \mathbb{R} : \eta = -3nn_1n_2 + o(\langle nn_1n_2\rangle^{\frac{1}{100}}) \text{ for some } n_1 \in \mathbb{Z} \text{ with } n = n_1 + n_2\}. \tag{63}$$

*Then, we have*

$$\int \langle \tau - n^3\rangle^{-\frac{3}{4}}\chi_{\Omega(n)}(\tau - n^3)d\tau \lesssim 1. \tag{64}$$

Note that (64) is stated with $\langle \tau - n^3\rangle^{-1}$ in [5]. However, by examining the proof of Lemma 7.4 in [5], one immediately sees that (64) is valid with $\langle \tau - n^3\rangle^{-\beta}$ for any $\beta > \frac{2}{3} + \frac{1}{100}$.

Then, (59) is estimated by

$$\lesssim \mathbb{E}\Big[\sum_{\substack{n,n_1\\n=n_1+n_2}}\int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{\langle n\rangle^{1-\alpha}d(n,\tau)}{\sigma_0^\alpha}\frac{\chi_{\Omega(n_1)}(\tau_1 - n_1^3)}{\sigma_1}\int_0^T |\phi_{n_1}(r)|d\beta_{n_1}(r)\frac{\langle n_2\rangle^{1-\alpha}c(n_2,\tau_2)}{\sigma_2^\alpha}\Big]$$

By Cauchy-Schwarz inequality and Ito isometry,

$$\lesssim \sum_{\substack{n,n_1\\n=n_1+n_2}}\int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{d(n,\tau)}{\sigma_0^{\alpha}}\frac{\chi_{\Omega(n_1)}(\tau_1 - n_1^3)\|\phi_{n_1}\|_{L^2[0,T]}}{\sigma_1^{\frac{1}{2}-\delta}\langle n_1\rangle^{\frac{1}{2}+\delta}}\frac{\|c(n_2,\tau_2)\|_{L^2(\Omega)}}{\sigma_2^\alpha} \tag{65}$$

By $L^4_{x,t}, L^2_{x,t}, L^4_{x,t}$-Hölder inequality along with Lemmata 3.3, 5.3, (16), (18), (49), and (56),

$$\lesssim T^\theta \|d\|_{L^2_{n,\tau}}\big\|\langle n_1\rangle^{-\frac{1}{2}-\delta}\|\phi_{n_1}\|_{L^2[0,T]}\|\chi_{\Omega(n_1)}(\tau_1 - n_1^3)\sigma_1^{-\frac{1}{2}+\delta}\|_{L^2_\tau}\big\|_{L^2_n}\|c\|_{L^2(\Omega;L^2_{n,\tau})}$$
$$\leq T^\theta\|\phi\|_{L^2([0,T];H^{-\frac{1}{2}-\delta})}\|u\|_{L^2(\Omega;X^{-(1-\alpha),\alpha})} \lesssim T^\theta\|\phi\|_{L^p([0,T];\widehat{b}^{-\alpha}_{p,\infty})}\|u\|_{L^2(\Omega;X^{-\alpha,\alpha}_{p,2})}.$$

Now, we are ready to prove Theorem 1.

*Proof of Theorem 1.* Fix mean zero $u_0 \in \widehat{b}^{-\alpha'}_{p,\infty}(\mathbb{T})$ and $\phi$ as in (3), where $\alpha' = \frac{1}{2} - \delta-$ with $\frac{p-2}{4p} < \delta < \frac{p-2}{2p}$ such that $(-\alpha')p < -1$. Consider sequences of initial data $u_0^N \in L^2(\mathbb{T})$ and diagonal covariance operator $\phi^N \in HS(L^2; L^2)$, given by

$$u_0^N = \mathbb{P}_{\leq N}u_0 = \sum_{|n|\leq N}\widehat{u}_0(n)e^{inx} \text{ and } \phi^N(t,\omega) := \mathrm{diag}(\phi_n(t,\omega); 0 < |n| \leq N) \tag{66}$$

