Multilinear Duality and Factorisation for Brascamp-Lieb-type Inequalities with applications

MULTILINEAR DUALITY AND FACTORISATION FOR BRASCAMP–LIEB-TYPE INEQUALITIES

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Abstract. We initiate the study of a duality theory which applies to norm inequalities for pointwise weighted geometric means of positive operators. The theory finds its expression in terms of certain pointwise factorisation properties of function spaces which are naturally associated to the norm inequality under consideration. We relate our theory to the Maurey–Nikisin–Stein theory of factorisation of operators, and present a fully multilinear version of Maurey's fundamental theorem on factorisation of operators through L^1. The development of the theory involves convex optimisation and minimax theory, functional-analytic considerations concerning the dual of L^∞, and the Yosida–Hewitt theory of finitely additive measures. We consider the connections of the theory with the theory of interpolation of operators. We discuss the ramifications of the theory in the context of concrete families of geometric inequalities, including Loomis–Whitney inequalities, Brascamp–Lieb inequalities and multilinear Kakeya inequalities.

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1. Introduction

In this paper we introduce and develop a general functional-analytic principle which gives a unifying framework for a range of multilinear phenomena that have recently arisen in a number of areas of mathematical analysis.

We shall be mainly concerned with norm inequalities for pointwise weighted geometric means

$$\prod_{j=1}^{d}(T_jf_j(x))^{\alpha_j}$$

of positive linear operators T_j defined on suitable spaces, where $\alpha_j \geq 0$ and $\sum_{j=1}^{d}\alpha_j = 1$. Before we describe the scope of our work in this paper, and to set the scene for our study, we briefly visit the analogous territory in the linear setting ($d = 1$) in order to help provide a context for what we are aiming to achieve. Throughout the whole paper we shall be dealing with real-valued rather than complex-valued functions.

1.1. **The linear setting.**

Let X and Y be measure spaces and let $T : L^p(Y) \to L^q(X)$ be a bounded linear operator, that is, it satisfies

$$\|Tf\|_q \leq A\|f\|_p \tag{1}$$

for all $f \in L^p(Y)$, for some $A > 0$. Here, $1 \leq p \leq \infty$ and $0 < q < \infty$. Since L^q is a Banach space only when $q \geq 1$, it is natural to focus separately on the regimes $q \geq 1$ and $0 < q < 1$.

(i) **Case** $q \geq 1$. Since $\|h\|_q = \max\{|\int hg| : \|g\|_{q'} = 1\}$, inequality (1) holds if and only if for all $f \in L^p$ and *all* $g \in L^{q'}$ we have

$$|\int(Tf)g| \leq A\|f\|_p\|g\|_{q'}. \tag{2}$$

Using the relation $\int(Tf)g = \int f(T^*g)$, this is in turn equivalent to the statement $\|T^*g\|_{p'} \leq A\|g\|_{q'}$ – that is the boundedness of the adjoint operator T^* between the dual spaces of L^q and L^p respectively, (at least when $1 < p, q < \infty$). We are therefore firmly in the terrain of classical linear duality theory, a theory whose utility and importance cannot be overstated. Notice that if $q = 1$ and T is also assumed to be positive (that is, $Tf \geq 0$ whenever $f \geq 0$), the equivalence of (1) and (2) is essentially without content since in this case it suffices to check on the function $g \equiv 1$.

(ii) **Case** $0 < q < 1$. Since $\|h\|_q = \min\{|\int hg| : \|g\|_{q'} = 1\}$, we have that (1) holds if and only if for all $f \in L^p$ *there exists* an (extended real-valued) $g \in L^{q'}$ such that (2) holds. Note that $q' < 0$ in this situation, so it is implicit that such a g satisfies $g(x) \neq 0$ almost everywhere. It is a remarkable result of Maurey, that, under certain conditions – such as positivity of T – given inequality (1), there exists a *single* $g \in L^{q'}$ with $\|g\|_{q'} = 1$ such that (2) holds for all $f \in L^p$. Such a result is an instance of the celebrated theory of factorisation of operators which is developed in [35]. Indeed, it is a case of *factorisation through* L^1 since the inequality

$$|\int(Tf)g| \leq A\|f\|_p$$

demonstrates that T may be factorised as $T = M_{g^{-1}} \circ S$ where $S = M_g \circ T$ satisfies $\|S\|_{L^p \to L^1} \leq A$ and $M_{g^{-1}}$, the operator of multiplication by g^{-1}, satisfies $\|M_{g^{-1}}\|_{L^1 \to L^q} = \|g\|_{q'}^{-1} = 1$.

Observe that there is no obvious point of direct contact between the two regimes $q \geq 1$ and $0 < q < 1$ in this linear setting.

The result of Maurey to which we refer falls within the wider scope of Maurey–Nikisin–Stein theory, which considers factorisation of operators in a broad variety of contexts. This includes consideration of non-positive operators, sublinear operators (for example maximal functions), operators with various domains and codomains, and factorisation through various weak- and strong-type spaces, often under some auxiliary hypotheses. The particular case of positive operators defined on normed lattices, taking values in L^q for $q < 1$, and factorising through (strong-type) L^1 was considered by Maurey, however, and for this reason we refer specifically to the Maurey theory rather than the broader Maurey–Nikisin–Stein theory. For an overview of this larger theory see [26], [27], [35] and [36].

1.2. **The multilinear setting.** The purpose of this paper is to develop duality and factorisation theories for certain classes of multilinear operators which are analogous to those that we have set out above in the linear setting. Amusingly, the notion of "factorisation" manifests itself in two distinct ways in our development. One of these is as a multilinear analogue of a formulation of a Maurey-type theorem as was briefly outlined in the discussion of the case $0 < q < 1$ above. The other is that our duality theory (corresponding to the case $1 \leq q < \infty$) will be expressed in terms of pointwise factorisation properties of certain spaces of functions. *Even simple instances of these pointwise factorisation results are new and striking: see Section 1.5.1 below.*

We begin by describing the scenario in which we shall work and the classes of operators we shall consider.

Let $(X, \mathrm{d}\mu)$ and $(Y_j, \mathrm{d}\nu_j)$, for $j = 1, \ldots, d$, be measure spaces,[1] let $\mathcal{S}(Y_j)$ denote the class of real-valued simple functions (i.e. finite linear combinations of characteristic functions of measurable sets of *finite* measure) on Y_j, and let $\mathcal{M}(X)$ denote the class of real-valued measurable functions on X. Let $T_1, \ldots, T_d$ be linear maps

$$T_j : \mathcal{S}(Y_j) \to \mathcal{M}(X).$$

We suppose throughout that the T_j are positive in the sense that if $f \geq 0$ almost everywhere on Y_j, then $T_j f \geq 0$ almost everywhere on X.

In this paper we shall be concerned with "multilinear" Lebesgue-space inequalities of the form

$$\left\| \prod_{j=1}^{d} (T_j F_j)^{\beta_j} \right\|_{L^q(X)} \leq C \prod_{j=1}^{d} \left\| F_j \right\|_{L^{p_j}(Y_j)}^{\beta_j} \tag{3}$$

where $0 < \beta_j < \infty$, $0 < p_j \leq \infty$[2]and $0 < q \leq \infty$.

These inequalities are to be interpreted in an *a priori* sense, with the F_j being nonnegative simple functions defined on Y_j. We are especially interested in the case that either the T_j are not bounded operators from $L^{p_j}(Y_j, \mathrm{d}\nu_j)$ to $L^q(X, \mathrm{d}\mu)$, or that they are bounded but do not enjoy effective bounds.

Strictly speaking such inequalities are multilinear only when each $\beta_j = 1$; we shall nevertheless abuse language and will refer to the inequalities under consideration as "multilinear". In fact the case when $\sum_{j=1}^d \beta_j = 1$ will play a special role in what follows. Of course we may always assume either that $q = 1$ or that $\sum_{j=1}^d \beta_j = 1$.

To fix ideas, we discuss some examples of inequalities falling under the scope of our study.

[1]Throughout the paper, when we refer to measure spaces X, Y or Y_j without explicit mention of the measure, it is implicit that the corresponding measures are μ, ν and ν_j respectively, unless the context demands otherwise.
[2]We shall soon focus on the case $p_j \geq 1$ and $\sum_j \beta_j = 1$.

1.3. **Examples.**

Example 1. [Hölder's inequality] The multilinear form of Hölder's inequality for nonnegative functions is simply

$$\int_X F_1(x)\cdots F_d(x)\mathrm{d}\mu(x) \le \|F_1\|_{L^{p_1}(X)}\cdots\|F_d\|_{L^{p_d}(X)}$$

where $p_j > 0$ and $\sum_{j=1}^d p_j^{-1} = 1$. This is of the form (3), with $T_j = I$ for all j, $q = 1$ and each $\beta_j = 1$. But, for any fixed set of positive exponents $\{\beta_j\}$, it is also trivially equivalent to the inequality

$$\|f_1^{\beta_1}\cdots f_d^{\beta_d}\|_q \le \|f_1\|_{q_1}^{\beta_1}\cdots\|f_d\|_{q_d}^{\beta_d}$$

for all $0 < q_j < \infty$ and $0 < q < \infty$ which satisfy $\sum_{j=1}^d \beta_j q_j^{-1} = q^{-1}$. In particular, there is an equivalent formulation of the multilinear Hölder inequality taking the form (3) with $\sum_{j=1}^d \beta_j = 1$. In fact there are many such equivalent forms, limited only by the requirement that $\sum_{j=1}^d \beta_j q_j^{-1} = q^{-1}$. Special cases of choices of exponents $\{\beta_j, q_j, q\}$ satisfying this condition are (i) β_j arbitrary subject to $\sum_{j=1}^d \beta_j = 1$, $q_j = 1$ for all j, and $q = 1$; and (ii) $\beta_j = d^{-1}$ for all j, q_j arbitrary subject to $\sum_{j=1}^d q_j^{-1} = 1$, and $q = d$. This observation demonstrates that we may expect that a given multilinear inequality might have *multiple* equivalent manifestations, each of the form (3), with $\sum_{j=1}^d \beta_j = 1$. In the context of the factorisation theory we shall develop, each manifestation of the inequality corresponds to a different factorisation property of associated function spaces. See Section 7.1 for further discussion.

Example 2. [Loomis–Whitney inequality] For $1 \le j \le n$ let $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be projection on the coordinate hyperplane perpendicular to the standard unit basis vector e_j; that is, $\pi_j x = (x_1, \dots, \widehat{x_j}, \dots, x_n)$. The Loomis–Whitney inequality [33] for nonnegative functions is

$$\int_{\mathbb{R}^n} F_1(\pi_1 x)\cdots F_n(\pi_n x)\,\mathrm{d}x \le \|F_1\|_{L^{n-1}(\mathbb{R}^{n-1})}\cdots\|F_n\|_{L^{n-1}(\mathbb{R}^{n-1})}.$$

For each $0 < p < \infty$, this is equivalent to the inequality

$$\|f_1(\pi_1 x)^{1/n}\cdots f_n(\pi_n x)^{1/n}\|_{L^{np/(n-1)}(\mathbb{R}^n)} \le \|f_1\|_{L^p(\mathbb{R}^{n-1})}^{1/n}\cdots\|f_n\|_{L^p(\mathbb{R}^{n-1})}^{1/n}.$$

Each of these inequalities is of the form (3) with $\sum_{j=1}^n \beta_j = 1$.

A very special case of the Loomis–Whitney inequality occurs in two dimensions where it becomes the trivial identity

$$\int_{\mathbb{R}^2} F_1(x_2)F_2(x_1)\,\mathrm{d}x_1\mathrm{d}x_2 = \int_{\mathbb{R}} F_1 \int_{\mathbb{R}} F_2.$$

In spite of its simplicity, this example will play an important guiding role for us. See Sections 6, 9.2, 9.3 and 10.2.1.

The Loomis–Whitney inequality has many variants – for example Finner's inequalities, the affine-invariant Loomis–Whitney inequality and the nonlinear Loomis–Whitney inequality. See [25], [11], and Sections 9.2 and 9.3.

Example 3. [Brascamp–Lieb inequalities] The class of Brascamp–Lieb inequalities includes the previous examples. Let $B_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ be linear surjections, $1 \le j \le d$. For $0 < p_j < \infty$ and F_j nonnegative we consider the Brascamp–Lieb inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^d F_j(B_j x)^{p_j}\,\mathrm{d}x \le C \prod_{j=1}^d \left(\int_{\mathbb{R}^{n_j}} F_j\right)^{p_j}. \tag{4}$$

It is not hard to see that in order for this inequality to hold with a finite constant C, it is necessary that $\sum_{j=1}^d p_j n_j = n$. It is known that the constant C is finite if and only if, in addition to $\sum_{j=1}^d p_j n_j = n$, it holds that

$$\dim V \leq \sum_{j=1}^{d} p_j \dim B_j V$$

for all V in the lattice of subspaces of $\mathbb{R}^n$ generated by $\{\ker B_j\}_{j=1}^d$. (See [8], [9] and [42].) From this one sees easily that $\cap_{j=1}^d \ker B_j = \{0\}$, $\sum_{j=1}^d p_j \geq 1$, and $p_j \leq 1$ are also necessary conditions for the finiteness of C. A celebrated theorem of Lieb [31] states that the value of the best constant C is obtained by checking the inequality on Gaussian inputs F_j. Lieb's theorem generalises Beckner's theorem [6] on extremisers for Young's convolution inequality.

Suppose that $0 < r_j < \infty$ and $0 < s < \infty$. Setting $F_j = f_j^{r_j}$ in (4) and taking s'th roots, we see that (4) is equivalent to

$$\| \prod_{j=1}^{d} f_j(B_j x)^{p_j r_j / s} \|_{L^s(\mathbb{R}^n)} \leq C^{1/s} \prod_{j=1}^{d} \|f_j\|_{L^{r_j}(\mathbb{R}^{n_j})}^{p_j r_j / s}.$$

If $\sum_{j=1}^d p_j r_j = s$ this is an inequality of the form (3) with $\sum_{j=1}^d \beta_j = 1$. In particular we can take $r_j = n_j$ and $s = n$ to obtain the equivalent form

$$\| \prod_{j=1}^{d} f_j(B_j x)^{p_j n_j / n} \|_{L^n(\mathbb{R}^n)} \leq C^{1/n} \prod_{j=1}^{d} \|f_j\|_{L^{n_j}(\mathbb{R}^{n_j})}^{p_j n_j / n};$$

or we can take $r_j = 1$ and $s = \sum_{j=1}^d p_j$ (recall that this number is at least 1 when the inequality is nontrivial) to obtain another equivalent form

$$\| \prod_{j=1}^{d} f_j(B_j x)^{p_j / s} \|_{L^s(\mathbb{R}^n)} \leq C^{1/s} \prod_{j=1}^{d} \|f_j\|_{L^1(\mathbb{R}^{n_j})}^{p_j / s}. \tag{5}$$

A special case of the class of Brascamp–Lieb inequalities is the class of *geometric* Brascamp–Lieb inequalities. Suppose that the linear surjections $B_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ satisfy

$$\sum_{j=1}^{d} p_j B_j^* B_j = I_n.$$

Then, by a result of Ball and Barthe, ([4] and [5], see also [8]) we have

$$\int_{\mathbb{R}^n} \prod_{j=1}^{d} F_j(B_j x)^{p_j} \, \mathrm{d}x \leq \prod_{j=1}^{d} \left(\int_{\mathbb{R}^{n_j}} F_j \right)^{p_j}, \tag{6}$$

and the sharp constant 1 is achieved by the standard Gaussians $F_j(y) = e^{-\pi |y|^2}$. Correspondingly, in the equivalent variants presented above, the constants are also 1. The geometric Brascamp–Lieb inequalities include a suitably reformulated version of the sharp Young inequality of Beckner [6]. See Section 1.5.1 and Section 10.1 for an application of the theory we present in the context of geometric Brascamp–Lieb inequalities.

Example 4. [Multilinear generalised Radon transforms] There is a vast literature on multilinear generalised Radon transforms into which we do not wish to enter. For us, this term will mean consideration of multilinear inequalities of the form (3) when the operators T_j take the form $T_j f = f \circ B_j$ for suitable mappings $B_j : X \to Y_j$. In most cases, X and Y_j will be endowed with a topological or smooth structure, and the mappings B_j will respect that structure in such a way that issues of measurability do not arise.

The class of multilinear generalised Radon transforms includes the Brascamp–Lieb inequalities. The most basic multilinear generalised Radon transform which is not included in the Brascamp–Lieb inequalities is probably the nonlinear Loomis–Whitney inequality. See Section 9.3 below.

Example 5. [Multilinear Kakeya inequalities] The Loomis–Whitney inequality of Example 2 is equivalent to

$$\int_{\mathbb{R}^n} \prod_{j=1}^n \left(\sum_{P_j \in \mathcal{P}_j} a_{P_j} \chi_{P_j}(x) \right)^{1/(n-1)} \mathrm{d}x \leq \prod_{j=1}^n \left(\sum_{P_j \in \mathcal{P}_j} a_{P_j} \right)^{1/(n-1)},$$

where $\mathcal{P}_j$ is a finite family of 1-tubes in $\mathbb{R}^n$ which are parallel to the j'th standard basis vector e_j, and the a_{P_j} are arbitrary positive numbers. (A 1-tube is simply a neighbourhood of a doubly infinite line in $\mathbb{R}^n$ which has $(n-1)$-dimensional cross-sectional area equal to 1.) Multilinear Kakeya inequalities have the same set-up, but now we allow the tubes in the family $\mathcal{P}_j$ to be *approximately* parallel to e_j, i.e. the direction $e(P) \in \mathbb{S}^{n-1}$ of the central axis of the tube $P \in \mathcal{P}_j$ must satisfy $|e(P) - e_j| \leq c_n$ where c_n is a small dimensional constant. Such inequalities have been studied in [10], [29], [16] and [22] and have proved to be very important over the last decade with significant applications in partial differential equations and especially in number theory – see for example [12], [13], [14] and [15]. The multilinear Kakeya inequality is the statement

$$\left\| \prod_{j=1}^n \left(\sum_{P_j \in \mathcal{P}_j} a_{P_j} \chi_{P_j}(x) \right)^{1/n} \right\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq C_n \prod_{j=1}^n \left(\sum_{P_j \in \mathcal{P}_j} a_{P_j} \right)^{1/n}.$$

This inequality is of the form (3) with $X = \mathbb{R}^n$, $q = n/(n-1)$, $Y_j = \mathcal{P}_j$ with counting measure, $p_j = 1$ for all j, $\beta_j = 1/n$ for all j, and $T((a_{P_j}))(x) = \sum_{P_j \in \mathcal{P}_j} a_{P_j} \chi_{P_j}(x)$. It was Guth's approach to such multilinear Kakeya inequalities in [29] which inspired the present paper.

The recent multilinear k_j-plane Kakeya inequalities, and indeed the even more general perturbed Brascamp–Lieb inequalities, both recently established by Zhang [45], also fit into our framework, the latter as a generalisation of inequality (5).

We shall return to consider these examples in some detail later in Part III. In particular we shall discuss the affine-invariant Loomis–Whitney inequality, the nonlinear Loomis–Whitney inequality and certain aspects of Brascamp–Lieb inequalities in the light of the theory we develop.

1.4. **The weighted geometric mean operator.** As we have just seen, all of our examples fit into the framework of inequality (3) with $\sum_{j=1}^d \beta_j = 1$, and with the L^{p_j} (and L^q) spaces in the Banach regime, i.e. with $p_j \geq 1$ (and $q \geq 1$). We shall therefore be concerned in this paper with norm inequalities for the *weighted geometric mean* operator

$$\mathcal{T}_\alpha : (f_1, \dots, f_d) \mapsto (T_1 f_1)^{\alpha_1} \cdots (T_d f_d)^{\alpha_d}$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$ and the α_j are positive numbers satisfying $\sum_{j=1}^d \alpha_j = 1$. That is, we shall consider inequalities of the form

$$\left\| \prod_{j=1}^d (T_j f_j)^{\alpha_j} \right\|_{L^q(X)} \leq A \prod_{j=1}^d \left\| f_j \right\|_{L^{p_j}(Y_j)}^{\alpha_j} \tag{7}$$

for nonnegative simple functions $f_j \in \mathcal{S}(Y_j)$, in the regime $p_j \geq 1$ and $q > 0$. While the case $q \geq 1$ is pertinent to our examples, we also wish to consider the case $0 < q < 1$ because this corresponds to the situation treated by Maurey in the linear setting. Throughout the paper, we use the quantities α_j to represent positive numbers whose sum is 1.

We have chosen to present our general theory for the weighted geometric mean operator $\mathcal{T}_\alpha$ – which is manifestly *not* linear in its arguments $f_1, \ldots, f_d$ – mainly because of the extra elegance and simplicity that such a treatment affords. Nevertheless, nearly all of the examples above also have equivalent strictly multilinear formulations. In particular, the multilinear Kakeya inequality of Example 5 can be re-cast as the manifestly multilinear

$$\Big\| \prod_{j=1}^{n} \Bigg(\sum_{P_j \in \mathcal{P}_j} \beta_{P_j} \chi_{P_j}(x) \Bigg) \Big\|_{L^{1/(n-1)}(\mathbb{R}^n)} \leq C \prod_{j=1}^{n} \Bigg(\sum_{P_j \in \mathcal{P}_j} \beta_{P_j} \Bigg).$$

The one class of examples that does not admit a genuinely multilinear reformulation consists of the perturbed Brascamp–Lieb inequalities which were briefly mentioned in Example 5.

Our first purpose in this paper is to propose and undertake a systematic study of the duality theory associated to the weighted geometric mean operator $\mathcal{T}_\alpha$ in the context of inequality (7) in the case $q \geq 1$, and some of its generalisations. It is hoped that the framework for this multilinear duality theory will in time have applications in a wide variety of contexts. Our second purpose is to establish suitable analogues of Maurey's theorems in the context of (7) in the case $0 < q < 1$. Interestingly, the case $q = 1$ will be central to our development of both the regimes $q \geq 1$ and $0 < q < 1$, unlike in the classical linear setting where the case $q = 1$ is essentially vacuous, and in which there appears to be no direct link between the two regimes $q \geq 1$ and $0 < q < 1$.

1.5. **A theory of multilinear duality – the regime $q \geq 1$.** We begin with the Banach regime $q \geq 1$.

One half of our duality theory – the 'easy' half – is largely contained in the following simple observation, the content of which is that if we have a certain pointwise factorisation property for the space $L^{q'}$, then the weighted geometric mean norm inequality (7) will hold.

Proposition 1.1. *Suppose that $T_j : L^{p_j}(Y_j) \to L^q(X)$ are positive linear operators, that $p_j, q \geq 1$ and that $\sum_{j=1}^d \alpha_j = 1$. Suppose that for every nonnegative $G \in L^{q'}(X)$ there exist nonnegative measurable functions g_j defined on X such that*

$$G(x) \leq \prod_{j=1}^{d} g_j(x)^{\alpha_j} \qquad \textit{a.e. on } X, \tag{8}$$

$$\textit{and} \qquad \Big\| T_j^* g_j \Big\|_{L^{p_j{}'}(Y_j)} \leq A \Big\| G \Big\|_{L^{q'}(X)} \qquad \textit{for all } j.$$

Then, for all nonnegative $f_j \in \mathcal{Y}_j$,

$$\Bigg\| \prod_{j=1}^{d} (T_j f_j)^{\alpha_j} \Bigg\|_{L^q(X)} \leq A \prod_{j=1}^{d} \Big\| f_j \Big\|_{L^{p_j}(Y_j)}^{\alpha_j};$$

that is, (7) holds, for all nonnegative $f_j \in L^{p_j}(Y_j)$.

For the (easy) proof and some discussion of this result, see the more general Proposition 2.1 below.

Rather surprisingly, the implication in Proposition 1.1 can be essentially reversed, and one of the main aims of this paper is to show that the factorisation property (8) enunciated in Proposition 1.1 is in fact *necessary* as well as sufficient for (7) to hold. This is the second half of the multilinear duality principle referred to in the abstract of the paper.

Before coming to this, however, we note that if there is a subset of X of positive measure upon which $T_j f_j$ vanishes for all $f_j \in L^{p_j}$, then this subset will play no role in the analysis of inequality (7). There is therefore no loss of generality in assuming such subsets do not exist. We formalise this notion by introducing the notion of **saturation** below.[3] In order to facilitate what follows later, we at the same time introduce the closely related notion of **strong saturation**, and also make the definitions in slightly greater generality than what is required by the current discussion. The definitions apply to linear operators $T : \mathcal{Y} \to \mathcal{M}(X)$, with $\mathcal{Y}$ a normed lattice and $(X, \mathrm{d}\mu)$ a measure space, which are positive in the sense that for every nonnegative $f \in \mathcal{Y}$ we have $Tf \geq 0$. (What is currently relevant is the fact that the space of simple functions defined on a measure space Y, together with the L^p norm for $p \geq 1$, forms a normed lattice.)

Definition 1.2. (i) We say that T **saturates** X if for each subset $E \subseteq X$ of positive measure, there exists a subset $E' \subseteq E$ with $\mu(E') > 0$ and a nonnegative $h \in \mathcal{Y}$ such that $Th > 0$ a.e. on E'.
(ii) We say that T **strongly saturates** X if there exists a nonnegative $h \in \mathcal{Y}$ such that Th is a.e. bounded away from 0 on X.

For further discussion of the relevance of these conditions, see Remarks 6 and 10 below. If T saturates a σ-finite measure space X, then there is an increasing and exhausting sequence of measurable subsets on each of which T is strongly saturating. For this and more, see Lemma 5.4 below.

Now we can state one of the main results of the paper:

Theorem 1.3. *Suppose that X and Y_j, for $j = 1, \ldots, d$, are measure spaces. Suppose that the linear operators $T_j : \mathcal{S}(Y_j) \to \mathcal{M}(X)$ are positive and that each T_j saturates X. Suppose that $p_j \geq 1$ for all j, $1 \leq q \leq \infty$ and $\sum_{j=1}^d \alpha_j = 1$. When $q = 1$ suppose additionally that X is σ-finite. Finally, suppose that*

$$\left\| \prod_{j=1}^d (T_j f_j)^{\alpha_j} \right\|_{L^q(X)} \leq A \prod_{j=1}^d \left\| f_j \right\|_{L^{p_j}(Y_j)}^{\alpha_j}$$

for all nonnegative simple functions f_j on Y_j, $1 \leq j \leq d$. Then for every nonnegative $G \in L^{q'}(X)$ there exist nonnegative measurable functions g_j on X such that

$$G(x) \leq \prod_{j=1}^d g_j(x)^{\alpha_j} \qquad \text{a.e. on } X, \tag{9}$$

and such that for each j,

$$\int_X g_j(x) T_j f_j(x) \mathrm{d}\mu(x) \leq A \|G\|_{L^{q'}} \|f_j\|_{p_j} \tag{10}$$

for all simple functions f_j on Y_j.

Remark 1. Note that we have used the formulation (10) instead of one explicitly involving T_j^* as we did in (8) because it is not immediately clear how T_j^* should be defined this context.

[3]For a related notion, see [44].

The special case of Theorem 1.3 corresponding to $q = 1$ and $G \equiv 1$ can be singled out:

Theorem 1.4. *Suppose that X and Y_j, for $j = 1, \dots, d$, are measure spaces, with X being σ-finite. Suppose that the operators $T_j : \mathcal{S}(Y_j) \to \mathcal{M}(X)$ are positive and that each T_j saturates X. Suppose that $p_j \geq 1$, $\sum_{j=1}^d \alpha_j = 1$ and that*

$$\int_X \prod_{j=1}^d (T_j f_j)^{\alpha_j} \mathrm{d}\mu \leq A \prod_{j=1}^d \left\| f_j \right\|_{L^{p_j}(Y_j)}^{\alpha_j}$$

for all nonnegative simple functions f_j on Y_j, $1 \leq j \leq d$. Then there exist nonnegative measurable functions g_j on X such that

$$1 \leq \prod_{j=1}^d g_j(x)^{\alpha_j} \qquad a.e.\ on\ X,$$

and such that for each j,

$$\int_X g_j(x) T_j f_j(x) \mathrm{d}\mu(x) \leq A \|f_j\|_{p_j} \tag{11}$$

for all simple functions f_j on Y_j.

In fact, Theorem 1.4 implies Theorem 1.3. Indeed, suppose that $1 < q \leq \infty$ and that

$$\left\| \prod_{j=1}^d (T_j f_j)^{\alpha_j} \right\|_{L^q(X)} \leq A \prod_{j=1}^d \left\| f_j \right\|_{L^{p_j}(Y_j)}^{\alpha_j}$$

for all nonnegative simple functions f_j on Y_j, $1 \leq j \leq d$. Then, for all nonnegative $G \in L^{q'}(X)$ with $\|G\|_{L^{q'}} = 1$, we have

$$\int_X \prod_{j=1}^d (T_j f_j)^{\alpha_j} G \, \mathrm{d}\mu \leq A \prod_{j=1}^d \left\| f_j \right\|_{L^{p_j}(Y_j)}^{\alpha_j}$$

for all nonnegative simple functions f_j on Y_j, $1 \leq j \leq d$. It is easy to see that if T_j saturates X with respect to the measure $\mathrm{d}\mu$, then it also does so with respect to $G \, \mathrm{d}\mu$. Now the measure $G \, \mathrm{d}\mu$ is σ-finite irrespective of whether $\mathrm{d}\mu$ is σ-finite measure. Therefore, by Theorem 1.4 applied with the measure $G \, \mathrm{d}\mu$ in place of $\mathrm{d}\mu$, there are nonnegative measurable functions γ_j such that

$$1 \leq \prod_{j=1}^d \gamma_j(x)^{\alpha_j} \qquad G \, \mathrm{d}\mu\text{-a.e. on } X,$$

and such that for each j,

$$\int_X \gamma_j(x) T_j f_j(x) G(x) \mathrm{d}\mu(x) \leq A \|f_j\|_{p_j}$$

for all simple functions f_j on Y_j. Setting $g_j = \gamma_j G$ gives the desired conclusion of Theorem 1.3 when $q > 1$. When $q = 1$, factorisation of the function 1 as in Theorem 1.4 immediately yields a corresponding factorisation of each $G \in L^\infty$.

The results described here will follow from the more general Theorem 2.2 below.

1.5.1. *An application to pointwise factorisation.* As an application of Theorem 1.3, we have the following sample result concerning pointwise factorisation of nonnegative functions in $L^2(\mathbb{R}^2)$:

Theorem 1.5. *Let v_1, v_2 and v_3 be unit vectors in $\mathbb{R}^2$ with angle $2\pi/3$ between each pair. Then, for every nonnegative $G \in L^2(\mathbb{R}^2)$, there exist nonnegative locally integrable functions g_1, g_2 and g_3 such that*

$$G(x) \leq g_1(x)^{1/3} g_2(x)^{1/3} g_3(x)^{1/3} \quad a.e.$$

and, for each j, for almost every line l in $\mathbb{R}^2$ which is parallel to v_j,

$$\int_l g_j \mathrm{d}\lambda \leq \|G\|_2$$

where $\mathrm{d}\lambda$ denotes Lebesgue measure on l.

For further details, and many more results of this nature, see Section 10.1 below.

1.6. **Multilinear Maurey-type factorisation – the regime $0 < q < 1$.** We now state a multilinear Maurey-type theorem.

Theorem 1.6. *Suppose that X and Y_j, for $j = 1, \dots, d$, are measure spaces and that X is σ-finite. Suppose that the operators $T_j : \mathcal{S}(Y_j) \to \mathcal{M}(X)$ are positive and that each T_j saturates X. Suppose that $p_j \geq 1$, $0 < q < 1$, $\sum_{j=1}^d \alpha_j = 1$ and that*

$$\left\| \prod_{j=1}^d (T_j f_j)^{\alpha_j} \right\|_{L^q(X)} \leq A \prod_{j=1}^d \left\| f_j \right\|_{L^{p_j}(Y_j)}^{\alpha_j} \tag{12}$$

for all nonnegative simple functions $f_j \in \mathcal{Y}_j$, $1 \leq j \leq d$. Then there exist nonnegative measurable functions g_j on X such that

$$\| \prod_{j=1}^d g_j(x)^{\alpha_j} \|_{L^{q'}(X)} = 1 \tag{13}$$

and such that for each j,

$$\int_X g_j(x) T_j f_j(x) \mathrm{d}\mu(x) \leq A \|f_j\|_{L^{p_j}(Y_j)} \tag{14}$$

for all simple functions f_j on Y_j.

We shall give the proof of this result, as a consequence of Theorem 1.4, in Section 2.3 below. Conversely, it is easy to see using Hölder's inequality that if there exist g_j such that (13) and (14) hold, then so does (12).

This result can be seen as a factorisation result in the spirit of Maurey: if we let $S_j f_j(x) = g_j(x) T_j f_j(x)$, $\mathcal{S}_\alpha(f_1, \dots, f_d) = \prod_{j=1}^d (S_j f_j)^{\alpha_j}$ and $g(x) = \prod_{j=1}^d g_j(x)^{\alpha_j}$, then

$$\mathcal{T}_\alpha = M_{g^{-1}} \circ \mathcal{S}_\alpha$$

where

$$\|S_j\|_{L^{p_j} \to L^1} \leq A$$

for all j, and

$$\|M_{g^{-1}}\|_{L^1 \to L^q} = \|g\|_{q'}^{-1} = 1.$$

In fact it has the rather strong conclusion that *each* S_j is bounded from $L^{p_j}(Y_j)$ to $L^1(\mathrm{d}\mu)$ with constant at most A (rather than the much weaker corresponding statement for the geometric mean $\mathcal{S}_\alpha$ alone).

Other, different, versions of multilinear Maurey-type theorems have been studied. See for example [38] and [23].

Remark 2. One may use the classical linear Maurey–Nikisin–Stein factorisation theory of positive operators ([35], Proposition 9) to upgrade conclusions (10) of Theorem 1.3, (11) of Theorem 1.4 and (14) of Theorem 1.6. Indeed, each of these conclusions states that T_j maps into a weighted L^1-space, and we can upgrade each to boundedness of T_j into a suitable weighted L^{p_j}-space. For example, in the context of Theorem 1.4, we may conclude that there exist nonnegative measurable functions ϕ_j on X such that

$$\int_X \left(\prod_{j=1}^d \phi_j(x)^{\alpha_j/p_j} \right)^{-1/(\sum_{j=1}^d \alpha_j/p_j{}')} \mathrm{d}\mu(x) \leq 1$$

and

$$\left(\int_X |T_j f_j|^{p_j} \phi_j \mathrm{d}\mu \right)^{1/p_j} \leq A \|f_j\|_{p_j}$$

for all simple f_j on Y_j. See also Remark 15 below. For stronger statements of this kind see the forthcoming [20].

1.7. **Structure of the paper.** The paper is divided into three parts.

In Part I (Sections 2–5) we present the theory of multilinear duality and factorisation and prove the main theorems.

In Section 2 we state and discuss the main results at some length. The principal result is Theorem 2.2. Taken together with Proposition 2.1, Theorem 2.2 forms the statement of the multilinear duality principle referred to in the abstract of the paper. (Theorem 1.3 and Proposition 1.1 presented in this introduction are more readily digested versions of Theorem 2.2 and Proposition 2.1 respectively.) The multilinear Maurey factorisation theorem, Theorem 2.3, is proved as a consequence of Theorem 2.2. (A more digestible version of Theorem 2.3 is found as Theorem 1.6 in this introduction.)

In Section 3 we give a proof of a finitistic case of Theorem 2.2 which recognises and emphasises its structure as a convex optimisation or minimax problem. This perspective sets the scene for the remainder of the theoretical part of the paper. In this case, none of the functional-analytic and measure-theoretic difficulties that we encounter later are present. However, Section 3 is not strictly speaking logically necessary for the development of the theory.

In Section 4 we begin to address the proof of Theorem 2.2. Our strategy will be to first consider the setting of finite measure spaces. Theorem 4.1 gives the main result in this case, and it represents a crucial step in the proof of Theorem 2.2. Already in Theorem 4.1 we are faced with substantial functional-analytic and measure-theoretic difficulties. These derive from the need to establish certain compactness statements necessary for the application of a minimax theorem. Briefly, they involve working with the dual space of $L^\infty(X)$, and dealing with various issues in the theory of finitely additive measures.

In Section 5 we give the details of the proofs of Theorem 4.1 and Theorem 2.2. We begin with a couple of technical but very important lemmas. Next, we pass to the proof of the finite-measure result, Theorem 4.1, via the minimax theory. Finally, for general σ-finite X, we "glue together" factorisations obtained for subsets X of finite measure via Theorem 4.1, and we obtain the factorisations needed for Theorem 2.2.

In a much shorter Part II, we begin to explore connections with other topics – in particular the theory of interpolation in Sections 6 and 7, and the extent to which the theory might apply in the context of more general multilinear operators in Section 8.

Finally, in Part III, we revisit the examples discussed earlier in this introduction in the light of the multilinear duality theory which has been developed. In Section 9 we give factorisation-based proofs of the affine-invariant Loomis–Whitney inequality (see Section 9.2) and the sharp nonlinear Loomis–Whitney inequality (see Section 9.3). In Section 10.1 we pose an interesting question related to the sharp Young convolution inequality and geometric Brascamp–Lieb inequalities, while in Section 10.2 we describe an algorithm for factorising the general Brascamp–Lieb inequality. In Section 11 we revisit the multilinear Kakeya inequality which inspired the paper in the light of the findings of Section 8, and make an observation about the size of the constant in the finite-field version of the multilinear Kakeya inequality which is derived from our methods.

1.8. **Future work.** In a sequel [20] to this paper, we broaden the scope of the multilinear duality theory from positive to potentially oscillatory multilinear inequalities. In particular, we extend the multilinear duality and factorisation theorem (Theorem 2.2) to non-positive operators, and indeed refine it when the normed spaces have certain additional geometric properties (p-convexity in case of positive operators, Rademacher type in case of non-positive operators). Moreover, in forthcoming work [21], we will give an alternative proof of Theorem 2.2 which bypasses the need to consider $(L^\infty)^*$.

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Part I. Statements and proofs of the theorems

2. Statement and discussion of the main results

It turns out that the theory we shall develop is most naturally presented in a more general setting. Moreover, limiting ourselves to the Lebesgue spaces L^{p_j} and L^q in the multilinear duality theory is unnecessarily restrictive. For example, one may wish to consider multilinear inequalities of the form (7) in which the L^{p_j} and L^q spaces are replaced by certain Lorentz spaces, Orlicz spaces or mixed-norm spaces, especially if the inequality under consideration is an endpoint inequality. We therefore introduce a more general framework in which we consider suitable spaces $\mathcal{X}$ and $\mathcal{Y}_j$ corresponding to $L^q(X)$ and $L^{p_j}(Y_j)$ respectively. Thus, for $\sum_{j=1}^d \alpha_j = 1$, which we recall is a standing convention, we now consider inequalities of the form

$$\left\| \prod_{j=1}^{d} (T_j f_j)^{\alpha_j} \right\|_{\mathcal{X}} \leq A \prod_{j=1}^{d} \left\| f_j \right\|_{\mathcal{Y}_j}^{\alpha_j}. \tag{15}$$

Each $\mathcal{Y}_j$ will be an (abstract) normed lattice – such as L^1 or L^{p_j} for $p_j \geq 1$. On the other hand, we will take $\mathcal{X}$ to be Banach space of locally integrable[4] functions defined on X, which contains the simple functions, and is such that if $f \in \mathcal{M}(X)$ and $g \in \mathcal{X}$ satisfy $|f(x)| \leq |g(x)|$ a.e., then $f \in \mathcal{X}$ and $\|f\|_{\mathcal{X}} \leq \|g\|_{\mathcal{X}}$. We may as well also assume that X is a complete measure space. Our measure space X will also be assumed to the σ-finite; these properties together then identify $\mathcal{X}$ as a **Köthe space**, see [32], Vol. II, p. 28. *We shall from now on assume that $\mathcal{X}$ is a Köthe space without further mention.* Natural examples of Köthe spaces include L^q for $1 \leq q \leq \infty$. We shall need a suitable primordial dual of $\mathcal{X}$, denoted by $\mathcal{X}'$, and defined to be

$$\mathcal{X}' = \{g \in \mathcal{M}(X) \,:\, \|g\|_{\mathcal{X}'} = \sup_{\|f\|_{\mathcal{X}} \leq 1} \int |fg| \,\mathrm{d}\mu < \infty\}.$$

The space $\mathcal{X}'$ is usually called the **Köthe dual** of $\mathcal{X}$. If $\mathcal{X} = L^q$ for $1 \leq q \leq \infty$, then $\mathcal{X}' = L^{q'}$ where $1/q + 1/q' = 1$. It is clear that $\mathcal{X}'$ is a linear space which contains the simple functions (as $\mathcal{X}$ is contained in the class of locally integrable functions) and is contained in the class of locally integrable functions (as $\mathcal{X}$ contains the simple functions). The quantity $\|g\|_{\mathcal{X}'}$ defines a norm on $\mathcal{X}'$. While by definition we always have the Hölder inequality

$$|\int fg \,\mathrm{d}\mu| \leq \|f\|_{\mathcal{X}} \|g\|_{\mathcal{X}'},$$

it may or may not be the case that $\mathcal{X}'$ is **norming** (for $\mathcal{X}$), i.e. that

$$\|f\|_{\mathcal{X}} = \sup\{\, |\int fg \,\mathrm{d}\mu| \,:\, \|g\|_{\mathcal{X}'} \leq 1\} \tag{16}$$

holds for all $f \in \mathcal{X}$.[5] The Köthe dual $\mathcal{X}'$ is always isometrically embedded in the norm-dual $\mathcal{X}^*$, but the two spaces may not coincide in general.

From now on, we shall adopt once and for all the convention that all named functions ($f, g, h, F, G, H, \beta, G, S, \psi$ etc., often adorned with subscripts) are assumed to be nonnegative. The two exceptions to this are the functions L and Λ appearing in the proofs of the main results.

2.1. **Duality theory – easy half.** The easy half of our duality theory is expressed in the following simple observation, the content of which is that if we have a certain factorisation property for the Köthe dual $\mathcal{X}'$, then the weighted geometric mean norm inequality (15) will hold.

Proposition 2.1. *Suppose that $\mathcal{X}$ is a Köthe space whose Köthe dual $\mathcal{X}'$ is norming, and that $\mathcal{Y}_j$ are normed lattices. Suppose that $T_j : \mathcal{Y}_j \to \mathcal{X}$ are positive linear operators. Suppose furthermore*

[4] That is, $\int_E |f| \mathrm{d}\mu < \infty$ whenever $\mu(E) < \infty$.

[5] By a result of Lorentz and Luxemburg (see [32], Vol. II, p. 29), if X is a Köthe space, $\mathcal{X}'$ is norming if and only if $\mathcal{X}$ has the so-called Fatou property, that is, whenever $f_n \in \mathcal{X}$ are such that $f_n \to f$ a.e., with $f_{n+1} \geq f_n \geq 0$, then $\|f_n\|_{\mathcal{X}} \to \|f\|_{\mathcal{X}}$. This is automatic when $\mathcal{X}$ is separable. If $\mathcal{X}$ is L^∞ then (16) holds by inspection since $\mathcal{X}'$ is simply L^1. We shall need the notion of norming only for Proposition 2.1.

that for every nonnegative $G \in \mathcal{X}'$ there exist nonnegative measurable functions g_j on X such that

$$G(x) \leq \prod_{j=1}^{d} g_j(x)^{\alpha_j} \qquad a.e. \text{ on } X, \tag{17}$$
$$\text{and} \qquad \Big\| T_j^* g_j \Big\|_{\mathcal{Y}_j^*} \leq A \Big\| G \Big\|_{\mathcal{X}'} \qquad \text{for all } j.$$

Then, for all nonnegative $f_j \in \mathcal{Y}_j$

$$\left\| \prod_{j=1}^{d} (T_j f_j)^{\alpha_j} \right\|_{\mathcal{X}} \leq A \prod_{j=1}^{d} \Big\| f_j \Big\|_{\mathcal{Y}_j}^{\alpha_j}.$$

That is, (15) holds, for all nonnegative $f_j \in \mathcal{Y}_j$.

Proof. Take $f_j \in \mathcal{Y}_j$ for $j = 1, \ldots, d$, and $G \in \mathcal{X}'$ with $\|G\|_{\mathcal{X}'} \leq 1$. Then

$$\begin{aligned}\int_X G(x) \prod_{j=1}^{d} (T_j f_j)^{\alpha_j} \mathrm{d}\mu(x) &\leq \int_X \prod_{j=1}^{d} g_j(x)^{\alpha_j} \prod_{j=1}^{d} T_j f_j(x)^{\alpha_j} \mathrm{d}\mu(x) = \int_X \prod_{j=1}^{d} (g_j(x) T_j f_j(x))^{\alpha_j} \mathrm{d}\mu(x) \\ &\leq \prod_{j=1}^{d} \left(\int_X g_j(x) T_j f_j(x) \mathrm{d}\mu(x) \right)^{\alpha_j} = \prod_{j=1}^{d} \big((T_j^* g_j)(f_j) \big)^{\alpha_j} \\ &\leq \prod_{j=1}^{d} \Big(\|T_j^* g_j\|_{\mathcal{Y}_j^*} \|f_j\|_{\mathcal{Y}_j} \Big)^{\alpha_j} \leq \prod_{j=1}^{d} \big(A \|G\|_{\mathcal{X}'} \|f_j\|_{\mathcal{Y}_j} \big)^{\alpha_j} \leq A \prod_{j=1}^{d} \|f_j\|_{\mathcal{Y}_j}^{\alpha_j}\end{aligned}$$

where the inequalities follow in order from the first condition of (17), Hölder's inequality, the second condition of (17), and the assumption that $\|G\|_{\mathcal{X}'} \leq 1$. The proposition now follows by taking the supremum over all such G, using the fact that $\mathcal{X}'$ is norming for $\mathcal{X}$. □

Remark 3. If the spaces $\mathcal{Y}_j$ are complete, the assumption that $T_j : \mathcal{Y}_j \to \mathcal{X}$ is positive automatically implies that T_j is bounded,[6] and so the adjoint operator T_j^* is well-defined. If not, we interpret the second condition of (17) as

$$\int_X g_j(x) T_j f_j(x) \mathrm{d}\mu(x) \leq A \|G\|_{\mathcal{X}'} \|f_j\|_{\mathcal{Y}_j} \tag{18}$$

for $f_j \in \mathcal{Y}_j$ and $j = 1, \ldots, d$, and we can still conclude the validity of (15) for functions $f_j \in \mathcal{Y}_j$, as the proof clearly demonstrates.

Remark 4. If the T_j are known to be bounded, it is immediate that (15) holds with A replaced by $\prod_{j=1}^{d} \|T_j\|^{\alpha_j}$.[7] However, the best constant A in (17) will in general be much smaller, and this assertion is the main content of Proposition 2.1.

Remark 5. Observe that Proposition 2.1 does not require any topological structure of the space X, only its nature as a measure space.

Remark 6. Notice that in order for the proof to go through, we only require that the factorisation property – i.e. the first condition of (17) – holds for those x which contribute to $\int_X G(x) \prod_{j=1}^{d} (T_j f_j)^{\alpha_j} \mathrm{d}\mu(x)$ for some functions $f_j \in \mathcal{Y}_j$. In other words, if a set $E \subseteq X$ with

[6] Indeed, if not, we can find nonnegative f_n with $\|f_n\| \leq 2^{-n}$ but $\|T_j f_n\| \geq 2^n$. So for each n, $2^n \leq \|T_j f_n\| \leq \|T_j(\sum_{n=1}^{\infty} f_n)\| \leq C$ for some finite C since $\sum_{n=1}^{\infty} f_n \in \mathcal{Y}_j$ (because $\mathcal{Y}_j$ is a Banach space). This is a contradiction.

[7] This follows since $\| \prod_{j=1}^{d} h_j^{\alpha_j} \| \leq \prod_{j=1}^{d} \|h_j\|^{\alpha_j}$ which in turn follows from the case where each $\|h_j\| = 1$, which itself follows by Young's numerical inequality and the triangle inequality.

$\mu(E) > 0$ has the property that for all choices f_j of nonnegative functions in $\mathcal{Y}_j$, $\prod_{j=1}^d (T_j f_j)(x)^{\alpha_j} = 0$ a.e. on E, then E will play no role in the analysis. There is therefore no loss of generality in assuming such sets do not exist. So we may assume without loss of generality that for all $E \subseteq X$ with $\mu(E) > 0$, there exist nonnegative $f_j \in \mathcal{Y}_j$ such that "$\prod_{j=1}^d T_j f_j(x)^{\alpha}_j = 0$ a.e. on E" fails – i.e. such that there exists $E' \subseteq E$, with $\mu(E') > 0$ such that $\prod_{j=1}^d T_j f_j(x)^{\alpha}_j > 0$ on E'. That is, we may assume that for all $E \subseteq X$ with $\mu(E) > 0$, there exists $E' \subseteq E$ with $\mu(E') > 0$, and, for each j, a nonnegative $f_j \in \mathcal{Y}_j$ such that for all $x \in E'$, $T_j f_j(x) > 0$. This condition is equivalent to the formally slightly weaker condition that for each j, for all $E \subseteq X$ with $\mu(E) > 0$, there exists $E' \subseteq E$ with $\mu(E') > 0$ and nonnegative $f_j \in \mathcal{Y}_j$ such that for all $x \in E'$, $T_j f_j(x) > 0$.[8] But this is simply the statement that each T_j saturates X, as in Definition 1.2. It is unsurprising that we will require saturation when it comes to formulating and proving the converse statement.

Remark 7. Note that in place of

$$\left\| T_j^* g_j \right\|_{\mathcal{Y}_j^*} \leq A \left\| G \right\|_{\mathcal{X}'} \qquad \text{for all } j.$$

we could have assumed the (formally weaker) condition

$$\prod_{j=1}^d \left\| T_j^* g_j \right\|_{\mathcal{Y}_j^*}^{\alpha_j} \leq A \left\| G \right\|_{\mathcal{X}'}.$$

(A homogeneity argument shows that the two conditions are indeed equivalent.)

Remark 8. Similarly, it suffices to suppose a formally weaker hypothesis ("weak factorisation"), namely that for every $G \in \mathcal{X}'$, there exist measurable functions g_{jk} on X such that

$$G(x) \leq \sum_k \prod_{j=1}^d g_{jk}(x)^{\alpha_j}$$

a.e. on X, and

$$\sum_k \left\| T_j^* g_{jk} \right\|_{\mathcal{Y}_j^*} \leq A \left\| G \right\|_{\mathcal{X}'}$$

for all j. But if this holds, and if we define $g_j = \sum_k g_{jk}$, Minkowski's inequality and Hölder's inequality yield (17). So this observation does not represent a genuine broadening of the scope of Proposition 2.1.

Remark 9. The argument for Proposition 2.1 was effectively given by Guth, in a less abstract form, in his proof the endpoint multilinear Kakeya inequality [29]. However, any strategy which includes an application Proposition 2.1 to establish an inequality of the form (15) involves the potentially difficult matter of first finding a suitable factorisation. Indeed, the main work of [29] consisted precisely in finding such. In this context see also [22].

2.2. **Duality theory – difficult half.** As suggested above, the implication in Proposition 2.1 can be essentially reversed, and a principal aim of this paper is to show that the factorisation property (17) enunciated in Proposition 2.1 is in fact *necessary* as well as sufficient for (15) to hold under very mild hypotheses. More precisely we prove:

Theorem 2.2 (Multilinear duality and factorisation theorem)**.** *Suppose that* $(X, \mathrm{d}\mu)$ *is a* σ*-finite measure space,* $\mathcal{X}$ *is a Köthe space of measurable functions on* X*,* $\mathcal{Y}_j$ *are normed lattices, and*

[8]If the latter condition holds, apply it to each j in turn to obtain the former condition.

$T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ are positive linear maps. Suppose that each T_j saturates X. Suppose that

$$\left\| \prod_{j=1}^{d} (T_j f_j)^{\alpha_j} \right\|_{\mathcal{X}} \leq A \prod_{j=1}^{d} \left\| f_j \right\|_{\mathcal{Y}_j}^{\alpha_j}$$

for all nonnegative $f_j \in \mathcal{Y}_j$, $1 \leq j \leq d$. Then there exists a weight function[9] *w on X such that for every nonnegative $G \in \mathcal{X}'$, there exist nonnegative measurable functions $g_j \in L^1(X, w\mathrm{d}\mu)$ such that*

$$G(x) \leq \prod_{j=1}^{d} g_j(x)^{\alpha_j} \qquad \text{a.e. on } X, \tag{19}$$

and such that for each j,

$$\int_X g_j(x) T_j f_j(x) \mathrm{d}\mu(x) \leq A \|G\|_{\mathcal{X}'} \|f_j\|_{\mathcal{Y}_j} \tag{20}$$

for all $f_j \in \mathcal{Y}_j$.