where $\phi_n$ is given in (3). Now, fix $\alpha = \frac{1}{2} - \delta > \alpha'$ as in (47). Note that such $u_0^N$ converges to $u_0$ in $\mathcal{F}L^{-\alpha,p}(\mathbb{T})$, and thus in $\widehat{b}^{-\alpha}_{p,\infty}(\mathbb{T})$. Also, $\phi^N$ converges to $\phi$ in $\mathcal{F}L^{-\frac{1}{2}-,p}(\mathbb{T})$ for each $t$ and $\omega$, and thus in $\widehat{b}^{-\frac{1}{2}-}_{p,\infty}(\mathbb{T})$. Then, by Monotone Convergence Theorem, $\phi^N$ converges to $\phi$ in $L^p([0,1]\times\Omega;\widehat{b}^{-\frac{1}{2}-}_{p,\infty})$. (Indeed, the convergence is in $L^\infty([0,1]\times\Omega;\widehat{b}^{-\frac{1}{2}-}_{p,\infty})$, since we have $|\phi_n(t,\omega)| = 1$ for all $n$, independent of $t\in\mathbb{R}$ and $\omega\in\Omega$.) Note that a slight loss of the regularity $-\alpha < -\alpha'$ was necessary since $u_0^N$ defined in (66) does not necessarily converge to $u_0$ in $\widehat{b}^{-\alpha'}_{p,\infty}(\mathbb{T})$ due to the $L^\infty$ nature of the norm over the dyadic blocks. We can avoid such a loss of the regularity if we start with $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$.

Now, let $\Gamma^N = \Gamma^N_{u_0^N}$ be the map defined by

$$\Gamma^N v = \Gamma^N_{u_0^N} v := S(t)u_0^N - \tfrac{1}{2}\mathcal{N}(v,v) + \eta\Phi^N, \tag{67}$$

where $\Phi^N$ is the stochastic convolution defined in (10) with the covariance operator $\phi^N$. By the well-posedness result in [8], there exists a unique global solution $u^N \in L^\infty(\mathbb{R}^+;L^2(\mathbb{T}))\cap C(\mathbb{R}^+;B^{0-}_{2,1}(\mathbb{T}))$ a.s. to (67) for each $N$ since $\phi^N \in HS(L^2;L^2)$.

Now, we put all the estimates together. Note that all the implicit constants are independent of $N$. Also, when there is no superscript $N$, it means that $N=\infty$. From Lemma 3.1, we have

$$\|S(t)u_0^N\|_{X^{s,b,T}_{p,2}} \le C_1\|u_0^N\|_{\widehat{b}^s_{p,\infty}} \tag{68}$$

for any $s,b\in\mathbb{R}$ with $C_1 = C_1(b)$. In particular, by taking $b > \frac{1}{2}$, we see that $S(t)u_0$ is continuous on $[0,T]$ with values in $\widehat{b}^s_{p,\infty}$. Also, by taking $b < \frac{1}{2}$, we gain a power of $T$. From the definition of $\mathcal{N}_j(\cdot,\cdot)$ and (57), we have

$$\|\mathcal{N}(u^N,u^N)\|_{X^{-\alpha,\alpha,T}_{p,2}} \le C_2T^{\theta_1}\|u^N\|^2_{X^{-\alpha,\alpha,T}_{p,2}} + 2\|\mathcal{N}_1(u^N,u^N)\|_{X^{-\alpha,\alpha,T}_{p,2}}. \tag{69}$$

Also, from (47) and (56), we have

$$\|\mathcal{N}_1(u^N,u^N)\|_{X^{-\alpha,1-\alpha,T}_{p,2}} \le I^N_{-\alpha,1-\alpha}. \tag{70}$$

Recall that $\eta\Phi\in X^{-\alpha,\alpha}_{p,2}$ a.s. from Proposition 4.1. Moreover, by defining $F_1^N$ and $F_2^N$ on $\mathbb{T}\times\mathbb{R}\times\Omega$ via their Fourier transforms:

$$\widehat{F_1^N}(n,\tau) = \langle n\rangle^{-\frac{1}{2}-\delta}(\sigma_0^{-\frac{3}{2}+\delta} + \sigma_0^{-\frac{1}{2}-\delta})\int_0^T |\phi_n(r)|d\beta_n(r), \quad\text{and} \tag{71}$$
$$\widehat{F_2^N}(n,\tau) = \langle n\rangle^{-\frac{1}{2}-\delta}\chi_{\Omega(n)}(\tau-n^3)\sigma_0^{-\frac{1}{2}+\delta}\int_0^T |\phi_n(r)|d\beta_n(r)$$

for $|n|\le N$, we have $F_1^N, F_2^N \in L^2(\Omega;L^2_{x,t})$ by Ito isometry and Lemma 5.3, which is basically shown in the estimate on (ii). See (61) and (65). Then, from (54) and the estimates on (i) and (ii), we have

$$I^N_{-\alpha,1-\alpha} \le C_3\big(T^{\theta_2}\|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}} I^N_{-\alpha,1-\alpha} + T^{\theta_3}\|u^N\|^3_{X^{-\alpha,\alpha,T}_{p,2}} + T^{\theta_4}L^N_\omega\|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}}\big), \tag{72}$$

where $L^N_\omega = L^N(F_1^N,F_2^N)(\omega) := \|F_1^N(\omega)\|_{L^2_{x,t}} + \|F_2^N(\omega)\|_{L^2_{x,t}} < \infty$ a.s. Moreover, $L^N_\omega$ is non-decreasing in $N$.