Remark 10. The hypothesis that each T_j saturates X is very natural as pointed out in Remark 6 above. Indeed, for the reasons set out there, without this hypothesis we cannot expect the conclusion to hold. Needless to say, it will play an important role in the proof of Theorem 2.2. In particular, the weight function w arises as a consequence of the saturation hypothesis. For its construction, see Section 5.3 below. If $\mu(X)$ is finite and the T_j strongly saturate X, we can take w to be the constant function 1, see Theorem 4.1 below.

Remark 11. In the case $d = 1$ the factorisation is trivial, and (20) is simply the usual duality relation corresponding to (2).

Remark 12. If there exist g_j satisfying (19) and (20), then by making one of the g_j smaller if necessary, we can find g_j satisfying (19) with *equality* in addition to (20).

Remark 13. We emphasise that the constant A appearing in (20) is *precisely* the constant A occuring in the hypothesis.

Remark 14. As in the case of Theorem 1.3, the general case of Theorem 2.2 follows from the special case in which $\mathcal{X} = L^1(X)$. Indeed, placing ourselves under the assumptions of the general case, let $G \in \mathcal{X}'$ have norm 1, and observe that by Hölder's inequality we have

$$\int_X \prod_{j=1}^{d} (T_j f_j)^{\alpha_j} G(x) \mathrm{d}\mu(x) \leq A \prod_{j=1}^{d} \left\| f_j \right\|_{\mathcal{Y}_j}^{\alpha_j}$$

for all $f_j \in \mathcal{Y}_j$, $1 \leq j \leq d$. This is the main hypothesis of the special case, but with respect to the measure $G\mathrm{d}\mu$ instead of $\mathrm{d}\mu$. It is easily verified that $G\mathrm{d}\mu$ is a σ-finite measure, and that if T_j saturates X with respect to $\mathrm{d}\mu$, it also does so likewise with respect to $G\mathrm{d}\mu$, and similarly for strong saturation. We may therefore conclude from the L^1 case of Theorem 2.2 that there exist nonnegative measurable γ_j such that

$$\prod_{j=1}^{d} \gamma_j(x)^{\alpha_j} \geq 1 \text{ a.e. } G\mathrm{d}\mu$$

and such that

$$\int_X \gamma_j(x) T_j f_j(x) G(x) \mathrm{d}\mu(x) \leq A \|f_j\|_{\mathcal{Y}_j}$$

[9] i.e. a measurable function w with $w(x) > 0$ a.e.

for all $f_j \in \mathcal{Y}_j$. Setting $g_j = G\gamma_j$, the easy observation that

$$\prod_{j=1}^{d} g_j(x)^{\alpha_j} \geq G(x) \text{ a.e. } \mathrm{d}\mu$$

completes the argument. (This argument does not directly place the g_j in a weighted L^1-space, but this feature can in any case be recovered from inequality (20).) However, this observation does not simplify the proof of Theorem 2.2, and we therefore establish the general case directly.

Remark 15. Following on from Remark 2 above, if the spaces $\mathcal{Y}_j$ are additionally supposed to be p_j-convex for some $p_j \geq 1$, we may use the classical linear Maurey–Nikisin–Stein theory for positive operators to upgrade conclusion (20) of Theorem 2.2 to boundedness of each T_j into a suitably weighted L^{p_j}-space. A similar remark applies in the context of Theorem 1.6 below. This perspective is further explored in [20].

The proof of Theorem 2.2 is highly nonconstructive and comes about as a result of duality methods in the theory of convex optimisation which ultimately rely upon a form of the minimax principle. For the details of the proof see Sections 3, 4 and 5 below. Nevertheless, in some cases, constructive factorisations can be given, and in other cases, the existence of the factorisation raises interesting questions and links with other areas of analysis. See Sections 6, 8, 9.2, 9.3 and 10.

2.3. **Multilinear Maurey-type theory.** In this section we state and prove a slight generalisation of Theorem 1.6, using the case $\mathcal{X} = L^1(X)$ of Theorem 2.2. Interestingly, the classical Maurey theorem follows easily from Theorem 2.2 specialised to the *bilinear* case $d = 2$ in which one of the normed lattices is one-dimensional. (Therefore, the case $d = 1$ of what follows is *not* trivial, in contrast to the situation for Theorem 2.2.)

Theorem 2.3 (Multilinear Maurey-type theorem)**.** *Suppose $(X, \mathrm{d}\mu)$ is a σ-finite measure space, $\mathcal{Y}_j$ are normed lattices, and $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ are positive linear maps. Suppose that each T_j saturates X. Let $0 < q < 1$, and suppose*

$$\left\| \prod_{j=1}^{d} (T_j f_j)^{\alpha_j} \right\|_{L^q(X)} \leq A \prod_{j=1}^{d} \left\| f_j \right\|_{\mathcal{Y}_j}^{\alpha_j} \tag{21}$$

for all nonnegative $f_j \in \mathcal{Y}_j$, $1 \leq j \leq d$. Then there exist nonnegative measurable functions g_j on X such that

$$\| \prod_{j=1}^{d} g_j(x)^{\alpha_j} \|_{L^{q'}(X)} = 1 \tag{22}$$

and such that for each j,

$$\int_X g_j(x) T_j f_j(x) \mathrm{d}\mu(x) \leq A \|f_j\|_{\mathcal{Y}_j} \tag{23}$$

for all $f_j \in \mathcal{Y}_j$.

It is an easy exercise using Hölder's inequality to show that if there exist g_j such that (22) and (23) hold, then (21) also holds. As in the case of Theorem 1.6, Theorem 2.3 admits an interpretation as a statement about factorisation of operators, see Section 1.6.

Proof. The main hypothesis is that

$$\int_X \prod_{j=1}^{d} (T_j f_j)^{\alpha_j q} \mathrm{d}\mu \leq A^q \prod_{j=1}^{d} \left\| f_j \right\|_{\mathcal{Y}_j}^{\alpha_j q}$$

for all $f_j \in \mathcal{Y}_j$. Let $\beta_j = \alpha_j q$ for $1 \leq j \leq d$ and let $\beta_{d+1} = 1 - \sum_{j=1}^{d} \beta_j = 1 - q > 0$. Let $Y_{d+1} = \{0\}$ and let $\mathcal{Y}_{d+1}$ be the trivial normed lattice $\mathbb{R}$ defined on the singleton measure space $\{0\}$. Let $T_{d+1} : \mathcal{Y}_{d+1} \to \mathcal{M}(X)$ be the linear map $\lambda \mapsto \lambda \mathbf{1}$ where $\mathbf{1}$ denotes the constant function taking the value 1 on X. Then we have

$$\int_X \prod_{j=1}^{d+1} (T_j f_j)^{\beta_j} \mathrm{d}\mu \leq A^q \prod_{j=1}^{d+1} \left\| f_j \right\|_{\mathcal{Y}_j}^{\beta_j}$$

for all $f_j \in \mathcal{Y}_j$. So by Theorem 2.2 in the case $\mathcal{X} = L^1(X)$ (and with $d+1$ in place of d, see also Remark 12 above), we conclude that there exist measurable functions $G_1, \ldots, G_{d+1}$ such that

$$\prod_{j=1}^{d+1} G_j(x)^{\beta_j} = 1 \qquad \text{a.e. on } X, \tag{24}$$

and such that for each $1 \leq j \leq d+1$,

$$\int_X G_j(x) T_j f_j(x) \mathrm{d}\mu(x) \leq A^q \|f_j\|_{\mathcal{Y}_j} \tag{25}$$

for all $f_j \in \mathcal{Y}_j$.

For $1 \leq j \leq d$, set $g_j(x) = A^{1-q} G_j(x)$; then (25) immediately gives (23) for $1 \leq j \leq d$. By (24) we have

$$\prod_{j=1}^{d} g_j(x)^{\alpha_j} = A^{1-q} G_{d+1}(x)^{(q-1)/q} \qquad \text{a.e. on } X,$$

while (25) for $j = d+1$ gives

$$\int_X G_{d+1}(x) \mathrm{d}\mu(x) \leq A^q.$$

Combining these last two relations gives (22) as desired. □

3. Discrete case: a convex optimisation problem

3.1. **Basic set-up.** The idea behind the proof of Theorem 2.2 is to view problem (19) and (20) as a convex optimisation problem. That is, we replace the number A in (20) by a variable K and seek to minimise over K. To illustrate how this works, we first prove the theorem in a model case when X and Y_j are finite sets endowed with counting measure and $\mathcal{Y}_j = L^1(Y_j)$ for $j = 1, \ldots, d$. One reason for doing this case first is that there are no measure-theoretic or functional-analytical difficulties to be dealt with in this setting, and indeed $\mathcal{X}' = \mathcal{X}^*$ is simply the class of all functions defined on X with the norm dual to that of $\mathcal{X}$. It therefore allows us to emphasise the nature of the problem as one concerning convex optimisation.

The minimisation problem we propose to examine now reads as follows. Fix $G : X \to [0, \infty)$ and consider

$$\gamma = \inf_{K, g_j} K$$

(26) $$\text{such that} \quad G(x) \le \prod_{j=1}^{d} g_j(x)^{\alpha_j} \quad \text{for all } x \in X \text{ and}$$

$$\max_{y_j \in Y_j} T_j^* g_j(y_j) \le K \|G\|_{\mathcal{X}^*} \quad \text{for all } j = 1, \dots, d.$$

We note that this is a convex optimisation problem since we are minimising a convex, in fact linear, function on the convex domain consisting of the $(d+1)$-tuples (K, g_j) satisfying the constraints in (26). The convexity of this domain follows from the fact that the second set of inequalities is linear in the arguments K and g_j, and the operation of taking the geometric mean on the right hand side of the first set of inequalities is a concave function. We note that the set of (K, g_j) satisfying the constraints in (26) is not empty and that we can in fact find (K, g_j) satisfying these constraints with strict inequality by taking each g_j to be $2G + 1$ and letting K be sufficiently large. Thus problem (26) satisfies what is known as *Slater's condition.* (We do not give full details here as the discussion will eventually be subsumed into that of the next section.) In particular we certainly have $\gamma < +\infty$.

We therefore follow a standard approach to convex optimisation problems, see for example [17]. We introduce Lagrange multipliers ψ and h_j, where $\psi : X \to \mathbb{R}_+$ (for the first set of constraints), and $h_j : Y_j \to \mathbb{R}_+$ (for the second set). Note that we are only interested case where these functions take nonnegative values since each of the constraints is an inequality constraint. We then introduce the Lagrangian functional

(27) $$L = K + \sum_{x \in X} \psi(x) \left(G(x) - \prod_{j=1}^{d} g_j(x)^{\alpha_j} \right) + \sum_{j=1}^{d} \sum_{y_j \in Y_j} h_j(y_j)(T_j^* g_j(y_j) - K\|G\|_{\mathcal{X}^*}).$$

We emphasise that this function and the corresponding one defined in the proof of the general case are the only functions which we allow to take negative values.

For nonnegative K, g_j, ψ, and h_j we now consider the two problems[10]

$$\gamma_{\mathcal{L}} = \inf_{K, g_j} \sup_{\psi, h_j} L \qquad \text{and} \qquad \eta = \sup_{\psi, h_j} \inf_{K, g_j} L$$

called the primal problem and the dual problem respectively. We shall show that (i) the problem for γ is identical to the problem for $\gamma_{\mathcal{L}}$, (ii) $\eta \le A$ where A is any number such that inequality (15) holds, and (iii) $\eta = \gamma_{\mathcal{L}}$ (it is obvious that $\eta \le \gamma_{\mathcal{L}}$). Finally, we show that the infimum in the definition of γ is attained, and this will complete the proof of the theorem in the special case.

3.2. **Identification of the problems for γ and $\gamma_{\mathcal{L}}$.** We begin by studying $\gamma_{\mathcal{L}}$. Fix $K \ge 0$ and g_j and consider $\sup_{\psi, h_j} L$. Suppose that any of the conditions in (26) is not satisfied at some point. Then take the relevant function ψ or h_j for some j to have value $t > 0$ at a point where an inequality fails and let all of the functions be zero everywhere else. Then let $t \to \infty$ and notice that $\sup_{\psi, h_j} L$ goes to $+\infty$ since t is multiplied by a positive number. So if $\sup_{\psi, h_j} L < +\infty$ we must have that the conditions of (26) are satisfied. Conversely, if these conditions are satisfied then all factors multiplying $\psi(x)$ and $h_j(y_j)$ for any x and y_j are non-positive so the supremum is attained by taking them all to equal 0. So, for each fixed (K, g_j), we have that $\sup_{\psi, h_j} L < +\infty$ if and only if the conditions in (26) hold, in which case, $\sup_{\psi, h_j} L = K$. Thus we see that the

[10] The subscript $\mathcal{L}$ in $\gamma_{\mathcal{L}}$ indicates that we are looking at the Lagrangian version of the problem as opposed to the original version which has γ without a subscript.

problem for $\gamma_{\mathcal{L}}$ is identical to problem (26), yielding $\gamma_{\mathcal{L}} = \gamma$. Moreover the infimum in the definition of γ is attained if and only if the infimum in the definition of $\gamma_{\mathcal{L}}$ is attained.

3.3. **Proof that $\eta \le A$.** We rearrange L as follows:

$$\begin{aligned} L = \sum_{x\in X} \psi(x)G(x) + K\left(1 - \|G\|_{\mathcal{X}^*} \sum_{j=1}^{d} \sum_{y_j \in Y_j} h_j(y_j)\right) \\ + \sum_{x\in X}\left(\sum_{j=1}^{d} g_j(x)T_jh_j(x) - \prod_{j=1}^{d} g_j(x)^{\alpha_j}\psi(x)\right) \end{aligned} \tag{28}$$

Let us fix ψ and h_j and consider $\inf_{K,g_j} L$. First of all, note that $\inf_{K,g_j} L = -\infty$ unless

$$\|G\|_{\mathcal{X}^*} \sum_{j=1}^{d} \sum_{y_j\in Y_j} h_j(y_j) \le 1 \tag{29}$$

since if this inequality fails then the term multiplying K in L is negative and so by taking $g_j = 0$ and letting K go to infinity we get that $\inf_{K,g_j} L = -\infty$. Also note that $\inf_{K,g_j} L = -\infty$ unless

$$\psi(x) \le \prod_{j=1}^{d} (\alpha_j^{-1} T_j h_j(x))^{\alpha_j} \tag{30}$$

for all $x \in X$. Seeing this is a matter of choosing $g_j(x)$ to balance the arithmetic-geometric mean inequality. Specifically, suppose that this condition (30) fails at a point x_0. Then we let $K = 0$ and $g_j(x) = 0$ for all $x \ne x_0$ and all $j = 1, \dots, d$. There are now two cases to consider. Firstly, if there exists an index j_0 such that $T_{j_0}h_{j_0}(x_0) = 0$ then we take $g_j(x_0) = 1$ for all $j \ne j_0$ and $g_{j_0}(x_0) = t > 1$. Then

$$L = \sum_{x\in X} \psi(x)G(x) + \sum_{\substack{j=1 \\ j\ne j_0}}^{d} T_jh_j(x_0) - t^{\alpha_{j_0}}\psi(x_0).$$

Since $\alpha_{j_0} > 0$ we can let t go to infinity and see that $\inf_{K,g_j} L = -\infty$. In the other case we have that $T_jh_j(x_0) > 0$ for all $j = 1, \dots, d$. Then we let

$$g_j(x_0) = t\alpha_j((T_jh_j)(x_0))^{-1} \prod_{j'=1}^{d} (\alpha_{j'}^{-1} T_{j'}h_{j'})(x_0)^{\alpha_{j'}}$$

and note that

$$L = \sum_{x\in X} \psi(x)G(x) + t\left(\prod_{j=1}^{d} (\alpha_j^{-1}T_jh_j(x_0))^{\alpha_j} - \psi(x_0)\right).$$

So by the assumption of the failure of (30) at x_0 we see that letting $t \to \infty$ yields $\inf_{K,g_j} L = -\infty$.

Conversely, if conditions (29) and (30) hold then the factor multiplying K is nonnegative and an application of the arithmetic-geometric mean inequality gives that for any choice of g_j then for each $x \in X$ the term in the second bracket of (28) is nonnegative, so we attain $\inf_{K,g_j} L$ by letting $K = 0$ and $g_j(x) = 0$ for all $x \in X$ and $j = 1, \dots, d$. Hence, for each fixed (ψ, h_j), $\inf_{K,g_j} L > -\infty$ if and only if ψ and h_j satisfy conditions (29) and (30), in which case $\inf_{K,g_j} L = \sum_{x\in X} \psi(x)G(x)$.

Noting that there always exist ψ and h_j satisfying conditions (29) and (30), we see that η is the solution to

$$\eta = \sup_{\psi, h_j} \sum_{x \in X} \psi(x) G(x)$$

$$\text{such that} \quad \psi(x) \leq \prod_{j=1}^{d} (\alpha_j^{-1} T_j h_j(x))^{\alpha_j} \quad \text{for all } x \in X \text{ and} \tag{31}$$

$$\|G\|_{\mathcal{X}^*} \sum_{j=1}^{d} \sum_{y_j \in Y_j} h_j(y_j) \leq 1.$$

For any ψ and h_j satisfying the conditions in (31) we can calculate

$$\sum_{x \in X} \psi(x) G(x) \leq \sum_{x \in X} \prod_{j=1}^{d} (\alpha_j^{-1} T_j h_j(x))^{\alpha_j} G(x) \leq \left\| \prod_{j=1}^{d} (\alpha_j^{-1} T_j h_j)^{\alpha_j} \right\|_{\mathcal{X}} \left\| G \right\|_{\mathcal{X}^*}$$
$$\leq A \prod_{j=1}^{d} \|\alpha_j^{-1} h_j\|_{\mathcal{Y}_j}^{\alpha_j} \|G\|_{\mathcal{X}^*} \leq A \sum_{j=1}^{d} \|h_j\|_1 \|G\|_{\mathcal{X}^*} \leq A$$

where the inequalities follow in order from the first condition of (31), the definition of the norm on $\mathcal{X}^*$, the inequality (15), the arithmetic-geometric mean inequality, and the second condition of (31). Taking the supremum now yields $\eta \leq A$.

3.4. **Proof that $\gamma_{\mathcal{L}} = \eta$ and existence of minimisers.** This is a minimax argument. As we have noted above, it is immediate that $\eta \leq \gamma_{\mathcal{L}}$ and this is referred to as weak duality. The other direction, giving $\gamma_{\mathcal{L}} = \eta$, is called strong duality and does not hold in general. However there are various conditions which guarantee strong duality, such as Slater's condition which is the condition that the original problem (26) is convex and there exists a point satisfying all of the constraints with strict inequality. See [17], p.226. We have noted above that Slater's condition holds in our setting. Moreover, Slater's condition guarantees the existence of a maximiser for the dual problem. However, we need optimisers for the primal problem. If for all $x \in X$ we have $T_j \mathbf{1}(x) > 0$ – which is simply the saturation hypothesis in our present case – then the set of g_j's which satisfy the constraints of (26) with $K = 2A$ will be compact, and therefore a minimiser will exist.

4. General case: overview of the proof

Let us now turn to the argument for Theorem 2.2 in the general case. It will entail substantial measure-theoretic and functional-analytic considerations not present in the case when X and Y_j are finite sets. While it is an attractive idea to try to establish Theorem 2.2 by approximating the general case by the discrete case, this does not seem a feasible route, even when $\mathcal{X}$ and $\mathcal{Y}_j$ are L^q and L^{p_j} spaces respectively, and a direct approach is therefore required. The bulk of the proof of Theorem 2.2 will be devoted to establishing a special case in which X is a finite measure space[11] and where we impose strong saturation on the T_j instead of saturation. This leads to the crucial conclusion that we can take the factors g_j to lie in $L^1(X, \mathrm{d}\mu)$. The result reads as follows:

Theorem 4.1. *Suppose X is a finite measure space, $\mathcal{X}$ is a Köthe space of functions defined on X, $\mathcal{Y}_j$ are normed lattices, and that the linear operators $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ are positive. Suppose*

[11] To be clear, a measure space $(X, \mathrm{d}\mu)$ with $\mu(X) < \infty$, not a finite set X with counting measure.

that each T_j strongly saturates X. Suppose that

$$\left\| \prod_{j=1}^{d} (T_j f_j)^{\alpha_j} \right\|_{\mathcal{X}} \leq A \prod_{j=1}^{d} \left\| f_j \right\|_{\mathcal{Y}_j}^{\alpha_j} \tag{32}$$

holds for all nonnegative $f_j \in \mathcal{Y}_j$, $1 \leq j \leq d$.[12] *Then for every nonnegative $G \in \mathcal{X}'$ there exist nonnegative functions $g_j \in L^1(X, \mathrm{d}\mu)$ such that*

$$G(x) \leq \prod_{j=1}^{d} g_j(x)^{\alpha_j} \qquad \text{a.e. on } X, \tag{33}$$

and such that for each j,

$$\int_X g_j(x) T_j f_j(x) \mathrm{d}\mu(x) \leq A \|G\|_{\mathcal{X}'} \|f_j\|_{\mathcal{Y}_j} \tag{34}$$

for all $f_j \in \mathcal{Y}_j$.

In the proof of this theorem we introduce an extended real-valued Lagrangian function $L(\Phi, \Psi)$ (where Φ corresponds to the variables (K, g_j) and Ψ corresponds to the variables (ψ, h_j) of the discrete model case discussed above). See Section 5.2 below for precise details of the definition of L. As in the model case, we relate $\sup_\Psi \inf_\Phi L$ (which we had previously called η) to problem (33) and (34) and $\inf_\Phi \sup_\Psi L$ (which we had previously called γ) to inequality (32). We then need to show that

$$\min_\Phi \sup_\Psi L(\Phi, \Psi) = \sup_\Psi \inf_\Phi L(\Phi, \Psi),$$

and for this we need to use the Lopsided Minimax Theorem (which can be found as Theorem 7 from Chapter 6.2 of [2]):

Theorem 4.2. *Suppose C and D are convex subsets of vector spaces and that C is endowed with a topology for which the vector space operations are continuous. Further, suppose that $L : C \times D \to \mathbb{R}$ satisfies*

(i) *$\Phi \mapsto L(\Phi, \Psi)$ is convex for all $\Psi \in D$;*
(ii) *$\Psi \mapsto L(\Phi, \Psi)$ is concave for all $\Phi \in C$;*
(iii) *$\Phi \mapsto L(\Phi, \Psi)$ is lower semicontinuous for all $\Psi \in D$; and*
(iv) *there exists a $\Psi_0 \in D$ such that the sublevel sets $\{\Phi \in C : L(\Phi, \Psi_0) \leq \lambda\}$ are compact for all sufficiently large $\lambda \in \mathbb{R}$.*

Then

$$\min_{\Phi \in C} \sup_{\Psi \in D} L(\Phi, \Psi) = \sup_{\Psi \in D} \inf_{\Phi \in C} L(\Phi, \Psi). \tag{35}$$

Remark 16. The existence of the minimum on the left-hand side of (35) is part of the conclusion: there exists a $\bar{\Phi} \in C$ such that $\sup_{\Psi \in D} L(\bar{\Phi}, \Psi) = \inf_{\Phi \in C} \sup_{\Psi \in D} L(\Phi, \Psi)$. Once we know that $\inf \sup = \sup \inf$ this is easy because (iii) tells us that the map $\Phi \mapsto \sup_{\Psi \in D} L(\Phi, \Psi)$ is lower semicontinuous, and (iv) then tells us that the sublevel sets

$$\{\Phi \in C : \sup_{\Psi \in D} L(\Phi, \Psi) \leq \lambda\} \subseteq \{\Phi \in C : L(\Phi, \Psi_0) \leq \lambda\}$$

are closed and compact, and hence $\Phi \mapsto \sup_{\Psi \in D} L(\Phi, \Psi)$ achieves its minimum on any such set. The fact that $\sup \inf \leq \inf \sup$ is trivial, so the main content of the theorem is that $\inf \sup \leq \sup \inf$.

[12] Notice that a hypothesis of strong saturation is unrealistic in the presence of inequality (32) unless X has finite measure.

Remark 17. There is nothing to stop both sides of (35) from being $+\infty$. Indeed, a nontrivial conclusion of the theorem is that if the right-hand side is finite, so is the left-hand side.

Remark 18. Traditional versions of minimax theorems assume that C itself is compact, rather than compactness of certain sublevel sets as condition (iv). However, in our case, we cannot, for the reasons set out below, expect C to be compact. *It is a remarkable feature of our analysis that the saturation hypothesis we must impose corresponds* **precisely** *to condition (iv) of the minimax theorem.*

Remark 19. The observant reader will have noticed that we have indicated our intention to introduce an extended real-valued Lagrangian L, but the minimax theorem applies only to real-valued Lagrangians. This mismatch necessitates a small detour which we wish to suppress here.[13] For details see Section 5.2.1 below.

It is a somewhat delicate matter to choose the vector space where we will locate the variables Φ featuring in the Lagrangian which we will use. Corresponding to the variables g_j occuring in the discrete model case of Section 3, we will now have variables S_j, which we would like to take to be elements of $L^1(X)_+$. (It is the weighted geometric mean of a particular collection of these which will ultimately furnish the desired factorisation.) However, it turns out to be helpful to instead allow, in the first instance, the S_j be elements of the larger space $L^\infty(X)_+^*$, that is, the positive cone of the dual of $L^\infty(X)$.[14] Thus we consider the vector space $\mathbb{R} \times L^\infty(X)^* \times \cdots \times L^\infty(X)^*$ and take C to be a suitable subset of the positive cone in this space. Ideally we would like to take C to be a *norm-bounded* convex subset and then use the Banach–Alaoglu theorem to assert compactness of C; but since we are not expecting any quantitative L^1 bounds on the functions S_j appearing in the factorisation, there is no natural norm-bounded set with which to work. Instead, we take C to be the whole positive cone $\mathbb{R}_+ \times L^\infty(X)_+^* \times \cdots \times L^\infty(X)_+^*$, endowed with the weak-star topology. The price for this is the need to verify hypothesis (iv) of Theorem 4.2. Fortunately this turns out to be not so difficult, and in fact is rather natural in our setting. Carrying out this process will yield some distinguished members of $L^\infty(X)_+^*$. However, working with the dual of $L^\infty(X)$ presents its own difficulties since some elements of $L^\infty(X)_+^*$ are quite exotic. Fortunately the theory of finitely additive measures comes to the rescue, and we will be able to show that elements satisfying the properties we require can be in fact be found in the smaller space $L^1(X)_+$. See Section 5.2 below for more details.

To set the scene for this, we recall three results of Yosida and Hewitt which can be found in [43]. The setting for each of these results is a σ-finite measure space $(X, \mathrm{d}\mu)$.

Theorem 4.3. *There is an isometric isomorphism between the space of finitely additive measures on X of finite total variation which are μ-absolutely continuous*[15] *and the space of bounded linear functionals on $L^\infty(X, \mathrm{d}\mu)$. For a finitely additive measure τ with these properties the corresponding element of $L^\infty(X, \mathrm{d}\mu)^*$ is given by $\tau(\psi) = \int_X \psi \, \mathrm{d}\tau$ (where the integral is the so-called Radon integral). Furthermore $L^1(X, \mathrm{d}\mu)$ embeds isometrically into $L^\infty(X, \mathrm{d}\mu)^*$ in such a way that the application of $g \in L^1(X, \mathrm{d}\mu)$ to an element of $\psi \in L^\infty(X, \mathrm{d}\mu)$ is given by $\int_X \psi g \, \mathrm{d}\mu$ where the integral is now the Lebesgue integral.*

[13] B. Ricceri has recently informed us (private communication) that Theorem 4.2 continues to hold when the Lagrangian is permitted to take the value $+\infty$. The detour takes no longer than establishing this more general minimax statement.

[14] Had we instead opted to work from the outset with $S_j \in L^1(X)_+$, we would have been forced to place unnatural topological conditions on X in order to identify $L^1(X)_+$ with a subspace of a dual space, and in any case we would have to work in the larger space of finite regular Borel measures on X in order to exploit weak-star compactness.

[15] This means that the finitely additive measure τ satisfies $\tau(E) = 0$ whenever $\mu(E) = 0$.

Theorem 4.4. *Any element $S \in L^\infty(X, \mathrm{d}\mu)^*$ can be written uniquely as $S = S_{\mathrm{ca}} + S_{\mathrm{pfa}}$ where S_{ca} is countably additive (and hence is given by integration against a function in $L^1(X, \mathrm{d}\mu)$) and S_{pfa} is purely finitely additive. Furthermore $S \geq 0$ if and only if $S_{\mathrm{ca}} \geq 0$ and $S_{\mathrm{pfa}} \geq 0$.*

We need not concern ourselves here with the definition of purely finitely additive measures, but, in order to be able to use these results, we do need a useful characterisation of which measures are purely finitely additive.

Theorem 4.5. *A nonnegative finitely additive measure τ which is μ-absolutely continuous is purely finitely additive if and only if for every nonnegative countably additive measure σ which is μ-absolutely continuous, every measurable set E and every pair of positive numbers δ_1 and δ_2, there is a measurable subset E' of E such that $\sigma(E') < \delta_1$ and $\tau(E \setminus E') < \delta_2$.*

We wish to remark that analysis related to the dual space of L^∞ has also been employed in a number of other contexts recently. See for example [3], [39], [41] and, in the financial mathematics literature, [34].

5. General case: details of the proof

5.1. **Preliminaries.** We shall first need two lemmas which will be useful for the proof of Theorem 4.1 and also that of Theorem 2.2 itself. The first one is they key technical tool which, in the context of Theorem 4.1, will allow us do induce existence of suitable integrable functions from existence of corresponding members of the dual of L^∞. We shall continue to assume that $\alpha_j > 0$ and that $\sum_{j=1}^d \alpha_j = 1$.

Lemma 5.1. *Let $(X, \mathrm{d}\mu)$ be a σ-finite measure space and suppose that $S_j \in L^\infty(X)^*_+$. Suppose that G is a measurable function such that*

$$\int_X G(x) \prod_{j=1}^d \beta_j^{\alpha_j}(x)\, \mathrm{d}\mu(x) \leq \sum_{j=1}^d \alpha_j S_j(\beta_j) \qquad \text{for all simple functions } \beta_j \text{ on } X. \tag{36}$$

If $(S_{j\mathrm{rn}})$ denotes the Radon–Nikodym derivative with respect to μ of the component of S_j which is countably additive, then

$$G(x) \leq \prod_{j=1}^d S_{j\mathrm{rn}}(x)^{\alpha_j} \quad \text{a.e. on } X. \tag{37}$$

Conversely, if G is such that (37) holds, then (36) holds.

Remark 20. This result extends the special case $d = 1$ which is implicit in Theorem 4.4.

Proof. The converse statement follows immediately from the arithmetic-geometric mean inequality, so we turn to the forward assertion. Suppose not. Then there exists a set E_0 with $\mu(E_0) > 0$ such that inequality (37) fails on E_0 and we can find an $\varepsilon > 0$ and $E_1 \subseteq E_0$ with $\mu(E_1) > 0$ such that

$$G(x) - \varepsilon > \prod_{j=1}^d S_{j\mathrm{rn}}^{\alpha_j}(x)$$

for all $x \in E_1$. Now take β_j to be simple functions supported on E_1. Then we get

$$
\begin{aligned}
&\int_{E_1} \prod_{j=1}^{d} S_{j\mathrm{rn}}^{\alpha_j}(x) \prod_{j=1}^{d} \beta_j^{\alpha_j}(x)\,\mathrm{d}\mu(x) + \varepsilon \int_{E_1} \prod_{j=1}^{d} \beta_j^{\alpha_j}(x)\,\mathrm{d}\mu(x) \\
&\qquad \le \int_{E_1} G(x) \prod_{j=1}^{d} \beta_j^{\alpha_j}(x)\,\mathrm{d}\mu(x) \le \sum_{j=1}^{d} \alpha_j S_j(\beta_j) \qquad (38)\\
&\qquad = \sum_{j=1}^{d} \alpha_j \int_{E_1} S_{j\mathrm{rn}}(x)\beta_j(x)\,\mathrm{d}\mu(x) + \sum_{j=1}^{d} \alpha_j \int_{E_1} \beta_j(x)\,\mathrm{d}\tau_{j\mathrm{pfa}}
\end{aligned}
$$

where $\tau_{j\mathrm{pfa}}$ is the purely finitely additive measure associated to the purely finitely additive component $S_{j\mathrm{pfa}}$ of S_j.

We can find a subset $E_2 \subseteq E_1$ with $\mu(E_2) > 0$ and a $C > 0$ such that $S_{j\mathrm{rn}}(x) \le C$ for all $x \in E_2$ and all j. There are now two cases to consider.

First, assume that there exists a subset $E_3 \subseteq E_2$ such that $\mu(E_3) > 0$ and an index j_0 such that $S_{j_0\mathrm{rn}}(x) = 0$ for all $x \in E_3$. Now let δ be small and positive (to be specified later) and E_4 a subset of E_3 with $0 < \mu(E_4) < \infty$ (also to be specified later[16]), and take $\beta_j = \delta\chi_{E_4}$ for $j \ne j_0$ and $\beta_{j_0} = \delta^{1-\alpha_{j_0}^{-1}}\chi_{E_4}$. This implies that $\prod_{j=1}^{d} \beta_j^{\alpha_j}(x) = 1$ for all $x \in E_4$ and so the top line of (38) equals $0 + \varepsilon\mu(E_4)$. The first term on the bottom line of (38) can be bounded by $C\delta\mu(E_4)$ since there is no contribution from the term with index j_0. The second term we can bound by $\delta^{1-\alpha_{j_0}^{-1}} \left(\sum_j \tau_{j\mathrm{pfa}}\right)(E_4)$. We have not chosen E_4 precisely yet. To do this we use Theorem 4.5 above.

Indeed, applying Theorem 4.5 with $\tau := \sum_j \tau_{j\mathrm{pfa}}$, $\sigma := \mu$, $E := E_3$ and $\delta_1 := \mu(E_3)/2$ gives that for all $\delta_2 > 0$ there is an $E_4 \subseteq E_3$ such that $\mu(E_3 \setminus E_4) < \mu(E_3)/2$ and $\tau(E_4) < \delta_2$. So (38) implies

$$\varepsilon\mu(E_4) \le C\delta\mu(E_4) + \delta^{1-\alpha_{j_0}^{-1}}\tau(E_4) \le C\delta\mu(E_4) + \delta^{1-\alpha_{j_0}^{-1}}\delta_2.$$

Now choose $\delta_2 = \varepsilon\mu(E_3)/4\delta^{1-\alpha_{j_0}^{-1}}$, so that for some $E_4 \subseteq E_3$ we have

$$\varepsilon\mu(E_4) \le C\delta\mu(E_4) + \varepsilon\mu(E_3)/4 \le C\delta\mu(E_4) + \varepsilon\mu(E_4)/2.$$

Finally, choosing $\delta < \varepsilon/(2C)$ yields a contradiction, since by construction $\mu(E_4) > 0$.

In the other case, we have that $S_{j\mathrm{rn}}(x) > 0$ for a.e. $x \in E_2$ and all j. Then we can find a subset $E_3 \subseteq E_2$ with $0 < \mu(E_3) < \infty$ and a number $c > 0$ such that $S_{j\mathrm{rn}}(x) \ge c$ for all $x \in E_3$ and all j. We define u_j on this set as

$$u_j(x) = S_{j\mathrm{rn}}(x)^{-1} \prod_{k=1}^{d} S_{k\mathrm{rn}}^{\alpha_k}(x)$$

and note that $u_j(x) \le c^{-1}C$ and that $\prod_j u_j^{\alpha_j}(x) = 1$. Since these functions are bounded then if we are given $\delta > 0$ we can find simple functions $\tilde\beta_j$ such that $u_j(x) - \delta \le \tilde\beta_j(x) \le u_j(x)$ for all $x \in E_3$. We may assume that $\tilde\beta_j(x) \ge cC^{-1}$ for all $x \in E_3$. Let us take $\beta_j = \tilde\beta_j\chi_{E_4}$ where E_4 is

[16]We are using σ-finiteness of μ to ensure that we can find such an E_4 with $\mu(E_4)$ finite.

a subset of E_3 to be chosen. Then for $x \in E_4$ we have that

$$0 \le \prod_{j=1}^d u_j^{\alpha_j}(x) - \prod_{j=1}^d \beta_j^{\alpha_j}(x) = \sum_{k=1}^d \left(\prod_{j=1}^k u_j^{\alpha_j}(x) \, (u_k^{\alpha_k}(x) - \beta_k^{\alpha_k}(x)) \prod_{j=k+1}^d \beta_j^{\alpha_j}(x) \right)$$
$$= \sum_{k=1}^d \left(\prod_{j=1}^k u_j^{\alpha_j}(x) \alpha_k \frac{\xi_k^{\alpha_k}(x)}{\xi_k(x)} \, (u_k(x) - \beta_k(x)) \prod_{j=k+1}^d \beta_j^{\alpha_j}(x) \right) \le dC^2 c^{-2} \delta.$$

Here $\xi_{j_0}(x)$ lies between $\beta_{j_0}(x)$ and $u_{j_0}(x)$ and we have used that $c/C \le \beta_j(x), \xi_j(x), u_j(x) \le C/c$. Now we can estimate the first term on the top line of (38) from below by

$$\int_{E_4} \prod_{j=1}^d S_{j\mathrm{rn}}^{\alpha_j}(x) \left(\prod_{j=1}^d u_j^{\alpha_j}(x) - dC^2 c^{-2} \delta \right) \mathrm{d}\mu(x) \ge \int_{E_4} \prod_{j=1}^d S_{j\mathrm{rn}}^{\alpha_j}(x) \, \mathrm{d}\mu(x) - \mu(E_4) dC^3 c^{-2} \delta,$$

and the second term on the top line of (38) we can estimate from below by $\varepsilon \mu(E_4)(1 - dC^2 c^{-2} \delta)$ since $\prod_j u_j^{\alpha_j}(x) = 1$ on E_4.

The first term on the bottom line of (38) we can estimate from above using $\beta_j \le u_j$ on E_4 and the definition of u_j by

$$\int_{E_4} \prod_{j=1}^d S_{j\mathrm{rn}}^{\alpha_j}(x) \, \mathrm{d}\mu(x).$$

The second term we can bound by $Cc^{-1} \left(\sum_j \tau_{j\mathrm{fa}} \right) (E_4)$.
Collecting this we have that

$$\int_{E_4} \prod_{j=1}^d S_{j\mathrm{rn}}^{\alpha_j}(x) \, \mathrm{d}\mu(x) - \mu(E_4) dC^3 c^{-2} \delta + \varepsilon \mu(E_4)(1 - dC^2 c^{-2} \delta)$$
$$\le \int_{E_4} \prod_{j=1}^d S_{j\mathrm{rn}}^{\alpha_j}(x) \, \mathrm{d}\mu(x) + Cc^{-1} \left(\sum_j \tau_{j\mathrm{pfa}} \right) (E_4).$$

The integrals cancel[17] and we get

$$\varepsilon \mu(E_4) \le \mu(E_4)(\varepsilon dC^2 c^{-2} \delta + dC^3 c^{-2} \delta) + Cc^{-1} \tau(E_4)$$

where $\tau = \sum_j \tau_{j\mathrm{pfa}}$. We can then choose E_4 and δ in much the same way as before to yield a contradiction. Note that δ will only depend on d, C, c and ε.

So in both cases we have a contradiction to the existence of E_0, and so (37) must hold.

□

It turns out that we shall need to consider the action of $S \in L^\infty(X, \mathrm{d}\mu)_+^*$ not just on $L^\infty(X)$, but on general nonnegative measurable functions in $\mathcal{M}(X)$. The reasons for this are explained in Section 5.2 below. To this end, we extend S to $\mathcal{M}(X)_+$ by declaring, for $F \in \mathcal{M}(X)_+$,

$$S(F) := \sup\{S(f) \, : \, 0 \le f \le F, f \in L^\infty(X)\} = \sup\{S(\phi) \, : \, 0 \le \phi \le F, \ \phi \text{ simple}\}.$$

Of course $S(F)$ will often now take the value $+\infty$.

The second lemma concerns continuity properties of this extension. Consider the map $S \mapsto S(F)$ for fixed $F \in \mathcal{M}(X)_+$ as S ranges over $L^\infty(X)_+^*$. If $F \in L^\infty(X)$ this map is norm continuous and hence weak-star continuous. For $F \in \mathcal{M}(X)_+$ we can assert less.

[17] Since $S_j \in (L^\infty)^*$ we have $S_{j\mathrm{rn}} \in L^1$, and the terms we are cancelling are indeed finite.

Lemma 5.2. *Fix $F \in \mathcal{M}(X)_+$. Then the map $S \mapsto S(F)$ from $L^\infty(X)^*_+$ to $\mathbb{R} \cup \{+\infty\}$ is weak-star lower semicontinuous.*

We remark that we have to be cautious here since $L^\infty(X)^*_+$ with the weak-star topology is not a metric space; so we cannot simply concern ourselves with sequential lower semicontinuity.

Proof. Let $S \in L^\infty(X)^*_+$. Either $S(F) = +\infty$ or $S(F) < +\infty$. Let us first deal with the latter case. We need to show that for every $\epsilon > 0$ there is a weak-star open neighbourhood U of S such that for $R \in U$ we have $R(F) \geq S(F) - \epsilon$.

Since $S(F) < +\infty$ there is an $f \in L^\infty(X)$ with $0 \leq f \leq F$ such that $S(f) > S(F) - \epsilon$. Let

$$U = \{R \in L^\infty(X)^*_+ \,:\, R(f) > S(F) - \epsilon\}.$$

Then $S \in U$, and U is weak-star open since for each $f \in L^\infty(X)$ the functional $R \mapsto R(f)$ is weak-star continuous. So for $R \in U$ we have

$$R(F) \geq R(f) > S(F) - \epsilon$$

which is what we needed.

Now we look at the case $S(F) = +\infty$. We now need to show that for every $N \in \mathbb{N}$ there is a weak-star open neighbourhood U of S such that for $R \in U$ we have $R(F) \geq N$.

Since $S(F) = +\infty$ there is an $f \in L^\infty(X)$ with $0 \leq f \leq F$ such that $S(f) > N$. Let

$$U = \{R \in L^\infty(X)^*_+ \,:\, R(f) > N\}.$$

Then $S \in U$, and U is weak-star open since for each $f \in L^\infty(X)$ the functional $R \mapsto R(f)$ is weak-star continuous. So for $R \in U$ we have

$$R(F) \geq R(f) > N$$

which is what we needed.

□

5.2. **Proof of Theorem 4.1.** Suppose we are in the situation in the statement of Theorem 4.1. In particular, we assume that $G \in \mathcal{X}'$, and we may clearly assume that $\|G\|_{\mathcal{X}'} \neq 0$.

We recall from Section 4 that we take C (in which we locate the variables $\Phi = (K, S_j)$) to be the positive cone $\mathbb{R}_+ \times (L^\infty(X)^*_+)^d$ in the vector space $\mathbb{R} \times (L^\infty(X)^*)^d$, and C is given the topology inherited from the product topology of the corresponding weak-star topologies. We take D (in which we locate the variables $\Psi = (\beta_j, h_j)$) to be the positive cone in the vector space $\mathcal{S}(X)^d \times \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_d$.

Therefore, for $K \in \mathbb{R}_+$, $S_j \in L^\infty(X)^*_+$, β_j simple functions on X and $h_j \in \mathcal{Y}_j$ we consider the functional

$$\begin{aligned} L = K + &\left(\int_X G(x) \prod_{j=1}^d \beta_j^{\alpha_j}(x)\, \mathrm{d}\mu(x) - \sum_{j=1}^d \alpha_j S_j(\beta_j) \right) \\ &+ \sum_{j=1}^d \left(S_j(T_j h_j) - K \|G\|_{\mathcal{X}'} \|h_j\|_{\mathcal{Y}_j} \right). \end{aligned}$$

Note that the integral term is well-defined since $G \in \mathcal{X}'$ (which is contained in $L^1(X, \mathrm{d}\mu)$ when μ is a finite measure, as we have previously observed) and the β_j are simple functions, and that the terms $S_j(\beta_j)$ are also well-defined since the β_j are bounded functions. The terms $S_j(T_j h_j)$ are well-defined via the extension of S_j to $\mathcal{M}(X)_+$ as discussed in Section 5.1 above. Thus

$L : C \times D \to \mathbb{R} \cup \{+\infty\}$ is well-defined and takes values in $\mathbb{R} \cup \{+\infty\}$, with the possible value $+\infty$ arising when $T_j h_j$ is not a *bounded* measurable function.

We next want to see how we can apply the minimax theorem, Theorem 4.2 to this Lagrangian. Recall from Remark 19 above that we have a problem in so doing, since our Lagrangian may take the value $+\infty$, while Theorem 4.2 requires that the Lagrangian be real-valued. To be clear, what we desire – and what we shall indeed obtain – is the conclusion

$$\min_{\Phi \in C} \sup_{\Psi \in D} L(\Phi, \Psi) = \sup_{\Psi \in D} \inf_{\Phi \in C} L(\Phi, \Psi)$$

of Theorem 4.2 in our case, but in order to achieve this we need to make a detour.

5.2.1. *A detour.* We now describe the necessary detour. This involves modifying the Lagrangian we have defined in order to make it real-valued, but without altering its essential purpose. The main technical difference is that instead of allowing S_j to act on the possibly unbounded $T_j h_j$, we have it act on an arbitrary nonnegative simple function ψ_j satisfying $\psi_j \leq T_j h_j$,

We therefore introduce a new Lagrangian $\Lambda : C \times \tilde{D} \to \mathbb{R}$, where C is as before, and where

$$\tilde{D} = \{(\beta_j, h_j, \psi_j) \in \mathcal{S}(X)^d \times \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_d \times \mathcal{S}(X)^d \ : \ \beta_j \geq 0,\ h_j \geq 0,\ 0 \leq \psi_j \leq T_j h_j\}.$$

Note that $\tilde{D}$ is convex.

For $(K, S_j) \in C$ and $(\beta_j, h_j, \psi_j) \in \tilde{D}$ we define

$$\Lambda = K + \left(\int_X G(x) \prod_{j=1}^{d} \beta_j^{\alpha_j}(x) \, \mathrm{d}\mu(x) - \sum_{j=1}^{d} \alpha_j S_j(\beta_j) \right) + \sum_{j=1}^{d} \left(S_j \psi_j - K \|G\|_{\mathcal{X}'} \|h_j\|_{\mathcal{Y}_j} \right).$$

Note that Λ is real-valued since $S_j \psi_j$ is real-valued. Moreover, note that by the definition of the extension of S_j to $\mathcal{M}(X)_+$, we have

$$L((K, S_j), (\beta_j, h_j)) = \sup_{\{\psi_j \,:\, \psi_j \leq T_j h_j\}} \Lambda((K, S_j), (\beta_j, h_j, \psi_j)). \tag{39}$$

We will momentarily check that the Lagrangian Λ satisfies the hypotheses of Theorem 4.2, but taking this as read for now, we deduce using (39) that

$$\min_{(K,S_j) \in C} \sup_{(\beta_j, h_j) \in D} L = \min_{(K,S_j) \in C} \sup_{(\beta_j, h_j, \psi_j) \in \tilde{D}} \Lambda = \sup_{(\beta_j, h_j, \psi_j) \in \tilde{D}} \inf_{(K,S_j) \in C} \Lambda.$$

But since trivially $\sup \inf \leq \inf \sup$, we have, using (39) once more,

$$\sup_{(\beta_j, h_j, \psi_j) \in \tilde{D}} \inf_{(K,S_j) \in C} \Lambda \leq \sup_{(\beta_j, h_j) \in D} \inf_{(K,S_j) \in C} \sup_{\psi_j \leq T_j h_j} \Lambda = \sup_{(\beta_j, h_j) \in D} \inf_{(K,S_j) \in C} L.$$

Combining the last two displays we obtain

$$\min_{(K,S_j) \in C} \sup_{(\beta_j, h_j) \in D} L \leq \sup_{(\beta_j, h_j) \in D} \inf_{(K,S_j) \in C} L.$$

Since the reverse inequality is once again trivial we conclude that

$$\min_{\Phi \in C} \sup_{\Psi \in D} L(\Phi, \Psi) = \sup_{\Psi \in D} \inf_{\Phi \in C} L(\Phi, \Psi)$$

as we needed.

Now we need to look at conditions (i) – (iv) of Theorem 4.2 in our case where Λ replaces L. Concerning (i), the map $S \mapsto S(F)$ is linear on $L^\infty(X)^*$ for each fixed $F \in \mathcal{S}(X)$. Therefore, for each fixed $\tilde{\Psi} \in \tilde{D}$, the map $\Phi \mapsto \Lambda(\Phi, \tilde{\Psi})$ is affine, thus convex on C. Concerning (ii), the map $F \mapsto S(F)$ is linear and hence concave on $\mathcal{S}(X)$ for each fixed $S \in L^\infty(X)^*_+$. Moreover the geometric mean is a concave operation and the map $h \mapsto \|h\|_{\mathcal{Y}_j}$ is convex. Therefore, for each fixed $\Phi \in C$, the map $\tilde{\Psi} \mapsto \Lambda(\Phi, \tilde{\Psi})$ is concave on $\tilde{D}$. Concerning (iii), this follows directly from the norm-continuity of $S \mapsto S(F)$ on $L^\infty(X)^*$ for each fixed $F \in \mathcal{S}(X)$.

Condition (iv) is more interesting, and it is in verification of this condition that we use the crucial strong saturation hypothesis of Theorem 4.1. We need to see that for some $\tilde{\Psi}_0 \in \tilde{D}$ the sublevel sets $\{\Phi \in C \,:\, L(\Phi, \tilde{\Psi}_0) \leq \lambda\}$ are compact for all sufficiently large λ. We will show that for a suitable choice of $\tilde{\Psi}_0$ these sets are norm-bounded, and from this the Banach–Alaoglu theorem will give us compactness.

We take $\tilde{\Psi}_0 = (\beta_j, h_j, \psi_j)$ to have $\beta_j = 0$ for all j. We take $h_j \in \mathcal{Y}_j$ such that $T_j h_j \geq c_0 > 0$ a.e. on X, as guaranteed by the hypothesis of Theorem 4.1. By multiplying by a suitable positive constant if necessary, we can certainly assume that $\sum_{j=1}^d \|h_j\|_{\mathcal{Y}_j} < (2\|G\|_{\mathcal{X}'})^{-1}$. Finally, we take $\psi_j = c_0 \mathbf{1}$ which satisfies $0 \leq \psi_j \leq T_j h_j$.

For such a choice of $\tilde{\Psi}_0$ we have

$$\Lambda((K, S_j), \tilde{\Psi}_0) = K\left(1 - \|G\|_{\mathcal{X}'} \sum_{j=1}^d \|h_j\|_{\mathcal{Y}_j}\right) + \sum_{j=1}^d S_j \psi_j \geq K/2 + c_0 \sum_{j=1}^d S_j \mathbf{1}.$$

Therefore, for given $\lambda > 0$,

$$\{(K, S_j) \in C \,:\, \Lambda((K, S_j), \tilde{\Psi}_0) \leq \lambda\} \subseteq [0, 2\lambda] \times \{S \in L^\infty(X)^*_+ \,:\, S(\mathbf{1}) \leq c_0^{-1}\lambda\}^d.$$

But it is easy to see that for $S \geq 0$, $S(\mathbf{1}) = \|S\|_{L^\infty(X)^*}$. Indeed, by definition we have

$$\|S\|_{L^\infty(X)^*} = \sup\{|S(u)| \,:\, u \in L^\infty(X), \|u\|_\infty \leq 1\}.$$

So let us take such a function u with $\|u\|_\infty \leq 1$. Since $S(-u) = -S(u)$ we may by choosing either u or $-u$ assume that $S(u) \geq 0$. We have that $u \leq 1$ a.e. and therefore the non-negativity of S gives us that $S(u) \leq S(\mathbf{1})$, as needed.