For fixed $R>0$, choose $T>0$ small such that $C_3T^{\theta_2}R \le \frac{1}{2}$. Then, from (72), we have

$$I^N_{-\alpha,1-\alpha} \le 2C_3\big(T^{\theta_3}\|u^N\|^3_{X^{-\alpha,\alpha,T}_{p,2}} + T^{\theta_4}L^N_\omega\|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}}\big), \tag{73}$$

for $\|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}} \le R$. From (67)∼(73), we have

$$\begin{aligned}\|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}} &= \|\Gamma^N u^N\|_{X_{p,2}^{-\alpha,\alpha,T}} \le C_1\|u_0^N\|_{\widehat{b}_{p,\infty}^{-\alpha}} + \tfrac{1}{2}C_2T^{\theta_1}\|u^N\|^2_{X_{p,2}^{-\alpha,\alpha,T}} \\ &+ 2C_3\big(T^{\theta_3}\|u^N\|^3_{X_{p,2}^{-\alpha,\alpha,T}} + T^{\theta_4}L_\omega^N\|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}}\big) + C_4\|\eta\Phi^N(\omega)\|_{X_{p,2}^{-\alpha,\alpha}},\end{aligned} \tag{74}$$

and

$$\begin{aligned}\|u^N-u^M\|_{X_{p,2}^{-\alpha,\alpha,T}} &= \|\Gamma^N u^N - \Gamma^M u^M\|_{X_{p,2}^{-\alpha,\alpha,T}} \\ &\le C_1\|u_0^N - u_0^M\|_{\widehat{b}_{p,\infty}^{-\alpha}} + \tfrac{1}{2}C_2T^{\theta_1}(\|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}} + \|u^M\|_{X_{p,2}^{-\alpha,\alpha,T}})\|u^N-u^M\|_{X_{p,2}^{-\alpha,\alpha,T}} \\ &+ C_5T^{\theta_3}\big(\|u^N\|^2_{X_{p,2}^{-\alpha,\alpha,T}} + \|u^M\|^2_{X_{p,2}^{-\alpha,\alpha,T}}\big)\|u^N-u^M\|_{X_{p,2}^{-\alpha,\alpha,T}} \\ &+ 2C_3T^{\theta_4}L_\omega^N\|u^N-u^M\|_{X_{p,2}^{-\alpha,\alpha,T}} + 2C_3T^{\theta_4}\widetilde{L}_\omega^{N,M}\|u^M\|_{X_{p,2}^{-\alpha,\alpha,T}} \\ &+ C_4\|\eta(\Phi^N-\Phi^M)\|_{X_{p,2}^{-\alpha,\alpha}},\end{aligned} \tag{75}$$

where

$$\widetilde{L}_\omega^{N,M} := \|F_1^N - F_1^M\|_{L^2_{x,t}} + \|F_2^N - F_2^M\|_{L^2_{x,t}}. \tag{76}$$

Note that in estimating the difference $\Gamma^N u^N - \Gamma^M u^M$ on $A_1$, one needs to consider

$$\widetilde{I}_{-\alpha,1-\alpha} := \|\mathcal{N}_1(u^N,u^N) - \mathcal{N}_1(u^M,u^M)\|_{-\alpha,1-\alpha} \tag{77}$$

as in [2]. We can follow the argument on pp.135-136 in [2], except for $R_\alpha$ defined in (58), yielding the third term on the right hand side of (75). As for $R_\alpha$, we can write

$$\mathcal{N}(\mathcal{N}(u,u),u) - \mathcal{N}(\mathcal{N}(v,v),v) = \mathcal{N}(\mathcal{N}(u+v,u-v),u) + \mathcal{N}(\mathcal{N}(v,v),u-v) \tag{78}$$

as in (3.4) in [2], and then we can repeat the computation done for $R_\alpha$ in Estimate on (i), also yielding the third term on the right hand side of (75).