Therefore

$$\{(K, S_j) \in C \,:\, \Lambda((K, S_j), \tilde{\Psi}_0) \leq \lambda\} \subseteq [0, 2\lambda] \times \{S \,:\, \|S\| \leq c_0^{-1}\lambda\}^d$$

is a norm-bounded, weak-star closed, hence weak-star compact subset of $\mathbb{R} \times (L^\infty(X)^*)^d$, by the Banach–Alaoglu theorem. This completes the verification of condition (iv) of Theorem 4.2 in our case, and we conclude that

$$\min_{\Phi \in C} \sup_{\Psi \in D} L(\Phi, \Psi) = \sup_{\Psi \in D} \inf_{\Phi \in C} L(\Phi, \Psi).$$

5.2.2. *Return to the main argument.* We may therefore conclude, by Theorem 4.2, that if for non-zero $G \in \mathcal{X}'$ fixed we define[18]

$$\gamma^*_{\mathcal{L}} = \inf_{(K,S_j) \in C} \sup_{(\beta_j, h_j) \in D} L \qquad \text{and} \qquad \eta = \sup_{(\beta_j, h_j) \in D} \inf_{(K,S_j) \in C} L$$

[18]The notation here is perhaps confusing. We shall consider four problems: γ, γ^*, $\gamma_{\mathcal{L}}$ and $\gamma^*_{\mathcal{L}}$. When there is no superscript we are dealing with the variant of the problem pertaining to L^1, and presence of the superscript $*$ denotes that we are dealing with the variant of the problem which pertains to $(L^\infty)^*$; when there is no subscript we are dealing with the original version of the problem, and presence of the subscript $\mathcal{L}$ denotes that we are dealing with the Lagrangian formulation. This is consistent with the notation we adopted in the treatment of the finite discrete case above; in that case there was no distinction between L^1 and $(L^\infty)^*$. We do not adorn η with either a superscript $*$ nor a subscript $\mathcal{L}$ since there is only one η-problem. Nevertheless we emphasise that the η-problem does indeed deal with the Langrangian formulation in the form pertaining to $(L^\infty)^*$.

then $\eta = \gamma^*_{\mathcal{L}}$ and the infimum in the problem for $\gamma^*_{\mathcal{L}}$ is achieved as a minimum. (It should be noted that we are not yet in a position to assert the finiteness of either of these numbers.)

In order to progress further, we shall also consider the problem

$$\gamma = \inf K$$

$$\text{such that} \quad G(x) \le \prod_{j=1}^{d} S_j(x)^{\alpha_j} \quad \text{a.e. on } X\text{, and} \tag{40}$$

$$\int_X S_j(x) T_j h_j(x)\,\mathrm{d}\mu(x) \le K\|G\|_{\mathcal{X}'}\|h_j\|_{\mathcal{Y}_j} \quad \text{for all } j \text{ and all } h_j \in \mathcal{Y}_j$$

where the S_j are taken to be in $L^1(X, \mathrm{d}\mu)$. We emphasise that this is the problem we really want to solve: if we can prove that $\gamma \le A$ and that minimisers exist, we will have our desired factorisation. Nevertheless, we should point out that it is not yet even clear that there exist (K, S_j) satisfying the constraints of (40). We shall be able to infer the existence of such (K, S_j), and hence the finiteness of γ, only from the conclusion of Theorem 4.1.

Our strategy is to show that (i) $\gamma^*_{\mathcal{L}} = \gamma$ and that if the problem for $\gamma^*_{\mathcal{L}}$ admits minimisers Φ, then the problem for γ also admits minimisers; and (ii) $0 \le \eta \le A$. Combining these with the minimax result $\gamma^*_{\mathcal{L}} = \eta$ and existence of minimsiers for $\gamma^*_{\mathcal{L}}$, we can conclude that the problem for γ admits minimisers and that $\gamma \le A$, which will conclude the proof of Theorem 4.1.

Proof that $\gamma = \gamma^*_{\mathcal{L}}$ and that existence of minimisers for $\gamma^*_{\mathcal{L}}$ implies existence of minimisers for γ. We begin by studying $\gamma^*_{\mathcal{L}}$ so let us consider for which $(K, S_j) \in C$ we have $\sup_{\beta_j, h_j} L < \infty$. Fix (K, S_j). First of all, suppose that S_j are such that there exists a tuple (β_j) such that

$$\int_X G(x) \prod_{j=1}^{d} \beta_j^{\alpha_j}(x)\,\mathrm{d}\mu(x) - \sum_{j=1}^{d} \alpha_j S_j(\beta_j) > 0.$$

Then by setting $h_j = 0$ and substituting $\beta_j \mapsto t\beta_j$ and letting $t \to \infty$ we see that the supremum is infinite. Therefore, if $\sup_{\beta_j, h_j} L < \infty$, we must have

$$\int_X G(x) \prod_{j=1}^{d} \beta_j^{\alpha_j}(x)\,\mathrm{d}\mu(x) \le \sum_{j=1}^{d} \alpha_j S_j(\beta_j) \qquad \text{for all simple functions } \beta_j, \tag{41}$$

which, by Lemma 5.1, is equivalent to

$$G(x) \le \prod_{j=1}^{d} S_{j\mathrm{rn}}(x)^{\alpha_j} \quad \text{a.e. on } X. \tag{42}$$

Now assume there exists a j_0 and an $h_{j_0} \in \mathcal{Y}_{j_0}$ such that $S_{j_0}(T_{j_0} h_{j_0}) > K\|G\|_{\mathcal{X}'}\|h_{j_0}\|_{\mathcal{Y}_{j_0}}$. Taking $\beta_j = 0$, and $h_j = 0$ for $j \ne j_0$ and multiplying h_{j_0} by a factor t which we send to infinity we again see that the supremum is infinite. Therefore, if $\sup_{\beta_j, h_j} L < \infty$, then we must also have

$$S_j(T_j h_j) \le K\|G\|_{\mathcal{X}'}\|h_j\|_{\mathcal{Y}_j} \tag{43}$$

for all nonnegative h_j and all j. From the positivity of $S_{j\mathrm{pfa}}$ we see that this implies

$$\int_X S_{j\mathrm{rn}}(x) T_j h_j(x)\,\mathrm{d}\mu(x) \le K\|G\|_{\mathcal{X}'}\|h_j\|_{\mathcal{Y}_j} \tag{44}$$

for all nonnegative h_j and all j.

On the other hand, if for fixed (K, S_j) conditions (42) and (43) are satisfied, then when we are looking for $\sup_{\beta_j, h_j} L$, we can do no better than taking $\beta_j = 0$ and $h_j = 0$ for all j. So for fixed (K, S_j), we have $\sup_{\beta_j,h_j} L < \infty$ if and only if conditions (42) and (43) hold, in which case $\sup_{\beta_j,h_j} L = K$. So the problem for $\gamma^*_{\mathcal{L}}$ is identical with the problem

$$\begin{aligned} \gamma^* &= \inf K \\ \text{such that} \quad G(x) &\leq \prod_j S_{j\mathrm{rn}}(x)^{\alpha_j} \qquad \text{a.e.,} \\ S_j(T_j h_j) &\leq K\|G\|_{\mathcal{X}'}\|h_j\|_{\mathcal{Y}_j} \quad \text{for all } j \text{ and all } h_j \in \mathcal{Y}_j, \end{aligned}$$

where we emphasise that the inf is taken over (K, S_j) with $S_j \in L^\infty(X)^*_+$.

Likewise, the problem

$$\gamma_{\mathcal{L}} := \inf_{(K,S_j)\in\mathbb{R}_+\times(L^1(X)_+)^d} \sup_{\beta_j,h_j} L$$

is identical with problem (40) for γ.

It is clear that $\gamma^*_{\mathcal{L}} \leq \gamma_{\mathcal{L}}$ as the infimum for the left hand side is over a larger set than for the right hand side.

Claim: $\gamma_{\mathcal{L}} \leq \gamma^*_{\mathcal{L}}$, and if minimisers $\Phi = (K, S_j)$ exist for problem $\gamma^*_{\mathcal{L}}$, they also exist for problem $\gamma_{\mathcal{L}}$.

Indeed, assume that $\gamma^*_{\mathcal{L}} < \infty$, let $\varepsilon > 0$ and let (K, S_j) with $S_j \in L^\infty(X)^*$ and satisfying conditions (42) and (43) be such that $K < \gamma^*_{\mathcal{L}} + \varepsilon$. Then the absolutely continuous component $S_{j\mathrm{rn}}$ satisfies (42) and (44), and so $(K, S_{j\mathrm{rn}})$ contributes to the infimum in the problem for $\gamma_{\mathcal{L}}$. Thus $\gamma_{\mathcal{L}} \leq \gamma^*_{\mathcal{L}} + \varepsilon$. Letting $\varepsilon \to 0$ establishes the first part of the claim. Now suppose that minimisers $\Phi = (K, S_j)$ exist for problem $\gamma^*_{\mathcal{L}}$. In particular this supposes that $\gamma^*_{\mathcal{L}} < \infty$. Let (K, S_j) with $S_j \in L^\infty(X)^*$ and satisfying conditions (42) and (43) be such that $K = \gamma^*_{\mathcal{L}}$. Then the absolutely continuous component $S_{j\mathrm{rn}}$ satisfies (42) and (44), and so $(K, S_{j\mathrm{rn}})$ contributes to and indeed achieves the infimum in the problem for $\gamma_{\mathcal{L}}$ (otherwise $\gamma_{\mathcal{L}}$ would be strictly less than $\gamma^*_{\mathcal{L}}$).

Summarising, the problems for γ and $\gamma_{\mathcal{L}}$ are equivalent; the problems for γ^* and $\gamma^*_{\mathcal{L}}$ are equivalent; $\gamma_{\mathcal{L}} = \gamma^*_{\mathcal{L}}$, and if extremisers exist for $\gamma^*_{\mathcal{L}}$, they also exist for $\gamma_{\mathcal{L}}$, and hence too for γ.

Proof that $0 \leq \eta \leq A$**.** We wish to carry out a similar analysis for $\inf_{K,S_j} L$, and for that we first of all rewrite L as

$$\begin{aligned} L = \int_X G(x) \prod_{j=1}^d \beta_j^{\alpha_j}(x)\, \mathrm{d}\mu(x) + K\left(1 - \|G\|_{\mathcal{X}'} \sum_{j=1}^d \|h_j\|_{\mathcal{Y}_j}\right) \\ + \sum_{j=1}^d S_j\left(T_j h_j - \alpha_j \beta_j\right). \end{aligned}$$

We consider for which $(\beta_j, h_j) \in D$ we have $\inf_{K,S_j} L > -\infty$.

First, by taking $S_j = 0$ for all $j = 1, \ldots, d$ and letting K to go infinity we see that if $\inf_{K,S_j} L > -\infty$ then we must have

$$\|G\|_{\mathcal{X}'} \sum_{j=1}^d \|h_j\|_{\mathcal{Y}_j} \leq 1. \tag{45}$$

Secondly, assume that there exists an index j_0 and a set $E \subseteq X$ with $\mu(E) > 0$ such that $T_{j_0} h_{j_0}(x) < \alpha_{j_0}\beta_{j_0}(x)$ for a.e. $x \in E$. Then by taking $S_{j_0} = t\chi_E \in L^1$, $S_j = 0$ for $j \neq j_0$ and

$K = 0$ and letting $t \to \infty$, then we see that $\inf_{K,S_j} L = -\infty$. Thus if $\inf_{K,S_j} L > -\infty$, we must also have

$$\alpha_j \beta_j(x) \le T_j h_j(x) \quad \text{a.e. on } X \text{ for all } j. \tag{46}$$

If conditions (45) and (46) are both satisfied we can do no better than take $K = 0$ and $S_j = 0$ for all j. So, for fixed (β_j, h_j), $\inf_{K,S_j} L > -\infty$ if and only if conditions (45) and (46) hold, in which case $\inf_{K,S_j} L = \int_X G(x) \prod_{j=1}^d \beta_j^{\alpha_j}(x) \mathrm{d}\mu(x)$. We can always find (β_j, h_j) such that conditions (45) and (46) hold, so $\eta = \sup_{\beta_j,h_j} \int_X G(x) \prod_{j=1}^d \beta_j^{\alpha_j}(x) \mathrm{d}\mu(x)$ subject to conditions (45) and (46). In particular this tells us that $\eta \ge 0$.

Let us now derive an upper bound for η. Examining the condition (46) on β_j we see that

$$\eta \le \sup_{h_j} \int_X G(x) \prod_{j=1}^d \left(\alpha_j^{-1} T_j h_j(x)\right)^{\alpha_j} \, \mathrm{d}\mu(x)$$

$$\text{such that} \quad \|G\|_{\mathcal{X}'} \sum_{j=1}^d \|h_j\|_{\mathcal{Y}_j} \le 1.$$

Clearly there exist functions $h_j \in \mathcal{Y}_j$ such that $\|G\|_{\mathcal{X}'} \sum_j \|h_j\|_{\mathcal{Y}_j} \le 1$, and for any such we have

$$\begin{aligned}
\int_X G(x) \prod_{j=1}^d \left(\alpha_j^{-1} T_j h_j(x)\right)^{\alpha_j} \, \mathrm{d}\mu(x) &\le \|G\|_{\mathcal{X}'} \| \prod_{j=1}^d \left(T_j(\alpha_j^{-1} h_j)(x)\right)^{\alpha_j} \|_{\mathcal{X}} \\
&\le \|G\|_{\mathcal{X}'} A \prod_{j=1}^d \|\alpha_j^{-1} h_j\|_{\mathcal{Y}_j}^{\alpha_j} \\
&\le \|G\|_{\mathcal{X}'} A \sum_{j=1}^d \alpha_j \|\alpha_j^{-1} h_j\|_{\mathcal{Y}_j} \\
&= \|G\|_{\mathcal{X}'} A \sum_{j=1}^d \|h_j\|_{\mathcal{Y}_j} \le A
\end{aligned}$$

by Hölder's inequality in the form $\int Gf \le \|G\|_{\mathcal{X}'} \|f\|_{\mathcal{X}}$, the multilinear inequality (15) which is our main hypothesis, the arithmetic-geometric mean inequality and finally the assumption on the h_j. This clearly implies $\eta \le A$, and thus concludes the proof of Theorem 4.1. □

5.3. **Consequences of saturation.** We give two lemmas needed for Theorem 2.2. These allow us to construct suitable exhausting sequences of subsets of X of finite measure, in order that we might apply Theorem 4.1. Then we construct the weight w of the statement of Theorem 2.2.

Lemma 5.3. *Let $(X, \mathrm{d}\mu)$ be a σ-finite measure space, and suppose that $\mathcal{P} \subseteq \mathcal{M}(X)_+$ has the property that for every measurable set $E \subseteq X$ with $\mu(E) > 0$, there exists an $f \in \mathcal{P}$ and a subset $E' \subseteq E$ with $\mu(E') > 0$, such that $f > 0$ a.e. on E'. Then there exists a countable subset $\{f_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}$ such that, with $E_n := \{x \in X \,:\, f_n(x) > 0\}$,*

$$\mu(X \setminus \bigcup_{n=1}^{\infty} E_n) = 0.$$

Proof. By exhausting X by a countable sequence of subsets, each of finite measure, we may assume that $\mu(X)$ is finite. We claim that for every $\epsilon > 0$ there is a finite subset $\{f_1, \dots, f_N\} \subseteq \mathcal{P}$

such that

$$\mu(X \setminus \bigcup_{n=1}^{N} E_n) < \epsilon.$$

Once we have this claim, we take the union of the finite subsets of $\mathcal{P}$ obtained for each $\epsilon = 1/m$, $m \in \mathbb{N}$, and we are finished.

Suppose, for a contradiction, that there is some $\epsilon > 0$ such that for all N, for all finite subfamilies $\{f_1, \dots, f_N\} \subseteq \mathcal{P}$ we have

$$\mu(X \setminus \bigcup_{n=1}^{N} E_n) \geq \epsilon > 0.$$

Let

$$t = \inf_N \inf_{\{f_1,\dots,f_N\} \subseteq \mathcal{P}} \mu(X \setminus \bigcup_{n=1}^{N} E_n).$$

Then $t \geq \epsilon > 0$ and also $t < \infty$ since $\mu(X)$ is finite. For $m \in \mathbb{N}$ let $\mathcal{P}_m = \{f_1, \dots, f_{N(m)}\}$ be such that

$$\mu(X \setminus \bigcup_{n=1}^{N(m)} E_n) \leq t + 1/m;$$

we may assume that $\mathcal{P}_m \subseteq \mathcal{P}_{m+1}$ for all m. Letting $m \to \infty$ we obtain

$$\mu(X \setminus \bigcup_{n=1}^{\infty} E_n) \leq t.$$

If $\mu(X \setminus \bigcup_{n=1}^{\infty} E_n) = 0$ we are done; otherwise $E = X \setminus \bigcup_{n=1}^{\infty} E_n$ has positive measure, and therefore, by hypothesis, there is a subset $E' \subseteq E$ with $\mu(E') = \delta > 0$ such that for some $f_0 \in \mathcal{P}$ we have $E' \subseteq E_0$. Then, (with the union now starting at $n = 0$),

$$\mu(X \setminus \bigcup_{n=0}^{N(m)} E_n) \leq t + 1/m - \delta.$$

If we choose $m > \delta^{-1}$, we then have

$$\mu(X \setminus \bigcup_{n=0}^{N(m)} E_n) < t,$$

in contradiction to the definition of t.

□

Lemma 5.4. *Let $(X, \mathrm{d}\mu)$ be a σ-finite measure space, $\mathcal{Y}$ a normed lattice, and suppose that $T : \mathcal{Y} \to \mathcal{M}(X)$ is a positive linear operator which saturates X. Then there is an increasing exhausting sequence of subsets (G_n) of X, each of finite measure, such that T strongly saturates each G_n. More precisely, there exists a sequence $(h_n) \subseteq \mathcal{Y}_+$ such that $h_{n+1} \geq h_n$ for all n, such that $\|h_n\|_{\mathcal{Y}} \leq 1$ for all n, and such that for all n, $Th_n(x) \geq 1/n$ for $x \in G_n$.*

Proof. Let $\mathcal{P} = T(\mathcal{Y}_+) \subseteq \mathcal{M}(X)_+$. The saturation hypothesis allows us to deduce from the previous lemma that there exists a sequence $h_n \in \mathcal{Y}_+$ such that if $E_n = \{Th_n > 0\}$, then $\{E_n\}_{n \in \mathbb{N}}$ covers X up to a set of measure zero. Letting

$$\tilde{h}_n = 2^{-1}\frac{h_1}{\|h_1\|_{\mathcal{Y}}} + 2^{-2}\frac{h_2}{\|h_2\|_{\mathcal{Y}}} + \cdots + 2^{-n}\frac{h_n}{\|h_n\|_{\mathcal{Y}}},$$

we see that we may additionally assume that $\|h_n\|_{\mathcal{Y}} \leq 1$ for all n, and that $h_{n+1} \geq h_n$. Thus without loss of generality $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$.

Since X is σ-finite there is an increasing sequence of subsets F_n of finite measure which exhausts X. Now we set

$$G_n := \{x : Th_n > 1/n\} \cap F_n.$$

Clearly $G_n \subseteq G_{n+1}$ for all n, and each G_n has finite measure. We check that $\{G_n\}$ is exhausting. Let $x \in X$. Then $x \in E_k$ for some k, i.e. $Th_k(x) > 0$, and therefore $Th_k(x) > 1/l$ for some $l \in \mathbb{N}$. Since X is exhausted by $\{F_m\}$ there is some m such that $x \in F_m$. Therefore, for n such that $n \geq \max\{k, l.m\}$, we have that $x \in G_n$. Finally, it is clear by definition that T strongly saturates G_n.

□

As an immediate consequence, we have:

Corollary 5.5. *Let $(X, \mathrm{d}\mu)$ be a σ-finite measure space and let $\mathcal{Y}_j$ be normed lattices for $1 \leq j \leq d$. Assume that $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ for $1 \leq j \leq d$ are positive linear operators, each of which saturates X. Then for each $1 \leq j \leq d$ there exists a sequence $(h_{j,n})_n \subset \mathcal{Y}_j$ such that $\|h_{j,n}\|_{\mathcal{Y}_j} \leq 1$, $h_{j,n} \leq h_{j,m}$ for $m \geq n$, and there exists an increasing and exhausting sequence of subsets $E_n \subseteq X$, each of finite measure, such that for each j and n, $T_j h_{j,n}(x) \geq 1/n$ for $x \in E_n$.*

With this in hand, we can now define the weight w referred to in Remark 10 above. Let $w_j(x)$ for $x \in E_m \setminus E_{m-1}$ be $T_j h_{j,m}(x)$, where we take $E_0 = \emptyset$. Define $w(x) = \min_j w_j(x)$. Note that w is a.e. positive and a.e. finite. (If the sets E_m stabilise in the sense that for some $M \in \mathbb{N}$, $E_M = X$ up to a set of measure zero, then $w \geq 1/M$, and we can simply take w to be 1).

5.4. **Proof of Theorem 2.2.** We will prove Theorem 2.2 by reducing it to Theorem 4.1. We will need the following lemma whose proof is an easy exercise in elementary point-set topology, and which is therefore omitted.

Lemma 5.6. *Let Z be a compact topological space and suppose (z_n) is an infinite sequence of distinct points in Z. Then there exists a point $z \in Z$ such that every open neighbourhood of z contains infinitely many z_n's.*

Proof of Theorem 2.2. We may assume that $A < \infty$ otherwise there is nothing to prove. Take a nonzero $G \in \mathcal{X}'$, and take E_n as in Corollary 5.5. For each m we can apply Theorem 4.1, with X replaced by E_m, to conclude that there exist $g_{j,m} \in L^1(E_m, \mathrm{d}\mu)$ such that

$$G(x) \leq \prod_{j=1}^{d} g_{j,m}(x)^{\alpha_j} \qquad \text{a.e. on } E_m, \tag{47}$$

and such that for each j,

$$\int_{E_m} g_{j,m}(x) T_j f_j(x) \mathrm{d}\mu(x) \leq A\|G\|_{\mathcal{X}'}\|f_j\|_{\mathcal{Y}_j} \tag{48}$$

for all $f_j \in \mathcal{Y}_j$.

If $E_M = X$ (up to a set of zero measure) for some M, we simply take $g_j = g_{j,M}$ and we are finished. So we may assume that the sets E_m do not stabilise, and therefore that there are infinitely many distinct $g_{j,m}$ for each j.

With w defined as in the previous subsection, let us now calculate

$$\begin{aligned}\|g_{j,m}\|_{L^1(w\,\mathrm{d}\mu)} &\leq \int_{E_m} g_{j,m}(x)\, w_j(x)\, \mathrm{d}\mu(x) = \sum_{n=0}^{m} \int_{E_n \setminus E_{n-1}} g_{j,m}(x)\, T_j h_{j,n}(x)\, \mathrm{d}\mu(x) \\ &\leq \int_{E_m} g_{j,m}(x)\, T_j h_{j,m}(x)\, \mathrm{d}\mu(x) \leq A\|G\|_{\mathcal{X}'}\|h_{j,m}\|_{\mathcal{Y}_j} \leq A\|G\|_{\mathcal{X}'}.\end{aligned}$$

Thus the functions $g_{j,m}$ all lie in a ball in $L^\infty(X,\, w\,\mathrm{d}\mu)^*$ which, by the Banach–Alaoglu theorem, is weak-star compact. It is therefore tempting to extract a weak-star convergent subsequence. However, we must resist this temptation since $L^\infty(X,\, w\,\mathrm{d}\mu)$ is not separable, and thus $L^\infty(X,\, w\,\mathrm{d}\mu)^*$ is not metrisable. We therefore proceed with some caution. We will use Lemma 5.6 as a substitute for the existence of weak-star convergent subsequences.

It is convenient to consider the vectors

$$\begin{aligned}\mathbf{g}_n = (g_{1,n}, \ldots, g_{d,n}) &\in L^1(X, w\,\mathrm{d}\mu) \times \cdots \times L^1(X, w\,\mathrm{d}\mu)\\ \subseteq L^\infty(X,\, w\,\mathrm{d}\mu)^* \times \cdots \times L^\infty(X,\, w\,\mathrm{d}\mu)^* &= (L^\infty(X,\, w\,\mathrm{d}\mu) \times \cdots \times L^\infty(X,\, w\,\mathrm{d}\mu))^*.\end{aligned}$$

By Lemma 5.6 there is a point $\mathbf{S} = (S_1, \ldots, S_d) \in (L^\infty(X,\, w\,\mathrm{d}\mu) \times \cdots \times L^\infty(X,\, w\,\mathrm{d}\mu))^*$ such that every weak-star open neighbourhood of $\mathbf{S}$ contains infinitely many of the $\mathbf{g}_n$.

Lemma 5.7. *Suppose* $(\mathbf{g}_n)$ *and* $\mathbf{S}$ *are as above.*

(a) If for some $\mathbf{q} \in \mathcal{M}(X, w\,\mathrm{d}\mu)^d_+$ *we have*

$$\mathbf{g}_n(\mathbf{q}) = \sum_{j=1}^d \int_X g_{j,n} q_j \, w\,\mathrm{d}\mu = \int_X \mathbf{g}_n \cdot \mathbf{q}\, w\,\mathrm{d}\mu \le K$$

for all sufficiently large n*, then*

$$\mathbf{S}(\mathbf{q}) \le K.$$

(b) If for some $\mathbf{q} \in L^\infty(X, w\,\mathrm{d}\mu)^d$ *we have*

$$\mathbf{g}_n(\mathbf{q}) = \sum_{j=1}^d \int_X g_{j,n} q_j w\,\mathrm{d}\mu = \int_X \mathbf{g}_n \cdot \mathbf{q}\, w\,\mathrm{d}\mu \ge L$$

for all sufficiently large n*, then*

$$\mathbf{S}(\mathbf{q}) \ge L.$$

Proof. (a) Suppose for a contradiction that $\mathbf{S}(\mathbf{q}) \ge K' > K$ for some finite K'. Let

$$U = \{\mathbf{R} \in ((L^\infty(w\,\mathrm{d}\mu))^d)^* \;:\; \mathbf{R}(\mathbf{q}) > (K + K')/2\}.$$

Then $\mathbf{S} \in U$, and U is weak-star open since for each $\mathbf{q} \in \mathcal{M}(w\,\mathrm{d}\mu)^d_+$ the functional $\mathbf{R} \mapsto \mathbf{R}(\mathbf{q})$ is weak-star lower semicontinuous, by (a vector-valued version of) Lemma 5.2. Thus U is an open neighbourhood of $\mathbf{S}$ in the weak-star topology. By the above remarks, U must contain infinitely many of the $(\mathbf{g}_n)$. But for all n sufficiently large,

$$\mathbf{g}_n(\mathbf{q}) = \int_X \mathbf{g}_n \cdot \mathbf{q}\, w\,\mathrm{d}\mu \le K < (K + K')/2$$

and so none of these $\mathbf{g}_n$ can be in U. This is a contradiction, and therefore $\mathbf{S}(\mathbf{q}) \le K$.