By definition of $u_0^N$, we have $2C_1\|u_0^N\|_{\widehat{b}_{p,\infty}^{-\alpha}} \le 2C_1\|u_0\|_{\widehat{b}_{p,\infty}^{-\alpha}} + \frac{1}{2}$ for $N$ sufficiently large. Also, since $\phi^N$ converges to $\phi$ in $L^p([0,1]\times\Omega; \widehat{b}_{p,\infty}^{-\alpha+})$, it follows from Corollary 4.2 and the estimate on (ii) -see (61), (62), and (65)- that $\mathbb{E}[\|\eta(\Phi^N-\Phi)\|_{X_{p,2}^{-\alpha,\alpha}}]$ and $\mathbb{E}[\widetilde{L}_\omega^{N,\infty}]$ defined in (76) converge to 0. Hence, $\|\eta(\Phi^N-\Phi)\|_{X_{p,2}^{-\alpha,\alpha}} + \widetilde{L}_\omega^{N,\infty} \to 0$ a.s. after selecting a subsequence (which we still denote with the index $N$.) Then, by Egoroff's theorem, given $\varepsilon > 0$, there exists a set $\Omega_\varepsilon$ with $\mathbb{P}(\Omega_\varepsilon^c) < 2^{-1}\varepsilon$ such that $\|\eta(\Phi^N-\Phi)\|_{X_{p,2}^{-\alpha,\alpha}} + \widetilde{L}_\omega^{N,\infty} \to 0$ uniformly in $\Omega_\varepsilon$. In particular, $2C_4\|\eta\Phi^N\|_{X_{p,2}^{-\alpha,\alpha}} \le 2C_4\|\eta\Phi\|_{X_{p,2}^{-\alpha,\alpha}} + \frac{1}{2}$ for large $N$ uniformly on $\Omega_\varepsilon$. In the following, we will work on $\Omega_\varepsilon$.

Now, let $R_\omega = 2(C_1\|u_0\|_{\widehat{b}_{p,\infty}^{-\alpha}} + C_4\|\eta\Phi(\omega)\|_{X_{p,2}^{-\alpha,\alpha}}) + 1$, and define the stopping time $T_\omega$ by

$$T_\omega = \inf\{T>0 : \max(C_3T^{\theta_2}R_\omega, P_1(T,R_\omega,\omega), P_2(T,R_\omega,\omega) \ge \tfrac{1}{2}\}, \tag{79}$$

where

$$\begin{cases} P_1(T,R_\omega,\omega) = \frac{1}{2}C_2T^{\theta_1}R_\omega + 2C_3T^{\theta_3}(R_\omega)^2 + 2C_3T^{\theta_4}L_\omega, & \text{from (74)} \\ P_2(T,R_\omega,\omega) = C_2T^{\theta_1}R_\omega + 2C_5T^{\theta_3}(R_\omega)^2 + 2C_3T^{\theta_4}L_\omega, & \text{from (75).} \end{cases}$$

The first condition in the definition of $T_\omega$ guarantees (73), and hence (74) and (75), for $\|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}} \leq R_\omega$. The second condition along with (74) indeed guarantees that

$$\|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}} \leq R_\omega \tag{80}$$

for $T \leq T_\omega$ from the following observation. Since we have the temporal regularity $b = \alpha < \frac{1}{2}$, we have $\|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}} = \|\chi_{[0,T]} u^N\|_{X^{-\alpha,\alpha}_{p,2}}$, where $\chi_{[0,T]}$ denotes the characteristic function of the time interval $[0,T]$. See Bourgain [3]. Hence, $\|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}}$ is continuous in $T$ since

$$\big| \|u^N\|_{X^{-\alpha,\alpha,T+\delta}_{p,2}} - \|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}} \big| \leq \|u^N\|_{X^{-\alpha,\alpha}_{p,2}[T,T+\delta]} \lesssim \delta^\theta \|u^N\|_{X^{0-,\frac{1}{2}}[T,T+\delta]} \tag{81}$$

for sufficiently small $\delta > 0$. Note that the last term in (81) is finite for small $\delta$ since the local-in-time solutions constructed in [8] are controlled in this norm (indeed in a stronger norm adapted to the Besov space $B^{0-}_{2,1}$.) Then, (80) follows from (74), the second condition in (79), and the continuity of the norm in $T$ since (80) clearly holds at $T = 0$.

From (75) along with the third condition in (79), we have

$$\begin{aligned} \|u^N - u^M\|_{X^{-\alpha,\alpha,T_\omega}_{p,2}} \leq 2C_1 \|u_0^N - u_0^M\|_{\widehat{b}^{-\alpha}_{p,\infty}} + 4C_3 T^{\theta_4} R_\omega \widetilde{L}^{N,M}_\omega \\ + 2C_4 \|\eta(\Phi^N - \Phi^M)\|_{X^{-\alpha,\alpha}_{p,2}}. \end{aligned} \tag{82}$$

The right hand side of (82) goes to 0 as $N, M \to \infty$ since $u_0^N$ is Cauchy in $\widehat{b}^{-\alpha}_{p,\infty}$ and $\|\eta(\Phi^N - \Phi^M)\|_{X^{-\alpha,\alpha}_{p,2}} + \widetilde{L}^{N,M}_\omega \to 0$ on $\Omega_\varepsilon$ uniformly in $N, M$. Let $u$ denote the limit in $X^{-\alpha,\alpha,T_\omega}_{p,2}$.

In the following, we give a brief discussion to show that the limit $u$ is a solution to (4). Clearly, $S(t)u_0^N$ and $\eta\Phi^N$ converge to $S(t)u_0$ and $\eta\Phi$ in $X^{-\alpha,\alpha,T_\omega}_{p,2}$. It follows from (57) that $\mathcal{N}_0(u^N, u^N)$ converges $\mathcal{N}_0(u,u)$ in $X^{-\alpha,\alpha,T_\omega}_{p,2}$. In view of (73), (75), and (77), we see that $\mathcal{N}_j(u^N, u^N)$ is Cauchy in a slightly stronger space $X^{-\alpha,1-\alpha,T_\omega}_{p,2}$, $j = 1, 2$. Let $v_j$ denote the corresponding limit. Thus, from (67), we have

$$u = S(t)u_0 - \tfrac{1}{2}\mathcal{N}_0(u,u) - \tfrac{1}{2}(v_1 + v_2) + \eta\Phi. \tag{83}$$

Now, we need to show that $\mathcal{N}_j(u^N, u^N)$ indeed converges to $\mathcal{N}_j(u,u)$, $j = 1, 2$. By symmetry, we only consider $\mathcal{N}_1(u,u) - \mathcal{N}_1(u^N, u^N)$. As before, we substitute (83) (and (67)) in the first factor $u$ (and $u^N$) of $\mathcal{N}_1(\cdot,\cdot)$, respectively. There are three contributions to consider.

• **(A)** Contribution from the stochastic terms: We have

$$\mathcal{N}_1(\eta\Phi, u) - \mathcal{N}_1(\eta\Phi^N, u^N) = \mathcal{N}_1(\eta(\Phi - \Phi^N), u) + \mathcal{N}_1(\eta\Phi^N, u - u^N). \tag{84}$$

From Estimate on (ii), we have

$$\|(84)\|_{X^{-\alpha,\alpha,T_\omega}_{p,2}} \lesssim \widetilde{L}^{N,\infty}_\omega \|u\|_{X^{-\alpha,\alpha,T_\omega}_{p,2}} + L^N_\omega \|u^N - u\|_{X^{-\alpha,\alpha,T_\omega}_{p,2}} \to 0$$

as $N \to \infty$, since $\|u\|_{X^{-\alpha,\alpha,T}_{p,2}} \leq R_\omega$ and $\widetilde{L}^{N,\infty}_\omega \to 0$ uniformly on $\Omega_\varepsilon$.

• **(B)** Contribution from $\mathcal{N}_0(\cdot,\cdot)$: In this case, we consider

$$\mathcal{N}_1(\mathcal{N}_0(u,u), u) - \mathcal{N}_1(\mathcal{N}_0(u^N, u^N), u^N). \tag{85}$$

Note that we have $\sigma_1 \geq \sigma_0, \sigma_2, \sigma_3, \sigma_4$ from the definition of $\mathcal{N}_1(\cdot,\cdot)$ and $\mathcal{N}_0(\cdot,\cdot)$. See (50) and (52). Indeed, we have $\sigma_1 \geq \sigma_0, \sigma_2$ since we are on $A_1$ defined in (45), and also $\sigma_1 \geq \sigma_3, \sigma_4$

since we are on the support of $\mathcal{N}_0(\cdot,\cdot)$ in the first factor of $\mathcal{N}_1(\cdot,\cdot)$. Once again, one can easily follow the argument on p.136 in [2] and show

$$\|(85)\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}} \lesssim \big(\|u^N\|^2_{X_{p,2}^{-\alpha,\alpha,T_\omega}} + \|u\|^2_{X_{p,2}^{-\alpha,\alpha,T_\omega}}\big)\|u^N-u\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}} \to 0.$$

In treating $R_\alpha - R_\alpha^N$ defined in (58), one needs to proceed as before, using (78) and Estimate on (i).