(b) Suppose for a contradiction that $\mathbf{S}(\mathbf{q}) = L' < L$. Let

$$U = \{\mathbf{R} \in ((L^\infty(w\,\mathrm{d}\mu))^d)^* \;:\; \mathbf{R}(\mathbf{q}) < (L + L')/2\}.$$

Then $\mathbf{S} \in U$, and U is weak-star open since for each $\mathbf{q} \in L^\infty(w\,\mathrm{d}\mu)^d_+$ the functional $\mathbf{R} \mapsto \mathbf{R}(\mathbf{q})$ is weak-star continuous. Thus U is an open neighbourhood of $\mathbf{S}$ in the weak-star topology. By the above remarks, U must contain infinitely many of the $(\mathbf{g}_n)$. But for all n sufficiently large,

$$\mathbf{g}_n(\mathbf{q}) = \int_X \mathbf{g}_n \cdot \mathbf{q}\, w\,\mathrm{d}\mu \ge L > (L + L')/2$$

and so none of these $\mathbf{g}_n$ can be in U. This is a contradiction, and therefore $\mathbf{S}(\mathbf{q}) \ge L$. □

We now wish to verify that the absolutely continuous components $(S_{j\mathrm{rn}})$ (where the Radon–Nikodym derivative is with respect to the measure $w\,\mathrm{d}\mu$) of (S_j) satisfy

$$G(x) \le \prod_{j=1}^{d} S_{j\mathrm{rn}}(x)^{\alpha_j} \quad \text{a.e. on } X\text{, and}$$
$$\int_X S_{j\mathrm{rn}}(x) T_j f_j(x)\,\mathrm{d}\mu(x) \le A\|G\|_{\mathcal{X}'}\|f_j\|_{\mathcal{Y}_j} \tag{49}$$

for all j and for all $f_j \in \mathcal{Y}_j$. Since we know that $S_j \in (L^\infty(X, w\,\mathrm{d}\mu))^*$, we will therefore have $S_{j\mathrm{rn}} \in L^1(w\,\mathrm{d}\mu)$, and this will conclude the proof of Theorem 2.2.

We may suppose that $\|G\|_{\mathcal{X}'} = 1$.

We look at the second inequality from (49) first. Fix m and consider

$$\int_{E_m} S_{jrn}(x)T_j f_j(x)\mathrm{d}\mu(x) = \int_X S_{jrn}(x)w(x)^{-1}\chi_{E_m}(x)T_j f_j(x)w(x)\mathrm{d}\mu(x)$$
$$\le S_j(w^{-1}\chi_{E_m}T_j f_j)$$

by positivity of each component in the Yosida–Hewitt decomposition of S_j, (recall Theorem 4.4). Now, for $n \ge m$,

$$\int g_{jn}(x)[w(x)^{-1}T_j f_j(x)\chi_{E_m}(x)]w(x)\mathrm{d}\mu(x) \le \int g_{jn}(x)T_j f_j(x)\mathrm{d}\mu(x) \le A\|f_j\|_{\mathcal{Y}_j},$$

so that by Lemma 5.7(a) (in the scalar case),

$$S_j(w^{-1}T_j f_j \chi_{E_m}) \le A\|f_j\|_{\mathcal{Y}_j}.$$

Thus

$$\int_{E_m} S_{jrn}(x)T_j f_j(x)\mathrm{d}\mu(x) \le A\|f_j\|_{\mathcal{Y}_j}$$

and we now let $m \to \infty$ to get the second inequality of (49).

Now we look at the first inequality from (49). By Lemma 5.1 (using the measure $w\,\mathrm{d}\mu$) and the fact that the E_m exhaust X, it suffices to show that for each fixed m, and all simple β_j,

$$\int_{E_m} G(x)\prod_{j=1}^{d}\beta_j(x)^{\alpha_j}w(x)\mathrm{d}\mu(x) \le \sum_{j=1}^{d}\alpha_j S_j(\beta_j).$$

Take $n \ge m$. By (47) and the arithmetic-geometric mean inequality, the left-hand side is at most

$$\int_{E_m}\prod_{j=1}^{d}[g_{jn}(x)\beta_j(x)]^{\alpha_j}w(x)\mathrm{d}\mu(x) \le \sum_{j=1}^{d}\alpha_j\int_{E_m} g_{jn}(x)\beta_j(x)w(x)\mathrm{d}\mu(x)$$

$$\le \sum_{j=1}^{d}\alpha_j\int_X g_{jn}(x)\beta_j(x)w(x)\mathrm{d}\mu(x) = \sum_{j=1}^{d}\int_X g_{jn}(x)[\alpha_j\beta_j(x)]w(x)\mathrm{d}\mu(x).$$

Thus for all $n \ge m$,

$$\sum_{j=1}^{d}\int_X g_{jn}(x)[\alpha_j\beta_j(x)]w(x)\mathrm{d}\mu(x) \ge \int_{E_m} G(x)\prod_{j=1}^{d}\beta_j(x)^{\alpha_j}w(x)\mathrm{d}\mu(x).$$

Since the simple functions β_j are bounded, Lemma 5.7(b) gives us that

$$\sum_{j=1}^{d} S_j(\alpha_j\beta_j) \geq \int_{E_m} G(x) \prod_{j=1}^{d} \beta_j(x)^{\alpha_j} w(x) \mathrm{d}\mu(x),$$

which is what we want.

This completes the proof of Theorem 2.2. □

Part II. Connections with other topics

6. Complex interpolation and factorisation

We begin by observing that the trivial identity of Example 2,

$$\int_{\mathbb{R}^2} f_1(x_2) f_2(x_1)\,\mathrm{d}x_1 \mathrm{d}x_2 = \int_{\mathbb{R}} f_1 \int_{\mathbb{R}} f_2,$$

immediately implies via Theorem 1.3 that, for every nonnegative $G \in L^2(\mathbb{R}^2)$, there exist nonnegative g_1 and g_2 such that

$$G(x) \leq \sqrt{g_1(x) g_2(x)} \quad \text{for almost every} \quad x \in \mathbb{R}^2$$

and

$$\operatorname{ess\,sup}_{x_2} \int g_1(x_1, x_2)\mathrm{d}x_1 \leq \|G\|_2 \quad \text{and} \quad \operatorname{ess\,sup}_{x_1} \int g_2(x_1, x_2)\mathrm{d}x_2 \leq \|G\|_2.$$

While it is not perhaps entirely obvious how to do this explicity (a point to which we return in Sections 9.2, 9.3 and 10.2.1 below), for now we want to point out that this example highlights the connection between our multilinear duality theory and the theory of interpolation of Banach spaces. In particular, we consider the *upper* method of complex interpolation of A. P. Calderón, [19].

Suppose that Z_0 and Z_1 are Banach lattices of measurable functions defined on some measure space. We define

$$Z_0^{1-\theta} Z_1^{\theta} = \{f \ : \ \text{there exist } f_j \in Z_j \text{ such that } |f| \leq |f_0|^{1-\theta}|f_1|^{\theta}\}$$

with

$$\|f\|_{Z_0^{1-\theta} Z_1^{\theta}} = \inf\{\|f_0\|_{Z_0}^{1-\theta} \|f_1\|_{Z_1}^{\theta}\},$$

the inf being taken over all possible decompositions of f. Under the assumption that the unit ball of $Z_0^{1-\theta} Z_1^{\theta}$ is closed in $Z_0 + Z_1$, Calderón showed that

$$Z_0^{1-\theta} Z_1^{\theta} = [Z_0, Z_1]^{\theta}$$

where $[Z_0, Z_1]^{\theta}$ is the interpolation space between Z_0 and Z_1 obtained by the upper complex method.

With this in mind, the factorisation statement in our example is tantamount to the statement

$$L^2(\mathbb{R}^2) \hookrightarrow [L^{\infty}_{x_1}(L^1_{x_2}), L^{\infty}_{x_2}(L^1_{x_1})]^{1/2}.$$

Many other special cases of our theory can be similarly expressed in the language of interpolation. We leave it to the interested reader to pursue this point of view more systematically.

In this particular example, there is further structure, see for example Pisier [37], (in which some of the ideas are attributed to Lust-Piquard). There it is established that we have

$$L^2(\mathbb{R}^2) = HS(L^2(\mathbb{R})) \hookrightarrow \mathcal{L}_{\text{reg}}(L^2) = [L^{\infty}_{x_1}(L^1_{x_2}), L^{\infty}_{x_2}(L^1_{x_1})]^{1/2}$$

where HS denotes the class of Hilbert–Schmidt operators and $\mathcal{L}_{\rm reg}(L^2)$ is the space of *regular* bounded linear operators on L^2. In rough terms, a regular bounded linear operator on L^2 is one such that if its kernel is $K(s,t)$, then $|K(s,t)|$ is also the kernel of a bounded linear operator.

The implicit factorisation arguments involved in establishing results of this type rely on the Hahn–Banach theorem or the Perron–Frobenius theorem, and are thus related to minimax theory; they are similarly non-constructive.

7. Factorisation and convexity

It is also natural to enquire about how factorisation and interpolation interact at the level of particular families of inequalities. For the sake of concreteness, suppose we are in the setting of multilinear generalised Radon transforms on euclidean spaces – so that $T_jF_j = F_j \circ B_j$ for suitable B_j. We shall suppress consideration of any of the technical hypotheses of Theorem 2.2 in what follows. Suppose that we have the pair of inequalities

$$\left\| \prod_{j=1}^d T_j F_j \right\|_{L^{q_k}} \leq A_k \prod_{j=1}^d \|F_j\|_{L^{p_{jk}}} \tag{50}$$

for $k = 0, 1$, where $q_k, p_{jk} \geq 1$.

Each of these has a family of corresponding equivalent factorisation statements, according to Theorem 2.2 and the remarks in Section 1.3. See also Section 7.1 below. After some changes of notation, one such equivalent pair of statements is as follows. For $k = 0, 1$, let $s_k := q_k \sum_j p_{jk}^{-1}$. Then for all nonnegative G_k $(k = 0, 1)$ such that $\int G_k^{s_k'} = 1$, there are nonnegative $g_{10}, \dots, g_{d0}$ and $g_{11}, \dots, g_{d1}$ such that

$$G_k(x) \leq \prod_{j=1}^d g_{jk}(x)^{q_k/p_{jk}s_k} \text{ a.e.} \tag{51}$$

and such that for all f_j with $\int f_j \leq 1$,

$$\int f_j(B_jx) g_{jk}(x) \mathrm{d}x \leq A_k^{q_k/s_k} \tag{52}$$

for $k = 0, 1$.

From (51) and (52) we shall deduce a factorisation statement which implies the natural interpolation statement

$$\left\| \prod_{j=1}^d T_j F_j \right\|_{L^{q_\theta}} \leq A_0^{1-\theta} A_1^{\theta} \prod_{j=1}^d \|F_j\|_{L^{p_{j\theta}}} \tag{53}$$

for $0 < \theta < 1$, where, as usual, $1/q_\theta = (1-\theta)/q_0 + \theta/q_1$, and similarly for $1/p_{j\theta}$.

Indeed, given a nonnegative G with $\int G = 1$, let $G_k = G^{1/s_k'}$. Taking convex combinations in (51) gives us

$$G(x) \leq \prod_{j=1}^d g_{j0}(x)^{q_0 s_0'(1-\theta)/p_{j0}s_0} g_{j1}(x)^{q_1 s_1' \theta/p_{j1}s_1} \text{ a.e.} \tag{54}$$

Next, we define

$$\gamma_j(\theta) := \frac{q_0 s_0'}{p_{j0}s_0}(1-\theta) + \frac{q_1 s_1'}{p_{j1}s_1}\theta$$

and define $g_{j\theta}$ by

$$g_{j\theta}^{\gamma_j(\theta)} := g_{j0}(x)^{q_0 s_0'(1-\theta)/p_{j0}s_0} g_{j1}(x)^{q_1 s_1'\theta/p_{j1}s_1}.$$

Then, by (52), we have

$$\int f_j(B_j x) g_{j\theta}(x)\mathrm{d}x = \int f_j(B_j x) g_{j0}(x)^{q_0 s_0'(1-\theta)/p_{j0}s_0\gamma_j(\theta)} g_{j1}(x)^{q_1 s_1'\theta/p_{j1}s_1\gamma_j(\theta)}\mathrm{d}x$$

$$\leq \left(\int f_j(B_j x) g_{j0}(x)\mathrm{d}x\right)^{q_0 s_0'(1-\theta)/p_{j0}s_0\gamma_j(\theta)} \left(\int f_j(B_j x) g_{j1}(x)\mathrm{d}x\right)^{q_1 s_1'\theta/p_{j1}s_1\gamma_j(\theta)}$$

by Hölder's inequality, since $\gamma_j(\theta)$ is defined precisely to ensure the two exponents on the right hand side here add to 1.

Therefore, if $\int f_j \leq 1$,

$$\int f_j(B_j x) g_{j\theta}(x)\mathrm{d}x \leq \left[A_0^{q_0/s_0}\right]^{q_0 s_0'(1-\theta)/p_{j0}s_0\gamma_j(\theta)} \left[A_1^{q_1/s_1}\right]^{q_1 s_1'\theta/p_{j1}s_1\gamma_j(\theta)}.$$

Now let $\beta_j(\theta) := \lambda(\theta)\gamma_j(\theta)$ where $\lambda(\theta)$ is defined so that $\sum_{j=1}^d \beta_j(\theta) = 1$. By the definition of s_0 and s_1 we have

$$\sum_{j=1}^d \gamma_j(\theta) = \sum_{j=1}^d \left(\frac{q_0 s_0'}{p_{j0}s_0}(1-\theta) + \frac{q_1 s_1'}{p_{j1}s_1}\theta\right) = (1-\theta)s_0' + \theta s_1'.$$

So, we take

$$\lambda(\theta) := \frac{1}{(1-\theta)s_0' + \theta s_1'}.$$

Now, bearing in mind Remark 7, we conclude that

$$\prod_{j=1}^d \left(\int f_j(B_j x) g_{j\theta}(x)\mathrm{d}x\right)^{\beta_j(\theta)} \leq \left[A_0^{q_0/s_0}\right]^{\sum_j q_0 s_0'(1-\theta)\beta_j(\theta)/p_{j0}s_0\gamma_j(\theta)} \left[A_1^{q_1/s_1}\right]^{\sum_j q_1 s_1'\theta\beta_j(\theta)/p_{j1}s_1\gamma_j(\theta)}$$

$$= \left[A_0^{q_0/s_0}\right]^{\lambda(\theta)\sum_j q_0 s_0'(1-\theta)/p_{j0}s_0} \left[A_1^{q_1/s_1}\right]^{\lambda(\theta)\sum_j q_1 s_1'\theta/p_{j1}s_1} = \left[A_0^{q_0/s_0}\right]^{\lambda(\theta)s_0'(1-\theta)} \left[A_1^{q_1/s_1}\right]^{\lambda(\theta)s_1'\theta}$$

$$= A_0^{\frac{s_0' q_0 \lambda(\theta)(1-\theta)}{s_0}} A_1^{\frac{s_1' q_1 \lambda(\theta)\theta}{s_1}} = \left\{\left[A_0^{\frac{s_0' q_0 \lambda(\theta)(1-\theta)}{s_0}} A_1^{\frac{s_1' q_1 \lambda(\theta)\theta}{s_1}}\right]^{S(\theta)/Q(\theta)}\right\}^{Q(\theta)/S(\theta)}$$

for a certain quantity $S(\theta)/Q(\theta)$ to which we turn our attention next. Indeed we define this quantity (*not* $S(\theta)$, $Q(\theta)$ separately), so that the exponents on A_0 and A_1 inside the curly brackets sum to 1. That is,

$$\frac{Q(\theta)}{S(\theta)} := \lambda(\theta)\left(\frac{s_0' q_0(1-\theta)}{s_0} + \frac{s_1' q_1 \theta}{s_1}\right).$$

Let us define these exponents of A_0 and A_1 as $1-\alpha(\theta)$ and $\alpha(\theta)$ respectively; that is, we define $\alpha(\theta)$ by

$$\alpha(\theta) := \frac{S(\theta)}{Q(\theta)}\lambda(\theta)\frac{s_1' q_1 \theta}{s_1}.$$

Next, we want the $\beta_j = \lambda\gamma_j$ to be of the form $\beta_j(\theta) = \frac{Q(\theta)}{P_j(\theta)S(\theta)}$ for certain $P_j(\theta)$; that is, $\frac{1}{P_j(\theta)} = \frac{S(\theta)\beta_j(\theta)}{Q(\theta)} = \frac{S(\theta)\lambda(\theta)\gamma_j(\theta)}{Q(\theta)}$. So, bearing in mind the definitions of γ_j and S/Q, we define $P_j(\theta)$ by

$$\frac{1}{P_j(\theta)} := \frac{\frac{q_0 s_0'}{p_{j0}s_0}(1-\theta) + \frac{q_1 s_1'}{p_{j1}s_1}\theta}{\frac{s_0' q_0(1-\theta)}{s_0} + \frac{s_1' q_1\theta}{s_1}}.$$

Finally, we define $Q(\theta)$ by

$$\frac{1}{Q(\theta)} := (1-\alpha(\theta))\frac{1}{q_0} + \alpha(\theta)\frac{1}{q_1}.$$

It is not hard to check that with all these definitions in place, we have, for each j,

$$\frac{1}{P_j(\theta)} = (1-\alpha(\theta))\frac{1}{p_{j0}} + \alpha(\theta)\frac{1}{p_{j1}}.$$

We therefore have that for each $0 \leq \theta \leq 1$, for all $G_\theta = G^{1/S'(\theta)}$ such that $\int G_\theta^{S'(\theta)} = 1$, there exist $g_{j\theta}$ such that

$$G_\theta(x) \leq \prod_{j=1}^d g_{j\theta}(x)^{Q(\theta)/P_j(\theta)S(\theta)}$$

and, for f_j such that $\int f_j \leq 1$,

$$\prod_{j=1}^d \left(\int f_j(B_j x) g_{j\theta}(x)\mathrm{d}x\right)^{Q(\theta)/P_j(\theta)S(\theta)} \leq \left(A_0^{1-\alpha(\theta)}A_1^{\alpha(\theta)}\right)^{Q(\theta)/S(\theta)}.$$

Note particularly that the exponents $Q(\theta)/P_j(\theta)S(\theta)$ sum to 1 since $\sum_{j=1}^d \beta_j = 1$.

Consequently, using the flexibility that Remark 7 affords us,

$$\left\|\prod_{j=1}^d T_j F_j\right\|_{L^{Q(\theta)}} \leq A_0^{1-\alpha(\theta)}A_1^{\alpha(\theta)}\prod_{j=1}^d \|F_j\|_{L^{P_j(\theta)}}$$

for $0 < \theta < 1$. Noting that the map $\alpha : [0,1] \to [0,1]$ is a surjection completes the argument proving (53).

The argument given here provides no insight into cases in which (53) might hold with a smaller constant than $A_0^{1-\theta}A_1^{\theta}$.

7.1. **Factorisation and multiple manifestations of generalised Radon transforms.** As we have observed in Section 1.3 there may be multiple equivalent manifestations of the same multilinear inequality. For concreteness, suppose that we are once again considering multilinear generalised Radon transforms on euclidean spaces so that $T_j f = f \circ B_j$ for suitable B_j. Then the two inequalities

$$\left\|\prod_{j=1}^d (T_j f_j)^{\alpha_j}\right\|_q \leq A \prod_{j=1}^d \left\|f_j\right\|_{p_j}^{\alpha_j}$$

and

$$\left\|\prod_{j=1}^d (T_j \tilde{f}_j)^{\tilde{\alpha}_j}\right\|_{\tilde{q}} \leq \tilde{A} \prod_{j=1}^d \left\|\tilde{f}_j\right\|_{\tilde{p}_j}^{\tilde{\alpha}_j}$$

(where we are imposing $\sum_{j=1}^d \alpha_j = 1 = \sum_{j=1}^d \tilde{\alpha}_j$) are clearly equivalent provided that $A^q = \tilde{A}^{\tilde{q}}$ and $\alpha_j\tilde{p}_j/\tilde{\alpha}_j p_j = \tilde{q}/q$ for all j. The corresponding factorisation statements

For all nonnegative $G \in L^{q'}$ there exist nonnegative locally integrable functions g_j such that

$$G(x) \le \prod_{j=1}^{d} g_j(x)^{\alpha_j} \qquad \text{a.e.}$$

and such that for each j, for all $f_j \in L^{p_j}$,

$$\int g_j(x) f_j(B_j x) \mathrm{d}x \le A \|G\|_{q'} \|f_j\|_{p_j}.$$

and

For all nonnegative $\tilde{G} \in L^{\tilde{q}'}$ there exist nonnegative locally integrable functions $\tilde{g}_j$ such that

$$\tilde{G}(x) \le \prod_{j=1}^{d} \tilde{g}_j(x)^{\tilde{\alpha}_j} \qquad \text{a.e.}$$

and such that for each j, for all $\tilde{f}_j \in L^{\tilde{p}_j}$,

$$\int \tilde{g}_j(x) \tilde{f}_j(B_j x) \mathrm{d}x \le \tilde{A} \|\tilde{G}\|_{L^{\tilde{q}'}} \|\tilde{f}_j\|_{\tilde{p}_j}.$$

are therefore also equivalent (subject to suitable hypotheses), by Proposition 1.1 and Theorem 1.3. However it is not immediately apparent whether this equivalence can be seen directly via changes of notation coupled with simple convexity arguments. In this connection the remarks in Section 5.7 of [17] may be helpful.

8. Factorisation and more general multilinear operators

The multilinear operators we have considered have a rather special form in so far as they are built out of a collection of positive linear operators by taking a pointwise geometric mean. One may ask to what extent the theory we have developed is valid for more general multilinear operators $T : \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_d \to \mathcal{X}$. In such a setting we will no longer be able to attribute different "weights" α_j to the different components $\mathcal{Y}_j$, and all of them will need to be treated on an equal footing.

For a nonnegative kernel K, let us therefore consider multilinear operators of the form

$$T(f_1, \dots, f_d)(x) = \int_{Y_d} \cdots \int_{Y_1} K(x, y_1, \dots, y_d) f_1(y_1) \dots f_d(y_d) \mathrm{d}\nu_1(y_1) \dots \mathrm{d}\nu_d(y_d)$$

and inequalities of the form

$$\left\| T(f_1, \dots, f_d)^{1/d} \right\|_{\mathcal{X}} \le A \prod_{j=1}^{d} \|f_j\|_{\mathcal{Y}_j}^{1/d}. \tag{55}$$

When K is of the form $K(x, y_1, \dots, y_d) = K_1(x, y_1) \dots K_d(x, y_d)$, these are the special cases of the inequalities (15) given by $\alpha_j = 1/d$ for $j = 1, \dots, d$. It is very natural to ask whether there is a general duality/factorisation result along the same lines as Proposition 2.1 and Theorem 2.2 which yields a necessary and sufficient condition for the validity of inequality (55).

As the reader will readily verify by following the proof of Proposition 2.1, inequality (55) does indeed hold (under hypotheses on $\mathcal{X}$ and $\mathcal{Y}_j$ similar to those of Proposition 2.1), if, for all $G \in \mathcal{X}'$

such that $\|G\|_{\mathcal{X}'} \leq 1$, we have that there there exist nonnegative functions g_j on $X \times Y_j$ such that

$$\begin{aligned} K(x, y_1, \ldots, y_d)^{1/d} G(x) &\leq \prod_{j=1}^{d} g_j(x, y_j)^{1/d} \quad \text{a.e.} \\ \text{and } \left\| \int_X g_j(x, \cdot) d\mu(x) \right\|_{\mathcal{Y}_j^*} &\leq A. \end{aligned} \tag{56}$$

This observation has proved very useful in multilinear Kakeya theory, see Section 11 below.

However, the converse is not true, namely inequality (55) does not in general imply the existence of S_j such that (56) holds even if we assume that the integral kernel K is invariant under permutations of the y-variables:

Proposition 8.1. *Let $d = 2$. Let $X = Y_1 = Y_2 = \{1, 2\} = \Omega$ with counting measure, $\mathcal{X} = L^4(\Omega)$, and $\mathcal{Y}_1 = \mathcal{Y}_2 = L^2(\Omega)$. There exists a bilinear $T : L^2(\Omega) \times L^2(\Omega) \to L^4(\Omega)$ such that (55) holds with $A = 2^{1/4}$ but such that (56) can only hold with $A \geq 2^{1/2}$.*

Proof. Let the integral kernel K of T satisfy

$$K(1, 1, 1) = K(2, 1, 1) = K(2, 2, 2) = 1$$

and let K equal zero otherwise. Let $f_1 = (a_1, a_2)$ and $f_2 = (b_1, b_2)$ and we assume $a_1^2 + a_2^2 = b_1^2 + b_2^2 = 1$. Then

$$T(f_1, f_2)(1) = a_1 b_1 \quad \text{and} \quad T(f_1, f_2)(2) = a_1 b_1 + a_2 b_2$$

so

$$\int T(f_1, f_2)(x)^2 \mathrm{d}x = (a_1 b_1)^2 + (a_1 b_1 + a_2 b_2)^2.$$

This is clearly maximised, subject to the normalisation constraints, by taking $a_1 = b_1 = 1$, $a_2 = b_2 = 0$ and the maximum is 2. So we see that the multilinear inequality (55) holds for this operator with $A = 2^{1/4}$.

For problem (56), consider $G = (0, 1)$. Then the non-trivial constraints are

$$1 \leq \sqrt{g_1(2, 1) g_2(2, 1)} \quad \text{and} \quad 1 \leq \sqrt{g_1(2, 2) g_2(2, 2)}$$

and

$$\begin{aligned} \sqrt{(g_1(1, 1) + g_1(2, 1))^2 + (g_1(1, 2) + g_1(2, 2))^2} &\leq A \\ \sqrt{(g_2(1, 1) + g_2(2, 1))^2 + (g_2(1, 2) + g_2(2, 2))^2} &\leq A. \end{aligned}$$

Using $uv \leq (u^2 + v^2)/2$ on the lower bounds gives

$$1 \leq (g_1(2, 1)^2 + g_2(2, 1)^2)/2 \quad \text{and} \quad 1 \leq (g_1(2, 2)^2 + g_2(2, 2)^2)/2,$$

so

$$2 \leq g_1(2, 1)^2 + g_2(2, 1)^2 \quad \text{and} \quad 2 \leq g_1(2, 2)^2 + g_2(2, 2)^2,$$

so

$$4 \leq g_1(2, 1)^2 + g_2(2, 1)^2 + g_1(2, 2)^2 + g_2(2, 2)^2,$$

and thus

$$2 \leq \max\{g_1(2, 1)^2 + g_1(2, 2)^2, g_2(2, 1)^2 + g_2(2, 2)^2\}$$

giving $A \geq 2^{1/2}$, which is strictly larger than $2^{1/4}$. So while inequality (55) holds in this case, there are G for which there are *no* g_j satisfying (56) with the same value of A. □

We invite the reader to use this idea to construct examples where (55) holds with $A = 1$ but for which (56) holds for no finite A.