• **(C)** Contribution from $v_j$ and $\mathcal{N}_j(u^N,u^N)$, $j=1$ or 2: By symmetry, assume $j=1$. In this case, we have $\sigma_1 \geq \sigma_0, \sigma_2$ but $\sigma_3 \geq \sigma_1, \sigma_4$. i.e. we control (54) by the first term on the right hand side. See (II.1) on p.126 in [2]. Now, we need to estimate

$$\begin{aligned}\mathcal{N}_1(v_1,u) - &\mathcal{N}_1(\mathcal{N}_1(u^N,u^N),u^N)\\ &= \mathcal{N}_1(v_1-\mathcal{N}_1(u^N,u^N),u) + \mathcal{N}_1(\mathcal{N}_1(u^N,u^N),u-u^N) =: \mathrm{I}+\mathrm{II}.\end{aligned} \tag{86}$$

Then, by proceeding as in [2] with (56) and (73), we have

$$\|\,\mathrm{II}\,\|_{X_{p,2}^{-\alpha,1-\alpha,T_\omega}} \lesssim I^N_{-\alpha,1-\alpha}\|u-u^N\|_{X^{-(1-\alpha),\alpha,T_\omega}} \lesssim \|u-u^N\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}} \to 0.$$

By proceeding as in (II.1) in [2] with $|n_1|^\alpha$ replaced by $|n_1|^{1-\alpha}$, followed by (56), we have

$$\begin{aligned}\|\,\mathrm{I}\,\|_{X_{p,2}^{-\alpha,1-\alpha,T_\omega}} &\lesssim \|v_1-\mathcal{N}_1(u^N,u^N)\|_{-(1-\alpha),1-\alpha}\|u\|_{-(1-\alpha),\alpha}\\ &\lesssim \|v_1-\mathcal{N}_1(u^N,u^N)\|_{X_{p,2}^{-\alpha,1-\alpha,T_\omega}}\|u\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}} \to 0\end{aligned}$$

since $v_1 = \lim_{N\to\infty}\mathcal{N}_1(u^N,u^N)$ in $X_{p,2}^{-\alpha,1-\alpha,T_\omega}$ by definition.

Hence, we have $u=\Gamma_{u_0}u$ for each $\omega\in\Omega_\varepsilon$. i.e. $u$ is a mild solution to (2) on $[0,T_\omega]$. Let $\Omega^{(1)}=\Omega_\varepsilon$. Now, we can recursively construct $\Omega^{(j+1)} \subset \Omega\setminus\bigcup_{k=1}^j \Omega^{(k)}$ for $j=1,2,\cdots$ with $\mathbb{P}(\Omega\setminus\bigcup_{k=1}^j\Omega^{(k)}) < 2^{-j}\varepsilon$ such that $\|\eta(\Phi^N-\Phi)\|_{X_{p,2}^{-\alpha,\alpha}}$ and $\widetilde{L}_\omega^{N,\infty}$ converge to 0 uniformly in each $\Omega^{(j)}$. Then, by repeating the argument, we can construct a solution $u$ on $\bigcup_{j=1}^\infty\Omega^{(j)}$. Note that $\mathbb{P}(\Omega\setminus\bigcup_{j=1}^\infty\Omega^{(j)})=0$.

We have constructed a solution $u$ to (2) in $X_{p,2}^{-\alpha,\alpha,T_\omega}$ with $u_0\in\widehat{b}^{-\alpha'}_{p,\infty}$. Since $u$ is a solution, the a priori estimate (74) holds with the regularity $(s,b)=(-\alpha',\alpha')$ in place of $(-\alpha,\alpha)$. Then, we easily see that $u\in X_{p,2}^{-\alpha',\alpha',T_\omega}$, by redefining $R_\omega$ and $T_\omega$ with this regularity. In the remaining of the paper, we work only with the spatial regularity $s=-\alpha'$, i.e. there is no approximating sequences any more. Hence, for notational simplicity, we will use $-\alpha$ in place of $-\alpha'$ to denote the spatial regularity of the solution in the following.

We still need to take care of several issues. Note that the temporal regularity $b=\alpha=\frac{1}{2}-\delta$ of the solution $u$ is less than $\frac{1}{2}$. In particular, we need to show that the solution $u$ is continuous from $[0,T_\omega]$ into $\widehat{b}^{-\alpha}_{p,\infty}$. We also need to show its uniqueness and continuous dependence on the initial data.