See [28] for a different approach to inequalities of the form (55), based upon considerations related to Schur's lemma rather than duality.

Part III. Examples and illustrations of the theory

In this part we revisit the examples in the introduction which motivated our study. We examine what insights our duality–factorisation results bring to, and have gained from, each of them. In some cases we reap the benefits of more direct and streamlined factorisation-based proofs of known inequalities. In others, an interesting challenge is posed – it can be argued that we cannot claim to have a full understanding of an inequality until we can exhibit its equivalent factorisation statement.

9. Classical inequalities revisited

9.1. **Hölder's inequality.** We observed above that the multilinear form of Hölder's inequality for nonnegative functions is equivalent, for any fixed set of exponents $\alpha_j > 0$ with $\sum_{j=1}^d \alpha_j = 1$, to

$$\|f_1^{\alpha_1} \cdots f_d^{\alpha_d}\|_q \leq \|f_1\|_{q_1}^{\alpha_1} \cdots \|f_d\|_{q_d}^{\alpha_d}$$

for any choice of indices $1 \leq q_j < \infty$ and $1 \leq q < \infty$ which satisfies $\sum_{j=1}^d \alpha_j q_j^{-1} = q^{-1}$.

By Theorem 2.2, each instance of this inequality is equivalent to the existence of a subfactorisation of any $G \in L^{q'}$ as

$$G(x) \leq \prod_{j=1}^d g_j(x)^{\alpha_j} \quad \text{a.e.}$$

where

$$\|g_j\|_{q_j'} \leq \|G\|_{q'}.$$

Taking $g_j = \lambda_j G^{\gamma_j}$ for appropriate λ_j and γ_j verifies this. In particular, if we take $q_j = q \geq 1$ for all j, then we can simply take $g_j = G$ for all j.

9.2. **The affine-invariant Loomis–Whitney inequality.** Recall that the Loomis–Whitney inequality is

$$\Big|\int_{\mathbb{R}^n} F_1(\pi_1 x) \cdots F_n(\pi_n x)\, \mathrm{d}x\Big| \leq \|F_1\|_{L^{n-1}(\mathbb{R}^{n-1})} \cdots \|F_n\|_{L^{n-1}(\mathbb{R}^{n-1})},$$

where $\pi_j x = (x_1, \dots, \widehat{x_j}, \dots, x_n)$ is projection onto the hyperplane perpendicular to the j'th standard basis vector e_j. For every $0 < p < \infty$ this is equivalent to the inequality

$$\|f_1(\pi_1 x)^{1/n} \cdots f_n(\pi_n x)^{1/n}\|_{L^{np/(n-1)}(\mathbb{R}^n)} \leq \|f_1\|_{L^p(\mathbb{R}^{n-1})}^{1/n} \cdots \|f_n\|_{L^p(\mathbb{R}^{n-1})}^{1/n}.$$

Each of these inequalities with $p \geq 1$ falls under the scope of our theory.

For example when $p = 1$ we have the equivalent formulation

$$\int_{\mathbb{R}^n} \prod_{j=1}^n f_j(\pi_j x)^{1/(n-1)} \mathrm{d}x \leq \prod_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} f_j\right)^{1/(n-1)}.$$

More generally, if $\pi_{\omega^\perp}$ represents orthogonal projection onto the hyperplane perpendicular to $\omega \in \mathbb{S}^{n-1}$, we have the affine-invariant Loomis–Whitney inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^n f_j(\pi_{\omega_j^\perp} x)^{1/(n-1)} \mathrm{d}x \leq (\omega_1 \wedge \cdots \wedge \omega_n)^{-1/(n-1)} \prod_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} f_j \right)^{1/(n-1)}, \tag{57}$$

where $(\omega_1 \wedge \cdots \wedge \omega_n)^{-1/(n-1)}$ is the best constant in the inequality. Here, $\omega_1 \wedge \cdots \wedge \omega_n$ is the modulus of the determinant of the matrix whose columns are $\omega_1, \ldots, \omega_n$, and it is the volume of the parallepiped whose sides are given by the vectors ω_j. (Clearly if we choose all the ω_j to be the same we cannot expect a finite constant, and the constant in general should reflect "quantitative linear independence" of the ω_j.)

We give a direct and elegant proof of (57) by explicitly establishing a suitable factorisation. Indeed, according to Proposition 1.1, it is sufficient that for every nonnegative $G \in L^n(\mathbb{R}^n)$ we can find $g_1, \ldots, g_n$ such that

$$G(x) = g_1(x)^{1/n} \cdots g_n(x)^{1/n} \text{ a.e.}$$

and, for all j and almost every x,

$$\int g_j(x + t\omega_j) \mathrm{d}t = (\omega_1 \wedge \cdots \wedge \omega_n)^{-1/n} \|G\|_n.$$

This is because for any $f : \mathbb{R}^{n-1} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$, writing $x \in \mathbb{R}^n$ as $x = u + t\omega_j$ with $u \in \omega_j^\perp$, we have

$$\begin{aligned}\int f(\pi_{\omega_j^\perp} x) g(x) \mathrm{d}x &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} f(\pi_{\omega_j^\perp}(u + t\omega_j)) g(u + t\omega_j) \mathrm{d}t \mathrm{d}u \\ &= \int_{\mathbb{R}^{n-1}} f(u) \left(\int_{\mathbb{R}} g(u + t\omega_j) \mathrm{d}t \right) \mathrm{d}u.\end{aligned}$$

Let $G : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative function which satisfies $\int_{\mathbb{R}^n} G(x)^n \mathrm{d}x = 1$. For $\omega_1, \ldots, \omega_n \in \mathbb{S}^{n-1}$ and $\xi \in \mathbb{R}^n$ let us first note that if we set, for $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$,

$$y(s) = \xi + s_1\omega_1 + \cdots + s_{n-1}\omega_{n-1} + s_n\omega_n,$$

then we have that the Jacobian map $\partial y / \partial s$ satisfies

$$|\det(\partial y/\partial s)| = \omega_1 \wedge \cdots \wedge \omega_n.$$

Therefore, for every $\xi \in \mathbb{R}^n$,

$$\begin{aligned}\int G(\xi + s_1\omega_1 + \cdots + s_{n-1}\omega_{n-1} + s_n\omega_n)^n \mathrm{d}s_1 \mathrm{d}s_2 \ldots \mathrm{d}s_n &= \int G(y(s))^n \mathrm{d}s \\ = \int G(y)^n \frac{1}{|\det(\partial y/\partial s)|} \mathrm{d}y &= (\omega_1 \wedge \cdots \wedge \omega_n)^{-1}.\end{aligned}$$

Secondly, $G(x)^n$ can be written (for a.e. x) as a telescoping product

$$\begin{aligned}&\frac{G(x)^n}{\int G(x + s_1\omega_1)^n \mathrm{d}s_1} \times \frac{\int G(x + s_1\omega_1)^n \mathrm{d}s_1}{\int G(x + s_1\omega_1 + s_2\omega_2)^n \mathrm{d}s_1 \mathrm{d}s_2} \times \cdots \\ &\times \frac{\int G(x + s_1\omega_1 + \cdots + s_{n-1}\omega_{n-1})^n \mathrm{d}s_1 \mathrm{d}s_2 \ldots \mathrm{d}s_{n-1}}{\int G(x + s_1\omega_1 + \cdots + s_{n-1}\omega_{n-1} + s_n\omega_n)^n \mathrm{d}s_1 \mathrm{d}s_2 \ldots \mathrm{d}s_n} \times (\omega_1 \wedge \cdots \wedge \omega_n)^{-1} \\ &:= g_1(x) \ldots g_n(x)\end{aligned}$$

where

$$g_j(x) = \frac{\int G(x + s_1\omega_1 + \cdots + s_{j-1}\omega_{j-1})^n \mathrm{d}s_1\mathrm{d}s_2 \ldots \mathrm{d}s_{j-1}}{\int G(x + s_1\omega_1 + \cdots + s_{j-1}\omega_{j-1} + s_j\omega_j)^n \mathrm{d}s_1\mathrm{d}s_2 \ldots \mathrm{d}s_j} \times (\omega_1 \wedge \cdots \wedge \omega_n)^{-1/n}.$$

If we replace x by $x + t\omega_j$ in this formula, the denominator is unchanged, and so if we then integrate with respect to t we immediately see that

$$\int g_j(x + t\omega_j)\mathrm{d}t = (\omega_1 \wedge \cdots \wedge \omega_n)^{-1/n}$$

identically for $x \in \mathbb{R}^n$, as we needed.

A similar approach works when we instead consider projections onto subspaces whose codimensions sum to n. Indeed, suppose that we have subspaces E_j of $\mathbb{R}^n$ with $\dim E_j = k_j$ and $\sum_{j=1}^d k_j = n$ and assume that $\mathbb{R}^n = E_1 + \cdots + E_d$ as an algebraic direct sum.

We identify a quantity which measures lack of orthogonality of these subspaces in the same way that the wedge product $\omega_1 \wedge \cdots \wedge \omega_n$ measures the degeneracy in the directions $\omega_1, \ldots, \omega_n \in \mathbb{S}^{n-1}$. Let $\{e_{j1}, e_{j2}, \ldots, e_{jk_j}\}$ be an orthonormal basis for E_j and define

$$E_1 \wedge \cdots \wedge E_d := \wedge_{j=1}^d \wedge_{k=1}^{k_j} e_{jk};$$

that is, $E_1 \wedge \cdots \wedge E_d$ is the absolute value of the determinant of the $n \times n$ matrix whose j'th block of k_j columns comprises an orthonormal basis for E_j. It is easily checked that this quantity is independent of the particular orthonormal bases chosen, and it can of course be defined in a more canonical and invariant way.

Proposition 9.1. *For E_j as above, let π_j be the projection whose kernel is E_j. Then we have the affine-invariant k_j-plane Loomis–Whitney inequality:*

$$\begin{aligned} &\int_{\mathbb{R}^n} f_1(\pi_1 x)^{1/(d-1)} \ldots f_d(\pi_d x)^{1/(d-1)}\mathrm{d}x \\ &\le (E_1 \wedge \cdots \wedge E_d)^{-1/(d-1)} \left(\int f_1\right)^{1/(d-1)} \cdots \left(\int f_d\right)^{1/(d-1)}. \end{aligned} \tag{58}$$

The proof via factorisation is formally the same as the case when $k_j = 1$ for all j, where now the roles of the variables $s_j \in \mathbb{R}^1$ are replaced by copies of $\mathbb{R}^{k_j}$. We leave the details to the reader.

In the special case of the trivial identity,

$$\int_{\mathbb{R}^2} F_1(x_2)F_2(x_1)\,\mathrm{d}x_1\mathrm{d}x_2 = \int_{\mathbb{R}} F_1 \int_{\mathbb{R}} F_2,$$

(see Section 6), a suitable factorisation of $G \in L^2(\mathbb{R}^2)$ with $\|G\|_2 = 1$ is given by $G(x)^2 = g_1(x)g_2(x)$ a.e., where

$$g_1(x_1, x_2) = \frac{G(x_1, x_2)^2}{\int_{\mathbb{R}} G(s, x_2)^2\mathrm{d}s}$$

and

$$g_2(x_1, x_2) = \int_{\mathbb{R}} G(s, x_2)^2\mathrm{d}s.$$

Note that this factorisation depends upon the order we have assigned to $\{1, 2\}$. On the other hand, given this ordering, the essentially unique way to write

$$G(x)^2 = g_1(x_1, x_2)g_2(x_2)$$

where $\|g_1(\cdot, x_2)\|_1 = 1$ for all x_2 and $\|g_2\|_1 = 1$ is as we have given. See Section 10.2.1, where this observation drives related issues.

There are many variants of the Loomis–Whitney inequality – for example Finner's inequalities [25] – which can likewise be established by the same factorisation method.

9.3. **The nonlinear Loomis–Whitney inequality.** Nonlinear Loomis–Whitney inequalities (and some multilinear generalised Radon transforms) can likewise be established by similar methods. In fact the first proof of the nonlinear Loomis–Whitney inequality with essentially the sharp constant was obtained via an explicit factorisation technique. We give the details.

Let V be an open neighbourhood of 0 in $\mathbb{R}^n$ and U an open neighbourhood of 0 in $\mathbb{R}^{n-1}$. Let $\pi : V \to U$ be a C^1 submersion onto U, and for $x \in V$ let $\omega(x)$ be the wedge product of the rows of $\mathrm{d}\pi(x)$. We assume that the fibres $\pi^{-1}(u)$ for $u \in U$ can be parametrised by C^1 curves $t \mapsto \gamma(t, x)$ in such a way that

- for all $x \in V$, $\gamma(0, x) = x$
- for all $x \in V$, for all t, $\pi\gamma(t, x) = \pi x$
- (semigroup property) for all $x \in V$, for all t and s,
$$\gamma(t, \gamma(s, x)) = \gamma(s + t, x)$$
- for all x and t, $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t, x) = \omega(\gamma(t, x))$.

The domain of each curve $\gamma(\cdot, x)$ will be an open interval I_x containing 0 which we largely suppress in what follows, but we stress that $\gamma(I_x, x)$ is the entire fibre containing x. In all the t-integrals below it is assumed that we are integrating over such maximal domains.

We note that under these assumptions, especially the last one, the co-area formula gives
$$\int_V f(\pi x)g(x)\mathrm{d}x = \int_U f(u)\left(\int g(\gamma(t, \tilde{u}))\mathrm{d}t\right)\mathrm{d}u$$
for any reasonable functions f and g.

We now assume that we have n submersions $\pi_1, \dots, \pi_n$ as above, and we assume that $\omega_1(0) \wedge \cdots \wedge \omega_n(0) \neq 0$. For each $x \in V$ we define the maps $\mathbf{t} \mapsto \Phi_x(\mathbf{t})$ by
$$\Phi_x : (t_1, \dots, t_n) \mapsto \gamma_1(t_1, \gamma_2(t_2, \dots, \gamma_n(t_n, x)) \dots)$$
which satisfy $\Phi_x(0) = x$ and also
$$|\det(D\Phi_x)(0)| = (\omega_1 \wedge \cdots \wedge \omega_n)(x) \neq 0$$
provided x is sufficiently close to 0.

We shall assume that V is sufficiently small so that for each $x \in V$, the map Φ_x is injective – as was pointed out in [11], even in two dimensions some global hypothesis of this sort is needed.

With the set-up above, for $x \in V$ let
$$W(x) := \inf_{\xi \in V} \det |(D\Phi_\xi)(\Phi_\xi^{-1}(x))|. \tag{59}$$

Note that $W(x) \leq \omega_1(x) \wedge \cdots \wedge \omega_n(x)$, (take $\xi = x$), and that $W(\Phi_x(\mathbf{t})) \leq |\det(D\Phi_x)(\mathbf{t})|$ for all x and $\mathbf{t}$.

For $1 \leq j \leq n$ and suitable F let
$$S_j(x) = \frac{\int \cdots \int F(\gamma_1(t_1, \gamma_2(t_2, \dots, \gamma_{j-1}(t_{j-1}, x)) \dots))\mathrm{d}t_{j-1} \dots \mathrm{d}t_1}{\int \cdots \int F(\gamma_1(t_1, \gamma_2(t_2, \dots, \gamma_j(t_j, x)) \dots))\mathrm{d}t_j \dots \mathrm{d}t_1}$$
(so that S_1 has no integrals in the numerator).

Then we have
$$S_j(\gamma_j(\tau, x)) = \frac{\int \cdots \int F(\gamma_1(t_1, \gamma_2(t_2, \dots, \gamma_{j-1}(t_{j-1}, \gamma_j(\tau, x))) \dots))\mathrm{d}t_{j-1} \dots \mathrm{d}t_1}{\int \cdots \int F(\gamma_1(t_1, \gamma_2(t_2, \dots, \gamma_j(t_j, \gamma_j(\tau, x))) \dots))\mathrm{d}t_j \dots \mathrm{d}t_1}.$$

We claim that for each j and each x,

$$\int S_j(\gamma_j(\tau, x))\mathrm{d}\tau = 1.$$

Indeed, notice that the denominator in the previous expression,

$$\int \cdots \int F(\gamma_1(t_1, \gamma_2(t_2, \ldots, \gamma_j(t_j, \gamma_j(\tau, x))) \ldots))\mathrm{d}t_j \ldots \mathrm{d}t_1,$$

equals

$$\begin{aligned}&\int \cdots \int F(\gamma_1(t_1, \gamma_2(t_2, \ldots, \gamma_j(t_j + \tau, x)) \ldots))\mathrm{d}t_j \ldots \mathrm{d}t_1\\ &= \int \cdots \int F(\gamma_1(t_1, \gamma_2(t_2, \ldots, \gamma_j(t_j, x)) \ldots))\mathrm{d}t_j \ldots \mathrm{d}t_1\end{aligned}$$

by the semigroup property, and is therefore independent of τ. So

$$\int S_j(\gamma_j(\tau, x))\mathrm{d}\tau = \frac{\int \cdots \int F(\gamma_1(t_1, \gamma_2(t_2, \ldots, \gamma_{j-1}(t_{j-1}, \gamma_j(\tau, x))) \ldots))\mathrm{d}t_{j-1} \ldots \mathrm{d}t_1 \mathrm{d}\tau}{\int \cdots \int F(\gamma_1(t_1, \gamma_2(t_2, \ldots, \gamma_j(t_j, x)) \ldots))\mathrm{d}t_j \ldots \mathrm{d}t_1}$$

which equals 1 by Fubini's theorem.

On the other hand,

$$\prod_{j=1}^{n} S_j(x) = \frac{F(x)}{\int \cdots \int F(\gamma_1(t_1, \gamma_2(t_2, \ldots, \gamma_n(t_n, x)) \ldots))\mathrm{d}t_n \ldots \mathrm{d}t_1},$$

so that

$$F(x) = \prod_{j=1}^{n} S_j(x) \int F(\Phi_x(\mathbf{t}))\mathrm{d}\mathbf{t}.$$

Taking $F(x) = S(x)^n W(x)$, we therefore have

$$\begin{aligned}S(x)^n W(x) &= \prod_{j=1}^{n} S_j(x) \int S(\Phi_x(\mathbf{t}))^n W(\Phi_x(\mathbf{t}))\mathrm{d}\mathbf{t}\\ &\leq \prod_{j=1}^{n} S_j(x) \int S(\Phi_x(\mathbf{t}))^n |\det(D\Phi_x)(\mathbf{t})|\mathrm{d}\mathbf{t} = \prod_{j=1}^{n} S_j(x) \int_V S(y)^n \mathrm{d}y\end{aligned}$$

since $W(\Phi_x(\mathbf{t})) \leq |\det(D\Phi_x)(\mathbf{t})|$ for all $\mathbf{t}$ and since each Φ_x is injective. We also have that for each j and each x,

$$\int_V f(\pi_j x) S_j(x)\mathrm{d}x = \int_{U_j} f(u) \left(\int S_j(\gamma_j(\tau, \tilde{u}))\mathrm{d}\tau \right) \mathrm{d}u = \int_{U_j} f.$$

By the easy half of the duality argument, this shows that for all nonnegative $f_j \in L^1(U_j)$ we have

$$\| \prod_{j=1}^{n} f_j(\pi_j x)^{1/n} W(x)^{1/n} \|_{L^{n/(n-1)}(V)} \leq \prod_{j=1}^{n} \left(\int_{U_j} f_j \right)^{1/n}.$$

Consequently we have:

Proposition 9.2. *Under the above assumptions, with W defined as in (59), we have*

$$\int_V \prod_{j=1}^{n} f_j(\pi_j x)^{1/(n-1)} W(x)^{1/(n-1)} \, \mathrm{d}x \leq \prod_{j=1}^{n} \left(\int_{U_j} f_j \right)^{1/(n-1)}.$$

Noting that $W(x) \leq \omega_1(x) \wedge \cdots \wedge \omega_n(x)$, one might ask whether

$$\int_V \prod_{j=1}^n f_j(\pi_j x)^{1/(n-1)} \omega_1(x) \wedge \cdots \wedge \omega_n(x)^{1/(n-1)} \, \mathrm{d}x \leq \prod_{j=1}^n \left(\int_{U_j} f_j \right)^{1/(n-1)}$$

holds for sufficiently small V.

As an immediate corollary of Proposition 9.2, we obtain the sharp form of the nonlinear Loomis–Whitney inequality of [11]:

Corollary 9.3. *Let V be an open neighbourhood of 0 in $\mathbb{R}^n$ and U an open neighbourhood of 0 in $\mathbb{R}^{n-1}$. For $1 \leq j \leq n$, let $\pi_j : V \to U$ be C^1 submersions onto U, and for $x \in V$ let $\omega_j(x)$ be the wedge product of the rows of $\mathrm{d}\pi_j$. Assume that $\omega_1(0) \wedge \cdots \wedge \omega_n(0) \neq 0$. Then, for all $\epsilon > 0$, there is a neighbourhood $V' \subseteq V$ of 0, such that for all f_j*

$$\int_{V'} \prod_{j=1}^n f_j(\pi_j x)^{1/(n-1)} \, \mathrm{d}x \leq (1+\epsilon) \omega_1(0) \wedge \cdots \wedge \omega_n(0)^{-1/(n-1)} \prod_{j=1}^n \left(\int_{U_j} f_j \right)^{1/(n-1)}$$

Proof. Given $\epsilon > 0$ we can choose V' sufficiently small that $\omega_1(0) \wedge \cdots \wedge \omega_n(0) \leq (1+\epsilon)^{n-1} W(x)$ for all $x \in V'$. □

Since the work in this section was presented in various public fora, Bennett et al have shown, using methods based on induction on scales, that any Brascamp–Lieb inequality has a corresponding nonlinear counterpart with the same loss in the constant of at most $(1+\epsilon)$. See [7].

10. Brascamp–Lieb inequalities revisited

We shall discuss the Brascamp–Lieb inequalities under two headings. Firstly we shall address geometric Brascamp–Lieb inequalities (where in particular we can identify the sharp constant and existence of Gaussian extremisers), and secondly we will examine general Brascamp–Lieb inequalities with a finite (but unquantified) constant.

10.1. **Geometric Brascamp–Lieb inequalities.** The next result is a direct application of Theorem 1.3 to the geometric Brascamp–Lieb inequalities of Example 3.

Theorem 10.1. *For $1 \leq j \leq d$ let V_j be a subspace of $\mathbb{R}^n$. Let $B_j : \mathbb{R}^n \to V_j$ be orthogonal projection. Suppose there exist p_j with $0 < p_j < \infty$ such that*

$$\sum_{j=1}^d p_j B_j^* B_j = I_n.$$

Let $1 \leq q_j < \infty$, and define $q = \sum_{j=1}^d p_j q_j$. Then for all $G \in L^{q'}(\mathbb{R}^n)$ there exist $g_1, \dots, g_d$ such that

$$G(x) \leq g_1(x)^{p_1 q_1/q} \dots g_d(x)^{p_d q_d/q} \quad a.e.$$

and, for all j,

$$\left\| \int_{V_j^\perp} g_j \right\|_{L^{q_j'}(V_j)} \leq \|G\|_{q'} .$$

One simply needs to note (see the discussion in Example 3) that under the hypothesis of this theorem,

$$\left\| \prod_{j=1}^d (f_j \circ B_j)^{p_j q_j/q} \right\|_{L^q(\mathbb{R}^n)} \leq \prod_{j=1}^d \|f_j\|_{L^{q_j}(V_j)}^{p_j q_j/q}$$

and $\sum_{j=1}^{d} p_j \geq 1$, and thus $q \geq 1$. Therefore Theorem 1.3 applies.

Except in some rather trivial cases[19] we do not know any such explicit factorisations with the sharp constant 1. For example, let v_1, v_2 and v_3 be unit vectors in $\mathbb{R}^2$ with angle $2\pi/3$ between each pair. Then, with B_j being orthogonal projection onto the span of v_j, we have

$$\frac{2}{3}(B_1^* B_1 + B_2^* B_2 + B_3^* B_3) = I_2.$$

Take $q_j = 1$ for each j so that $q = 2$. Consequently, for all $G \in L^2(\mathbb{R}^2)$, there exist g_1, g_2, g_3 such that

$$G(x) \leq g_1(x)^{1/3} g_2(x)^{1/3} g_3(x)^{1/3} \quad \text{a.e.}$$

and, for each j,

$$\operatorname{ess\,sup}_s \int_{\mathbb{R}} g_j(sv_j + tv_j^{\perp}) \mathrm{d}t \leq \|G\|_2.$$

Even in such simple cases as this the factorisation is not yet understood explicitly.

10.2. **General Brascamp–Lieb inequalities.** On the other hand, under the conditions

$$\sum_{j=1}^{d} p_j \dim \operatorname{im} B_j = n \tag{60}$$

and

$$\dim V \leq \sum_{j=1}^{d} p_j \dim B_j V \tag{61}$$

for all V in the lattice of subspaces of $\mathbb{R}^n$ generated by $\{\ker B_j\}_{j=1}^{d}$, we now indicate how to construct semi-explicit factorisations yielding the finiteness of the constant C in (4). We use the term "semi-explicit" because the construction is algorithmic in nature. Notwithstanding, we give an informal discursive treatment rather than a collection of flow-charts. We assume throughout the discussion that the B_j are nonzero mappings, that is, $n_j = \operatorname{rank}(B_j) \geq 1$ for each j. (If some $B_j = 0$ it plays no role in inequality (4), nor in (60) or (61), and it can simply be dropped.) When $n = 1$ matters quickly reduce to consideration of Hölder's inequality, which is treated in Section 9.1 above, so we shall focus on what happens when $n \geq 2$.