From Proposition 4.5, $\eta\Phi\in C([0,T_\omega];\widehat{b}^{-\alpha}_{p,\infty})$ a.s. Also, it follows from (68) with $b=\frac{1}{2}+\delta$, (70), (73), and symmetry on $\sigma_1$ and $\sigma_2$, that

$$S(t)u_0+\mathcal{N}_1(u,u)+\mathcal{N}_2(u,u)\in X_{p,2}^{-\alpha,\frac{1}{2}+\delta,T_\omega}\subset C([0,T_\omega];\widehat{b}^{-\alpha}_{p,\infty})$$

a.s. Now, we consider $\mathcal{N}_0(u,u)$, i.e. when $\sigma_0=\mathrm{MAX}$. Note that the contribution comes only from $\mathcal{M}_2(u,u)$ defined in (51). Let $\mathcal{N}_3(u,u)$ denotes the contribution of $\mathcal{N}_0(u,u)$ on $\{\max(\sigma_1,\sigma_2)\gtrsim\langle nn_1n_2\rangle^{\frac{1}{100}}\}$, and $\mathcal{N}_4(u,u)=\mathcal{N}_0(u,u)-\mathcal{N}_3(u,u)$.

• **Case (a):** First, we consider $\mathcal{N}_3(u,u)$. i.e. $\max(\sigma_1,\sigma_2) \gtrsim \langle nn_1n_2\rangle^{\frac{1}{100}}$. Say $\sigma_1 \gtrsim \langle nn_1n_2\rangle^{\frac{1}{100}}$. Then, by Lemma 3.2 and (15), we have

$$\|\mathcal{N}_3(u,u)\|_{X_{p,2}^{-\alpha,\frac{1}{2}+\delta,T_\omega}} \lesssim \|\partial_x(u^2)\|_{X_{p,2}^{-\alpha,-\frac{1}{2}+\delta,T_\omega}} \lesssim \|\partial_x(u^2)\|_{X^{-\alpha,-\frac{1}{2}+\delta,T_\omega}}$$

Then, by duality and (44), we have

$$= \sup_{\|d\|_{L^2_{n,\tau}}=1} \sum_{\substack{n,n_1\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{\langle n\rangle^{1-\alpha} d(n,\tau)}{\sigma_0^{\frac{1}{2}-\delta}} \prod_{j=1}^2 \frac{\langle n_j\rangle^{1-\alpha} c(n_j,\tau_j)}{\sigma_j^\alpha} d\tau d\tau_1$$

$$\lesssim \sup_{\|d\|_{L^2_{n,\tau}}=1} \sum_{\substack{n,n_1\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d(n,\tau)\frac{c(n_1,\tau_1)}{\sigma_1^{\alpha-200\delta}}\frac{c(n_2,\tau_2)}{\sigma_2^\alpha} d\tau d\tau_1$$

where $c(n,\tau)$ is defined in (49). Then, by $L^2_{x,t}, L^4_{x,t}, L^4_{x,t}$-Hölder inequality along with Lemma 3.3, (49), and (56),

$$\leq \|c\|^2_{L^2_{n,\tau}} \leq \|u\|^2_{X^{-(1-\alpha),\alpha}} \lesssim \|u\|^2_{X_{p,2}^{-\alpha,\alpha}} < \infty.$$

• **Case (b):** Now, consider $\mathcal{N}_4(u,u)$. i.e. $\max(\sigma_1,\sigma_2) \ll \langle nn_1n_2\rangle^{\frac{1}{100}}$. Note that it suffices to show that $\mathcal{N}_0(u,u) \in X_{p,1}^{-\alpha,0,T_\omega}$, since $X_{p,1}^{-\alpha,0,T_\omega} \subset C([0,T_\omega]; \widehat{b}^{-\alpha}_{p,\infty})$. Then, by Cauchy-Schwarz inequality, Lemma 5.3 and duality, we have

$$\begin{aligned}
\|\mathcal{N}_4(u,u)\|_{X_{p,1}^{-\alpha,0,T_\omega}} &\leq \|\partial_x(u^2)\|_{X_{2,1}^{-\alpha,-1,T_\omega}} \leq \Big\| \|\langle n\rangle^{-\alpha}\langle \tau-n^3\rangle^{-1}\chi_{\Omega(n)}(\tau-n^3)\widehat{\partial_x(u^2)}(n,\tau)\|_{L^1_\tau}\Big\|_{L^2_n}\\
&\leq \|\langle\tau-n^3\rangle^{-\frac{1}{2}+\delta}\chi_{\Omega(n)}(\tau-n^3)\|_{L^2_\tau}\|\partial_x(u^2)\|_{-\alpha,-\frac{1}{2}-\delta}\\
&\lesssim \sup_{\|d\|_{L^2_{n,\tau}}=1} \sum_{\substack{n,n_1\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{\langle n\rangle^{1-\alpha} d(n,\tau)}{\sigma_0^{\frac{1}{2}+\delta}} \prod_{j=1}^2 \frac{\langle n_j\rangle^{1-\alpha} c(n_j,\tau_j)}{\sigma_j^\alpha} d\tau d\tau_1\\
&\lesssim \sup_{\|d\|_{L^2_{n,\tau}}=1} \sum_{\substack{n,n_1\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d(n,\tau)\frac{c(n_1,\tau_1)}{\sigma_1^{\alpha}}\frac{c(n_2,\tau_2)}{\sigma_2^\alpha} d\tau d\tau_1.
\end{aligned}$$

The rest follows as before. Hence, the solution $u$ is continuous from $[0,T_\omega]$ to $\widehat{b}^{-\alpha}_{p,\infty}$.

Lastly, we show the uniqueness and the continuous dependence of the solutions on the initial data. Let $u$ and $v$ be the mild solutions of (2) on $[0,T_\omega]$ with initial data $u_0$ and $v_0$ respectively. i.e.

$$u - v = \Gamma_{u_0}u - \Gamma_{v_0}v = S(t)(u_0-v_0) - \tfrac{1}{2}\big(\mathcal{N}(u,u)-\mathcal{N}(v,v)\big), \tag{87}$$

where $\Gamma$ is defined in (67). Moreover, assume that

$$\|u_0\|_{\widehat{b}^{-\alpha}_{p,\infty}}, \|v_0\|_{\widehat{b}^{-\alpha}_{p,\infty}}, \|u\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}}, \|v\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}} \leq R. \tag{88}$$

Let $\widetilde{\mathcal{N}}_j(u,v) := -\frac{1}{2}\big(\mathcal{N}_j(u,u) - \mathcal{N}_j(v,v)\big)$ for $j = 1,\cdots,4$. First, note that $\|\widetilde{\mathcal{N}}_4(u,v)\|_{X_{p,1}^{-\alpha,\varepsilon,T_\omega}} \lesssim R^2 < \infty$ from (a slight variation of) Case (b), and we have

$$\|(u-v)-\widetilde{\mathcal{N}}_4(u,v)\|_{X_{p,1}^{-\alpha,\varepsilon,T_\omega}} \leq \Big\|S(t)(u_0-v_0) + \sum_{j=1}^3 \widetilde{\mathcal{N}}_j(u,v)\Big\|_{X_{p,2}^{-\alpha,\frac{1}{2}+\delta,T_\omega}} \lesssim C_1(R) < \infty$$

by Cauchy-Schwarz inequality with $\varepsilon < \delta$, followed by (68), (70), (73), Case (a), and (88). Then, by interpolation and Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|u-v\|_{C([0,T_\omega];\widehat{b}^{-\alpha}_{p,\infty})} &\lesssim \|u-v\|_{X^{-\alpha,0,T_\omega}_{p,1}} \lesssim \|u-v\|^{\beta}_{X^{-\alpha,-\delta-,T_\omega}_{p,1}} \|u-v\|^{1-\beta}_{X^{-\alpha,\varepsilon,T_\omega}_{p,1}} \\ &\lesssim C_2(R)\|u-v\|^{\beta}_{X^{-\alpha,\frac{1}{2}-\delta,T_\omega}_{p,2}} \end{aligned} \tag{89}$$

with $\beta = \frac{\varepsilon}{\varepsilon+\delta+} \in (0,1)$. From (68) and the nonlinear estimates (see (69), (73), (75), (77)), we have

$$\|u-v\|_{X^{-\alpha,\frac{1}{2}-\delta,T_\omega}_{p,2}} \lesssim \|u_0-v_0\|_{\widehat{b}^{-\alpha}_{p,\infty}} + C_3(R)T^{\theta}_{\omega}\|u-v\|_{X^{-\alpha,\frac{1}{2}-\delta,T_\omega}_{p,2}}.$$

Hence, for sufficiently small $T>0$, we have

$$\|u-v\|_{X^{-\alpha,\frac{1}{2}-\delta,T_\omega}_{p,2}} \lesssim \|u_0-v_0\|_{\widehat{b}^{-\alpha}_{p,\infty}}. \tag{90}$$

Therefore, it follows from (89) and (90) that the solution map is Hölder continuous with the bound

$$\|u-v\|_{C([0,T_\omega];\widehat{b}^{-\alpha}_{p,\infty})} \le C_4(R)\|u_0-v_0\|^{\beta}_{\widehat{b}^{-\alpha}_{p,\infty}}.$$

In particular, the solution is unique. This completes the proof of Theorem 1. □

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