We now sketch how this is done, and we begin with a couple of definitions from [8] and [9]. Given a collection of linear surjections $\{B_j\}$, its *Brascamp–Lieb polytope* is defined by

$$\mathcal{P}(\{B_j\}) = \{(p_1, \ldots, p_d) \in [0, \infty)^d \,:\, \dim V \leq \sum_{j=1}^{d} p_j \dim B_j V \text{ for all subspaces } V\}.$$

This is manifestly a closed convex set, and, as has been previously noted, is contained in $[0, 1]^d$, and is therefore the convex hull of its extreme points. Given data $\{B_j\}$ and $\{p_j\}$, a *critical subspace* is a nontrivial proper subspace V of $\mathbb{R}^n$ for which

$$\dim V = \sum_{j=1}^{d} p_j \dim B_j V. \tag{62}$$

The construction of the factorisations hinges on the question of existence or non-existence of critical subspaces for the problem with data $\{B_j, p_j\}$. Indeed, if there is a critical subspace V for $\{B_j, p_j\}$, then the problem of factorising a function on $\mathbb{R}^n$ decomposes into two factorisation subproblems on the spaces V and $V^{\perp}$, each of which has positive but *strictly smaller* dimension

[19] For example when the V_j are mutually orthogonal and $p_j = 1$ for all j.

than n.[20] This will allow us in effect to induct on the parameter n. These subproblems inherit the same $\{p_j\}$ and have "new" B_j which are related to the "old" B_j in a precise way. The two subproblems inherit the conditions corresponding to (60) and (61): indeed, (60) for each of the two subproblems holds precisely because the subspace V is critical, and we shall make crucial use of this fact. We isolate the details of how this works – in particular how factorisations for the two subproblems combine to give a factorisation for the original problem – in Section 10.2.1 below.

On the other hand, if there is no critical subspace for the problem $\{B_j, p_j\}$, then $(p_1, \dots, p_d)$ lies in the interior of $\mathcal{P}(\{B_j\})$. To establish a factorisation for the problem in this case, it therefore suffices to (i) establish factorisations for the extreme points of $\mathcal{P}(\{B_j\})$ and (ii) to show, given factorisations at the extreme points, how to establish factorisations at all interior points of $\mathcal{P}(\{B_j\})$. Point (ii) is tantamount to showing that factorisations behave well under multilinear interpolation, and this we have already successfully addressed separately in Section 7.

To deal with point (i), we consider the Brascamp–Lieb problems at the extreme points $(\tilde{p}_1, \dots, \tilde{p}_d)$ of $\mathcal{P}(\{B_j\})$,[21] and, at each of them, ask the same question – does there exist a critical subspace? Since $(\tilde{p}_1, \dots, \tilde{p}_d)$ is an extreme point of $\mathcal{P}(\{B_j\})$, there will certainly be subspaces V of $\mathbb{R}^n$ satisfying

$$\dim V = \sum_{j=1}^{d} \tilde{p}_j \dim B_j V. \tag{63}$$

If there is a *nontrivial and proper* such subspace, we have a critical subspace for the problem $\{B_j, \tilde{p}_j\}$, and we can proceed as above, in effect going around the loop. The only remaining possibility is that the only subspaces V of $\mathbb{R}^n$ satisfying (63) are $\{0\}$ and $\mathbb{R}^n$ itself.

We are thus left to deal with the special case of our original problem in which $(p_1, \dots, p_d)$ is an extreme point of $\mathcal{P}(\{B_j\})$, but for which the only subspaces of V of $\mathbb{R}^n$ satisfying (62) are $\{0\}$ and $\mathbb{R}^n$ itself. Matters quickly reduce to rather trivial considerations. Indeed, in this situation, $\mathcal{P}(\{B_j\})$ consists precisely of those $(p_1, \dots, p_d) \in [0, \infty)^d$ lying on the hyperplane $\sum_{j=1}^d p_j n_j = n$, and its extreme points are precisely those of the form $(0, \dots, 0, n/n_j, 0, \dots, 0)$. Since $n_j \leq n$ always, and since $\mathcal{P}(\{B_j\}) \subseteq [0,1]^d$, the only circumstances in which this case arises is when $n_j = n$ for all j. In this case, our Brascamp–Lieb problem at an extreme point of $\mathcal{P}(\{B_j\})$ is necessarily of the form (modulo permutations of the coordinate axes)

$$\int_{\mathbb{R}^n} f_1(B_1 x)^1 f_2(B_2 x)^0 \dots f_d(B_d x)^0 \mathrm{d}x \leq C \left(\int_{\mathbb{R}^n} f_1 \right)^1 \left(\int_{\mathbb{R}^n} f_2 \right)^0 \dots \left(\int_{\mathbb{R}^n} f_d \right)^0$$

or, equivalently,

$$\int_{\mathbb{R}^n} f_1(B_1 x) \mathrm{d}x \leq C \left(\int_{\mathbb{R}^n} f_1 \right)$$

where B_1 is invertible. This of course holds with equality with $C = (\det B_1)^{-1}$, and a trivial factorisation applies.

Running the machine described above in reverse will thus eventually furnish a factorisation in the general case, and, indeed, the only possible loss in terms of sharp constants occurs at steps where interpolation is employed.

[20] To facilitate the discussion which follows we should strictly speaking replace the roles of $\mathbb{R}^n$ and $\mathbb{R}^{n_j}$ by those of abstract n- and n_j-dimensional real Hilbert spaces respectively.

[21] An algorithm for locating these extreme points can be found in [42].

10.2.1. *Factorisation in the presence of a critical subspace.* We give the details needed to close the argument set out above in the presence of a critical subspace. The only place we use criticality is that it implies that (60) and (61) hold for the two subproblems which arise – see [8]. Since these are the necessary and sufficient conditions for finiteness of the constant, we may assume that factorisations for the two subproblems exist. (Formally we proceed by induction on n, and the case $n = 1$ is trivial.)

Let $B_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ be linear surjections. Suppose that U is a nontrivial proper subspace of $\mathbb{R}^n$. (As indicated above, we do not assume that it is a critical subspace.) Define $\tilde{B}_j : U \to B_jU$ and $\tilde{\tilde{B}}_j : U^\perp \to (B_jU)^\perp$ by

$$\tilde{B}_j(x) = B_jx$$

and

$$\tilde{\tilde{B}}_j(y) = \Pi_{(B_jU)^\perp}B_jy.$$

If some $\tilde{B}_j$ or $\tilde{\tilde{B}}_j$ is zero we can simply discard it. (It cannot be the case that every $\tilde{B}_j$ is zero, for if this happened, we would have $U \subseteq \cap_{j=1}^d \ker B_j$, and, as we have noted previously, a necessary condition for finiteness of the Brascamp–Lieb constant is that $\cap_{j=1}^d \ker B_j = \{0\}$. For similar reasons it cannot be the case that every $\tilde{\tilde{B}}_j$ is zero.)

Also define $\Gamma_j : U^\perp \to B_jU$ by

$$\Gamma_j(y) = \Pi_{(B_jU)}B_jy.$$

Here, Π_W denotes orthogonal projection onto a subspace W. So for $x \in U$ and $y \in U^\perp$,

$$B_j(x + y) = \tilde{B}_jx + \tilde{\tilde{B}}_jy + \Gamma_jy = \left(\tilde{B}_jx + \Gamma_jy\right) + \tilde{\tilde{B}}_jy \in B_jU \oplus (B_jU)^\perp.$$

The two Brascamp–Lieb subproblems arising can be written in the form

$$\int_U \prod_{j=1}^d f_j(\tilde{B}_jx)^{p_j/p}\mathrm{d}x \le C\prod_{j=1}^d \left(\int f_j\right)^{p_j/p}$$

and

$$\int_{U^\perp} \prod_{j=1}^d f_j(\tilde{\tilde{B}}_jx)^{p_j/p}\mathrm{d}x \le C\prod_{j=1}^d \left(\int f_j\right)^{p_j/p}$$

where $p = \sum_j p_j \ge 1$. With $\alpha_j = p_j/p$, we are entitled to suppose that the following two corresponding factorisation statments hold:

For all $H \in L^{p'}(U)$ of norm 1 there exist $H_1, \dots, H_d$ such that

$$H(x) \le \prod_{j=1}^d H_j(x)^{\alpha_j}$$

and, for all $\phi \in L^1(B_jU)$ of norm at most 1,

$$\int_U \phi(\tilde{B}_jx)H_j(x)\mathrm{d}x \le K_1;$$

For all $M \in L^{p'}(U^\perp)$ of norm 1 there exist $M_1, \dots, M_d$ such that

$$M(y) \le \prod_{j=1}^d M_j(y)^{\alpha_j}$$

and, for all $\psi \in L^1((B_jU)^\perp)$ of norm at most 1,

$$\int_{U^\perp} \psi(\tilde{\tilde{B}}_j y) M_j(y) \mathrm{d}y \leq K_2.$$

Given $G \in L^{p'}(\mathbb{R}^n)$ of norm 1, we want to subfactorise it as

$$G(z) \leq \prod_{j=1}^d G_j(z)^{\alpha_j}$$

such that for all $f \in L^1(\mathbb{R}^{n_j})$ of norm at most 1,

$$\int_{\mathbb{R}^n} f(B_j z) G_j(z) \mathrm{d}z \leq K_1 K_2.$$

This is a factorisation statement corresponding to the problem

$$\int_{\mathbb{R}^n} \prod_{j=1}^d f_j(B_j x)^{p_j/p} \mathrm{d}x \leq C \prod_{j=1}^d \left(\int f_j \right)^{p_j/p}.$$

If we can do this, then the procedure described above for factorising Brascamp–Lieb problems closes.

We begin by writing $G \in L^{p'}$ of norm 1 as

$$G(x,y) = H_y(x) M(y) \tag{64}$$

where $\|H_y\|_{p'} = 1$ for all y and $\|M\|_{p'} = 1$. We will then factorise M and each H_y as above, and combine the factorisations to obtain a suitable factorisation for G.

Indeed, defining H_y and M by

$$G(x,y) = \frac{G(x,y)}{\left(\int G(x,y)^{p'} \mathrm{d}x\right)^{1/p'}} \left(\int G(x,y)^{p'} \mathrm{d}x \right)^{1/p'} := H_y(x) M(y)$$

is essentially the unique way to achieve (64) with the desired conditions.[22]

Therefore,

$$G(x,y) \leq \prod_{j=1}^d [H_{jy}(x) M_j(y)]^{\alpha_j} := \prod_{j=1}^d G_j(x,y)^{\alpha_j}$$

where for all $y \in U^\perp$, for all $\phi \in L^1(B_jU)$ of norm at most 1,

$$\int_U \phi(\tilde{B}_j x) H_{jy}(x) \mathrm{d}x \leq K_1$$

and where for all $\psi \in L^1((B_jU)^\perp)$ of norm at most 1,

$$\int_{U^\perp} \psi(\tilde{\tilde{B}}_j y) M_j(y) \mathrm{d}y \leq K_2.$$

We want to show that for all $f \in L^1(\mathbb{R}^{n_j})$ of norm at most 1,

$$\int_{\mathbb{R}^n} f(B_j z) G_j(z) \mathrm{d}z = \int_{U^\perp} \int_U f(B_j(x,y)) H_{jy}(x) M_j(y) \mathrm{d}x \mathrm{d}y \leq K_1 K_2.$$

[22]Indeed, suppose G is in the mixed-norm space $L^r_{\mathrm{d}y}(L^s_{\mathrm{d}x})$ and we want to write $G(x,y) = H(x,y)M(y)$ where $\|M\|_r = \|G\|_{L^r(L^s)}$ and where $\|H(\cdot,y)\|_s = 1$ for all y. Integrating $G(x,y)^s = H(x,y)^s M(y)^s$ with respect to x shows that the only way to do this is to take $M(y) = \|G(\cdot,y)\|_s$ and $H(x,y) = G(x,y)/\|G(\cdot,y)\|_s$. See the remarks at the end of Section 9.2.

Fix $y \in U^\perp$ and write the inner integral over U as

$$\int_U f(B_j(x,y))H_{jy}(x)\mathrm{d}x = \int_U f(B_jx + B_jy)H_{jy}(x)\mathrm{d}x.$$

Now $f(B_jx+B_jy) = f((\tilde{B}_jx+\Gamma_jy)+\tilde{\tilde{B}}_jy)$. For $w \in B_jU$ and $\xi \in (B_jU)^\perp$ let $\phi_\xi(w) := f(w+\xi)$. Therefore $f(B_jx + B_jy) = \phi_{\tilde{\tilde{B}}_jy}(\tilde{B}_jx + \Gamma_jy) = (\tau_{(\Gamma_jy)}\phi_{\tilde{\tilde{B}}_jy})(\tilde{B}_jx)$, where $(\tau_\eta\chi)(\cdot) = \chi(\cdot + \eta)$ denotes translation by η. So,

$$\int_U f(B_j(x,y))H_{jy}(x)\mathrm{d}x = \int_U (\tau_{(\Gamma_jy)}\phi_{\tilde{\tilde{B}}_jy})(\tilde{B}_jx)H_{jy}(x)\mathrm{d}x \leq K_1\|\tau_{(\Gamma_jy)}\phi_{\tilde{\tilde{B}}_jy}\|_1$$

by what we are assuming.

Now, by translation invariance, $\|\tau_{(\Gamma_jy)}\phi_{\tilde{\tilde{B}}_jy}\|_1 = \|\phi_{\tilde{\tilde{B}}_jy}\|_1$. Therefore, letting $\psi(\xi) := \|\phi_\xi\|_1$ for $\xi \in (B_jU)^\perp$,

$$\int_{\mathbb{R}^n} f(B_jz)G_j(z)\mathrm{d}z \leq K_1 \int_{U^\perp} \psi(\tilde{\tilde{B}}_jy)M_j(y)\mathrm{d}y \leq K_1K_2\|\psi\|_1.$$

Finally,

$$\|\psi\|_1 = \int_{(B_jU)^\perp} \left(\int_{B_jU} \phi_\xi(w)\mathrm{d}w \right) \mathrm{d}\xi = \int_{\mathbb{R}^{n_j}} f(z)\mathrm{d}z = 1,$$

and this gives what we wanted.

11. Multilinear Kakeya inequalities revisited

Recall that we have families $\mathcal{P}_j$ of 1-tubes in $\mathbb{R}^n$, and for $P \in \mathcal{P}_j$, its direction $e(P) \in \mathbb{S}^{n-1}$ satisfies $|e(P) - e_j| \leq c_n$ where c_n is a small dimensional constant. The multilinear Kakeya theorem of Guth [29] (see also [22]) states that

$$\Big\| \prod_{j=1}^n \Big(\sum_{P_j \in \mathcal{P}_j} a_{P_j}\chi_{P_j}(x) \Big)^{1/n} \Big\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq C_n \prod_{j=1}^n \Big(\sum_{P_j \in \mathcal{P}_j} a_{P_j} \Big)^{1/n}.$$

This inequality is of the form (7) with $X = \mathbb{R}^n$, $q = n/(n-1)$, $Y_j = \mathcal{P}_j$ with counting measure, $p_j = 1$ for all j, $\alpha_j = 1/n$ for all j, and $T((a_{P_j}))(x) = \sum_{P_j \in \mathcal{P}_j} a_{P_j}\chi_{P_j}(x)$.

Guth proved this result essentially by establishing a suitable subfactorisation for each nonnegative $M \in L^n(\mathbb{R}^n)$, and then applying Proposition 1.1. His subfactorisation is described in terms of an auxiliary polynomial p of 'low' degree dominated by $\|M\|_n$, whose zero set Z_p has 'large' **visibility** on each unit cube Q of $\mathbb{R}^n$ in the sense that $\mathrm{vis}(Z_p \cap Q) \gtrsim \int_Q M$. We do not enter into the details of the definition of visibility, nor into how this gives the desired subfactorisation, but instead refer the reader to [29] and [22]. (In the latter paper the approach using Proposition 1.1 is explicit while in the former it is implicit. And one should note that the definition of visibility used in [22] is a power of the original one used in [29].) It was the shock of seeing such an unlikely functional-analytic method succeed which inspired us to study the general question of necessity of factorisation as taken up in this paper. In hindsight, our linkage of subfactorisation of functions with Maurey's theory of factorisation of operators helps place Guth's method in perspective.

Bourgain and Guth in [16] established an affine invariant form of the multilinear Kakeya inequality, removing the hypothesis that $|e(P) - e_j| \leq c_n$ for $P \in \mathcal{P}_j$, at the price of inserting a damping factor on the left-hand side which is consistent with the affine-invariant Loomis–Whitney inequality of Section 9.2. That is, they proved that for $\mathcal{P}_j$ arbitrary families of 1-tubes,

$$\int_{\mathbb{R}^n}\left(\sum_{P_1\in\mathcal{P}_1} a_{P_1}\chi_{P_1}(x)\cdots\sum_{P_n\in\mathcal{P}_n} a_{P_n}\chi_{P_n}(x)e(P_1)\wedge\cdots\wedge e(P_n)\right)^{1/(n-1)}\mathrm{d}x$$
$$\leq C_n\prod_{j=1}^n\left(\sum_{P_j\in\mathcal{P}_j} a_{P_j}\right)^{1/(n-1)}.$$

As the reader will readily verify (using the same argument as in the proof of Proposition 1.1, see also Section 8 above), in order to establish this, it suffices to show that for every nonnegative $M\in L^n(\mathbb{R}^n)$ which is constant on unit cubes in a standard lattice $\mathcal{Q}$, there exist nonnegative functions $S_j:\mathcal{Q}\times\mathcal{P}_j\to\mathbb{R}$ such that

$$M(Q)\lesssim\frac{S_1(Q,P_1)^{1/n}\cdots S_n(Q,P_n)^{1/n}}{e(P_1)\wedge\cdots\wedge e(P_n)^{1/n}}$$

whenever the 1-tubes P_j meet at Q, and, for all j, for all $P_j\in\mathcal{P}_j$,

$$\sum_{Q\in\mathcal{Q},\,Q\cap P_j\neq\emptyset} S_j(Q,P_j)\lesssim\left(\sum_{Q\in\mathcal{Q}} M(Q)^n\right)^{1/n}.$$

And indeed this is what Bourgain and Guth essentially did (see also [22]). It is therefore very tempting to ask whether, in analogy with the situation of Theorem 1.3, this method is *guaranteed* to work in so far as the statement of the affine-invariant multilinear Kakeya inequality automatically implies the existence of a subfactorisation as in the last two displayed inequalities. Unfortunately, as we have established above in Section 8, there is no such general functional-analytic principle which guarantees this.

The recent multilinear Kakeya k_j-plane inequalities, and indeed the even more general perturbed Brascamp–Lieb inequalities, both recently established by Zhang [45], also fit into the framework we consider, the latter as a generalisation of inequality (5).

11.1. **The finite field multilinear Kakeya inequality.** Zhang [46] has recently solved the discrete analogue of the multilinear Kakeya problem. Let $\mathbb{F}$ be a field and let $\mathcal{L}_j$ be arbitrary families of lines in $\mathbb{F}^n$. For $l_j\in\mathcal{L}_j$ declare $e(l_1)\wedge\cdots\wedge e(l_n)$ to be 1 if the vectors $\{e(l_j)\}$ are linearly independent and to be 0 otherwise. Zhang has proved that for a certain C_n depending only on n,

$$\sum_{x\in\mathbb{F}^n}\left(\sum_{l_1\in\mathcal{L}_1} a_{l_1}\chi_{l_1}(x)\cdots\sum_{l_n\in\mathcal{L}_n} a_{l_n}\chi_{l_n}(x)e(l_1)\wedge\cdots\wedge e(l_n)\right)^{1/(n-1)}$$
$$\leq C_n\prod_{j=1}^n\left(\sum_{l_j\in\mathcal{L}_j} a_{l_j}\right)^{1/(n-1)}. \tag{65}$$

When $n=2$ the constant $C_2=1$, as is readily verified using $1/(n-1)=1$ and changing the order of summation on the left-hand side. Moreover, for general n, if all the lines in $\mathcal{L}_j$ are parallel to some fixed vector y_j with $\{y_j\}_{j=1}^n$ linearly independent, the constant is likewise 1, since matters can then be reduced to the classical Loomis–Whitney inequality via an invertible linear transformation of $\mathbb{F}^n$, (or one can write down a suitable factorisation as in Example 9.2).

The presence of the factor $e(l_1)\wedge\cdots\wedge e(l_n)$ in (65) precludes any assertion that (65) is equivalent to a factorisation statement: see Section 8 above. If however the $\mathcal{L}_j$ are presumed to satisfy the property that if $(l_1,\ldots,l_n)\in\mathcal{L}_1\times\cdots\times\mathcal{L}_n$, then the directions $\{e(l_1),\ldots,e(l_n)\}$ are linearly independent, we have that the term $e(l_1)\wedge\cdots\wedge e(l_n)$ is identically 1, and the result then falls under the scope of Theorem 2.2.

In particular, when $n=2$ and we have two finite families of lines $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathbb{F}^2$ such that no line in $\mathcal{L}_1$ is parallel to any line in $\mathcal{L}_2$, this holds. Hence we obtain:

Proposition 11.1. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be finite families of lines in $\mathbb{F}^2$ such that no line in $\mathcal{L}_1$ is parallel to any line in $\mathcal{L}_2$. Let $J\subseteq\mathbb{F}^2$ be the set of points where some $l_1\in\mathcal{L}_1$ meets an $l_2\in\mathcal{L}_2$. Suppose $\sum_{x\in J}G(x)^2=1$. Then there exist $g_1,g_2:J\to\mathbb{R}_+$ such that for all $x\in J$*

$$G(x)=\sqrt{g_1(x)g_2(x)}$$

and, moreover, for all $l_j\in\mathcal{L}_j$, $j=1,2$,

$$\sum_{x\in J\cap l_j}g_j(x)\le 1.$$

In spite of the extreme simplicity of the original problem, no procedure for coming to an explicit such factorisation is currently known.

Jon Bennett had asked whether, even in higher dimensions, the constant C_n in the finite field multilinear Kakeya inequality might still be 1. This is true in the case of $\mathbb{F}_2^3$. However, this turns out to have been over-optimistic, and we have:

Proposition 11.2. *Suppose the discrete multilinear Kakeya inequality (65) holds in the case $n=3$ for $\mathbb{F}=\mathbb{F}_3$. Then $C_3>1.04$.*

We remark that Tidor, Yu and Zhao [40] have very recently established numerical values for the constants in Zhang's theorem, and in particular they show that $C_3\le\sqrt{6}$.

Proof. We construct an example. In this example, for each $j=1,2,3$, we nominate two directions, and the family $\mathcal{L}_j$ will consist of all lines with one of these directions. The two directions for each j will be chosen so that each of the eight choices of one direction from each of the three families results in a linearly independent set of directions, so that the terms $e(l_1)\wedge e(l_2)\wedge e(l_3)$ are all 1. Each family of coefficients a – as a function defined on $\mathcal{L}_j$ and more properly denoted by a_j – is defined to be supported on three lines from $\mathcal{L}_j$ in such a way that the x-summand on the left-hand side of (65) is non-zero at five points. Each a will take nonzero values in $\{1,2\}$ and thus each line under consideration will have a *weight* equal to 1 or 2. For each j we shall have that two of the three lines pass through two of these five points and the remaining line passes through the remaining point.

More concretely, let

- $\mathcal{L}_1$ be the lines with direction $(1,1,0)$ or $(2,1,1)$;
- $\mathcal{L}_2$ be the lines with direction $(0,1,0)$ or $(0,1,1)$; and
- $\mathcal{L}_3$ be the lines with direction $(1,0,1)$ or $(0,0,1)$.

It is straightforward to verify that the directions of any three lines, one from each collection, span $\mathbb{F}_3^3$.

We now proceed to properly define the coefficients a. We denote by a_j the function whose domain is $\mathcal{L}_j$, and which is defined as follows:

- Let a_1 be
  - 2 on the line with direction $(1,1,0)$ passing through $(0,2,2)$ and $(2,1,2)$,

  - 2 on the line with direction $(2, 1, 1)$ passing through $(0, 2, 1)$ and $(2, 0, 2)$,
  - 1 on the line with direction $(1, 1, 0)$ through $(0, 0, 0)$, and
  - 0 on other lines of $\mathcal{L}_1$.
- Let a_2 be
  - 2 on the line with direction $(0, 1, 0)$ passing through $(2, 0, 2)$ and $(2, 1, 2)$,
  - 2 on the line with direction $(0, 1, 1)$ passing through $(0, 0, 0)$ and $(0, 2, 2)$,
  - 1 on the line with direction $(0, 1, 0)$ through $(0, 2, 1)$, and
  - 0 on other lines of $\mathcal{L}_2$.
- Let a_3 be
  - 2 on the line with direction $(0, 0, 1)$ passing through $(0, 2, 1)$ and $(0, 2, 2)$,
  - 2 on the line with direction $(1, 0, 1)$ passing through $(0, 0, 0)$ and $(2, 0, 2)$,
  - 1 on the line with direction $(0, 0, 1)$ through $(2, 1, 2)$, and
  - 0 for other lines of of $\mathcal{L}_3$.

Each $\mathcal{L}_j$ has two lines of a-value or weight 2 and one of weight 1.

We can see that the only points where lines from all three families intersect are the five points mentioned, namely $(0, 0, 0)$, $(0, 2, 1)$, $(0, 2, 2)$, $(2, 0, 2)$ and $(2, 1, 2)$. At the three points $(0, 0, 0)$, $(0, 2, 1)$ and $(2, 1, 2)$ we have two lines of weight 2 and one of weight 1 meeting; at the two points $(0, 2, 2)$ and $(2, 0, 2)$ we have three lines of weight 2 meeting. So the value of the x-summand on the left-hand side of (65) is 2 at the three points $(0, 0, 0)$, $(0, 2, 1)$ and $(2, 1, 2)$, and is $2^{3/2}$ at the two points $(0, 2, 2)$ and $(2, 0, 2)$. The left-hand side adds up to $3 \cdot 2 + 2 \cdot 2^{3/2} > 11.65$. The value of the right-hand side of (65) is $C_3 \cdot 5^{3/2} \leq C_3 \cdot 11.19$. This shows that $C_3 \geq \frac{6+2^{3/2}}{5^{3/2}} > 11.65/11.19 > 1.04 > 1$.

□

A counterexample to the conjecture that (65) holds with $C_n = 1$ was first found by use of the duality theory developed above, which, as we have mentioned, is valid under the assumption that any n-tuple of lines taken from $\mathcal{L}_1 \times \cdots \times \mathcal{L}_n$ has linearly independent directions. To explain why this route was taken, let us assume that we are considering a finite field of size q. If we let $\mathcal{L}_j$ consist of all lines with directions in some given set of size r then the input to (65), namely the tuple $(a_1, \ldots, a_n)$ belongs to a real vector space of dimension $nq^{n-1}r$. The input to problem (17) is the function G which belongs to a real vector space of dimension q^n. In our case we have $n = q = 3$ and $r = 2$ so the input to the problem (17) belongs to a smaller vector space than the input to (65). The additional cost of solving the convex optimisation problem compared with the cost of simply evaluating each side of (65) does not significantly alter the balance of cost.

The solution to the convex optimisation problem was found using the software package CVXOPT [1], which yields the solution for both the primal and dual problems. The solution to the dual problem was then slightly simplified by hand for neater exposition and this is what is presented here.

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